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Feng, Xiaoyi; Xie, Yangyang; Wang, Shouyang; Yan, Houmin

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Optimal structure of joint inventory-pricing management with dual suppliers and different lead times

Xiaoyi Feng [a, 1], Yangyang Xie [b, *, 1], Shouyang Wang [b, c, d], Houmin Yan [d]

[a] Department of Risk Strategy, The Hongkong and Shanghai Banking Corporation Limited, Hong Kong, China
[b] International Institute of Finance, School of Management, University of Science and Technology of China, Hefei, Anhui, 230026, China
[c] University of the Chinese Academy of Sciences, Beijing, 100049, China
[d] Department of Management Sciences, College of Business, City University of Hong Kong, Hong Kong, China

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ABSTRACT

We consider a joint inventory-pricing control problem in a single-product, periodic-review, dual-supplier inventory system. The two suppliers have different lead times. One expedited supplier offers instantaneous replenishment, and one regular supplier requires an L-period lead time for delivery. The supply quantity is stochastic and the demand is price-dependent. For the expedited inventory replenishment, we characterize the optimal policy as a state-dependent almost-threshold policy by extending the stochastically linear in mid-point to a multidimensional setting. To investigate the optimal regular inventory replenishment and pricing policy, we propose the notions of partially stochastic translation (PST) and increasing partially stochastic translation (IPST), which help in obtaining the antimultimodularity preservation in dynamic programming problems. We provide properties, sufficient conditions, and examples for PST and IPST functions. By applying PST and IPST, we obtain the antimultimodularity of the profit functions. The antimultimodular profit functions ensure that the optimal regular ordering quantity and the optimal price are monotone in the current inventory level and outstanding order quantities. Moreover, we reveal that as the time interval increases, the effects of previous outstanding orders on the optimal regular ordering and pricing decisions are decreasing and increasing, respectively. PST and IPST also enable us to further characterize the optimal expedited ordering quantity as decreasing in the inventory level. However, the optimal expedited ordering quantity can be non-monotone with respect to the outstanding order quantities, as shown in the example.

1. Introduction

This study investigates a periodic-review, joint inventory-pricing management problem with dual supply sources. In each period, the retailer places two replenishment orders, i.e., an expedited order with immediate replenishment and a regular

* Corresponding author.
E-mail address: xieyclio@ustc.edu.cn (Y. Xie).
[1] Xiaoyi Feng and Yangyang Xie contributed equally to this work.

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order that requires an L-period lead time. At the same time, a one-period list price is determined to coordinate the demand with the inventory stock such that the maximum expected total profit is attained.

We characterize the optimal expedited order as a state-dependent almost-threshold policy with the joint concavity of the profit functions. The proof of concavity preservation is achieved through extending the notion of stochastically linear in mid-point (SL(mp)) to a multidimensional setting. However, the joint concavity is not enough to characterize the optimal regular ordering and pricing policies. It invites us to develop the terminologies of partially stochastic translation (PST) and increasing partially stochastic translation (IPST), which are applied to the proof of antimultimodularity preservation in dynamic programming problems. We demonstrate a necessary and sufficient condition for antimultimodularity, and provide properties, sufficient conditions, and examples for the PST and IPST functions. With the antimultimodularity of the profit functions, we discover that the optimal regular ordering quantity and list price are monotone with respect to the current inventory level and outstanding order quantities. Moreover, the optimal regular ordering quantity is more sensitive to the outstanding order quantities than to the current inventory level. Among the outstanding orders, the optimal regular ordering quantity is more sensitive to those placed more recently than to those placed earlier. On the contrary, the optimal price is more sensitive to the current inventory level than to the outstanding order quantities. In addition, the optimal price is more sensitive to earlier placed outstanding orders. It implies that the effects of previous outstanding orders on the current regular ordering decision are decreasing over the time (i.e., as the time interval increases), whereas their effects on the pricing decision are increasing over the time. The antimultimodularity also helps in further characterizing the optimal expedited ordering quantity as monotone decreasing in the inventory level. However, differing from the optimal regular ordering quantity, the optimal expedited ordering quantity cannot be monotone in the outstanding order quantities, as illustrated in Example 2.

Our work lies within two main streams of research. The first stream concerns joint inventory and pricing decision making. Federgruen and Heching (1999) first consider the optimal joint pricing and inventory decisions in a multi-period setting and propose the well-known base-stock list-price policy. Subsequent variants include the work on fixed ordering costs by Chen and Simchi-Levi (2004), the work on random yield rates by Li and Zheng (2006), the extension with both the ordering costs and lost sales by Song et al. (2009), the model of a multiechelon inventory system by Chao and Zhou (2009), and the work on positive lead times by Pang et al. (2012). Several recent developments include the batch ordering model (Yang et al., 2014), quasi-concave model (Chen and Zhang, 2014), model with reference effect (Güler et al., 2014), and model with perishable products (Chen et al., 2014). All of these studies aim to characterize the optimal inventory and pricing policy under various specified constraints. The second stream of research, which is originated from Barankin (1961), focuses on inventory management with multiple suppliers. Fukuda (1964), Whittemore and Saunders (1977), and Feng et al. (2006) try to specify the conditions under which the optimal policy is a base-stock policy. Afterwards, researchers in this field focus on developing comprehensive optimal inventory policies under general settings, such as the monotone structure with ordered change rates proposed by Li and Yu (2014) and Hua et al. (2015), and the two-reorder-points policy developed by Tan et al. (2016). Due to the complexity of the multiple supply inventory problem, numerous heuristic policies have been developed, such as the single-index policy (Scheller-Wolf et al., 2007), dual-index policy (Veeraraghavan and Scheller-Wolf, 2008), and weighted dual-index policy (Sheopuri et al., 2010).

Compared with the literature above, we incorporate suppliers with different lead times in the joint inventory-pricing control problem, and investigate the optimal joint dual inventory ordering and pricing policy. The most relevant works that deal with lead times and price-dependent demand are Pang et al. (2012), Zhou and Chao (2014), Gong et al. (2014), and Feng and Shanthikumar (2017). Pang et al. (2012) consider a single-supplier, joint inventory-pricing model with positive lead times. Our model differs from theirs by introducing an additional expedited supplier and relaxing the perfectly reliable replenishment assumption. Therefore, we rely on the structure of joint concavity together with antimultimodularity to characterize the optimal dual order and pricing decisions under different sets of assumptions. Zhou and Chao (2014) and Gong et al. (2014) address dual supply systems with 0- and 1-period lead times. The key difference between our model and theirs is the dimension of the state variables. The post-order inventory position, which is used as the unidimensional state variable in their model, has to be expanded to record the outstanding order quantities when considering lead times longer than one period. This dimensional augmentation makes the traditional concavity and modularity analyses inapplicable, and therefore invalidates the base-stock structure obtained in Zhou and Chao (2014) and the monotone structure developed in Gong et al. (2014). Finally, our model differs from Feng and Shanthikumar (2017), because we consider dual suppliers with different lead times, whereas they consider multiple suppliers all with zero lead time. The zero-lead-time suppliers in Feng and Shanthikumar (2017) lead to a single state dynamic programming problem. However, the L-period-lead-time supplier in this study requires us to consider multidimensional state space. The joint concavity derived with the SL(mp) proposed by Feng and Shanthikumar (2017) only suffices to characterize the optimal policy of the expedited supplier. To further characterize the optimal regular ordering policy and optimal pricing policy, we propose the notions of PST and IPST, which are applied to the proof of the antimultimodularity preservation rather than the concavity preservation.

The remainder of this paper is organized as follows. Section 2 formulates the dual supply model. In Section 3, we characterize the optimal policy for expedited orders with the joint concavity derived by SL(mp), and review the notion of antimultimodularity as preparation for investigating the optimal policies for regulars orders and pricing. In Section 4, we formally define PST and IPST, and demonstrate properties, sufficient conditions, and examples for PST and IPST functions. Section 5 applies the concepts of PST and IPST to the proof of the antimultimodularity preservation in the joint inventory-pricing problem with dual suppliers. The antimultimodularity of the profit functions enable us to characterize the optimal policies for regular orders and pricing. Section 6 summarizes the paper.

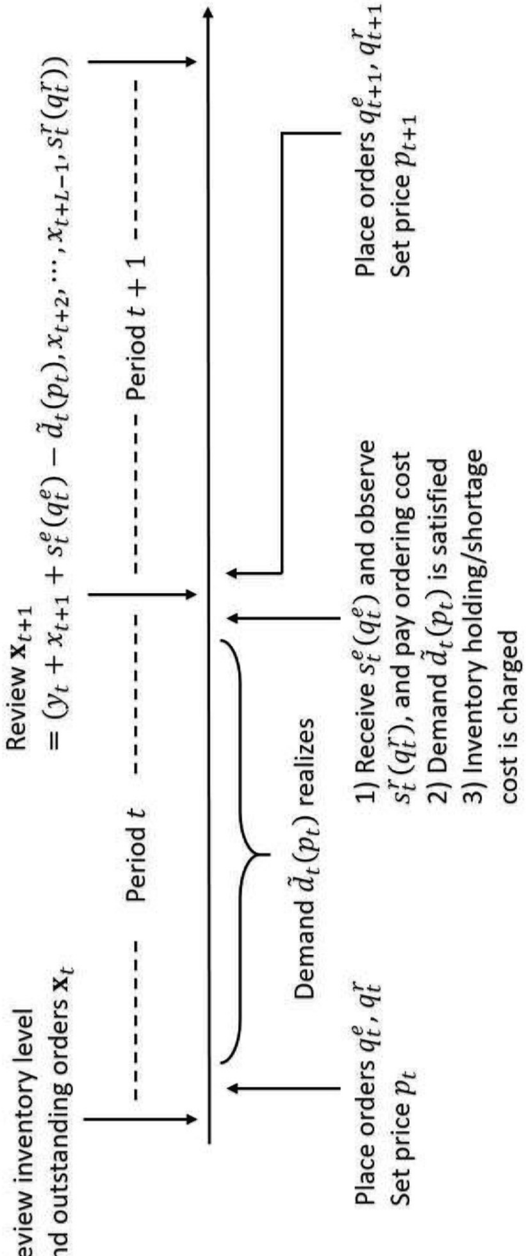


Fig. 1. Time line.

2. Model formulation

Consider a single-product, finite-horizon, periodic-review inventory system with price-sensitive demands and dual suppliers. The expedited supplier offers instantaneous replenishment that arrives within the current period, and the regular supplier offers the same product with an L-period replenishment lead time. Both replenishments have stochastic yield rates that are realized before the end of the period in which the ordering decisions are made. Hence, it is a dual supply system with lead times of 0- and L-periods. Let T be the total number of periods, and denote t as the time indicator, i.e., $t = 1, 2, \cdots, T$. The sequence of events is as follows (1) At the beginning of period t, the regular order placed at period $t-L+1$ arrives, and the inventory level and outstanding order quantities are reviewed. (2) Simultaneous dual ordering and pricing decisions are made. (3) The stochastic yield rates for both orders are realized, and the ordering costs are paid depending on the determined delivery quantities. (4) The expedited order arrives and the price-dependent demand is satisfied. (5) The remaining inventory is held in stock for the next period and incurs a holding cost; the excessive demand is fully backlogged and incurs a cost for the loss of goodwill. The timeline of the model is illustrated in Fig. 1. The notations used in this paper are summarized in Table 1.

The system dynamic equation can be expressed with the realized demand $\tilde{d}_t(p_t)$ and dual supply $s_t^e(q_t^e)$ and $s_t^r(q_t^r)$ as follow.

$$\mathbf{x}_{t+1} = \left(y_t + x_{t+1} + s_t^e(q_t^e) - \tilde{d}_t(p_t), x_{t+2}, \cdots, x_{t+L-1}, s_t^r(q_t^r) \right).$$

Note that in the following, we omit the subscript t in the demand and supply functions to make the notations in the proofs more concise. All the subsequent analyses in this study hold for time-variant demand and supply functions, through adding subscript t to the corresponding notations in the proofs. To facilitate the structural analysis, we impose the following assumption on the demand functions.

Assumption 1. The expected demand rate $d_t = \mathbb{E}[\tilde{D}(p_t)]$ is finite, continuous, and strictly decreasing in p_t. Define an inverse mapping $p_t = P(d_t)$ for the d_t lying in $[\underline{d}, \overline{d}] \triangleq [\mathbb{E}[\tilde{D}(\overline{p})], \mathbb{E}[\tilde{D}(\underline{p})]]$. The expected revenue $R(d_t) = d_t \cdot P(d_t)$ is concave in d_t.

As a commonly used technique in the literature, we take the expected demand rate d_t as the decision variable instead of pricing p_t. Let $D(d_t) \triangleq \tilde{D}(P(d_t))$ denote the demand as a function of the expected demand rate d_t. We also impose the following assumption on the supply functions.

Assumption 2. For $i \in \{e, r\}$, $S^i(0) =_{a.s.} 0$, and $\mu^i(q^i) = \mathbb{E}[S^i(q^i)]$ is finite, continuous, and strictly increasing in q^i.

The monotonicity of the expectations provides the possibility for variable transformation, and thus transforms the original nonconcave problem into a concave one. We further impose Assumption 3 on the inventory holding/shortage costs, as in Feng and Shi (2012) and Feng and Shanthikumar (2017).

Assumption 3. $H(x)$ is non-negative, continuous, and convex in x. Moreover, $H(0) = 0$, $|H(x_1) - H(x_2)| \le c^H |x_1 - x_2|$ for some finite c^H, and $\lim_{x \to \pm\infty} H(x) = \infty$.

Denote $\mathscr{A} = \Re_+^2 \times [\underline{d}, \overline{d}]$ as the feasible decision space in each period, where $\Re_+ = [0, +\infty)$. Similar to the assumptions in Gong et al. (2014) and Chen and Tan (2016), the ordering cost depends on the delivered quantities, i.e., the ordering decisions q_t^e and q_t^r incur the expected ordering cost of $c^e \mathbb{E}[S^e(q_t^e)] + c^r \mathbb{E}[S^r(q_t^r)]$ in the current period. Let $\alpha \in (0, 1]$ denote the discount factor. With replacing $\tilde{D}(p_t)$ with $D(d_t)$, the dual supply inventory-pricing control problem is formulated as follows.

$$\begin{aligned} V_t(\mathbf{x}_t) \quad = \quad & \max_{(q_t^e, q_t^r, d_t) \in \mathscr{A}} R(d_t) - c^e \mathbb{E}\left[S^e(q_t^e)\right] - c^r \mathbb{E}\left[S^r(q_t^r)\right] - \mathbb{E}\left[H\left(y_t + S^e(q_t^e) - D(d_t)\right)\right] \\ & + \alpha \mathbb{E}\left[V_{t+1}\left(y_t + x_{t+1} + S^e(q_t^e) - D(d_t), x_{t+2}, \cdots, x_{t+L-1}, S^r(q_t^r)\right)\right]. \end{aligned} \tag{1}$$

We assume that the leftover inventory and the order arriving after period T have no salvage value, i.e., $V_{T+1}(\mathbf{x}_{T+1}) \equiv 0$. Our structural analysis can be carried to the infinite horizon model with discount factor $\alpha \in (0, 1)$ and stationary demand and

Table 1
Notations.

y_t	The inventory level at the beginning of period t.
q_t^e / q_t^r	The expedited/regular order placed in period t.
$S_t^e(q_t^e)/S_t^r(q_t^r)$	The stochastic supply depending on the expedited/regular order at period t; the corresponding realization is denoted by $s_t^e(q_t^e)/s_t^r(q_t^r)$.
c^e / c^r	The unit inventory ordering cost for the expedited/regular order.
p_t	The price charged per unit of product sold in period t, $p_t \in [\underline{p}, \overline{p}]$.
$\tilde{D}_t(p_t)$	The stochastic demand in period t depending on the list price p_t; the realization of which is denoted as $\tilde{d}_t(p_t)$.
$H(x)$	The inventory holding or shortage cost incurred with an ending inventory level of x.
x_{t+i}	$x_{t+i} = s^r(q_{t-L+i}^r)$ is the realization of the regular order placed in period $t-L+i$, $i = 1, 2, \cdots, L-1$, which arrives at the beginning of period $t+i$.
$\mathbf{x}_t$	$\mathbf{x}_t = (y_t, x_{t+1}, \cdots, x_{t+L-1})$ is the inventory level and outstanding order vector, representing the system state at the beginning of period t.

supply functions by using the standard convergence arguments in Schäl (1993), if Conditions (B) and (W) in his work are satisfied.

3. Preliminaries

In this section, we first prove the joint concavity of the profit function in problem (1) with respect to the inventory level and outstanding order vector $\mathbf{x}_t = (y_t, x_{t+1}, \cdots, x_{t+L-1})$, through applying the stochastically linear in mid-point (SL(mp)) proposed by Feng and Shanthikumar (2017). Then, we characterize the optimal policy for the expedited ordering decision and analyze the effects of supply randomness on the optimal inventory and pricing decisions. Lastly, we briefly review the definition of antimultimodularity and provide an easy-to-verify necessary and sufficient condition for antimultimodularity, which plays an important role in deriving the sufficient conditions for PST/IPST functions and in characterizing the optimal joint dual ordering and pricing policy.

Definition 1. (Stochastically Linear in)

A stochastic function $\{\mathbf{Y}(\mathbf{x}) : \mathscr{X} \to \Re^n\}$ defined on a convex set $\mathscr{X}$ is called SL(mp), written as $\mathbf{Y}(\mathbf{x}) \in SL(mp)$, if for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{X}$, there exist $\widehat{\mathbf{Y}}(\mathbf{x}_1)$ and $\widehat{\mathbf{Y}}(\mathbf{x}_2)$ defined on a common probability space such that $\widehat{\mathbf{Y}}(\mathbf{x}_i) \overset{d}{=} \mathbf{Y}(\mathbf{x}_i)$ for $i = 1, 2$, and $\frac{\widehat{\mathbf{Y}}(\mathbf{x}_1)+\widehat{\mathbf{Y}}(\mathbf{x}_2)}{2} \le_{cv} \mathbf{Y}\left(\frac{\mathbf{x}_1+\mathbf{x}_2}{2}\right)$.

Here, $\le_{cv}$ is the multivariate concave order (details in Shaked and Shanthikumar, 2007). For handling models with multi-dimensional states and decision variables such as problem (1), we derive the following properties of SL(mp).

Lemma 1. *(i) If $\{\mathbf{Y}(\mathbf{x}) : \mathscr{X} \to \Re^n\} \in SL(mp)$, then any affine transformation $A \cdot \mathbf{Y}(\mathbf{x}) + \mathbf{b}$ is also $SL(mp)$. (ii) If for $i \in \{1, 2, \cdots, n\}$, $\{Y_i(\mathbf{x}_i) : \mathscr{X}_i \to \Re\} \in SL(mp)$ and they are independent of each other, then $\{\sum_{i=1}^n Y_i(\mathbf{x}_i) : \mathscr{X}_1 \times \cdots \times \mathscr{X}_n \to \Re\} \in SL(mp)$. (iii) If for $i \in \{1, 2, \cdots, n\}$, $\{Y_i(\mathbf{x}) : \mathscr{X} \to \Re\} \in SL(mp)$ and they are independent of each other, then the vector-valued function $\mathbf{Y}(\mathbf{x}) \triangleq (Y_1(\mathbf{x}), \cdots, Y_n(\mathbf{x}))^T \in SL(mp)$. (iv) If $\{\mathbf{Y}(\mathbf{x}) : \mathscr{X} \to \Re^n\} \in SL(mp)$, then for any concave function $\varphi : \Re^n \to \Re$, $\mathbb{E}[\varphi(\mathbf{Y}(\mathbf{x}))]$ is jointly concave in $\mathbf{x}$.*

Parts (i) and (ii) in Lemma 1 imply that SL(mp) is preserved under linear combinations. Part (iii) indicates that a vector-valued function is SL(mp) if each of its elements is SL(mp) and the elements are independent from each other. Part (iv) is equivalent to the fact that SL(mp) implies stochastic linearity, which plays a key role in deriving the joint concavity.

For $i \in \{e, r\}$, let $Q^i(\mu^i)$ denote the inverse function of $\mu^i(q^i) = \mathbb{E}[S^i(q^i)]$. We rewrite the stochastic supply as the function of its mean, i.e., $\tilde{S}^i(\mu^i) = S^i(Q^i(\mu^i))$. Denote J_t as the profit-to-go function from period t to T, which is a function of the transformed decision variables (μ_t^e, μ_t^r, d_t). Then, the DP problem (1) can be rewritten as follows.

$$V_t(\mathbf{x}_t) = \max_{\substack{\mu_t^i \in [0, \overline{\mu}^i], \text{for } i \in \{e,r\} \\ d_t \in [\underline{d}, \overline{d}]}} J_t(\mathbf{x}_t, \mu_t^e, \mu_t^r, d_t), \tag{2}$$

$$\begin{aligned} J_t(\mathbf{x}_t, \mu_t^e, \mu_t^r, d_t) = \; & R(d_t) - c^e \mu_t^e - c^r \mu_t^r - \mathbb{E}\left[H\left(y_t + \tilde{S}^e(\mu_t^e) - D(d_t)\right)\right] \\ & + \alpha \mathbb{E}\left[V_{t+1}\left(y_t + x_{t+1} + \tilde{S}^e(\mu_t^e) - D(d_t), x_{t+2}, \cdots, x_{t+L-1}, \tilde{S}^r(\mu_t^r)\right)\right], \end{aligned} \tag{3}$$

where $\overline{\mu}^i = \mathbb{E}(S^i(\infty)), i \in \{e, r\}$. Note that if $\mathbb{E}(S^i(\infty)) = \infty$, then the feasible range of μ_t^i is defined as $[0, +\infty)$. By assuming that the transformed demand and supply functions are SL(mp), we are ready to prove that V_t and J_t are both jointly concave with respect to their arguments.

Lemma 2. *If $D(d_t)$, $\tilde{S}^e(\mu_t^e)$, and $\tilde{S}^r(\mu_t^r)$ are SL(mp), and they are independent from each other, then $V_t(\mathbf{x}_t)$ and $J_t(\mathbf{x}_t, \mu_t^e, \mu_t^r, d_t)$ are both jointly concave for $t = 1, 2, \cdots, T$.*

Let $\mathbf{x}_{t+} = (x_{t+1}, \cdots, x_{t+L-1})$ denote the outstanding order vector, and $(q_t^{e*}(\mathbf{x}_t), q_t^{r*}(\mathbf{x}_t), d_t^*(\mathbf{x}_t))$ denote the optimal policy at period t. If there exist multiple maximizers, we choose the smallest one by lexicographical order. Then, the optimal policy for expedited order has a state-dependent almost-threshold structure.

Theorem 1. *Assume that there exists a constant C such that $\tilde{S}^e(\mu^e) \le_{as} C\mu^e$, then for any given outstanding order quantity vector $\mathbf{x}_{t+}$, there exists a threshold $\overline{y}_t(\mathbf{x}_{t+})$ such that*

$$q_t^{e*}(\mathbf{x}_t) = \begin{cases} >0, & \text{if } y_t \in \left(-\infty, \bar{y}_t(\mathbf{x}_{t+})\right) \setminus \Theta_t, \\ =0, & \text{if } y_t \in \left[\bar{y}_t\left(\mathbf{x}_{t+}\right), \infty\right) \cup \Theta_t \end{cases}$$

where Θ_t is a set with Lebesque measure of 0. If we further assume that for any t, $V_t(\mathbf{x}_t)$ is partially differentiable with respect to its first argument, then $\Theta_t = \varnothing$ and a state-dependent strict-threshold policy is optimal.

The assumption on $\tilde{S}^e(\mu^e)$ means that the actual delivered quantity is almost surely bounded by a multiple of the expected delivery quantity. Theorem 1 indicates that the optimal policy for the expedited order is of a threshold type. The threshold depends on the outstanding order quantities. The optimal policy is to place a positive expedited order if the inventory level lies below the threshold; otherwise, no orders should be placed with the expedited supplier, except for a set with Lebesque measure of 0. We name the policy as "state-dependent almost-threshold policy", which is a natural high-dimensional generalization of the almost-threshold policy in Feng and Shanthikumar (2017).

Remark 1. *The effect of supply randomness (e.g., the standard deviation of the random supply) on the optimal policy is non-monotone, depending on the demand distribution. Take the following single-supplier problem as an example.*

$$\pi(d,\mu) = p(d)\mathbb{E}\min\{D(d), S(\mu)\} - h\mathbb{E}(S(\mu) - D(d))^+ - g\mathbb{E}(D(d) - S(\mu))^+ - c\mu,$$

where d is the mean demand, μ is the mean supply, $p(d)$ is the price as a function of the mean demand, h is the unit holding cost, and g is the unit lost-sales penalty. Let $F(\tau; d)$ and $G(\eta; \mu)$ denote the CDFs of $D(d)$ and $S(\mu)$, respectively. Define $\overline{F}(\tau; d) = 1 - F(\tau; d)$ and $\overline{G}(\eta; \mu) = 1 - G(\eta; \mu)$. Then,

$$\pi(d,\mu) = p(d)\left[\int_0^{+\infty}\int_0^{\tau} \eta f(\tau;d)g(\eta;\mu)\mathrm{d}\eta\mathrm{d}\tau + \int_0^{+\infty}\int_{\tau}^{+\infty} \tau f(\tau;d)g(\eta;\mu)\mathrm{d}\eta\mathrm{d}\tau\right]$$

$$-h\int_0^{+\infty}\int_{\tau}^{+\infty}(\eta-\tau)f(\tau;d)g(\eta;\mu)\mathrm{d}\eta\mathrm{d}\tau - g\int_0^{+\infty}\int_0^{\tau}(\tau-\eta)f(\tau;d)g(\eta;\mu)\mathrm{d}\eta\mathrm{d}\tau - c\mu$$

$$= [p(d)+h+g]\int_0^{+\infty}\overline{F}(\tau;d)\overline{G}(\tau;\mu)\mathrm{d}\tau - (h+c)\mu - gd.$$

The randomness of the supply quantity is reflected by the term $\overline{G}(\tau; \mu)$. To investigate the effect of the supply randomness, we consider a simple case in which the retailer only makes the inventory replenishment decision μ. Assume that $S(\mu)$ follows a two-point distribution with mean μ and standard deviation σ, i.e., $\Pr(S=\mu-\sigma) = 0.5$ and $\Pr(S=\mu+\sigma) = 0.5$ where $\mu \geq \sigma$. The effect of supply randomness is reflected by σ in this example. Then,

$$\overline{G}(\eta;\mu) = \begin{cases} 0, & \eta \geq \mu+\sigma, \\ 0.5, & \mu-\sigma \leq \eta < \mu+\sigma, \\ 1, & \eta < \mu-\sigma. \end{cases}$$

$$\pi(d,\mu) = [p(d)+h+g]\left[\frac{1}{2}\int_{\mu-\sigma}^{\mu+\sigma}\overline{F}(\tau;d)\mathrm{d}\tau + \int_0^{\mu-\sigma}\overline{F}(\tau;d)\mathrm{d}\tau\right] - (h+c)\mu - gd.$$

Because the example focuses on the inventory decision, we write $\overline{F}(\tau; d)$ as $\overline{F}(\tau)$ when no ambiguity arises. The first-order condition of μ is

$$[p(d)+h+g]\left[\frac{1}{2}\overline{F}(\mu+\sigma) + \frac{1}{2}\overline{F}(\mu-\sigma)\right] - h - c = 0.$$

The optimal μ is a function of σ, denoted by $\widehat{\mu}(\sigma)$. Taking derivatives with respect to σ on both sides, we obtain $(\widehat{\mu}'+1)f(\widehat{\mu}+\sigma) + (\widehat{\mu}'-1)f(\widehat{\mu}-\sigma) = 0$, i.e.,

$$\widehat{\mu}'(\sigma) = \frac{f(\widehat{\mu}-\sigma)-f(\widehat{\mu}+\sigma)}{f(\widehat{\mu}-\sigma)+f(\widehat{\mu}+\sigma)}.$$

It is easy to check that $\widehat{\mu}$ may be a non-monotone function for certain demand probability density function $f(\cdot)$. Hence, the effect of supply randomness (i.e., σ in this example) on the optimal inventory replenishment decision μ can be either positive or negative, depending on the demand distribution. Likewise, the effect of supply randomness on the optimal pricing decision is also non-monotone.

With respect to the regular order, unfortunately, there is no similar structure under the assumptions in Lemma 2*. The reason can be found from* equation (3)*, i.e., the joint concavity of the profit function J_t implies a substitution relationship between the expedited order and inventory level. However, it provides little information about the relationship between the regular order and the state variables. To track how the inventory level and outstanding order quantities influence the optimal regular ordering quantity, we rely on a structural property called antimultimodularity, which implies a substitution relationship between the arguments in a multivariate function.*

Antimultimodular functions are first proposed by Murota (2005)*, in which the functions are defined on integer space $\mathbf{Z}^n$.* Li and Yu (2014) *relax the definition space to real space.*

Definition 2. (*Antimultimodularity,* *Li and Yu, 2014*)

A function $f(\mathbf{x})$ defined on V is antimultimodular if $\varphi(\mathbf{x},y)=f(x_1-y,x_2-x_1,x_3-x_2,\cdots x_n-x_{n-1})$ is supermodular on $S=\{(\mathbf{x},y)\in\Re^{n+1},(x_1-y,x_2-x_1,\cdots x_n-x_{n-1})\in V\}$.

For a twice-differentiable function $g(\mathbf{v})$, $\mathbf{v}\in\Re^n$, define $g_i(\mathbf{v})=\frac{\partial}{\partial v_i}g(\mathbf{v})$, $g_{i,j}(\mathbf{v})=\frac{\partial^2}{\partial v_i\partial v_j}g(\mathbf{v})$. Then, a necessary and sufficient condition for the antimultimodularity of $g(\mathbf{v})$ is.

Condition 1.

$$\begin{aligned}
&g_{i,j}+g_{i+1,j+1}\ge g_{i,j+1}+g_{i+1,j}, \text{for } i\ne j, i\le n-1, j\le n-1,\\
&g_{n,n}\le g_{n,n-1}\cdots\le g_{n,1}\le 0,\\
&g_{n,n}\le g_{n-1,n}\cdots\le g_{1,n}\le 0,\\
&g_{1,1}\le g_{2,1}\le\cdots\le g_{n,1}\le 0, \text{and}\\
&g_{1,1}\le g_{1,2}\le\cdots\le g_{1,n}\le 0.
\end{aligned}$$

Condition 1 *reveals that the antimultimodularity of a function is equal to a set of inequalities between adjacent second-order derivatives of the function. As discussed in* Pang et al. (2012) *and* Li and Yu (2014)*, antimultimodular and $L^\natural$-concave functions are related through a certain variable transformation. In this paper, we use antimultimodularity rather than $L^\natural$-concavity because (i) the application of antimultimodularity analysis to problems with substitute variables is more intuitive and avoids the detour state transformation procedure and (ii) the properties of antimultimodularity in Lemma* 8 *in the appendix inspire us to propose the notion of PST.*

4. PST and sufficient conditions

In this section, we formally define partially stochastic translation (PST). As an extension of PST, we introduce the notion of increasing PST (IPST). Moreover, we provide sufficient conditions and derive properties for PST and IPST, which are applied to deriving the antimultimodularity of the profit function in the joint inventory-pricing problem with dual suppliers.

4.1. Condition and properties of PST

Definition 3. (Partially Stochastic Translation)

A stochastic function $\{Y(x), x\in\mathscr{X}\subset\Re\}$ is called a Partially Stochastic Translation, written as $Y(x)\in PST$, if there exists an extension $\{\widehat{Y}(x), x\in\Re\}$ defined on the same probability space such that $\widehat{Y}(x)=_{a.s.}Y(x)$ for $x\in\mathscr{X}$, and for any antimultimodular function $g(\mathbf{v})$ defined on $\Re^n$, $\mathbb{E}[g(\mathbf{v}+\widehat{Y}(x)\mathbf{e}_1)]$ defined on $\Re^{n+1}$ is antimultimodular in $(x,\mathbf{v})$, where $\mathbf{e}_1$ is the first unit vector of dimension n.

Remark 2. Definition 3 is equivalent to the definition with antimultimodular being replaced by multimodular and all other statements remaining the same.

The concept of PST works in proving the antimultimodularity preservation for the profit functions. The antimultimodularity leads to the monotonicity of the optimal regular ordering quantity and list price with respect to the ordered change rate. The definition implies that, if we add a PST function $Y(x)$ to the left-end argument of an arbitrary antimultimodular function, then the antimultimodularity is preserved under the expectation, with respect to the variable sequence beginning with x. Similar to stochastic linearity, we name this definition PST because any deterministic translation function $Y(x) = x + C$ satisfies this definition. In fact, the deterministic translation function is better behaved because adding it to any argument preserves the antimultimodularity. Therefore, under the expectation, any stochastic function $Y(x) \in PST$ behaves partially like a translation function. Recall that $g(\mathbf{v})$ is antimultimodular if and only if $-g(\mathbf{v})$ is multimodular. Hence, this definition can also be explained with multimodularity. In problems concerning multimodularity, such as the model without the pricing decision, it is more convenient to use the definition in Remark 2.

Let $\widehat{Y}(x) = \phi(x, Z)$, where $\phi(x,z)$ is a deterministic function of (x,z) and Z is a random variable with support $\mathscr{Z} \in \Re$ (following Theorem 1 in Feng and Shanthikumar, 2017). Let $\phi_x(x,z) = \frac{\partial}{\partial x}\phi(x,z)$ and $\phi_z(x,z) = \frac{\partial}{\partial z}\phi(x,z)$. Then, a sufficient condition for PST is.

Condition 2. $0 \le \phi_x(x,z) \le 1$, for all $x \in \mathscr{X}, z \in \mathscr{Z}$, and

$$\mathbb{E}\Big[g_{1,1}(\mathbf{v} + \phi(x,Z)\mathbf{e}_1)\Big(\phi_x(x,Z) - \phi_x^2(x,Z)\Big)\Big] \ge \mathbb{E}[g_1(\mathbf{v} + \phi(x,Z)\mathbf{e}_1)\phi_{xx}(x,Z)],$$

for any antimultimodular function $g(\mathbf{v})$ and $x \in \mathscr{X}$.

The following are some properties of PST functions.

Lemma 3. *(i) If $\{Y(x),\ x \in \mathscr{X}\} \in PST$, then $Z(x) = -Y(-x)$ defined on $-\mathscr{X}$ is PST. (ii) Suppose $\{Y(x),\ x \in \mathscr{X}\} \in PST$, $\{Z(y), y \in \mathscr{Y}\} \in PST$ and independent from Y, $\{Y(x), x \in \mathscr{X}\} \subset \mathscr{Y}$; then, $\{Z(Y(x)), x \in \mathscr{X}\} \in PST$. (iii) Suppose $\{Y(x), x \in \mathscr{X}\} \in$ PST. Then, for any antimultimodular function $g(\mathbf{v})$ defined on $\Re^n$, (a) $\mathbb{E}[g(\mathbf{v} + Y(x)\mathbf{e}_1)]$ is antimultimodular in $(x, \mathbf{v})$; (b) $\mathbb{E}[g(\mathbf{v} + Y(x)\mathbf{e}_n))$ is antimultimodular in $(\mathbf{v}, x)$; (c) $\mathbb{E}[g(Y(x), v_2, \cdots, v_n)]$ is antimultimodular in $(x, v_2, \cdots, v_n)$; and (d) $\mathbb{E}[g(v_1, \cdots, v_{n-1}, Y(x))]$ is antimultimodular in $(v_1, \cdots, v_{n-1}, x)$.*

Parts (i) and (ii) illustrate the math operators which preserve the PST property. Part (iii) indicates that under the expectation, adding a PST function on the left- or right-end argument of an antimultimodular function preserves the antimultimodularity, and the preservation also holds for replacing the argument with a PST function.

4.2. Condition and properties of IPST

In addition to PST, we introduce the notions of increasing PST and decreasing PST, which are relaxed versions of PST and can be applied to the proof of the antimultimodularity preservation as well.

Definition 4. (Increasing Partially Stochastic Translation)

A stochastic function $Y(x), x \in \mathscr{X} \subset \Re$ is called an increasing partially stochastic translation, written as $Y(x) \in$ *IPST*, if there exists an extension $\{\widehat{Y}(x), x \subset \Re\}$ defined on the same probability space such that $\widehat{Y}(x) =_{a.s.} Y(x)$ for $x \in \mathscr{X}$, and for any first-argument increasing antimultimodular function $g(\mathbf{v})$ defined on $\Re^n$ with bounded first- and second-order derivatives, $\mathbb{E}[g(\mathbf{v} + \widehat{Y}(x)\mathbf{e}_1)]$ defined on $\Re^{n+1}$ is antimultimodular in $(x, \mathbf{v})$, where $\mathbf{e}_1$ is the first unit vector of dimension n.

Lemma 4. *If a stochastic function $Y(x) \in PST$, then $Y(x) \in IPST$.*

Lemma 4 *implies that the PST function set is a subset of the IPST function set. Hence, a sufficient condition for PST is also a sufficient condition for IPST.*

From Condition 1, *we know that for any antimultimodular $g(\mathbf{v})$, $g_{1,1}(\mathbf{v}) \le 0$. Let L, M, and N denote the lower bound of $g_{1,1}(\mathbf{v})$, lower bound of $g_{1,1}(\mathbf{v})$, and upper bound of $g_{1,1}(\mathbf{v})$, respectively. Then, a sufficient condition for IPST is.*

Condition 3. $0 \le \phi_x(x,z) \le 1$, $\phi_{xx}(x,z) \le 0$, *and* $\mathbb{E}\phi_x(x,Z) - \mathbb{E}\phi_x^2(x,Z) \le -\frac{M}{L}\mathbb{E}\phi_{xx}(x,Z)$, *for all $x \in \mathscr{X}$ and $z \in \mathscr{Z}$.*

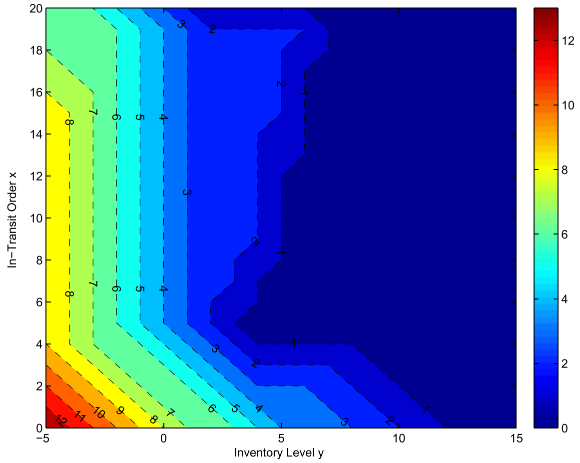
The main part of the sufficient condition is an inequality between the first-order and second-order derivatives of the resulting function with the bounds of the derivatives as coefficients. The following are some examples of IPST functions, which include additive, multiplicative, logarithmic, and exponential form random functions. In the example, we assume $\mathscr{X} \subset [0, +\infty)$ and $\mathscr{Z} \subset [0, +\infty)$.

Example 1. (Examples of PST and IPST functions)

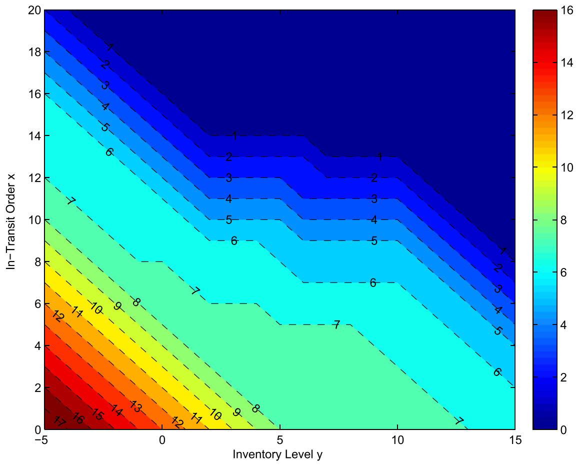
(i) $\phi(x,Z) = x + Z$, which is PST and IPST.
(ii) $\phi(x,Z) = xZ$ *with Z being a Bernoulli random variable,* which is PST and IPST.
(iii) $\phi(x,Z) = \ln(a + x + Z)$ *with* $a \ge 1$ *and* $\mathbb{E}\frac{-\frac{M}{L} + a - 1 + x + Z}{(a + x + Z)^2} \le 0$, which is IPST.
(iv) $\phi(x,Z) = a - e^{-b-x-Z}$ *with* $b \ge 0$ *and* $\mathbb{E}e^{-b-x-Z}\left(1 - \frac{M}{L} - e^{-b-x-Z}\right) \le 0$, which is IPST.

The lemma below summarizes the properties of IPST, which are used in the proof of antimultimodularity preservation in Theorems 3.

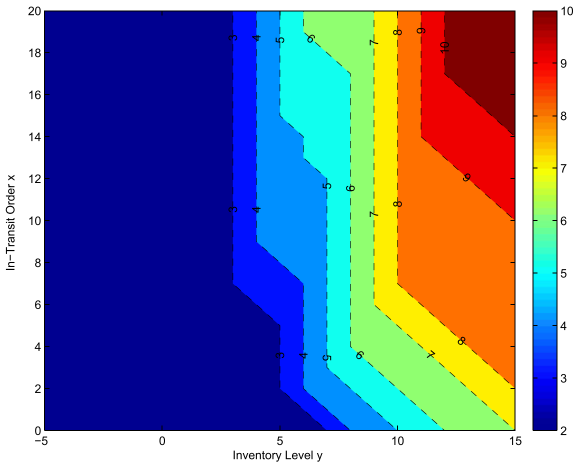
Lemma 5. *(i) If $\{Y(x), x \in \mathscr{X}\} \in IPST$, then $Z(x) = -Y(-x)$ defined on $-\mathscr{X}$ is IPST. (ii) Suppose $\{Y(x), x \in \mathscr{X}\} \in IPST$, $\{Z(y), y \in \mathscr{Y}\} \in IPST$, Y and Z are independent, and $\{Y(x), x \in \mathscr{X}\} \subset \mathscr{Y}$; then, $\{Z(Y(x)), x \in \mathscr{X}\} \in IPST$. (iii) Suppose $\{Y(x), x \in \mathscr{X}\} \in$ IPST. Then, for any first-argument increasing antimultimodular function $g(\mathbf{v})$ defined on $\Re^n$, (a) $\mathbb{E}[g(\mathbf{v} + Y(x)\mathbf{e}_1)]$ is antimultimodular in*



(a) q_1^{e*}



(b) q_1^{r*}



(c) d_1^{*}

Fig. 2. *Optimal Policy of* Example 2.

$(x,\mathbf{v})$; (b) $\mathbb{E}[g(\mathbf{v}+Y(x)\mathbf{e}_n))$ is antimultimodular in $(\mathbf{v},x)$; (c) $\mathbb{E}[g(Y(x),v_2,\cdots,v_n)]$ is antimultimodular in $(x,v_2,\cdots,v_n)$; and (d) $\mathbb{E}[g(v_1,\cdots,v_{n-1},Y(x))]$ is antimultimodular in $(v_1,\cdots,v_{n-1},x)$.

5. Applications of PST

In this section, we apply PST and IPST to the proof of antimultimodularity preservation. The application of PST is on the basic model, i.e., problem (1). The application of IPST requires further assumptions on the basic model.

5.1. Application of PST

To apply PST, we make a transformation on the decision variables in problem (1). Let $\widehat{d}_t = -d_t$, $\widehat{D}_t(\widehat{d}_t) = -D_t(-\widehat{d}_t)$, and $\tilde{R}(\widehat{d}_t) = R(-\widehat{d}_t)$. Let $\tilde{\mathscr{A}}_t = \Re_+^2 \times [-\overline{d}, -\underline{d}]$ denote the transformed action space. Then, the transformed DP problem becomes

$$V_t(\mathbf{x}_t) = \max_{(q_t^e, q_t^r, \widehat{d}_t) \in \tilde{\mathscr{A}}} \tilde{J}_t\left(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r\right), \tag{4}$$

where

$$\begin{aligned}\tilde{J}_t\left(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r\right) = {} & \tilde{R}\left(\widehat{d}_t\right) - c^e \mathbb{E}[S^e(q_t^e)] - c^r \mathbb{E}[S^r(q_t^r)] - \mathbb{E}\left[H\left(y_t + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right)\right)\right] \\ & + \alpha \mathbb{E}\left[\tilde{V}_{t+1}\left(y_t + x_{t+1} + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right), x_{t+2}, \cdots, x_{t+L-1}, S^r(q_t^r)\right)\right].\end{aligned} \tag{5}$$

Theorem 2. *Assume that $S^e(q_t^e)$ and $S^r(q_t^r)$ are both PST, $D(d_t)$ has an additive form, and they are independent of each other. Then, for each $t \in \{1, 2, \cdots, T\}$, (i) $\tilde{J}_t(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$ is antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$, and (ii) $V_t(\mathbf{x}_t)$ is antimultimodular in $\mathbf{x}_t$.*

With PST and its properties in Lemma 3, we are able to prove the preservation of the antimultimodularity of $\tilde{J}_t(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$ and $V_t(\mathbf{x}_t)$. The antimultimodularity provides more structural information among multiple arguments than the joint concavity proved in Lemma 2, and thus enables us to characterize the optimal regular ordering quantity and pricing strategy, which are not explored in Theorem 1.

5.2. Application of IPST

In this subsection, to apply the notion of IPST, we assume that the inventory purchasing cost linearly depends on the ordering quantity and $H(x)$ is decreasing convex. Decreasing convex $H(x)$ implies that the shortage cost is convex and the inventory holding cost is negligible. Then, the dynamic equations of the dual supply inventory-pricing problem become

$$V_t(\mathbf{x}_t) = \max_{(q_t^e, q_t^r, \widehat{d}_t) \in \tilde{\mathscr{A}}} \tilde{J}_t\left(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r\right), \tag{6}$$

where

$$\begin{aligned}\tilde{J}_t\left(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r\right) = {} & \tilde{R}\left(\widehat{d}_t\right) - c^e q_t^e - c^r q_t^r - \mathbb{E}\left[H\left(y_t + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right)\right)\right] \\ & + \alpha \mathbb{E}\left[\tilde{V}_{t+1}\left(y_t + x_{t+1} + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right), x_{t+2}, \cdots, x_{t+L-1}, S^r(q_t^r)\right)\right].\end{aligned} \tag{7}$$

We prove the preservation of antimultimodularity with IPST through backward induction. Assumption 3 ensures that $|H'(x)|$ is bounded by c_H. If $H(x)$ is twice continuous differentiable, then from the continuity of $H''(x)$ and boundedness of $H'(x)$, we derive that $|H''(x)|$ is also bounded by a finite constant. With the boundedness of the second-order derivatives of $\tilde{R}(\tilde{d}_t)$, $\mathbb{E}S^e(q_t^e)$, $\mathbb{E}S^r(q_t^r)$, $H(x)$, and $\tilde{V}_T(\mathbf{x}_T)$, we obtain the boundedness of the second-order derivatives of $\tilde{J}_t(q_t^e, \tilde{d}_t, \mathbf{x}_t, q_t^r)$ and $\tilde{V}_t(\mathbf{x}_t)$ by backward induction. In addition, by applying the properties of IPST in Lemma 5, we obtain the antimultimodularity of $J_t(\mathbf{x}_t, q_t^e, q_t^r, d_t)$ and $V_t(\mathbf{x}_t)$ as follows.

Theorem 3. *Assume that $S^e(q_t^e)$ and $S^r(q_t^r)$ are both IPST, $D(d_t)$ has an additive form, and they are independent of each other. Then, for $t \in \{1, 2, \cdots, T\}$, (i) $\tilde{J}_t(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$ is antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$, and (ii) $V_t(\mathbf{x}_t)$ is antimultimodular in $\mathbf{x}_t$.*

5.3. Joint dual ordering and pricing policy

With the antimultimodularity of the profit functions derived with PST and IPST, we are able to characterize the optimal regular ordering and pricing policies.

Theorem 4. *Under the assumptions in Theorem 2, $q_t^{r*}(\mathbf{x}_t)$ is decreasing in $\mathbf{x}_t$, and $d_t^*(\mathbf{x}_t)$ is increasing in $\mathbf{x}_t$. Moreover, the following inequalities hold.*

$$-1 \le \Delta_{x_{t+L-1}} q_t^{r*}(\mathbf{x}_t) \le \Delta_{x_{t+L-2}} q_t^{r*}(\mathbf{x}_t) \le \cdots \le \Delta_{x_{t+1}} q_t^{r*}(\mathbf{x}_t) \le \Delta_{y_t} q_t^{r*}(\mathbf{x}_t) \le 0, \tag{8}$$

$$0 \le \Delta_{x_{t+L-1}} d_t^*(\mathbf{x}_t) \le \Delta_{x_{t+L-2}} d_t^*(\mathbf{x}_t) \le \cdots \le \Delta_{x_{t+1}} d_t^*(\mathbf{x}_t) \le \Delta_{y_t} d_t^*(\mathbf{x}_t) \le 1. \tag{9}$$

Inequality (8) indicates that the optimal regular ordering quantity is decreasing in the current inventory level and outstanding order quantities. Moreover, the optimal regular ordering quantity is more sensitive to the outstanding order quantities than to the current inventory level. This suggests that the outstanding orders have greater effects on a retailer's optimal regular ordering quantity, compared with the current inventory level. Among the outstanding orders, the optimal regular ordering quantity is more sensitive to those placed more recently than to those placed earlier.

Inequality (9) reveals that the optimal demand rate is increasing in the inventory level and outstanding order quantities. The optimal demand rate is more sensitive to the inventory level than to the outstanding order quantities, and more sensitive to earlier placed outstanding orders than to more recently placed outstanding orders. Moreover, all of the change rates are bounded by 1 or − 1, which indicates that a unit change in the inventory level or outstanding order quantity causes at most a unit change in the optimal regular ordering quantity and demand rate. This property can be used to substantially reduce computational complexity when calculating the exact optimal policy. For example, in the discrete-state simulation, if we have obtained the optimal policy (q_t^{r*}, d_t^*) for a certain state $\mathbf{x}_t^1$, then with Theorem 4, the optimal (q_t^r, d_t) for its adjacent state $\mathbf{x}_t^2 = \mathbf{x}_t^1 + \mathbf{e}_i$ must be within one of the four candidates $\{(q_t^{r*}, d_t^*), (q_t^{r*} - 1, d_t^*), (q_t^{r*}, d_t^* + 1), (q_t^{r*} - 1, d_t^* + 1)\}$, where $\mathbf{e}_i$ is the i-th unit vector of dimension L. Hence, we only need to calculate and compare the value functions of the four candidates rather than search the entire decision space. Let $\mathbf{x}_t^3$ be an arbitrary state, $||\mathbf{x}||_1$ denote the 1-norm of vector $\mathbf{x}$, and assume that given (q_t^r, d_t), the calculation of the corresponding optimal q_t^e can be done within a constant time. Then, the computational complexity of searching the optimal (q_t^e, q_t^r, d_t) corresponding to state $\mathbf{x}_t^3$ is $O(||\mathbf{x}_t^3 - \mathbf{x}_t^1||_1)$ with Theorem 4, which is more efficient than traversing the feasible region of the decision variables.

Recall that $p_t = P(d_t)$ is strictly decreasing in d_t. Denote the optimal pricing strategy as $p_t^*(\mathbf{x}_t)$. Then, we obtain

$$\Delta_{y_t} p_t^*(\mathbf{x}_t) \le \Delta_{x_{t+1}} p_t^*(\mathbf{x}_t) \le \cdots \le \Delta_{x_{t+L-1}} p_t^*(\mathbf{x}_t) \le 0$$

directly from inequality (9). *The above inequality implies that the optimal price is decreasing in the inventory level and outstanding order quantities. In addition, the optimal price is more sensitive to the current inventory level than to the outstanding order quantities, and more sensitive to earlier placed outstanding orders than to more recently placed outstanding orders. The reason is that the earlier the order is, the sooner the replenishment arrives. A closer replenishment clearly has a greater effect on the current-period pricing decision. As inventory is carried over periods, a later arriving replenishment has a longer effect on the current-period inventory decision and therefore its effect is more sensitive. The antimultimodularity also helps in further characterizing the optimal expedited ordering policy.*

Corollary 1. *Under the assumptions in Theorem 2, for any given outstanding order $\mathbf{x}_{t+}$, the optimal expedited order q_t^{e*} is decreasing in inventory level y_t. With the result in Theorem 1, there exists a threshold $\bar{y}_t(\mathbf{x}_{t+})$ such that*

$$q_t^{e*}(\mathbf{x}_t) = \begin{cases} > 0, & \text{if } y_t \in \left(-\infty, \bar{y}_t(\mathbf{x}_{t+}) \right), \\ = 0, & \text{if } y_t \in \left[\bar{y}_t\left(\mathbf{x}_{t+}\right), \infty \right) \end{cases}$$

Corollary 1 reveals that the state-dependent almost-threshold structure still holds for the expedited orders under PST assumption, because antimultimodularity implies joint concavity for continuous functions. Moreover, with the antimultimodular profit functions, we are able to characterize the optimal structure of the expedited order one step further as monotone decreasing in the inventory level, which is illustrated in Example 2. Thus, the optimal joint dual ordering and pricing policy has a mixed form, as together characterized by Theorem 4 and Corollary 1.

Note that the variable sequence of antimultimodularity cannot be altered arbitrarily. Therefore, the optimal structure obtained for p_t and q_t^r cannot be carried to q_t^e, i.e., there is no similar monotonicity for the expedited ordering quantity with respect to the outstanding ordering quantities. A counterexample is given in Example 2 below.

Example 2. (*Non-monotonicity of the optimal expedited order*) *Consider* $L = 2$, $T = 3$, $\alpha = 1$, *and* $H(x) = 2 \cdot x^{+} + 40 \cdot x^{-}$. *The inventory ordering costs* $c^{e} = 3$ *and* $c^{r} = 2$. *The supply functions* $S^{e}(q^{e}) = Z_1 q^{e}$ *and* $S^{r}(q^{r}) = Z_2 q^{r}$, *where* Z_1 *and* Z_2 *are Bernoulli random variables with success probabilities* 0.2 *and* 0.8, *respectively. The demand function has an additive form, i.e.,* $D(d) = d + \varepsilon$ *where* ε *takes its value from* $\{-2, 0, 2\}$ *with probability* $\{0.2, 0.6, 0.2\}$, *respectively. The price* $P(d) = 15 - d$.

Fig. 2 *illustrates the optimal policy at period* 1. *The state variables* (y, x) *denote the inventory level and outstanding order, respectively.* Fig. 2(b) and (c) *illustrate the monotone structures of* q^{r*} *and* d^{*}. *In particular, the horizontal contour line segments in* Fig. 2(b) *indicate that* q^{r*} *is more sensitive to the outstanding order than the inventory level, while the vertical contour line segments in* Fig. 2(c) *indicate that* d^{*} *is more sensitive to the inventory level than the outstanding order. Moreover,* Fig. 2(a) *indicates that although* q^{e*} *is decreasing in the inventory level, it is not necessarily decreasing in the outstanding order.*

6. Concluding remarks

The main contribution of this study is summarized as follows.

State-Dependent Almost Threshold-Policy and Non-monotonicity for Expedited Ordering Decision. We characterize the optimal expedited ordering policy as a state-dependent almost-threshold policy, with the joint concave profit functions derived by extending the notion of SL(mp) to a multidimensional setting. The threshold of the optimal policy depends on the outstanding quantities of the regular orders. It is optimal to place a positive order if the inventory level lies below the threshold; otherwise, no orders should be placed, except for a set with a Lebesque measure of 0. However, the joint concavity is not enough to characterize the optimal regular ordering and pricing policies, which invites us to propose the notions of PST and IPST to establish the antimultimodularity of the profit functions. With the antimultimodular profit functions, we are able to further characterize the optimal expedited ordering quantity as decreasing in the current inventory level. However, the optimal expedited ordering quantity can be non-monotone with respect to the outstanding order quantities, as illustrated in Example 2.

Partially Stochastic Translation Functions and Antimultimodularity Preservation. To characterize the optimal joint dual ordering and pricing policy, we define a class of stochastic functions as partially stochastic translation (PST) functions. PST functions play an important role in proving the antimultimodularity preservation in dynamic programming problems. As an extension to PST, we also propose a relaxed version of PST, i.e., increasing partially stochastic translation (IPST) functions. We show that IPST can be applied to obtaining the antimultimodularity under different model settings, and the PST function set is a subset of the intersection of the IPST function set. Moreover, we demonstrate properties, sufficient conditions, and examples for the PST and IPST functions.

Optimal Policy for Regular Ordering and Pricing Decisions and Sensitivity Analysis. With the proposed PST and IPST, we are able to derive the antimultimodularity of the profit function in the joint inventory-pricing control model with dual suppliers and different lead times. Based on the antimultimodularity, both the optimal regular ordering quantity and optimal price are decreasing in the inventory level and outstanding order quantities. In addition, the optimal regular ordering quantity is more sensitive to outstanding orders placed more recently than to those orders placed earlier and than to the current inventory level. On the contrary, the optimal price is more sensitive to the inventory level than to the outstanding orders, and is more sensitive to the outstanding orders placed earlier than to those placed more recently. It implies that as the time interval increases, the effects of the outstanding orders on the optimal regular ordering quantity and optimal price are decreasing and increasing, respectively. The reason is that the arrival time of earlier orders is closer, and closer replenishments have greater effects on the current-period pricing decision. On the other hand, unlike the price, the inventory is carried over periods. Later replenishments have longer effects on the current-period inventory decision, and hence their effects are more sensitive.

The following are possible extensions to the model in this study.

General Demand and Supply Functions. In this study, we assume that the supply functions are PST and the demand functions are additive. If we assume that the demand functions are PST and there is no random yield in the supply, the antimultimodularity of the profit functions can be similarly derived. However, the variable sequence of the antimultimodularity starts with the demand rate. Hence, in this case, the optimal price may not be monotone with an ordered change rate structure, whereas the optimal expedited and regular ordering policies preserve the structure. For dealing with the general demand functions in (Chen and Simchi-Levi, 2004) or the supply functions in (Li and Zheng, 2006; Feng and Shi, 2012), the antimultimodularity structure is no longer preserved because the functions are not PST. However, the joint concavity still holds because these functions can be transformed to be SL(mp). As a result, the optimal expedited order still has a state-dependent almost-threshold structure.

Two Arbitrary Lead Times. For the dual sourcing problem with two arbitrary lead times and no pricing decision, Li and Yu (2014) arrange the two ordering decisions in the left- and right ending positions in the variable sequence, respectively, and use multimodularity to characterize the optimal policy. However, after introducing the pricing decision, the decision variables cannot be allocated to the ending positions simultaneously. Thus, no antimultimodularity structure can be derived. Although we can obtain the joint concavity of the profit functions, it does not help in characterizing the structure of the optimal policy as a threshold type. We expect new techniques to be developed to analyze this challenging joint inventory-pricing model with general lead times, and therefore leave this problem for future research.

Conflict of interest

The authors declare no conflict of interest.

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Appendix A. Proofs

Appendix A.1 Proof of Lemma 1

(i). The proof is immediate from the definition of SL(mp) and the fact that any concave function composites an affine function preserves the concavity.

(ii). We only prove the property for the case $n = 2$. Then the property for the case $n > 2$ can be verified by induction. Suppose $Y_1(\mathbf{x}),\ Y_2(\mathbf{z}) \in SL(mp)$, then for any $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathscr{X},\ \mathbf{z}_1,\ \mathbf{z}_2 \in \mathscr{Z}$, there exist $\widehat{Y}_1(\mathbf{x}_1) \overset{d}{=} Y_1(\mathbf{x}_1)$, $\widehat{Y}_1(\mathbf{x}_2) \overset{d}{=} Y_1(\mathbf{x}_2)$, $\widehat{Y}_2(\mathbf{z}_1) \overset{d}{=} Y_2(\mathbf{z}_1)$ and $\widehat{Y}_2(\mathbf{z}_2) \overset{d}{=} Y_2(\mathbf{z}_2)$ defined on a common probability space such that

$$\frac{\widehat{Y}_1(\mathbf{x}_1) + \widehat{Y}_1(\mathbf{x}_2)}{2} \le_{cv} Y_1\left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right), \tag{A.1}$$

$$\frac{\widehat{Y}_2(\mathbf{z}_1) + \widehat{Y}_2(\mathbf{z}_2)}{2} \le_{cv} Y_2\left(\frac{\mathbf{z}_1 + \mathbf{z}_2}{2}\right). \tag{A.2}$$

Especially, we could take $\widehat{Y}_1(\mathbf{x}_1), \widehat{Y}_1(\mathbf{x}_2)$ independent from $\widehat{Y}_2(\mathbf{z}_1), \widehat{Y}_2(\mathbf{z}_2)$ since we assume that $Y_1(\cdot)$ is independent from $Y_2(\cdot)$. Then by Theorem 3.A.12 (d) in Shaked and Shanthikumar [2007], we have

$$\frac{\widehat{Y}_1(\mathbf{x}_1) + \widehat{Y}_1(\mathbf{x}_2)}{2} + \frac{\widehat{Y}_2(\mathbf{z}_1) + \widehat{Y}_2(\mathbf{z}_2)}{2} \le_{cv} Y_1\left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right) + Y_2\left(\frac{\mathbf{z}_1 + \mathbf{z}_2}{2}\right). \tag{A.3}$$

Therefore, for $Y_3(\mathbf{x}, \mathbf{z}) = Y_1(\mathbf{x}) + Y_2(\mathbf{z})$ and pairs $(\mathbf{x}_1, \mathbf{z}_1)$, $(\mathbf{x}_2, \mathbf{z}_2)$, we can define $\widehat{Y}_3(\mathbf{x}_1, \mathbf{z}_1) = \widehat{Y}_1(\mathbf{x}_1) + \widehat{Y}_2(\mathbf{z}_1)$, and $\widehat{Y}_3(\mathbf{x}_2, \mathbf{z}_2) = \widehat{Y}_1(\mathbf{x}_2) + \widehat{Y}_2(\mathbf{z}_2)$ to illustrate that $Y_3 \in SL(mp)$.

(iii). For any $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{X}$, since $Y_i(\mathbf{x}) \in SL(mp)$, we can take $\widehat{Y}_i(\mathbf{x}_1) \overset{d}{=} Y_i(\mathbf{x}_1)$, $\widehat{Y}_i(\mathbf{x}_2) \overset{d}{=} Y_i(\mathbf{x}_2)$ that are independent among i such that

$$\frac{\widehat{Y}_i(\mathbf{x}_1) + \widehat{Y}_i(\mathbf{x}_2)}{2} \le_{cv} Y_i\left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right) \tag{A.4}$$

for $i \in \{1, 2, \cdots, n\}$. Then by theorem 7.A.8 in Shaked and Shanthikumar (2007), we have

$$\begin{pmatrix} \frac{\widehat{Y}_1(\mathbf{x}_1) + \widehat{Y}_1(\mathbf{x}_2)}{2} \\ \vdots \\ \frac{\widehat{Y}_n(\mathbf{x}_1) + \widehat{Y}_n(\mathbf{x}_1)}{2} \end{pmatrix} \le_{cv} \begin{pmatrix} Y_1\left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right) \\ \vdots \\ Y_n\left(\frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right) \end{pmatrix}. \tag{A.5}$$

Therefore, define $\widehat{\mathbf{Y}}(\mathbf{x}_i) = (\widehat{Y}_1(\mathbf{x}_i), \cdots, \widehat{Y}_n(\mathbf{x}_i))^T$ for $i = 1, 2$, and we have proven that the vector valued function $\mathbf{Y}(\mathbf{x}) \in SL(mp)$.

(iv). The proof is referred to the proof of Lemma 2 in Feng and Shanthikumar (2017).

Appendix A.2 Proof of Lemma 2

First note that $V_{T+1}(\mathbf{x}_{T+1}) \equiv 0$ is jointly concave. Suppose that V_{k+1} is concave, we intend to prove the concavity of J_k and V_k. By Lemma 1 (i) (ii), $y_t + x_{t+1} + \tilde{S}^e(\mu_t^e) - D(d_t) \in SL(mp)$ since each term is SL(mp) and independent from others. Therefore, by Lemma 1 (iii), the L dimensional vector $(y_t + x_{t+1} + \tilde{S}^e(\mu_t^e) - D(d_t), x_{t+2}, \cdots, x_{t+L-1}, \tilde{S}^r(\mu_t^r)) \in SL(mp)$. Therefore, the last term in J_k is jointly concave by the inductive hypothesis and Lemma 1 (iv).

The concavity of the first three terms in J_k is directly verified under the Assumption 1. Moreover, the forth term in J_k is jointly concave since $-H(\cdot)$ is assumed to be concave and $y_t + \tilde{S}^e(\mu_t^e) - D(d_t) \in SL(mp)$. Therefore, we have proven the joint concavity of J_k. Finally, $V_k(\mathbf{x}_k)$ is jointly concave since the concavity is preserved under partial maximization.

Appendix A.3 Proof of Theorem 1

The proof of Theorem 1 relies on Lemma 6 and Lemma 7.

Lemma 6. *For any given outstanding order $\mathbf{x}_{t+}$, let $\mu^{e*}(y_t, \mathbf{x}_{t+})$ denotes the optimal expedited order in period t, with the inventory level y_t. If there exists an interval (y_1, y_2) such that $\mu^{e*}(y_t, \mathbf{x}_{t+}) > 0$ for $y_t \in (y_1, y_2)$ and $\mu^{e*}(y_1, \mathbf{x}_{t+}) = 0$, then*

$$\lim_{a \to 0^+} \frac{V_t(y_t + a, \mathbf{x}_{t+}) - V_t(y_t, \mathbf{x}_{t+})}{a} > c^e \tag{A.6}$$

for $y_t \le y_1$.

A.3.1. Proof of Lemma 6

Given $\mathbf{x}_{t+}$, we simplify the notations $V_t(y_t, \mathbf{x}_{t+})$ as $V_t(y_t)$, $\mu_t^{e*}(y_t, \mathbf{x}_{t+})$, $\mu_t^{r*}(y_t, \mathbf{x}_{t+})$, $d_t^*(y_t, \mathbf{x}_{t+})$ as $\mu_t^{e*}(y_t)$, $\mu_t^{r*}(y_t)$, $d_t^*(y_t)$ respectively. Since J_t is jointly concave and such concavity can be easily extended to $\Re^5$, it must be a continuous function. Therefore, the set constituted by the optimal strategy is upper semicontinuous by maximum theorem, which leads to the result that $\mu_t^{e*}(y_t)$ is continuous in y_t. Combining with the assumption that $\mu^{e*}(y_1) = 0$ and $S^e(\mu^e) \le_{as} C \cdot \mu^e$, we can take sufficiently small $\eta > 0$ such that $y_1 + \eta + S^e(\mu^{e*}(y_1 + \eta)) < y_2$ except for a set with lebesque measure of 0. Now we prove that $\lim_{a \to 0^+}(V_t(y_1 + \eta + a) - V_t(y_1 + \eta))/a > c^e$.

Falsely presume that $\lim_{a \to 0^+}(V_t(y_1 + \eta + a) - V_t(y_1 + \eta))/a \le c^e$, then by the concavity of V_t we have

$$c^e \cdot a \ge V_t(y_1 + \eta + a) - V_t(y_1 + \eta) \tag{A.7}$$

for all $a \ge 0$. For expression simplicity, we define

$$\varphi(z, x) = -\mathbb{E}[H(z - D(d^*(x)))] + \alpha \mathbb{E}\left[V_{t+1}\left(z + x_{t+1} - D(d^*(x)), x_{t+2}, \cdots, x_{t+L-1}, S^r(\mu^{r*}(x))\right)\right]. \tag{A.8}$$

Substitute a in inequality (A.7) with any realization of the stochastic expedited supply $s^e(\mu^{e*}(y_1 + \eta))$. Abbreviate $\eta + s^e(\mu^{e*}(y_1 + \eta))$ as η', then the following inequalities hold.

$$\begin{aligned} c^e \cdot s^e(\mu^{e*}(y_1 + \eta)) \ge\ & V_t(y_1 + \eta') - V_t(y_1 + \eta) \\ =\ & J_t(y_1 + \eta', \mu^{e*}(y_1 + \eta'), \mu^{r*}(y_1 + \eta'), d^*(y_1 + \eta')) \\ & - J_t(x_1 + \eta, \mu^{e*}(y_1 + \eta), \mu^{r*}(y_1 + \eta), d^*(y_1 + \eta)) \\ >_{as}\ & J_t(y_1 + \eta', 0, \mu^{r*}(y_1 + \eta), d^*(y_1 + \eta)) \\ & - J_t(y_1 + \eta, \mu^{e*}(y_1 + \eta), \mu^{r*}(y_1 + \eta), d^*(y_1 + \eta)) \\ =\ & \varphi(y_1 + \eta + s^e(\mu^{e*}(y_1 + \eta)), y_1 + \eta) - \mathbb{E}\left[\varphi(y_1 + \eta + S^e(\mu^{e*}(y_1 + \eta)), y_1 + \eta)\right] \\ & + c^e \cdot \mu^{e*}(y_1 + \eta)) \end{aligned} \tag{A.9}$$

The strict inequality holds due to the fact that $\mu^{e*}(y_1 + \eta') > 0$ almost surely. Therefore, with probability 1, the revenue associated with the policy $(q^{e*}(y_1 + \eta'), q^{r*}(y_1 + \eta'), d^*(y_1 + \eta'))$ must be strictly greater than the revenue with policy $(0, q^{r*}(y_1 + \eta), d^*(y_1 + \eta))$. Then, taking expectation to both side of (A.9) yields to the contradiction $c^e > c^e$. Therefore, $\lim_{a \to 0^+}(V_t(y_1 + \eta + a) - V_t(y_1 + \eta))/a > c^e$. Combining with the concavity of $V_t(y_t)$, we further have

$$\lim_{a \to 0^+} \frac{V_t(x + a) - V_t(x)}{a} > c^e \tag{A.10}$$

for all $x < y_1 + \eta$.

Lemma 7. *For any given outstanding order $\mathbf{x}_{t+}$, If $\partial \tilde{V}(y_t, \mathbf{x}_{t+})/\partial y_t$ exists on $y_t = \hat{y}$, and $\mu^{e*}(\hat{y}, \mathbf{x}_{t+}) = 0$, then*

$$\frac{\partial \tilde{V}(y_t, \mathbf{x}_{t+})}{\partial y_t}\Big|_{y_t=\widehat{y}} \leq c^e. \tag{A.11}$$

A.3.2. Proof of Lemma 7

Since $\mathbf{x}_{t+}$ is fixed, we can simplify the notations of V_t, μ_t^{e*}, μ_t^{r*}, d_t^* in the same way as in the proof of Lemma 6 and define $\varphi(z,x)$ as in equation (A.8). Then, the desired partial derivative is represented as normal derivative on $\widehat{y}$. By envelope theorem, we have

$$\begin{aligned} V_t{}'(\widehat{y}) &= \frac{\partial \mathbb{E}\left[\varphi\left(y + \tilde{S}^e(\mu^{e*}(\widehat{y})), \widehat{y}\right)\right]}{\partial y}\Big|_{y=\widehat{y}} \\ &= \frac{\partial \varphi(y, \widehat{y})}{\partial y}\Big|_{y=\widehat{y}} \\ &= \lim_{\Delta \to 0^+} \frac{\varphi(y+\Delta, \widehat{y}) - \varphi(y, \widehat{y})}{\Delta}\Big|_{y=\widehat{y}} \\ &= \lim_{\Delta \to 0^+} \frac{\mathbb{E}\left[\varphi\left(y + \tilde{S}^e(\Delta), \widehat{y}\right)\right] - \varphi(y, \widehat{y})}{\Delta}\Big|_{y=\widehat{y}} \\ &= \lim_{\Delta \to 0^+} \frac{J_t(y, \mathbf{x}_{t+}, \Delta, \mu^{r*}(\widehat{y}), d^*(\widehat{y})) + c^e\Delta - J_t(y, \mathbf{x}_{t+}, 0, \mu^{r*}(\widehat{y}), d^*(\widehat{y}))}{\Delta}\Big|_{y=\widehat{y}} \\ &\leq c^e. \end{aligned} \tag{A.12}$$

The second equality and the last inequality is due to the fact that $\mu^{e*}(\widehat{y}) = 0$. The forth equality holds since we have

$$\begin{aligned} \lim_{\Delta \to 0^+} \frac{\mathbb{E}\left[\varphi\left(y + \tilde{S}^e(\Delta), \widehat{y}\right)\right] - \varphi(y, \widehat{y})}{\Delta} &= \lim_{\Delta \to 0^+} \int_0^{c \cdot \Delta} \frac{\varphi(y+z, \widehat{y}) - \varphi(y, \widehat{y})}{z} \frac{z}{\Delta} \mathrm{d}F_{\tilde{S}^e(\Delta)(z)} \\ &= \lim_{\Delta \to 0^+} \frac{\varphi(y+\Delta, \widehat{y}) - \varphi(y, \widehat{y})}{\Delta} \frac{\mathbb{E}\left[\tilde{S}^e(\Delta)\right]}{\Delta} \\ &= \lim_{\Delta \to 0^+} \frac{\varphi(y+\Delta, \widehat{y}) - \varphi(y, \widehat{y})}{\Delta}. \end{aligned}$$

First note that $\mu^e(q^e)$ is strictly increasing in q^e and $\mu^e(0) = 0$, we only need to show that μ^{e*} has a state-dependent almost-threshold structure. For any fixed outstanding order $\mathbf{x}_{t+}$, simplify the notations of μ_t^{e*}, μ_t^{r*}, d_t^* as functions of y_t. Combining Assumption 3 and the fact that the mean demand is finite, it is not difficult to show that $\mu_t^{e*}(y_t) = 0$ for large enough y_t by using induction. Define $\overline{y}_t(\mathbf{x}_{t+})$ as the minimal value such that $\mu_t^{e*}(y_t) = 0$ for $y_t \in [\overline{y}_t(\mathbf{x}_{t+}), \infty)$. $\overline{y}_t(\mathbf{x}_{t+})$ is attainable since $\mu_t^{e*}(y_t)$ is a continuous function.

If $\mu^{e*}(y_t)$ is strictly positive on $(-\infty, \overline{y}_t(\mathbf{x}_{t+}))$ except for countable points, then the proof is done. Otherwise, there must exists an interval $[y_1, y_2] \subset (-\infty, \overline{y}_t(\mathbf{x}_{t+}))$ with positive length, such that $\mu^{e*}(y_t) = 0$ on $[y_1, y_2]$. We will show contradiction in this situation as follows.

First, we can extend the interval $[y_1, y_2]$, such that $\exists \delta > 0$, and $\mu^{e*}(y_t) > 0$ for $y_t \in (y_2, y_2 + \delta)$. That is to say, we push y_2 to the rightmost bound that leads to a zero optimal expedited order. Then, from Lemma 6,

$$\lim_{a \to 0^+} \frac{V_t(y_t + a, \mathbf{x}_{t+}) - V_t(y_t, \mathbf{x}_{t+})}{a} > c^e \tag{A.13}$$

for $y_t < y_2$. Since $V_t(y_t, \mathbf{x}_{t+})$ is jointly concave, it must be differentiable in y_t except for countable points. Therefore, we could choose $\widehat{y} \in [y_1, y_2]$ such that $\frac{\partial V_t(y_t, \mathbf{x}_{t+})}{\partial y_t}$ exists on $y_t = \widehat{y}$. Then, Lemma 7 requires that $\frac{\partial V_t(y_t, \mathbf{x}_{t+})}{\partial y_t}\Big|_{y_t=\widehat{y}} \leq c^e$, which contradicts the inequality (A.13).

If $V_t(\mathbf{x}_t)$ is partially differentiable with respect to y_t, even for the case $y_1 = y_2$, i.e., the interval degenerates to a singleton, the contradiction still exists for $\frac{\partial V_t(y_t, \mathbf{x}_{t+})}{\partial y_t}\Big|_{y_t=y_1}$. Therefore, we must have $\Theta_t = \varnothing$, and the optimal policy is a state-dependent strict-threshold policy.

Appendix A.4 Proof of Condition 1

If function $g(\mathbf{v})$, $\mathbf{v}\in\Re^n$, *is antimultimodular, then* $\varphi(\mathbf{v},y)=g(v_1-y,v_2-v_1,v_3-v_2,\cdots,v_n-v_{n-1})$ *is supermodular on* $(\mathbf{v},y)$. *Define* $[f+g](\mathbf{v})=f(\mathbf{v})+g(\mathbf{v})$. *First, we calculate the derivatives of* $\varphi(\mathbf{v},y)$. *The first-order derivatives are*

$$\frac{\partial}{\partial v_i}\varphi(\mathbf{v},y)$$

$$=\begin{cases}[g_i-g_{i+1}](v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i\le n-1,\\ g_n(v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i=n,\end{cases}$$

$$\frac{\partial}{\partial y}\varphi(\mathbf{v},y)=-g_1(v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}).$$

The second-order derivatives are

$$\frac{\partial^2}{\partial v_i\partial v_j}\varphi(\mathbf{v},y)$$

$$=\begin{cases}\left[g_{i,j}-g_{i,j+1}-g_{i+1,j}+g_{i+1,j+1}\right](v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i\le n-1, j\le n-1,\\ \left[g_{n,j}-g_{n,j+1}\right](v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i=n, j\le n-1,\\ \left[g_{i,n}-g_{i+1,n}\right](v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i\le n-1, j=n\\ g_{n,n}(v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i=n, j=n,\end{cases}$$

$$\frac{\partial^2}{\partial v_i\partial y}\varphi(\mathbf{v},y)$$

$$=\begin{cases}\left[-g_{i,1}+g_{i+1,1}\right](v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i\le n-1,\\ -g_{n,1}(v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i=n,\end{cases}$$

$$\frac{\partial^2}{\partial y\partial v_i}\varphi(\mathbf{v},y)$$

$$=\begin{cases}\left[-g_{1,i}+g_{1,i+1}\right](v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i\le n-1,\\ -g_{1,n}(v_1-y,v_2-v_1,\cdots,v_n-v_{n-1}), & \text{for } i=n.\end{cases}$$

If g is twice continuously differentiable, then the antimultimodularity of $g(\mathbf{v})$ *is equivalent to*

$$\frac{\partial^2\varphi}{\partial v_i\partial v_j}(\mathbf{v},y)\ge 0,\ \text{for all } i\neq j, \tag{A.14}$$

$$\frac{\partial^2}{\partial v_i\partial y}\varphi(\mathbf{v},y)\ge 0,\ \text{for all } i, \tag{A.15}$$

$$\frac{\partial^2}{\partial y\partial v_i}\varphi(\mathbf{v},y)\ge 0,\ \text{for all } i. \tag{A.16}$$

With the expressions of the second-order derivatives of φ, *condition* (A.14) *is inequivalent to*

$$g_{i,j}-g_{i,j+1}-g_{i+1,j}+g_{i+1,j+1}\ge 0,\ \text{for } i\neq j, i\le n-1, j\le n-1,$$

$$g_{n,j}-g_{n,j+1}\ge 0, j\le n-1,$$

$$g_{i,n}-g_{i+1,n}\ge 0, i\le n-1.$$

Condition (*A*.15) *is inequivalent to*

$$-g_{i,1}+g_{i+1,1}\ge 0,\ \text{for } i\le n-1,$$

$$-g_{n,1} \ge 0.$$

Condition (A.16) *is inequivalent to*

$$-g_{1,i} + g_{1,i+1} \ge 0, \text{ for } i \le n-1,$$

$$-g_{1,n} \ge 0.$$

To summarize, conditions (A.14), (A.15), *and* (A.16) *could be equivalently rewritten as*

$$g_{i,j} + g_{i+1,j+1} \ge g_{i,j+1} + g_{i+1,j}, \text{ for } i \ne j, i \le n-1, j \le n-1, \tag{A.17}$$

$$g_{n,1} \ge g_{n,2} \ge \cdots \ge g_{n,n}, \tag{A.18}$$

$$g_{1,n} \ge g_{2,n} \ge \cdots \ge g_{n,n}, \tag{A.19}$$

$$0 \ge g_{n,1} \ge g_{n-1,1} \ge \cdots \ge g_{1,1}, \tag{A.20}$$

$$0 \ge g_{1,n} \ge g_{1,n-1} \ge \cdots \ge g_{1,1}. \tag{A.21}$$

Appendix A.5 Proof of Condition 2

Let $\psi(x,\mathbf{v}) = \mathbb{E}[g(\mathbf{v} + \widehat{Y}(x)\mathbf{e}_1)]$, $(x,\mathbf{v}) \in \mathbb{R}^{n+1}$. *With the definition of PST,* $\psi(x,\mathbf{v})$ *should be antimultimodular in* $(x,\mathbf{v})$. *That is, we need to prove that* $\psi(x,\mathbf{v})$ *satisfies*

$$\psi_{i,j} + \psi_{i+1,j+1} \ge \psi_{i,j+1} + \psi_{i+1,j}, \text{ for } i \ne j, i \le n, j \le n, \tag{A.22}$$

$$\psi_{n+1,1} \ge \psi_{n+1,2} \ge \cdots \ge \psi_{n+1,n+1}, \tag{A.23}$$

$$\psi_{1,n+1} \ge \psi_{2,n+1} \ge \cdots \ge \psi_{n+1,n+1}. \tag{A.24}$$

$$0 \ge \psi_{n+1,1} \ge \psi_{n,1} \ge \cdots \ge \psi_{1,1}, \tag{A.25}$$

$$0 \ge \psi_{1,n+1} \ge \psi_{1,n} \ge \cdots \ge \psi_{1,1}. \tag{A.26}$$

First, we write the derivatives of ψ *in terms of* g. *Let* $\widehat{Y}(x) = \phi(x,Z)$, *where* $\phi(x,z)$ *is a deterministic function of* (x,z) (*following* Theorem 1 *in* Feng and Shanthikumar, 2016) *and* Z *is a random variable with support* $\mathcal{Z} \in \mathbb{R}$. *Let* $\phi_x(x,z) = \frac{\partial}{\partial x}\phi(x,z)$ *and* $\phi_z(x,z) = \frac{\partial}{\partial z}\phi(x,z)$.

$$\psi(x,\mathbf{v}) = \mathbb{E}[g(\mathbf{v} + \widehat{Y}(x)\mathbf{e}_1)] = \mathbb{E}[g(\mathbf{v} + \phi(x,Z)\mathbf{e}_1)]$$

With assuming the order of integral and differential is exchangeable, we obtain

$$\psi_i(x,\mathbf{v}) = \begin{cases} \dfrac{\partial}{\partial x}\mathbb{E}[g(\mathbf{v} + \phi(x,Z)\mathbf{e}_1)], & \text{for } i = 1, \\ \dfrac{\partial}{\partial v_{i-1}}\mathbb{E}[g(\mathbf{v} + \phi(x,Z)\mathbf{e}_1)], & \text{for } 2 \le i \le n+1, \end{cases}$$

$$= \begin{cases} \mathbb{E}[g_1(\mathbf{v} + \phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)], & \text{for } i = 1, \\ \mathbb{E}[g_{i-1}(\mathbf{v} + \phi(x,Z)\mathbf{e}_1)], & \text{for } 2 \le i \le n+1. \end{cases}$$

$$\psi_{i,j}(x,\mathbf{v})$$

$$= \begin{cases} \mathbb{E}\Big[g_1(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_{xx}(x,Z)+g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x^2(x,Z)\Big], & \text{for } i=1, j=1, \\ \mathbb{E}\Big[g_{i-1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big], & \text{for } 2\le i\le n+1, j=1, \\ \mathbb{E}\Big[g_{1,j-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big], & \text{for } i=1, 2\le j\le n+1, \\ \mathbb{E}\Big[g_{i-1,j-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big], & \text{for } 2\le i\le n+1, 2\le j\le n+1. \end{cases}$$

Then, condition (A.22)

$$\psi_{i,j}+\psi_{i+1,j+1}\ge\psi_{i,j+1}+\psi_{i+1,j}, \text{ for } i\ne j, i\le n, j\le n,$$

becomes

$$\mathbb{E}\Big[g_{1,j-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big]+\mathbb{E}\Big[g_{1,j}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]$$

$$\ge\mathbb{E}\Big[g_{1,j}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big]+\mathbb{E}\Big[g_{1,j-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big], \text{for } i=1, 2\le j\le n,$$

$$\mathbb{E}\Big[g_{i-1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big]+\mathbb{E}\Big[g_{i,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]$$

$$\ge\mathbb{E}\Big[g_{i-1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]+\mathbb{E}\Big[g_{i,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big], \text{for } 2\le i\le n, j=1,$$

$$\mathbb{E}\Big[g_{i-1,j-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]+\mathbb{E}\Big[g_{i,j}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]$$

$$\ge\mathbb{E}\Big[g_{i-1,j}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]+\mathbb{E}\Big[g_{i,j-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big], \text{for } 2\le i\le n, 2\le j\le n, i\ne j.$$

The last inequality is ensured by (A.17).
Condition (A.23)

$$\psi_{n+1,1}\ge\psi_{n+1,2}\ge\cdots\ge\psi_{n+1,n+1},$$

becomes

$$\mathbb{E}\Big[g_{n,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big]$$

$$\ge\mathbb{E}\Big[g_{n,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]$$

$$\ge\mathbb{E}\Big[g_{n,2}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]$$

$$\ge\cdots$$

$$\ge\mathbb{E}\Big[g_{n,n}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big].$$

Except the first inequality, all the inequalities are ensured by (A.18).
Condition (A.24)

$$\psi_{1,n+1}\ge\psi_{2,n+1}\ge\cdots\ge\psi_{n+1,n+1},$$

becomes

$$\mathbb{E}\Big[g_{1,n}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\Big]$$

$$\ge\mathbb{E}\Big[g_{1,n}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big]$$

$$\geq \mathbb{E}\left[g_{2,n}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\right]$$
$$\geq \cdots$$
$$\geq \mathbb{E}\left[g_{n,n}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\right].$$

Except the first inequality, all the inequalities are ensured by (A.19).
To prove conditions (A.25) *and* (A.26), *we need the following assumption.*

Assumption 4. $\phi_x \geq 0$, for all x, z.
Condition (A.25)

$$0 \geq \psi_{n+1,1} \geq \psi_{n,1} \geq \cdots \geq \psi_{1,1},$$

becomes

$$0 \geq \mathbb{E}\left[g_{n,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\right]$$
$$\geq \mathbb{E}\left[g_{n-1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\right]$$
$$\geq \cdots$$
$$\geq \mathbb{E}\left[g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\right]$$
$$\geq \mathbb{E}\left[g_1(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_{xx}(x,Z)+g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x^2(x,Z)\right].$$

With Assumption 4, *except the last inequality, all the inequalities are ensured by* (A.20).
Condition (A.26)

$$0 \geq \psi_{1,n+1} \geq \psi_{1,n} \geq \cdots \geq \psi_{1,1},$$

becomes

$$0 \geq \mathbb{E}\left[g_{1,n}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\right]$$
$$\geq \mathbb{E}\left[g_{1,n-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\right]$$
$$\geq \cdots$$
$$\geq \mathbb{E}\left[g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x(x,Z)\right]$$
$$\geq \mathbb{E}\left[g_1(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_{xx}(x,Z)+g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_x^2(x,Z)\right].$$

With Assumption 4, *except the last inequality, all the inequalities are ensured by* (A.21).
To summarize, for proving conditions (A.22) (A.26), *under* Assumption 4, *we only need to prove the following inequalities.*

Assumption 5. $\phi_x \leq 1$, for all x, z.

$$\begin{aligned} & \mathbb{E}\left[g_{1,j-1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)(\phi_x(x,Z)-1)\right] \\ \geq \ & \mathbb{E}\left[g_{1,j}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)(\phi_x(x,Z)-1)\right], \text{for } i=1, 2 \leq j \leq n, \end{aligned} \tag{A.27}$$

$$\begin{aligned} & \mathbb{E}\left[g_{i-1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)(\phi_x(x,Z)-1)\right] \\ \geq \ & \mathbb{E}\left[g_{i,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)(\phi_x(x,Z)-1)\right], \text{for } 2 \leq i \leq n, j=1, \end{aligned} \tag{A.28}$$

$$\mathbb{E}\Big[g_{n,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)(\phi_x(x,Z)-1)\Big]\geq 0, \tag{A.29}$$

$$\mathbb{E}\Big[g_{1,n}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)(\phi_x(x,Z)-1)\Big]\geq 0, \tag{A.30}$$

$$\mathbb{E}\Big[g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big(\phi_x(x,Z)-\phi_x^2(x,Z)\Big)\Big]\geq \mathbb{E}[g_1(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_{xx}(x,Z)]. \tag{A.31}$$

With (A.20) *and* (A.21), *and* Assumption 5, *conditions* (A.27) (A.30) *are satisfied. Thus, a sufficient condition for PST is* $0\leq\phi_x(x,z)\leq 1$, *for all* $x\in\mathscr{X}, z\in\mathscr{Z}$, *and*

$$\mathbb{E}\Big[g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big(\phi_x(x,Z)-\phi_x^2(x,Z)\Big)\Big]\geq \mathbb{E}[g_1(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_{xx}(x,Z)],$$

for any antimultimodular function $g(\mathbf{v})$ and $x\in\mathscr{X}$.

Appendix A.6 Lemma 8 and Its Proof

To prove Lemma 3, we first summarize the key properties of antimultimodular functions in Lemma 8 below.

Lemma 8. (i) If $g(\mathbf{v})$ and $f(\mathbf{v})$ are both antimultimodular and $\alpha>0$, then $\alpha g(\mathbf{v})$, $g(-\mathbf{v})$, $f(\mathbf{v})+g(\mathbf{v})$, and $\tilde{g}(\mathbf{v})=g(v_n,v_{n-1},\cdots,v_1)$ are all antimultimodular. (ii) If $g(\mathbf{v},D)$ is antimultimodular in $\mathbf{v}$ for any realization of random variable D, then $\mathbb{E}[g(v,D)]$ is antimultimodular in $\mathbf{v}$. (iii) If $f:\Re\to\Re$ is concave, then $f(\mathbf{v})=f(v_1+\cdots+v_n)$ is antimultimodular. Especially, a one-dimensional function is antimultimodular if and only if it is concave. (iv) If $g(\mathbf{v})$ is antimultimodular, then $g(v_1,\cdots,v_{i-1},w_1+\cdots+w_m,v_{i+1},\cdots v_n)$ is antimultimodular in $(v_1,\cdots,v_{i-1},w_1,\cdots,w_m,v_{i+1},\cdots,v_n)$. (v) If $g(\mathbf{v})$ is antimultimodular, then for any partition $\mathbf{v}=(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$, $h(\mathbf{v}_1,\mathbf{v}_3)=g(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)|_{\mathbf{v}_2=\widehat{\mathbf{v}}_2}$ is antimultimodular in $(\mathbf{v}_1,\mathbf{v}_3)$. (vi) If $g(\mathbf{v},\mathbf{w})$ is an anti-multimodular function defined on polyhedron $V=\{(\mathbf{v},\mathbf{w})|\mathbf{v}\in S\subset\Re^n, \mathbf{w}\in A(\mathbf{v})\subset\Re^m\}$, then $f(\mathbf{v})=\max_{\mathbf{w}\in A(\mathbf{v})}g(\mathbf{v},\mathbf{w})$ is antimultimodular on S. (vii) If $g(\mathbf{v},\zeta)$ is antimultimodular on polyhedron V. Let $\zeta^*(\mathbf{v})$ denote the smallest value of ζ that maximizes $g(\mathbf{v},\zeta)$. Then, $\zeta^*(\mathbf{v})$ is decreasing in $\mathbf{v}$, and $-1\leq\Delta_{v_n}\zeta^*\leq\Delta_{v_{n-1}}\zeta^*\leq\cdots\leq\Delta_{v_1}\zeta^*\leq 0$, where $\Delta_{v_i}\zeta^*(\mathbf{v})=(\zeta^*(\mathbf{v}+\delta\mathbf{e}_i)-\zeta^*(\mathbf{v}))/\delta$, δ is a small positive number, and $\mathbf{e}_i$ is a vector in which the i-th element equals 1 and all the others equal 0.

All of the above properties hold for antimultimodular functions defined on a discrete state space as well. Except for part (v), the proofs of the properties are analogous to Lemma 2 and Theorem 1 in Li and Yu (2014). The proof of Lemma 8(v) is as follows.

Proof. Denote $\mathbf{v}_1=(v_1,\cdots,v_i)$, $\mathbf{v}_2=(v_{i+1},\cdots,v_j)$, $\mathbf{v}_3=(v_{j+1},\cdots,v_n)$, where $1<i<j<n$. We intend to show that for any $\widehat{\mathbf{v}}_2=(\widehat{v}_{i+1},\cdots,\widehat{v}_j)$, $g(\mathbf{v}_1,\widehat{\mathbf{v}}_2,\mathbf{v}_3)$ is antimultimodular in $(\mathbf{v}_1,\mathbf{v}_3)$. This is equivalent with the property that

$$h(y,\mathbf{v}_1,\mathbf{v}_3)=g\Big(v_1-y,v_2-v_1,\cdots,v_i-v_{i-1},\widehat{v}_{i+1},\cdots,\widehat{v}_j,v_{j+1}-v_i,\cdots,v_n-v_{n-1}\Big)$$

is supermodular in $(y,\mathbf{v}_1,\mathbf{v}_3)$. From the antimultimodularity of g, we know that $w(y,\mathbf{v})=g(v_1-y,v_2-v_1,\cdots,v_n-v_{n-1})$ is supermodular on $\Re^{n+1}$. Therefore, restricting w on $\{(y,\mathbf{v}),v_{i+1}=v_i+\widehat{v}_{i+1},v_{i+2}=v_i+\widehat{v}_{i+1}+\widehat{v}_{i+2},\cdots,v_j=v_i+\widehat{v}_{i+1}+\cdots+\widehat{v}_j\}$ preserves the supermodularity, i.e.,

$$g\Big(v_1-y,v_2-v_1,\cdots,v_i-v_{i-1},\widehat{v}_{i+1},\cdots,\widehat{v}_j,v_{j+1}-v_i-\sum_{k=i+1}^{j}\widehat{v}_k,v_{j+2}-v_{j+1},\cdots,v_n-v_{n-1}\Big)$$

is supermodular in $(y,\mathbf{v}_1,\mathbf{v}_3)$. This implies the supermodularity of $h(y,\mathbf{v}_1,\mathbf{v}_3)$. ∎

Parts (i) to (v) of Lemma 8 exposit functional operators that preserve antimultimodularity and provide tools for deriving the antimultimodularity of the one-period profit function. In particular, part (v) is used to obtain the monotonicity of the optimal expedited order under the structure of antimultimodularity in Corollary 1. Part (vi) indicates that anti-multimodularity is preserved under partial maximization of the right-side variables, which guarantees the carry-over of antimultimodularity during the dynamic programming iterations. Part (vii) characterizes that the maximizer for the last variable in an antimultimodular function is decreasing in the other variables, with higher change rates in the variables closer to it in sequence. In addition, the change rate is always bounded by -1 and 0. This property is the key to the characterization of the optimal regular ordering quantity and price.

With part (i), we know that parts (vi) and (vii) also hold under the altered variable sequence, i.e., $\max_{\mathbf{w}\in A(\mathbf{v})}\tilde{g}(\mathbf{w},\mathbf{v})$ is antimultimodular in $\mathbf{v}$, $\zeta^*(\mathbf{v})$ as the smallest maximizer of $\tilde{g}(\zeta,\mathbf{v})$ is non-increasing in $\mathbf{v}$, and $-1\leq\Delta_{v_1}\zeta^*\leq\Delta_{v_2}\zeta^*\leq\cdots\leq\Delta_{v_n}\zeta^*\leq 0$. Note that antimultimodularity must be defined with a specified variable sequence. Parts

(vi) and (vii) are applicable only in the cases where we are maximizing variables located on the left/right end in the sequence. There is no similar result for the maximization of the variables in the middle.

Appendix A.7 Proof of Lemma 3

(i) For any antimultimodular function $g(\mathbf{v}):\Re^n\to\Re$, we know that $g(-\mathbf{v})$ is also antimultimodular by Lemma 8 (i). From the definition of PST, there exists $\{\widehat{Y}(x), x\in\Re\}$ satisfying $\widehat{Y}(x)=_{a.s.}Y(x)$ for $x\in\mathscr{X}$, such that $\mathbb{E}[g(-\mathbf{v}-\widehat{Y}(x)\mathbf{e}_1)]$ is antimultimodular in $(x,\mathbf{v})$. Then, employing Lemma 8 (i) again, $\mathbb{E}[g(\mathbf{v}-\widehat{Y}(-x)\mathbf{e}_1)]$ is antimultimodular. Therefore, we can define $\widehat{Z}(x)=-\widehat{Y}(-x)$ hence $Z(x)\in PST$.

(ii) Denote $\widehat{Y}(x)$ and $\widehat{Z}(y)$ as the extensions for $Y(x)$ and $Z(y)$ that satisfies the definition of *PST*, respectively. Then, for any antimultimodular function $g(\mathbf{v}):\Re^n\to\Re$, $\mathbb{E}[g(\mathbf{v}+\widehat{Z}(y)\mathbf{e}_1)]$ is antimultimodular in $(y,\mathbf{v})$. From the result in (iii) and the independency, $E_Y[E_Z[g(\mathbf{v}+\widehat{Z}(\widehat{Y}(x))\mathbf{e}_1)]]$ is antimultimodular in $(x,\mathbf{v})$. Therefore, $\widehat{Z}(\widehat{Y}(x))$ is the desired extension for $Z(Y(x))$ hence $Z(Y(x))\in PST$.

(iii) We only prove (a) and (c), then (b) and (d) are immediate from Lemma 8 (i).

(a). Denote $\widehat{Y}(x):\Re\to\Re$ as the extension of $Y(x)$ that satisfies the definition of *PST*, then for any antimultimodular function $g(\mathbf{v}):\Re^n\to\Re$, $\mathbb{E}[g(\mathbf{v}+\widehat{Y}(x)\mathbf{e}_1)]$ defined on $\Re^{n+1}$ is antimultimodular in $(x,\mathbf{v})$. Restricting it on $\Re^n\times\mathscr{X}$ and recall that $\widehat{Y}(x)=_{a.s.}Y(x)$ on $\mathscr{X}$, we obtain the desired antimultimodularity of $\mathbb{E}[g(\mathbf{v}+Y(x)\mathbf{e}_1)]$.

(c). From the result in (iii) (a), we know that $\mathbb{E}[g(\mathbf{v}+Y(x)\mathbf{e}_1)]$ is antimultimodular in $(x,\mathbf{v})$. This is equivalent with: $h(x,y,\mathbf{v})=\mathbb{E}[g(Y(x-y)+v_1-x,v_2-v_1,\cdots,v_n-v_{n-1})]$ is supermodular. Therefore, restricting h on $\{(x,y,v_1,\cdots,v_n),v_1=x\}$ preserves the supermodularity, thus we have $\mathbb{E}[g(Y(x),v_2,\cdots,v_n)]$ is antimultimodular in $(x, v_2,\cdots,v_n)$.

Appendix A.8 Proof of Conditions 3

For first-argument increasing function g, we have $g_1\geq 0$. Then, with assuming $\phi_{xx}\leq 0$, the right-hand side of condition (A.31) is a negative value. From Assumption 4 and 5, we know $0\leq\phi_x(x,z)\leq 1$, which implies $\phi_x(x,z)-\phi_x^2(x,z)\geq 0$. From (A.20), we have $g_{1,1}\leq 0$. Then, we derive

$$\mathbb{E}\Big[g_{1,1}(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\Big(\phi_x(x,Z)-\phi_x^2(x,Z)\Big)\Big]\geq 0\geq\mathbb{E}[g_1(\mathbf{v}+\phi(x,Z)\mathbf{e}_1)\phi_{xx}(x,Z)].$$

Therefore, a sufficient conditions for IPST is $0\leq\phi_x(x,z)\leq 1$, $\phi_{xx}(x,z)\leq 0$, and $\mathbb{E}\phi_x(x,Z)-\mathbb{E}\phi_x^2(x,Z)\leq-\frac{M}{L}\mathbb{E}\phi_{xx}(x,Z)$, for all $x\in\mathscr{X}$ and $z\in\mathscr{Z}$.

Appendix A.9 Proof of Lemma 4

The lemma is proved directly from the definitions of PST and IPST.

Appendix A.10 Proof of Lemmas 5

(i) For any first-argument increasing antimultimodular function $g(\mathbf{v}):\Re^n\to\Re$, we know that $g(-\mathbf{v})$ is also antimultimodular by Lemma 8 (i). From the definition of *IPST*, there exists $\{\widehat{Y}(x), x\in\Re\}$ satisfying $\widehat{Y}(x)=_{a.s.}Y(x)$ for $x\in\mathscr{X}$, such that $\mathbb{E}[g(-\mathbf{v}-\widehat{Y}(x)\mathbf{e}_1)]$ is antimultimodular in $(x,\mathbf{v})$. Then, by employing Lemma 8 (i) again, $\mathbb{E}[g(\mathbf{v}-\widehat{Y}(-x)\mathbf{e}_1)]$ is antimultimodular. We can define $\widehat{Z}(x)=-\widehat{Y}(-x)$ and therefore $Z(x)\in IPST$.

(ii) Denote $\widehat{Y}(x)$ and $\widehat{Z}(y)$ as the extensions for $Y(x)$ and $Z(y)$ that satisfies the definition of *IPST*, respectively. Then, for any first-argument increasing antimultimodular function $g(\mathbf{v}):\Re^n\to\Re$, $\mathbb{E}[g(\mathbf{v}+\widehat{Z}(y)\mathbf{e}_1)]$ is antimultimodular in $(y,\mathbf{v})$. From the result in (iii) and the independency, $E_Y[E_Z[g(\mathbf{v}+\widehat{Z}(\widehat{Y}(x))\mathbf{e}_1)]]$ is antimultimodular in $(x,\mathbf{v})$. Therefore, $\widehat{Z}(\widehat{Y}(x))$ is the desired extension for $Z(Y(x))$ hence $Z(Y(x))\in IPST$.

(iii) We only prove (a) and (c), then (b) and (d) are immediate from Lemma 8 (i).

(a). Denote $\widehat{Y}(x):\Re\to\Re$ as the extension of $Y(x)$ that satisfies the definition of *IPST*, then for any first-argument increasing antimultimodular function $g(\mathbf{v}):\Re^n\to\Re$, $\mathbb{E}[g(\mathbf{v}+\widehat{Y}(x)\mathbf{e}_1)]$ defined on $\Re^{n+1}$ is antimultimodular in $(x,\mathbf{v})$. Restricting it on $\Re^n\times\mathscr{X}$ and recalling that $\widehat{Y}(x)=_{a.s.}Y(x)$ on $\mathscr{X}$, we obtain the desired antimultimodularity of $\mathbb{E}[g(\mathbf{v}+Y(x)\mathbf{e}_1)]$.

(c). From the result in (iii) (a), we know that $\mathbb{E}[g(\mathbf{v}+Y(x)\mathbf{e}_1)]$ is antimultimodular in $(x,\mathbf{v})$. This is equivalent to that $h(x,y,\mathbf{v})=\mathbb{E}[g(Y(x-y)+v_1-x,v_2-v_1,\cdots,v_n-v_{n-1})]$ is supermodular in $(x,y,\mathbf{v})$. Setting $v_1=x$, we obtain that $\mathbb{E}[g(Y(x-y),v_2-x,v_3-v_2,\cdots,v_n-v_{n-1})]$ is supermodular in $(x, v_2, v_3, \cdots, v_n, y)$. Then, by the definition of antimultimodularity, $\mathbb{E}[g(Y(x),v_2,\cdots,v_n)]$ is antimultimodular in $(x,v_2,\cdots,v_n)$.

Appendix A.11 Proof of Theorem 2

First note that the definition domain of V_t and J_t are all polyhedron that is compatible with antimultimodularity. Replace the decision variable d_t by $-d_t$, and denote $\widehat{d}_t = -d_t$, $\widehat{D}_t(\widehat{d}_t) = -D_t(-\widehat{d}_t)$, and $\tilde{R}(\widehat{d}_t) = R(-\widehat{d}_t)$. Let $\tilde{\mathscr{A}}_t = \Re_+^2 \times [-\overline{d}, -\underline{d}]$ as the transformed action space. Moreover, we extend the definition domain of $V_t(\mathbf{x})$ to $\Re^L$ in the natural way and denote the extended function as $\widehat{V}_t(\mathbf{x})$. When restricting on $\Re \times \Re_+^{L-1}$, $\widehat{V}_t$ coincides with V_t. Then, we can rewrite the transformed DP problem as follows.

$$\widehat{V}_t(\mathbf{x}_t) = \max_{(q_t^e, q_t^r, \widehat{d}_t) \in \tilde{\mathscr{A}}} \widehat{J}_t\left(\mathbf{x}_t, q_t^e, q_t^r, \widehat{d}_t\right), \tag{A.32}$$

$$\widehat{J}_t\left(\mathbf{x}_t, q_t^e, q_t^r, \widehat{d}_t\right) = \tilde{R}\left(\widehat{d}_t\right) - c^e \mathbb{E}[S^e(q_t^e)] - c^r \mathbb{E}[S^r(q_t^r)] - \mathbb{E}\left[H\left(y_t + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right)\right)\right]$$
$$+\alpha\mathbb{E}\left[\widehat{V}_{t+1}\left(y_t + x_{t+1} + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right), x_{t+2}, \cdots, x_{t+L-1}, S^r(q_t^r)\right)\right]. \tag{A.33}$$

First note that $\widehat{V}_{T+1} \equiv 0$ hence is antimultimodular in $\mathbf{x}_{T+1}$. Suppose $\widehat{V}_{k+1}$ is antimultimodular, we intend to show that both $\widehat{J}_k$ and $\widehat{V}_k$ are antimultimodular.

By Assumption 1, $R(d_k)$ is concave in d_k, hence $\tilde{R}(\widehat{d}_t)$ is also concave in $\widehat{d}_t$. By Lemma 8 (iii), $\tilde{R}(\widehat{d}_t)$ is antimultimodular in $\widehat{d}_t$ therefore also antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$. The second and third terms in $\widehat{J}_k$ are antimultimodular because the fact that linear functions are concave and $S^e(q_t^e)$, $S^r(q_t^r) \in PST$.

Furthermore, $-H$ is concave therefore also antimultimodular. From Lemma 3 (iii), we know that $\mathbb{E}[-H(y_k + \widehat{D}(\widehat{d}_k + S^e(q_k^e)))]$ is antimultimodular in $(q_k^e, \widehat{d}_k, y_k)$. Combining with the fact that $D(d)$ has an additive form, we have $-\mathbb{E}[H(y_k + S^e(q_k^e) + \widehat{D}(\widehat{d}_k))]$ is antimultimodular in $(q_k^e, \widehat{d}_k, \mathbf{x}_k, q_k^r)$. Analogously, $\mathbb{E}[\widehat{V}_{k+1}(y_k + x_{k+1} + S^e(q_k^e) + \widehat{D}(\widehat{d}_k), x_{k+2}, \cdots, x_{k+L-1}, S^r(q_k^r)))$ is also antimultimodular in $(q_k^e, \widehat{d}_k, \mathbf{x}_k, q_k^r)$ by Lemma 3 (iii) and the inductive hypothesis. Summarize all above results, $\widehat{J}_t$ is antimultimodular in the desired sequences. Then, by Lemma 8 (vi), $\widehat{V}_k$ is antimultimodular in $\mathbf{x}_k$. Using induction, we have verified the antimultimodularity of $\widehat{J}_t$ and $\widehat{V}_t$ for each t. Finally, Restricting $\mathbf{x}_t$ on $\Re \times \Re_+^{L-1}$ yields to the desired antimultimodularity of J_t and V_t.

Appendix A.12 Proof of Theorem 3

First, we extend the definition domain of $V_t(\mathbf{x})$ to $\Re^L$ in the natural way and denote the extended function as $\tilde{V}_t(\mathbf{x})$. When restricting on $\Re \times \Re_+^{L-1}$, $\tilde{V}_t$ coincides with V_t. Then, the transformed DP problem can be rewritten as follows.

$$\tilde{V}_t(\mathbf{x}_t) = \max_{(q_t^e, q_t^r, \widehat{d}_t) \in \tilde{\mathscr{A}}} \tilde{J}_t\left(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r\right), \tag{A.34}$$

$$\text{where } \tilde{J}_t\left(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r\right) = \tilde{R}\left(\widehat{d}_t\right) - c^e q_t^e - c^r q_t^r - \mathbb{E}\left[H\left(y_t + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right)\right)\right]$$
$$+\alpha\mathbb{E}\left[\tilde{V}_{t+1}\left(y_t + x_{t+1} + S^e(q_t^e) + \widehat{D}\left(\widehat{d}_t\right), x_{t+2}, \cdots, x_{t+L-1}, S^r(q_t^r)\right)\right]. \tag{A.35}$$

$\tilde{V}_{T+1}$ is antimultimodular in $\mathbf{x}_{T+1}$ because $\tilde{V}_{T+1} \equiv 0$. Suppose that the antimultimodularity of $\tilde{V}_{t+1}$ has been obtained, we next show that both $\tilde{J}_t$ and $\tilde{V}_t$. are antimultimodular.

From Assumption 1, $R(d_t)$ is ccave in d_t, and therefore $\tilde{R}(\widehat{d}_t) = R(-\widehat{d}_t)$ is concave in $\widehat{d}_t$. By Lemma 8 (iii), $\tilde{R}(\widehat{d}_t)$ is antimultimodular in $\widehat{d}_t$ and therefore also antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$. $-c^e q_t^e$ is antimultimodular in q_t^e and $-c^r q_t^r$ is antimultimodular in q_t^r, because linear functions are concave.

Furthermore, $-H$ is increasing concave and therefore antimultimodular. With Lemma 5 (iii) and $S^e \in IPST$, we know that $\mathbb{E}[-H(y_t + \widehat{D}(\widehat{d}_t + S^e(q_t^e)))]$ is antimultimodular in $(q_t^e, \widehat{d}_t, y_t)$. Combining with that $D(d_t)$ has an additive form, we obtain that $-\mathbb{E}[H(y_t + S^e(q_t^e) + \widehat{D}(\widehat{d}_t))]$ is antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$. Analogously, $\mathbb{E}[\tilde{V}_{t+1}(y_t + x_{t+1} + S^e(q_t^e) + \widehat{D}(\widehat{d}_t), x_{k+2}, \cdots, x_{k+L-1}, S^r(q_t^r)$)) is also antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$ by Lemma 5 (iii) and the inductive hypothesis.

Summarizing all the results above, $\tilde{J}_t$ is antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$. Then, by Lemma 8 (vi), $\tilde{V}_t$ is antimultimodular in $\mathbf{x}_t$. Using induction, we have verified the antimultimodularity of $\tilde{J}_t$ and $\tilde{V}_t$ for each t. Finally, Restricting $\mathbf{x}_t$ on $\Re \times \Re_+^{L-1}$ leads to the desired antimultimodularity of J_t and V_t.

Appendix A.13 Proof of Theorem 4

From Theorem 2 we see that $\tilde{J}_t(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$ is antimultimodular in $(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$. Then by Lemma 8 (vi), $J_t^1(\mathbf{x}_t, q_t^r) = \max_{q_t^e \geq 0, \widehat{d}_t \in [\underline{d}, \overline{d}]} \tilde{J}_t(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$ is antimultimodular in $(\mathbf{x}_t, q_t^r)$, and $J_t^2(d_t, \mathbf{x}_t) = \max_{q_t^e \geq 0, q_t^r \geq 0} \tilde{J}_t(q_t^e, \widehat{d}_t, \mathbf{x}_t, q_t^r)$ is antimultimodular in $(\widehat{d}_t, \mathbf{x}_t)$. Note that J_t is antimultimodular implies that it is jointly concave. Therefore, $(q_t^{e*}(\mathbf{x}_t), q_t^{r*}(\mathbf{x}_t), d_t^*(\mathbf{x}_t))$ can be obtained by maximizing J_t^1 or J_t^2. Finally, we can derive the desired monotonicity and change rate inequality for $q_t^{r*}(\mathbf{x}_t)$ and $d_t^*(\mathbf{x}_t)$ by using Lemma 8 (vii).

Appendix A.14 Proof of Corollary 1

Since $S^e(q^e)$, $S^r(q^r)$ are PST and $D(d)$ has an additive form, we can extend their definition domain to $\Re$ and denote as $\widehat{S}^e(q^e)$, $\widehat{S}^r(q^r)$ and $\widehat{D}(d)$. By using similar techniques in the proof of Theorem 2, we can prove that

$$\widehat{J}\prime_t(\mathbf{x}_t, q_t^e, q_t^r, d_t) = R(d_t) - c^e \mathbb{E}\left[\widehat{S}^e(q_t^e)\right] - c^r \mathbb{E}\left[\widehat{S}^r(q_t^r)\right] - \mathbb{E}\left[H\left(y_t + \widehat{S}^e(q_t^e) - \widehat{D}(d_t)\right)\right]$$

$$+\alpha \mathbb{E}\left[\widehat{V}_{t+1}\left(y_t + x_{t+1} + \widehat{S}^e(q_t^e) - \widehat{D}(d_t), x_{t+2}, \cdots, x_{t+L-1}, \widehat{S}^r(q_t^r)\right)\right]$$

defined on $\Re^{L+3}$ is antimultimodular in $(q_t^e, y_t, -d_t, x_{t+1}, \cdots, x_{t+L-1}, q_t^r)$. Then, by Lemma 8 (v), after fixing $(x_{t+1}, \cdots, x_{t+L-1}) = \mathbf{x}_{t+}$, $\widehat{J}\prime_t$ is antimultimodular in $(q_t^e, y_t, -d_t, q_t^r)$. Restricting it on $\{q_t^e \geq 0, q_t^r \geq 0, d_t \in [\underline{d}, \overline{d}]\}$ yields to the result that $J(\mathbf{x}_t, q_t^e, q_t^r, d_t)$ is antimultimodular in $(q_t^e, y_t, -d_t, q_t^r)$ after fixing $(x_{t+1}, \cdots, x_{t+L-1})$. Therefore,

$$J_t^3(q_t^e, y_t) = \max_{q_t^r \geq 0, d_t \in [\underline{d}, \overline{d}]} J_t(\mathbf{x}_t, q_t^e, q_t^r, d_t)|_{(x_{t+1}, \cdots, x_{t+L-1}) = \mathbf{x}_{t+}}$$

is antimultimodular in (q_t^e, y_t). As a consequence, for any given outstanding order $\mathbf{x}_{t+}$, $q_t^{e*}(\mathbf{x}_t)$ is decreasing in y_t, and the state-dependent almost-threshold policy proposed in Theorem 1 turns to be a state-dependent monotone threshold policy.

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