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Non-Iterative DOA Estimation Using Discrete Fourier Transform Interpolation

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ABSTRACT A fast and accurate non-iterative direction-of-arrival (DOA) estimation algorithm for multiple targets in additive white Gaussian noise is devised in this paper. The proposed estimator makes use of the two highest magnitudes discrete Fourier transform (DFT) coefficients of the input data and two of their associated neighboring bins, resulting in a deterministic complexity of $O(N(1 + \log(N)))$ with $N$ being the number of sensors. The bias and mean squares error of the DOA estimates are analyzed. The simulation results are presented to validate the correctness of theoretical derivation and demonstrate the superiority of the devised estimator over several conventional DOA estimators.

INDEX TERMS Direction-of-arrival, single snapshot, interpolation, discrete Fourier transform.

I. INTRODUCTION

Direction-of-arrival (DOA) estimation can be found in numerous areas such as single-input single-output [1], multiple-input single-output radar/sonar range-Doppler imaging [2] and array processing [3]–[8], which refers to accurately finding the locations of sources using a finite set of noisy observations in terms of either parametric or nonparametric methodologies [9], [10]. In the former one, the signal is assumed to be described as a known function, and no assumptions are made on the signal in the latter approach. The parametric algorithms usually allow the derivation of the optimal estimators, but the performance may deteriorate when the assumed signal model and actual one are mismatched. Although the nonparametric ones may not provide the optimum estimation performance, it can be utilized in more applications even when there is no prior knowledge of the signal.

Classical nonparametric methods such as Capon [11]–[13], multiple signal classification (MUSIC) [14], [15] and estimation of signal parameters via rotation invariance techniques (ESPRIT) [16] are popular solvers. However, they involve extensive computational cost because peak searching is required. Amplitude and phase estimator (APES) is suggested in [17] and [18], which can accurately locate multiple sources using multiple snapshots. Although these methods can obtain accurate DOA estimation in the case of high signal-to-noise ratio (SNR) or numerous snapshots, their computational complexity is very high.

To alleviate the computational complexity problem, discrete Fourier transform (DFT) on noisy measurements can be applied. Several two-step algorithms have been suggested [19]–[21], where the coarse DOA estimates determined in the DFT step are refined in the second step by a spectral peak interpolation method in the spectral domain. Although the two-step approach can achieve high estimation accuracy with low computational complexity, it has not yet been applied to multiple-target DOA estimation. In [22], the observed time-domain sequence is first preprocessed by different windows, and the largest-magnitude and its neighbors of the DFT coefficients of windowed data are utilized to provide an unbiased estimation. Candan [23] proposes an unbiased DFT interpolation method, which employs the Taylor series expansion (TSE) on the largest-magnitude DFT coefficient and the neighbor bins. As the approximation of TSE is utilized, it cannot be extended to the multiple-target DOA estimation directly. Furthermore, [24] suggests an iterative interpolation method, which uses all DFT coefficients.
For existing interpolation methods, they are devised according to the single-target model and thus cannot be applied directly to multiple-target scenario. A DOA estimator known as non-iterative fast method (NIFM) is devised in this paper, which can estimate the DOAs with low and deterministic complexity. A two-target scenario is taken as an illustration, which can be applied in the area of the long linear microphone array [25]. Our method is based on the two-step approach provided in [26], where the fine-tune procedure is illustrated, which can be applied in the area of the long linear array [25]. Our method is based on the two-step approach provided in [26].

The observed single-snapshot data of the \( n \)-th target, \( y_n \), is given by:

\[
y_n = \sum_{p=1}^{P} A_p e^{j2\pi nd \cos(\theta_p)/\lambda} + q_n,
\]

where \( \theta_p \in [0^\circ, 180^\circ] \) is the azimuth angle corresponding to the \( p \)-th target, \( \lambda \) denotes the wavelength, \( d \) is the distance between two adjacent sensors with the value of \( \lambda/2 \) [28] and \( q_n \) is the independent identically distributed (IID) complex noise term following the zero-mean white Gaussian distribution with unknown variance \( \sigma^2 \). Our task is estimating \( \{\theta_p\}_{p=1}^{P} \) from observations \( \{y_n\}_{n=0}^{N-1} \). Let \( \varpi_p = -\pi \cos(\theta_p) \), the signal model in (1) can be rewritten as:

\[
y_n = \sum_{p=1}^{P} A_p e^{j\varpi_p n} + q_n. \tag{2}
\]

Since \( \varpi_p \) and \( \theta_p \) are one-to-one mapping relationship, the DOA estimation task is converted to finding \( \{\varpi_p\}_{p=1}^{P} \) from observations \( \{y_n\}_{n=0}^{N-1} \).

Consider the \( N \)-DFT on \( \{y_n\}_{n=0}^{N-1} \). The \( k \)-th DFT coefficient, referred to as \( Y_k \), is expressed as:

\[
Y_k = \sum_{n=0}^{N-1} y_n \exp(-j\omega_k n), \quad k = 0, 1, \ldots, N-1. \tag{3}
\]

The signal component is given by:

\[
S_k = \sum_{p=1}^{P} A_p \exp \left( j \frac{\varpi_p - \omega_k}{2} (N-1) \right) \frac{\sin \left( \frac{\varpi_p - \omega_k}{2} N \right)}{\sin \left( \frac{\varpi - \omega_k}{2} N \right)}, \tag{4}
\]

where \( \omega_k = -\pi + \frac{2\pi k}{N} \) and \( Q_k \) denote the noise components associated with the DFT coefficients. Let \( L_p \) \( (p = 1, 2, \ldots, P) \) be the \( P \)-th largest-magnitude peak indices among \( \{Y_k\}_{k=0}^{N-1} \), so that the true values of \( \varpi_p \) are represented as:

\[
\varpi_p = -\pi + \frac{2\pi (L_p + \delta_p)}{N}, \quad p = 1, 2, \ldots, P, \tag{5}
\]

where \(-0.5 \leq \delta_p \leq 0.5 \) denote the offsets between the index of \( \varpi_p \) from the bins at \( L_p \), respectively. Since \( \{L_p\}_{p=1}^{P} \) are obtained according to DFT, the task is now converted to estimate \( \{\delta_p\}_{p=1}^{P} \) from observations \( \{y_n\}_{n=0}^{N-1} \). Employing (4)–(5), the DFT coefficient \( S_k \) corresponding to \( m \)-th \( (m = 1, 2, \ldots, P) \) peak and its neighbors has the form of:

\[
S_{L_m} = \sum_{p=1}^{P} \left\{ A_p \exp \left( j \frac{\pi (N-1)}{N} (L_p + \delta_p) \right) \right\}, \tag{6}
\]

\[
S_{L_{m-1}} = \sum_{p=1}^{P} \left\{ A_p \exp \left( j \frac{\pi (N-1)}{N} (L_p + \delta_p + 1) \right) \right\}, \tag{7}
\]

\[
S_{L_{m+1}} = \sum_{p=1}^{P} \left\{ A_p \exp \left( j \frac{\pi (N-1)}{N} (L_p + \delta_p - 1) \right) \right\}. \tag{8}
\]

where \( L_{pm} = L_p - L_m \) and \( \cdot \) denote the scalar product operator. In the scenario of large \( N \), with the use of the fact that

**FIGURE 1.** Illustration of uniform linear array.
\[ \exp(\pi(K + x)) \sin(\pi(K + x)) = \exp(\pi x) \sin(\pi x), \] 
(6)--(8) can be simplified as

\[ S_{Lm} = \sum_{p=1}^{P} A_p \exp\left(j\pi \delta_p\right) \sin\left(\frac{\pi \delta_p}{N} (L_{pm} + \delta_p)\right), \]
(9)

\[ S_{Lm-1} = \sum_{p=1}^{P} A_p \exp\left(j\pi \delta_p\right) \sin\left(\frac{\pi \delta_p}{N} (L_{pm} + \delta_p + 1)\right), \]
(10)

\[ S_{Lm+1} = \sum_{p=1}^{P} A_p \exp\left(j\pi \delta_p\right) \sin\left(\frac{\pi \delta_p}{N} (L_{pm} + \delta_p - 1)\right). \]
(11)

Furthermore, (9)--(11) are

\[ \prod_{\ell=1}^{P} \sin\left(\frac{\pi}{N} (L_{\ell m} + \delta_\ell)\right) S_{p_{m-1}} \]
(12)

\[ = \sum_{p=1}^{P} A_p \exp\left(j\pi \delta_p\right) \sin\left(\frac{\pi \delta_p}{N} (L_{\ell m} + \delta_\ell)\right), \]

\[ \prod_{\ell=1}^{P} \sin\left(\frac{\pi}{N} (L_{\ell m} + \delta_\ell + 1)\right) S_{p_{m-1}} \]
(13)

\[ = \sum_{p=1}^{P} A_p \exp\left(j\pi \delta_p\right) \sin\left(\frac{\pi \delta_p}{N} (L_{\ell m} + \delta_\ell + 1)\right), \]

\[ \prod_{\ell=1}^{P} \sin\left(\frac{\pi}{N} (L_{\ell m} + \delta_\ell - 1)\right) S_{p_{m+1}} \]
(14)

\[ = \sum_{p=1}^{P} A_p \exp\left(j\pi \delta_p\right) \sin\left(\frac{\pi \delta_p}{N} (L_{\ell m} + \delta_\ell - 1)\right). \]

According to the product-to-sum property of trigonometric functions, there exists three coefficients \(a_0, a_1\) and \(a_2\) such that

\[ a_1 \prod_{\ell=1}^{P} \sin\left(\frac{\pi}{N} (L_{\ell m} + \delta_\ell)\right) S_{p_m} \]

\[ + a_2 \prod_{\ell=1}^{P} \sin\left(\frac{\pi}{N} (L_{\ell m} + \delta_\ell + 1)\right) S_{p_{m-1}} \]

\[ = a_0 \prod_{\ell=1}^{P} \sin\left(\frac{\pi}{N} (L_{\ell m} + \delta_\ell - 1)\right) S_{p_{m+1}}, \]
(15)

where \(-0.5 \leq \delta_1, \delta_2 \leq 0.5\) denote the offsets between the true values from the bin values at \(L_1\) and \(L_2\), respectively. As \(L_1\) and \(L_2\) are straightforwardly obtained by DFT, the estimation task is converted to finding \(\delta_1\) and \(\delta_2\). Let \(\Delta \delta = \delta_2 - \delta_1\) and \(L = L_2 - L_1\). Considering the first two terms in (3), the DFTs on \(s_{\nu_{m}}\) with peaks at \(S_{\nu_{1}}\) and \(S_{\nu_{2}}\) are

\[ S_{\nu_{1}} = A_1 N + A_2 \exp\left(j\left(\frac{N-1}{2N} \pi \Delta \delta - \frac{\pi}{N} \left(1 + \frac{\nu_{1}}{N}\right)\right)\right) \sin\left(\frac{\pi \nu_{1}}{N} (\Delta \delta + L)\right), \]
(18)

\[ S_{\nu_{2}} = A_2 N + A_1 \exp\left(-j\left(\frac{N-1}{2N} \pi \Delta \delta - \frac{\pi}{N} \left(1 + \frac{\nu_{2}}{N}\right)\right)\right) \sin\left(\frac{\pi \nu_{2}}{N} (\Delta \delta + L)\right). \]
(19)

It can be seen from (18) and (19) that \(S_{\nu_{p}}\) (\(p = 1, 2\)) are influenced by both \(\nu_{1}\) and \(\nu_{2}\). Note that the existing interpolation schemes with assumption of single-target model cannot provide the satisfactory estimation performance since they ignore this information.

In the absence of noise term \(Q_\delta\), the first peak and its neighboring bins, namely, \(L_1\)-th, \((L_1 - 1)\)-th and \((L_1 + 1)\)-th DFT coefficients, are given by

\[ Y_{L_1} = \frac{\gamma_1}{\sin\left(\frac{\pi \delta_1}{N}\right)} + \frac{\gamma_2}{\sin\left(\frac{\pi (L_1 + \delta_2)}{N}\right)}, \]
(20)

\[ Y_{L_1 - 1} = \exp\left(-j\frac{\pi}{N}\right) \left(\frac{\gamma_1}{\sin\left(\frac{\pi (\delta_1) + 1}{N}\right)} + \frac{\gamma_2}{\sin\left(\frac{\pi (L_1 + \delta_2)}{N}\right)}\right), \]
(21)

\[ Y_{L_1 + 1} = \exp\left(j\frac{\pi}{N}\right) \left(\frac{\gamma_1}{\sin\left(\frac{\pi (\delta_1 - 1)}{N}\right)} + \frac{\gamma_2}{\sin\left(\frac{\pi (L_1 + \delta_2 - 1)}{N}\right)}\right). \]
(22)

where

\[ \gamma_1 = A_1 \exp\left(j\pi \delta_1 \frac{N - 1}{N}\right) \sin(\pi \delta_1), \]
(23)

\[ \gamma_2 = B_2 \exp\left(j\pi \frac{L_2 + \delta_2}{N}\right) \sin(\pi \delta_2), \]
(24)

and \(B_2 = A_2 \exp\left(-j\frac{\pi}{N}\right)\). Then (20)--(22) satisfy

\[ \sin\left(\frac{\pi \delta_1}{N}\right) \sin\left(\frac{\pi (L_1 + \delta_2)}{N}\right) \exp\left(j\frac{\pi}{N}\right) Y_{L_1 - 1} \]

\[ + \sin\left(\frac{\pi (\delta_1 - 1)}{N}\right) \sin\left(\frac{\pi (L_1 + \delta_2 - 1)}{N}\right) \exp\left(-j\frac{\pi}{N}\right) Y_{L_1 + 1} \]

\[ = 2 \cos\left(\frac{\pi}{N}\right) \sin\left(\frac{\pi \delta_1}{N}\right) \sin\left(\frac{\pi (L_1 + \delta_2)}{N}\right) Y_{L_1}. \]
(25)
Similarly, the neighbors of the $L_2$-th bin satisfy the following relationship:

$$\sin\left(\frac{\pi(\delta_1 + 1 - L)}{N}\right) \sin\left(\frac{\pi(\delta_2 + 1)}{N}\right) \exp\left(\frac{j\pi}{N} Y_{L_2-1}\right) + \sin\left(\frac{\pi(\delta_1 - 1 - L)}{N}\right) \sin\left(\frac{\pi(\delta_2 - 1)}{N}\right) \exp\left(-\frac{j\pi}{N} Y_{L_2+1}\right) = 2 \cos\left(\frac{\pi}{N}\right) \sin\left(\frac{\pi(\delta_1 - L)}{N}\right) \sin\left(\frac{\pi\delta_2}{N}\right) Y_{L_2}. \tag{26}$$

Let $\mu_1 = \tan\left(\frac{\pi\delta_1}{N}\right)$ and $\mu_2 = \tan\left(\frac{\pi\delta_2}{N}\right)$. Knowing that $\mu_1$ and $\mu_2$ are real numbers, therefore, we can consider the real part of the DFT coefficients alone as

$$\cos \{\mu_1\} \text{ and } \cos \{\mu_2\} \text{ on (25)--(26), we have}$$

$$r_0 = \text{Re}\{Y_{L_1}\}, \quad \ell_0 = \text{Re}\{Y_{L_2}\},$$

$$r_1 = \exp\left(\frac{\pi}{N} Y_{L_1-1}\right), \quad r_2 = \exp\left(-\frac{\pi}{N} Y_{L_1+1}\right),$$

$$\ell_1 = \exp\left(\frac{\pi}{N} Y_{L_2-1}\right), \quad \ell_2 = \exp\left(-\frac{\pi}{N} Y_{L_2+1}\right),$$

$$C_1 = \tan\left(\frac{\pi}{N}\right), \quad C_2 = \tan\left(\frac{\pi(L + 1)}{N}\right),$$

$$C_3 = \tan\left(\frac{\pi(L - 1)}{N}\right), \quad C_4 = \tan\left(\frac{\pi L}{N}\right),$$

$$D_1 = \cos\left(\frac{\pi L}{N}\right), \quad D_2 = \cos\left(\frac{\pi(L + 1)}{N}\right),$$

$$D_3 = \cos\left(\frac{\pi(L - 1)}{N}\right), \quad B = \frac{2D_1}{D_2 D_3}. \tag{28}$$

where $\text{Re}\{x\}$ denotes the real part of $x$. To remove the term $\mu_1\mu_2$, (27), as shown at the bottom of this page can be simplified as two quadratic equations, which are

$$\mu_2^2 (x^T Az) + \mu_2 (x^T Bz) + x^T Cz = 0, \tag{30}$$

$$\mu_1^2 (x^T Dz) + \mu_1 (x^T Wz) + x^T Qz = 0, \tag{31}$$

where (32)–(38), shown at the top of the next page.

Solving (30) and (31) using the roots of quadratic equation allows us to estimate $\mu_1$ and $\mu_2$, namely, $\hat{\mu}_1$ and $\hat{\mu}_2$ as

$$\hat{\mu}_2 = \frac{-x^T Bz + \sqrt{(x^T Bz)^2 - 4(x^T Az)(x^T Cz)}}{2(x^T Az)}, \tag{39}$$

$$\hat{\mu}_1 = \frac{-x^T Wz + \sqrt{(x^T Wz)^2 - 4(x^T Dz)(x^T Qz)}}{2(x^T Dz)}. \tag{40}$$

Since $|\mu_1|$ and $|\mu_2|$ ($|\mu_1|, |\mu_2| < \tan\left(\frac{\pi}{2\ell_0}\right)$) are small, we select the smaller root as the estimates.

After $\hat{\mu}_1$ and $\hat{\mu}_2$ have been estimated, combining (27)–(40), as shown at the bottom of this page, the estimates of $\theta_1$ and $\theta_2$, referred to as $\hat{\theta}_1$ and $\hat{\theta}_2$, are then:

$$\hat{\theta}_1 = \cos^{-1}\left(1 - \frac{2L_1}{N} - 2\tan^{-1}\left(\frac{\hat{\mu}_1}{\pi}\right)\right), \tag{41}$$

$$\hat{\theta}_2 = \cos^{-1}\left(1 - \frac{2L_2}{N} - 2\tan^{-1}\left(\frac{\hat{\mu}_2}{\pi}\right)\right). \tag{42}$$

To further improve the performance, we divide $\delta_1$ or $\delta_2$ into different subrange without additional computational complexity. Actually, we can divide the subrange into any ranges, however, increasing the number of subranges will also result in the higher computational complexity, which is not desirable. Taking nine subranges as an example, the accuracy and computational cost will be as high as performing the algorithm twice. Since the iterative application of the proposed algorithm is equivalent to using more subranges, we choose dividing the offset into three subranges.

To choose the subrange, it must satisfy three conditions: first, the union of all subranges should cover the whole ranges of each offset; second, the length of subrange should be equal to each other to avoid biased estimation; finally, the overlapped subrange is not a good choice, since it results in biased estimation. Based on the above discussions, we choose the subranges with boundaries of 0.25 and $-0.25$. Take $\delta_1$ as an illustration, and the case for $\delta_2$ follows similarly. The three cases are

$$\delta_1 \in \begin{cases} 
-0.5, & -0.25 > |Y_{L_1-0.25}| > |Y_{L_1}|, \\
0.25, & |Y_{L_1+0.25}| > |Y_{L_1}|, \\
-0.25, & 0.25 > |Y_{L_1-0.25}| > |Y_{L_1}|, \\
otherwise.
\end{cases} \tag{43}$$

The reduced estimation range of $\delta_1$ or $\delta_2$ will help to reduce the MSE of $\hat{\theta}_1$ and $\hat{\theta}_2$. The details of the NIFM algorithm are shown in Tables 1 and 2.

<table>
<thead>
<tr>
<th>TABLE 1. Estimation algorithm with $\theta_1$ in three scenarios.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I) Obtain $Y_0$ using N-DFT and find peak index $L_1$;</td>
</tr>
<tr>
<td>(II) Choose the location of $\delta_1$ using (27);</td>
</tr>
<tr>
<td>(III) Obtain $\hat{\theta}_1$ using (27)–(40);</td>
</tr>
<tr>
<td>For $\delta_1 \in [-0.25, 0.25]$</td>
</tr>
<tr>
<td>(I) Obtain $Y_0$ using (27)–(40);</td>
</tr>
<tr>
<td>(II) Obtain $\hat{\theta}_1$ using (27)–(40);</td>
</tr>
<tr>
<td>(III) Update observations $y_n$ using $y_n \exp\left(j \frac{\pi}{N} \theta_1\right)$;</td>
</tr>
<tr>
<td>(IV) Calculate new $Y_{L_1}$ and $Y_{L_2}$ using (27)–(40);</td>
</tr>
<tr>
<td>(V) Obtain $\hat{\theta}_1$ using (27)–(40);</td>
</tr>
<tr>
<td>For $\delta_1 \in [0.25, 0.5]$</td>
</tr>
<tr>
<td>(I) Obtain $Y_0$ using (27)–(40);</td>
</tr>
<tr>
<td>(II) Update observations $y_n$ using $y_n \exp\left(-j \frac{\pi}{N} \theta_1\right)$;</td>
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</tr>
<tr>
<td>(V) Obtain $\hat{\theta}_1$ using (27)–(40);</td>
</tr>
</tbody>
</table>
III. PERFORMANCE AND COMPUTATIONAL COMPLEXITY ANALYSIS

A. BIAS AND VARIANCE ANALYSIS

In this section, we analyze the bias and variance of \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \). To simplify the problem, our discussion focuses on the range of \( \delta_1, \delta_2 \in [-0.25, 0.25] \). For the other subranges such as \( \delta_1, \delta_2 \in [-0.5, -0.25] \), according to the subrange criterion in Section 2, we introduce two new variables \( \delta_{1\text{new}} = \delta_1 + 0.5 \) and \( \delta_{2\text{new}} = \delta_2 + 0.5 \) \( (\delta_{1\text{new}}, \delta_{2\text{new}} \in [0, 0.25]) \). With the use of analysis in this section as well as \( \delta_{1\text{new}} \) and \( \delta_{2\text{new}} \), the variances of \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \), corresponding to the case of \( \delta_1, \delta_2 \in [-0.5, -0.25] \), can be derived.

Since \( \mu_1 = \tan \left( \frac{\pi \delta_1}{N} \right) \) and \( \mu_2 = \tan \left( \frac{\pi \delta_2}{N} \right) \), the variances of \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) can be derived by those of \( \hat{\mu}_1 \) and \( \hat{\mu}_2 \). To analyze the performance of \( \hat{\mu}_2 \), we shall first note that \( \hat{\mu}_2 \) is actually obtained by solving

\[
 f(\hat{\mu}_2) = 0,
\]

where \( f(\hat{\mu}_2) = \mu_2^2 (x^T A z) + \mu_2 (x^T B z) + x^T C z \). According to the definitions in (32)–(38), we have

\[
 B = AH + FA, \quad C = FAH.
\]
TABLE 2. Estimation algorithm with $\theta_2$ in three scenarios.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>Obtain $Y_2$ using N-DFT and find peak index $L_2$;</td>
</tr>
<tr>
<td>(II)</td>
<td>Choose the location of $\theta_2$ using (??);</td>
</tr>
<tr>
<td>For $\theta_2 \in [-0.25, 0.25]$</td>
<td>Obtain $\hat{\theta}_2$ using (??)–(??);</td>
</tr>
<tr>
<td>(III)</td>
<td>Obtain $\hat{\theta}_2$ using (??)–(??);</td>
</tr>
<tr>
<td>For $\theta_2 \in [-0.5, -0.25]$</td>
<td>Update observations $y_n$, using $y_n \exp (j \frac{\pi}{2} \frac{n}{N})$;</td>
</tr>
<tr>
<td>(III.1)</td>
<td>Calculate new $Y_{L_2}$ and $Y_{L_2=1}$ using (??)–(??);</td>
</tr>
<tr>
<td>(III.2)</td>
<td>Obtain $\hat{\theta}_2$ using (??)–(??);</td>
</tr>
<tr>
<td>(III.3)</td>
<td>Update observations $y_n$, using $y_n \exp (-j \frac{\pi}{2} \frac{n}{N})$;</td>
</tr>
<tr>
<td>(III)</td>
<td>Calculate new $Y_{L_2}$ and $Y_{L_2=1}$ using (??)–(??);</td>
</tr>
</tbody>
</table>

where

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & -C_1 \end{bmatrix}, \quad H = \begin{bmatrix} C_4 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}. \quad (46)$$

Combining (45)–(46), $f(\hat{\mu}_2)$ is given by

$$f(\hat{\mu}_2) = x^T K z \quad (47)$$

where $K = (\mu_2 I_3 + F)A(\mu_2 I_3 + H)$ with $I_3$ denoting the $3 \times 3$ identity matrix.

Utilizing [29], the bias and MSE of $\hat{\mu}_2$ can be calculated by TSE on $f(\hat{\mu}_2)$ [30]:

$$Bias(\hat{\mu}_2) = E(\hat{\mu}_2) - \mu_2 \approx -E[f(\mu_2)] \quad (48)$$

$$MSE(\hat{\mu}_2) = E[(\hat{\mu}_2 - \mu_2)^2] \approx E\left[\frac{f(\mu_2)}{f'(\mu_2)}\right]^2, \quad (49)$$

where $f'(\mu_2)$ is the first-order derivative of $f(\mu_2)$ on $\mu_2$ and $E(\cdot)$ denotes the expectation operator.

We rewrite $x$ and $z$ in (32) as $x = x_s + x_q$ and $z = z_s + z_q$, where $x_s$ and $z_s$ denote the signal parts, while $x_q$ and $z_q$ are the noise parts. It is also worth pointing out that $x_q$ and $z_q$ are IID noise terms with variance $\sigma^2$. Then $f(\mu_2)$ in (47) can be expressed as

$$f(\mu_2) = x_s^T K z_s + x_s^T K z_q + x_q^T K z_q + x_q^T K z_q. \quad (50)$$

According to the Appendix, we have

$$x_s^T K z_s = 0. \quad (51)$$

Utilizing (25)–(26) and the definition of $f(\mu_2)$, we obtain

$$E[f(\mu_2)] = x_s^T K z_s = 0, \quad (52)$$

$$E[f'(\mu_2)] = x_s^T L z_s, \quad (53)$$

$$E[f'(\mu_2)]^2 = \sigma^2 (N\sigma^2 tr(K^T K) + x_s^T K K^T x_s \quad (54)$$

where $tr(\cdot)$ enotes the matrix trace and

$$L = A(\mu_2 I_3 + H) + (\mu_2 I_3 + F)A. \quad (55)$$

According to (52) and (54), (48) is calculated as

$$Bias(\hat{\mu}_2) \approx 0. \quad (56)$$

With the use of (49) and (53)–(54), $MSE(\hat{\mu}_2)$ is expressed as

$$MSE(\hat{\mu}_2) \approx \sigma^2 N \sigma^2 tr(K^T K x_s) + x_s^T K K^T x_s + N z_s^T K K z_s \quad (57)$$

Similarly, the bias and MSE of $\hat{\mu}_1$, are

$$Bias(\hat{\mu}_1) \approx 0, \quad (58)$$

$$MSE(\hat{\mu}_1) \approx \frac{\sigma^2 N \sigma^2 tr(M^T M) + x_s^T M M^T x_s + N z_s^T M M^T z_s}{(x_s^T T z_s)^2} \quad (59)$$

where

$$M = G_1 \times D \times G_2, \quad (60)$$

$$T = D \times G_2 + G_1 \times D. \quad (61)$$

where

$$G_1 = \begin{bmatrix} \mu_1 - C_4 & 0 & 0 \\ 0 & \mu_1 - C_3 & 0 \\ 0 & 0 & \mu_1 - C_2 \end{bmatrix}, \quad (62)$$

$$G_2 = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 + C_1 & 0 \\ 0 & 0 & \mu_1 - C_1 \end{bmatrix}. \quad (63)$$
Employing the definition in (41)–(42) and according to [31], the MSEs of $\hat{\theta}_1$ and $\hat{\theta}_2$, referred to as $\text{MSE}(\hat{\theta}_1)$ and $\text{MSE}(\hat{\theta}_2)$, have the forms of

$$\text{MSE}(\hat{\theta}_1) = \frac{4}{(1 + \mu_1^2)^2 \left( \pi^2 - (\pi \theta_1)^2 \right)} \text{MSE}(\hat{\mu}_1).$$

(64)

$$\text{MSE}(\hat{\theta}_2) = \frac{4}{(1 + \mu_2^2)^2 \left( \pi^2 - (\pi \theta_2)^2 \right)} \text{MSE}(\hat{\mu}_2).$$

(65)

**B. COMPUTATIONAL COMPLEXITY ANALYSIS**

The estimation steps of NIFM are summarized as follows:

1. Calculate the spectrum of observations using fast Fourier transform (FFT) with $O(N \log_2(N))$ flops and search for the two magnitude peaks with $O(N)$;
2. Estimate $\mu_1$ and $\mu_2$ using (39) and (40) twice;
3. Estimate the two DOA values $\hat{\theta}_1$ and $\hat{\theta}_2$ using Tables 1 and 2.

It is vivid that the dominant computational complexity is the FFT, and the peak search, which makes the algorithm complexity to be proportional to $O(N(1 + \log_2(N)))$.

**IV. SIMULATIONS**

To verify the performance of the interpolation formulas, computer simulations have been conducted. We employ the empirical MSEs and absolute biases of $\hat{\theta}_1$ and $\hat{\theta}_2$ as the performance metrics, which are defined as $E((\theta_p - \hat{\theta}_p)^2)$ and $|\hat{\theta}_p - E(\theta_p)|$ with $p = 1, 2$, respectively. Two sources locating
at $\theta_1 = 158^\circ$ and $\theta_2 = 126^\circ$ are considered, whose powers are 1 and 4, respectively. And hence, the corresponding peak indices in model (2) are $L_1 = 2$ and $L_2 = 6$, while the offsets are $\delta_1 = -0.12$ and $\delta_2 = 0.018$. The Cramér-Rao lower bound (CRLB) [32] is included as the benchmark while comparisons with Jacobsen [22], Candan [23], MUSIC [33] and ESPRIT [10] methods are also provided. For Jacobsen algorithm, Hanning window is utilized with scaling parameter $0.55$. All results are simulated using Matlab running on Intel(R) Core(TM) i7-4790 CPU@3.60GHz and Windows 7 for 1000 Monte Carlo trials with $N = 24$ sensors.

First of all, we investigate the performance of the proposed method in different noise conditions. The MSEs and absolute biases of $\hat{\theta}_1$ and $\hat{\theta}_2$ versus SNR are plotted in Figures 2 and 3.

It can be observed in Figure 2 that although the proposed method is suboptimal, it can approach CRLB faster than other optimal estimators in the case of low SNR. Figure 3 also verifies this finding since our approach can provide stable estimates when SNR $> 5$ dB, but those of the other methods are only reliable for SNR $> 10$ dB. It is noted that Jacobsen and Candan algorithms are biased in estimating $\theta_1$ and $\theta_2$, since they are devised with the assumption of single-target model.

Second, the MSEs and the computational cost versus sensor number $N$ are plotted in Figures 4 and 5 for the cases of SNR = 12 dB with $N \in [20, 80]$. Here the parameters $L_1, L_2, \delta_1$ and $\delta_2$ are same with the previous experiment. Here, the DOAs are $\theta_1 \in [156^\circ, 168^\circ]$ and $\theta_2 \in [120^\circ, 151^\circ]$. The stopwatch timer is utilized to measure the run times of all methods. It is indicated that in the case of nearly optimal estimation performance, the complexity of the proposed method is significantly lower than the MUSIC and ESPRIT. While the performance of NIFM is more accurate than the Jacobsen and Candan schemes with approximately the same computational cost. Moreover, as the number of sensors increases, the computational complexity of NIFM almost does not change while other methods grow linearly in high rate.

Third, the estimation performance for different $\theta_1$ and $\theta_2$ is examined with $L_1 = 2$, $L_2 = 6$ and the SNR is 12 dB. For different $\theta_1$ in $[151^\circ, 163^\circ]$, $\theta_2$ is fixed to $126^\circ$, while in the case of different $\theta_2$ in $[152^\circ, 156^\circ]$, $\theta_1 = 168^\circ$. It is shown in Figure 6 that the MSEs of all methods approach CRLB in all values of $\theta_1$ and $\theta_2$, except the Jacobsen and Candan algorithms. With the use of the criterion in Table 1, the gap between the proposed method and CRLB is less than 3.5 dB in Figure 6, and does not change much as $\theta_1$ and $\theta_2$ change.
Finally, the estimation performance in the case of different DOAs is studied. We vary \( \theta_2 \) from 95° to 139°, when \( \theta_1 \) is fixed to 158°. Figure 7 shows that in the scenario of closely-spaced peaks, our method can also perform well, indicating the stability of the proposed method. It is worth pointing out that in the case of increasing \( \theta_2 \), the MSEs of the Jacobsen and Candan algorithms become larger. This is because these two methods assume that the signal model is single-target and ignore the influence in dual-target scenario.

In summary, in the scenarios of different SNR, \( N \), \( \theta_1 \) and \( \theta_2 \), the proposed method is nearly optimal, and it has the lowest computational complexity with unbiased estimation performance.

V. CONCLUSION

In this paper, a fast and simple DOA estimator, referred to as NIFM, using the magnitudes of DFT are developed, which has lower computational complexity than that of existing methods. Theoretical analysis is investigated to show the near optimality of the proposed algorithm. Computer simulations show that the performance of the proposed algorithm is similar to that of Jacobsen, Candan, MUSIC and ESPRIT, while it can achieve much lower computational complexity. In particular, the computational complexity of NIFM is not sensitive to data set size increment, when compared to the other algorithms discussed in this paper. It is worth pointing that although our work focuses on the two-target DOA estimation in single-snapshot, it can be also extended to the scenario of multiple targets, even with multiple snapshots.

APPENDIX

Utilizing (32), \( x_t \), and \( z_t \), are

\[
x_t = \begin{bmatrix}
Re \left\{ S_{L_2} \right\} \\
Re \left\{ \exp \left( j \frac{\pi}{N} \right) S_{L_2-1} \right\} \\
Re \left\{ \exp \left( -j \frac{\pi}{N} \right) S_{L_2+1} \right\}
\end{bmatrix}
\]

(66)

\[
z_t = \begin{bmatrix}
Re \left\{ S_{L_1} \right\} \\
Re \left\{ \exp \left( j \frac{\pi}{N} \right) S_{L_1-1} \right\} \\
Re \left\{ \exp \left( -j \frac{\pi}{N} \right) S_{L_1+1} \right\}
\end{bmatrix}
\]

(67)

where \( S_i \) has the form of

\[
S_{L_1} = \frac{\gamma_1}{\sin \left( \frac{\pi \delta_1}{N} \right)} + \frac{\gamma_2}{\sin \left( \frac{\pi (L+\delta_2)}{N} \right)},
\]

\[
S_{L_{1-1}} = \exp \left( j \frac{\pi}{N} \right) \left( \frac{\gamma_1}{\sin \left( \frac{\pi (\delta_1+1)}{N} \right)} + \frac{\gamma_2}{\sin \left( \frac{\pi (L+\delta_2+1)}{N} \right)} \right),
\]

where \( \gamma_1 \) and \( \gamma_2 \) are given by

\[
\gamma_1 = \frac{C_1 S_{L_1} - C_4 S_{L_2}}{C_3 S_{L_1} - C_4 S_{L_2}},
\]

\[
\gamma_2 = \frac{C_1 S_{L_1} - C_4 S_{L_2}}{C_3 S_{L_1} - C_4 S_{L_2}}.
\]

Employing the definitions of \( \mu_1 \) and \( \mu_2 \) as well as (28)–(29) and (72)–(73), we have (74), shown at the top of this page, where

\[
S_{L_1+1} = \exp \left( j \frac{\pi}{N} \right) \left( \frac{\gamma_1}{\sin \left( \frac{\pi (\delta_1+1)}{N} \right)} + \frac{\gamma_2}{\sin \left( \frac{\pi (L+\delta_2+1)}{N} \right)} \right),
\]

(74)

\[
S_{L_2} = \frac{\alpha_1}{\sin \left( \frac{\pi (L+\delta_1)}{N} \right)} + \frac{\alpha_2}{\sin \left( \frac{\pi \delta_1}{N} \right)},
\]

(68)

\[
S_{L_{2-1}} = \exp \left( j \frac{\pi}{N} \right) \left( \frac{\alpha_1}{\sin \left( \frac{\pi (L+\delta_1+1)}{N} \right)} + \frac{\alpha_2}{\sin \left( \frac{\pi (\delta_1+1)}{N} \right)} \right),
\]

(72)

\[
S_{L_{2+1}} = \exp \left( j \frac{\pi}{N} \right) \left( \frac{\alpha_1}{\sin \left( \frac{\pi (L+\delta_1+1)}{N} \right)} + \frac{\alpha_2}{\sin \left( \frac{\pi (\delta_1+1)}{N} \right)} \right),
\]

(69)

with

\[
\alpha_1 = A_1 \exp \left( j \frac{\pi \delta_1}{N} \right) \exp \left( j \frac{\pi (L \delta_1 - 1)}{N} \right) \sin(\pi \delta_1),
\]

(70)

\[
\alpha_2 = A_2 \exp \left( j \frac{\pi \delta_2}{N} \right) \exp \left( j \frac{\pi (L \delta_2 - 1)}{N} \right) \sin(\pi \delta_2).
\]

(71)
To remove the term $\mu_1^2$, with the use of (66)-(67), (74) can be rewritten as

$$\mu_2^2 (x^T A z_x) + \mu_2 (x^T B z_x) + x^T C z_x = 0, \quad (75)$$

Since $x^T K z_x = \mu_2^2 (x^T A z_x) + \mu_2 (x^T B z_x) + x^T C z_x$, we have

$$x^T K z_x = 0. \quad (76)$$

REFERENCES


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