Optimal Contracts for Time-Inconsistent Consumers with Heterogeneous Beliefs

Gao, Buqu; Guo, Liang

Published in:
Management Science

Published: 01/01/2025

Document Version:
Post-print, also known as Accepted Author Manuscript, Peer-reviewed or Author Final version

Publication record in CityU Scholars:
Go to record

Published version (DOI):
10.1287/mnsc.2022.02653

Publication details:
Gao, B., & Guo, L. (2025). Optimal Contracts for Time-Inconsistent Consumers with Heterogeneous Beliefs. *Management Science*, *71*(1), 861-878. https://doi.org/10.1287/mnsc.2022.02653

Gao, B., & Guo, L. (2024). Optimal Contracts for Time-Inconsistent Consumers with Heterogenous Beliefs. *Management Science, 71*(1), 861-878.

https://doi.org/10.1287/mnsc.2022.02653

Optimal Contracts for Time-Inconsistent Consumers with Heterogeneous Beliefs*

Buqu Gao† Liang Guo‡

Abstract

In many markets (e.g., cell phones, video games), firms offer menus of contracts that include some tariffs charging per-usage prices above marginal cost and others below marginal cost. We term this puzzling phenomenon as two-sided deviations from marginal-cost pricing, and present a potential explanation based on two well-recognized consumer characteristics. The first one is that consumers' actual consumptions may depart systematically from their initial plans (i.e., time-inconsistent preferences). In addition, consumers can be either sophisticated or naive in their beliefs about their time inconsistency (i.e., heterogenous beliefs). We characterize properties of the optimal contracts for a firm to screen the time-inconsistent consumers with heterogeneous beliefs. We articulate the conditions under which the optimal menu may account for two-sided deviations from marginal-cost pricing. We also show that, contrary to intuition, a higher degree of time inconsistency may reduce firm profit and increase social welfare. Meanwhile, reducing consumer naivete may harm the society. Moreover, we confirm that our main results are robust to the presence of time-consistent consumers.

Keywords: time inconsistency, sophistication, naivete, screening, contract design

*For valuable comments and discussions, we thank Rahul Deb, Taosong Deng, Randy Ellis, Larry Epstein, Keith Ericson, Daniel Houser, Kevin Lang, Youming Liu, Bart Lipman, Chiara Margaria, Michael Manove, Lingfeng Que, Albert Ma, Dilip Mookherjee, Jawwad Noor, Juan Ortner, Alex White, Tak-Yuen Wong, Jianye Yan, Xing Zhang, Xin Zhao, and workshop participants at Boston University and the 42nd Symposium of the Spanish Economic Association. An earlier version of this paper forms Chapter 1 of Gao's PhD dissertation at Boston University, and she is especially grateful to Dilip Mookherjee for his guidance. This work was supported by Hunan Provincial Natural Science Foundation of China, Grant 2022JJ40102.

†School of Economics & Trade, Hunan University, Changsha, Hunan, China. Email: buqu@hnu.edu.cn

‡Department of Marketing, City University of Hong Kong, Hong Kong SAR, China. Email: liang.guo@cityu.edu.hk

1 Introduction

In many markets, firms offer menus of advance contracts to consumers. These contracts require consumers to trade off upfront payments and future contingent payments based on their predicted future usage. A menu of advance contracts often includes some tariffs charging per-usage prices above marginal cost and others below marginal cost. We call this phenomenon *two-sided deviations from marginal-cost pricing*, and illustrate it in the following examples. The marginal costs in these examples, which we conjecture to be positive, come from either direct service and maintainance or wholesale prices charged by upstream firms.

Cell phones. Most US and Canadian cell phone carriers offer international roaming plans that only charge daily/monthly fixed fees for unlimited talk, text, and even data. They also offer "pay as you go" options, which charge zero fixed fee but very expensive per-usage prices.

Video games. Video game platforms such as Microsoft's Xbox offer monthly subscription plans to users. The subscription fee costs \$10–\$18/month. Subscribers can play any game, including new releases, in the library at no extra cost. The users can also opt out of the subscription plan and buy games individually from the Microsoft Store. The sales price of a game usually falls in the range \$50–\$80.

Onboard Wi-Fi. Mainstream US and Canadian airlines offer in-flight Wi-Fi service with single-use options, as well as monthly/annual plans that charge zero price per flight and high fixed fees, around \$50/month and \$599/year.

Theme parks. Theme parks often offer single-visit tickets, as well as seasonal/annual passes that allow unlimited visits within the time frame. Visitors need to reserve their tickets prior to their visits. The seasonal/annual passes cannot be cancelled and hence can be regarded as options with high fixed fees and zero per-visit price. The single-visit tickets are (partially) refundable for cancellation. The non-refundable part corresponds to a low fixed fee. The refundable part, which is indeed contingent payment, corresponds to a high per-visit price.

The existing models do not provide a compelling explanation for such pricing patterns. For homogeneous consumers, the basic economic principle suggests the optimal per-usage price equal marginal cost when charging upfront fees is viable. Thus, consumption is efficient, total surplus is maximized, and the firm with sufficient market power can extract the full surplus with the upfront fee. Allowing consumer heterogeneity does not solve the problem, as standard screening models only predict one-sided upward pricing deviation. To be clear, screening consumers' willingness-to-pay suggests that the high-demand consumers should receive the efficient quantity, meaning that they should be charged at the marginal cost; the low-demand consumers should receive a rationed quantity, which is implemented by a price above marginal cost.

A widely accepted behavioral tendency in making intertemporal decisions is that people often have self-control problems. Based on the theory of time-inconsistent consumer preferences, DellaVigna and Malmendier (2004; hereafter DM) give a systematic account of the deviations from marginal-cost pricing in advance contracts. They assume that consumers care more about immediate gratification. Thus consumers are tempted to overconsume leisure goods that, in their definition, feature short-run returns and long-run costs. DM show that irrespective of whether consumers are sophisticated (i.e., fully aware of the self-control problem) or naive (i.e., underestimate the self-control problem), it is optimal to price leisure goods above marginal cost in two-part tariffs, as long as the firm can perfectly observe consumer types.[1] Still, the existence of tariffs charging below marginal cost remains to be explained.

In reality, firms cannot immediately detect consumer beliefs, i.e., whether consumers are sophisticated or naive. Because sophisticated consumers can foresee future usage, they are able to choose the contract that best suits their demand. Naive consumers, on the contrary, are more likely to favor a backloaded contract with high contingent price due to their underestimation of demand. We investigate the screening mechanism of consumer beliefs, with the goal of providing a possible and unified explanation of the two-sided deviations from marginal-cost pricing observed in an array of markets. Naturally, we ask the following questions. Can the two-sided deviations from marginal-cost pricing arise when screening consumer beliefs of time inconsistency? Is it a necessary feature and, if not, under what condition does it arise? More generally, what does the optimal menu of contracts look like?

We develope a two-period monopolistic screening model to address these questions. Period 0 is the contracting stage and Period 1 is the purchase/consumption stage. We define goods that are systematically overvalued at the consumption stage relative to the planning stage as *temptation goods*.[2] Consumers contract with a monopolistic firm in Period 0, so that they can buy one unit of temptation good in Period 1. A contract consists of a fixed fee and a per-usage price. Our model has three main features: valuation uncertainty, time-inconsistent consumer preference, and heterogeneous consumer beliefs. First, the consumers face uncertainty in their value of future consumption when signning contracts in Period 0. Moving to Period 1, the consumers privately learn their individual valuation before choosing whether to consume at the per-usage price. Second, the consumers have time-inconsistent preference: for temptation goods, the Period-1 consumption is valued systematically higher in Period 1 than in Period 0. To capture the time-inconsistent preference, we introduce a distortion function that specifies how the valuation for Period-1 consumption may increase from Period 0 to Period 1. Lastly, the consumer type may be sophisticated or naive, which the firm cannot identify. The sophisticated consumers are fully aware of the distortion in their valuation, whereas the naive consumers are fully unaware.

Asymmetric information on consumer beliefs creates an adverse-selection problem for the firm. The naive consumers do not consider the utility loss from overconsuming temptation goods relative to their plans and hence value any contract more than the sophisticated consumers. To ensure that the naive consumers do not choose the contract for the sophisticated consumers, the firm must give the naive consumers an information rent. A key instrument in our analysis is the illusional

[1] Guo (2023) shows that seemingly naive beliefs as revealed by ex ante contract choice can be rationalized by motivations to deal with limited memory.

[2] Similarly, we can define aversion goods as those that are undervalued at the consumption stage relative to the planning stage. Prominent examples include health clubs and product repairs, which feature immediate or more salient costs. The analysis of aversion goods is symmetric. For clarity's sake, we will focus on temptation goods in the basic model and discuss aversion goods in an extension.

surplus, defined as the increment in consumer expected utility due to naivete. We show that it can be profitable to screen the consumers by offering a menu of contracts. The contract for the naive consumers is associated with a high illusional surplus, leading to exploitation of the naive consumers. The contract for the sophisticated consumers comes with a low illusional surplus, so that the naive consumers find it unattractive (or, put differently, information rent to the naive consumers is effectively curtailed).

Our paper has two main contributions. First, we characterize the optimal menu of contracts used to screen consumer beliefs of time inconsistency. The existence of the naive consumers exerts informational externality on the sophisticated consumers, whose optimal price may then be upward or downward distorted. We find that properties of the optimal contract depend on two key parameters: the proportion of the sophisticated consumers and the slope of the distortion function. The proportion of the sophisticated consumers determines the firm's weight on efficiency vs. rent extraction, the essential tradeoff of our model, and hence affects the optimal contract. The slope of the distortion function determines how price would change the naive consumers' misprediction in future demand. It then affects how the illusional surplus would respond to price changes, a key mechanism that the firm would leverage to reduce information rent.

Second, we show that screening consumer beliefs of time inconsistency can potentially explain the two-sided deviations from marginal-cost pricing. In the context of temptation goods, the optimal price for the naive consumers is undistorted relative to the symmetric information setting and hence above marginal cost. The optimal price for the sophisticated consumers can be below marginal cost, leading to overconsumption. Instead of regulating the sophisticated consumers who actually demand regulation, the firm indulges them. However, the two-sided deviations from marginal-cost pricing is not a necessary feature of the optimal menu of contracts. We predict that this pricing pattern is more likely when the marginal production cost is lower, the proportion of the sophisticated consumers is smaller, and the distortion function is steeper. The rough intuition is that the firm has stronger incentive to indulge the sophisticated consumers under these conditions, as overproduction becomes less costly and reducing information rent becomes more rewarding.

Our comparative statics regarding unconventional consumer attributes–the degree of time inconsistency and the proportion of the sophisticated consumers–unveils their impact on profit and welfare in the screening setting and cautions unintended consequences of policy interventions. We show that the consumers' degree of time inconsistency may negatively affect firm profit and positively affect consumer surplus and social welfare. Since the level of illusional surplus shifts up under a higher degree of time inconsistency, the firm's gain from exploiting naivete and the loss from paying the information rent both increase. If the latter effect dominates, the firm is less motivated to discriminate consumers and focuses more on production efficiency. As a result, firm profit decreases, and both consumer surplus and social welfare are enhanced. We also find that educating consumers may decrease social welfare. The reason is that the optimal contract is socially inefficient not only for the naive consumers (due to the firm's exploitation), but also for the sophisticated ones (because of the need for screening). When the latter effect is stronger, increasing the proportion of the sophisticated consumers may lower social welfare.

Our paper is structured as follows. Section 1.1 discusses related literature. Section 2 describes the model setup. Section 3 analyzes two benchmarks: one with only time-consistent consumers, and the other with time-inconsistent consumers and symmetric information on consumer beliefs. Section 4 characterizes the menu of optimal contracts under asymmetric information on the beliefs of time-inconsistent consumers. Section 5 considers two extensions. The first one includes time-consistent consumers in the market and the second one investigates the market for aversion goods.

1.1 Related Literature

This paper relates to three strands of literature. First, it builds on the literature on time-inconsistent preferences (e.g., Strotz 1955; Phelps and Pollak 1968; Akerlof 1991; Laibson 1997; O'Donoghue and Rabin 1999a,b). Strotz (1955) investigates general time-inconsistent preferences without giving a specific functional form. Popular ways to formulate time inconsistency are quasi-hyperbolic discounting (Phelps and Pollak 1968; Laibson 1997; O'Donoghue and Rabin 1999a,b) and salience of certain payoff aspects given contexts (Akerlof 1991). We capture time inconsistency by a linear distortion function that formulates how the valuation of a consumption may change depending on when it is assessed. Our setting is general enough to generate various qualitatively different equilibrium outcomes, which may not arise under these popular formulations of time inconsistency.

A conjugate issue in modelling time-inconsistent preference is people's self-awareness of future time inconsistency. Strotz (1955) introduces the distinction between sophistication and naivete, but his formal analysis considers sophistication only. Many times, the modellers consider either sophistication (Laibson 1997) or naivete (Akerlof 1991; O'Donoghue and Rabin 1999b). Those having both sophistication and naivete usually contrast them in parallel worlds, without considering the potential externality of one group on the other (O'Donoghue and Rabin 1999a; Quah and Strulovici 2013; Freeman 2021).

Second, this paper contributes to a growing body of literature that studies pricing contracts with nonstandard consumer preferences or biased beliefs. (see, e.g., DellaVigna and Malmendier 2004; Lim and Ho 2007; Lim 2010; Kuksov and Wang 2014; Guo 2015; Heidhues and Kőszegi 2017). Extensive reviews are provided by Ellison (2006), Ho, Lim, and Camerer (2006), and Heidhues and Kőszegi (2018).

Within this literature are works on contracting with time-inconsistent consumers (e.g., DellaVigna and Malmendier 2004; Eliaz and Spiegler 2006; Gottlieb 2008; Heidhues and Kőszegi 2010; Li, Yan, and Xiao 2014; Gottlieb and Zhang 2021). The most closely related work is that of DellaVigna and Malmendier (2004). They define leisure goods with the feature of short-run benefits and long-run costs. Their definition is in line with their adoption of quasi-hyperbolic discounting to model time inconsistency. Our definition of temptation goods is in spirit the same but not constrained by timing of costs and benefits. DellaVigna and Malmendier (2004) consider either sophisticated or naive consumers separately but not together, equivalent to assuming observable beliefs. Their result shows that it is always profit-maximizing to price temptation goods above marginal cost in two-part tariffs, irrespective of whether the consumers are naive or sophisticated. In contrast, we show that asymmetric information on consumer beliefs can generate two-sided deviations from marginal-cost pricing.

Li, Yan, and Xiao (2014) study the screening contracts along the classic dimension of asymmetric information–consumers' willingness-to-pay, assuming that all consumers are time-inconsistent and sophisticated. Their result shows that high-demand consumers choose membership plans with below-marginal-cost prices for aversion goods (such as gym visits), and low-demand consumers choose pay-as-you-go with above-marginal-cost prices. However, they focus on aversion goods and their setting cannot be readily adapted to generate two-sided deviations from marginal-cost pricing for temptation goods.

Similar to our setting, Eliaz and Spiegler (2006) consider asymmetric information about the consumers' naivete of time inconsistency. However, they abstract away uncertainty in consumers' intrinsic valuation. Therefore, a single action ought to be chosen by all consumers from the firm's view, although the naive consumers falsely believe there is a positive probability of choosing a different action. In their model, the naive consumers do not exert any informational externality on the sophisticated, despite information asymmetry. In our model, the consumers make the consumption decision by responding to realized intrinsic valuation and the sophisticated consumers' actual consumption can be severely distorted to reduce the naive consumers' information rent.

Gottlieb and Zhang (2021) assume naive time-inconsistent consumers can engage in long-term contractual relationship with firms. They investigate the long-run welfare effects under various market conditions, including the settings i) when the firm has marker power and ii) when there is an additional group of sophisticated time-inconsistent consumers that the competitive firms cannot identify. Their results on both settings show vanishing welfare loss as the contracting horizon goes to infinity. Compared to them, we highlight finite-horizon welfare consequences induced by time inconsistency and naivete, in a setting integrating the firm's market power and asymmetric information on consumer beliefs.

Li and Jiang (2022) analyze a market with naive time-inconsistent consumers and vertically differentiated firms. The firms sell durable aversion goods. They compare the firms' profits under dynamic pricing with those under static pricing with commitment. Esteban and Miyagawa (2006) and Esteban, Miyagawa, and Shum (2007) study monopolistic pricing when consumers' preferences exhibit temptation and self-control. They use the Gul-Pesendorfer utility to capture the consumers' self-control problem and their consumers are all sophisticated.

Not in a time inconsistency framework, some studies consider other types of consumer biases or nonstandard preferences. Grubb (2009) investigates the pricing implications when consumers have satiated utility and underestimate the variance of future demand. His result shows that the optimal contract can be approximated by three-part tariffs with below-marginal-cost price within an allowance and steep price beyond the allowance. Gabaix and Laibson (2006) and Heidhues and Kőszegi (2017) consider consumers' naivete of product attributes or pricing schemes in that they are unaware of add-on prices. Gabaix and Laibson (2006) show that the firms shroud add-on prices in equilibrium and thus compete by subsidizing basic goods and charging high add-on prices. Heidhues and Kőszegi (2017) study how horizontally differentiated firms engage in third-degree discrimination based on external information on consumer naivete. While these works show pricing deviations from marginal cost, their issues are different from the coexistence of above- and below-marginal-cost pricing in different tariffs in a menu.

Finally, this paper relates to the literature on mechanism design (e.g., Mussa and Rosen 1978; Myerson 1981; Maskin and Riley 1984) and, in particular, sequential screening (e.g., Baron and Besanko 1984; Courty and Li 2000). Earlier studies are mostly concerned with asymmetric information between a designer and heterogeneous participants, all time-consistent. Our two-stage contractual setup is similiar to that of Courty and Li (2000), but the asymmetric information in their setting originates from the time-consistent agents' private knowledge about the distribution of future valuation. Galperti (2015) studies the optimal design of commitment devices with time-inconsistent agents. He assumes that the agents are all sophisticated and heterogeneous in their degree of time inconsistency. His focus is on the truncation or expansion of the agents' options given an interval of feasible actions.

2 Model Setup

We define temptation goods, such as cell phones and video games, as goods that are overvalued by consumers at the consumption stage relative to the planning stage. Symmetrically, aversion goods, such as health clubs and product repairs, are defined as goods undervalued by consumers at the consumption stage relative to the planning stage. We focus on temptation goods here and will take care of aversion goods later as an extension.

Consider a two-period model with a monopolistic firm and a unit measure of consumers. All are risk-neutral. The firm produces a temptation good at marginal cost a. The consumers can buy and consume one unit of the temptation good if they contract with the firm in advance. We refer to the contracting stage as Period 0 and the purchase/consumption stage as Period 1.[3] A contract consists of an upfront fee L and a per-usage price p. The consumers pay L to the firm if they accept the contract in Period 0, and pay an additional p if they decide to consume in Period 1. In both periods, the consumers have access to an outside option, whose value is normalized to zero.

Our model has three main features. First, the consumers face uncertainty in their value of consumption. To be specific, the consumers' value of consumption depend on state v, which may reflect individual taste shocks. In Period 0, the consumers do not know their individual state but only its distribution. In Period 1, the consumers privately observe their own realized state before choosing whether to consume. For simplicity, assume the distribution of v is uniform on $[\underline{v}, \bar{v}]$,

[3]As in the literature (e.g., DM; Eliaz and Spiegler 2006), to model time-inconsistent preference, it is necessary to separate the contracting and the consumption periods. Nevertheless, our setup can be readily extended to allow for multiple consumption periods that are independent from each other. Our insights can still be retained qualitatively, even if some (but not all) of the consumptions occur immediately after contracting in Period 0. However, if all consumptions happened in the contracting period (which is unrealistic for advance contracts in practice), the issue of time inconsistency would be immaterial.

with density function $f(v) = \frac{1}{\bar{v}-\underline{v}}$ and CDF $F(v) = \frac{v-\underline{v}}{\bar{v}-\underline{v}}$. The expectation of v is denoted by $E[v] = \int_{\underline{v}}^{\bar{v}} v dF(v)$.

The second main feature is that the consumers have time-inconsistent preference. We capture the time-inconsistent preference by a linear distortion function $h(v) = \alpha v + \tau$. The distortion function represents how the state-dependent valuation for (Period-1) consumption may change as the time passes from Period 0 to Period 1. In particular, given any state v, the (Period-1) consumption would be valued as v when the consumers are in Period 0 but as $h(v)$ when they come to Period 1.[4] Assume $\alpha > 0$ so that the preference ordering is preserved under the distortion. For temptation goods, $h(v) \geq v$ on $[\underline{v}, \bar{v}]$. The difference between Period-1 and Period-0 valuations, $h(v) - v$, represents the extent of time inconsistency. Depending on whether $\alpha > 1$ or $\alpha < 1$, the extent of time inconsistency may be increasing or decreasing in v. As will be made clear, this parameter α is a key driver of our results.[5] The intercept $\tau \geq \max\{(1-\alpha)\bar{v}, (1-\alpha)\underline{v}\}$ reflects a uniform shift in the time-inconsistent valuation. We say time inconsistency is higher under h_1 than under h_0 if $h_1(v) \geq h_0(v)$ for all $v \in [\underline{v}, \bar{v}]$ and $h_1(v) > h_0(v)$ at least for some $v \in [\underline{v}, \bar{v}]$. The class of linear distortion can accommodate the following cases that are frequently adopted in literature.

Example 1. Proportional distortion: $h(v) = \alpha v$, with $\underline{v} \geq 0$ and $\alpha > 1$. The parameter α can be viewed as a salience factor for immediate return. Akerlof (1991), Wu, Ramachandran, and Krishnan (2014), and Galperti (2015) use this specification to model time inconsistency.

Example 2. Uniform distortion: $h(v) = v + \tau$, with $\tau > 0$. This case can capture the three-period specification of quasi-hyperbolic discounting in DM, Gottlieb (2008), and Li, Yan, and Xiao (2014). Let β denote the short-run discount factor and δ denote the long-run discount factor. The temptation good brings a constant Period-2 cost c and a random Period-1 benefit b. Before signing the contract in Period 0, the consumers weigh the discounted utility of consuming in Period 1, $\beta\delta(b-\delta c)$, against the discounted price $\beta\delta p$. However, when it comes to Period 1, the consumers weigh the discounted utility of (immediate) consumption, $b - \beta\delta c$, against the price p. The consumers' decision process can be equivalently captured by taking $v = b - \delta c$ and $h(v) = v + \delta(1-\beta)c$ in our setup.[6]

The third main feature of our model is that the consumer type may be sophisticated or naive. The sophisticated type (indexed by s) takes proportion $\gamma_s \in (0, 1)$ and the naive type (indexed by n) takes proportion $\gamma_n = 1 - \gamma_s$. The sophisticated consumers are fully aware of their time inconsistency and correctly predict that they will consume in Period 1 if $h(v) \geq p$. Their perceived

[4] Note that the state v is realized only in Period 1 so that the consumers have uncertainty in Period 0 about the valuation of consumption (which is equal to v). In other words, when the consumers are in Period 0, their expected valuation for the consumption should be $E[v]$, the expectation of the random variable v.

[5] The cases of $\alpha > 1$ and $\alpha < 1$ imply that the correlation between the product valuation (v) and the extent of time inconsistency ($h(v) - v$) is positive or negative, respectively. Therefore, if we can measure empirically the extent of time inconsistency or the change of product valuations over time, we can estimate α empirically.

[6] Note that considering time discounting does not change our analysis/results at all as long as the firm and the consumers share the same discount factor δ. That is, we can re-interpret all the payoff-relevant variables/parameters that are incurred in Period 1 (i.e., v, p, τ, and a) as their Period-0 discounted equivalents (i.e., multiply each of these variables/parameters by the same discount factor δ), and our model would remain unchanged qualitatively.

utility of contract (L, p) in Period 0 is

$$U_s(L, p) = -L + \int_{h^{-1}(p)}^{\bar{v}} (v - p)\, dF(v) \tag{1}$$

In contrast, the naive consumers are fully unaware of his time inconsistency. They forecast (Period-1) consumption based on the belief that their valuation in Period 1 remains the same as in Period 0. Thus the naive consumers mispredict that they will consume whenever $v \geq p$, underestimating the likelihood of consumption. Their perceived utility of contract (L, p) in Period 0 is

$$U_n(L, p) = -L + \int_{p}^{\bar{v}} (v - p)\, dF(v) \tag{2}$$

The time-consistent firm anticipates the actual consumer behavior. It is without loss to assume that the firm offers a menu of two contracts, one for each type. The Period-0 expected profit from offering (L_s, p_s) to the sophisticated type and (L_n, p_n) to the naive type is given by

$$\Pi((L_j, p_j)_{j=s,n}) = \gamma_s \left[L_s + \int_{h^{-1}(p_s)}^{\bar{v}} (p_s - a)\, dF(v) \right] + \gamma_n \left[L_n + \int_{h^{-1}(p_n)}^{\bar{v}} (p_n - a)\, dF(v) \right] \tag{3}$$

We assume the marginal cost $a \in (\underline{v}, E[v])$ such that, from the long-run perspective, consumption is on average efficient but not in every state. The assumption $a < E[v]$ can simplify matters by ensuring that both consumer types would always be served (see Lemma 2) and thus the optimal menu would involve more than one tariff.

To facilitate our analysis, we introduce two functions:

$$T(p) = \int_{h^{-1}(p)}^{\bar{v}} (v - a) dF(v)$$
$$I(p) = \int_{h^{-1}(p)}^{p} -(v - p) dF(v)$$

$T(p)$ is defined as the total surplus of a contract with per-usage price p. It represents the contract's social value in Period 0, given the actual consumption decision.[7] The shape of T is shown in Figure 1. When $p \geq h(\bar{v})$, no trade occurs and the corresponding total surplus is zero. When $p \leq h(\underline{v})$, the good is traded with probability 1, and the corresponding total surplus is $E[v] - a$. Moreover, $T(p)$ is single-peaked at $h(a)$, which we call the *efficient price*. We impose Assumption 1 throughout our analysis.

Assumption 1. *The efficient price falls in the range of Period-0 valuation: $h(a) \in (\underline{v}, \bar{v})$.*

$I(p)$ is defined as the illusional surplus of a contract with per-usage price p. It captures the additional utility perceived by the naive type relative to the sophisticated type, i.e., the differ-

[7]The ex ante criterion for welfare evaluation is commonly used in the literature studying time-inconsistent preferences. See, e.g., Akerlof (1991), O'Donoghue and Rabin (1999), Eliaz and Spiegler (2006), Heidhues and Koszegi (2010), Galperti (2015).

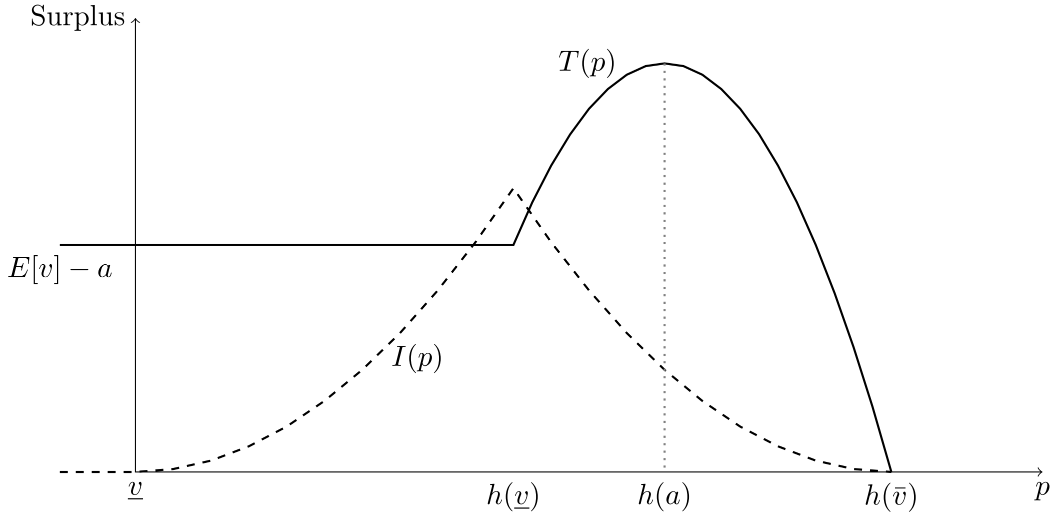

Figure 1: Plot of total surplus $T(p)$ and illusional surplus $I(p)$ for temptation goods. $[\underline{v}, \bar{v}] = [0, 1]$, $h(v) = 0.5v + 0.5$, and $a = 0.4$.

ence between $U_n(L, p)$ in equation (2) and $U_s(L, p)$ in equation (1), due to misprediction of future consumption. As the naive consumers believe they are time-consistent, we can also view $I(p)$ as the utility loss of the sophisticated time-inconsistent consumers relative to the time-consistent consumers, due to overconsumption of the temptation good. $I(p)$ is positive on $(\underline{v}, h(\bar{v}))$ and zero elsewhere (see Figure 1).

Lemma 1. *I is single-peaked. On the regular interval $[h(\underline{v}), \bar{v}]$, I is increasing if $\alpha > 1$, decreasing if $\alpha < 1$, and constant if $\alpha = 1$.*

Lemma 1 shows that I's shape is mainly driven by the slope α of the distortion function. With uniformly distributed v, $I(p) = \frac{1}{2(\bar{v}-\underline{v})}(p - h^{-1}(p))^2$ on the regular interval $[h(\underline{v}), \bar{v}]$.[8] The regular interval is sufficiently wide and contains $h(a)$, under Assumption 1. A nice property on this interval is that I is positively determined by the extent of misprediction in future demand $\frac{1}{\bar{v}-\underline{v}}(p - h^{-1}(p))$, measured by the difference between the actual Period-1 demand $1 - F(h^{-1}(p)) = \frac{1}{\bar{v}-\underline{v}}(\bar{v} - h^{-1}(p))$ and the naive type's anticipated Period-1 demand $1 - F(p) = \frac{1}{\bar{v}-\underline{v}}(\bar{v} - p)$. Under linear distortion $h(v) = \alpha v + \tau$, the extent of misprediction in future demand is proportional to $p - h^{-1}(p) = (1 - \frac{1}{\alpha})p + \frac{\tau}{\alpha}$. This expression increases in p when $\alpha > 1$, decreases in p when $\alpha < 1$, and stays constant when $\alpha = 1$. Thus the illusional surplus also behaves in this way on the regular interval.

Finally, we define $\mu(L, p) = -L + \int_{h^{-1}(p)}^{\bar{v}} (v - p)\, dF(v)$ as the consumer surplus generated by the contract (L, p). It is the expected utility calculated using the actual consumption decision and hence represents the ex ante surplus the firm concedes to either type of consumers. Note that $\mu(L, p)$ coincides with the sophisticated consumers' perceived utility, given by $U_s(L, p)$ in equation (1).

[8]Although I is also positive on $(\underline{v}, h(\underline{v}))$ and $(\bar{v}, h(\bar{v}))$, it is either independent or has less intuitive connection to h on these intervals for the technical reason: we assume the distribution of v has a bounded support $[\underline{v}, \bar{v}]$, and thus $h^{-1}(p)$ hits the lower bound of the distribution on $(\underline{v}, h(\underline{v}))$ and p hits the upper bound of the distribution on $(\bar{v}, h(\bar{v}))$.

With some abuse of notation, we redefine a contract as (μ, p), where μ is the consumer surplus of the contract and p is the price. There is a one-to-one correspondence between (μ, p) and (L, p), since L is automatically pinned down given μ and p. Therefore, we can take the firm's contract design problem as choosing (μ, p). When the contract is defined in this way, the sophisticated consumers' perceived utility is μ, and naive consumers' perceived utility is $\mu + I(p)$. Note that the naive consumers falsely believe that the illusional surplus I can be obtained on top of the actual surplus μ. Firm profit in equation (3) becomes $\gamma_s[T(p_s) - \mu_s] + \gamma_n[T(p_n) - \mu_n]$, which is just the total surplus net of consumer surplus, weighted by the proportion of each type.

3 Benchmarks

In this section, we analyze two benchmarks. The first benchmark considers the market involving only time-consistent consumers. The second one assumes that the firm can identify each consumer's type and thus treat the sophisticated and naive consumers independently.

3.1 Market with Time-Consistent Consumers

In this subsection, we assume the consumers are time-consistent, i.e., $h(v) = v$. Then, there is no distinction between the sophisticated type and the naive type. The illusional surplus vanishes and consumer utility equals consumer surplus. Moreover, the total surplus attains its maximum at $p = a$. The monopolistic firm chooses (μ, p) that maximizes profit $T(p) - \mu$ subject to the consumers' participation constraint $\mu \geq 0$. At the optimum, the firm sets $p = a$ to maximize the total surplus and then extracts all of it by setting $\mu = 0$. As a result, marginal-cost pricing is optimal when the consumers are time-consistent.

3.2 Market with Time-Inconsistent Consumers and Observable Beliefs

Now we assume the consumers are time-inconsistent and the firm can observe their types/beliefs. The firm then treats each type of consumers separately, equivalent to operating in two independent markets, one market with sophisticated consumers and the other with naive consumers. For the sophisticated consumers, the firm chooses (μ_s, p_s) to maximize profit $T(p_s) - \mu_s$ subject to their participation constraint $\mu_s \geq 0$. Similarly, for the naive consumers, the firm chooses (μ_n, p_n) to maximize $T(p_n) - \mu_n$ subject to their participation constraint $\mu_n + I(p_n) \geq 0$. In either case, the participation constraint is binding at the optimum. Call the optimal contract in this setting the *first-best* contract and denote it by superscript FB.

Observation 1. *Suppose the firm can identify each consumer's type. The first-best contract for the sophisticated type satisfies* $p_s^{FB} = \arg\max T(p_s)$ *and* $\mu_s^{FB} = 0$*. The first-best contract for the naive type satisfies* $p_n^{FB} = \arg\max T(p_n) + I(p_n)$ *and* $\mu_n^{FB} = -I(p_n^{FB})$*.*

Observation 1 can be interpreted as follows. The firm chooses the price to maximize the extractable surplus and then extracts all of it, making the participation constraint just binding. The extractable surplus is $T(p_s)$ from contracting with the sophisticated consumers. The extractable surplus is $T(p_n) + I(p_n)$ from contracting with the naive consumers. Note also that, although the perceived utility is zero for both consumer types under their respective optimal contract, the naive consumers are exploited and get negative consumer surplus.

Proposition 1. *The first-best prices p_s^{FB} and p_n^{FB} are above marginal cost a. Moreover, $p_s^{FB} = h(a) < p_n^{FB}$ if $\alpha > 1$; $p_s^{FB} = h(a) > p_n^{FB}$ if $\alpha < 1$; and $p_s^{FB} = p_n^{FB} = h(a)$ if $\alpha = 1$.*

Proposition 1 shows that the first-best price for both consumer types are above marginal cost. This result is consistent with DellaVigna and Malmendier (2004). Overpricing can maximize the total surplus from the sophisticated consumers' contract by correcting their overconsumption tendency. In the meantime, overpricing exploits consumer naivete. Since the naive consumers consume more than what they anticipate, the firm makes more profit by tilting payments backwards, leading to high p_n^{FB}.

Proposition 1 also ranks the first-best prices between the different types of consumers. The first-best price for the sophisticated consumers is always efficient. However, the first-best price for the naive consumers is above the efficient price when $\alpha > 1$, below the efficient price when $\alpha < 1$, and equals the efficient price when $\alpha = 1$. Notably, the firm has incentive to distort p_n^{FB} to increase illusional surplus. Under Assumption 1, the efficient price belongs to the regular interval $[h(\underline{v}), \bar{v}]$. The ranking between the efficient price and p_n^{FB} thus follows from the monotonicity of illusional surplus in Lemma 1. Although the good is tempting in nature, it is consumed efficiently when $\alpha = 1$ and even less than the efficient level when $\alpha > 1$ due to an overly high price.

4 Market with Time-Inconsistent Consumers and Unobservable Beliefs

From now on, we analyze the market with time-inconsistent consumers and assume the firm cannot observe their types/beliefs. The firm's profit-maximization problem is to choose (μ_s, p_s) and (μ_n, p_n) that solve

$$\max \gamma_s[T(p_s) - \mu_s] + \gamma_n[T(p_n) - \mu_n]$$

s.t.

$$\mu_s \geq 0 \qquad (PC_s)$$

$$\mu_n + I(p_n) \geq 0 \qquad (PC_n)$$

$$\mu_s \geq \mu_n \qquad (IC_s)$$

$$\mu_n + I(p_n) \geq \mu_s + I(p_s) \qquad (IC_n)$$

The participation constraint PC_j requires that the consumers of type j perceive nonnegative utility from contracting with the firm; the incentive compatibility constraint IC_j requires that the consumers of type j perceive a weakly higher utility from choosing their contract than choosing the contract for the other type.

4.1 Characterization of the Optimal Contracts

Recall from Section 2 that given any contract (μ, p), the naive type perceives an additional $I(p)$ relative to the sophisticated type. Since $I(p)$ is nonnegative, the naive type is the high type in the classic monopolistic screening setting (see Chapter 2 of Bolton and Dewatripont 2005). Suppose the firm continued to offer the menu of the first-best contracts. Then the naive type would choose (μ_s^{FB}, p_s^{FB}), because their perceived utility would be $I(p_s^{FB}) > 0$ from (μ_s^{FB}, p_s^{FB}) and only zero from (μ_n^{FB}, p_n^{FB}). Therefore, the first-best contracts violate the constraint IC_n. To prevent the naive type from mimicking the sophisticated type, the firm needs to modify the contract terms by paying the naive type an information rent.

To solve the menu of optimal contracts, we can reduce the firm's constraints as follows. First, PC_n is nonbinding, due to IC_n and PC_s. Second, IC_n is binding; otherwise, the firm could gain by decreasing μ_n without violating any constraint. Third, PC_s is also binding; otherwise, decreasing μ_s and μ_n by the same amount would raise firm profit without violating any constraint. Lastly, IC_s is nonbinding. To see why, note first that IC_s is equivalent to $\mu_n \leq 0$ due to the binding PC_s. However, any contract menu with $\mu_n > 0$ is strictly dominated by the pooling contract $(\mu^{pool}, p^{pool}) = (0, h(a))$, as this pooling contract attains the maximum total surplus and concedes less consumer surplus.

Using the only binding constraints PC_s and IC_n, we can reduce the firm's problem into

$$\max \gamma_s T(p_s) + \gamma_n [T(p_n) + I(p_n) - I(p_s)]$$

In the firm's objective, the first term is the total surplus from contracting with the sophisticated type, weighted by the proportion γ_s. The second term is the expected profit from the naive type, whose proportion is γ_n. Inside the bracket is the surplus that can be extracted from the naive type: $T(p_n)$ is the total surplus from contracting with the naive type, $I(p_n)$ is the gain from exploiting naviete, and $I(p_s)$ is the information rent paid to the naive type.

Observation 2. *Suppose the firm cannot identify the consumers' type. The optimal menu of contracts is characterized by* $p_s^* = \arg\max \gamma_s T(p_s) - \gamma_n I(p_s)$, $p_n^* = \arg\max T(p_n) + I(p_n)$, $\mu_s^* = 0$, *and* $\mu_n^* = I(p_s^*) - I(p_n^*)$.

Observation 2 has the following implications. First, $p_n^* = p_n^{FB}$. Indeed, as the naive type corresponds to the high type in our setting, the general no-distortion-at-the-top result applies. Second, the firm needs to balance two incentives in choosing the optimal p_s. On one hand, it wants to enhance the total surplus to be extracted from the sophisticated type, $T(p_s)$. On the other hand, it wants to reduce the information rent paid to the naive type, $I(p_s)$. The first incentive is increasing in the proportion of the sophisticated type γ_s, whereas the second one is increasing in $\gamma_n = 1 - \gamma_s$. Moreover, at the optimum, the sophisticated consumers get zero consumer surplus/perceived utility, as their participation constraint is binding. In comparison, the consumer surplus for the naive type is $\mu_n^* \leq 0$ and their perceived utility is $\mu_n^* + I(p_n^*) = I(p_s^*) \geq 0$. The information rent exactly equals the naive type's perceived utility $I(p_s^*)$.

4.2 Properties of the Optimal Contracts

The solution for the optimal contracts may exhibit three different types of outcomes.[9] First, the firm can serve both consumer types with a pooling contract and hence does not discriminate between them. Second, the firm can serve both consumer types with a screening menu of contracts. Alternatively, the firm can get the total extractable surplus from serving one consumer type, while excluding the other type. Lemma 2 shows that the last type of outcome is suboptimal.

Lemma 2. *It is optimal to serve both types of consumers, i.e., $p_s^*, p_n^* < h(\bar{v})$.*

Exclusion never happens under the optimal contract in our setting in the sense that both types consume with positive probability. If exclusion ever occurred, the sophisticated consumers would be excluded first, because they value any given contract less than the naive consumers. The firm would exclude the sophisticated consumers if serving them decreased the firm's profit, i.e., $\gamma_s T(p_s) - \gamma_n I(p_s) < 0$, for any p_s. However, this clearly does not happen: setting $p_s = \underline{v}$ guarantees that $\gamma_s T(\underline{v}) - \gamma_n I(\underline{v}) = \gamma_s(E[v] - a) > 0$.

Proposition 2. *(pooling vs. separating) The optimal menu of contracts is pooling if $\alpha = 1$ and $\gamma_s \geq \hat{\gamma}_s$ for some $\hat{\gamma}_s \in (0, 1)$. The optimal menu of contracts is separating if $\alpha \neq 1$ or $\gamma_s < \hat{\gamma}_s$. Moreover, $\hat{\gamma}_s$ is decreasing in marginal cost; and if time inconsistency is higher under h_1 than under h_0, $\hat{\gamma}_s$ is larger under h_1 than under h_0.*

Proposition 2 examines the optimality of pooling. The firm's total profit is reduced to $T(p)$ when $p_s = p_n = p$ under a pooling contract. The optimal pooling contract then offers $p^{pool} = h(a)$, the efficient price, and concedes zero surplus to the consumers. Proposition 2 first shows that a pooling optimum at the efficient price arises if the distortion is uniform, i.e., $\alpha = 1$, and the proportion of the sophisticated type is sufficiently large. It indicates that asymmetric information on consumer beliefs need not preclude the optimal contract from being efficient. Proposition 2 also identifies conditions under which the menu of optimal contracts is separating. In particular, when $\alpha \neq 1$, the illusional surplus is not flat around p^{pool} by Assumption 1 and Lemma 1. Thus, manipulating the per-usage price around p^{pool} leads to a first-order change of the illusional surplus and only a second-order loss of the total surplus. The firm can then set p_s and p_n around p^{pool} in opposite directions to enlarge (reduce) the illusional surplus of the naive (sophisticated) consumers' contract, which necessarily leads to a separating optimum. In addition, when $\gamma_s < \hat{\gamma}_s$, the efficiency loss of distorting the sophisticated consumers' contract is small relative to the gain of reducing the information rent to the naive consumers. The firm can then turn to the reserve option $p_s = \underline{v}$, which guarantees positive

[9] We must have $L_s^* \leq L_n^*$ if and only if $p_s^* \geq p_n^*$. This is due to binding IC_n and the fact that the naive type's perceived utility, defined in equation (2), is decreasing in both L and p. We focus our analysis on p_s^* and p_n^*.

profit and entails zero information rent. In this case, we obtain a separating optimum again. We also find that $\hat{\gamma}_s$ is decreasing in marginal cost and increasing in time inconsistency. Intuitively, the firm is more likely to offer $p_s = \underline{v}$ when marginal cost decreases and time inconsistency increases, because indulging the sophisticated type becomes less costly and curtailing information rent becomes more urgent, respectively.

Proposition 3. *(price distortion) The optimal price p_s^* is downward distorted relative to p_s^{FB} if $\alpha > 1$ or $\gamma_s < \hat{\gamma}_s$. The optimal p_s^* is upward distorted relative to p_s^{FB} if $\alpha < 1$ and $\gamma_s \geq \hat{\gamma}_s$.*

Proposition 3 shows that p_s^* may be upward or downward distorted relative to $p_s^{FB} = h(a)$. When the proportion of the sophisticated consumers is sufficiently small, i.e., $\gamma_s < \hat{\gamma}_s$, the firm prioritizes to mitigate information rent and simply offers the reserve option $p_s = \underline{v}$. Downward distortion of p_s^* necessarily occurs. In comparison, when $\gamma_s \geq \hat{\gamma}_s$, both upward and downward distortion of p_s^* are possible. The direction of price distortion depends on the monotonicity of the illusional surplus, which in turn depends on the slope α of the distortion function. When $\alpha > 1$ ($\alpha < 1$), the firm optimally distorts p_s^{FB} below (above) the first best to reduce the illusional surplus–and hence the information rent–associated with the sophisticated consumers' contract.

Proposition 4. *(ranking reversal) If $\alpha < 1$ and $\gamma_s < \hat{\gamma}_s$, the ranking between p_s^* and p_n^* is reversed relative to the first best; in particular, $p_s^* < p_n^*$ and $p_s^{FB} > p_n^{FB}$. If $\alpha \geq 1$ or $\gamma_s \geq \hat{\gamma}_s$, the ranking between p_s^* and p_n^* is preserved and their gap is enlarged, relative to the first best.*

Proposition 4 shows that the ranking of the two optimal prices can be reversed relative to the first best, meaning that asymmetric information on consumer beliefs can qualitatively modify the firm's optimal contract terms. In our setting, ranking reversal happens when the slope of distortion and the proportion of the sophisticated type are both small. When $\alpha < 1$, the illusional surplus is decreasing over the regular interval containing the efficient price $h(a)$. Thus, the firm's incentive to exploit naivete drives p_n^{FB} below $p_s^{FB} = h(a)$ by Proposition 1. In the asymmetric information setting, the firm's incentive to reduce the naive type's information rent decreases in γ_s. When $\gamma_s < \hat{\gamma}_s$, the firm optimally chooses the reserve option $p_s = \underline{v}$ to minimize information rent. But the optimal p_n^* remains to be first best and above $\underline{v}$ under asymmetric information. Therefore, given a combination of $\alpha < 1$ and $\gamma_s < \hat{\gamma}_s$, the price ranking is reversed.

Proposition 5. *(two-sided deviations from marginal-cost pricing) The optimal prices deviate from marginal cost a in two sides if $\gamma_s < \tilde{\gamma}_s$ for some $\tilde{\gamma}_s \in [\hat{\gamma}_s, 1)$; in particular, $p_s^* < a < p_n^*$. The optimal prices are both above marginal cost if $\gamma_s \geq \tilde{\gamma}_s$. Moreover, $\tilde{\gamma}_s$ is decreasing in marginal cost and increasing in the slope α of the distortion function.*

Proposition 5 shows that the optimal menu can include one contract with marginal price above marginal cost and the other contract with marginal price below marginal cost. We know from Proposition 1 that the first-best prices p_s^{FB} and p_n^{FB} are both above marginal cost. Because the naive type corresponds to the high type, the optimal price for this type is not distorted, i.e., $p_n^* = p_n^{FB} > a$. We also know that p_s^* is distorted relative to the first best, to reduce the naive type's the information rent. To obtain two-sided pricing deviations from marginal cost, we need sufficiently strong downward distortion of p_s^* so that $p_s^* < a$. It requires the proportion of the sophisticated type–and hence the firm's weight on efficiency–to be sufficiently small.

The firm is more likely to offer below-marginal-cost price to the sophisticated consumers when marginal cost is small, because overpricing would exclude too many profitable trades in this case. We also find that p_s^* tends to go below marginal cost when the slope α of the distortion function is large, holding fixed the uniform shift τ.[10] It implies that the sophisticated type is more likely to be indulged for higher time inconsistency induced by α. To understand this result, first consider $\alpha \leq 1$. We know from Proposition 2 and 3 that when $\gamma_s \geq \hat{\gamma}_s$, p_s^* features either no distortion or upward distortion relative to $p_s^{FB} > a$. Thus, the only possibility for below-marginal-cost pricing is to choose the reserve option $p_s = \underline{v}$, which is optimal when $\gamma_s < \hat{\gamma}_s$. Therefore, the cutoff $\tilde{\gamma}_s = \hat{\gamma}_s$ increases in time inconsistency and hence in α. In the case $\alpha > 1$, it is possible to have interior p_s^* below marginal cost when the firm's screening motive is sufficiently strong. The logic is then different from the previous case with below-marginal-cost p_s^* only at the boundary. Recall that the naive type's misprediction in future demand is proportional to $(1 - \frac{1}{\alpha})p + \frac{\tau}{\alpha}$. Increasing α further above unity makes the extent of misprediction more sensitive to price. Then relative to the total surplus, the illusional surplus becomes more responsive to price changes, raising the firm's motive to manipulate interior prices for screening.

4.3 Profit, Consumer Surplus, and Welfare

We take the ex ante perspective to evaluate welfare. In addition, we consider the (naive) consumers' actual surplus, rather than their perceived utility, in evaluating their equilibrium welfare. Under the optimal menu of contracts, firm profit Π^*, social welfare W^*, and consumer surplus $\{\mu_s^*, \mu_n^*\}$ are given by

$$\Pi^* = \gamma_s T(p_s^*) + \gamma_n[T(p_n^*) + I(p_n^*) - I(p_s^*)] \qquad \mu_s^* = 0$$
$$W^* = \gamma_s T(p_s^*) + \gamma_n T(p_n^*) \qquad \mu_n^* = I(p_s^*) - I(p_n^*) \leq 0$$

Now we investigate the comparative statics of the equilibrium profit, consumer surplus, and social welfare. We first examine the impact of varying the degree of time inconsistency. Recall that time inconsistency is higher under h_1 than under h_0 if $h_1(v) \geq h_0(v)$ for all $v \in [\underline{v}, \bar{v}]$ and $h_1(v) > h_0(v)$ at least for some $v \in [\underline{v}, \bar{v}]$. For linear distortion $h(v) = \alpha v + \tau$, the increasing time inconsistency may come from a greater α, a greater τ, or a combination of changes in α and τ, as long as the above condition is satisfied.

Proposition 6. *Suppose time inconsistency is higher under h_1 than under h_0, $h_1(\bar{v}) - \bar{v} < \bar{v} - a$, and γ_s is sufficiently large.*

[10] Note that, although we have $\tilde{\gamma}_s$ increasing in α, $\tilde{\gamma}_s$ may not increase in time inconsistency. For example, consider $[\underline{v}, \bar{v}] = [0, 1]$, $a = 0.4$, and two distortion functions $h_1(v) = 1.2v + 0.1$ and $h_0(v) = 1.3v$. Obviously, h_1 induces higher time inconsistency than h_0. But we can compute the cutoff $\tilde{\gamma}_{si} = 1 - \frac{1}{\alpha_i}$, which is higher under h_0. See Appendix for more details.

(i) If either $\alpha_1 > \alpha_0 \geq 1$ or $\alpha_1 < \alpha_0 \leq 1$, firm profit is higher under h_1, consumer surplus and social welfare are lower under h_1.

(ii) If α_1 is sufficiently close to unity and either $\alpha_0 > \alpha_1 \geq 1$ or $\alpha_0 < \alpha_1 \leq 1$, firm profit is lower under h_1, consumer surplus and social welfare are higher under h_1.

Proposition 6 shows that higher time inconsistency may increase (decrease) firm profit if the slope α of the distortion function becomes more (less) distant to unity. An increase in time inconsistency enhances the illusional surplus, which exerts two countervailing effects on the firm's equilibrium profit. On the positive side, the naive type's underestimation of his future consumption would be increasingly exploited. On the negative side, the naive type would receive a higher information rent. Firm profit increases if the former effect dominates and vice versa. The total effect on firm profit is closely related to α, the slope of the distortion function. Recall that the naive type's misprediction in future demand is proportional to $(1 - \frac{1}{\alpha})p + \frac{\tau}{\alpha}$. The misprediction is more sensitive to price if α is more different from unity. It implies that relative to the total surplus, the illusional surplus is more responsive to price changes, making screening more effective. As a result, firm profit increases. The intuition is symmetric when α gets much closer to unity. In this case, the illusional surplus becomes relatively less responsive to price changes. Hence, screening is more costly and firm profit decreases.

Proposition 6 also suggests that higher time inconsistency can be beneficial to the (naive) consumers and the whole society. This happens in the situation described in part (ii). As we have argued, the illusional surplus becomes relatively less responsive to price changes in this case. Then distorting prices away from the efficient level brings reduced benefit to the firm. In other words, the firm is less motivated to either exploit the naive consumers or screen the different consumer types. As a result, the naive consumers are better off and the social welfare increases.

Proposition 7. *Suppose the proportion of sophisticated consumers increases to γ_s'.*

(i) Firm profit decreases.

(ii) Consumer surplus increases for the initially naive consumers and does not change for the initially sophisticated consumers.

(iii) Social welfare may increase or decrease. In particular, social welfare decreases if $\gamma_s' < \hat{\gamma}_s$ and either $\alpha \leq 1$ or $a \geq \hat{a}$.

Proposition 7 shows the consequence of correcting the consumers' naive belief. Firm profit decreases in the proportion of the sophisticated consumers, as naivete directly constributes to the firm's extractable surplus. As a result, the profit-maximizing firm is unwilling to educate consumers, even at zero cost. Education does not affect the initially sophisticated consumers' surplus because their participation constraint is always binding. In comparison, education makes the initially naive consumers better off in two aspects. First, the naive consumers who become sophisticated get higher consumer surplus because they can avoid being exploited. Second, consumer surplus of those remaining naive is also higher, although still negative. The reason is that, in response to a larger proportion of the sophisticated type, the firm focuses more on production efficiency and less on information rent extraction.

Proposition 7 also shows that increasing the proportion of the sophisticated type may reduce social welfare. Suppose the proportion of the sophisticated type is small after education, i.e., $\gamma_s' < \hat{\gamma}_s$. Then the firm's incentive to reduce information rent is strong and p_s^* is severely downward distorted to reach $\underline{v}$. This must also be true before education, when the firm's information-rent-reduction incentive is even stronger. In addition, p_n^* is unaffected by education, according to Observation 2. Thus, under the condition $\gamma_s' < \hat{\gamma}_s$, education has no impact on either $T(p_n^*)$ or $T(p_s^*)$. It follows that a smaller proportion of the naive consumers decreases social welfare if the equilibrium total surplus generated from a naive consumer is higher than that from a sophisticated consumer, i.e., $T(p_n^*) > T(p_s^*) = T(\underline{v})$. This is guaranteed when either $\alpha \leq 1$ or $a \geq \hat{a}$. When $\alpha \leq 1$, the optimal prices are ranked as $\underline{v} = p_s^* < p_n^* < h(a)$. Although both types of consumers overconsume relative to the efficient level, the sophisticated type overconsumes more severely. Thus, marginally increasing the proportion of the sophisticated type decreases social welfare. In addition, when a is not too small, the social cost of indulging the sophisticated type can be very high and hence a larger proportion of the sophisticated type reduces social welfare. Our result cautions the unintended consequence of campaigns and public policies aiming at increasing consumer sophistication, which are often believed to be welfare-improving.

5 Extensions

5.1 The Time-Consistent Type

So far we have only considered the time-inconsistent consumers. In this subsection, we consider the existence of time-consistent consumers. The proportions of the sophisticated, the naive, and the time-consistent consumers add up to one. We do not impose any additional restriction on the consumer composition. Thus, one can always set the proportion of a particular type to be zero and consider the market interaction with the remaining two types. We show that our main results are robust when the time-consistent consumer exists as a third type. However, the two-sided deviations from marginal-cost pricing would not arise if we consider a setup with the time-consistent consumers and either the sophisticated or the naive consumers (but not both).

Because the time-consistent type and the naive type share the same Period-0 perceived utility, they are pooled to the same contract and get the same information rent. Let $\gamma_c = 1 - \gamma_s - \gamma_n$ be the proportion of the time-consistent type and $G(p) = \int_p^{h^{-1}(p)} (v-a)dF(v)$ be the difference in total surplus generated by the time-consistent type and the time-inconsistent type. The firm chooses (μ_s, p_s) and (μ_n, p_n) to solve the following profit-maximization problem:

$$\max \gamma_s[T(p_s) - \mu_s] + \gamma_n[T(p_n) - \mu_n] + \gamma_c[T(p_n) + G(p_n) - \mu_n - I(p_n)]$$

s.t.

$$\mu_s \geq 0 \qquad (PC_s)$$

$$\mu_n + I(p_n) \geq 0 \qquad (PC_n)$$

$$\mu_s \geq \mu_n \qquad (IC_s)$$

$$\mu_n + I(p_n) \geq \mu_s + I(p_s) \qquad (IC_n)$$

In the firm's objective, the first and second components are the expected profit from the sophisticated type and the naive type, respectively. The last component is the expected profit from the time-consistent type: inside the bracket is the total surplus from contracting with the time-consistent type, $T(p_n)+G(p_n)$, net of the consumer surplus conceded to this type, $\mu_n+I(p_n)$. The constraints have the same interpretation as before and, following the reasoning in Section 4.1, we can reduce them to $\mu_s = 0$ and $\mu_n + I(p_n) = \mu_s + I(p_s)$. Therefore, the firm's problem can be transformed into an unconstrained optimization:

$$\max \gamma_s T(p_s) + \gamma_n[T(p_n) + I(p_n) - I(p_s)] + \gamma_c[T(p_n) + G(p_n) - I(p_s)]$$

We use superscript tc to denote the optimal prices in the current setting. The optimal price charged to the sophisticated type is $p_s^{tc} = \arg\max \gamma_s T(p_s) - (\gamma_n + \gamma_c)I(p_s)$. Analogous to the basic model, p_s^{tc} needs to enhance the total surplus from serving the sophisticated type, while controlling the information rent to both the naive and the time-consistent type. Holding γ_s constant, p_s^{tc} is independent of the ratio of the naive type and the time-consistent type. Therefore, our results on the optimal p_s^* in the basic model extend to the current setting.

The naive type and the time-consistent type cannot be differentiated at Period 0. The optimal price charged to them is $p_n^{tc} = \arg\max \gamma_n[T(p_n)+I(p_n)]+\gamma_c[T(p_n)+G(p_n)]$. Besides the total surplus from these two types $\gamma_n T(p_n) + \gamma_c[T(p_n) + G(p_n)]$, the firm also considers the naive type's illusional surplus $\gamma_n I(p_n)$ as part of the extractable surplus. Since $T(p_n) + G(p_n)$ attains its maximum at a, we have $p_n^{tc} = a$ when $\gamma_n = 0$. It implies that absent the naive type, at most one contract from the optimal menu has price divergent from marginal cost. Following a continuity argument, p_n^{tc} takes an intermediate value between a and p_n^* for $\gamma_n \in (0, 1-\gamma_s)$. Moreover, when γ_n is large, p_n^{tc} is close to p_n^* and inherits its properties.

5.2 Aversion Goods

In this subsection, we investigate the market for aversion goods. In our definition, aversion goods are systematically undervalued at the consumption stage relative to the planning stage. Examples are health clubs and product repairs, which require immediate or more salient cost of effort or time.[11] According to DellaVigna and Malmendier (2004), aversion goods should be priced below

[11] Evidence shows that health club members tend to visit the health club less often than they had planned while enrolling. Members also exhibit substantial naivete in that they could have saved \$600 on average during their membership if they chose pay-per-visit plan instead. See DellaVigna and Malmendier (2006) and Garon, Masse, and

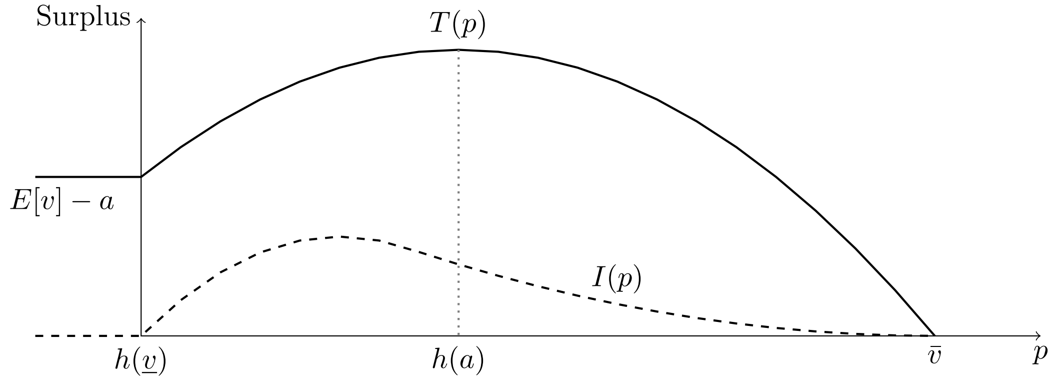

Figure 2: Plot of total surplus $T(p)$ and illusional surplus $I(p)$ for aversion goods. $[\underline{v}, \bar{v}] = [0, 1]$, $h(v) = 1.5v - 0.5$, and $a = 0.4$.

marginal cost when the firm can identify consumer beliefs. However, we observe that health clubs offer pay-per-visit options along with monthly/annual plans with high upfront fee and zero per-visit price. Also, durable-good sellers often offer regular repair service, as well as extended warranties that charge an upfront premium and a price per repairment much lower than market value.[12]

Our framework is readily applicable to the aversion good setting, except that we need to modify the assumption for distortion function. Specifically, we need $h(v) \leq v$ on $[\underline{v}, \bar{v}]$, i.e., the consumer undervalues aversion goods in the short run relative to the long run. The sophisticated type still correctly anticipates his future consumption. The interpretation of consumer naivete is the same as in Section 2: the naive type thinks he will consume whenever $v \geq p$ but actually consumes when $h(v) \geq p$. Notably, the naive type overestimates the likelihood of consumption for aversion goods. The expressions for consumer utility and firm profit are exactly the same as equation (1)–(3). We continue to use $T(p)$ and $I(p)$ to facilitate our analysis. Figure 2 illustrates the two functions in the aversion good setting.

Lemma 3. *In the aversion good market, I is single-peaked. On the regular interval $[\underline{v}, h(\bar{v})]$, I is increasing if $\alpha < 1$, decreasing if $\alpha > 1$, and constant if $\alpha = 1$.*

Lemma 3 is the counterpart of Lemma 1 in the aversion good market. For aversion goods, the extent of misprediction in future demand is proportional to $h^{-1}(p) - p = (\frac{1}{\alpha} - 1)p - \frac{\tau}{\alpha}$, which increases in p if $\alpha < 1$, decreases in p if $\alpha > 1$, and constant in p if $\alpha = 1$. Thus the illusional surplus, positively determined by the extent of misprediction, also behaves in this way on the regular interval. In the aversion good setting, the regular interval $[\underline{v}, h(\bar{v})]$ is sufficiently wide and contains $h(a)$ by Assumption 1. The following proposition shows that the main pricing properties for aversion goods are analogous to those for temptation goods. In particular, both upward and downward price

Michaud (2015) for more discussion.

[12] Apple Inc. offers AppleCare+ to provide additional coverage of its products beyond the standard one-year warranty. For example, AppleCare+ for a 13" MacBook Pro costs \$89.99 annually or \$249 for 3 years. The coverage includes unlimited incidents of accidental damage protection, each subject to a service fee of \$99 for screen damage or external enclosure damage, or \$299 for other accidental damage. Without AppleCare+, the cost to replace or repair a 13″ MacBook Pro screen is \$329-\$549 for most modern models.

distortions are possible, and the price ranking may be reversed in comparison to the first best. The intuition is analogous to Proposition 3–4.

Proposition 8. *In the aversion good market, the optimal menu of contracts satisfies the following properties:*

(i) (price distortion) p_s^* *is downward distorted relative to* p_s^{FB} *if* $\alpha < 1$ *or* $\gamma_s < \hat{\gamma}_s$. p_s^* *is upward distorted relative to* p_s^{FB} *if* $\alpha > 1$ *and* $\gamma_s \geq \hat{\gamma}_s$.

(ii) (ranking reversal) The ranking between p_s^* *and* p_n^* *is reversed relative to the ranking between* p_s^{FB} *and* p_n^{FB} *if* $\alpha > 1$ *and* $\gamma_s < \hat{\gamma}_s$*; in particular,* $p_s^* < p_n^*$ *and* $p_s^{FB} > p_n^{FB}$*. Otherwise, the ranking between* p_s^* *and* p_n^* *is preserved and their gap is enlarged, relative to the first best.*

However, in the aversion good market, a large proportion of the naive consumers need not lead to two-sided deviations from marginal-cost pricing. We can show that, analogous to Proposition 1, the first-best price for aversion goods is always below marginal cost. The intuition is simple: underpricing regulates the sophisticated consumers and exploits the naive consumers. By no distortion at the top, p_n^* remains the first best despite asymmetric information on consumer beliefs. When γ_s is very small, p_s^* is downward distorted by Proposition 8, and hence p_s^* and p_n^* are both below marginal cost. A very large γ_s fails to generate two-sided deviations from marginal-cost pricing either. This is because for large γ_s, p_s^* is close to the first-best level $p_s^{FB} = h(a)$, which is again below marginal cost. Therefore, an intermediate γ_s is required for the menu of optimal contracts to offer both below- and above-marginal-cost prices.

Given that p_n^* is the first best and stays below marginal cost, we must have p_s^* above marginal cost when two-sided deviations from marginal-cost pricing arise. The result indicates the direction of consumer seperation in the aversion good market. For example, in the health club industry, our result suggests that the naive consumers enroll in membership plans following their overestimation of future visits, whereas the sophisticated consumers choose pay-per-visit.

6 Conclusion

In this paper, we study the firm's optimal contract design when the consumers hold heterogeneous beliefs about their time-inconsistent preferences. We show that it can be profitable to screen the consumers based on their naviete. Since the naive consumers expect future utilization of any contract in a rosier way, the firm can exploit naivete by providing an inefficient contract associated with a high illusional surplus but a low consumer surplus. In the meantime, the firm's regulatory incentive is undermined: the contract designed for the sophisticated consumers is also inefficient and associated with a low illusional surplus because the firm wants to make it unattractive to the naive consumers. The optimal price for the naive consumers is always the first best, whereas the optimal price for the sophisticated consumers may be upward or downward distorted. Moreover, the informational distortion can reverse the ranking of the two optimal prices relative to the first best. Although we illustrate our points in the setting of uniform state distribution and linear distortion, Observation 2 provides the way to compute the optimal prices in a more general setting.

The per-usage prices for the naive and the sophisticated consumers may well diverge from marginal cost. Interestingly, the optimal menu can involve one contract with marginal price above marginal cost and the other contract with marginal price below marginal cost. Consider a monopolistic temptation-good market. It can be profit-maximizing to offer the sophisticated consumers a price below marginal cost with a large upfront fee and offer the naive consumers a price above marginal cost with a small upfront fee. Consequently, the firm exploits the naive consumers and profits from their unexpected heavy usage. The firm also indulges the sophisticated consumers by having them overconsume the temptation good. This is in sharp contrast to the first-best monopolistic pricing (e.g., DellaVigna and Malmendier 2004): in a market of sophisticated consumers, the temptation-good seller efficiently regulates their consumption and always adopts above-marginal-cost pricing.

Although the consumers' time inconsistency is necessary for discriminatory contracting to be feasible in our model, we find that a higher time inconsistency is not necessarily good for the firm. In some cases, increasing time inconsistency makes the illusional surplus harder to manipulate. Then the firm can only screen the consumers less effectively, leading to reduced profit. Following this logic, consumer surplus and social welfare may increase in time inconsistency because of reduced informational distortion. Furthermore, informing the consumers of their naivete is not necessarily good for society, as a larger proportion of the sophisticated consumers can exacerbate inefficiency due to asymmetric information. In the extensions, we show that our main results are robust in the presence of the time-consistent consumers and our analysis is largely symmetric for aversion goods. Our findings provide new insights into exploitation and regulation issues related to time inconsistency.

References

Akerlof GA (1991) Procrastination and obedience. *The American Economic Review* 81(2):1–19.

Baron DP, Besanko D (1984) Regulation and information in a continuing relationship. *Information Economics and Policy* 1(3):267–302.

Bolton P, Dewatripont M (2005) *Contract theory* (MIT Press).

Courty P, Li H (2000) Sequential screening. *The Review of Economic Studies* 67(4):697–717.

DellaVigna S, Malmendier U (2004) Contract design and self-control: Theory and evidence. *The Quarterly Journal of Economics* 119(2):353–402.

DellaVigna S, Malmendier U (2006) Paying not to go to the gym. *The American Economic Review* 96(3):694–719.

Eliaz K, Spiegler R (2006) Contracting with diversely naive agents. *The Review of Economic Studies* 73(3):689–714.

Ellison G (2006) Bounded rationality in industrial organization. *Econometric Society Monographs* 42:142.

Esteban S, Miyagawa E (2006) Temptation, self-control, and competitive nonlinear pricing. *Economics Letters* 90(3):348–355.

Esteban S, Miyagawa E, Shum M (2007) Nonlinear pricing with self-control preferences. *Journal of Economic Theory* 135(1):306–338.

Freeman DJ (2021) Revealing naïveté and sophistication from procrastination and preproperation. *American Economic Journal: Microeconomics* 13(2):402–438.

Gabaix X, Laibson D (2006) Shrouded attributes, consumer myopia, and information suppression in competitive markets. *The Quarterly Journal of Economics* 121(2):505–540.

Galperti S (2015) Commitment, flexibility, and optimal screening of time inconsistency. *Econometrica* 83(4):1425–1465.

Garon JD, Masse A, Michaud PC (2015) Health club attendance, expectations and self-control. *Journal of Economic Behavior & Organization* 119:364–374.

Gottlieb D (2008) Competition over time-inconsistent consumers. *Journal of Public Economic Theory* 10(4):673–684.

Gottlieb D, Zhang X (2021) Long-term contracting with time-inconsistent agents. *Econometrica* 89(2):793–824.

Grubb MD (2009) Selling to overconfident consumers. *The American Economic Review* 99(5):1770–1807.

Guo L (2015) Inequity aversion and fair selling. *Journal of Marketing Research* 52(1):77–89.

Guo L (2023) The mnemonomics of contractual screening. *Management Science* 69(3):1739–1757.

Heidhues P, Kőszegi B (2010) Exploiting naivete about self-control in the credit market. *The American Economic Review* 100(5):2279–2303.

Heidhues P, Kőszegi B (2017) Naivete-based discrimination. *The Quarterly Journal of Economics* 132(2):1019–1054.

Heidhues P, Kőszegi B (2018) *Behavioral Industrial Organization*, chapter 6, 517–612 (Elsevier B.V.).

Ho TH, Lim N, Camerer CF (2006) Modeling the psychology of consumer and firm behavior with behavioral economics. *Journal of marketing Research* 43(3):307–331.

Kuksov D, Wang K (2014) The bright side of loss aversion in dynamic and competitive markets. *Marketing Science* 33(5):693–711.

Laibson D (1997) Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics* 112(2):443–478.

Li L, Jiang L (2022) How should firms adapt pricing strategies when consumers are time-inconsistent? *Production and Operations Management* 31(9):3457–3473.

Li S, Yan J, Xiao B (2014) Contract design and self-control with asymmetric information. *Economic Inquiry* 52(2):618–624.

Lim N (2010) Social loss aversion and optimal contest design. *Journal of Marketing Research* 47(4):777–787.

Lim N, Ho TH (2007) Designing price contracts for boundedly rational customers: Does the number of blocks matter? *Marketing Science* 26(3):312–326.

Maskin E, Riley J (1984) Monopoly with incomplete information. *The RAND Journal of Economics* 15(2):171–196.

Mussa M, Rosen S (1978) Monopoly and product quality. *Journal of Economic Theory* 18(2):301–317.

Myerson RB (1981) Optimal auction design. *Mathematics of Operations Research* 6(1):58–73.

O'Donoghue T, Rabin M (1999a) Doing it now or later. *American Economic Review* 89(1):103–124.

O'Donoghue T, Rabin M (1999b) Incentives for procrastinators. *The Quarterly Journal of Economics* 114(3):769–816.

O'Donoghue T, Rabin M (2001) Choice and procrastination. *The Quarterly Journal of Economics* 116(1):121–160.

Phelps ES, Pollak RA (1968) On second-best national saving and game-equilibrium growth. *The Review of Economic Studies* 35(2):185–199.

Quah JKH, Strulovici B (2013) Discounting, values, and decisions. *Journal of Political Economy* 121(5):896–939.

Strotz RH (1955) Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies* 23(3):165–180.

Wu Y, Ramachandran K, Krishnan V (2014) Managing cost salience and procrastination in projects: Compensation and team composition. *Production and Operations Management* 23(8):1299–1311.

Appendix

To begin with, we give the expression of total surplus $T(p)$ and illusional surplus $I(p)$, as well as their derivatives, under the specification that $v \sim U_{[\underline{v},\bar{v}]}$ and $h(v) = \alpha v + \tau$. On the relevant interval $[\underline{v}, h(\bar{v})]$,

$$T(p) = \begin{cases} \frac{1}{2(\bar{v}-\underline{v})}[(\bar{v}-a)^2 - (a - \frac{p-\tau}{\alpha})^2] & \text{if } p \in [h(\underline{v}), h(\bar{v})] \\ \frac{1}{2(\bar{v}-\underline{v})}[(\bar{v}-a)^2 - (a-\underline{v})^2] & \text{if } p \in [\underline{v}, h(\underline{v})) \end{cases} \tag{4}$$

$$T'(p) = \begin{cases} \frac{1}{\bar{v}-\underline{v}} \cdot \frac{1}{\alpha}(a - \frac{p-\tau}{\alpha}) & \text{if } p \in [h(\underline{v}), h(\bar{v})] \\ 0 & \text{if } p \in [\underline{v}, h(\underline{v})) \end{cases} \tag{5}$$

$$I(p) = \begin{cases} \frac{1}{2(\bar{v}-\underline{v})}(p-\underline{v})^2 & \text{if } p \in [\underline{v}, h(\underline{v})) \\ \frac{1}{2(\bar{v}-\underline{v})}[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}]^2 & \text{if } p \in [h(\underline{v}), \bar{v}] \\ \frac{1}{2(\bar{v}-\underline{v})}[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}]^2 - \frac{1}{2(\bar{v}-\underline{v})}(p-\bar{v})^2 & \text{if } p \in (\bar{v}, h(\bar{v})] \end{cases} \tag{6}$$

$$I'(p) = \begin{cases} \frac{1}{\bar{v}-\underline{v}}(p-\bar{v}) & \text{if } p \in [\underline{v}, h(\underline{v})) \\ \frac{1}{\bar{v}-\underline{v}}(1-\frac{1}{\alpha})[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}] & \text{if } p \in [h(\underline{v}), \bar{v}] \\ \frac{1}{\bar{v}-\underline{v}}(1-\frac{1}{\alpha})[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}] - \frac{1}{\bar{v}-\underline{v}}(p-\bar{v}) & \text{if } p \in (\bar{v}, h(\bar{v})] \end{cases} \tag{7}$$

These functions will be called upon repeatedly in the following proofs.

Proof of Lemma 1. By definition, $p - h^{-1}(p) = (1-\frac{1}{\alpha})p + \frac{\tau}{\alpha} > 0$ for temptation goods. In the following, we shall refer to the expression of $I'(p)$ in equation (7) and discuss in three cases. Case $\alpha > 1$: I is increasing on $(\underline{v}, \bar{v})$ and concave on $(\bar{v}, h(\bar{v}))$. Case $\alpha < 1$: I is increasing on $(\underline{v}, h(\underline{v}))$ and decreasing on $(h(\underline{v}), h(\bar{v}))$. Case $\alpha = 1$: I is increasing on $(\underline{v}, h(\underline{v}))$, constant on $(h(\underline{v}), \bar{v})$, and decreasing on $(\bar{v}, h(\bar{v}))$. ■

Proof of Proposition 1. Since T is peaked at $h(a)$, we have $p_s^{FB} = \arg\max T = h(a)$. We also know $h(a) > a$ for temptation goods. Hence, $p_s^{FB} > a$. Next, we compare $p_n^{FB} = \arg\max T + I$ with marginal cost a. For any $p \leq \underline{v}$, $[T+I](p) = E[v] - a$. For any $\underline{v} < p \leq a$, $T + I$ is strictly increasing, since

$$\begin{aligned} &[T' + I'](p) \\ =&F(p) - F(h^{-1}(p)) - (p-a)\frac{f(h^{-1}(p))}{h'(h^{-1}(p))} \\ >&0 \end{aligned}$$

Therefore, we must have $p_n^{FB} > a$.

Finally, we compare p_n^{FB} with the efficient price $h(a)$. We know $p_n^{FB} \notin (h(\bar{v}), +\infty)$ as $T + I$ is zero on this interval. We can also rule out $p_n^{FB} \in (-\infty, \underline{v})$ as it is obviously dominated by any

$p_n \in (h(\underline{v}), h(\bar{v}))$. By Assumption 1 and Lemma 1, when $\alpha > 1$, $T + I$ is strictly increasing on $(\underline{v}, h(a) + \varepsilon)$ for small $\varepsilon > 0$ and thus $p_n^{FB} > h(a)$. When $\alpha < 1$, $T + I$ is strictly decreasing on $(h(a) - \varepsilon, h(\bar{v}))$ for small $\varepsilon > 0$ and thus $p_n^{FB} < h(a)$. When $\alpha = 1$, $T + I$ peaks at $h(a)$ and thus $p_n^{FB} = h(a)$. ■

Before we proceed to prove our main results, we first prove the following lemma.

Lemma 4. *There exists a threshold $\hat{\gamma}_s \in (0, 1)$ such that the sophisticated consumers always consume if $\gamma_s < \hat{\gamma}_s$ and the sophisticated consumers may not consume if $\gamma_s \geq \hat{\gamma}_s$. Moreover, $\hat{\gamma}_s$ is decreasing in marginal cost; and if time inconsistency is higher under h_1 than under h_0, $\hat{\gamma}_s$ is larger under h_1 than under h_0.*

Proof of Lemma 4. Recall that $p_s^* = \arg\max \gamma_s T - \gamma_n I$. Define $\bar{p} \in (h(a), h(\bar{v}))$ by $T(\bar{p}) = E[v] - a$. Then $T(p) \leq T(\bar{p}) = T(\underline{v})$ and $I(p) \geq 0 = I(\underline{v})$ for any $p \geq \bar{p}$. Thus, any $p_s \geq \bar{p}$ is dominated by $p_s = \underline{v}$. We also know that for any $p \in (\underline{v}, h(\underline{v})]$, $T(p) = T(\underline{v})$ and $I(p) > 0 = I(\underline{v})$. Thus, any $p_s \in (\underline{v}, h(\underline{v})]$ is also dominated by $p_s = \underline{v}$. In addition, the firm is indifferent between any p_s below $\underline{v}$. The set of remaining candidates for p_s^* is thus $\{\underline{v}\} \cup (h(\underline{v}), \bar{p})$. Let $p_s^{int}(\gamma_s)$ be the locally optimal price for the sophisticated type under γ_s when price is restricted to the interval $[h(\underline{v}), \bar{p}]$. Then $\gamma_s T(p_s^{int}(\gamma_s)) - \gamma_n I(p_s^{int}(\gamma_s)) < \gamma_s T(\underline{v}) - \gamma_n I(\underline{v})$ if and only if $\gamma_s < \hat{\gamma}_s$ for some $\hat{\gamma}_s \in (0, 1)$. This can be proved as follows. Define

$$H(\gamma_s) = [\gamma_s T(p_s^{int}(\gamma_s)) - (1 - \gamma_s) I(p_s^{int}(\gamma_s))] - [\gamma_s T(\underline{v}) - (1 - \gamma_s) I(\underline{v})]$$

Then $H(1) > 0 > H(0)$, since $H(1) = T(h(a)) - T(\underline{v})$ and $H(0) = -I(p_s^{int}(0))$. If $p_s^{int}(\gamma_s)$ is interior on $[h(\underline{v}), \bar{p}]$, $H'(\gamma_s) = T(p_s^{int}(\gamma_s)) + I(p_s^{int}(\gamma_s)) - T(\underline{v}) > 0$ by the envelope theorem. If $p_s^{int}(\gamma_s)$ hits the boundary, either $h(\underline{v})$ or $\bar{p}$, it means that the firm's priority is to reduce $\gamma_n I$. Hence, $p_s^{int}(\gamma_s')$ must stay at the boundary for all $\gamma_s' < \gamma_s$. Obviously, when $p_s^{int}(\gamma_s)$ hits the boundary, $H(\gamma_s)$ is negative and increasing. Now we have shown that $H(\gamma_s)$ is increasing when $p_s^{int}(\gamma_s)$ is either interior or at the boundary. We must have $H(\gamma_s)$ increasing globally since it is continuous. Together with the fact that $H(1) > 0 > H(0)$, we can establish the existence of $\hat{\gamma}_s \in (0, 1)$.

Next, we want to show that $\hat{\gamma}_s$ is decreasing in marginal cost a. Obviously,

$$\frac{\partial H(\hat{\gamma}_s; a)}{\partial a} = \hat{\gamma}_s F(h^{-1}(p_s^{int}(\hat{\gamma}_s))) \geq 0$$

By the implicit function theorem, $\frac{\partial \hat{\gamma}_s}{\partial a} = -\frac{\partial H(\hat{\gamma}_s; a)/\partial a}{\partial H(\hat{\gamma}_s; a)/\partial \hat{\gamma}_s} \leq 0$.

Finally, we want to show that $\hat{\gamma}_s$ is increasing in time inconsistency. We already know $H(\hat{\gamma}_s) = 0$ and $H(\gamma_s) < 0$ whenever $p_s^{int}(\gamma_s)$ hits the boundary. Thus, $p_s^{int}(\hat{\gamma}_s)$ must be interior. Using the optimality of $p_s^{int}(\hat{\gamma}_s)$, we must have $\hat{\gamma}_s T'(p_s^{int}(\hat{\gamma}_s)) - (1-\hat{\gamma}_s)I'(p_s^{int}(\hat{\gamma}_s)) = 0$. It then implies

$$\begin{aligned}&(1-\hat{\gamma}_s)[F(p) - F(h^{-1}(p))]|_{p=p_s^{int}(\hat{\gamma}_s)}\\ =&\frac{\partial h^{-1}(p)}{\partial p} f(h^{-1}(p))[-\hat{\gamma}_s(h^{-1}(p)-a) + (1-\hat{\gamma}_s)(p-h^{-1}(p))]|_{p=p_s^{int}(\hat{\gamma}_s)}\end{aligned} \tag{8}$$

Let σ index the degree of time inconsistency and suppose $\alpha(\sigma)$ and $\tau(\sigma)$ are differentiable. We have $h(v;\sigma) = \alpha(\sigma)v + \tau(\sigma)$ increasing in σ, for every $v \in [\underline{v}, \bar{v}]$. Then we must have $\frac{\partial h^{-1}(p;\sigma)}{\partial \sigma} \leq 0$. We can also get

$$\frac{\partial H(\hat{\gamma}_s;\sigma)}{\partial \sigma} = \frac{\partial h^{-1}(p;\sigma)}{\partial \sigma} f(h^{-1}(p;\sigma))[-\hat{\gamma}_s(h^{-1}(p;\sigma)-a) + (1-\hat{\gamma}_s)(p-h^{-1}(p;\sigma))]|_{p=p_s^{int}(\hat{\gamma}_s)}$$

Substituting equation (8) into $\frac{\partial H(\hat{\gamma}_s;\sigma)}{\partial \sigma}$, we obtain

$$\frac{\partial H(\hat{\gamma}_s;\sigma)}{\partial \sigma} = \frac{\partial h^{-1}(p;\sigma)}{\partial \sigma} \frac{(1-\hat{\gamma}_s)[F(p) - F(h^{-1}(p;\sigma))]}{\partial h^{-1}(p;\sigma)/\partial p}|_{p=p_s^{int}(\hat{\gamma}_s)} \leq 0$$

By the implicit function theorem, $\frac{\partial \hat{\gamma}_s}{\partial \sigma} = -\frac{\partial H(\hat{\gamma}_s;\sigma)/\partial \sigma}{\partial H(\hat{\gamma}_s;\sigma)/\partial \hat{\gamma}_s} \geq 0$. ■

Proof of Proposition 2. We already know that the optimal pooling contract offers $p^{pool} = h(a)$. Thus, the optimal contract is pooling if and only if $p_n^* = p_s^* = h(a)$. Since $p_n^* = p_n^{FB}$, we know from Proposition 1 that $p_n^* = h(a)$ if and only if $\alpha = 1$. To characterize the condition for pooling, we just need to compare p_s^* with $h(a)$ when $\alpha = 1$. We already know from Lemma 4 that $p_s^* = \underline{v}$ when $\gamma_s < \hat{\gamma}_s$. In the following, we prove $p_s^* = h(a)$ when $\gamma_s \geq \hat{\gamma}_s$ and $\alpha = 1$.

Recall that $p_s^* = \arg\max \gamma_s T - \gamma_n I$. When $\alpha = 1$, the peak of I is $[h(\underline{v}), \bar{v}]$, an interval covering $h(a)$ by Assumption 1. Thus, the maximum of $\gamma_s T - \gamma_n I$ on $[h(\underline{v}), \bar{v}]$ is attained at $h(a)$. From equation (4)–(7), we know that for $\alpha = 1$, the first-order condition for an interior solution $\hat{p}_s \in (\bar{v}, h(\bar{v})]$ is

$$\begin{aligned}&\gamma_s T'(\hat{p}_s) - \gamma_n I'(\hat{p}_s)\\ =&\frac{1}{\bar{v} - \underline{v}}[\gamma_s(a - \hat{p}_s + \tau) + \gamma_n(\hat{p}_s - \bar{v})] = 0\end{aligned}$$

We can solve $\hat{p}_s = \frac{\gamma_s(a+\tau) - \gamma_n \bar{v}}{\gamma_s - \gamma_n}$. The second-order condition requires $\gamma_n - \gamma_s < 0$. When the second-order condition does not hold, the globally optimal $p_s^* \notin (\bar{v}, h(\bar{v})]$. When the second-order condition holds, $\hat{p}_s > \bar{v}$ is equivalent to $a + \tau > \bar{v}$, which contradicts Assumption 1. Therefore, whenever $\alpha = 1$ and $\gamma_s \geq \hat{\gamma}_s$, we must have $p_s^* = h(a)$.

Based on our analysis, $p_s^* = p_n^* = p^{pool}$ if $\alpha = 1$ and $\gamma_s \geq \hat{\gamma}_s$; $p_s^* \neq p_n^*$ if $\alpha \neq 1$ or $\gamma_s < \hat{\gamma}_s$. The monotonicity of $\hat{\gamma}_s$ follows from Lemma 4. ■

Proof of Proposition 3. Recall that $p_s^* = \arg\max \gamma_s T - \gamma_n I$. By Lemma 1, we know p_s^* is upward distorted if $p_s^* > h(\underline{v})$ and $\alpha < 1$. Moreover, by Lemma 4, we know $p_s^* > h(\underline{v})$ if $\gamma_s \geq \hat{\gamma}_s$. Therefore, p_s^* is upward distorted if $\alpha < 1$ and $\gamma_s \geq \hat{\gamma}_s$.

To characterize the condition for downward distortion of p_s^*, we first prove that $p_s^* \leq \bar{v}$ when $\alpha \geq 1$. Obviously, $p_s^* < h(\bar{v})$. It remains to show $p_s^* \notin (\bar{v}, h(\bar{v}))$. From equation (4)–(7), we know the first-order condition for an interior solution $\hat{p}_s \in (\bar{v}, h(\bar{v}))$ is

$$\begin{aligned}&\gamma_s T'(\hat{p}_s) - \gamma_n I'(\hat{p}_s)\\ =&\frac{1}{\bar{v}-\underline{v}}\{\gamma_s \frac{1}{\alpha}(a - \frac{\hat{p}_s - \tau}{\alpha}) - \gamma_n(1-\frac{1}{\alpha})[(1-\frac{1}{\alpha})\hat{p}_s + \frac{\tau}{\alpha}] + \gamma_n(\hat{p}_s - \bar{v})\}\\ =&0\end{aligned}$$

We can solve $\hat{p}_s = \frac{\gamma_s(\alpha a+\tau)-\gamma_n(\alpha-1)\tau-\gamma_n\alpha^2\bar{v}}{\gamma_s+\gamma_n(\alpha-1)^2-\gamma_n\alpha^2}$. The second-order condition requires $-[\gamma_s + \gamma_n(\alpha-1)^2 - \gamma_n\alpha^2] < 0$. When the second-order condition does not hold, the globally optimal $p_s^* \notin (\bar{v}, h(\bar{v}))$. When the second-order condition holds, $\hat{p}_s > \bar{v}$ is equivalent to

$$\gamma_s(\alpha a + \tau) - \gamma_n(\alpha-1)\tau > [\gamma_s + \gamma_n(\alpha-1)^2]\bar{v} \tag{9}$$

When $\alpha \geq 1$, we have $-\gamma_n(\alpha-1)\tau \leq \gamma_n(\alpha-1)^2\bar{v}$. Moreover, $\gamma_s(\alpha a + \tau) < \gamma_s\bar{v}$ by Assumption 1. But then inequality (9) fails to hold. Therefore, we also have the globally optimal $p_s^* \notin (\bar{v}, h(\bar{v}))$.

Now we have shown that $p_s^* \leq \bar{v}$ when $\alpha \geq 1$. By Assumption 1 and Lemma 1, we know that p_s^* is downward distorted if $\alpha > 1$. Moreover, from Lemma 4, we also know that $p_s^* = \underline{v}$ and hence downward distorted if $\gamma_s < \hat{\gamma}_s$. Therefore, p_s^* is downward distorted if $\alpha > 1$ or $\gamma_s < \hat{\gamma}_s$. ■

Proof of Proposition 4. When $\alpha < 1$, $p_n^{FB} < p_s^{FB} = h(a)$ by Proposition 1. When $\gamma_s < \hat{\gamma}_s$, $p_s^* = h(\underline{v}) < p_n^*$ by Lemma 4. Therefore, price ranking is reversed relative to the first best when $\alpha < 1$ and $\gamma_s < \hat{\gamma}_s$.

Next, we prove that when $\alpha \geq 1$ or $\gamma_s \geq \hat{\gamma}_s$, price ranking is preserved and the price gap increases relative to the first best. When $\alpha \geq 1$, $p_n^{FB} \geq p_s^{FB} = h(a)$ by Proposition 1. We also know from Proposition 3 that p_s^* is (weakly) downward distorted. Therefore, $p_n^* = p_n^{FB} \geq p_s^{FB} \geq p_s^*$. When $\alpha < 1$ and $\gamma_s \geq \hat{\gamma}_s$, p_s^* is upward distorted. Therefore, $p_n^* = p_n^{FB} < p_s^{FB} < p_s^*$. ■

Proof of Proposition 5. From Observation 2, we know $p_n^* = p_n^{FB}$, which must be above marginal cost a. In the following, we just need to compare p_s^* and a.

Case 1: Suppose $\alpha \leq 1$. By Proposition 3, $p_s^* = \underline{v} < a$ whenever $\gamma_s < \hat{\gamma}_s$ and $p_s^* \geq h(a) > a$ whenever $\gamma_s \geq \hat{\gamma}_s$. We can set $\tilde{\gamma}_s = \hat{\gamma}_s$.

Case 2: Suppose $\alpha > 1$ and $h(\underline{v}) \geq a$. The only possibility for $p_s^* < a$ is to set $p_s = \underline{v}$, which is optimal when $\gamma_s < \hat{\gamma}_s$ by Lemma 4. We can set $\tilde{\gamma}_s = \hat{\gamma}_s$.

Case 3: Suppose $\alpha > 1$ and $h(\underline{v}) < a$. We already know from Proposition 3 that $p_s^* < p_s^{FB} = h(a)$ when $\alpha > 1$. Obviously, $p_s \in (\underline{v}, h(\underline{v}))$ is suboptimal. Then p_s^* is either at $\underline{v}$ or on $(h(\underline{v}), h(a))$. Let $p_s^{int}(\gamma_s)$ be the locally optimal price for sophisticated consumers under γ_s when restricted to the interval $[h(\underline{v}), h(a)]$. For $\gamma_s' > \gamma_s$, the tradeoff between efficiency and rent extraction is ameliorated. Thus, whenever $\gamma_s' > \hat{\gamma}_s$, $p_s^{int}(\gamma_s')$ is closer to the efficient price $h(a)$ relative to $p_s^{int}(\gamma_s)$, i.e., $p_s^{int}(\gamma_s) \leq p_s^{int}(\gamma_s') \leq h(a)$. It implies that $p_s^{int}(\gamma_s)$ is increasing. We further have $p_s^{int}(1) = h(a)$. Therefore, $p_s^{int}(\gamma_s) < a$ if and only if $\gamma_s < \eta$ for some $\eta \in [0, 1)$. Moreover, η is given by $\eta T'(a) - (1-\eta)I'(a) = 0$. We can solve that $\eta = 1 - \frac{1}{\alpha}$. Therefore, $p_s^* < a$ if and only if $\gamma_s < \max\{\hat{\gamma}_s, \eta\}$. We can set $\tilde{\gamma}_s = \max\{\hat{\gamma}_s, \eta\}$.

In the following, we examine the monotonicity of $\tilde{\gamma}_s$ with respect to a and α. Within each of the three cases above, it is easy to show that $\tilde{\gamma}_s$ is monotone in a and α. But we also need to take care of the monotonicity at the boundaries of the three cases.

First, we want to show that $\tilde{\gamma}_s$ is decreasing in a. We know from Lemma 4 that $\hat{\gamma}_s$ is decreasing in a. When $\alpha \leq 1$, $\tilde{\gamma}_s$ is obviously decreasing in a. Now consider $\alpha > 1$. Within Case 2 and 3, $\tilde{\gamma}_s$ is decreasing in a. When $a = h(\underline{v})$, $T(a) = T(\underline{v})$ and $I(a) > I(\underline{v})$. It follows that $p_s = a$ is dominated by $p_s = \underline{v}$ at $a = h(\underline{v})$. But then $\tilde{\gamma}_s = \hat{\gamma}_s > \eta$ at $a = h(\underline{v})$. Therefore, $\tilde{\gamma}_s$ is continuous in a. We must have $\tilde{\gamma}_s$ decreasing in a when $\alpha > 1$.

Next, we want to show that $\tilde{\gamma}_s$ is increasing in α. By Lemma 4, $\hat{\gamma}_s$ is increasing in time inconsistency and hence in α. Moreover, η is increasing in α. Therefore, within each of the three cases, $\tilde{\gamma}_s$ is increasing in α. When $\alpha = \frac{a-\tau}{\underline{v}}$ (i.e., $h(\underline{v}) = a$), we have $\tilde{\gamma}_s = \hat{\gamma}_s > \eta$ following the previous argument. When $\alpha = 1$, $\tilde{\gamma}_s = \hat{\gamma}_s > 0 = \eta$. Therefore, $\tilde{\gamma}_s$ is continuous in α. Based on the analysis above, we must have $\tilde{\gamma}_s$ increasing in α.

To see that $\tilde{\gamma}_s$ may not increase in time inconsistency induced by higher τ, we just need to give an example. Since η is increasing in α for $\alpha \in (1, \frac{a-\tau}{\underline{v}})$ and independent of τ, we need to give an example of h_0 and h_1 satisfying $h_0 \leq h_1$, $\alpha_0 > \alpha_1$, and the cutoff $\tilde{\gamma}_{si}$ takes $\eta_i = 1 - \frac{1}{\alpha_i}$ for $i \in \{0, 1\}$. We first identify the condition for $\hat{\gamma}_s < \eta$ so that $\tilde{\gamma}_s = \eta$. Recall that η is the proportion of the sophisticated type that makes $p_s = a$ the optimal price, whereas $\hat{\gamma}_s$ is the cutoff proportion of the sophisticated type below which $p_s = \underline{v}$ is optimal. Thus, $\hat{\gamma}_s < \eta$ if and only if $\eta T(a) - (1-\eta)I(a) > \eta T(\underline{v}) - (1-\eta)I(\underline{v})$. Substituting in $\eta = 1 - \frac{1}{\alpha}$, we can simplify this condition to

$$\alpha(\alpha - 1)(a - \underline{v})^2 > [(\alpha - 1)a + \tau]^2 \tag{10}$$

Let $[\underline{v}, \bar{v}] = [0, 1]$ and $a = 0.4$. Take two distortion functions as $h_1(v) = 1.2v + 0.1$ and $h_0(v) = 1.3v$. Since $h_1 \geq h_0$ on $[0, 1]$, h_1 induces higher time inconsistency than h_0. It is also easy to check that they satisfy condition (10), $\alpha_i > 1$, and $h_i(\underline{v}) < a$. Therefore, the cutoff proportion $\tilde{\gamma}_{si} = 1 - \frac{1}{\alpha_i}$ is higher under h_0, since $\alpha_0 > \alpha_1$. ■

Proof of Proposition 6. We first use a change of variable to concentrate the effect of varying time inconsistency on the illusional surplus I. Let $x = h^{-1}(p)$. Define

$$\tilde{I}(x) \equiv I(h(x)) = \int_x^{h(x)} (h(x) - v) dF(v)$$
$$\tilde{T}(x) \equiv T(h(x)) = \int_x^{\bar{v}} (v - a) dF(v)$$

In the following proof, variables with subscript $i \in \{0, 1\}$ correspond to the setting with distortion function h_i. Notably, $\tilde{T}(x)$ is independent of the degree of time inconsistency. Thus, we abbreviate both $\tilde{T}_0$ and $\tilde{T}_1$ by $\tilde{T}$. Moreover, for $x \in [\underline{v}, h_i^{-1}(\bar{v})]$,

$$\tilde{I}_i'(x) = \frac{1}{\bar{v} - \underline{v}} (\alpha_i - 1)(h_i(x) - x)$$

We have assumed $h_1 \geq h_0$ on $[\underline{v}, \bar{v}]$ with the inequality being strict for at least some $v \in [\underline{v}, \bar{v}]$. Obviously, when either $\alpha_1 > \alpha_0 \geq 1$ or $\alpha_1 < \alpha_0 \leq 1$, $|\tilde{I}_1'(x)| > |\tilde{I}_0'(x)|$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. In comparison, when α_1 is sufficiently close to 1 and either $\alpha_0 > \alpha_1 \geq 1$ or $\alpha_0 < \alpha_1 \leq 1$, $|\tilde{I}_1'(x)| < |\tilde{I}_0'(x)|$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. The rest of the proof consists of three steps.

Step 1: We want to show that x_{is}^* and x_{in}^* are on $[\underline{v}, h_i^{-1}(\bar{v})]$ when $h_1(\bar{v}) - \bar{v} < \bar{v} - a$ and γ_s is sufficiently large. Assumption 1 implies that the interval $[\underline{v}, h_i^{-1}(\bar{v})]$ contains a, the peak of $\tilde{T}$. Thus, it is obvious that x_{is}^*, as the maximizer of $\gamma_s \tilde{T} - \gamma_n \tilde{I}$, belongs to $[\underline{v}, h_i^{-1}(\bar{v})]$ for large γ_s. Next we prove $x_{in}^* \in [\underline{v}, h_i^{-1}(\bar{v})]$. Suppose $p_{in}^* > \bar{v}$. From the first-order condition $T'(p_{in}^*) + I_i'(p_{in}^*) = 0$ on the interval $(\bar{v}, h_i(\bar{v}))$, we can solve that $p_{in}^* = \frac{a+\tau+\alpha_i \bar{v}}{2}$. Then $p_{in}^* > \bar{v}$ is equivalent to $\alpha_i \bar{v} + \tau - \bar{v} > \bar{v} - a$, which contradicts the assumption that $h_1(\bar{v}) - \bar{v} \leq \bar{v} - a$. Therefore, $p_{in}^* \leq \bar{v}$. It is also obvious that $p_{in}^* \geq h_i(\underline{v})$. We then have $p_{in}^* \in [h_i(\underline{v}), \bar{v}]$ and hence $x_{in}^* \in [\underline{v}, h_i^{-1}(\bar{v})]$.

Step 2: We prove when either $\alpha_1 > \alpha_0 \geq 1$ or $\alpha_1 < \alpha_0 \leq 1$, x_{1s}^* and x_{1n}^* deviate more from a relative to x_{0s}^* and x_{0n}^*, respectively. We also prove when α_1 is sufficiently close to 1 and either $\alpha_0 > \alpha_1 \geq 1$ or $\alpha_0 < \alpha_1 \leq 1$, x_{1s}^* and x_{1n}^* deviate less from a relative to x_{0s}^* and x_{0n}^*, respectively. The proof goes as follows.

Case $\alpha_i \geq 1$: Recall that $p_n^* = p_n^{FB}$. By Proposition 1 and 3, $x_{is}^* \leq a \leq x_{in}^*$. By Lemma 1, $\tilde{T}'(x_{in}^*) \leq 0 \leq \tilde{I}_i'(x_{in}^*)$ and both $\tilde{T}'(x_{is}^*)$ and $\tilde{I}_i'(x_{is}^*)$ are nonnegative. Suppose $\alpha_1 > \alpha_0 \geq 1$, then we have $\tilde{I}_1'(x) > \tilde{I}_0'(x) \geq 0$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. But then $\tilde{T}'(x_{1n}^*) + \tilde{I}_0'(x_{1n}^*) < 0$ and $\gamma_s \tilde{T}'(x_{1s}^*) - \gamma_n \tilde{I}_0'(x_{1s}^*) > 0$. It follows that $x_{1s}^* < x_{0s}^* \leq a \leq x_{0n}^* < x_{1n}^*$. Suppose $\alpha_0 > \alpha_1 \geq 1$ and α_1 is sufficiently close to 1, then we have $\tilde{I}_1'(x) < \tilde{I}_0'(x) \leq 0$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. In this case, $\tilde{T}'(x_{1n}^*) + \tilde{I}_0'(x_{1n}^*) > 0$ and $\gamma_s \tilde{T}'(x_{1s}^*) - \gamma_n \tilde{I}_0'(x_{1s}^*) < 0$. It follows that $x_{0s}^* < x_{1s}^* \leq a \leq x_{1n}^* < x_{0n}^*$.

Case $\alpha_i \leq 1$: By Proposition 1 and 3, $x_{in}^* \leq a \leq x_{is}^*$. By Lemma 1, $\tilde{T}'(x_{in}^*) \geq 0 \geq \tilde{I}_i'(x_{in}^*)$ and both $\tilde{T}'(x_{is}^*)$ and $\tilde{I}_i'(x_{is}^*)$ are nonpositive. Suppose $\alpha_1 < \alpha_0 \leq 1$, then we have $\tilde{I}_1'(x) < \tilde{I}_0'(x) \leq 0$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. But then $\tilde{T}'(x_{1n}^*) + \tilde{I}_0'(x_{1n}^*) > 0$ and $\gamma_s \tilde{T}'(x_{1s}^*) - \gamma_n \tilde{I}_0'(x_{1s}^*) < 0$. It follows that $x_{1n}^* < x_{0n}^* \leq a \leq x_{0s}^* < x_{1s}^*$. Suppose $\alpha_0 < \alpha_1 \leq 1$ and α_1 is sufficiently close to 1, then we have $\tilde{I}_0'(x) < \tilde{I}_1'(x) \leq 0$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. But then $\tilde{T}'(x_{1n}^*) + \tilde{I}_0'(x_{1n}^*) < 0$ and $\gamma_s \tilde{T}'(x_{1s}^*) - \gamma_n \tilde{I}_0'(x_{1s}^*) > 0$. It follows that $x_{0n}^* < x_{1n}^* \leq a \leq x_{1s}^* < x_{0s}^*$.

Step 3: We prove when x^*_{1j} deviates more from a relative to x^*_{0j} and $x^*_{ij} \in [\underline{v}, h_i^{-1}(\bar{v})]$ for $i \in \{0, 1\}$ and $j \in \{s, n\}$, firm profit is higher under h_1, consumer surplus and social welfare are lower under h_1. We also prove when x^*_{1j} deviates less from a relative to x^*_{0j} and $x^*_{ij} \in [\underline{v}, h_i^{-1}(\bar{v})]$ for $i \in \{0, 1\}$ and $j \in \{s, n\}$, firm profit is lower under h_1, consumer surplus and social welfare are higher under h_1. The proof goes as follows.

From Step 2, we know x^*_{1j} deviates more from a relative to x^*_{0j} when $|\tilde{I}'_1(x)| > |\tilde{I}'_0(x)|$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. Then $|\tilde{I}'_1(x)| > |\tilde{I}'_0(x)|$ between x^*_{0s} and x^*_{0n} by the result of Step 1. It implies $\tilde{I}_1(x^*_{0n}) - \tilde{I}_1(x^*_{0s}) \geq \tilde{I}_0(x^*_{0n}) - \tilde{I}_0(x^*_{0s})$. Hence, we obtain

$$\begin{aligned}
\Pi^*_1 &= \gamma_s \tilde{T}(x^*_{1s}) + \gamma_n[\tilde{T}(x^*_{1n}) + \tilde{I}_1(x^*_{1n}) - \tilde{I}_1(x^*_{1s})] \\
&\geq \gamma_s \tilde{T}(x^*_{0s}) + \gamma_n[\tilde{T}(x^*_{0n}) + \tilde{I}_1(x^*_{0n}) - \tilde{I}_1(x^*_{0s})] \\
&\geq \gamma_s \tilde{T}(x^*_{0s}) + \gamma_n[\tilde{T}(x^*_{0n}) + \tilde{I}_0(x^*_{0n}) - \tilde{I}_0(x^*_{0s})] = \Pi^*_0
\end{aligned}$$

where the first inequality is due to the optimality of x^*_{1j}. The consumer surplus for the sophisticated type CS^*_{is} is always zero. For the naive type,

$$\begin{aligned}
-CS^*_{0n} &= \tilde{I}_0(x^*_{0n}) - \tilde{I}_0(x^*_{0s}) \\
&\leq \tilde{I}_1(x^*_{0n}) - \tilde{I}_1(x^*_{0s}) \\
&< \tilde{I}_1(x^*_{1n}) - \tilde{I}_1(x^*_{1s}) = -CS^*_{1n}
\end{aligned}$$

where the second inequality holds because x^*_{1j} deviates more from a relative to x^*_{0j}. As to the social welfare,

$$\begin{aligned}
W^*_1 &= \gamma_s \tilde{T}(x^*_{1s}) + \gamma_n \tilde{T}(x^*_{1n}) \\
&< \gamma_s \tilde{T}(x^*_{0s}) + \gamma_n \tilde{T}(x^*_{0n}) = W^*_0
\end{aligned}$$

where the inequality holds because x^*_{1j} deviates more from a relative to x^*_{0j}.

We also know that x^*_{1j} deviates less from a relative to x^*_{0j} when $|\tilde{I}'_1(x)| < |\tilde{I}'_0(x)|$ on $[\underline{v}, h_1^{-1}(\bar{v})]$. Then $|\tilde{I}'_1(x)| < |\tilde{I}'_0(x)|$ between x^*_{1s} and x^*_{1n} by the result of Step 1. It implies $\tilde{I}_1(x^*_{1n}) - \tilde{I}_1(x^*_{1s}) \leq \tilde{I}_0(x^*_{1n}) - \tilde{I}_0(x^*_{1s})$. Hence, we obtain

$$\begin{aligned}
\Pi^*_1 &= \gamma_s \tilde{T}(x^*_{1s}) + \gamma_n[\tilde{T}(x^*_{1n}) + \tilde{I}_1(x^*_{1n}) - \tilde{I}_1(x^*_{1s})] \\
&\leq \gamma_s \tilde{T}(x^*_{1s}) + \gamma_n[\tilde{T}(x^*_{1n}) + \tilde{I}_0(x^*_{1n}) - \tilde{I}_0(x^*_{1s})] \\
&\leq \gamma_s \tilde{T}(x^*_{0s}) + \gamma_n[\tilde{T}(x^*_{0n}) + \tilde{I}_0(x^*_{0n}) - \tilde{I}_0(x^*_{0s})] = \Pi^*_0
\end{aligned}$$

where the second inequality is due to the optimality of x^*_{0j}. The consumer surplus for the sophisticated type CS^*_{is} is always zero. For the naive type,

$$\begin{aligned}-CS^*_{0n} &= \tilde{I}_0(x^*_{0n}) - \tilde{I}_0(x^*_{0s}) \\ &> \tilde{I}_0(x^*_{1n}) - \tilde{I}_0(x^*_{1s}) \\ &\geq \tilde{I}_1(x^*_{1n}) - \tilde{I}_1(x^*_{1s}) = -CS^*_{1n}\end{aligned}$$

where the first inequality holds because x^*_{1j} deviates less from a relative to x^*_{0j}. As to the social welfare,

$$\begin{aligned}W^*_1 &= \gamma_s \tilde{T}(x^*_{1s}) + \gamma_n \tilde{T}(x^*_{1n}) \\ &> \gamma_s \tilde{T}(x^*_{0s}) + \gamma_n \tilde{T}(x^*_{0n}) = W^*_0\end{aligned}$$

where the inequality holds because x^*_{1j} deviates less from a relative to x^*_{0j}. ■

Proof of Proposition 7. (i) With slight abuse of notation, we use $\Pi^*(\gamma_s)$ to denote the firm's optimal profit when the proportion of the sophisticated type is γ_s. We also use $p^*_s(\gamma_s)$ to denote the optimal price for the sophisticated type under γ_s. It is obvious that p^*_n is independent of γ_s. Recall that $\Pi^*(\gamma_s) = \gamma_s T(p^*_s(\gamma_s)) + \gamma_n[T(p^*_n) + I(p^*_n) - I(p^*_s(\gamma_s))]$. If $p^*_s(\gamma_s) \in (h(\underline{v}), h(\bar{v}))$, by the envelope theorem,

$$\frac{d\Pi^*(\gamma_s)}{d\gamma_s} = T(p^*_s(\gamma_s)) - [T(p^*_n) + I(p^*_n) - I(p^*_s(\gamma_s))]$$

If $p^*_s(\gamma_s) \notin (h(\underline{v}), h(\bar{v}))$, it must equal $\underline{v}$. Moreover, $p^*_s(\gamma'_s) = \underline{v}$ for all $\gamma'_s < \gamma_s$. Then

$$\frac{d\Pi^*(\gamma_s)}{d\gamma_s} = T(\underline{v}) - [T(p^*_n) + I(p^*_n) - I(\underline{v})]$$

In both cases, we know the profit from a naive consumer $T(p^*_n) + I(p^*_n) - I(p^*_s(\gamma_s))$ must be greater than the profit from a sophisticated consumer $T(p^*_s(\gamma_s))$, because otherwise the firm would simply offer $p^*_s(\gamma_s)$ to both types. Therefore, $\frac{d\Pi^*(\gamma_s)}{d\gamma_s} < 0$ in both cases. Since $\Pi^*(\gamma_s)$ is continuous, we must have $\Pi^*(\gamma_s)$ decreasing in γ_s.

(ii) It is obvious that the initially sophisticated consumers' surplus is zero and independent of γ_s, because their participation constraints are always binding. The consumer surplus for the naive consumers is $\mu^*_n(\gamma_s) = I(p^*_s(\gamma_s)) - I(p^*_n) < 0$. Thus the consumer surplus for the naive consumers who later become sophisticated increases from negative to zero. Next, we want to show the consumer surplus of the remaining naive consumers also increases. Since p^*_n is independent of γ_s, so is $I(p^*_n)$. However, in choosing $p^*_s(\gamma_s)$, the firm maximizes $\gamma_s T(p_s) - \gamma_n I(p_s)$, and hence $I(p^*_s(\gamma_s))$ must increase in response to a higher γ_s. Therefore, $\mu^*_n(\gamma_s) = I(p^*_s(\gamma_s)) - I(p^*_n)$ increases in γ_s.

(iii) Since p^*_n is independent of γ_s, so is $T(p^*_n)$. In contrast, the firm maximizes $\gamma_s T(p_s) - \gamma_n I(p_s)$ when choosing $p^*_s(\gamma_s)$, and hence $T(p^*_s(\gamma_s))$ increases in response to a higher γ_s. If $T(p^*_s(\gamma_s)) > T(p^*_n)$, then for any $\gamma'_s > \gamma_s$, $T(p^*_s(\gamma'_s)) \geq T(p^*_s(\gamma_s)) > T(p^*_n)$, which results in $W^*(\gamma'_s) > W^*(\gamma_s)$. If $T(p^*_s(\gamma_s)) < T(p^*_n)$, then it is possible that for a local increase of γ_s, $W^*(\gamma'_s) < W^*(\gamma_s)$.

When $\gamma_s < \gamma_s' < \hat{\gamma}_s$, $p_s^*(\gamma_s) = p_s^*(\gamma_s') = \underline{v}$. Then $T(p_s^*(\gamma_s)) = T(p_s^*(\gamma_s')) = T(\underline{v})$. If $\alpha \le 1$, $p_n^* \in (h(\underline{v}), h(a))$ and hence $T(p_n^*) > T(\underline{v})$. Next, consider $\alpha > 1$. Let z denote the peak of I. Then $z \in (\bar{v}, h(\bar{v}))$ and is given by $I'(z) = 0$. Using equation (7), we can solve $z = \frac{(\alpha-1)\tau+\alpha^2\bar{v}}{2\alpha-1}$. Let $\bar{p} \in (h(a), h(\bar{v}))$ be the price that $T(\bar{p}) = T(\underline{v})$. Using equation (4), we can solve $\bar{p} = \alpha(2a-\underline{v})+\tau$. To obtain $T(p_n^*) > T(\underline{v})$, it is sufficient that $z < \bar{p}$, or equivalently $a > \hat{a} \equiv \frac{1}{2}[\underline{v} + \frac{\alpha}{2\alpha-1}h^{-1}(\bar{v}) + \frac{\alpha-1}{2\alpha-1}\bar{v}]$. Therefore, if $\gamma_s < \gamma_s' < \hat{\gamma}_s$ and either $\alpha \le 1$ or $a > \hat{a}$, we have

$$\begin{aligned} W^*(\gamma_s') &= \gamma_s' T(\underline{v}) + (1-\gamma_s')T(p_n^*) \\ &< \gamma_s T(\underline{v}) + (1-\gamma_s)T(p_n^*) = W^*(\gamma_s) \end{aligned}$$

■

Proof of Lemma 3. On the relevant interval $[h(\underline{v}), \bar{v}]$,

$$T(p) = \begin{cases} \frac{1}{2(\bar{v}-\underline{v})}[(\bar{v}-a)^2 - (a - \frac{p-\tau}{\alpha})^2] & \text{if } p \in [h(\underline{v}), h(\bar{v})] \\ 0 & \text{if } p \in (h(\bar{v}), \bar{v}] \end{cases} \tag{11}$$

$$T'(p) = \begin{cases} \frac{1}{\bar{v}-\underline{v}} \cdot \frac{1}{\alpha}(a - \frac{p-\tau}{\alpha}) & \text{if } p \in [h(\underline{v}), h(\bar{v})] \\ 0 & \text{if } p \in (h(\bar{v}), \bar{v}] \end{cases} \tag{12}$$

$$I(p) = \begin{cases} \frac{1}{2(\bar{v}-\underline{v})}[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}]^2 - \frac{1}{2(\bar{v}-\underline{v})}(p-\underline{v})^2 & \text{if } p \in [h(\underline{v}), \underline{v}) \\ \frac{1}{2(\bar{v}-\underline{v})}[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}]^2 & \text{if } p \in [\underline{v}, h(\bar{v})] \\ \frac{1}{2(\bar{v}-\underline{v})}(\bar{v}-p)^2 & \text{if } p \in (h(\bar{v}), \bar{v}] \end{cases} \tag{13}$$

$$I'(p) = \begin{cases} \frac{1}{\bar{v}-\underline{v}}(1-\frac{1}{\alpha})[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}] - \frac{1}{(\bar{v}-\underline{v})}(p-\underline{v}) & \text{if } p \in [h(\underline{v}), \underline{v}) \\ \frac{1}{\bar{v}-\underline{v}}(1-\frac{1}{\alpha})[(1-\frac{1}{\alpha})p + \frac{\tau}{\alpha}] & \text{if } p \in [\underline{v}, h(\bar{v})] \\ \frac{1}{\bar{v}-\underline{v}}(p-\bar{v}) & \text{if } p \in (h(\bar{v}), \bar{v}] \end{cases} \tag{14}$$

By definition, $p - h^{-1}(p) = (1-\frac{1}{\alpha})p + \frac{\tau}{\alpha} < 0$ for aversion goods. We discuss in three cases. Case $\alpha < 1$: I is increasing on $(h(\underline{v}), h(\bar{v}))$ and decreasing on $(h(\bar{v}), \bar{v})$. Case $\alpha > 1$: I is concave on $(h(\underline{v}), \underline{v})$ and decreasing on $(\underline{v}, \bar{v})$. Case $\alpha = 1$: I is increasing on $(h(\underline{v}), \underline{v})$, constant on $(\underline{v}, h(\bar{v}))$, and decreasing on $(h(\bar{v}), \bar{v})$. ■

Proof of Proposition 8. (i) Recall that $p_s^* = \arg\max \gamma_s T - \gamma_n I$. Since $T(p) = 0$ and $I(p) \ge 0$ for $p \ge h(\bar{v})$, we must have $p_s^* < h(\bar{v})$. By Lemma 4, we know $p_s^* = h(\underline{v})$ without loss when $\gamma_s < \hat{\gamma}_s$ and $p_s^* \in (h(\underline{v}), h(\bar{v}))$ when $\gamma_s \ge \hat{\gamma}_s$.[13] Moreover, by Assumption 1 and Lemma 3, we know p_s^* is downward distorted if $\alpha < 1$. Therefore, p_s^* is downward distorted if $\alpha < 1$ or $\gamma_s < \hat{\gamma}_s$.

[13] The proof of Lemma 4 also needs to be adapted to the aversion good setting, which is straightforward and hence omitted here.

To characterize the condition for upward distortion of p_s^*, we first prove that $p_s^* \geq \underline{v}$ when $\alpha > 1$. Given equation (5) and (14), the first-order condition for an interior solution $\hat{p}_s \in (h(\underline{v}), \underline{v})$ is

$$
\begin{aligned}
&\gamma_s T'(\hat{p}_s) - \gamma_n I'(\hat{p}_s) \\
=&\frac{1}{\bar{v} - \underline{v}}\{\gamma_s \frac{1}{\alpha}(a - \frac{\hat{p}_s - \tau}{\alpha}) - \gamma_n(1 - \frac{1}{\alpha})[(1 - \frac{1}{\alpha})\hat{p}_s + \frac{\tau}{\alpha}] + \gamma_n(\hat{p}_s - \underline{v})\} \\
=&0
\end{aligned}
$$

We can solve $\hat{p}_s = \frac{\gamma_s(\alpha a+\tau)-\gamma_n(\alpha-1)\tau-\gamma_n\alpha^2\underline{v}}{\gamma_s+\gamma_n(\alpha-1)^2-\gamma_n\alpha^2}$. The second-order condition requires $-[\gamma_s + \gamma_n(\alpha-1)^2 - \gamma_n\alpha^2] < 0$. When the second-order condition does not hold, the globally optimal $p_s^* \notin (h(\underline{v}), \underline{v})$. When the second-order condition holds, $\hat{p}_s < \underline{v}$ is equivalent to

$$\gamma_s(\alpha a + \tau) - \gamma_n(\alpha - 1)\tau < [\gamma_s + \gamma_n(\alpha - 1)^2]\underline{v} \tag{15}$$

When $\alpha > 1$, we have $-\gamma_n(\alpha - 1)\tau \geq \gamma_n(\alpha - 1)^2\underline{v}$. By Assumption 1, $\gamma_s(\alpha a + \tau) > \gamma_s\underline{v}$. But then the inequality (15) fails to hold. Therefore, $p_s^* \notin (h(\underline{v}), \underline{v})$ when $\alpha > 1$.

Now we have shown $p_s^* \in (\underline{v}, h(\bar{v}))$ when $\alpha > 1$. By Assumption 1 and Lemma 3–4, we know p_s^* is upward distorted if $\alpha > 1$ and $\gamma_s \geq \hat{\gamma}_s$.

(ii) When $\alpha > 1$, $p_n^{FB} < p_s^{FB} = h(a)$ by Assumption 1 and Lemma 3. When $\gamma_s < \hat{\gamma}_s$, $p_s^* = h(\underline{v}) < p_n^*$ by Lemma 4. Therefore, price ranking is reversed relative to the first best when $\alpha > 1$ and $\gamma_s < \hat{\gamma}_s$.

Next, we prove that when $\alpha \leq 1$ or $\gamma_s \geq \hat{\gamma}_s$, price ranking is preserved and the price gap increases relative to the first best. When $\alpha \leq 1$, $p_n^{FB} \geq p_s^{FB} = h(a)$ by Assumption 1 and Lemma 3. We also know from part (i) that p_s^* is (weakly) downward distorted. Therefore, $p_n^* = p_n^{FB} \geq p_s^{FB} \geq p_s^*$. When $\alpha > 1$ and $\gamma_s \geq \hat{\gamma}_s$, p_s^* is upward distorted. Therefore, $p_n^* = p_n^{FB} < p_s^{FB} < p_s^*$. ■

Web Appendix of "Optimal Contracts for Time-Inconsistent Consumers with Heterogeneous Beliefs"

Buqu Gao Liang Guo

Perfect Competition with Heterogeneous Consumer Beliefs

In this subsection, we consider a perfectly competitive market. As in the main model, the consumers are time-inconsistent and can be either sophisticated or naive. The firms cannot observe the consumers' type. In this perfectly competitive market, the consumers contract with the firm that gives them the highest perceived utility. A firm's problem is to maximize the perceived utility for each consumer type subject to their participation constraints and incentive compatibility constraints, as well as the zero-profit constraints. The zero-profit constraints require that $\mu_s = T(p_s)$ and $\mu_n = T(p_n)$. It follows that the perceived utility is $T(p_s)$ for the sophisticated type and $T(p_n) + I(p_n)$ for the naive type. Based on the above analysis, each firm's problem is to choose p_s and p_n that solve

$$\max \gamma_s T(p_s) + \gamma_n [T(p_n) + I(p_n)]$$

s.t.

$$T(p_s) \geq 0 \qquad (PC_s)$$

$$T(p_n) + I(p_n) \geq 0 \qquad (PC_n)$$

$$T(p_s) \geq T(p_n) \qquad (IC_s)$$

$$T(p_n) + I(p_n) \geq T(p_s) + I(p_s) \qquad (IC_n)$$

PC_j is the participation constraint for type j, which requires that the type-j consumers perceive nonnegative utility from contracting with the firm. IC_j is the incentive compatibility constraint for type j, which requires that the type-j consumers perceive a weakly higher utility from choosing their contract than choosing the contract for the other type. We can first ignore the participation constraints and solve that the optimal prices are p_s^{FB} and p_n^{FB}. This is because the first-best prices, by definition, maximize the perceived utility for each consumer type and automatically satisfy the incentive constraints. We can verify that the participation constraints are satisfied at p_s^{FB} and p_n^{FB}, making the two prices optimal in the firm's original problem. It follows that asymmetric information on consumer beliefs does not generate any distortion under perfect competition. The optimal prices are again above marginal cost for temptation goods.