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ASYMPTOTICS AND TOTAL INTEGRALS OF THE $\mathrm{P}_{\mathrm{I}}^2$ TRITRONQUÉE SOLUTION AND ITS HAMILTONIAN *

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Abstract. We study the tritronquée solution $u(x,t)$ of the $\mathrm{P}_{\mathrm{I}}^2$ equation, the second member of the Painlevé I hierarchy. This particular solution is also known as the Gurevich-Pitaevskii solution of the KdV equation. It is pole-free on the real line and has various applications in mathematical physics. We obtain a full asymptotic expansion of $u(x,t)$ as $x \to \pm\infty$, uniformly for the parameter t in a large interval. Based on this result, we successfully derive the total integrals of $u(x,t)$ and the associated Hamiltonian with respect to $x \in \mathbb{R}$. Surprisingly, although $u(x,t)$ exhibits significant differences between $t > 0$ and $t < 0$, both integrals equal zero for all t.

Key words. Painlevé I hierarchy; KdV equation; full asymptotic expansion; total integrals; Riemann-Hilbert method.

AMS subject classifications. 33C10, 33E17, 34M55, 41A60.

1. Introduction.

The second member in the Painlevé I hierarchy, also referred to as the $\mathrm{P}_{\mathrm{I}}^2$ equation, is the following fourth-order ordinary differential equation

$$u_{xxxx} + 10u_x^2 + 20uu_{xx} + 40(u^3 - 6tu + 6x) = 0 \tag{1.1}$$

with $x \in \mathbb{C}$ and $t \in \mathbb{R}$. Like the Painlevé I equation

$$u_{xx} = 6u^2 + x, \tag{1.2}$$

general solutions $u(x,t)$ of $\mathrm{P}_{\mathrm{I}}^2$ are meromorphic in x, and have infinitely many poles in the complex-x plane; see [28]. However, researchers have identified a class of pole-free solutions $u(x,t)$ for x on the real axis. These pole-free solutions play a significant role in various topics of mathematical physics.

In the study of string theory, Brézin et al. [5] and Moore [27] discovered that, the equation (1.1) (with $t = 0$) possesses a special solution which is pole-free on the real line and which satisfies the following asymptotic behavior:

$$u(x, t = 0) \sim \mp|6x|^{\frac{1}{3}}, \qquad \text{as } x \to \pm\infty. \tag{1.3}$$

Later, uniqueness of the real pole-free solution $u(x, t = 0)$ was proven by Kapaev [22]. For general $t \neq 0$, Dubrovin [16] conjectured that there exists a unique real pole-free solution $u(x,t)$ of (1.1) as well. The existence of the real pole-free solution $u(x,t)$ to $\mathrm{P}_{\mathrm{I}}^2$ was proved in Claeys and Vanlessen [10] for any $t \in \mathbb{R}$. The main motivation to study this solution in [16] is that, Dubrovin suggested that the aforementioned particular solution $u(x,t)$ describes the universal asymptotics for Hamiltonian perturbations

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of hyperbolic equations near the point of gradient catastrophe for the unperturbed equation. Nowadays, this is known as the *universality conjecture* for Hamiltonian PDEs; see also [18]. As far as we know, the conjecture has only been verified for the Korteweg-de Vries equation (KdV)

$$u_t + uu_x + \frac{1}{12}u_{xxx} = 0 \tag{1.4}$$

and its hierarchy in [8, 9]. It is worth mentioning that, Dubrovin also formulated a similar universality conjecture for equations of the elliptic type. In this case, near a point of elliptic umbilic catastrophe, the universal asymptotics is described by the tritronquée solution to the Painlevé I equation (1.2); see the developments related to this conjecture in [2, 25]. Recently, in the study of two-dimensional dispersive shock waves, Dubrovin et al. [17] made another conjecture that, the asymptotic description of solutions to the generalized Kadomtsev-Petviashvili (KP) equations can also be given in terms of the real pole-free solution of P_I^2. This solution is also found to describe certain critical behavior in nonlinear waves; see [26]. It is interesting to note that, if $u(x,t)$ is a solution of P_I^2, it is a solution of the KdV equation as well. Then, in a different context, the particular pole-free solution $u(x,t)$ is referred as the Gurevich-Pitaevskii solution of the KdV equation; cf. [29].

The real pole-free solution $u(x,t)$ of P_I^2 also appears in random matrix theory and orthogonal polynomials. It is well-known that, local eigenvalue statistics of various random matrix ensembles exhibit universality properties when the matrix size n tends to infinity; see [24] and references therein. For the random Hermitian invariant ensembles, the limiting eigenvalue correlation kernels are expressed in terms of the sine function and the Airy function, in the bulk or at the soft edge of the spectrum, respectively. If the limiting mean eigenvalue density vanishes like a power 5/2 near a singular edge point, the local eigenvalues correlation kernel is given in terms of functions associated with the real pole-free solutions of P_I^2; see Claeys and Vanlessen [11]. Moreover, the real pole-free solution $u(x,t)$ appears explicitly in the second term of the asymptotic expansions for recurrence coefficients of the associated orthogonal polynomials; cf. [11].

Due to the importance of the real pole-free solution, there has been considerable interest in studying its properties. For example, besides the existence, it is also shown in [10] that $u(x,t)$ satisfies the following asymptotic behavior:

$$u(x,t) = \frac{1}{2}z_0|x|^{\frac{1}{3}} + O(|x|^{-2}), \qquad \text{as } x \to \pm\infty, \tag{1.5}$$

for fixed t, where z_0 is the real solution of

$$z_0^3 = -48\,\mathrm{sgn}(x) + 24z_0|x|^{-2/3}t.$$

In [6], Claeys extended the above asymptotic results by considering $u(x,t)$ when x and t tend to infinity simultaneously. Depending on the precise scaling of x and t, both algebraic and elliptic asymptotics were derived. Recently, Grava et al. [21] studied properties of $u(x,t)$ in the complex-x plane. They showed that, $u(x,t)$ is not only real and regular on the real line, but also has an extension to the complex plane with uniform algebraic asymptotics in large sectors. As $u(x,t)$ shares some similar features as the well-known *tritronquée solution* to the Painlevé I equation (1.2), they call $u(x,t)$ the tritronquée solution of P_I^2 and conjectured it to be pole-free in large sectors in the complex-x plane; see the precise statement of the conjecture in [21, Conjecture 1.1]. For readers who are interested in other members in the Painlevé I hierarchy, we refer to [7, 14].

In this paper, we intend to deepen our understanding about the tritronquée solution $u(x,t)$ of the P_I^2 equation by further investigating its asymptotics, as well as the asymptotics of its associated Hamiltonians $H_1(x,t)$ and $H_2(x,t)$. These two Hamiltonians are defined explicitly as follows:

$$H_1(x,t) = xu + \frac{1}{24}u^4 - \frac{1}{2}tu^2 + \frac{1}{24}uu_x^2 + \frac{1}{240}u_x u_{xxx} - \frac{1}{480}u_{xx}^2, \tag{1.6}$$

$$\begin{aligned} H_2(x,t) = {} & \frac{1}{1920}u_{xxx}^2 + \frac{1}{80}uu_x u_{xxx} + \frac{1}{16}u^2u_x^2 + \frac{1}{10}u^5 + \frac{1}{24}u^3u_{xx} + \frac{1}{240}uu_{xx}^2 \\ & - \frac{1}{480}u_x^2u_{xx} - \frac{1}{4}u_x + \frac{3}{2}xu^2 + \frac{1}{4}xu_{xx} - tu^3 - \frac{1}{4}tuu_{xx} + \frac{1}{8}tu_x^2. \end{aligned} \tag{1.7}$$

Note that, the P_I^2 equation is the compatibility conditions of the following Hamiltonian system in two time variables (cf. [21, 31]),

$$\frac{dq_j}{dx} = \frac{\partial \mathcal{H}_1}{\partial p_j}, \qquad \frac{dp_j}{dx} = -\frac{\partial \mathcal{H}_1}{\partial q_j}, \quad j = 1,2,$$
$$\frac{dq_j}{dt} = \frac{\partial \mathcal{H}_2}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial \mathcal{H}_2}{\partial q_j}, \quad j = 1,2,$$

with $\mathcal{H}_1 = H_1, \mathcal{H}_2 = -\frac{H_2}{3}$, $q_1 = u$, $p_1 = \frac{1}{240}(u_{xxx} + 8uu_x)$, $q_2 = \frac{1}{240}(u_{xx} + 6u^2)$ and $p_2 = u_x$. We will derive a full asymptotic expansion of $u(x,t)$, uniformly for $t|x|^{-\frac{2}{3}} \in (-\infty, M]$ with a constant $M \in (0, 2^{-\frac{2}{3}}3^{-\frac{1}{3}})$. This expansion can be utilized to obtain the total integrals of $u(x,t)$ and $H_1(x,t)$ for x on the real line. The investigation of the total integrations of $u(x,t)$ and $H_1(x,t)$ is partially motivated by eigenvalue spacing problems in random matrix theory. It has been realized that, total integrals of several Painlevé transcendents or associated Hamiltonians are related to large gap asymptotics in certain random matrix models. Take the well-known Tracy-Widom distribution as a concrete example, which is explicitly expressed as

$$F_{\mathrm{TW}}(x) = \exp\left(-\frac{1}{2}\int_x^{+\infty}(y-x)(q_{\mathrm{HM}}(y))^2dy\right), \qquad x \in \mathbb{R}, \tag{1.8}$$

where $q_{\mathrm{HM}}(y)$ is the Hastings-McLeod solution of the Painlevé II equation; see [32]. The above formula can be rewritten as

$$F_{\mathrm{TW}}(x) = \exp\left(-\int_x^{+\infty} H(y)dy\right), \tag{1.9}$$

where $H(x) = -2^{1/3}H_{\mathrm{PII}}(-2^{1/3}x)$ with H_{PII} being the Hamiltonian for the Painlevé II equation; see Forrester and Witte [20], Tracy and Widom [32]. Therefore, the large gap asymptotics of $F_{\mathrm{TW}}(x)$, including the constant term, can be achieved by deriving the total integrals of the Hastings-McLeod solution $q_{\mathrm{HM}}(y)$ or the associated Hamiltonian; see Baik et al. [1]. This idea has been successfully applied to study large gap asymptotics for other matrix models; for example, see [3, 4, 13].

The rest of this paper is organized as follows. In Section 2, we state main results for the full asymptotic expansion of $u(x,t)$ in Theorem 2.1 and the total integrals of $u(x,t)$ and $H_1(x,t)$ in Theorem 2.8. They are obtained via a steepest descent analysis of the associated Riemann-Hilbert (RH) problem. The detailed analysis is conducted in Section 3. Last, we prove Theorem 2.1 and Theorem 2.8 in Section 4.

2. Main results.

2.1. Asymptotics of $u(x,t)$ and the associated Hamiltonians. We first obtain a full asymptotic expansion for the tritronquée solution $u(x,t)$ of $\mathrm{P}_{\mathrm{I}}^2$ as $x \to \pm\infty$ below.

THEOREM 2.1. *Let $\mu = tx^{-\frac{2}{3}}$, and $z_\pm := z_\pm(\mu)$ be the unique real root of*

$$z_\pm^3 - 24\mu z_\pm \pm 48 = 0. \tag{2.1}$$

Then, for any fixed $M \in (0, 2^{-\frac{2}{3}}3^{-\frac{1}{3}})$, we have

$$u(x,t) = \frac{z_\pm}{2}(\pm x)^{\frac{1}{3}} + E(x,\mu) \tag{2.2}$$

with

$$E(x,\mu) \sim |x|^{\frac{1}{3}} \sum_{k=1}^{\infty} \frac{e_k^\pm(\mu)}{|x|^{\frac{7k}{3}}} \tag{2.3}$$

as $x \to \pm\infty$, where $e_k^\pm(\mu), k = 1, 2, \cdots,$ are bounded uniformly for $\mu \in (-\infty, M]$. Moreover, for arbitrary $k = 1, 2, \cdots,$ we have

$$e_k^\pm(\mu) = \mathcal{O}(\mu^{-2}) \qquad \text{as } \mu \to -\infty. \tag{2.4}$$

The leading asymptotics of $u(x,t)$ as $x \to \pm\infty$ have been derived in [10, 21] for fixed t. When $t \to \pm\infty$ and $x = s|t|^{\frac{3}{2}}$, a similar (leading) asymptotic behavior of $u(x,t)$ is provided in [6] as follows:

$$u(s|t|^{\frac{3}{2}}, t) = \frac{z_0(s)}{2}|t|^{\frac{1}{2}} + \mathcal{O}(|t|^{-1}), \text{ as } \begin{cases} t \to +\infty, s \in \mathbb{C} \setminus \left[-2\sqrt{3}, \frac{2\sqrt{15}}{27}\right], \\ t \to -\infty, s \in \mathbb{R}, \end{cases} \tag{2.5}$$

where $z_0(s)$ is the unique real root of $z_0^3 - 24 \cdot \mathrm{sgn}(t) z_0 + 48s = 0$. A full asymptotic expansion of $u(x,t)$ is also given in [30] as follows:

$$u(x,t) \sim -6^{\frac{1}{3}} x^{\frac{1}{3}} - 2 \cdot 6^{-\frac{1}{3}} t x^{-\frac{1}{3}} + \sum_{j=2}^{\infty} \frac{P_j(t)}{x^{\frac{j}{3}}} \qquad \text{as } x \to \pm\infty, \tag{2.6}$$

with t being bounded. Note that, in the above formula and afterwards, $x^{\frac{1}{3}} = -|x|^{\frac{1}{3}}$ when $x < 0$. Our expansion (2.2) is superior to (2.6) in the sense that it holds in a larger region of t. However, when comparing the above expansion with (2.2)-(2.3), it is important to note the dependence of $z_\pm$ and $e_k^\pm(\mu)$ on x. Consequently, when we re-expand (2.2) for fixed t, terms other than $|x|^{-\frac{7k}{3}}$ may arise in the asymptotic expansion; see Eq. (2.15) below as an illustration.

Remark 2.2. The coefficients $e_k^\pm(\mu)$ in (2.2) can be explicitly constructed term by term; see the derivation in Section 4. More precisely, we show that $e_k^\pm(\mu)$ are actually polynomials in terms of $(z_\pm^2 - 8\mu)^{-1}$; for example,

$$e_1^\pm(\mu) = -\frac{64}{3}(z_\pm^2 - 8\mu)^{-3} + \frac{256 z_\pm^2}{3}(z_\pm^2 - 8\mu)^{-4}. \tag{2.7}$$

By choosing $M \in (0, 2^{-\frac{2}{3}}3^{-\frac{1}{3}})$, one can ensure that $z_\pm^2 - 8\mu$ is bounded away from 0 for all $\mu \in (-\infty, M]$. This implies that $e_k^\pm(\mu), k = 1, 2, \cdots$, are analytic with respect to μ in the neighborhood of $(-\infty, M]$, and

$$E(x,\mu) = \mathcal{O}((z_\pm^2 - 8\mu)^{-2}x^{-2}) \qquad \text{as } x \to \pm\infty. \tag{2.8}$$

Moreover, we will show that there exists $\epsilon_0 > 0$ such that (2.8) holds as $|x| \to \infty$ uniform for $\arg x \in [-\epsilon_0, \epsilon_0] \cup [3\pi - \epsilon_0, 3\pi + \epsilon_0]$. For more details, please refer to Remark 3.4 and Appendix A. Note that the tritronquée solution $u(x,t)$ is real when x lies on the real line; see [10, 21]. When considering x as a variable in the complex plane, we choose $\arg x = 3\pi$ to ensure that terms such as $x^{\frac{1}{3}}$ in the asymptotic expansion (2.2) are real for $x < 0$.

Differentiating both sides of (2.2) with respect to x (keeping in mind that both $z_\pm$ and μ depend on x) and making use of the Cauchy's integral formula, one gets asymptotics for derivatives of $u(x,t)$ as follows.

Corollary 2.3. *Under the same conditions as in Theorem 2.1, we have*

$$u_x(x,t) = \frac{-8}{z_\pm^2 - 8\mu}|x|^{-\frac{2}{3}} + \mathcal{O}((z_\pm^2 - 8\mu)^{-2}|x|^{-3}), \tag{2.9}$$

$$u_{xx}(x,t) = \mathcal{O}((z_\pm^2 - 8\mu)^{-2}|x|^{-\frac{5}{3}}), \tag{2.10}$$

$$u_{xxx}(x,t) = \mathcal{O}((z_\pm^2 - 8\mu)^{-2}|x|^{-\frac{8}{3}}), \tag{2.11}$$

as $x \to \pm\infty$, uniformly for all $\mu \in (-\infty, M]$.

The proof of Corollary 2.3 will be provided in Appendix B. Next, substituting (2.2) and (2.9)-(2.11) into (1.6), we obtain asymptotics of the Hamiltonians via a straightforward computation.

Corollary 2.4. *Let $z_\pm$ be given in (2.1), and the Hamiltonians $H_1(x,t), H_2(x,t)$ defined in (1.6) and (1.7), respectively, then we have, as $x \to \pm\infty$,*

$$H_1(x,t) = \left(\pm\frac{1}{4}z_\pm - \frac{z_\pm^4}{384}\right)|x|^{\frac{4}{3}} + \frac{4z_\pm}{3}\left(\frac{1}{z_\pm^2 - 8\mu}\right)^2 |x|^{-1} + \mathcal{O}(|x|^{-\frac{10}{3}}), \tag{2.12}$$

$$H_2(x,t) = \left(\frac{z_\pm^5}{320} \pm \frac{3z_\pm^2}{8} - \frac{\mu z_\pm^3}{8}\right)|x|^{\frac{5}{3}} + \frac{3z_\pm^2 - 8\mu}{(z_\pm^2 - 8\mu)^2}|x|^{-\frac{2}{3}} + \mathcal{O}(|x|^{-3}), \tag{2.13}$$

uniformly for all $\mu \in (-\infty, M]$.

Remark 2.5. By a similar argument as in Remark 2.2, the error terms in (2.12) and (2.13) are also analytic with respect to μ and can be rewritten as $\mathcal{O}((z_\pm^2 - 8\mu)^{-2}|x|^{-\frac{10}{3}}), \mathcal{O}((z_\pm^2 - 8\mu)^{-2}|x|^{-3})$ respectively. As the error estimation is valid for $\mu \in (-\infty, M]$, *i.e.* $t \in (-\infty, M|x|^{\frac{2}{3}}]$, we can get a well-controlled error after integrating the expansion (2.12) with respect to t from $-\infty$ to any positive fixed constant. This property will play an important role in our derivation for the total integral of $H_1(x,t)$ below.

Remark 2.6. Since the above asymptotic results for $u(x,t)$ and the associated Hamiltonians $H_{1,2}(x,t)$ hold uniformly for all $\mu \in (-\infty, M]$, they are valid for any fixed t. Indeed, when t is fixed, then $\mu \to 0$ as $x \to \pm\infty$. From the definition of $z_\pm$ in (2.1), we find that

$$z_+(\mu) = -z_-(\mu) = -2 \cdot 6^{\frac{1}{3}} - 4 \cdot 6^{-\frac{1}{3}}\mu + \frac{4 \cdot 6^{\frac{1}{3}}}{27}\mu^3 + \frac{4 \cdot 6^{\frac{2}{3}}}{81}\mu^4 + \mathcal{O}(\mu^5) \tag{2.14}$$

as $\mu \to 0$. Substituting this formula into (2.2) and (2.12) respectively, we immediately obtain

$$u(x,t) = -6^{\frac{1}{3}}x^{\frac{1}{3}} - 2\cdot 6^{-\frac{1}{3}}tx^{-\frac{1}{3}} + \frac{2\cdot 6^{\frac{1}{3}}}{27}t^3x^{-\frac{5}{3}} + \frac{1}{36x^2} + \mathcal{O}(x^{-\frac{7}{3}}) \tag{2.15}$$

and

$$H_1(x,t) = -\frac{3}{4}6^{\frac{1}{3}}x^{\frac{4}{3}} - 3\cdot 6^{-\frac{1}{3}}tx^{\frac{2}{3}} - t^2 - \frac{6^{\frac{1}{3}}}{9}t^3x^{-\frac{2}{3}} - \frac{1}{36x} + \mathcal{O}(x^{-\frac{4}{3}}) \tag{2.16}$$

as $x \to \pm\infty$ for any fixed t.

2.2. Total integrals of $u(x,t)$ and $H_1(x,t)$. It has been shown in [10] that $u(x,t)$ is pole-free on the real-x axis for any $t \in \mathbb{R}$. From its definition in (1.6), it is easy to see that $H_1(x,t)$ is pole-free for real x as well. This makes it possible for us to compute their total integrals for $x \in \mathbb{R}$. There are two matters we need to consider. First, according to the asymptotic expansions in (2.2) and (2.12), both functions $u(x,t)$ and $H(x,t)$ do not decay sufficiently fast as $x \to \pm\infty$. Then, certain terms need to be deducted to make the integrals convergent. Second, as we don't have enough information about $u(x,t)$ and $H(x,t)$ when x is finite, the asymptotics in (2.2) and (2.12) only are not enough for us to establish the desired total integrals. Some differential identities are required to overcome this obstacle; for example, see some similar ideas in [1, 13, 23].

For the problem we are addressing, the first differential identity we need is

$$\frac{\partial H_1(x,t)}{\partial x} = u(x,t), \tag{2.17}$$

which is known in the literature. Next, keeping in mind that $u(x,t)$ also satisfies the KdV equation (1.4), a straightforward computation gives us an additional differential identity

$$\frac{\partial H_1(x,t)}{\partial t} = -\frac{1}{3}\frac{\partial H_2(x,t)}{\partial x} - \frac{1}{12}u_{xx}(x,t) = -\frac{1}{2}u(x,t)^2 - \frac{1}{12}u_{xx}(x,t); \tag{2.18}$$

see the explicit expression of $H_1(x,t)$ and $H_2(x,t)$ in (1.6) and (1.7). As the asymptotics of $H_1(x,t)$ is given in (2.16), it is easy for us to obtain the total integral of $u(x,t)$ with the aid of (2.17). Derivation of the total integral of $H_1(x,t)$ is more involved. Let us denote

$$I(t) = \int_{X_1}^{X_2} H_1(x,t)dx \tag{2.19}$$

with two arbitrary constants X_k, $k = 1,2$. Then, with the second differential identity (2.18), we have

$$\begin{aligned}\frac{dI(t)}{dt} &= \int_{X_1}^{X_2}\frac{\partial H_1(x,t)}{\partial t}dx = \int_{X_1}^{X_2}\left[-\frac{1}{3}\frac{\partial H_2(x,t)}{\partial x} - \frac{1}{12}u_{xx}(x,t)\right]dx \\ &= \frac{1}{3}H_2(X_1,t) + \frac{1}{12}u_x(X_1,t) - \frac{1}{3}H_2(X_2,t) - \frac{1}{12}u_x(X_2,t).\end{aligned} \tag{2.20}$$

Integrating both sides of the last formula from $-\infty$ to t, we get

$$I(t) = I(-\infty) + \int_{-\infty}^{t}\left(\frac{1}{3}H_2(X_1,\tau) + \frac{1}{12}u_x(X_1,\tau) - \frac{1}{3}H_2(X_2,\tau) - \frac{1}{12}u_x(X_2,\tau)\right)d\tau. \tag{2.21}$$

In such a way, we transform an integral of $H_1(x,t)$ with respect to x into a new one with respect to t. Using uniform asymptotics for both $u(x,t)$ and $H_2(x,t)$ as $x \to \pm\infty$, we successfully establish the total integral of $H_1(x,t)$.

Remark 2.7. To get a total integral of $H_1(x,t)$, we will let $X_1 \to -\infty$ and $X_2 \to +\infty$. By (2.21), this requires the asymptotics for both $u_x(x,\tau)$ and $H_2(x,\tau)$ as $x \to \pm\infty$ hold uniformly for all $\tau \in (-\infty, t]$. It is one of the motivations why we first establish the corresponding asymptotic expansion of $u(x,t)$ in Theorem 2.1.

We are now in position to state our second main results about the total integrals of $u(x,t)$ and the associated Hamiltonian $H_1(x,t)$.

THEOREM 2.8. *Let $u(x,t)$ be the tritronquée solution of the P_I^2 equation* (1.1) *and $H_1(x,t)$ be the associated Hamiltonian given in* (1.6). *Then, for any fixed $t \in \mathbb{R}$, we have*

$$\int_{-\infty}^{+\infty} \left(u(x,t) + 6^{\frac{1}{3}} x^{\frac{1}{3}} + 2 \cdot 6^{-\frac{1}{3}} t x^{-\frac{1}{3}} \right) dx = 0 \tag{2.22}$$

and

$$\int_{-\infty}^{+\infty} \left(H_1(x,t) + \frac{3}{4} 6^{\frac{1}{3}} x^{\frac{4}{3}} + 3 \cdot 6^{-\frac{1}{3}} t x^{\frac{2}{3}} + t^2 + \frac{6^{\frac{1}{3}} t^3}{9} x^{-\frac{2}{3}} + \frac{x}{36(x^2+1)} \right) dx = 0. \tag{2.23}$$

Remark 2.9. Due to the asymptotics of $u(x,t)$ and $H_1(x,t)$ given in (2.15) and (2.16), some leading terms are deducted to ensure the convergence of the integral in the above theorem. Moreover, in (2.23), we adopt the term $\frac{x}{36(x^2+1)}$, instead of $\frac{1}{36x}$, to ensure that the integral is convergent at $x = 0$. It should also be noted that both $u(x,t)$ and $H_1(x,t)$ are functions of (x,t), which implies that their total integrals about x should dependent on t in general. Surprisingly, the right-hand side of (2.22) and (2.23) are both zero, despite the fact that $u(x,t)$ behaves dramatically differently for $t > 0$ and $t < 0$; see [6] or [21, Fig. 19].

Remark 2.10. Motivated by the applications of (1.8) and (1.9) in random matrix theory, we are also exploring potential applications of the total integrals in Theorem 2.8. In an ongoing investigation [12], we study the Fredholm determinants associated with the $\mathrm{P}_{\mathrm{I}}^2$ kernel $K^{\mathrm{crit,III}}(u,v;s,t)$. This kernel, which was defined in Claeys and Vanlessen [11, Eq. (1.34)], characterizes the behavior of eigenvalues near a singular edge point, where the limiting mean density vanishes like a power of $5/2$. With the notation in the present paper, the parameters s and t in $K^{\mathrm{crit,III}}(u,v;s,t)$ are given by x and t, respectively. In the large gap asymptotic expansion of the Fredholm determinant associated with $K^{\mathrm{crit,III}}(u,v;x,t)$, we find that the $\mathcal{O}(1)$ term involves the well-known constant $\chi = \frac{1}{24}\ln 2 + \zeta'(-1)$ and an integral of the Hamiltonian $H_1(x,t)$ as follows:

$$\int_{-\infty}^{x} \left(H_1(\xi,t) + \frac{3}{4} 6^{\frac{1}{3}} \xi^{\frac{4}{3}} + 3 \cdot 6^{-\frac{1}{3}} t \xi^{\frac{2}{3}} + t^2 + \frac{6^{\frac{1}{3}} t^3}{9} \xi^{-\frac{2}{3}} + \frac{\xi}{36(\xi^2+1)} \right) d\xi. \tag{2.24}$$

As $x \to +\infty$, one can observe a phase transition of the matrix model in [11] at two different levels. At the level of the limiting kernel, one can see the kernel $K^{\mathrm{crit,III}}(u,v;x,t)$ tends to the Airy kernel. More precisely, let

$$u = x^{\frac{1}{3}} \left(z_+ + \frac{\tilde{u}}{x^{\frac{7}{9}}} \right), \quad v = x^{\frac{1}{3}} \left(z_+ + \frac{\tilde{v}}{x^{\frac{7}{9}}} \right) \tag{2.25}$$

with z_+ defined in (2.1), we have

$$\lim_{x\to+\infty} \tilde{c}_1 x^{-\frac{4}{9}} K^{\mathrm{crit,III}}(u,v;x,t) = \frac{\mathrm{Ai}(\tilde{c}_2\tilde{u})\mathrm{Ai}'(\tilde{c}_2\tilde{v}) - \mathrm{Ai}(\tilde{c}_2\tilde{v})\mathrm{Ai}'(\tilde{c}_2\tilde{u})}{\tilde{u}-\tilde{v}}, \tag{2.26}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are certain constants. At the level of large gap asymptotics, the total integral (2.23) can be employed to demonstrate the phase transition towards the asymptotics of the Airy kernel determinant. Notably, as the total integral (2.23) equals zero, we precisely recover the constant term $\chi = \frac{1}{24}\ln 2 + \zeta'(-1)$ appearing in the asymptotics of the Airy kernel determinant.

Remark 2.11. We perform a numerical check on our total integrals, which confirms the reliability of our results in Theorem 2.8. We divide the interval $(-\infty,+\infty)$ into three parts: $(-\infty,-10]$, $[-10,10]$, and $[10,+\infty)$. The integrals over $(-\infty,-10]$ and $[10,+\infty)$ are directly computed using the asymptotic expansion of $u(x,t)$ in (2.6). Next, we determine the solution $u(x,t)$ on the interval $[-10,10]$ by employing a boundary value problem solver for ordinary differential equations (ODEs) in Matlab. The truncation ($j = 228$) of the asymptotic expansion (2.6) is set as the boundary value. Once $u(x,t)$ is computed, the corresponding integral over $[-10,10]$ is evaluated accordingly. In Table 1, it can be observed that the accuracy of the numerical results is lower for $t > 0$ compared to $t < 0$. The decrease in accuracy for $t > 0$ is due to the accumulated error of the boundary value problem solver when the solution exhibits rapid oscillations. To enhance accuracy, one can refer to [21].

Values of t	The integral in (2.22)	The integral in (2.23)
-1	-0.0000000000495	-0.0000000227081
-0.5	-0.0000000029662	-0.0000000255779
0	-0.0000001170272	0.0000000638861
0.5	0.0000077322141	-0.0000053153475
1	-0.0002015970845	0.0006676097951

TABLE 1

A few numerical tests on the total integrals in Theorem 2.8.

By differentiating both sides of (2.23) with respect to t and taking into account the differential identity in (2.18), we also derive the total integral of $u(x,t)^2$.

COROLLARY 2.12. *For any fixed $t \in \mathbb{R}$, we have*

$$\int_{-\infty}^{+\infty} \left(u(x,t)^2 - 6^{\frac{2}{3}} x^{\frac{2}{3}} - 4t - 4\cdot 6^{-\frac{2}{3}} t^2 x^{-\frac{2}{3}} \right) dx = 0. \tag{2.27}$$

Since $\frac{\partial H_1(x,t)}{\partial x} = u(x,t)$, integration by parts gives us

$$\int_{X_1}^{X_2} x\cdot u(x,t)dx = X_2 H_1(X_2,t) - X_1 H_1(X_1,t) - \int_{X_1}^{X_2} H_1(x,t)dx. \tag{2.28}$$

Taking limits $X_1 \to -\infty$, $X_2 \to +\infty$, and making use of the asymptotic of $u(x,t)$ in (2.15) and the total integral of $H_1(x,t)$ in (2.23), we obtain a total integral for $xu(x,t)$.

COROLLARY 2.13. *For any fixed $t \in \mathbb{R}$, we have*

$$\int_{-\infty}^{+\infty} \left(x\cdot u(x,t) + 6^{\frac{1}{3}} x^{\frac{4}{3}} + 2\cdot 6^{-\frac{1}{3}} t x^{\frac{2}{3}} - \frac{x}{36(x^2+1)} \right) dx = 0. \tag{2.29}$$

3. RH analysis for the Painlevé I hierarchy. In this section, we perform RH analysis to obtain the uniform asymptotic expansion stated in Theorem 2.1. The process of the RH analysis is very similar as that in [6, 10]. However, our goal is to derive a complete asymptotic expansion and demonstrate that the coefficients $e_k^{\pm}(\mu)$ are analytic in a neighborhood of $(-\infty, M]$ with respect to μ. Therefore, we will pay more attention to the influence of μ in the RH analysis. To keep it concise, we will only present the essential steps here and refer to [6, 10] for more detailed information. Additionally, we will only analyze the case of $x \to +\infty$, as it is similar to the case of $x \to -\infty$.

3.1. The steepest descent analysis. The tritronquée solution $u(x,t)$ of $\mathrm{P}_{\mathrm{I}}^2$ equation is associated with a RH problem as follows; see [19] for more detailed information about the Painlevé equations and RH problems.

RH problem for $Y(\lambda)$

(Y1) $Y(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \Gamma$, where $\Gamma = \cup_{k=1}^{4} \Gamma_4$ is described in Fig. 1;

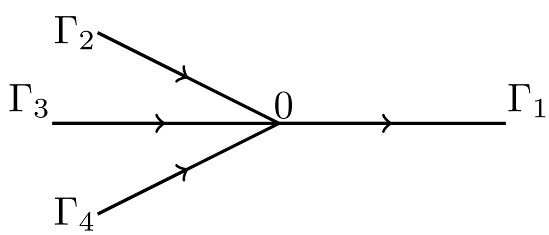

FIG. 1. *Contours for the RH problem for Y*

(Y2) $Y(\lambda)$ satisfies the following jump conditions on Γ:

$$Y_+(\lambda) = Y_-(\lambda) \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \lambda \in \Gamma_1, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \lambda \in \Gamma_2 \cup \Gamma_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \lambda \in \Gamma_3; \end{cases} \tag{3.1}$$

(Y3) When $\lambda \to \infty$, the asymptotic expansion of $Y(\lambda)$ is

$$Y(\lambda) \sim \left(I + \sum_{k=1}^{\infty} A_k(x,t) \lambda^{-k} \right) \lambda^{-\frac{1}{4}\sigma_3} N e^{-\theta(\lambda;x,t)\sigma_3}, \tag{3.2}$$

where

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.3}$$

and

$$\theta(\lambda; x, t) = \frac{1}{105}\lambda^{\frac{7}{2}} - \frac{1}{3} t \lambda^{\frac{3}{2}} + x \lambda^{\frac{1}{2}}. \tag{3.4}$$

In the above formulas, the branch is chosen such that $\arg \lambda \in (-\pi, \pi)$.

It has been proved in [10] that the RH problem for $Y(\lambda)$ is uniquely solvable for all real values of x and t. The tritronquée solution $u(x,t)$ for the P_I^2 equation (1.1) is given explicitly by

$$u(x,t) = 2(A_1)_{11} - (A_1)_{12}^2, \tag{3.5}$$

where $A_1 = A_1(x,t)$ is the coefficient matrix in (3.2), and $(A_1)_{ij}$ denotes its (i,j)-entry.

To investigate the asymptotics of the RH problem for $Y(\lambda)$ as $x \to +\infty$, let us first make the rescaling $t = \mu x^{\frac{2}{3}}$ and $\lambda = \zeta x^{\frac{1}{3}}$. Then, the function $\theta(\cdot)$ in (3.4) can be written as

$$\theta(x^{\frac{1}{3}}\zeta; x, t) = x^{\frac{7}{6}}\left(\frac{1}{105}\zeta^{\frac{7}{2}} - \frac{\mu}{3}\zeta^{\frac{3}{2}} + \zeta^{\frac{1}{2}}\right). \tag{3.6}$$

Next, in order to normalize the large-λ behavior in (3.2), we introduce the following g-function:

$$g(\zeta) = c_1(\zeta - z_+)^{\frac{7}{2}} + c_2(\zeta - z_+)^{\frac{5}{2}} + c_3(\zeta - z_+)^{\frac{3}{2}}, \qquad \zeta \in \mathbb{C}\setminus(-\infty, z_+] \tag{3.7}$$

with $\arg(\zeta - z_+) \in (-\pi, \pi)$, where $z_+ = z_+(\mu)$ is the unique real root of $z^3 - 24\mu z + 48 = 0$. The constants c_k in the above formula are chosen to be

$$c_1 = \frac{1}{105}, \qquad c_2 = \frac{z_+}{30}, \qquad c_3 = -\frac{\mu}{3} + \frac{1}{24}z_+^2, \tag{3.8}$$

such that

$$x^{\frac{7}{6}}g(\zeta) = \theta(x^{\frac{1}{3}}\zeta; x, t) + d_1\zeta^{-\frac{1}{2}} + \mathcal{O}(\zeta^{-\frac{3}{2}}), \qquad \text{as } \zeta \to \infty, \tag{3.9}$$

with d_1 being a constant independent of ζ.

To facilitate our subsequent analysis, let us list some important properties of $g(\zeta)$ below.

Lemma 3.1. *Given any constant M with $0 < M < M_+ = 3^{\frac{5}{3}}2^{-\frac{2}{3}}5^{-\frac{1}{3}}$, we have, for all $\mu \in (-\infty, M]$,*
(i) $g(\zeta) > 0$ *when* $\zeta > z_+$;
(ii) $\operatorname{Re} g_\pm(\zeta) \equiv 0$ *and* $\operatorname{Im} g_\pm(\zeta) \neq 0$ *when* $\zeta < z_+$;
(iii) $\operatorname{Im} g_+'(\zeta) > 0$ *when* $\zeta < z_+$;
(iv) $\frac{1}{|g(\zeta)|} = \mathcal{O}(z_+^2 - 8\mu)^{-1}$ *as* $\mu \to -\infty$ *uniformly for all* ζ *bounded away from* z_+.

Proof. It is easily seen from its definition in (3.7) that $g(\zeta)$ is real when $\zeta > z_+$ and $\operatorname{Re} g_\pm(\zeta) = 0$ when $\zeta < z_+$. Next, we establish (i) and (ii) by proving z_+ is the unique real zero of $g(\zeta)$. This can be justified by imposing the condition $c_2^2 - 4c_1c_3 < 0$, which implies $\mu < \frac{3}{80}z_+^2$. Recalling $\mu = \frac{z_+^3+48}{24z_+}$, then we have $-480^{\frac{1}{3}} < z_+ < 0$. This agrees with the condition $\mu \in (-\infty, M]$. In such a way, we prove (i) and (ii).

To prove (iii), we first have from (3.7) that

$$g'(\zeta) = (\zeta - z_+)^{\frac{1}{2}}\left[\frac{1}{30}(\zeta - z_+)^2 + \frac{1}{12}z_+(\zeta - z_+) - \left(\frac{\mu}{2} - \frac{z_+^2}{16}\right)\right]. \tag{3.10}$$

Based on the derivation in the previous paragraph, it is clear that $z_+ < 0$ and $\mu < \frac{3}{80}z_+^2$. Then, we get

$$\operatorname{Im} g_+'(\zeta) \geq |z_+ - \zeta|^{\frac{1}{2}}\left(\frac{z_+^2}{16} - \frac{\mu}{2}\right) > 0, \qquad \text{when } \zeta < z_+. \tag{3.11}$$

Last, it follows from (3.7) that $\frac{1}{|g(\zeta)|} = \mathcal{O}(c_3^{-1})$ as $\mu \to \infty$. Recalling the definition of c_3 in (3.8), we immediately get (iv). ∎

Remark 3.2. From (iii) in the above lemma, a straightforward derivation using the Cauchy-Riemann equations implies that there exists a constant $0 < \theta_0 < \frac{2\pi}{7}$, such that $\operatorname{Re} g(\zeta) < 0$ when $\arg(\zeta - z_+) = \pm\pi + \theta_0$.

Remark 3.3. When considering the asymptotics as $x \to -\infty$, the role of z_+ will be replaced by z_-, the real root of $z^3 - 24\mu z - 48 = 0$. In this case, the upper bound of M in the above lemma should be replaced by $M_- = 2^{-\frac{2}{3}}3^{-\frac{1}{3}}$. Therefore, to ensure the asymptotic expansions in Theorem 2.1 are valid as $x \to \pm\infty$, we set $M \in (0, 2^{-\frac{2}{3}}3^{-\frac{1}{3}})$.

Normalization: $Y \mapsto T$**.** Define

$$T(\zeta) = x^{\frac{1}{12}\sigma_3} \begin{pmatrix} 1 & 0 \\ d_1 x^{\frac{1}{6}} & 1 \end{pmatrix} \begin{cases} Y(x^{\frac{1}{3}}\zeta) e^{x^{\frac{7}{6}} g(\zeta)\sigma_3}, & \zeta \in I \cup II \cup III, \\ Y(x^{\frac{1}{3}}\zeta) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} e^{x^{\frac{7}{6}} g(\zeta)\sigma_3}, & \zeta \in I' \cup I''. \end{cases} \tag{3.12}$$

where d_1 is given in (3.9) and the regions I, I', I'', II, III are described in Fig. 2. A direct calculation from the RH problem for $Y(\lambda)$ leads to the following RH problem for $T(\zeta)$.

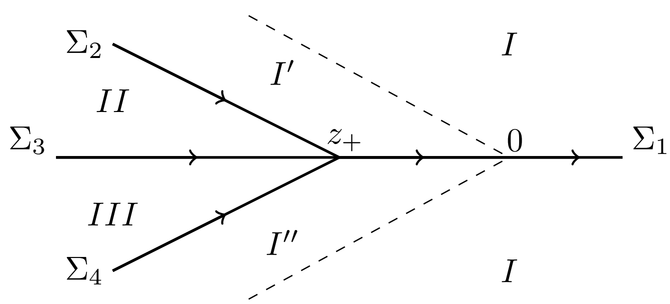

FIG. 2. *Regions and contours for the RH problem for T*

RH problem for $T(\zeta)$

(T1) $T(\zeta)$ is analytic in $\mathbb{C} \setminus \Sigma$, where $\Sigma = \cup_{k=1}^{4} \Sigma_k$ is described in Fig. 2;

(T2) $T(\zeta)$ satisfies the following jump conditions on Σ:

$$T_+(\zeta) = T_-(\zeta) \begin{cases} \begin{pmatrix} 1 & e^{-2x^{\frac{7}{6}} g(\zeta)} \\ 0 & 1 \end{pmatrix}, & \zeta \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ e^{2x^{\frac{7}{6}} g(\zeta)} & 1 \end{pmatrix}, & \zeta \in \Sigma_2 \cup \Sigma_4, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in \Sigma_3. \end{cases} \tag{3.13}$$

(T3) As $\zeta \to \infty$, we have

$$T(\zeta) = \left(I + \frac{T_1(x,t)}{\zeta} + \mathcal{O}(\zeta^{-2}) \right) \zeta^{-\frac{1}{4}\sigma_3} N, \tag{3.14}$$

with
(3.15)
$$T_1(x,t) = \begin{pmatrix} \frac{d_1^2}{2} + x^{-\frac{1}{3}}(A_1)_{11} - d_1 x^{-\frac{1}{6}}(A_1)_{12} & x^{-\frac{1}{6}}(A_1)_{12} - d_1 \\ * & * \end{pmatrix},$$

where A_1 is given in (3.2) and $*$ denotes some unimportant entries.

A combination of (3.5) and (3.15) yields that

$$u(x,t) = 2x^{\frac{1}{3}}(T_1)_{11} - x^{\frac{1}{3}}(T_1)_{12}^2. \tag{3.16}$$

Global parametrix. From Lemma 3.1 and Remark 3.2, one can see the jump matrices in (3.13) tend to the identity matrix when $x \to +\infty$, except for the one on Σ_3. This gives us a global parametrix for $P^{(\infty)}(\zeta)$, only possessing a jump on $(-\infty, z_+)$.

RH problem for $P^{(\infty)}(\zeta)$

- $P^{(\infty)}(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus (-\infty, z_+]$;
- For $\zeta \in (-\infty, z_+)$, we have $P_+^{(\infty)}(\zeta) = P_-^{(\infty)}(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;
- As $\zeta \to \infty$, we have $P^{(\infty)}(\zeta) = \left(I + \mathcal{O}(\zeta^{-1})\right) \zeta^{-\frac{1}{4}\sigma_3} N$.

It is easy to check that the above RH problem possesses a solution as follows

$$P^{(\infty)}(\zeta) = (\zeta - z_+)^{-\frac{\sigma_3}{4}} N, \qquad \zeta \in \mathbb{C} \setminus (-\infty, z_+]. \tag{3.17}$$

Moreover, via a straightforward computation, we have from (3.14) and the above formula

$$T(\zeta) P^{(\infty)}(\zeta)^{-1} = I + \left(T_1 - \frac{z_+ \sigma_3}{4}\right) \frac{1}{\zeta} + \mathcal{O}(\zeta^{-2}) \qquad \text{as } \zeta \to \infty. \tag{3.18}$$

Local parametrix near z_+**.** Denote $U(z_+, \delta)$ by the neighborhood $\{\zeta : |\zeta - z_+| \le \delta\}$, where δ is a small and fixed constant. In $U(z_+, \delta)$, a local parametrix $P(\zeta)$ is constructed to approximate $T(\zeta)$ in terms of the Airy functions; for example, see [6, Section 2.4]. More precisely, we have

$$P(\zeta) = (\zeta - z_+)^{-\frac{\sigma_3}{4}} \left(x^{\frac{7}{9}} f(\zeta)\right)^{\frac{\sigma_3}{4}} M(x^{\frac{7}{9}} f(\zeta)), \tag{3.19}$$

where

$$f(\zeta) = \left(\frac{3}{2} g(\zeta)\right)^{\frac{2}{3}} \tag{3.20}$$

is a conformal mapping in $U(z_+, \delta)$ and $M(\cdot)$ is the standard Airy parametrix; see the explicit RH problem for M in [10, Section 2.4]. Here, we omit its details and only record its large-z behaviour below:

$$M(z) \sim \left(I + \sum_{k=1}^{\infty} B_k z^{-k}\right) z^{-\frac{1}{4}\sigma_3} N, \qquad \text{as } z \to \infty, \tag{3.21}$$

where N is given in (3.2) and

$$B_{3k-2} = \begin{pmatrix} 0 & 0 \\ t_{2k-1} & 0 \end{pmatrix}, \qquad B_{3k-1} = \begin{pmatrix} 0 & \hat{t}_{2k-1} \\ 0 & 0 \end{pmatrix}, \qquad B_{3k} = \begin{pmatrix} \hat{t}_{2k} & 0 \\ 0 & t_{2k} \end{pmatrix} \tag{3.22}$$

with

$$\hat{t}_k = \frac{\Gamma(3k + 1/2)}{36^k k! \Gamma(k + 1/2)}, \qquad t_k = -\frac{6k+1}{6k-1} \hat{t}_k. \tag{3.23}$$

Final transformation $T \mapsto R$. The final transformation is defined as

$$R(\zeta) = \begin{cases} T(\zeta)P^{(\infty)}(\zeta)^{-1}, & \text{for } \zeta \in (\mathbb{C} \setminus U(z_+,\delta)) \setminus \Sigma, \\ T(\zeta)P(\zeta)^{-1}, & \text{for } \zeta \in U(z_+,\delta) \setminus \Sigma. \end{cases} \tag{3.24}$$

Then $R(\zeta)$ satisfies the following RH problem.

RH problem for $R(\zeta)$

(R1) $R(\zeta)$ is analytic in $\mathbb{C} \setminus \Sigma_R$, where $\Sigma_R = \Sigma_{R,1} \cup \Sigma_{R,2} \cup \Sigma_{R,4} \cup \partial U(z_+,\delta)$ is described in Fig. 3;

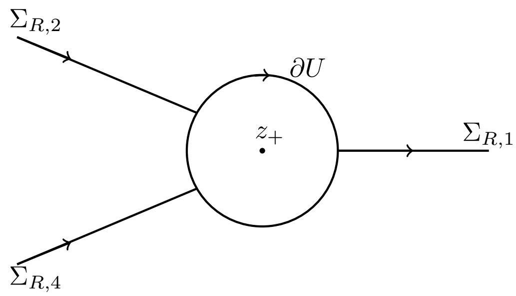

FIG. 3. *Contours for the RH problem for R.*

(R2) When $\zeta \in \Sigma_R$, we have

$$R_+(\zeta) = R_-(\zeta)J_R(\zeta) \tag{3.25}$$

where

(3.26)
$$J_R(\zeta) = \begin{cases} P^{(\infty)}(\zeta)\begin{pmatrix} 1 & e^{-2x^{\frac{7}{6}}g(\zeta)} \\ 0 & 1 \end{pmatrix} P^{(\infty)}(\zeta)^{-1}, & \zeta \in \Sigma_{R,1}, \\ P^{(\infty)}(\zeta)\begin{pmatrix} 1 & 0 \\ e^{2x^{\frac{7}{6}}g(\zeta)} & 1 \end{pmatrix} P^{(\infty)}(\zeta)^{-1}, & \zeta \in \Sigma_{R,2} \cup \Sigma_{R,4}, \\ P(\zeta)P^{(\infty)}(\zeta)^{-1}, & \zeta \in \partial U(z_+,\delta); \end{cases}$$

(R3) As $\zeta \to \infty$, we have

$$R(\zeta) = I + \frac{C_1(x,t)}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \text{where } C_1 = T_1 - \frac{z_+\sigma_3}{4}. \tag{3.27}$$

Remark 3.4. In the preceding RH analysis, it is possible to relax the restriction that x is real and instead assume that $\arg x \in [-\epsilon_0, \epsilon_0]$ for certain small $\epsilon_0 > 0$. Since z_+ becomes complex when $\arg x \neq 0$ (see the definition of z_+ in (2.1)), we need to shift the contours in Figures 2-3 accordingly. Apart from the contour deformation, we only need to show that, there exist $\epsilon_0 > 0$ and $\rho_0 > 0$ such that $\operatorname{Re} g(\zeta) < 0$ for $\zeta \in \Sigma_{R,2} \cup \Sigma_{R,4}$ and $\operatorname{Re} g(\zeta) > 0$ for $\zeta \in \Sigma_{R,1}$, uniformly for all $|x| \geq \rho_0$ with $\arg x \in [-\epsilon_0, \epsilon_0]$. Due to the complexity of the proof, we provide it in Appendix A.

Next, we will conduct a detailed asymptotic analysis on $R(\zeta)$ to establish a full asymptotic expansion (2.2) in Theorem 2.1, as well as the analyticity of the coefficients $e_k^{\pm}(\mu)$. This part is new comparing with the existing RH analysis related to the tritronquée solution $u(x,t)$.

3.2. Asymptotics of $R(\zeta)$. As $x \to +\infty$, the jump matrix $J_R(\zeta)$ in (3.25) decays exponentially to the identical matrix uniformly on $\Sigma_{R,1} \cup \Sigma_{R,2} \cup \Sigma_{R,4}$. On $\partial U(z_+, \delta)$, it follows from (3.17), (3.19), (3.21) and (3.26) that

$$J_R(\zeta) = P(\zeta)P^{(\infty)}(\zeta)^{-1} \sim I + \sum_{k=1}^{\infty} Q_k(\zeta) x^{-\frac{7}{6}k}, \qquad \text{as } x \to +\infty, \tag{3.28}$$

with

$$\begin{aligned} Q_{2k}(\zeta) =& B_{3k} \left(\frac{3}{2} g(\zeta)\right)^{-2k} = \begin{pmatrix} \hat{t}_{2k} & 0 \\ 0 & t_{2k} \end{pmatrix} \left(\frac{3}{2} g(\zeta)\right)^{-2k}, \\ Q_{2k-1}(\zeta) =& \left(\frac{B_{3k-1}}{(\zeta - z_+)^{\frac{1}{2}}} + B_{3k-2}(\zeta - z_+)^{\frac{1}{2}}\right) \left(\frac{3}{2} g(\zeta)\right)^{-2k+1} \\ =& \begin{pmatrix} 0 & \frac{\hat{t}_{2k-1}}{(\zeta - z_+)^{\frac{1}{2}}} \\ t_{2k-1}(\zeta - z_+)^{\frac{1}{2}} & 0 \end{pmatrix} \left(\frac{3}{2} g(\zeta)\right)^{-2k+1}, \end{aligned} \tag{3.29}$$

where t_k and $\hat{t}_k$ are constants given in (3.23). This implies that the RH problem for R is a small norm problem when $x \to +\infty$. By a standard argument for RH problems [15], $R(\zeta)$ exists when x is large enough and admits an asymptotic expansion as follows:

$$R(\zeta) \sim I + \sum_{k=1}^{\infty} \frac{R_k(\zeta)}{x^{\frac{7}{6}k}}, \qquad \text{as } x \to +\infty, \tag{3.30}$$

uniformly for $\zeta \in \mathbb{C} \setminus \Sigma_R$.

It is possible to get more information about $R_k(\zeta)$ in the above expansion. Substituting (3.28) and (3.30) into (3.25), we have, for $k \in \mathbb{N}^+$,

$$R_{k,+}(\zeta) = R_{k,-}(\zeta) + \sum_{m=0}^{k-1} R_{m,-}(\zeta) Q_{k-m}(\zeta), \qquad \zeta \in \partial U(z_+, \delta), \tag{3.31}$$

with $R_0(\zeta) \equiv I$. First, we have the following result.

LEMMA 3.5. *For any $k \in \mathbb{N}^+$, $R_{2k}(\zeta)$ is a diagonal matrix and $R_{2k-1}(\zeta)$ is an anti-diagonal matrix.*

Proof. Note that the coefficient matrices $Q_k(\zeta)$ in (3.28) possess diagonal and anti-diagonal structures; see (3.29). Using (3.31), we obtain the lemma by mathematical induction. ∎

The next lemma gives us the analyticity of $R_k(\zeta)$ with respect to μ in a neighborhood of $(-\infty, M]$.

LEMMA 3.6. *Denote $n_{2k} = 3k$, $n_{2k-1} = 3k - 1$ for all $k \in \mathbb{N}^+$. We have the following representation for the coefficients $R_k(\zeta)$ in (3.30):*

$$R_k(\zeta) = \begin{cases} \sum_{m=1}^{n_k} \frac{R_k^{(-m)}}{(\zeta - z_+)^m}, & \zeta \in \mathbb{C} \setminus \overline{U(z_+, \delta)}, \\ \sum_{m=0}^{+\infty} R_k^{(m)} (\zeta - z_+)^m, & \zeta \in U(z_+, \delta), \end{cases} \tag{3.32}$$

where $R_k^{(m)}$ are polynomials of $\frac{1}{c_3}$ and analytic with respect to μ in a neighborhood of $(-\infty, M]$ for all $k \in \mathbb{N}^+$ and $m = -n_k, -n_k+1, \cdots$. In particular, we have

$$R_1^{(-1)} = \begin{pmatrix} 0 & -\frac{2c_2\hat{t}_1}{3c_3^2} \\ \frac{2t_1}{3c_3} & 0 \end{pmatrix}, \tag{3.33}$$

$$R_2^{(-1)} = \begin{pmatrix} \frac{4\hat{t}_2(3c_2^2-2c_1c_3)-4t_1\hat{t}_1(c_2^2-c_1c_3)}{9c_3^4} & 0 \\ 0 & \frac{4t_2(3c_2^2-2c_1c_3)-4t_1\hat{t}_1(2c_2^2-c_1c_3)}{9c_3^4} \end{pmatrix}, \tag{3.34}$$

where c_k, t_k and $\hat{t}_k$ are given in (3.8) and (3.23), respectively.

Proof. When $k=1$, the equation (3.31) is simply $R_{1,+}(\zeta) = R_{1,-}(\zeta) + Q_1(\zeta)$ for $\zeta \in \partial U(z_+, \delta)$. It then follows from the Plemelj formula that

$$R_1(\zeta) = \frac{1}{2\pi i}\oint_{\partial U(z_+,\delta)} \frac{Q_1(s)}{s-\zeta} ds, \tag{3.35}$$

where the orientation of $\partial U(z_+, \delta)$ is depicted in Fig. 3. From (3.7) and (3.29), one can see that $Q_1(\zeta)$ has a pole of order 2 at $\zeta = z_+$. Denote its Laurent series at $\zeta = z_+$ by

$$Q_1(\zeta) = \sum_{m=-2}^{+\infty} \frac{Q_1^{(m)}}{(\zeta - z_+)^m}, \qquad |z - z_+| > 0, \tag{3.36}$$

where $Q_1^{(m)}, m = -2, -1, 0, \cdots$, are polynomials of $\frac{1}{c_3}$. Combining the above two formulas, we have

$$R_1(\zeta) = \begin{cases} \frac{R_1^{(-2)}}{(\zeta-z_+)^2} + \frac{R_1^{(-1)}}{\zeta - z_+}, & \zeta \in \mathbb{C} \setminus \overline{U(z_+,\delta)}, \\ \sum_{m=0}^{+\infty} R_1^{(m)}(\zeta - z_+)^m, & \zeta \in U(z_+, \delta), \end{cases} \tag{3.37}$$

where $R_1^{(m)} = Q_1^{(m)}$ for $m = -1, -2$ and $R_1^{(m)} = -Q_1^{(m)}$ for $m = 0, 1, 2, \cdots$. In particular, $R_1^{(-1)}$ is given explicitly in (3.33). Moreover, since c_3 is bounded away from zero when μ lies in a fixed neighborhood of $(-\infty, M]$, all the coefficients $R_1^{(m)}$ are analytic functions of μ for $\mu \in (-\infty, M]$.

The case for $k > 1$ is similar. First, it follows from the definition of $g(\zeta)$ in (3.7) that

$$\begin{aligned} \left(\frac{3}{2}g(\zeta)\right)^{-2k} &= \sum_{i=-3k}^{\infty} g_{2k,i}(\zeta - z_+)^i, \\ \left(\frac{3}{2}g(\zeta)\right)^{-2k+1} &= (\zeta - z_+)^{-\frac{1}{2}} \sum_{i=-3k}^{\infty} g_{2k-1,i}(\zeta - z_+)^i, \end{aligned} \tag{3.38}$$

where the coefficients $g_{k,m}$ are all polynomials of $\frac{1}{c_3}$ with degree at least 2, and analytic functions of μ in the neighborhood of $(-\infty, M]$. Note that $g_{2k-1,-3k} = 0$. It then follows (3.29) that $Q_k(\zeta)$ possesses a pole of order n_k at $\zeta = z_+$. Using the above formula, we also have the Laurent series for $Q_k(\zeta)$ below

$$Q_k(\zeta) = \sum_{m=-n_k}^{+\infty} Q_k^{(m)}(\zeta - z_+)^m, \qquad |z - z_+| > 0, \tag{3.39}$$

with
(3.40)

$$Q_{2k}^{(m)} = \begin{pmatrix} \hat{t}_{2k} g_{2k,m} & 0 \\ 0 & t_{2k} g_{2k,m} \end{pmatrix}, \quad Q_{2k-1}^{(m)} = \begin{pmatrix} 0 & \hat{t}_{2k-1} g_{2k-1,m+1} \\ t_{2k-1} g_{2k-1,m} & 0 \end{pmatrix},$$

for all $m \geq -n_k$. From (3.31), we have

$$R_k(\zeta) = \frac{1}{2\pi i} \oint_{\partial U(z_+,\delta)} \frac{Q_k(s) + \sum_{m=0}^{k-1} R_{m,-}(s) Q_{k-m}(s)}{s-\zeta} ds. \tag{3.41}$$

A straightforward computation yields

$$R_k(\zeta) = \begin{cases} \sum_{m=1}^{n_k} \frac{R_k^{(-m)}}{(\zeta - z_+)^m}, & \zeta \in \mathbb{C} \setminus \overline{U(z_+,\delta)}, \\ \sum_{m=0}^{+\infty} R_k^{(m)} (\zeta - z_+)^m, & \zeta \in U(z_+,\delta), \end{cases} \tag{3.42}$$

with the coefficients given as

$$R_k^{(m)} = Q_k^{(m)} + \sum_{j=1}^{k-1} \left(\sum_{i=0}^{p_j+m} R_j^{(i)} Q_{k-j}^{(-i+m)} \right), \quad k \geq 2, \tag{3.43}$$

for all $m \geq -n_k$. As $g_{k,m}$ are all analytic functions of μ in the neighborhood of $(-\infty, M]$, by mathematical induction, we can conclude that $R_k^{(m)}$ are all polynomials of $\frac{1}{c_3}$ and analytic for $\mu \in (-\infty, M]$. In particular, we have

$$R_2^{(-1)} = Q_2^{(-1)} + R_1^{(0)} Q_1^{(-1)} + R_1^{(1)} Q_1^{(-2)} = Q_2^{(-1)} - Q_1^{(0)} Q_1^{(-1)} - Q_1^{(1)} Q_1^{(-2)}, \tag{3.44}$$

which gives us (3.34). This finishes the proof of the lemma. ∎

Remark 3.7. Besides (3.33) and (3.34), it is possible to get explicit formulas of $R_k^{(-1)}$ for all $k \geq 3$ with the aid of (3.40) and the recursion formula (3.43). Moreover, using (3.30) and (3.32), we get the following approximation for the $\frac{1}{\zeta}$-coefficient C_1 in (3.27):

$$C_1(x,t) \sim \sum_{k=1}^{\infty} \frac{R_k^{(-1)}}{x^{\frac{7k}{6}}}, \qquad \text{as } x \to +\infty. \tag{3.45}$$

Due to Lemma 3.5, one can see that $R_{2k}^{(-1)}$ and $R_{2k-1}^{(-1)}$ are diagonal and anti-diagonal matrices, respectively.

4. Proof of main results.

In the last section, we first derive uniform asymptotics for the tritronquée solution $u(x,t)$ in Theorem 2.1. Then, we establish the total integrals for $u(x,t)$ and the associated Hamiltonian $H_1(x,t)$ in Theorem 2.8.

Proof of Theorem 2.1: First, recalling (3.16) and (3.27), we have

$$u(x,t) = 2x^{\frac{1}{3}} \left(\frac{z_+}{4} + (C_1)_{11} \right) - x^{\frac{1}{3}} (C_1)_{12}^2 = \frac{z_+}{2} x^{\frac{1}{3}} + E(x,\mu). \tag{4.1}$$

It then follows from (3.45) and the above formula that

$$E(x,\mu) \sim 2x^{\frac{1}{3}}\left(\sum_{k=1}^{\infty}\frac{(R_{2k}^{(-1)})_{11}}{x^{\frac{7k}{3}}}\right) - x^{\frac{1}{3}}\left(\sum_{k=1}^{\infty}\frac{(R_{2k-1}^{(-1)})_{12}}{x^{\frac{(2k-1)7}{6}}}\right)^2, \qquad \text{as } x\to+\infty, \tag{4.2}$$

This gives us the asymptotic expansion (2.3) with the coefficients given below

$$e_k^+(\mu) := e_k(z_+,\mu) = 2(R_{2k}^{(-1)})_{11} - \sum_{i=1}^{k}(R_{2i-1}^{(-1)})_{12}(R_{2(k-i)+1}^{(-1)})_{12}, \quad k\ge 1. \tag{4.3}$$

When $k=1$, with the explicit expressions of $R_1^{(-1)}$ and $R_2^{(-1)}$ in (3.33) and (3.34), we obtain

$$e_1^+(\mu) = 2(R_2^{(-1)})_{11} - (R_1^{(-1)})_{12}^2 = -\frac{64}{3}(z_+^2-8\mu)^{-3} + \frac{256z_+^2}{3}(z_+^2-8\mu)^{-4}. \tag{4.4}$$

Next, we show that $e_k^+(\mu)$ are uniformly bounded for $\mu\in(-\infty,M]$. To see this, we recall Lemma 3.6, which indicates that all the coefficients $R_k^{(m)}$ with $k\in\mathbb{N}^+$ and $m=-n_k,-n_k+1,\cdots$, are polynomials of $\frac{1}{c_3}$. In particular, $\deg R_1^{(-1)}\ge 1$ and $\deg R_k^{(-1)}\ge 2$ for $k\ge 2$. This implies that, there exists a constant $\tilde{C}$, such that

$$|R_1^{(-1)}| \le \tilde{C}|(z_+^2-8\mu)^{-1}| \quad \text{and} \quad |R_k^{(-1)}|\le \tilde{C}(z_+^2-8\mu)^{-2}, \quad k\ge 2, \tag{4.5}$$

uniformly for all $\mu\in(-\infty,M]$. Substituting the above approximations into (4.3), we can see that all $e_k^+(\mu)$ are bounded uniformly for all $\mu\in(-\infty,M]$. Moreover, they satisfy the approximation (2.4) as $\mu\to-\infty$.

When $x\to-\infty$, the RH analysis is similar to the case when $x\to+\infty$. One only needs to replace z_+ by z_-; see the definitions of $z_\pm$ in (2.1). We have the asymptotic expansion (2.2), with z_+ replaced by z_-. The coefficients $e_k^-(\mu)$ share the same properties as $e_k^+(\mu)$.

This finishes the proof of the theorem.

Next, we will establish the total integrals.

Proof of Theorem 2.8: When t is fixed, in view of the asymptotics of $u(x,t)$ in (2.15), we add the factor $6^{\frac{1}{3}}x^{\frac{1}{3}}+2\cdot 6^{-\frac{1}{3}}tx^{-\frac{1}{3}}$ on both sides of (2.17). Then, integrating with respect to x from X_1 and X_2, we have

$$\begin{aligned}&\int_{X_1}^{X_2}\left(u(x,t)+\sqrt[3]{6}x^{\frac{1}{3}}+\frac{2t}{\sqrt[3]{6}}x^{-\frac{1}{3}}\right)dx\\ =&H_1(X_2,t)+\frac{3}{4}X_2^{\frac{4}{3}}+3\cdot 6^{-\frac{1}{3}}tX_2^{\frac{2}{3}} - H_1(X_1,t)-\frac{3}{4}X_1^{\frac{4}{3}}-3\cdot 6^{-\frac{1}{3}}tX_1^{\frac{2}{3}}.\end{aligned} \tag{4.6}$$

Taking the limits $X_1\to-\infty$ and $X_2\to+\infty$ and making use of the asymptotic behavior of $H_1(x,t)$ in (2.16), we immediately get (2.22).

Now, we turn to the total integral of $H_1(x,t)$, and consider $I(t)$ in (2.19) and its alternative form (2.21). According to [21, Remark 4.3], one can see that, for any fixed x,

$$u(x,t) = \frac{x}{t} + \mathcal{O}(-t)^{-4}, \qquad \text{as } t\to-\infty. \tag{4.7}$$

It then follows from (1.6) that $H_1(x,t) = \mathcal{O}(-t)^{-1}$ as $t \to -\infty$, hence $I(-\infty) = 0$. For any fixed t, let us rewrite (2.21) as

$$I(t) = -I(X_2,t) + I(X_1,t) - \frac{1}{24}\log|T_2| + \frac{1}{24}\log|T_1|, \tag{4.8}$$

where T_1 and T_2 are two negative constants chosen to satisfy the relation $T_1|X_1|^{-\frac{2}{3}} = T_2|X_2|^{-\frac{2}{3}}$, and

$$\begin{aligned} I(X_k,t) = &\int_{T_k}^{t} \left[\frac{H_2(X_k,\tau)}{3} + \frac{u_x(X_k,\tau)}{12}\right] d\tau \\ &+ \int_{-\infty}^{T_k} \left[\frac{H_2(X_k,\tau)}{3} + \frac{u_x(X_k,\tau)}{12} - \frac{1}{24\tau}\right] d\tau \end{aligned} \tag{4.9}$$

with $k = 1, 2$. Note that, to ensure the convergence at $\tau = -\infty$, an additional factor $\frac{1}{24\tau}$ is introduced in the above integrand.

Next, we study $I(X_2,t)$ and $I(X_1,t)$ as $X_2 \to +\infty$ and $X_1 \to -\infty$, respectively. Recall the asymptotics of $u_x(x,t)$ and $H_2(x,t)$ in (2.9) and (2.13), and the fact that both $z_\pm$ and μ depend on t. Let us slightly abuse the notation by setting $\mu = \tau|X_2|^{-\frac{2}{3}}$ and $z_+ := z_+(\tau|X_2|^{-\frac{2}{3}})$. As discussed in Remark 2.5, the error terms in (2.9) and (2.13) are still $o(1)$ after integrating with respect to t from $-\infty$ to any positive fixed constant. Then, we have

$$\begin{aligned} I(X_2,t) = &\frac{1}{3}\int_{-\infty}^{t} \left[\left(\frac{z_+^5}{320} + \frac{3z_+^2}{8} - \frac{\mu z_+^3}{8}\right)|X_2|^{\frac{5}{3}}\right] d\tau + \frac{1}{3}\int_{T_2}^{t} \left[\frac{z_+^2+8\mu}{(z_+^2-8\mu)^2}|X_2|^{-\frac{2}{3}}\right] d\tau \\ &+ \frac{1}{3}\int_{-\infty}^{T_2} \left[\frac{z_+^2+8\mu}{(z_+^2-8\mu)^2}|X_2|^{-\frac{2}{3}} - \frac{1}{8\tau}\right] d\tau + o(1), \qquad \text{as } X_2 \to +\infty. \end{aligned} \tag{4.10}$$

Introduce a change of variables $z = z_+(\tau|X_2|^{-\frac{2}{3}})$ in the above integral. Recalling $\mu = \frac{z_+^2}{24} + \frac{2}{z_+}$ (cf. (2.1)), we have

$$\frac{d\tau}{dz} = \frac{d\tau}{d\mu}\cdot\frac{d\mu}{dz} = |X_2|^{\frac{2}{3}}\left(\frac{z}{12} - \frac{2}{z^2}\right). \tag{4.11}$$

Based on the first paragraph in the proof of Lemma 3.1, we know

$$-480^{\frac{1}{3}} < z_+(\tau|X_2|^{-\frac{2}{3}}) < 0$$

for all $\tau \in (-\infty, t)$. This gives us $\frac{d\tau}{dz} < 0$ for all $\tau \in (-\infty,t)$ and $z_+(\tau|X_2|^{-\frac{2}{3}}) \to 0-$ as $\tau \to -\infty$. Let us introduce two more notations $z_0^+ = z_+(t|X_2|^{-\frac{2}{3}})$ and $\tilde{z}_2 = z_+(T_2|X_2|^{-\frac{2}{3}})$. Then, the integral (4.10) becomes

$$\begin{aligned} I(X_2,t) = &\frac{1}{3}|X_2|^{\frac{7}{3}}\int_0^{z_0^+} \left[\left(\frac{z^5}{320} + \frac{3z^2}{8} - \left(\frac{z^2}{24} + \frac{2}{z}\right)\frac{z^3}{8}\right)\right]\left(\frac{z}{12} - \frac{2}{z^2}\right) dz \\ &+ \frac{1}{3}\int_{\tilde{z}_2}^{z_0^+} \left[\frac{z^2+8\mu}{8z(z^2-8\mu)}\right] dz + \frac{1}{4}\int_0^{\tilde{z}_2} \left[\frac{1}{(z^2-8\mu)\mu}\right] dz + o(1). \end{aligned} \tag{4.12}$$

Using $\mu = \frac{z_+^2}{24} + \frac{2}{z_+}$ again to replace the variable μ in the above formula, we get

$$I(X_2,t) = \frac{1}{3}|X_2|^{\frac{7}{3}} \int_0^{z_0^+} \left[\left(\frac{z^5}{320} + \frac{3z^2}{8} - \left(\frac{z^2}{24} + \frac{2}{z}\right)\frac{z^3}{8}\right)\right]\left(\frac{z}{12} - \frac{2}{z^2}\right) dz \tag{4.13}$$

$$+ \frac{1}{3}\int_{\tilde{z}_2}^{z_0^+} \left[\frac{z^3+12}{4z(z^3-24)}\right] dz + \int_0^{\tilde{z}_2} \left[\frac{9z^2}{(z^3-24)(z^3+48)}\right] dz + o(1) \tag{4.14}$$

as $X_2 \to +\infty$. In a similar way, with $z_0^- = z_-(t|X_1|^{-\frac{2}{3}})$ and $\tilde{z}_1 = z_-(T_1|X_1|^{-\frac{2}{3}})$, we have

$$I(X_1,t) = \frac{1}{3}|X_1|^{\frac{7}{3}} \int_0^{z_0^-} \left[\left(\frac{z^5}{320} - \frac{3z^2}{8} - \left(\frac{z^2}{24} - \frac{2}{z}\right)\frac{z^3}{8}\right)\right]\left(\frac{z}{12} + \frac{2}{z^2}\right) dz \tag{4.15}$$

$$+ \frac{1}{3}\int_{\tilde{z}_1}^{z_0^-} \left[\frac{z^3-12}{4z(z^3+24)}\right] dz + \int_0^{\tilde{z}_1} \left[\frac{-9z^2}{(z^3+24)(z^3-48)}\right] dz + o(1) \tag{4.16}$$

as $X_1 \to -\infty$. Since $z_+(\mu) = -z_-(\mu)$ for any $\mu \in (-\infty, M]$, we get

$$\tilde{z}_1 = z_-(T_1|X_1|^{-\frac{2}{3}}) = -z_+(T_2|X_2|^{-\frac{2}{3}}) = -\tilde{z}_2$$

due to our choice $T_1|X_1|^{-\frac{2}{3}} = T_2|X_2|^{-\frac{2}{3}}$. This means that the third integral in (4.13) and (4.15) are actually the same. In the meantime, the second integral in (4.13) and (4.15) only differ by a $o(1)$-factor when $|X_{1,2}| \to \infty$ because $z_0^\pm \sim \mp 2\cdot 6^{\frac{1}{3}} + o(1)$. Then, when considering $I(X_1,t) - I(X_2,t)$, only the first integrals make a contribution. As a consequence, combining (4.8), (4.13) and (4.15), we have

$$\begin{aligned} I(t) =& \frac{|X_2|^{\frac{7}{3}}}{3}\int_0^{z_0^+} \frac{(z^3+24)(z^3+60)}{5760} dz \\ &+ \frac{|X_1|^{\frac{7}{3}}}{3}\int_0^{z_0^-} \frac{(z^3-24)(z^3-60)}{5760} dz + \frac{1}{36}\log\frac{|X_1|}{|X_2|} + o(1), \end{aligned} \tag{4.17}$$

as $X_1 \to -\infty$ and $X_2 \to +\infty$. Obviously, the remaining integrals in the above formula can be computed explicitly. When t is fixed, we further expand $z_0^+ = z_+(t|X_2|^{-\frac{2}{3}})$ and $z_0^- = z_-(t|X_1|^{-\frac{2}{3}})$ as $|X_{1,2}| \to \infty$ (cf. (2.14)) to obtain

(4.18)
$$\begin{aligned} I(t) =& \frac{9}{28}|X_1|^{\frac{7}{3}} + \frac{9}{5}6^{-\frac{1}{3}}t|X_1|^{\frac{5}{3}} + t^2|X_1| + \frac{6^{\frac{1}{3}}t^3}{3}|X_1|^{\frac{1}{3}} + \frac{1}{36}\log|X_1| \\ &- \left(\frac{9}{28}|X_2|^{\frac{7}{3}} + \frac{9}{5}6^{-\frac{1}{3}}t|X_2|^{\frac{5}{3}} + t^2|X_2| + \frac{6^{\frac{1}{3}}t^3}{3}|X_2|^{\frac{1}{3}} + \frac{1}{36}\log|X_2|\right) + o(1) \end{aligned}$$

as $X_1 \to -\infty$ and $X_2 \to +\infty$. The above approximation for the integral $I(t)$ does not really make sense unless we modify the integrand in (2.19) and replace $H_1(x,t)$ by $H_1(x,t) + \frac{3}{4}6^{\frac{1}{3}}x^{\frac{4}{3}} + 3\cdot 6^{-\frac{1}{3}}tx^{\frac{2}{3}} + t^2 + \frac{6^{\frac{1}{3}}t^3}{9}x^{-\frac{2}{3}} + \frac{x}{36(x^2+1)}$. More precisely, it follows from (2.19) and (4.18) that

(4.19)
$$\int_{X_1}^{X_2}\left(H_1(x,t) + \frac{3}{4}6^{\frac{1}{3}}x^{\frac{4}{3}} + 3\cdot 6^{-\frac{1}{3}}tx^{\frac{2}{3}} + t^2 + \frac{6^{\frac{1}{3}}t^3}{9}x^{-\frac{2}{3}} + \frac{x}{36(x^2+1)}\right) dx = o(1),$$

as $X_1 \to -\infty$ and $X_2 \to +\infty$. This formula gives us the desired total integral in (2.23), which finishes the proof of Theorem 2.8.

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Appendix A. The g-function with complex x.

Recalling Lemma 3.1 and Fig. 3, one can see that there exist positive constants ρ and c_0 such that

$$\begin{aligned} \operatorname{Re} g(\zeta) &< -c_0 < 0, \qquad \zeta \in \Sigma_{R,2} \cup \Sigma_{R,4}, \\ \operatorname{Re} g(\zeta) &> c_0 > 0, \qquad \zeta \in \Sigma_{R,1}, \end{aligned} \tag{A.1}$$

when $|x| > \rho$ with $\arg x = 0$, uniformly for all $t \le M|x|^{\frac{2}{3}}$. In this appendix, we consider the function $g(\zeta)$ defined in (3.7) as a function of x as well, more precisely,

$$g(\zeta) := g(\zeta; x) = g(\zeta; r, \varphi) \qquad \text{with } r = |x| \text{ and } \varphi = \arg x. \tag{A.2}$$

We will show that there exist $\rho_0 > 0$ and $\epsilon_0 > 0$ such that

$$\begin{aligned} \operatorname{Re} g(\zeta; r, \varphi) &< -\frac{1}{2} c_0, \qquad \zeta \in \Sigma_{R,2} \cup \Sigma_{R,4}, \\ \operatorname{Re} g(\zeta; r, \varphi) &> \frac{1}{2} c_0, \qquad \zeta \in \Sigma_{R,1} \end{aligned} \tag{A.3}$$

when $r > \rho_0$ and $\varphi \in [-\epsilon_0, \epsilon_0]$, uniformly for all $t \le M|x|^{\frac{2}{3}}$.

First, from the definition of z_+ in (2.1), we get

$$z_+ = -\left[(576 - 512\mu^3)^{\frac{1}{2}} + 24\right]^{\frac{1}{3}} - \frac{8\mu}{\left[(576 - 512\mu^3)^{\frac{1}{2}} + 24\right]^{\frac{1}{3}}}, \tag{A.4}$$

which is analytic in μ for $\mu \in \mathbb{C} \setminus \left[2^{-1} \cdot 3^{\frac{2}{3}}, +\infty\right)$. As $\mu = t x^{-\frac{2}{3}}$, this implies that z_+ is analytic with respect to x when $|x| > 1$ with $t \le M|x|^{\frac{2}{3}}$. Since $\mu < \frac{3}{80} z_+^2$ and $\arg c_3 = \arg\left(-\frac{\mu}{3} + \frac{z_+^2}{24}\right) = 0$ when $\arg x = 0$, there exist $\epsilon_1 > 0$ and $\rho_1 > 0$ such that $\arg c_3 \in [-\frac{1}{2}\theta_0, \frac{1}{2}\theta_0]$ for all $|x| > \rho_1$ and $\arg x \in [-\epsilon_1, \epsilon_1]$, where $0 < \theta_0 < \frac{2\pi}{7}$ is the constant discussed in Remark 3.2. Observing that $\arg(\zeta - z_+) = 0, \pi - \theta_0$ and $-\pi + \theta_0$ for $\zeta \in \Sigma_1, \Sigma_2$ and Σ_4, respectively, we have

$$\begin{aligned} &\arg\left(c_3(\zeta - z_+)^{\frac{3}{2}}\right) \in \left[\frac{3\pi}{2} - 2\theta_0, \frac{3\pi}{2} - \theta_0\right], \qquad \zeta \in \Sigma_{R,2}, \\ &\arg\left(c_3(\zeta - z_+)^{\frac{3}{2}}\right) \in \left[-\frac{3\pi}{2} + \theta_0, -\frac{3\pi}{2} + 2\theta_0\right], \qquad \zeta \in \Sigma_{R,4}, \\ &\arg\left(c_3(\zeta - z_+)^{\frac{3}{2}}\right) \in \left[-\frac{1}{2}\theta_0, \frac{1}{2}\theta_0\right], \qquad \zeta \in \Sigma_{R,1}, \end{aligned} \tag{A.5}$$

when $|x| > \rho_1$ with $\arg x \in [-\epsilon_1, \epsilon_1]$, uniformly for all $t \le M|x|^{\frac{2}{3}}$. This yields

$$\begin{aligned} \operatorname{Re}\left(c_3(\zeta - z_+)^{\frac{3}{2}}\right) &< 0, \qquad \zeta \in \Sigma_{R,2} \cup \Sigma_{R,4}, \\ \operatorname{Re}\left(c_3(\zeta - z_+)^{\frac{3}{2}}\right) &> 0, \qquad \zeta \in \Sigma_{R,1}. \end{aligned} \tag{A.6}$$

when $|x| > \rho_1$ with $\arg x \in [-\epsilon_1, \epsilon_1]$.

With the explicit formula of z_+ in (A.4), one can see z_+ is bounded when $|x| \geq \rho_1$ and $t \leq M|x|^{\frac{2}{3}}$ for $\arg x \in [-\epsilon_1, \epsilon_1]$. By (3.8), c_2 is also bounded. Then, since $c_1(\zeta - z_+)^{\frac{7}{2}} + c_2(\zeta - z_+)^{\frac{5}{2}} \sim c_1\zeta^{\frac{7}{2}}$ as $\zeta \to \infty$, there exists $\delta_1 > 0$, independent of x and t, such that, when $|\zeta| \geq \delta_1$,

$$\begin{aligned} &\operatorname{Re}\left(c_1(\zeta - z_+)^{\frac{7}{2}} + c_2(\zeta - z_+)^{\frac{5}{2}}\right) < -\frac{1}{2}c_0, \quad \zeta \in \Sigma_{R,2} \cup \Sigma_{R,4}, \\ &\operatorname{Re}\left(c_1(\zeta - z_+)^{\frac{7}{2}} + c_2(\zeta - z_+)^{\frac{5}{2}}\right) > \frac{1}{2}c_0, \quad \zeta \in \Sigma_{R,1}. \end{aligned} \tag{A.7}$$

When $|\zeta| \leq \delta_1$, since $g(\zeta; x)$ is analytic with respect to both ζ and x in

$$\Omega = \{\arg(\zeta - z_+) \in (-\pi, \pi), |x| > \rho_1, \arg x \in (-\epsilon_1, \epsilon_1)\},$$

then $g(\zeta; x) = g(\zeta; r, \varphi)$ is continuous with respect to $\varphi = \arg x$. Hence, there exists $\epsilon_2 > 0$ such that

$$|\operatorname{Re} g(\zeta; r, \varphi) - \operatorname{Re} g(\zeta; r, 0)| \leq |g(\zeta; r, \varphi) - g(\zeta; r, 0)| \leq \frac{c_0}{2} \tag{A.8}$$

for any $\varphi \in [-\epsilon_2, \epsilon_2]$. It then follows that, when $\varphi \in [-\epsilon_2, \epsilon_2]$,

$$\begin{aligned} &\operatorname{Re} g(\zeta; r, \varphi) < \operatorname{Re} g(\zeta; r, 0) + \frac{c_0}{2}, \qquad \zeta \in \Sigma_{R,2} \cup \Sigma_{R,4}, \\ &\operatorname{Re} g(\zeta; r, \varphi) > \operatorname{Re} g(\zeta; r, 0) - \frac{c_0}{2}, \qquad \zeta \in \Sigma_{R,1}. \end{aligned} \tag{A.9}$$

Set $\rho_0 = \rho_1$ and $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, then a combination of (A.6), (A.7) and (A.9) gives us (A.3).

Remark A.1. With (A.3), we can show that the whole RH analysis can be successfully conducted when $|x| \to \infty$ with $\arg x \in [-\epsilon_0, \epsilon_0]$. When $|x| \to \infty$ with $\arg x \in [3\pi - \epsilon_0, 3\pi + \epsilon_0]$, the analysis is similar except for replacing z_+ by z_- and M_+ by M_-, respectively. Hence, from (4.1) and (4.2), we have

$$\begin{aligned} &E(x, \mu) = \mathcal{O}((z_+^2 - 8\mu)^{-2}|x|^{-2}), \qquad |x| \to \infty, \ \arg x \in [-\epsilon_0, \epsilon_0], \\ &E(x, \mu) = \mathcal{O}((z_-^2 - 8\mu)^{-2}|x|^{-2}), \qquad |x| \to \infty, \ \arg x \in [3\pi - \epsilon_0, 3\pi + \epsilon_0]. \end{aligned} \tag{A.10}$$

uniformly for $t \leq M|x|^{\frac{2}{3}}$.

Appendix B. Proof of Corollary 2.3. To obtain the asymptotic approximation for $u_x(x,t), u_{xx}(x,t)$ and $u_{xxx}(x,t)$ in Corollary 2.3, we need to show

$$\frac{d^k E(x, \mu(x))}{dx^k} = \mathcal{O}((z_\pm^2 - 8\mu(x))^{-2}|x|^{-2-k}), \qquad x \to \pm\infty, \ k = 1, 2, 3, \tag{B.1}$$

uniformly for all $t \leq M|x|^{\frac{2}{3}}$. Note that $z_\pm$ and μ are functions of x. Since (A.10) holds for $\arg x \in [-\epsilon_0, \epsilon_0]$ and $\arg x \in [3\pi - \epsilon_0, 3\pi + \epsilon_0]$, we introduce a circle $\mathcal{C}_\pm$ (when $\pm x$ is large enough) in the complex plane, centered at $z = x$ with radius $\tilde{r} = \frac{1}{2}|x| \sin \epsilon_0$. Then, according to the Cauchy's integral formula, we have

(B.2)

$$\begin{aligned} \left|\frac{d^k E(x, \mu(x))}{dx^k}\right| &= \left|\frac{1}{2\pi i} \int_{\mathcal{C}_\pm} \frac{E(z, \mu(z))}{(z - x)^{1+k}} dz\right| \\ &\leq \frac{2}{\pi |x|^{1+k} \sin^2 \epsilon_0} \max_{z \in \mathcal{C}_\pm}\{|E(z, \mu(z))|\} = \mathcal{O}((z_\pm^2 - 8\mu(x))^{-2}|x|^{-2-k}) \end{aligned}$$

as $x \to \pm\infty$.

Finally, differentiating both sides of (2.2) and keeping in mind that $z_{\pm}$, μ are both functions of x, we obtain (2.9)-(2.11).

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