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Research Article

Numerical Solutions to Optimal Portfolio Selection and Consumption Strategies under Stochastic Volatility

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Based on the method of dynamic programming, this paper uses analysis methods governed by the nonlinear and inhomogeneous partial differential equation to study modern portfolio management problems with stochastic volatility, incomplete markets, limited investment scope, and constant relative risk aversion (CRRA). In this paper, a three-level Crank–Nicolson finite difference scheme is used to determine numerical solutions under this general setting. One of the main contributions of this paper is to apply this three-level technology to solve the portfolio selection problem. In addition, we have used a technique to deal with the nonlinear term, which is another novelty in performing the Crank–Nicolson algorithm. The Crank–Nicolson algorithm has also been extended to third-order accuracy by performing Richardson's extrapolation. The accuracy of the proposed algorithm is much higher than the traditional finite difference method. Lastly, experiments are conducted to show the performance of the proposed algorithm.

1. Introduction

How to optimally allocate assets and optimally consume are extremely important and difficult topics in portfolio management [1–3]. These topics are important not only for theoretical consideration but also for applications in the financial industry. Early studies usually assumed the volatility of the risky asset to be a constant. However, in recent years, researchers found that volatility should be modeled as stochastic rather than deterministic [4–7]. This adds further complication to the problem. The optimal asset allocation and optimal consumption strategies are governed by the Hamilton–Jacobi–Bellman (HJB) equation. Due to the nonlinearity and inhomogeneity of this partial differential equation, no exact solution has been found. Furthermore, even numerical solutions are not available. In this paper, we present an accurate and efficient numerical method for solving this equation and generate the first set of accurate numerical solutions for this problem.

Due to the importance of portfolio selection under stochastic volatilities, several important theoretical works have been carried out, and exact solutions have been obtained under certain special settings, such as no consumption [8–10], complete markets which means that the stock movement and the volatility movement are either perfectly correlated or perfectly anticorrelated [9–12], or when investors have unit elasticity of intertemporal substitution of consumption [13].

In this paper, we consider this optimal stochastic control problem under a general setting: stochastic volatility, incomplete markets, finite investment horizons, and CRRA utility. Our numerical method combines a three-level Crank–Nicolson scheme and Richardson's extrapolation technique. The Crank–Nicolson scheme has second-order accuracy in terms of discretization error, and Richardson's extrapolation technique further improves the accuracy. We verify that our numerical method is accurate and efficient.

This paper is organized as follows. In Section 2, we describe the model for financial market, the stochastic control optimization procedure, and the governing HJB equation for the optimal asset allocation and consumption strategies. In Section 3, we present our numerical method for solving the HJB equation. In Section 4, we verify the accuracy and the efficiency of our numerical method and present accurate numerical solutions for the optimal asset allocation strategy and the optimal consumption strategy. In the last section, we present our conclusions.

2. Financial Market and Stochastic Control

We consider a market consisting of one riskless asset B_t, whose price is governed by

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \tag{1}$$

with a constant risk-free interest rate r and a risky asset S_t modeled as

$$\mathrm{d}S_t = S_t\left[\mu(v_t) + \sigma(v_t)\mathrm{d}W_t^S\right]. \tag{2}$$

In (2), $\mu(v_t)$ and $\sigma(v_t)$ are the return and the stochastic volatility of the stock price S_t, respectively. $v_t = \sigma_t^2$ is the stochastic variance of S_t. Empirical studies show presence of mean reversion in the stock movements [14]. Heston model [5] is selected for v_t, namely,

$$\mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \xi\sqrt{v_t}\mathrm{d}W_t^v. \tag{3}$$

Here, $\mathrm{d}W_t^v$ and $\mathrm{d}W_t^S$ are the increments of the Wiener processes under a probability P. The correlation between $\mathrm{d}W_t^v$ and $\mathrm{d}W_t^S$ is ρ, namely, $\mathrm{Corr}((\mathrm{d}v_t/v_t), (\mathrm{d}S_t/S_t)) = \rho\mathrm{d}t$. We assume ρ is a constant. In (3), θ is the long-run average variance (i.e., as t tends to infinity, the expected value of v_t tends to θ), κ is the rate at which v_t reverts to θ, and ξ is the volatility of the stock variance v_t. The parameters κ, θ, ξ are positive constants and need to satisfy the Feller condition, $2\kappa\theta > \xi^2$, to ensure that v_t is strictly positive. The risk premia is defined as

$$A = \frac{\mu_t - r}{\sigma_t^2} \equiv \frac{\mu_t - r}{v_t}. \tag{4}$$

Following [1, 5, 15–17], we assume A is a constant. This means the stock excess return is proportional to the stock variance.

Consider an investor who has an initial wealth w_0 and needs to determine strategies for asset allocation and consumption over an investment horizon $[0, T]$. Let w_t be the investor's wealth at time t. The strategies consist of an asset allocation rate φ_t and a consumption rate c_t, which mean he/she allocates $\varphi_t w_t$ to the risky asset and $(1 - \varphi_t)w_t$ to the riskless asset at time t and consumes $c_t\mathrm{d}t$ over the time interval $[t, t + \mathrm{d}t]$. Thus, under the strategies φ_t and c_t, the wealth process is governed by

$$\mathrm{d}w_t = \frac{\varphi_t w_t}{S_t}\mathrm{d}S_t + (1 - \varphi_t)w_t r\mathrm{d}t - c_t\mathrm{d}t. \tag{5}$$

The goal is to maximize the expected utilities over the investment horizon, namely,

$$\sup_{\varphi_t, c_t} E\left[\int_0^T \alpha e^{-\beta t}u_1(c_t)\mathrm{d}t + (1-\alpha)e^{-\beta T}u_2(w_T)\right]. \tag{6}$$

In (6), φ_t and c_t are control variables for this optimization problem. E is the expectation operator under the probability P. β is the subjective discount rate, namely, the time preference of the investor. The larger β is, the more weight the investor puts on the present than on the future. The parameter α determines the relative importance between intertemporal consumption and the terminal wealth. $u_1(\cdot)$ and $u_2(\cdot)$ are the investor's utility functions which measure the investor's degree of satisfaction with the outcomes from intertemporal consumption and terminal wealth, respectively.

CRRA utility functions have been widely adopted for modeling investors' behavior. Therefore, we adopt the CRRA utility function for $u_1(\cdot)$ and $u_2(\cdot)$:

$$\begin{cases} u_1(c_t) = \dfrac{a_c c_t^{1-\gamma}}{1-\gamma}, & \text{for } \gamma \neq 1, \\ a_c \log(c_t), & \text{for } \gamma = 1, \end{cases}$$
$$\begin{cases} u_2(w_T) = \dfrac{a_w w_T^{1-\gamma}}{1-\gamma}, & \text{for } \gamma \neq 1, \\ a_w \log(w_T), & \text{for } \gamma = 1, \end{cases} \tag{7}$$

where γ, a_c, and a_w are positive constants. Since $u_1(\cdot)$ and $u_2(\cdot)$ stand for the intertemporal consumption utility and the terminal wealth utility of the same investor, we use the same γ in $u_1(\cdot)$ and $u_2(\cdot)$. However, a_c and a_w can be different since c and w have different dimensions.

Let $V(t, w, v)$ be the value function of problem (6), which is given by

$$\begin{aligned} V(t, w_t, v_t) = \sup_{\varphi_t, c_t} E\bigg[\int_t^T \alpha e^{-\beta t'}u_1(c_t')\mathrm{d}t' \\ + (1-\alpha)e^{-\beta T}u_2(w_T) \,\Big|\, \mathscr{F}_t\bigg], \end{aligned} \tag{8}$$

where $\mathscr{F}_T$ is the filtration associated to the stochastic processes in this problem. The terminal condition is obtained by setting $t = T$ in (8):

$$V(T, w_T, v_T) = (1-\alpha)e^{-\beta T}u_2(w_T). \tag{9}$$

Based on the HJB dynamic programming procedure, V is governed by

$$\begin{aligned} 0 = \sup_{\varphi, c}\bigg[&\alpha e^{-\beta t}u_1(c) + V_t + (rw + \varphi w A v - c)V_w \\ &+ \kappa(\theta - v)V_v + \frac{1}{2}\varphi^2 w^2 v V_{ww} + \varphi w \rho \xi v V_{wv} + \frac{1}{2}\xi^2 v V_{vv}\bigg], \end{aligned} \tag{10}$$

with the optimal strategies φ^* and c^* determined by

$$\varphi^* = -\frac{AV_w + \rho\xi V_{wv}}{wV_{ww}}, \tag{11}$$

$$c^* = \left(\frac{V_w}{a_c \alpha e^{-\beta t}}\right)^{-(1/\gamma)}. \tag{12}$$

After substituting expressions (11) and (12) into (10), one obtains an equation for the value function V:

$$\frac{\gamma}{1-\gamma}\left(a_c \alpha e^{-\beta t}\right)^{(1/\gamma)} V_w^{(1-\gamma/-\gamma)} + rwV_w + V_t + \kappa(\theta - v)V_v + \frac{1}{2}\xi^2 v V_{vv} - \frac{1}{2} v \frac{\left(AV_w + \rho\xi V_{wv}\right)^2}{V_{ww}} = 0. \tag{13}$$

Based on the terminal condition and the scaling property of (13), it is reasonable to guess that

$$V(\tau, w, v) = e^{-\beta(T-\tau)} \frac{a_w w^{1-\gamma}}{1-\gamma} f(\tau, v)^\gamma, \tag{14}$$

where $\tau = T - t$, (13) becomes

$$\begin{aligned} 0 = &-f_\tau + \frac{1}{2}\xi^2 v f_{vv} + \left(\kappa(\theta - v) + \frac{1-\gamma}{\gamma} A\rho\xi v\right) f_v \\ &- \frac{1}{2}(1-\gamma)\left(1-\rho^2\right)\xi^2 v \frac{f_v^2}{f} \\ &+ \left(\frac{(1-\gamma)A^2}{2\gamma^2} v + \frac{(1-\gamma)r}{\gamma} - \frac{\beta}{\gamma}\right) f + \left(\frac{\alpha a_c}{a_w}\right)^{(1/\gamma)}, \end{aligned} \tag{15}$$

with

$$f(0, v) = (1-\alpha)^{(1/\gamma)}, \tag{16}$$

and (11) and (12) become

$$\varphi^* = \frac{A}{\gamma} + \rho\xi \frac{f_v}{f}, \tag{17}$$

$$\frac{c^*}{w} = \left(\frac{\alpha a_c}{a_w}\right)^{(1/\gamma)} f^{-1}. \tag{18}$$

Equation (15) is a nonlinear and inhomogeneous partial differential equation. Since no closed-form solution is available for this equation, numerical computation plays a critical role for studying this important practical problem in modern finance. However, there are even no numerical solutions available in the literature.

3. Numerical Method

In this section, we develop a numerical method for solving (15). For the sake of conciseness of our expressions, we rewrite (15) as

$$\begin{aligned} &-f_\tau + a_1 v f_{vv} + \left(a_2 v + a_3\right) f_v + a_4 v \frac{f_v^2}{f} + \left(a_5 v + a_6\right) f \\ &+ \alpha^{(1/\gamma)} a_7 = 0, \end{aligned} \tag{19}$$

with the initial condition $f(0, v) = (1-\alpha)^{(1/\gamma)}$, where

$$\begin{aligned} a_1 &= \frac{1}{2}\xi^2, \\ a_2 &= \frac{1-\gamma}{\gamma} A\rho\xi - \kappa, \\ a_3 &= \kappa\theta, \\ a_4 &= -\frac{1}{2}(1-\gamma)\xi^2\left(1-\rho^2\right), \\ a_5 &= \frac{1-\gamma}{2\gamma^2} A^2, \\ a_6 &= \frac{(1-\gamma)r}{\gamma} - \frac{\beta}{\gamma}, \\ a_7 &= \left(\frac{a_c}{a_w}\right)^{(1/\gamma)}. \end{aligned} \tag{20}$$

3.1. Crank–Nicolson Scheme and Richardson's Extrapolation. We use a three-level Crank–Nicolson scheme (see [18–21]) of second-order accuracy to solve the nonlinear and inhomogeneous partial differential equation given by (19) and use Richardson's extrapolation technique for further improving accuracy. Numerically, one can only solve (19) over a finite domain $v \in [0, v_{\max}]$. Since the boundary conditions at $v = 0$ and at $v = v_{\max}$ are not known, we use one-sided difference method at these two numerical boundaries. Step sizes $\Delta\tau$ and Δv are used to discretize τ and v, respectively. Thus, $\tau = n\Delta\tau$ and $v = m\Delta v$. We adopt the standard notation $f_m^n = f(n\Delta\tau, m\Delta v)$.

The three-level Crank–Nicolson scheme involves the levels $n-1$, n, and $n+1$. It is straightforward to discretize all linear terms in (19) with second-order errors, namely,

$$\begin{aligned} \left(f_\tau\right)_m^n &= \frac{f_m^{n+1} - f_m^{n-1}}{2\Delta\tau} + O\left(\Delta\tau^2\right), \\ \left(f_v\right)_m^n &= \frac{1}{2}\left(\frac{f_{m+1}^{n+1} - f_{m-1}^{n+1}}{2\Delta v} + \frac{f_{m+1}^{n-1} - f_{m-1}^{n-1}}{2\Delta v}\right) + O\left(\Delta\tau^2\right) \\ &\quad + O\left(\Delta v^2\right), \\ \left(f_{vv}\right)_m^n &= \frac{1}{2}\left(\frac{f_{m+1}^{n+1} - 2f_m^{n+1} + f_{m-1}^{n+1}}{\Delta v^2} + \frac{f_{m+1}^{n-1} - 2f_m^{n-1} + f_{m-1}^{n-1}}{\Delta v^2}\right) \\ &\quad + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right). \end{aligned} \tag{21}$$

The nonlinear term f_v^2/f has two factors f_v/f and f_v. We discretize the factor f_v/f at level n and approximate the factor f_v as an average between f_v at level $n-1$ and that at level $n+1$, namely,

$$\begin{aligned}\left(\frac{f_v^2}{f}\right)_m^n &= \left(\frac{f_v}{f}\right)_m^n \frac{1}{2}\left(\left(f_v\right)_m^{n+1} + \left(f_v\right)_m^{n-1}\right) + O\left(\Delta\tau^2\right) \\ &= \frac{f_{m+1}^n - f_{m-1}^n}{4\Delta v f_m^n}\left(\frac{f_{m+1}^{n+1} - f_{m-1}^{n+1}}{2\Delta v} + \frac{f_{m+1}^{n-1} - f_{m-1}^{n-1}}{2\Delta v}\right) \\ &\quad + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right).\end{aligned} \tag{22}$$

This discretization scheme leads to a set of linear equations. Based on the expressions given by (21) and (22), equation (19) can be discretized as

$$e_1(m) f_{m-1}^{n+1} + e_2(m) f_m^{n+1} + e_3(m) f_{m+1}^{n+1} = e_4(m) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \tag{23}$$

for $0 < m < M$ and $n \geq 2$, where M is the maximal value of m and

$$\begin{aligned} e_1(m) &= \frac{a_1 m}{\Delta v} - \frac{\lambda(m)}{2\Delta v}, \\ e_2(m) &= \frac{-2a_1 m}{\Delta v} - \frac{1}{\Delta\tau} + a_5 m\Delta v + a_6, \\ e_3(m) &= \frac{a_1 m}{\Delta v} + \frac{\lambda(m)}{2\Delta v}, \\ e_4(m) &= -\frac{1}{\Delta\tau} f_m^{n-1} - 2\alpha^{(1/\gamma)} a_7 - \frac{a_1 m}{\Delta v}\left(f_{m+1}^{n-1} - 2f_m^{n-1} + f_{m-1}^{n-1}\right) \\ &\quad - \frac{\lambda(m)}{2\Delta v}\left(f_{m+1}^{n-1} - f_{m-1}^{n-1}\right) - \left(a_5 m\Delta v + a_6\right) f_m^{n-1}, \end{aligned} \tag{24}$$

with

$$\lambda(m) = a_2 m\Delta v + a_3 + \frac{a_4 m\left(f_{m+1}^n - f_{m-1}^n\right)}{2f_m^n}. \tag{25}$$

Since (23) is not applicable to the boundaries at $m = 0$ and $m = M$, we used one-sided difference technique to discretize (19) and obtained the boundary equations in the following. It is straightforward to show that, at $m = 0$, we have

$$\begin{aligned} f_v(0,\tau) &= \frac{1}{4\Delta v}\left(-3f_0^{n+1} + 4f_1^{n+1} - f_2^{n+1} - 3f_0^{n-1} + 4f_1^{n-1} - f_2^{n-1}\right) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \\ f_{vv}(0,\tau) &= \frac{1}{2\Delta v^2}\left(2f_0^{n+1} - 5f_1^{n+1} + 4f_2^{n+1} - f_3^{n+1} + 2f_0^{n-1} - 5f_1^{n-1} + 4f_2^{n-1} - f_3^{n-1}\right) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \\ \frac{f_v^2}{f}(0,\tau) &= \frac{-3f_0^n + 4f_1^n - f_2^n}{8\Delta v^2 f_0^n}\left(-3f_0^{n+1} + 4f_1^{n+1} - f_2^{n+1} - 3f_0^{n-1} + 4f_1^{n-1} - f_2^{n-1}\right) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \end{aligned} \tag{26}$$

and at $m = M$, we have

$$\begin{aligned} f_v(v_{\max},\tau) &= \frac{1}{4\Delta v}\left(3f_M^{n+1} - 4f_{M-1}^{n+1} + f_{M-2}^{n+1} + 3f_M^{n-1} - 4f_{M-1}^{n-1} + f_{M-2}^{n-1}\right) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \\ f_{vv}(v_{\max},\tau) &= \frac{1}{2\Delta v^2}\left(2f_M^{n+1} - 5f_{M-1}^{n+1} + 4f_{M-2}^{n+1} - f_{M-3}^{n+1} + 2f_M^{n-1} - 5f_{M-1}^{n-1} + 4f_{M-2}^{n-1} - f_{M-3}^{n-1}\right) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \\ \frac{f_v^2}{f}(v_{\max},\tau) &= \frac{3f_M^n - 4f_{M-1}^n + f_{M-2}^n}{8\Delta v^2 f_M^n}\left(3f_M^{n+1} - 4f_{M-1}^{n+1} + f_{M-2}^{n+1} + 3f_M^{n-1} - 4f_{M-1}^{n-1} + f_{M-2}^{n-1}\right) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right). \end{aligned} \tag{27}$$

By substituting these expressions into (19), we have

$$d_1(0) f_0^{n+1} + d_2(0) f_1^{n+1} + d_3(0) f_2^{n+1} + d_4(0) f_3^{n+1} = d_5(0) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \tag{28}$$

$$d_4(M) f_{M-3}^{n+1} + d_3(M) f_{M-2}^{n+1} + d_2(M) f_{M-1}^{n+1} + d_1(M) f_M^{n+1} = d_5(M) + O\left(\Delta\tau^2\right) + O\left(\Delta v^2\right), \tag{29}$$

where

$$
\begin{aligned}
d_1(0) &= -\frac{3\lambda(0)}{2\Delta v} - \frac{1}{\Delta\tau} + a_6, \\
d_2(0) &= \frac{2\lambda(0)}{\Delta v}, \\
d_3(0) &= -\frac{\lambda(0)}{2\Delta v}, \\
d_4(0) &= 0, \\
d_5(0) &= -\frac{1}{\Delta\tau} f_0^{n-1} - 2\alpha^{\frac{1}{\gamma}} a_7 - \frac{\lambda(0)}{2\Delta v}\left(-3 f_0^{n-1} + 4 f_1^{n-1} - f_2^{n-1}\right) - a_6 f_0^{n-1},
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
d_1(M) &= \frac{2a_1 M}{\Delta v} + \frac{3\lambda(M)}{2\Delta v} - \frac{1}{\Delta\tau} + a_5 M\Delta v + a_6, \\
d_2(M) &= \frac{-5a_1 M}{\Delta v} - \frac{2\lambda(M)}{\Delta v}, \\
d_3(M) &= \frac{4a_1 M}{\Delta v} + \frac{\lambda(M)}{2\Delta v}, \\
d_4(M) &= \frac{-a_1 M}{\Delta v}, \\
d_5(M) &= -\frac{1}{\Delta\tau} f_M^{n-1} - 2\alpha^{(1/\gamma)} a_7 - \frac{a_1 M}{\Delta v}\left(2 f_M^{n-1} - 5 f_{M-1}^{n-1} + 4 f_{M-2}^{n-1} - f_{M-3}^{n-1}\right) \\
&\quad - \frac{\lambda(M)}{2\Delta v}\left(3 f_M^{n-1} - 4 f_{M-1}^{n-1} + f_{M-2}^{n-1}\right) - \left(a_5 M\Delta v + a_6\right) f_M^{n-1},
\end{aligned}
\tag{31}
$$

with

$$
\begin{aligned}
\lambda(0) &= a_3, \\
\lambda(M) &= a_2 M\Delta v + a_3 + \frac{a_4 M\left(3 f_M^n - 4 f_{M-1}^n + f_{M-2}^n\right)}{2 f_M^n}.
\end{aligned}
\tag{32}
$$

From (23), (28), and (29), the numerical solution of (19) for $n \geq 2$ is determined by the following system of linear equations:

$$
\begin{bmatrix}
d_1(0) & d_2(0) & d_3(0) & d_4(0) & & \\
e_1(1) & e_2(1) & e_3(1) & & & \\
 & e_1(2) & e_2(2) & e_3(2) & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & e_1(M-1) & e_2(M-1) & e_3(M-1) \\
 & & d_4(M) & d_3(M) & d_2(M) & d_1(M)
\end{bmatrix}
\begin{bmatrix}
f_0^{n+1} \\ f_1^{n+1} \\ f_2^{n+1} \\ \vdots \\ f_{M-1}^{n+1} \\ f_M^{n+1}
\end{bmatrix}
=
\begin{bmatrix}
d_5(0) \\ e_4(1) \\ e_4(2) \\ \vdots \\ e_4(M-1) \\ d_5(M)
\end{bmatrix}.
\tag{33}
$$

After eliminating $d_3(0)$, $d_4(0)$, $d_3(M)$, and $d_4(M)$, (33) can be transformed into the following tridiagonal matrix form for $n \geq 2$:

$$\begin{bmatrix} e_2(0) & e_3(0) & & & & \\ e_1(1) & e_2(1) & e_3(1) & & & \\ & e_1(2) & e_2(2) & e_3(2) & & \\ & & \ddots & \ddots & \ddots & \\ & & & e_1(M-1) & e_2(M-1) & e_3(M-1) \\ & & & & e_1(M) & e_2(M) \end{bmatrix} \begin{bmatrix} f_0^{n+1} \\ f_1^{n+1} \\ f_2^{n+1} \\ \vdots \\ f_{M-1}^{n+1} \\ f_M^{n+1} \end{bmatrix} = \begin{bmatrix} e_4(0) \\ e_4(1) \\ e_4(2) \\ \vdots \\ e_4(M-1) \\ e_4(M) \end{bmatrix}, \quad (34)$$

where

$$\begin{aligned} e_2(0) &= d_1(0) - \frac{e_3(2)d_3(0) - e_2(2)d_4(0)}{e_3(2)e_3(1)} e_1(1), \\ e_3(0) &= d_2(0) - \frac{d_4(0)}{e_3(2)} e_1(2) - \frac{e_3(2)d_3(0) - e_2(2)d_4(0)}{e_3(2)e_3(1)} e_2(1), \\ e_4(0) &= d_5(0) - \frac{d_4(0)}{e_3(2)} e_4(2) - \frac{e_3(2)d_3(0) - e_2(2)d_4(0)}{e_3(2)e_3(1)} e_4(1), \\ e_1(M) &= d_2(M) - \frac{d_4(M)}{e_1(M-2)} e_3(M-2) - \frac{e_1(M-2)d_3(M) - e_2(M-2)d_4(M)}{e_1(M-2)e_1(M-1)} e_2(M-1), \\ e_2(M) &= d_1(M) - \frac{e_1(M-2)d_3(M) - e_2(M-2)d_4(M)}{e_1(M-2)e_1(M-1)} e_3(M-1), \\ e_4(M) &= d_5(M) - \frac{d_4(M)}{e_1(M-2)} e_4(M-2) - \frac{e_1(M-2)d_3(M) - e_2(M-2)d_4(M)}{e_1(M-2)e_1(M-1)} e_4(M-1). \end{aligned} \quad (35)$$

The method given by (21) and (22) is a two-step method, namely, f^{n+1} depends on f^n and f^{n-1}. At the zeroth step, f^0 is given by initial condition (16), namely, $f_m^0 = (1-\alpha)^{(1/\gamma)}$, for $0 \leq m \leq M$. We now determine f^1, the solution at the first step. Performing Taylor expansion on $f(\Delta\tau, v)$ at $\tau = 0$ gives

$$\begin{aligned} f(\Delta\tau, v) = {} & f(0, v) + f_\tau(0, v)\Delta\tau + \frac{1}{2} f_{\tau\tau}(0, v)\Delta\tau^2 \\ & + \frac{1}{6} f_{\tau\tau\tau}(0, v)\Delta\tau^3 + O(\Delta\tau^4), \end{aligned} \quad (36)$$

where $f(0, v)$ is given by (16) and $f_\tau(0, v)$, $f_{\tau\tau}(0, v)$, and $f_{\tau\tau\tau}(0, v)$ can be determined analytically from (19):

$$\begin{aligned} f_\tau(0, v) &= (a_5 v + a_6) f(0, v) + \alpha^{(1/\gamma)} a_7, \\ f_{\tau\tau}(0, v) &= \left(a_5^2 v^2 + (a_2 a_5 + 2a_5 a_6) v + a_3 a_5 + a_6^2\right) f(0, v) + (a_5 v + a_6)\alpha^{(1/\gamma)} a_7, \\ f_{\tau\tau\tau}(0, v) &= \left[a_5^3 v^3 + \left(3a_2 a_5^2 + 3a_6 a_5^2\right) v^2 + \left(a_5 a_2^2 + 3a_5 a_6 a_2 + 2a_1 a_5^2 + 3a_3 a_5^2 + 2a_4 a_5^2 + 3a_5 a_6^2\right) v + a_2 a_3 a_5 + 3a_3 a_6 a_5 + a_6^3\right] \\ & \quad \cdot f(0, v) + (a_2 v + a_3) a_5 \alpha^{(1/\gamma)} a_7 + (a_5 v + a_6)^2 \alpha^{(1/\gamma)} a_7. \end{aligned} \quad (37)$$

The details of derivations for $f_\tau(0,\nu)$, $f_{\tau\tau}(0,\nu)$, and $f_{\tau\tau\tau}(0,\nu)$ are given in Appendix A. From (36), f_m^1 is given by

$$f_m^1 = f_m^0 + f_\tau(0,m\Delta\nu)\Delta\tau + \frac{1}{2}f_{\tau\tau}(0,m\Delta\nu)\Delta\tau^2 + \frac{1}{6}f_{\tau\tau\tau}(0,m\Delta\nu)\Delta\tau^3, \tag{38}$$

with an error of $O(\Delta\tau^4)$.

Knowing f, the numerical solutions of optimal portfolio and consumption rules can be obtained from (17) and (18):

$$\varphi_m^{n*} = \frac{A}{\gamma} + \rho\xi\frac{(f_\nu)_m^n}{f_m^n}, \tag{39}$$

$$\frac{c_m^{n*}}{w} = \left(\frac{\alpha a_c}{a_w}\right)^{(1/\gamma)}\left(f_m^n\right)^{-1}, \tag{40}$$

for $0 \le m \le M$, where f_m^n is given by (34) and $(f_\nu)_m^n$ is given by

$$(f_\nu)_m^n = \begin{cases} \dfrac{-3f_0^n + 4f_1^n - f_2^n}{2\Delta\nu}, & \text{for } m = 0, \\[2ex] \dfrac{f_{m+1}^n - f_{m-1}^n}{2\Delta\nu}, & \text{for } 0 < m < M, \\[2ex] \dfrac{3f_M^n - 4f_{M-1}^n + f_{M-2}^n}{2\Delta\nu}, & \text{for } m = M. \end{cases} \tag{41}$$

In summary, our numerical solutions for f_m^0 and f_m^1 are determined by (16) and (38), respectively, and the numerical solutions for f_m^n with $n \ge 2$ are determined by (34). The numerical solution of f obtained by the three-level Crank–Nicolson scheme has an accuracy of $O(\Delta\tau^2) + O(\Delta\nu^2)$.

3.2. Performing Richardson's Extrapolation. To further improve the accuracy of the numerical method, we apply Richardson's extrapolation technique to f. We will choose $\Delta\nu$ proportional to $\Delta\tau$. Let $f(\tau_n, \nu_m, \Delta\tau)$ represent f_m^n obtained by (34) with a step size $\Delta\tau$. Then,

$$f(\tau_n, \nu_m, \Delta\tau) = f_{\text{exact}}(\tau,\nu) + C_1\Delta\tau^2 + C_2\Delta\tau^3 + O\left(\Delta\tau^4\right), \tag{42}$$

where f_{exact} is the exact value. We perform two computations with the step sizes $\Delta\tau$ and $\Delta\tau/2$, respectively. Then, we have the following two equations:

$$f(\tau_n, \nu_m, \Delta\tau) = f_{\text{exact}}(\tau,\nu) + C_1\Delta\tau^2 + C_2\Delta\tau^3 + O\left(\Delta\tau^4\right), \tag{43}$$

$$f\left(\tau_{2n}, \nu_{2m}, \frac{\Delta\tau}{2}\right) = f_{\text{exact}}(\tau,\nu) + \frac{C_1}{2^2}\left(\Delta\tau^2\right) + \frac{C_2}{2^3}\Delta\tau^3 + O\left(\Delta\tau^4\right). \tag{44}$$

From (43) and (44), we solve f_{exact} and obtain an expression based on Richardson's extrapolation technique:

$$\begin{aligned} f_{\text{extpl.}}(n\Delta\tau, m\Delta\nu) &= \frac{4}{3}f\left(\tau_{2n}, \nu_{2m}, \frac{\Delta\tau}{2}\right) - \frac{1}{3}f(\tau_n, \nu_m, \Delta\tau) \\ &= f_{\text{exact}} + O\left(\Delta\tau^3\right). \end{aligned} \tag{45}$$

After substituting $f_{\text{extpl.}}$ into (39) and (40), we obtain the expressions for φ^* and c^*/w with an accuracy of $O(\Delta\tau^3)$:

$$\varphi^*_{\text{extpl.}} = \frac{A}{\gamma} + \rho\xi\frac{\left(f_{\text{extpl.}}\right)_\nu}{f_{\text{extpl.}}}, \tag{46}$$

$$\frac{c^*_{\text{extpl.}}}{w} = \left(\frac{\alpha a_c}{a_w}\right)^{(1/\gamma)}\left(f_{\text{extpl.}}\right)^{-1}, \tag{47}$$

where $f_{\text{extpl.}}$ is given by (45) and $(f_{\text{extpl.}})_\nu$ is given by

$$\left(\left(f_{\text{extpl.}}\right)_\nu\right)_m^n = \begin{cases} \dfrac{1}{6\Delta\nu}\left[-9\left(f_{\text{extpl.}}\right)_m^n + 16\left(f_{\text{extpl.}}\right)_{m+1}^n - 8\left(f_{\text{extpl.}}\right)_{m+2}^n + \left(f_{\text{extpl.}}\right)_{m+4}^n\right], & \text{for } m \le 1, \\[2ex] \dfrac{1}{6\Delta\nu}\left[\left(f_{\text{extpl.}}\right)_{m-2}^n - 4\left(f_{\text{extpl.}}\right)_{m-1}^n + 4\left(f_{\text{extpl.}}\right)_{m+1}^n - \left(f_{\text{extpl.}}\right)_{m+2}^n\right], & \text{for } 1 < m < M-1, \\[2ex] \dfrac{1}{6\Delta\nu}\left[9\left(f_{\text{extpl.}}\right)_m^n - 16\left(f_{\text{extpl.}}\right)_{m-1}^n + 8\left(f_{\text{extpl.}}\right)_{m-2}^n - \left(f_{\text{extpl.}}\right)_{m-4}^n\right], & \text{for } m \ge M-1. \end{cases} \tag{48}$$

For the purpose of giving a quick understanding of our method, Figure 1 presents a flowchart of the algorithm for solving (19) We also summarize the procedure in words in the following for obtaining the numerical solutions of f, φ^*, c^*, $f_{\text{extpl.}}$, $\varphi^*_{\text{extpl.}}$, and $c^*_{\text{extpl.}}$. Here, we choose $\Delta\nu = \Delta\tau$.

(i) Step 1: initialize f by initial condition (16), namely, $f_m^0 = (1-\alpha)^{(1/\gamma)}$, for $0 \le m \le M$.

(ii) Step 2: initialize f_m^1 by (38) for $0 \le m \le M$.

(iii) Step 3: for $n > 2$, knowing f_m^{n-1} and f_m^n, for $0 \le m \le M$, f_m^{n+1} can be determined from (34), which is in a tridiagonal form and can be easily and efficiently solved. f_m^n has an accuracy of $O(\Delta\tau^2)$.

(iv) Step 4: to obtain φ^* and c^*, we substitute f_m^n from Steps 1–3 into (39) and (40). This provides the

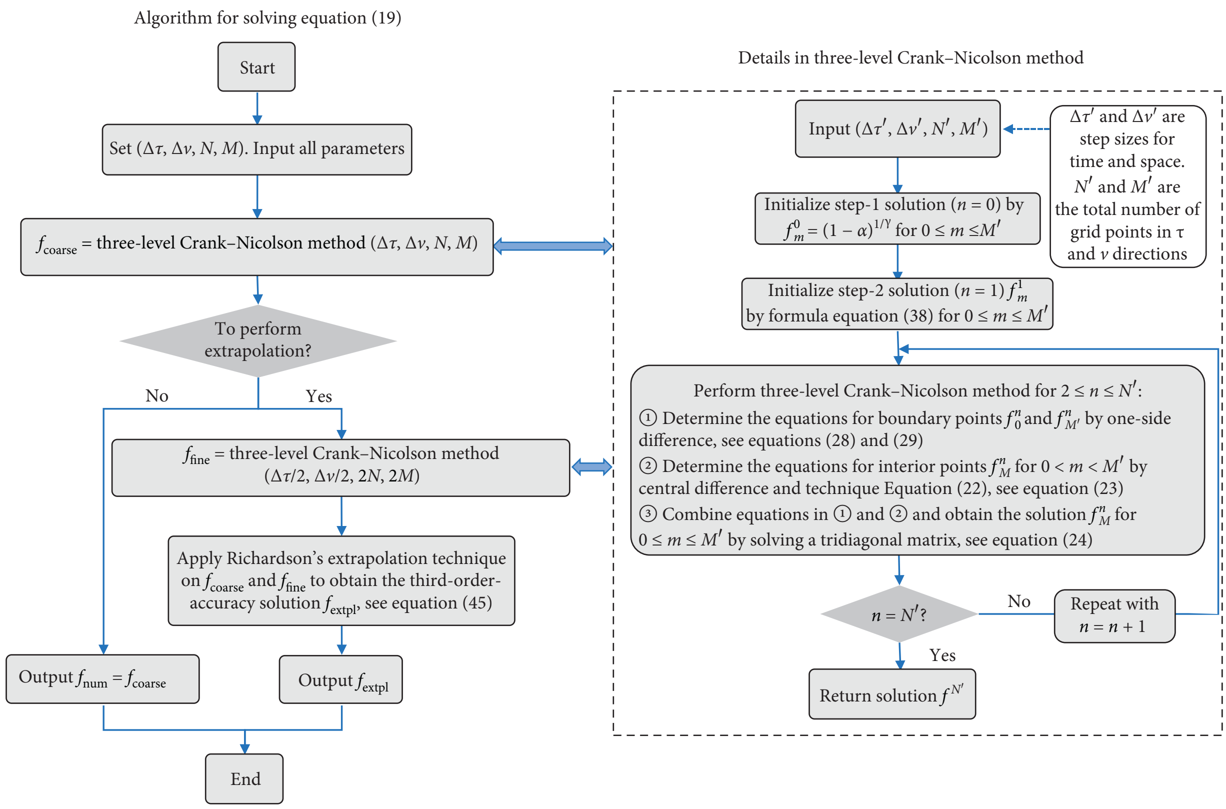

Figure 1: Flowchart of the algorithm.

numerical solutions for optimal strategies φ_m^{*n} and c_m^{*n} without extrapolation, which have accuracy of $O(\Delta\tau^2)$.

(v) Step 5: to obtain $f_{\text{extpl.}}$, $\varphi^*_{\text{extpl.}}$, and $c^*_{\text{extpl.}}$, we repeat Steps 1–3 with the step size $\Delta\tau/2$ to obtain $f(\tau_{2n}, v_{2m}, (\Delta\tau/2))$. Then, from (45), (46), and (47), we obtain $f_{\text{extpl.}}$, $\varphi^*_{\text{extpl.}}$, and $c^*_{\text{extpl.}}$, all of which have accuracy of $O(\Delta\tau^3)$.

In the next section, we will verify the accuracy of the numerical solutions without extrapolation and those with extrapolation.

4. Validation Study of the Numerical Method

Equation (19) is an inhomogeneous equation. However, since the inhomogeneous term $\alpha^{(1/\gamma)}a_7$ affects neither the stability nor the accuracy of the three-level Crank–Nicolson method, it is sufficient to conduct validation studies for the corresponding homogeneous equation, namely, for the case of $\alpha = 0$. Let $\hat{f}$ be the solution of the homogeneous equation of (19), namely, the case of $\alpha = 0$. Then, $\hat{f}$ satisfies

$$-\left(\hat{f}\right)_\tau + a_1 v \left(\hat{f}\right)_{vv} + \left(a_2 v + a_3\right)\left(\hat{f}\right)_v + a_4 v \frac{\left(\hat{f}\right)_v^2}{\hat{f}} + \left(a_5 v + a_6\right)\hat{f} = 0, \tag{49}$$

with the initial condition $\hat{f}(0, v) = 1$. Following the procedure outlined in [10], the exact solution for $\hat{f}$ can be obtained. After expressing $\hat{f}(\tau, v)$ as

$$\hat{f}(\tau, v) = e^{h_1(\tau)v + h_2(\tau)}, \tag{50}$$

from (49), $h_1(\tau)$ and $h_2(\tau)$ are governed by

$$\begin{aligned} h_1'(\tau) - (a_1 + a_4)h_1(\tau)^2 - a_2 h_1(\tau) - a_5 &= 0, \quad \text{with } h_1(0) = 0, \\ h_2'(\tau) - a_3 h_1(\tau) - a_6 &= 0, \quad \text{with } h_2(0) = 0, \end{aligned} \tag{51}$$

and the solutions are

$$h_1(\tau)=\begin{cases}\dfrac{2a_5\left(e^{\sqrt{\Delta}\,\tau}-1\right)}{a_2+\sqrt{\Delta}-\left(a_2-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\,\tau}}, & \text{for } \Delta>0,\\[2ex] -\dfrac{2a_5\tau}{a_2\tau-2}, & \text{for } \Delta=0,\\[2ex] \dfrac{2a_5}{\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\,\tau/2\right)-\left(a_2/\sqrt{-\Delta}\right)\right)}, & \text{for } \Delta<0,\end{cases} \tag{52}$$

$$h_2(\tau)=\begin{cases}-\dfrac{a_3}{a_8}\log\left|\dfrac{\left(a_2+\sqrt{\Delta}\right)e^{-\sqrt{\Delta}\,\tau}-a_2+\sqrt{\Delta}}{2\sqrt{\Delta}}\right|+\left(a_6-\dfrac{a_3\sqrt{\Delta}}{2a_8}-\dfrac{a_2a_3}{2a_8}\right)\tau, & \text{for } \Delta>0,\\[2ex] -\dfrac{a_3}{a_8}\log\left|1-\dfrac{a_2\tau}{2}\right|+\left(a_6-\dfrac{a_2a_3}{2a_8}\right)\tau, & \text{for } \Delta=0,\\[2ex] -\dfrac{a_3}{a_8}\log\left|\cos\left(\dfrac{\sqrt{-\Delta}\,\tau}{2}\right)-\dfrac{a_2}{\sqrt{-\Delta}}\sin\left(\dfrac{\sqrt{-\Delta}\,\tau}{2}\right)\right|+\left(a_6-\dfrac{a_2a_3}{2a_8}\right)\tau, & \text{for } \Delta<0,\end{cases} \tag{53}$$

where $a_8 = a_1 + a_4$ and $\Delta = a_2^2 - 4a_5a_8$. From (17) and (18), we obtain the exact solutions of optimal strategies $\widehat{\varphi}^*$ and $\widehat{c}^*$ for the case of $\alpha = 0$:

$$\widehat{\varphi}^*_{\text{exact}} = \frac{A}{\gamma} + \rho\xi h_1(\tau), \tag{54}$$

$$\frac{\widehat{c}^*_{\text{exact}}}{w} = 0, \tag{55}$$

where $h_1(\tau)$ is given by (52).

The exact solutions $\widehat{f}_{\text{exact}}$, $\widehat{\varphi}^*_{\text{exact}}$, and $\widehat{c}^*_{\text{exact}}$ for the case of $\alpha = 0$ given by (50), (54), and (55) offer a benchmark for testing the accuracy of our numerical solutions. We show that our numerical solutions are accurate and efficient for $\alpha = 0$. Since neither the inhomogeneous term $\alpha^{(1/\gamma)}a_7$ nor the constant initial condition $(1-\alpha)^{(1/\gamma)}$ affects the stability or the accuracy of a numerical method, the accuracy and the stability of the method remain valid for $\alpha \neq 0$. The numerical results for $\alpha \neq 0$ are presented at the end of this section.

4.1. Numerical Validation. To set parameters for numerical validation, we use the estimation values of the parameters $\kappa, \theta, \xi, \rho, A, \beta$, and γ given in [4, 15, 22, 23] and the historical records of r. These values are listed in Table 1.

We note that since a_w determines the wealth scale and a_w/a_c determines the temporal scale, without loss of generality, we choose $a_c = a_w = 1$ in this study.

For the range of the state variables t and v, we consider $T \leq 100$. Based on the historical records of the Chicago Board Options Exchange Volatility Index, a popular measure of the implied volatility of S&P 500 index options, we examine the numerical solutions for the instantaneous volatility $\sigma_t = \sqrt{v_t}$ in the interval $[0.1, 0.8]$ (to eliminate possible influence from the numerical boundary, the authors choose $v_{\max} = 2$ in their numerical computations). Since wealth w does not appear in (19), (17), and (18), its value is irrelevant in our study.

In Table 2, we show the comparison between $\widehat{f}_{\text{exact}}$, $\widehat{f}_{\text{num.}}$, and $\widehat{f}_{\text{extpl.}}$. $\widehat{f}_{\text{exact}}$ is the exact solution of (49). When $\alpha = 0$, $\widehat{f}_{\text{num.}}$ given by (34) is the numerical solution of (49) without performing Richardson's extrapolation and $\widehat{f}_{\text{extpl.}}$ given by (45) is the numerical solution of (49) after performing Richardson's extrapolation. The relative errors in $\widehat{f}_{\text{num.}}$ and $\widehat{f}_{\text{extpl.}}$, namely, $|(\widehat{f}_{\text{num.}} - \widehat{f}_{\text{exact}})/\widehat{f}_{\text{exact}}|$ and $|(\widehat{f}_{\text{extpl.}} - \widehat{f}_{\text{exact}})/\widehat{f}_{\text{exact}}|$, are shown in the last two columns of Table 2.

In Table 3, we show the comparison between the exact solution $\widehat{\varphi}^*_{\text{exact}}$ given by (54), the numerical solution $\widehat{\varphi}^*_{\text{num.}}$ determined from (39) without performing Richardson's extrapolation, and $\widehat{\varphi}^*_{\text{extpl.}}$ determined from (46) after performing Richardson's extrapolation. The relative errors in $\widehat{\varphi}^*_{\text{num.}}$ and $\widehat{\varphi}^*_{\text{extpl.}}$, namely, $|(\widehat{\varphi}^*_{\text{num.}} - \widehat{\varphi}^*_{\text{exact}})/\widehat{\varphi}^*_{\text{exact}}|$ and $|(\widehat{\varphi}^*_{\text{extpl.}} - \widehat{\varphi}^*_{\text{exact}})/\widehat{\varphi}^*_{\text{exact}}|$, are shown in the last two columns of Table 3.

There is no error in c^* since both numerical and theoretical values of c^* are zero.

In Table 4, we show the global relative errors and the computational times of $\widehat{f}_{\text{num.}}$ and $\widehat{f}_{\text{extpl.}}$ for different values of $\Delta\tau$ and Δv. The global relative error in $\widehat{f}_{\text{num.}}$ is defined as the maximum of the local relative errors between $\widehat{f}_{\text{exact}}$ and $\widehat{f}_{\text{num.}}$ in the domain $0 \leq \tau \leq 100$ and $0 \leq \sqrt{v} = \sigma \leq 0.8$. The global relative error in $\widehat{f}_{\text{extpl.}}$ is defined in the same way.

Table 4 confirms that our numerical solutions $\widehat{f}_{\text{num.}}$ and $\widehat{\varphi}^*_{\text{num.}}$ have accuracy at orders of $\Delta\tau^2$ and that $\widehat{f}_{\text{extpl.}}$ and $\widehat{\varphi}^*_{\text{extpl.}}$ have accuracy at orders of $\Delta\tau^3$. Therefore, the extrapolation technique does improve the accuracy. For the same order of accuracy, the application of Richardson's extrapolation technique significantly saves the computational time.

4.2. Extensive Sets of Parameters. In this section, some extensive sets of parameters are used to further validate the proposed method. Since this system contains several

Table 1: Values for the parameters $\kappa, \theta, \xi, \rho, A, r, \beta, \gamma, a_c$, and a_w.

Parameter	κ	θ	ξ	ρ	A	r	β	γ	a_c	a_w
Value	1.6048	0.0464	0.3796	−76.70%	1.55	1%	0.06	2	1	1

Table 2: Comparison between the exact solution $\widehat{f}_{\text{exact}}$, the numerical solution without Richardson's extrapolation $\widehat{f}_{\text{num.}}$, and the numerical solution after Richardson's extrapolation $\widehat{f}_{\text{extpl.}}$. The relative errors in numerical solutions are shown in the last two columns. Here, the numerical solutions are obtained with $\Delta\tau = \Delta\nu = 0.01$.

τ	$\sigma = \sqrt{\nu}$	$\widehat{f}_{\text{exact}}$	$\widehat{f}_{\text{num.}}$	$\widehat{f}_{\text{extpl.}}$	$\lvert(\widehat{f}_{\text{num.}} - \widehat{f}_{\text{exact}})/\widehat{f}_{\text{exact}}\rvert$	$\lvert(\widehat{f}_{\text{extpl.}} - \widehat{f}_{\text{exact}})/\widehat{f}_{\text{exact}}\rvert$
0.1	0.1	0.9961202112	0.9961202904	0.9961202112	7.9×10^{-8}	2.1×10^{-12}
0.1	0.4	0.9919378844	0.9919376809	0.9919378844	2.1×10^{-7}	6.2×10^{-12}
0.1	0.8	0.9786720788	0.9786706845	0.9786720789	1.4×10^{-6}	8.1×10^{-11}
1	0.1	0.9569365332	0.9569367514	0.9569365330	2.3×10^{-7}	1.3×10^{-10}
1	0.4	0.9339516557	0.9339511052	0.9339516557	5.9×10^{-7}	1.4×10^{-11}
1	0.8	0.8640451414	0.8640421623	0.8640451415	3.4×10^{-6}	8.9×10^{-11}
10	0.1	0.6062531863	0.6062531702	0.6062531839	2.6×10^{-8}	4.0×10^{-9}
10	0.4	0.5870696178	0.5870696136	0.5870696159	7.1×10^{-9}	3.2×10^{-9}
10	0.8	0.5296676303	0.5296676662	0.5296676288	6.8×10^{-8}	2.7×10^{-9}
100	0.1	0.0061763000	0.0061762986	0.0061762997	2.2×10^{-7}	4.2×10^{-8}
100	0.4	0.0059808642	0.0059808630	0.0059808640	2.0×10^{-7}	4.1×10^{-8}
100	0.8	0.0053960721	0.0053960714	0.0053960718	1.3×10^{-7}	4.0×10^{-8}

Table 3: Comparison between the exact solution $\widehat{\varphi}^*_{\text{exact}}$, the numerical solution without Richardson's extrapolation $\widehat{\varphi}^*_{\text{num.}}$, and the numerical solution after Richardson's extrapolation $\widehat{\varphi}^*_{\text{extpl.}}$. The relative errors in numerical solutions are shown in the last two columns. Here, the numerical solutions are obtained with $\Delta\tau = \Delta\nu = 0.01$.

τ	$\sigma = \sqrt{\nu}$	$\widehat{\varphi}^*_{\text{exact}}$	$\widehat{\varphi}^*_{\text{num.}}$	$\widehat{\varphi}^*_{\text{extpl.}}$	$\lvert(\widehat{\varphi}^*_{\text{num.}} - \widehat{\varphi}^*_{\text{exact}})/\widehat{\varphi}^*_{\text{exact}}\rvert$	$\lvert(\widehat{\varphi}^*_{\text{extpl.}} - \widehat{\varphi}^*_{\text{exact}})/\widehat{\varphi}^*_{\text{exact}}\rvert$
0.1	0.1	0.7831667609	0.7831672706	0.7831667609	6.5×10^{-7}	1.5×10^{-11}
0.1	0.4	0.7831667609	0.7831673568	0.7831667609	7.6×10^{-7}	2.8×10^{-11}
0.1	0.8	0.7831667609	0.7831676487	0.7831667608	1.1×10^{-6}	9.1×10^{-11}
1	0.1	0.8221908761	0.8221924687	0.8221908741	1.9×10^{-6}	2.4×10^{-9}
1	0.4	0.8221908761	0.8221925174	0.8221908760	2.0×10^{-6}	6.6×10^{-11}
1	0.8	0.8221908761	0.8221927448	0.8221908760	2.3×10^{-6}	5.8×10^{-11}
10	0.1	0.8374121549	0.8374122233	0.8374121478	8.2×10^{-8}	8.6×10^{-9}
10	0.4	0.8374121549	0.8374121589	0.8374121544	4.8×10^{-9}	6.4×10^{-10}
10	0.8	0.8374121549	0.8374121565	0.8374121548	1.9×10^{-9}	1.6×10^{-10}
100	0.1	0.8374121969	0.8374122652	0.8374121897	8.2×10^{-8}	8.6×10^{-9}
100	0.4	0.8374121969	0.8374122009	0.8374121964	4.7×10^{-9}	6.4×10^{-10}
100	0.8	0.8374121969	0.8374121985	0.8374121968	1.9×10^{-9}	1.6×10^{-10}

Table 4: Maximum relative errors and computational times of $\widehat{f}_{\text{num.}}$ and $\widehat{f}_{\text{extpl.}}$ in the domain $0 \le \tau \le 100$ and $0 \le \sqrt{\nu} = \sigma \le 0.8$.

Step size		$\widehat{f}_{\text{num.}}$		$\widehat{f}_{\text{extpl.}}$	
$\Delta\tau$	$\Delta\nu$	Max. rel. err.	Comp. time	Max. rel. err.	Comp. time
0.02	0.02	1.5×10^{-5}	0.39	3.3×10^{-7}	0.84
0.01	0.01	3.8×10^{-6}	1.52	4.2×10^{-8}	3.31
0.005	0.005	9.6×10^{-7}	6.02	5.2×10^{-9}	13.18
0.0025	0.0025	2.4×10^{-7}	24.02	6.3×10^{-10}	52.54

parameters, we vary only one of the parameters at a time and keep other parameters at their benchmark values as shown in Table 1. Extensive sets are given in Table 5.

By choosing the step size $\Delta\tau = 0.01$ and $\Delta\nu = 0.01$, the proposed algorithm is conducted, and the relative errors are recorded both before and after performing the extrapolation technique. In Tables 6 and 7, we show the maximum relative errors of the proposed algorithm before and after performing the extrapolation technique, respectively, in the domain $0 \le \tau \le 100$ and $0 \le \sqrt{\nu} = \sigma \le 0.8$. It can be found that the extrapolation technique does improve one-order accuracy for the ten sets of parameters in Table 5.

By taking half of the step size above, namely, choosing $\Delta\tau = 0.005$ and $\Delta\nu = 0.005$, the proposed algorithm is

Table 5: Values for the parameters $\kappa, \theta, \xi, \rho, A, r, \beta, \gamma, a_c$, and a_w.

Parameter	κ	θ	ξ	ρ	A	r	β	γ	a_c	a_w
Set 1	2.5	0.0464	0.3796	−76.70%	1.55	1%	0.06	2	1	1
Set 2	1.6048	0.08	0.3796	−76.70%	1.55	1%	0.06	2	1	1
Set 3	1.6048	0.0464	0.6	−76.70%	1.55	1%	0.06	2	1	1
Set 4	1.6048	0.0464	0.3796	−50%	1.55	1%	0.06	2	1	1
Set 5	1.6048	0.0464	0.3796	−76.70%	1.25	1%	0.06	2	1	1
Set 6	1.6048	0.0464	0.3796	−76.70%	1.55	5%	0.06	2	1	1
Set 7	1.6048	0.0464	0.3796	−76.70%	1.55	1%	0.15	2	1	1
Set 8	1.6048	0.0464	0.3796	−76.70%	1.55	1%	0.06	5	1	1
Set 9	1.6048	0.0464	0.3796	−76.70%	1.55	1%	0.06	2	2	1
Set 10	1.6048	0.0464	0.3796	−76.70%	1.55	1%	0.06	2	1	2

Table 6: Maximum relative error for $\widehat{f}_{\text{num.}}$ and $\widehat{\varphi}_{\text{num.}}$ with $\Delta\tau = 0.01$ and $\Delta\nu = 0.01$.

Set	Maximum relative error	
	$\widehat{f}_{\text{num.}}$	$\widehat{\varphi}_{\text{num.}}$
Set 1	5.8×10^{-6}	3.9×10^{-6}
Set 2	3.6×10^{-6}	2.7×10^{-6}
Set 3	3.6×10^{-6}	3.9×10^{-6}
Set 4	4.0×10^{-6}	1.8×10^{-6}
Set 5	2.4×10^{-6}	2.0×10^{-6}
Set 6	4.0×10^{-6}	2.8×10^{-6}
Set 7	4.2×10^{-6}	2.9×10^{-6}
Set 8	2.1×10^{-6}	1.4×10^{-6}
Set 9	3.8×10^{-6}	2.7×10^{-6}
Set 10	3.8×10^{-6}	2.7×10^{-6}

Table 7: Maximum relative error for $\widehat{f}_{\text{extpl.}}$ and $\widehat{\varphi}_{\text{extpl.}}$ with $\Delta\tau = 0.01$ and $\Delta\nu = 0.01$.

Set	Maximum relative error	
	$\widehat{f}_{\text{extpl.}}$	$\widehat{\varphi}_{\text{extpl.}}$
Set 1	1.2×10^{-7}	9.3×10^{-8}
Set 2	5.0×10^{-8}	4.3×10^{-8}
Set 3	9.5×10^{-8}	5.9×10^{-8}
Set 4	5.9×10^{-8}	3.1×10^{-8}
Set 5	3.3×10^{-8}	3.1×10^{-8}
Set 6	5.6×10^{-8}	4.5×10^{-8}
Set 7	5.8×10^{-8}	4.7×10^{-8}
Set 8	2.5×10^{-8}	2.02×10^{-8}
Set 9	5.4×10^{-8}	4.4×10^{-8}
Set 10	5.4×10^{-8}	4.4×10^{-8}

Table 8: Maximum relative error for $\widehat{f}_{\text{num.}}$ and $\widehat{\varphi}_{\text{num.}}$ with $\Delta\tau = 0.005$ and $\Delta\nu = 0.005$.

Set	Maximum relative error	
	$\widehat{f}_{\text{num.}}$	$\widehat{\varphi}_{\text{num.}}$
Set 1	1.4×10^{-6}	9.7×10^{-7}
Set 2	9.1×10^{-7}	6.7×10^{-7}
Set 3	9.0×10^{-7}	9.8×10^{-7}
Set 4	1.0×10^{-6}	4.6×10^{-7}
Set 5	6.0×10^{-7}	5.0×10^{-7}
Set 6	10.0×10^{-7}	6.9×10^{-7}
Set 7	1.0×10^{-6}	7.2×10^{-7}
Set 8	5.1×10^{-7}	3.5×10^{-7}
Set 9	9.6×10^{-7}	6.7×10^{-7}
Set 10	9.6×10^{-7}	6.7×10^{-7}

Table 9: Maximum relative error for $\widehat{f}_{\text{extpl.}}$ and $\widehat{\varphi}_{\text{extpl.}}$ with $\Delta\tau = 0.005$ and $\Delta\nu = 0.005$.

Set	Maximum relative error	
	$\widehat{f}_{\text{extpl.}}$	$\widehat{\varphi}_{\text{extpl.}}$
Set 1	1.6×10^{-8}	1.2×10^{-8}
Set 2	6.4×10^{-9}	7.5×10^{-9}
Set 3	2.1×10^{-8}	7.6×10^{-9}
Set 4	7.6×10^{-9}	4.0×10^{-9}
Set 5	4.2×10^{-9}	4.0×10^{-9}
Set 6	7.1×10^{-9}	5.8×10^{-9}
Set 7	7.4×10^{-9}	6.0×10^{-9}
Set 8	3.2×10^{-9}	2.5×10^{-9}
Set 9	6.9×10^{-9}	5.6×10^{-9}
Set 10	6.9×10^{-9}	5.6×10^{-9}

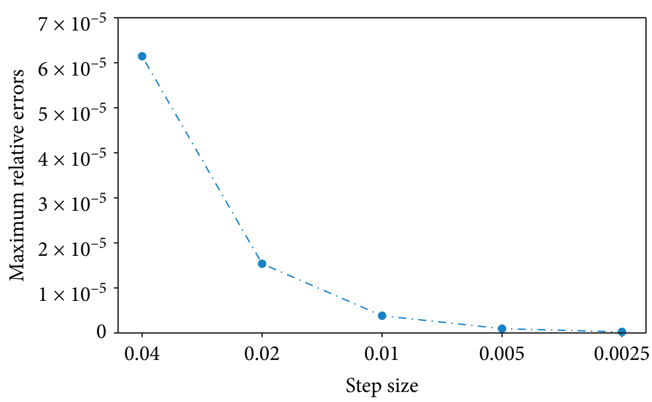

Figure 2: Maximum relative errors for decreasing step size.

Table 10: Maximum relative errors and convergence orders of $\widehat{f}_{\text{num.}}$ in the domain $0 \le \tau \le 100$ and $0 \le \sqrt{\nu} = \sigma \le 0.8$.

Step size		Maximum relative errors	Convergence order
$\Delta\tau$	$\Delta\nu$		
0.02	0.02	1.53×10^{-5}	2.0
0.01	0.01	3.83×10^{-6}	2.0
0.005	0.005	9.58×10^{-7}	2.0
0.0025	0.0025	2.40×10^{-7}	2.0

conducted, and the relative errors are recorded both before and after performing the extrapolation technique. In Tables 8 and 9, we show the maximum relative errors of the proposed algorithm before and after performing the extrapolation technique, respectively, in the domain $0 \le \tau \le 100$ and $0 \le \sqrt{\nu} = \sigma \le 0.8$. Tables 6–9 show that the proposed algorithm before and after performing the extrapolation technique has second-order accuracy and third-order accuracy, respectively, for the extensive sets of parameters.

4.3. Evidence for Stability and Convergence. To provide the evidence for stability, we calculate the maximum relative errors of $\widehat{f}_{\text{num.}}$ for $\Delta\tau = \Delta\nu = 0.04, 0.02, 0.01, 0.005, 0.0025$, respectively. The results are shown in Figure 2. One can see that, as the step size goes to zero, the maximum relative errors tend to zero, which numerically verifies the stability of the proposed algorithm.

To provide the evidence for convergence, let $e(\Delta\tau, \Delta\nu)$ be the maximum relative error; then, the convergence order is given by

$$\text{convergence order} = \log_2\left(\frac{e(\Delta\tau, \Delta\nu)}{e((\Delta\tau/2), (\Delta\nu/2))}\right). \tag{56}$$

In Table 10, the maximum relative errors and convergence orders of $\widehat{f}_{\text{num.}}$ for different values of $\Delta\tau$ and $\Delta\nu$ are given. It shows that the convergence orders are always equal to 2.0 as $\Delta\tau$ and $\Delta\nu$ go to zero, which numerically verifies the convergence of the proposed algorithm.

Table 11: Numerical solutions for f, φ^*, and c^*/w after performing Richardson's extrapolation. Here, $\alpha = 0.1$ and $\Delta\tau = \Delta\nu = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma = \sqrt{\nu}$	$f_{\text{extpl.}}$	$\varphi^*_{\text{extpl.}}$	$c^*_{\text{extpl.}}/w$
0.1	0.1	0.976564446	0.783037744	0.323816587
0.1	0.4	0.972528955	0.783037394	0.325160258
0.1	0.8	0.959728271	0.783036268	0.329497187
1	0.1	1.217443571	0.817505616	0.259747370
1	0.4	1.191090583	0.817448428	0.265494305
1	0.8	1.110757357	0.817260409	0.284695630
10	0.1	3.073495770	0.833010306	0.102888629
10	0.4	2.983046823	0.832947671	0.106008315
10	0.8	2.711715965	0.832739549	0.116615372
100	0.1	6.227692985	0.835240145	0.050777674
100	0.4	6.037435062	0.835206694	0.052377833
100	0.8	5.467451073	0.835095101	0.057838243

Table 12: Numerical solutions for f, φ^*, and c^*/w after performing Richardson's extrapolation. Here, $\alpha = 0.5$ and $\Delta\tau = \Delta\nu = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma = \sqrt{\nu}$	$f_{\text{extpl.}}$	$\varphi^*_{\text{extpl.}}$	$c^*_{\text{extpl.}}/w$
0.1	0.1	0.774937774	0.782803210	0.912469110
0.1	0.4	0.771828844	0.782802269	0.916144540
0.1	0.8	0.761966328	0.782799238	0.928002663
1	0.1	1.368974100	0.812872895	0.516523126
1	0.4	1.342553293	0.812781707	0.526688054
1	0.8	1.261787768	0.812484452	0.560400726
10	0.1	6.015181176	0.832382642	0.117553696
10	0.4	5.840061390	0.832312731	0.121078656
10	0.8	5.314527422	0.832080689	0.133051676
100	0.1	13.916810422	0.835238781	0.050809543
100	0.4	13.491657157	0.835205311	0.052410669
100	0.8	12.217961221	0.835093653	0.057874368

Table 13: Numerical solutions for f, φ^*, and c^*/w after performing Richardson's extrapolation. Here, $\alpha = 0.9$ and $\Delta\tau = \Delta\nu = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma = \sqrt{\nu}$	$f_{\text{extpl.}}$	$\varphi^*_{\text{extpl.}}$	$c^*_{\text{extpl.}}/w$
0.1	0.1	0.409686387	0.782244153	2.315632955
0.1	0.4	0.408160455	0.782242030	2.324290085
0.1	0.8	0.403318534	0.782235205	2.352193658
1	0.1	1.231451909	0.808291630	0.770377869
1	0.4	1.210542696	0.808188721	0.783684294
1	0.8	1.146400499	0.807855696	0.827532175
10	0.1	7.686789813	0.832131657	0.123417359
10	0.4	7.463974966	0.832058953	0.127101618
10	0.8	6.795202083	0.831817743	0.139610756
100	0.1	18.667454349	0.835238326	0.050820175
100	0.4	18.097174990	0.835204850	0.052421624
100	0.8	16.388702420	0.835093170	0.057886419

4.4. Numerical Results for $\alpha \neq 0$. We have confirmed the accuracy of our numerical solutions for $\alpha = 0$. This guarantees that our numerical solution $f_{\text{extpl.}}$ for $\alpha \neq 0$ will also have an accuracy of $O(\Delta\tau^3)$. In Tables 11–13, we present the numerical solutions of $f_{\text{extpl.}}$, $\varphi^*_{\text{extpl.}}$, and $c^*_{\text{extpl.}}/w$ for $\alpha = 0.1, 0.5$, and 0.9 with step sizes $\Delta\tau = \Delta\nu = 0.001$. In Tables 14 and 15, we show the results for $\gamma = 1$ and 10 with $\alpha = 0.5$ and $\Delta\tau = \Delta\nu = 0.001$. All other parameter values in Tables 11–15 are the same as the ones in Table 1. All digits shown in Tables 11–15 are exact, in the sense that they do not change when we further refine the values of $\Delta\tau$ and $\Delta\nu$. Therefore, we have provided the first set of exact numerical

Table 14: Numerical solutions for f, φ^*, and c^*/w after performing Richardson's extrapolation. Here, $\alpha = 0.5$, $\gamma = 1$, and $\Delta\tau = \Delta\nu = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma = \sqrt{\nu}$	$f_{\text{extpl.}}$	$\varphi^*_{\text{extpl.}}$	$c^*_{\text{extpl.}}/w$
0.1	0.1	0.546859282	1.550000000	0.914311994
0.1	0.4	0.546859282	1.550000000	0.914311994
0.1	0.8	0.546859282	1.550000000	0.914311994
1	0.1	0.956177820	1.550000000	0.522915288
1	0.4	0.956177820	1.550000000	0.522915288
1	0.8	0.956177820	1.550000000	0.522915288
10	0.1	4.034308851	1.550000000	0.123936966
10	0.4	4.034308851	1.550000000	0.123936966
10	0.8	4.034308851	1.550000000	0.123936966
100	0.1	8.313916441	1.550000000	0.060140128
100	0.4	8.313916441	1.550000000	0.060140128
100	0.8	8.313916441	1.550000000	0.060140128

Table 15: Numerical solutions for f, φ^*, and c^*/w after performing Richardson's extrapolation. Here, $\alpha = 0.5$, $\gamma = 10$, and $\Delta\tau = \Delta\nu = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma = \sqrt{\nu}$	$f_{\text{extpl.}}$	$\varphi^*_{\text{extpl.}}$	$c^*_{\text{extpl.}}/w$
0.1	0.1	1.024730849	0.157833805	0.910515178
0.1	0.4	1.023235912	0.157833681	0.911845432
0.1	0.8	1.018467192	0.157833282	0.916114922
1	0.1	1.840789858	0.169544364	0.506865565
1	0.4	1.827054636	0.169530453	0.510676021
1	0.8	1.783873146	0.169485620	0.523037747
10	0.1	9.182254704	0.178575988	0.101612624
10	0.4	9.071426914	0.178564461	0.102854049
10	0.8	8.726039146	0.178527017	0.106925144
100	0.1	38.546073475	0.180136006	0.024205656
100	0.4	38.050155049	0.180132872	0.024521135
100	0.8	36.506088964	0.180122662	0.025558284

data for optimal asset allocation and consumption strategies.

5. Conclusion

In this paper, we study the portfolio selection problem in a general setting under CRRA utility functions: stochastic volatility, incomplete markets, finite investment horizons, and consumption choice. To the best of our knowledge, no explicit solution or numerical result is available in the literature for this setting. We present an accurate and efficient numerical method for optimal asset allocation and optimal consumption strategies. The optimal strategies are depended on a solution to a nonlinear and inhomogeneous partial differential equation which is derived from the portfolio selection problem. A three-level Crank–Nicolson finite difference scheme, which has second-order accuracy, is used to determine numerical solutions. In addition, we have used a technique to deal with the nonlinear term, which is one of our main contributions. We believe that the technique to deal with the nonlinear term could be applied to other similar numerical problems. The Crank–Nicolson algorithm also has been extended to third-order accuracy by performing Richardson's extrapolation. Some experiments are conducted to verify the performance of the proposed algorithm. Based on this algorithm, we present the first set of accurate numerical solutions of optimal strategies. Since the portfolio selection problem under stochastic volatility is an important issue in modern finance, the proposed algorithm will be useful for further theoretical research and for applications in the financial industry.

Appendix

A. Derivations for $f_\tau(0,\nu)$, $f_{\tau\tau}(0,\nu)$, and $f_{\tau\tau\tau}(0,\nu)$

From (19), we obtain

$$f_\tau = a_1 \nu f_{\nu\nu} + (a_2\nu + a_3) f_\nu + a_4 \nu \frac{f_\nu^2}{f} + (a_5\nu + a_6) f + \alpha^{(1/\gamma)} a_7. \tag{A.1}$$

By taking the derivative of (A.1) with respect to τ, we obtain

$$f_{\tau\tau} = a_1 \nu f_{\nu\nu\tau} + (a_2\nu + a_3) f_{\nu\tau} + a_4 \nu \frac{2 f_\nu f_{\nu\tau}}{f} - a_4 \nu \frac{f_\nu^2}{f^2} f_\tau + (a_5\nu + a_6) f_\tau. \tag{A.2}$$

By taking the derivative of (A.2) with respect to τ, we obtain

$$\begin{aligned} f_{\tau\tau\tau} = {} & a_1 v f_{vv\tau\tau} + (a_2 v + a_3) f_{v\tau\tau} + a_4 v \frac{2 f_{v\tau}^2}{f} + a_4 v \frac{2 f_v f_{v\tau\tau}}{f} \\ & - 4 a_4 v \frac{f_v f_{v\tau}}{f^2} f_\tau + 2 a_4 v \frac{f_v^2}{f^3} f_\tau^2 - a_4 v \frac{f_v^2}{f^2} f_{\tau\tau} \\ & + (a_5 v + a_6) f_{\tau\tau}. \end{aligned} \tag{A.3}$$

By setting $\tau = 0$ in (A.1) and using the initial condition $f(0, v) = (1-\alpha)^{(1/\gamma)}$, we obtain

$$f_\tau (0, v) = (a_5 v + a_6) f(0, v) + \alpha^{(1/\gamma)} a_7. \tag{A.4}$$

By setting $\tau = 0$ in (A.2) and using the initial condition $f(0, v) = (1-\alpha)^{(1/\gamma)}$ and (A.4), we obtain

$$\begin{aligned} f_{\tau\tau}(0, v) = {} & (a_5^2 v^2 + (a_2 a_5 + 2 a_5 a_6) v + a_3 a_5 + a_6^2) f(0, v) \\ & + (a_5 v + a_6) \alpha^{(1/\gamma)} a_7. \end{aligned} \tag{A.5}$$

By setting $\tau = 0$ in (A.3) and using the initial condition $f(0, v) = (1-\alpha)^{(1/\gamma)}$, (A.4), and (A.5), we obtain

$$f_{\tau\tau\tau}(0, v) = a_5^3 v^3 + a_2 a_3 a_5 + 3 a_3 a_6 a_5 + a_6^3. \tag{A.6}$$

B. HARA Utility Setting

In this appendix, the portfolio selection problem under HARA utility setting is considered. We would like to show that the proposed method could also be applied to this problem under HARA utility setting.

We assume that HARA utility functions $u_1(\cdot)$ and $u_2(\cdot)$ are used for consumption utility and terminal wealth utility, respectively:

$$\begin{aligned} u_1(c_t) &= \frac{a_c (1-\gamma)}{\gamma} \left(\frac{c_t}{1-\gamma} + b_c \right)^\gamma, \\ u_2(w_T) &= \frac{a_w (1-\gamma)}{\gamma} \left(\frac{w_T}{1-\gamma} + b_w \right)^\gamma. \end{aligned} \tag{B.1}$$

The market setting and investors' objective are the same as before which are given by (2)–(6). Following the procedure in this paper, the value function satisfies the following equation:

$$\begin{aligned} 0 = {} & \frac{(1-\gamma)^2}{\gamma} \left(a_c \alpha e^{-\beta t} \right)^{(1/1-\gamma)} V_w^{(\gamma/\gamma-1)} + ((1-\gamma) b_c + r w) V_w \\ & + V_t + \kappa(\theta - v) V_v + \frac{1}{2} \xi^2 v V_{vv} - \frac{1}{2} v \frac{(A V_w + \rho \xi V_{wv})^2}{V_{ww}}, \end{aligned} \tag{B.2}$$

and optimal strategies are given by

$$\begin{aligned} \varphi^* &= -\frac{A V_w + \rho \xi V_{wv}}{w V_{ww}}, \\ c^* &= (1-\gamma) \left[\left(\frac{V_w}{a_c \alpha e^{-\beta t}} \right)^{(1/\gamma - 1)} - b_c \right]. \end{aligned} \tag{B.3}$$

Based on the terminal condition and the scaling property of this problem, a reasonable trial solution form for V is assumed:

$$V(\tau, w, v) = e^{-\beta (T-\tau)} \frac{a_w (1-\gamma)}{\gamma} \left(\frac{w}{1-\gamma} + g(\tau) \right)^\gamma f(\tau, v)^{1-\gamma}, \tag{B.4}$$

where $g(\tau)$ and $f(\tau, v)$ need to be determined and $\tau = T - t$. The initial conditions of g and f are

$$\begin{aligned} g(0) &= b_w, \\ f(0, v) &= (1-\alpha)^{(1/1-\gamma)}. \end{aligned} \tag{B.5}$$

To determine f and g, we substitute expression (B.4) into (B.2) and obtain

$$\begin{aligned} & -f_\tau + \frac{1}{2} \xi^2 v f_{vv} + \left(\kappa(\theta - v) + \frac{\gamma}{1-\gamma} A \rho \xi v \right) f_v \\ & - \frac{1}{2} \gamma (1-\rho^2) \xi^2 v \frac{f_v^2}{f} \\ & + \left(\frac{A^2 \gamma}{2(1-\gamma)^2} v + \frac{\gamma r}{1-\gamma} - \frac{\beta}{1-\gamma} \right) f + \left(\frac{\alpha a_c}{a_w} \right)^{(1/1-\gamma)} \\ & - \frac{\gamma}{1-\gamma} \left(\frac{g'(\tau) + r g(\tau) - b_c}{(w/1-\gamma) + g} \right) f = 0. \end{aligned} \tag{B.6}$$

By setting

$$g'(\tau) + r g(\tau) - b_c = 0, \tag{B.7}$$

equation (B.6) becomes an equation involving f only, namely,

$$\begin{aligned} & -f_\tau + \frac{1}{2} \xi^2 v f_{vv} + \left(\kappa(\theta - v) + \frac{\gamma}{1-\gamma} A \rho \xi v \right) f_v \\ & - \frac{1}{2} \gamma (1-\rho^2) \xi^2 v \frac{f_v^2}{f} \\ & + \left(\frac{A^2 \gamma}{2(1-\gamma)^2} v + \frac{\gamma r}{1-\gamma} - \frac{\beta}{1-\gamma} \right) f + \left(\frac{\alpha a_c}{a_w} \right)^{(1/1-\gamma)} = 0. \end{aligned} \tag{B.8}$$

Thus, we have achieved dimension reduction by removing w dependence in (B.2). The solution of g is obtained from (B.7) and (B.5):

$$g(\tau) = b_w e^{-r\tau} + \frac{b_c}{r} (1 - e^{-r\tau}). \tag{B.9}$$

Equation (B.8) has the same form as (19), which could be solved numerically by applying the proposed algorithm the same as that in Section 3. Then, the optimal strategies could be obtained:

$$\varphi^{*} = \left(1 + \frac{1-\gamma}{w} g(\tau)\right)\left(\frac{A}{1-\gamma} + \rho\xi \frac{f_{v}}{f}\right),$$

$$\frac{c^{*}}{w} = \left(\frac{\alpha a_{c}}{a_{w}}\right)^{(1/1-\gamma)} f^{-1} + \frac{1-\gamma}{w}\left(\left(\frac{\alpha a_{c}}{a_{w}}\right)^{(1/1-\gamma)} g(\tau) f^{-1} - b_{c}\right). \tag{B.10}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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