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Article

Some Iterative Properties of \((\mathcal{F}_1,\mathcal{F}_2)\)-Chaos in Non-Autonomous Discrete Systems

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Abstract: This paper is concerned with invariance \((\mathcal{F}_1,\mathcal{F}_2)\)-scrambled sets under iterations. The main results are an extension of the compound invariance of Li–Yorke chaos and distributional chaos. New definitions of \((\mathcal{F}_1,\mathcal{F}_2)\)-scrambled sets in non-autonomous discrete systems are given. For a positive integer \(k\), the properties \(P(k)\) and \(Q(k)\) of Furstenberg families are introduced. It is shown that, for any positive integer \(k\), for any \(s \in [0, 1]\), Furstenberg family \(\mathcal{M}(s)\) has properties \(P(k)\) and \(Q(k)\), where \(\mathcal{M}(s)\) denotes the family of all infinite subsets of \(\mathbb{Z}^+\) whose upper density is not less than \(s\). Then, the following conclusion is obtained. \(D\) is an \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set of \((X, f_{1,n})\) if and only if \(D\) is an \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set of \((X, f_{1,\infty})\).

Keywords: nonautonomous discrete system; Furstenberg family; scrambled sets; chaos

MSC2010: 37B55, 37D45, 54H20

1. Introduction

Chaotic properties of a dynamical system have been extensively discussed since the introduction of the term chaos by Li and Yorke in 1975 [1] and Devaney in 1989 [2]. To describe some kind of unpredictability in the evolution of a dynamical system, other definitions of chaos have also been proposed, such as generic chaos [3], dense chaos [4], Li–Yorke sensitivity [5], and so on. An important generalization of Li–Yorke chaos is distributional chaos, which is given in 1994 by B. Schweizer and J. Smital [6]. Then, theories related to scrambled sets are discussed extensively (see [7–12] and others). In 1997, the Furstenberg family was introduced by E. Akin [13]. J. Xiong, F. Tan described chaos with a couple of Furstenberg Families. \((\mathcal{F}_1,\mathcal{F}_2)\)-chaos has also been defined [14]. Moreover, \(\mathcal{F}\)-sensitivity was given in [15] and shadowing properties were discussed in [16]. Most existing papers studied the chaoticity in autonomous discrete systems \((X, f)\). However, if a sequence of perturbations to a system are described by different functions, then there are a sequence of maps to describe them, giving rise to non-autonomous systems. Non-autonomous discrete systems were precisely introduced in [17], in connection with non-autonomous difference equations (see [18,19] and some references therein).

Let \((X, r)\) (briefly, \(X\)) be a compact metric space and consider a sequence of continuous maps \(f_n : X \to X, n \in \mathbb{N}\), denoted by \(f_{1,\infty} = (f_1, f_2, \ldots)\). This sequence defines a non-autonomous discrete system \((X, f_{1,\infty})\). The orbit of any point \(x \in X\) is given by the sequence \((f_1^n(x)) = Orb(x, f_{1,\infty})\), where \(f_1^n = f_0 \circ \cdots \circ f_1\) for \(n \geq 1\), and \(f_0\) is the identity map.

For \(m \in \mathbb{N}\), define

\[
\begin{align*}
\mathcal{F}_1 &= f_m \circ \cdots \circ f_1, \\
\mathcal{F}_2 &= f_{2m} \circ \cdots \circ f_{m+1}, \\
&\vdots \\
\mathcal{F}_p &= f_{pm} \circ \cdots \circ f_{(p-1)m+1}, \ldots
\end{align*}
\]
Call \((X, g_{1,\infty})\) a compound system of \((X, f_{1,\infty})\).

Also, denote \(g_{1,\infty}\) by \(f_{1,\infty}^\uparrow\) and denote \(f_n^k = f_{n+k-1} \circ \cdots \circ f_n\) for \(n \geq 1\). By [5], if \((f_n)_{n=1}^\infty\) converges uniformly to a map \(f\). Then, for any \(m \geq 2\) \((m \in \mathbb{N})\), the sequence \((f_n^{m+1})_{n=1}^\infty\) converges uniformly to \(f^m\).

In the present work, some notions relating to Furstenberg families and properties \(P(k)\), \(Q(k)\) are recalled in Sections 2 and 3. Section 4 states some definitions about \(\mu\)-scrambled sets of a Furstenberg family. Also, denote \(\{s\} = [0, 1] = \{0, 1, 2, \ldots\}\). Thus, \(\bigcap_{n \geq 1} \bigcup_{k \geq 1} [f_n^k] = \{s\}\) implies \(f_{1,\infty}\) is \(\mathbb{Z}^+\)-chaos. In Section 5, it is shown that the collection of all infinite subsets of \(\mathbb{Z}\) has the same \(\mu\)-chaos.

### 2. Furstenberg Families

Let \(\mathcal{P}\) be the collection of all subsets of the positive integers set \(\mathbb{Z}^+ = \{0, 1, 2, \ldots\}\). A collection \(\mathcal{F} \subset \mathcal{P}\) is called a Furstenberg family if it is hereditary upwards, i.e., \(F_1 \subset F_2\) and \(F_1 \in \mathcal{F}\) imply \(F_2 \in \mathcal{F}\). Obviously, the collection of all infinite subsets of \(\mathbb{Z}^+\) is a Furstenberg family, denoted by \(B\).

Define the dual family \(k\mathcal{F}\) of a Furstenberg family \(\mathcal{F}\) by

\[
\forall \mathcal{F} \in \mathcal{P}, \; k \mathcal{F} = \{F \in \mathcal{P} : \mathbb{Z}^+ - F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset \text{ for any } F' \in \mathcal{F}\}.
\]

It is clear that \(k\mathcal{F}\) is a Furstenberg family and \(k(k\mathcal{F}) = \mathcal{F}\) (see [13]).

For \(F \in \mathcal{P}\), \(i \in \mathbb{Z}^+\), let \(F - i = \{j - i \geq 0 : j \in F\}\) and \(F + i = \{j + i \geq 0 : j \in F\}\). The following shows a class of Furstenberg families which is related to upper density.

Let \(F \subset \mathcal{P}\). The upper density and the lower density of \(F\) are defined as follows:

\[
\overline{\mu}(F) = \lim\sup_{n \to \infty} \frac{\#(F \cap \{0, 1, \ldots, n - 1\})}{n}, \quad \underline{\mu}(F) = \lim\inf_{n \to \infty} \frac{\#(F \cap \{0, 1, \ldots, n - 1\})}{n},
\]

where \(\#(A)\) denotes the cardinality of the set \(A\).

For any \(s \in [0, 1]\), set \(\overline{\mathcal{M}}(s) = \{F \in B : \overline{\mu}(F) \geq s\}\). The following shows a class of Furstenberg families which is related to upper density.

**Proposition 1.** For any \(s \in [0, 1]\), \(\overline{\mathcal{M}}(s)\) is shift-invariant Furstenberg family. And \(\overline{\mathcal{M}}(0) = B\).

**Proof.**

(i) Let \(F_1, F_2 \in \overline{\mathcal{M}}(s)\), then, \(\forall n \in \mathbb{N}\) (where \(\mathbb{N} = \{1, 2, 3, \ldots\}\)),

\[
\overline{\mu}(F_1) = \lim\sup_{n \to \infty} \frac{\#(F_1 \cap \{0, 1, \ldots, n - 1\})}{n} \leq \lim\sup_{n \to \infty} \frac{\#(F_2 \cap \{0, 1, \ldots, n - 1\})}{n} = \overline{\mu}(F_2)
\]

Thus, \(F_1 \in \overline{\mathcal{M}}(s)\) (i.e., \(\overline{\mu}(F_1) \geq s\)) implies \(F_2 \in \overline{\mathcal{M}}(s)\) (i.e., \(\overline{\mu}(F_1) \geq s\)). So, \(\overline{\mathcal{M}}(s) \forall s \in [0, 1]\) are Furstenberg families.
(ii) Let $F \in \overline{M}(s)$, that is, $\overline{\pi}(F) = \limsup_{n \to \infty} \frac{\#(F \cap [0,1,\ldots,n-1])}{n} \geq s$. Denote $F = \{t_1, t_2, \cdots\}$(where $t_k \in \mathbb{Z}^+$, $t_1 < t_2 (k_1 < k_2)$), then $F + i = \{t_1 + i, t_2 + i, \cdots\}$ and $F - i = \{t_1 - i, t_2 - i, \cdots\}$ for any $i \in \mathbb{Z}^+$.

$$\limsup_{n \to \infty} \frac{\#((F + i) \cap \{0, 1, \ldots, n - 1\})}{n} = \limsup_{n \to \infty} \frac{\#(\{t_1 + i, t_2 + i, \cdots\} \cap \{0, 1, \ldots, n - 1\})}{n}$$

$$= \limsup_{n \to \infty} \frac{\#(\{t_1, t_2, \cdots\} \cap \{0, 1, \ldots, n - 1\})}{n} = \overline{\pi}(F) \geq s$$

and

$$\limsup_{n \to \infty} \frac{\#((F - i) \cap \{0, 1, \ldots, n - 1\})}{n} \geq \limsup_{n \to \infty} \frac{\#(F \cap \{0, 1, \ldots, n - 1\}) - i}{n} = \overline{\pi}(F) \geq s$$

So, $\overline{M}(s)$ is shift-invariant.

(iii) Obviously,

$$\overline{M}(0) = \{F \in \mathcal{B} : \overline{\pi}(F) \geq 0\} = \{F \in \mathcal{B} : \limsup_{n \to \infty} \frac{\#(F \cap \{0, 1, \ldots, n - 1\})}{n} \geq 0\} = \mathcal{B}.$$

This completes the proof.

\[ \square \]

3. Properties $P(k)$, $Q(k)$ of Furstenberg Families

Definition 1. Let $k$ be a positive integer and $\mathcal{F}$ be a Furstenberg family.

(1) For any $F \in \mathcal{F}$, if there exists an integer $j \in \{0, 1, \cdots, k - 1\}$ such that $F_{kj} = \{i \in \mathbb{Z}^+ : ki + j \in F\} \in \mathcal{F}$, we say $\mathcal{F}$ have property $P(k)$.

(2) If $F_k = \{ki + j \in \mathbb{Z}^+ : j \in \{0, 1, \cdots, k - 1\}, i \in F\} \in \mathcal{F}$, we say $\mathcal{F}$ have property $Q(k)$.

The following proposition is given by [24]. For completeness, we give the proofs.

Proposition 2. For any $s \in [0, 1]$ and any $k \in \mathbb{Z}^+$, $\overline{M}(s)$ have properties $P(k)$ and $Q(k)$.

Proof.

(1) If $k = 1$, $\forall F \in \overline{M}(s)$, $F_{1,0} = \{i \in \mathbb{Z}^+ : i \in F\} = F$, i.e., there exists an integer $j = 0$ such that $F_{kj} \in \overline{M}(s)$. The following will discuss the case $k > 1$.

If $s = 0$, $\overline{M}(0) = \mathcal{B}$. $\forall F \in \mathcal{B}$, $\forall k \in \mathbb{Z}^+$, obviously, there exist $j \in \{0, 1, \ldots, k - 1\}$ such that $F_{kj} \in \mathcal{B}$.

If $0 < s \leq 1$, suppose properties $P(k)$ does not hold. Then there exists a $F \in \overline{M}(s)$ such that $\overline{\pi}(F_{kj}) < s$ for every $j \in \{0, 1, \ldots, k - 1\}$.

For any $j \in \{0, 1, \ldots, k - 1\}$, put $\varepsilon_j > 0$ which satisfied $\overline{\pi}(F_{kj}) < s - \varepsilon_j$. One can find a sufficiently large number $N$ such that, $n \geq N$, $\#(F_{kj}) < n(s - \varepsilon_j)$ (where $\#(F_{kj})$ denotes the cardinality of the set $F_{kj} \cap \{0, 1, \ldots, n - 1\}$). Then $\#(F_{kj}^c) > n - n(s - \varepsilon_j)$, where $F_{kj}^c$ denotes the complementary set of $F_{kj}$.

Give an integer $m = kn + l_m > KN$, $l_m \in \{0, 1, \ldots, k - 1\}$. By the definition of $F_{kj}$, $ki + j \notin F$ if $i \notin F_{kj}$. And $ki_1 + j_1 \neq ki_2 + j_2$ if $i_1, i_2 \in \{0, 1, \ldots, n - 1\}, j_1, j_2 \in \{0, 1, \ldots, k - 1\}$ and $j_1 \neq j_2$. Then

$$\#(F^c) \geq \sum_{j=0}^{k-1} \#(F_{kj}^c) > \sum_{j=0}^{k-1} (n - n(s - \varepsilon_j)).$$
So,
\[ \#_m(F) < m - \sum_{j=0}^{k-1} (n - n(s - \epsilon_j)). \]

Put \( \epsilon = \min\{\epsilon_j : j = 0, 1, \ldots, k-1\} \), then
\[
\underline{\mu}(F) = \limsup_{n \to \infty} \frac{\#_m(F)}{m} \leq \lim_{n \to \infty} \frac{m - \sum_{j=0}^{k-1} (n - n(s - \epsilon_j))}{m} \leq \lim_{n \to \infty} \frac{m - k(n - n(s - \epsilon))}{m} = \lim_{n \to \infty} \frac{kn + l_m - kn + kn(s - \epsilon)}{kn + l_m} = s - \epsilon < s
\]

This contradicts to \( \underline{\mu}(F) \geq s \).

(2) Similarly, just consider the case \( k > 1, 0 < s \leq 1 \).

Suppose properties \( Q(k) \) does not hold. Then there exists an integer \( F \in \overline{M}(s) \) such that \( \underline{\mu}(F_k) < s \).

Put \( \epsilon > 0 \) which satisfied \( \underline{\mu}(F_k) < s - \epsilon \). One can find a sufficiently large number \( N \) such that, \( m \geq N, \#_m(F_k) < m(s - \epsilon) \). Give a \( m = kn + l_m > kN(m \geq N), l_m \in \{0, 1, \ldots, k - 1\} \). By the definition of \( F_k, ki+j \in F_k(j \in \{0, 1, \ldots, k - 1\}) \) if \( i \in F \). And \( ki+1 \neq k_2+j \) if \( i_1 \neq i_2 \) and \( j_1, j_2 \in \{0, 1, \ldots, k - 1\} \). Then
\[
k(\#_m(F)) \leq \#_m(F_k) < m(s - \epsilon).
\]

So,
\[
\underline{\mu}(F) \leq \lim_{n \to \infty} \frac{m(s - \epsilon)}{kn} = \lim_{n \to \infty} \frac{(kn + l_m)(s - \epsilon)}{kn} = s - \epsilon \leq s.
\]

This contradicts to \( \underline{\mu}(F) \geq s \).

This completes the proof.

\[ \square \]

4. \((F_1, F_2)\)-Chaos in Non-Autonomous Systems

Now, we state the definition of \((F_1, F_2)\)-chaos in nonautonomous systems.

**Definition 2.** Let \((X, \rho)\) be a compact metric space, \(F_1\) and \(F_2\) are two Furstenberg families. \(D \subset X\) is called a \((F_1, F_2)\)-scrambled set of \((X, f_{1,\infty})\) (briefly, \(f_{1,\infty}\)), if \(\forall x \neq y \in D\), the following two conditions are satisfied:

(i) \( \forall t > 0, \{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < t\} \in F_1 \);

(ii) \( \exists \delta > 0, \{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) > \delta\} \in F_2. \)

The pair \((x, y)\) which satisfies the above two conditions is called an \((F_1, F_2)\)-scrambled pair of \(f_{1,\infty}\).

\(f_{1,\infty}\) is said to be \((F_1, F_2)\)-chaotic if there exists an uncountable \((F_1, F_2)\)-scrambled set of \(f_{1,\infty}\). If \(F_1 = F_2 = F\), \(f_{1,\infty}\) is said to be \(F\)-chaotic and \((x, y)\) is an \(F\)-scrambled pair. \(f_{1,\infty}\) is said to be strong \((F_1, F_2)\)-chaotic if there are some \(\delta > 0\) and an uncountable subset \(D \subset X\) such that for any \(x, y \in D\) with \(x \neq y\), the following two conditions holds:

(i) \( \{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < t\} \in F_1 \) for all \( t > 0 \);

(ii) \( \{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) > \delta\} \in F_2. \)

\(f_{1,\infty}\) is said to be strong \(F\)-chaos if it is strong \((F_1, F_2)\)-chaotic and \(F_1 = F_2 = F\).

Let us recall the definitions of Li-Yorke chaos and distributional chaos in non-autonomous systems (see [25,26]).
**Definition 3.** Assume that \((X, f_{1,\infty})\) is a non-autonomous discrete system. If \(x, y \in X\) with \(x \neq y\), \((x, y)\) is called a Li–Yorke pair if

\[
\limsup_{n \to \infty} \rho(f^n_1(x), f^n_1(y)) > 0 \quad \text{and} \quad \liminf_{n \to \infty} \rho(f^n_1(x), f^n_1(y)) = 0.
\]

The set \(D \subset X\) is called a Li–Yorke scrambled set if all points \(x, y \in D\) with \(x \neq y\), \((x, y)\) is a Li–Yorke pair. \(f_{1,\infty}\) is Li–Yorke chaotic if \(X\) contains an uncountable Li–Yorke scrambled set.

Assume that \((X, f_{1,\infty})\) is a non-autonomous discrete system. For any pair of points \(x, y \in X\), define the upper and lower (distance) distributional functions generated by \(f_{1,\infty}\) as

\[
F^+_x(t, f_{1,\infty}) = \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi([0,t]) (\rho(f^n_1(x), f^n_1(y)))
\]

and

\[
F_x(t, f_{1,\infty}) = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi([0,t]) (\rho(f^n_1(x), f^n_1(y)))
\]

respectively. Where \(\chi([0,t])\) is the characteristic function of the set \([0,t]\), i.e., \(\chi([0,t])(a) = 1\) when \(a \in [0, t)\) or \(\chi([0,t])(a) = 0\) when \(a \not\in [0, t)\).

**Definition 4.** \(f_{1,\infty}\) is distributionally chaotic if exists an uncountable subset \(D \subset X\) such that for any pair of distinct points \(x, y \in D\), we have that \(F^+_x(t, f_{1,\infty}) = 1\) for all \(t > 0\) and \(F_x(t, f_{1,\infty}) = 0\) for some \(\delta > 0\).

The set \(D\) is a distributionally scrambled set and the pair \((x, y)\) a distributionally chaotic pair.

It is not difficult to obtain that the pair \((x, y)\) is a \((\mathcal{M}(0), \mathcal{M}(0))\)-scrambled pair if and only if \((x, y)\) is a Li–Yorke scrambled pair, and the pair \((x, y)\) is a \((\mathcal{M}(1), \mathcal{M}(1))\)-scrambled pair if and only if \((x, y)\) is a distributionally scrambled pair. In fact,

\[
\mathcal{M}(0) = B, \mathcal{M}(1) = \{F \in B : \limsup_{n \to \infty} \frac{\#(F \cap \{1, 2, \ldots, n\})}{n} = 1\}.
\]

Then, \(\{n \in \mathbb{N} : \rho(f^n_1(x), f^n_1(y)) < t\} \in \mathcal{M}(0)\) for any \(t > 0\) and \(\{n \in \mathbb{N} : \rho(f^n_1(x), f^n_1(y)) > \delta\} \in \mathcal{M}(0)\) for some \(\delta > 0\) is equivalent to that \(\limsup_{n \to \infty} \rho(f^n_1(x), f^n_1(y)) > 0\) and \(\liminf_{n \to \infty} \rho(f^n_1(x), f^n_1(y)) = 0\). \(\{n \in \mathbb{N} : \rho(f^n_1(x), f^n_1(y)) < t\} \in \mathcal{M}(1)\) for any \(t > 0\) and \(\{n \in \mathbb{N} : \rho(f^n_1(x), f^n_1(y)) > \delta\} \in \mathcal{M}(1)\) for some \(\delta > 0\) is equivalent to that \(F^+_x(t, f_{1,\infty}) = 1\) and \(F_x(\delta, f_{1,\infty}) = 0\).

Hence, \((\mathcal{M}(0), \mathcal{M}(0))\)-chaos is Li–Yorke chaos and \((\mathcal{M}(1), \mathcal{M}(1))\)-chaos is distributional Chaos.

5. Main Results

**Theorem 1.** Let \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are two Furstenberg families with property \(P(k)\), where \(k\) is a positive integer. \(\mathcal{F}_1\) is positive shift-invariant. If the system \((X, f_{1,\infty})\) is \((\mathcal{F}_1, \mathcal{F}_2)\)-chaos, then the system \((X, f_{1,\infty}^{[k]})\) is \((\mathcal{F}_1, \mathcal{F}_2)\)-chaos too.

**Proof.** If \(D\) is an \((\mathcal{F}_1, \mathcal{F}_2)\)-scrambled set of \(f_{1,\infty}\), the following proves that \(D\) is an \((\mathcal{F}_1, \mathcal{F}_2)\)-scrambled set of \(f_{1,\infty}^{[k]}\).

(i) Since \(X\) is compact and \(f_i (i \in \mathbb{N})\) are continuous, then, for any \(j \in \{1, 2, \ldots, k - 1\}\), \(f_{s_1}, \ldots, f_{s_{k-j}}\) are uniformly continuous (where \(f_{s_1}, \ldots, f_{s_{k-j}}\) are freely chosen from the sequence \(f_j (i \in \mathbb{N})\)). That is, for any \(\delta > 0\), there exists a \(\delta^* > 0\), \(\forall a, b \in X, \rho(a, b) < \delta^*\) implies \(\rho(f_{s_{k-j}} \circ \cdots \circ f_{s_1}(a), f_{s_{k-j}} \circ \cdots \circ f_{s_1}(b)) < \delta (j = 1, 2, \ldots, k - 1)\).
Let $\text{Theorem 2.}$

(i) Since $\text{Entropy}$ $f$

Proof.

By the selection of $\delta^*$, we have

$$F = \{n \in \mathbb{N} : \rho(f_1^n(x), f_1^n(y)) < \delta^* \} \in \mathcal{F}_1.$$ 

And because $\mathcal{F}_1$ have property $P(k)$, there exists some $j \in \{1, 2, \ldots, k-1\}$ such that

$$F_{k,j} = \{i \in \mathbb{N} : ki + j \in F \} = \{i \in \mathbb{N} : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) < \delta^* \} \in \mathcal{F}_1.$$ 

(ii) Since $\text{Entropy}$ $f$

By the selection of $\delta^*$, we put $s_r = ki + j + r(r = 1, 2, \ldots, k-j)$, then

$$F_{k,j} \subset \{i \in \mathbb{N} : \rho(f_1^{ki+j+k-i}(x), f_1^{ki+j+k-i}(y)) < \delta \} = \{i \in \mathbb{N} : \rho(f_1^{ki+i}(x), f_1^{ki+i}(y)) < \delta \}.$$ 

Write $F_{k,j} + 1 = \{i + 1 : i \in \mathbb{N} : ki + j \in F, i \in \mathbb{N} \} (\forall j = 1, 2, \ldots, k-1)$, then $F_{k,j} + 1 \subset \{i \in \mathbb{N} : \rho(f_1^{ki}(x), f_1^{ki}(y)) < \delta \}$.

By the positive shift-invariant of $\mathcal{F}_1$ and $F_{k,j} \in \mathcal{F}_1$, we have $F_{k,j} + 1 \in \mathcal{F}_1$. And with the hereditary upwards of $\mathcal{F}_1$, for any $x, y \in D : x \neq y$, $\forall \delta > 0$, $\{i \in \mathbb{N} : \rho(f_1^{ki}(x), f_1^{ki}(y)) < \delta \} \in \mathcal{F}_1$.

Since $\mathcal{F}_1$ is compact and $X$ is an $\{F_1, F_2\}$-scrambled set of $f_1$, then, for the above $x, y \in D(x \neq y)$, $\exists \delta > 0$, such that $E = \{i \in \mathbb{N} : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \delta \} \in \mathcal{F}_2$. And because $\mathcal{F}_2$ have property $P(k)$, then, there exists some $j \in \{1, 2, \ldots, k-1\}$ such that

$$E_{k,j} = \{i \in \mathbb{N} : ki + j \in E \} = \{i \in \mathbb{N} : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \epsilon^* \} \in \mathcal{F}_2.$$ 

$X$ is compact and $f_1(i \in \mathbb{N})$ are continuous, then, for any $j \in \{1, 2, \ldots, k-1\}$, $f_1, \ldots, f_1$ are uniformly continuous (where $f_1, \ldots, f_1$ are freely chosen from the sequence $f(i \in \mathbb{N})$). For the above $\epsilon^* > 0$, $\forall \epsilon > 0$, $\forall p, q \in X$ satisfied $\rho(p, q) \leq \epsilon$, inequality $\rho(f_1 \circ \cdots \circ f_1(p), f_1 \circ \cdots \circ f_1(q)) \leq \epsilon^*$ holds.

The following will prove that $\{i \in \mathbb{N} : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \epsilon \} \in \mathcal{F}_2$.

Suppose $\{i \in \mathbb{N} : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \epsilon \} \notin \mathcal{F}_2$, then

$$Z^+ - \{i \in Z^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \epsilon \} = \{i \in Z^+ : \rho(f_1^{ki}(x), f_1^{ki}(y)) \leq \epsilon \} \in kF_2.$$ 

By the selection of $\epsilon^*$, we put $s_r = ki + j + r(r = 1, 2, \ldots, j)$, then

$$\{i \in \mathbb{N} : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) \leq \epsilon^* \} \in kF_2.$$ 

So,

$$\{i \in \mathbb{N} : \rho(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \epsilon^* \} \notin kF_2,$$

This contradicts $E_{k,j} \in \mathcal{F}_2$.

Hence, for $x \neq y \in D$ in (i), there exists a $\epsilon > 0$ such that $\{i \in \mathbb{N} : \rho(f_1^{ki}(x), f_1^{ki}(y)) > \epsilon \} \in \mathcal{F}_2$.

Combining with (i) and (ii), $f_1^{[k]}$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos.

This completes the proof.

\[\square\]

**Theorem 2.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Furstenberg families with property $Q(k)$, where $k$ is a positive integer. $\mathcal{F}_2$ is negative shift-invariant. If the system $(X, f_1^{[k]})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, then the system $(X, f_1^{[k]})$ is $(\mathcal{F}_1, \mathcal{F}_2)$-chaos too.

**Proof.** If $D$ is a $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_1^{[k]}$, the following prove that $D$ is a $(\mathcal{F}_1, \mathcal{F}_2)$-scrambled set of $f_1^{[k]}$. 

(i) Similar to Theorem 1, for any \( j \in \{1, 2, \ldots, k - 1\} \), \( f_{s_1}, \ldots, f_{s_j} \) are uniformly continuous (where \( f_{s_1}, \ldots, f_{s_j} \) are freely chosen from the sequence \( f_i (i \in \mathbb{N}) \)). That is, for any \( \delta > 0 \), there exists a \( \delta^* > 0 \), \( \forall a, b \in X \), \( \rho(a, b) < \delta^* \) implies \( \rho(f_{s_1} \circ \cdots \circ f_{s_j}(a), f_{s_1} \circ \cdots \circ f_{s_j}(b)) < \delta \) (\( j = 1, 2, \ldots, k - 1 \)).

For any pair of distinct points \( x, y \in D \), for the above \( \delta^* \), one has

\[ F = \{ n \in \mathbb{Z}^+ : \rho(f_1^{kn}(x), f_1^{kn}(y)) < \delta^* \} \subseteq F_1. \]

By the selection of \( \delta^* \), for \( \forall n \in F \), \( \forall j \in \{1, 2, \ldots, k - 1\} \), put \( s_r = k_i + j + r (r = 1, 2, \ldots, j) \), then \( \rho(f_1^{kn+j}(x), f_1^{kn+j}(y)) < \delta \). And because \( F_1 \) have property Q(k), then

\[ F_k = \{ kn + j \in \mathbb{Z}^+ : j = 1, 2, \ldots, k - 1, n \in F \} \subseteq F_1. \]

Notice that \( F_k \subseteq \{ m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) < \delta \} \), then \( \{ m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) < \delta \} \subseteq F_1. \)

(ii) Since \( D \) is an \((F_1, F_2)\)-scrambled set of \( f_{1,\infty}^{[k]} \), then, for the above \( x, y \in D (x \neq y) \), there exist \( \epsilon^* > 0 \), such that \( E = \{ n \in \mathbb{Z}^+ : \rho(f_1^{kn}(x), f_1^{kn}(y)) > \epsilon^* \} \subseteq F_2. \)

For any \( j \in \{1, 2, \ldots, k - 1\} \), \( f_{s_1}, \ldots, f_{s_j} \) are uniformly continuous (where \( f_{s_1}, \ldots, f_{s_j} \) are freely chosen from the sequence \( f_i (i \in \mathbb{N}) \)), then, for the above \( \epsilon^* > 0 \), there exist \( \epsilon > 0 \) such that

\[ \rho(p, q) < \epsilon \] (\( p, q \in X \)) implies \( \rho(f_{s_1} \circ \cdots \circ f_{s_j}(p), f_{s_1} \circ \cdots \circ f_{s_j}(q)) \leq \epsilon^* \).

That is, \( \rho(f_1^{k}(p), f_1^{k}(q)) > \epsilon^* \) implies \( \rho(f_1^{kn}(x), f_1^{kn}(y)) > \epsilon (j = 1, 2, \ldots, k - 1) \).

\[ \forall n \in E, \forall j = 1, 2, \ldots, k - 1, \text{ put } s_r = k(n - 1) + r(r = 1, 2, \ldots, j), \text{ then } \]

\[ \rho(f_1^{k(n-1)+j}(x), f_1^{k(n-1)+j}(y)) > \epsilon. \]

Since \( F_2 \) is negative shift-invariant, then \( E - 1 \subseteq F_2 \). And because \( F_2 \) have property Q(k), then \( (E - 1)_k \subseteq F_2 \), i.e., \( \{ k(n - 1) + j \in \mathbb{Z}^+ : n - 1 \in E - 1, j = 1, 2, \ldots, k - 1 \} \subseteq F_2 \).

Combining \( (E - 1)_k \subseteq \{ m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) > \epsilon \} \) with the hereditary upwards of \( F_2 \), we have \( \{ m \in \mathbb{Z}^+ : \rho(f_1^m(x), f_1^m(y)) > \epsilon \} \subseteq F_2. \)

By (i) and (ii), \( D \) is an \((F_1, F_2)\)-scrambled set of \( f_{1,\infty}^{[k]} \).

This completes the proof.

\( \square \)

Similarly, the following corollaries hold.

**Corollary 1.** Let \( F_1 \) and \( F_2 \) are two Furstenberg families with property P(k), where k is a positive integer. \( F_1 \) is positive shift-invariant. If the system \((X, f_{1,\infty})\) is \( F \)-chaos (strong \((F_1, F_2)\)-chaos, or strong \( F \)-chaos), then the system \((X, f_{1,\infty}^{[k]})\) is \( F \)-chaos (strong \((F_1, F_2)\)-chaos, or strong \( F \)-chaos).

**Corollary 2.** Let \( F_1 \) and \( F_2 \) are two Furstenberg families with property Q(k), where k is a positive integer. \( F_2 \) is negative shift-invariant. If the system \((X, f_{1,\infty}^{[k]})\) is \( F \)-chaos (strong \((F_1, F_2)\)-chaos, or strong \( F \)-chaos), then the system \((X, f_{1,\infty})\) is \( F \)-chaos (strong \((F_1, F_2)\)-chaos, or strong \( F \)-chaos).

Combining with Propositions 1 and 2, Theorems 1 and 2, and Corollaries 1 and 2, the following conclusions are obtained.

**Theorem 3.** Let \( s \) and \( t \) are arbitrary two numbers in \([0, 1]\), then

1. If \( D \) is an \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set (or strong \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set) of \( f_{1,\infty}^{[k]} \), then, for every \( k \in \mathbb{Z}^+ \), \( D \) is an \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set (or strong \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set) of \( f_{1,\infty}^{[k]} \).
2. For some positive integer \( k \), if \( D \) is an \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set (or strong \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set) of \( f_{1,\infty}^{[k]} \), then \( D \) is an \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set (or strong \((\mathcal{M}(s), \mathcal{M}(t))\)-scrambled set) of \( f_{1,\infty}^{[k]} \).
Proof.

(1) By Proposition 1, $\mathcal{M}(s)$ is shift-invariant (obviously positive shift-invariant). And because $\mathcal{M}(s), \mathcal{M}(t)$ are two Furstenberg families with property $P(k)$ (Proposition 2). Then, according to the proof of Theorem 1, if $D$ is an $((\mathcal{M}(s), \mathcal{M}(t))$-scrambled set of $f_{1,\infty}$, then, for every $k \in \mathbb{Z}^+$, $D$ is an $((\mathcal{M}(s), \mathcal{M}(t))$-scrambled set of $f_{1,\infty}^k$.

(2) In the same way, (2) holds.

This completes the proof.

With the preparations in Section 4, we have

**Corollary 3.**

(1) If $D$ is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}$, then, for every $k \in \mathbb{Z}^+$, $D$ is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}^k$.

(2) For some positive integer $k$, if $D$ is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}^k$, then, $D$ is a Li–Yorke scrambled set (or distributionally scrambled set) of $f_{1,\infty}$.

**Remark 1.** In the non-autonomous systems, the iterative properties of Li–Yorke chaos and distributional chaos are discussed in [25,26] before. The conclusions in Corollary 3 remains consistent with them.

This paper has presented several properties of $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, strong $(\mathcal{F}_1, \mathcal{F}_2)$-chaos, and strong $\mathcal{F}$-chaos. There are some other problems, such as generically $\mathcal{F}$-chaos and $\mathcal{F}$-sensitivity, to discuss. Moreover, property $P(k)$ is closely related to congruence theory. Follow this line, one can consider other Furstenberg families which consist of number sets with some special characteristics.

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**References**


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