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Strong Cosmic Censorship with bounded curvature

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Abstract

In this paper we propose a weaker version of Penrose's much heeded Strong Cosmic Censorship (SCC) conjecture, asserting inextendability of maximal Cauchy developments by manifolds with Lipschitz continuous Lorentzian metrics and Riemann curvature bounded in L^p. Lipschitz continuity is the threshold regularity for causal structures, while curvature bounds rule out infinite tidal accelerations, arguing for physical significance of this weaker SCC conjecture. The main result of this paper, under the assumption that no extensions exist with higher connection regularity $W^{1,p}_{\text{loc}}$, proves in the affirmative this *SCC conjecture with bounded curvature* for p sufficiently large, ($p > 4$ to address uniform bounds, $p > 2$ without uniform bounds).

Keywords: Strong Cosmic Censorship, optimal metric regularity, Lipschitz continuous metrics, uniform curvature bounds, elliptic partial differential equations

1. Introduction

In this paper we propose and address a weaker form of Penrose's *Strong Cosmic Censorship* (SCC) conjecture, subject to bounded curvature. For this we consider solutions of the Einstein equations of General Relativity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

where $g_{\mu\nu}$ is a Lorentzian metric on spacetime, $R_{\mu\nu}$ its Ricci tensor, R its scalar curvature, $T_{\mu\nu}$ the energy momentum tensor of matter fields and $\Lambda \in \mathbb{R}$ the cosmological constant [11]. We begin by stating Penrose's original Strong Cosmic Censorship conjecture [25], which is deeply related to the question of whether the Einstein equations are deterministic and which has been subject of a large number of investigations over the last 25 years, (see [1, 3, 5, 6, 8–10, 13, 14, 20, 22–24, 26, 37] and references therein).

Conjecture 1.1 (penrose's SCC conjecture). For generic compact, asymptotically flat or asymptotically anti-de Sitter initial data, the maximal Cauchy development of (1.1) is inextendable as a manifold with Lorentzian metric in C^0.

Identifying a suitable notion of 'genericity' of data (beyond the absence of symmetries) is part of the problem, and recent work of Kehle shows validity of the SCC conjecture may depend on the notion of genericity [20]. For the purposes of this paper, we can leave the notion of genericity of data open, and we do not need to specify the matter fields in (1.1). We refer from now on to generic compact, asymptotically flat or asymptotically anti-de Sitter data simply as *generic data*.

Dafermos and Luk recently disproved conjecture 1.1 in the case $\Lambda \geqslant 0$, under the assumption that the exterior Kerr spacetime is stable [9]. The following weaker form of Penrose's SCC conjecture, introduced by Christodoulou [4], remains open.

Conjecture 1.2 (christodoulou's SCC conjecture). For generic data, the maximal Cauchy development of (1.1) is inextendible as a manifold with C^0 Lorentzian metric and metric connection Γ locally in L^2.

The L^2 connection regularity of conjecture 1.2 is the minimal regularity for which weak solutions of the Einstein equations can be introduced, and is therefore a well motivated lower threshold regularity for physical significance of extensions. In this paper we propose the following weaker form of conjectures 1.1 and 1.2, addressing Lipschitz continuous ($C^{0,1}$) Lorentzian metrics subject to bounded curvature.

Conjecture 1.3 (SCC conjecture with Lp curvature bound). For generic data, the maximal Cauchy development of (1.1) is inextendible as a manifold with $C^{0,1}$ Lorentzian metric and Riemann curvature locally in L^p, some $p \geqslant 2$.

Conjecture 1.3 entertains the possibility that suitable curvature bounds could serve as a criterion for physical relevance of extensions of maximal Cauchy developments, keeping in mind that boundedness of the Riemann curvature in L^∞ rules out infinite tidal accelerations by the Jacobi equations of geodesic deviation; and this interpretation extends when measuring boundedness in the sense of the L^p integral norm[1]. Moreover, Lipschitz continuity of Lorentzian metrics has been established by Chruciel and Grant as the minimal regularity for causal structures to be well-defined [7, theorem 1.20], which places Lipschitz continuity of Lorentzian metrics as a threshold regularity for physical significance[2]. We do not speculate in this paper whether conjecture 1.3 is true or false. Instead, the main results of this paper (theorems 2.1 and 2.2 below) provide a step for establishing conjecture 1.3 in the affirmative. This is summarized in a non-technical way as follows:

[1] Note that existence of geodesic curves, and hence the Jacobi equation, can be made sense of in a standard way for connections in L^p when the curvature is bounded in L^p [36].

[2] Inextendability with Lipschitz metrics was recently also addressed in [15, 38].

Main Result (non-technical statement): Assume a maximal Cauchy development $(\mathcal{M}, g)$ of (1.1) is inextendable as a Lorentzian manifold with metric connection Γ locally in $W^{1,p}$, for some $p > 2$, $\mathcal{M}$ four dimensional[3]. Then, by theorem 2.1, $\mathcal{M}$ is inextendable as a manifold with Lorentzian metric in $C^{0,1}$ and Riemann curvature locally bounded in L^p. Moreover, by theorem 2.2, 'generic' families of extensions with uniform $C^{0,1}$-bounds on metrics and uniform L^p-bounds on their curvature are ruled out, assuming no extensions with uniform $W^{1,p}$-bounds on connections exist, $p > 4$.

Because proving inextendability by hyperbolic PDE methods is more feasible at higher regularity [23], the results in this paper provide a useful step in validating conjecture 1.3. Since $W^{1,p}$ is contained in $W^{1,2}$ on bounded domains for any $p \geqslant 2$, theorems 2.1 and 2.2 can readily be combined with results on the Cauchy problem with L^2 based Sobolev spaces which, at the forefront of lowest regularities [21], fit to our incoming assumptions. Our main results do not depend on a specific notion of 'genericity' for the data, (which is only relevant for establishing the inextendability at higher regularity, assumed in theorems 2.1 and 2.2, by hyperbolic PDE methods applied to the Einstein equations), making our results more feasible to apply. The proof of theorems 2.1 and 2.2 is given in section 4. It is based on the author's joint work with Blake Temple in [33], where we prove that affine connections can always be regularized by coordinate transformation to one derivative above their Riemann curvature in L^p. Our result in [33] is based on the discovery of a system of elliptic PDE's which determines the regularizing transformation, (as summarized in section 3).

2. Statement of results

Let $\mathcal{O}$ be some open set of generic data. Each datum in $\mathcal{O}$ is a tuple of a Riemannian metric h_θ on a 3-D manifold Σ and a symmetric $(0,2)$-tensor K_θ [2], (K_θ is the second fundamental form when embedding Σ into spacetime), which we represent as $\theta \in \mathcal{O}$ for brevity. To each datum $\theta \in \mathcal{O}$, we consider the maximal Cauchy development $(\mathcal{M}_\theta, g_\theta)$ of the Einstein equation (1.1), which is a 4-dimensional manifold $\mathcal{M}_\theta$ endowed with a Lorentzian metric g_θ that solves (1.1). We denote extensions of $(\mathcal{M}_\theta, g_\theta)$ by $(\mathcal{M}^{\text{ext}}_\theta, g_\theta)$. To state our main theorems, we now introduce what we call *bubble extensions*. That is, we assume each extension $(\mathcal{M}^{\text{ext}}_\theta, g_\theta)$ is a manifold $\mathcal{M}^{\text{ext}}_\theta$ endowed with a Lorentzian metric g_θ, such that $\mathcal{M}^{\text{ext}}_\theta \setminus \mathring{\mathcal{M}}_\theta$ can be covered by a single coordinate chart $(\Omega_\theta, x^\mu_\theta)$ of $\mathcal{M}^{\text{ext}}_\theta$. Assume further one can identify all such charts Ω_θ with some open set $\Omega \subset \mathbb{R}^4$, bounded with smooth boundary, on which a coordinate system x^μ is given, and we assume one can identify all coordinates x^μ_θ with x^μ. We refer to each such extension $\mathcal{M}^{\text{ext}}_\theta$ as a *bubble extension* and to $\left(\mathcal{M}^{\text{ext}}_\theta\right)_{\theta \in \mathcal{O}}$ as a *set of bubble extensions*. Note that only the metric regularity of local extensions is of relevance for our main result, but not that of the metrics g_θ on $\mathcal{M}_\theta$, which can be taken as weak solutions of (1.1) with $g_\theta \in W^{1,2}_{\text{loc}}$. We now state the two main results of this paper.

Theorem 2.1. *Let $(\mathcal{M}_\theta, g_\theta)$ be the maximal Cauchy development of some datum $\theta \in \mathcal{O}$. Assume $\mathcal{M}_\theta$ is inextendable in the sense that no bubble extension exists with connection $\Gamma_\theta \in W^{1,p}(\Omega)$, some $p > 2$. Then no bubble extension $\mathcal{M}^{ext}_\theta$ exists with Hölder continuous*

[3] The Sobolev space $W^{1,p}$ of functions with L^p integrable weak derivatives is introduced in section 2.1.

Lorentzian metric $g_\theta \in C^{0,\alpha}(\Omega)$*, metric connection* $\Gamma_\theta \in L^{2p}(\Omega)$ *and Riemann curvature* $\mathrm{Riem}(\Gamma_\theta) \in L^p(\Omega)$[4].

Theorem 2.1 rules out extensions with Lipschitz continuous metrics, as asserted in its non-technical statement in section 1, because Lipschitz continuous functions are also Hölder continuous, and since derivatives of Lipschitz continuous functions are always in L^∞ and thus L^p integrable over bounded domains for any $p \leqslant \infty$, see [12, chapter 5.8]. The incoming assumption of theorem 2.1 is rather strong, and is indeed incorrect on the interior Kerr spacetime [17]. For this reason it is more relevant to consider families of extensions of generic data $\mathcal{O}$ subject to uniform bounds on their metrics independent of $\theta \in \mathcal{O}$. Our refinement of theorem 2.1, establishing such uniform bounds, requires the stronger assumption that $p > 4$, and implies inextendability with Lipschitz continuous metrics with L^p bounded curvature.

Theorem 2.2. *Consider a set of generic data* $\mathcal{O}$ *with maximal Cauchy developments* $(\mathcal{M}_\theta, g_\theta)$ *for each* $\theta \in \mathcal{O}$*. Assume the Cauchy developments* $\mathcal{M}_\theta$ *are inextendable in the sense that no set of bubble extensions exists with connections* $\Gamma_\theta \in W^{1,p}(\Omega)$*, some* $p > 4$*, subject to the uniform bound*

$$\|\Gamma_\theta\|_{W^{1,p}(\Omega)} \leqslant C, \tag{2.1}$$

for all $\theta \in \mathcal{O}$*, for some constant* $C > 0$ *independent of* $\theta \in \mathcal{O}$*. Then no set of bubble extensions* $\left(\mathcal{M}^{ext}_\theta, g_\theta\right)_{\theta\in\mathcal{O}}$ *exists, with Lorentzian metrics* $g_\theta \in C^{0,1}(\Omega)$ *and connections* $\Gamma_\theta \in L^\infty(\Omega)$*, subject to the uniform bound*[5]

$$\|g_\theta\|_{L^\infty(\Omega)} + \|\Gamma_\theta\|_{L^\infty(\Omega)} + \|\mathrm{Riem}\,(\Gamma_\theta)\,\|_{L^p(\Omega)} \leqslant M, \tag{2.2}$$

for all $\theta \in \mathcal{O}$*, for some constant* $M > 0$ *independent of* θ*.*

2.1. Coordinate based Sobolev norms and spaces

We denote with $W^{m,p}(\Omega)$ the Sobolev space of functions with weak derivatives in $L^p(\Omega)$, (the space of functions whose p-th power is integrable), up to order m, ($m \geqslant 0$, $1 \leqslant p \leqslant \infty$). We say a tensor or a connection is in $W^{m,p}(\Omega)$ in some coordinate chart $\Omega \subset \mathbb{R}^4$, if all its components are functions in $W^{m,p}(\Omega)$ in their respective coordinate representation. Correspondingly, we take Sobolev norms $\|\cdot\|_{W^{m,p}(\Omega)}$ component-wise on tensors and connections in a coordinate representation, based on a fixed coordinate system x. (Usually x denotes the coordinate system of a bubble extension, unless otherwise stated.) For example, the $W^{1,p}$-norm on connection components Γ is

[4] By Christoffel's formula a metric tensor is always exactly one derivative more regular than its connection, e.g. L^p connection regularity implies its metric is in $W^{1,p}$, (the Sobolev space of functions with first derivatives in L^p, cf section 2.1). By Morrey's inequality, a function in $W^{1,p}$ with $p > n$, (n denoting the dimension of its domain), is Hölder continuous with exponent $\alpha = 1 - \frac{n}{p}$. Thus, in theorem 2.1, $\Gamma_\theta \in L^{2p}(\Omega)$ implies $g_\theta \in W^{1,2p}$, which in turn implies $g_\theta \in C^{0,\alpha}(\Omega)$ for $\alpha = 1 - \frac{2}{p}$.

[5] Since metric derivatives and connection components are in one-to-one correspondence by Christoffel's formula, it follows that the L^∞ bound on g_θ and Γ_θ in (2.2) is equivalent to a $W^{1,\infty}$ bound on g_θ, which in turn is equivalent to a Lipschitz bound on g_θ, see [12, chapter 5.8]. A physical interpretation of the bounds in (2.2) is that the L^p norm on the connection gives upper bounds on the gravitational forces, and the L^p norm on the curvature gives upper bounds on the tidal accelerations, both as measured by the observer with coordinate system x and averaged out in the L^p sense over a spacetime region. However, since the L^p norms are not invariant, the values of these norms as well as the constant M do not appear to have any interpretation as physical scalars like charge or mass.

$$\|\Gamma\|_{W^{1,p}(\Omega)} \equiv \|\Gamma\|_{L^p(\Omega)} + \sum_{\rho=0,\ldots,3} \|\partial_\rho \Gamma\|_{L^p(\Omega)}, \|\Gamma\|_{L^p(\Omega)} \equiv \sum_{\sigma,\mu,\nu} \|\Gamma^\sigma_{\mu\nu}\|_{L^p(\Omega)}, \tag{2.3}$$

where $\partial_\rho \equiv \frac{\partial}{\partial x^\rho}$ denotes partial differentiation in x-coordinates, taken component-wise on tensors and connections, and integration is taken with respect to the volume element of the Euclidean metric in x-coordinates. Note, $W^{m,p}$ regularity of tensors is preserved under $W^{m+1,p}$ coordinate transformations, and $W^{m,p}$ regularity of connections is preserved under $W^{m+2,p}$ coordinate transformations. However, the value of $W^{m,p}$-norms is coordinate dependent, a problematic issue in Lorentzian geometry circumvented here by restricting consideration to bubble extensions.

2.2. *Strategy of proof*

The proof of theorems 2.1 and 2.2 is based on our optimal regularity result in [33]. We now explain the idea of proof for theorem 2.2. Assume for contradiction there exist a bubble extension $\left(\mathcal{M}^{\text{ext}}_\theta\right)_{\theta\in\mathcal{O}}$ with metric connections $\Gamma_\theta \in L^\infty(\Omega)$ and curvature $\text{Riem}(\Gamma_\theta) \in L^p(\Omega)$ subject to the uniform bound (2.2). Then the optimal regularity result in [33] implies that, locally, there exists a coordinate transformation which maps Γ_θ to optimal regularity, $\Gamma_\theta \in W^{1,p}$, with uniform $W^{1,p}$ bounds in the new coordinate system. This can then be shown to contradict our assumption in theorem 2.2 that no such extension with $\Gamma_\theta \in W^{1,p}$ exists. We give the detailed proofs of theorems 2.1 and 2.2 in section 4. For this, we state in section 3 our optimal regularity result in [33] (as theorem 3.1), and outline its proof based on the *Regularity Transformation (RT-) equations*, a system of partial differential equations (PDE's), *elliptic* regardless of metric regularity.

2.3. *Remark on the Einstein equations*

The Einstein equations (1.1) play a formal role in this paper, used to have context for the notion of maximal Cauchy development. For this reason, and since the results in [33] are independent of the Einstein equations, we do not need to specify the sources in (1.1). Moreover, we do not require the $W^{1,p}$ extensions in theorems 2.1 and 2.2 to be solutions of (1.1), which again is possible because the results in [33] are independent of the Einstein equations. However, one can built in the constraint that the extensions solve the Einstein equations. That is, assuming no $W^{1,p}$ extensions exist as solutions of (1.1), it follows that no extensions exist with L^∞ connections and L^p curvature which solve (1.1), (we address here the connection regularity of theorem 2.2 as an example), because the regularizing coordinate transformation provided by the RT-equations [33, theorem3.1] maps solutions of (1.1) again to solutions of (1.1), cf section 2.2. In this way one can include the Einstein equations into the statements of theorems 2.1 and 2.2.

2.4. *Remark on higher regularities and dimensions*

Our earlier results in [31] yield optimal regularity at higher levels of Sobolev regularity, furnishing coordinate transformations which map non-optimal connections in $W^{m,p}$ with Riemann curvature in $W^{m,p}$, $(m \geqslant 1,\ p > 4)$, to optimal connection regularity $W^{m+1,p}$. By applying this results in [31], one can extend theorems 2.1 and 2.2 to assert inextendability of maximal Cauchy developments to Lorentzian manifolds with connections of regularity $W^{m,p}$ and Riemann curvature bounded in $W^{m,p}$, under the assumption of inextendability to Lorentzian manifolds with connections of regularity $W^{m+1,p}$, $(m \geqslant 1, p > 4)$. Moreover, since the results in [31, 33] hold for any dimension $n \geqslant 2$, theorems 2.1 and 2.2 extend to spacetimes of higher dimensions, assuming $p > n/2$ in theorem 2.1 and $p > n$ in theorem 2.2.

3. Optimal regularity in Lorentzian geometry by the RT-equations

We now introduce the optimal regularity result [33, theorem 3.1], due to Blake Temple and the author, on which the proof of theorems 2.1 and 2.2 is based. To state the theorem, consider a fixed chart (Ω, x) on a 4-dimensional manifold $\mathcal{M}$, such that $\Omega_x \equiv x(\Omega) \subset \mathbb{R}^4$, (the image of Ω under the coordinate map), is open and bounded with smooth boundary[6]. Let Γ_x denote the collection of components of an affine connection Γ in x-coordinates, $\Gamma_x \equiv \Gamma^k_{ij}(x)$. We view Γ_x as a matrix valued 1-form in x-coordinates, $(\Gamma_x)^\mu_\nu \equiv (\Gamma_x)^\mu_{\nu j} dx^j$. Let $d\Gamma_x$ denote its exterior derivative, $d(\Gamma_x)^\mu_\nu \equiv \partial_i (\Gamma_x)^\mu_{\nu j} dx^i \wedge dx^j$, (using μ, ν to denote matrix indices). Writing the Riemann curvature tensor as a matrix valued 2-form, $\mathrm{Riem}(\Gamma_x) = d\Gamma_x + \Gamma_x \wedge \Gamma_x$, it follows that the assumption $\Gamma_x \in L^{2p}(\Omega_x)$ and $\mathrm{Riem}(\Gamma_x) \in L^p(\Omega_x)$ is equivalent to the assumption $\Gamma_x \in L^{2p}(\Omega_x)$ and $d\Gamma_x \in L^p(\Omega_x)$; we henceforth assume the latter. We now state the version of theorem 3.1 in [33] relevant for this paper.

Theorem 3.1. *Assume $\Gamma_x \in L^{2p}(\Omega_x)$ and $d\Gamma_x \in L^p(\Omega_x)$ in x-coordinates, for some $p > 2$. Let $M > 0$ be a constant such that*

$$\|\Gamma_x\|_{L^{2p}(\Omega_x)} + \|d\Gamma_x\|_{L^p(\Omega_x)} \leqslant M. \tag{3.1}$$

Then for any point $q \in \Omega$ there exists a neighborhood $\Omega' \subset \Omega$ of q, and a coordinate transformation $x \to y$ with Jacobian $J = \frac{\partial y}{\partial x} \in W^{1,2p}(\Omega'_x)$, such that the connection components Γ_y in y-coordinates exhibit optimal regularity $\Gamma_y \in W^{1,p}(\Omega'_y)$, where $\Omega_y \equiv y(\Omega)$. Moreover, Γ_y satisfies the uniform bound

$$\|\Gamma_y\|_{W^{1,p}(\Omega'_y)} \leqslant C(M), \tag{3.2}$$

and the Jacobian J and its inverse J^{-1}, both measured in x-coordinates, satisfy

$$\|J\|_{W^{1,2p}(\Omega'_x)} + \|J^{-1}\|_{W^{1,2p}(\Omega'_x)} \leqslant C(M), \tag{3.3}$$

for some constant $C(M) > 0$ depending only on Ω', p and M. Furthermore, if $p > 4$, and if $\|\Gamma_x\|_{L^\infty(\Omega_x)} \leqslant M$ in addition to the curvature bound (3.1), then the Euclidean volume of Ω'_x is bounded from below by $1/M$[7].

Theorem 3.1 in [33] applies to general (affine) connections on the tangent bundle of an n-dimensional manifold $\mathcal{M}$ with L^p bounded curvature, and extends the classical optimal regularity result of Kazdan–DeTurck [19] from Riemannian to Lorentzian metrics and to affine connections. The proof of theorem 3.1 in [33] was a long time coming [29–32], motivated by earlier work on non-optimal Lorentzian metrics of shock wave solutions of the Einstein–Euler equations [16, 18, 27, 28], all summarized in the RSPA article [35], (including the extension to vector bundles and Yang–Mills gauge theories in [34]). The main idea for establishing theorem 3.1 was to derive from the connection transformation law a non-invariant system of *elliptic* PDE's on the regularizing Jacobian J as an unknown, an idea motivated by the Riemann-flat

[6] Our use here of Ω, as a chart on $\mathcal{M}$, slightly differs from the use of $\Omega \subset \mathbb{R}^4$ in section 2, (where we identified $\Omega \equiv \Omega_x \subset \mathbb{R}^4$), but this ambiguity is irrelevant since our result and methods are local.

[7] The neighborhood Ω' can be taken as $\Omega' = \Omega \cap B_r(q)$, where $B_r(q)$ is the Euclidean ball of radius r in x-coordinates. The radius r is uniform of the order of $1/M$, as long as $\|\Gamma\|_{L^\infty(\Omega)} \leqslant M$.

condition in [29]. This idea lead to the formulation of the *RT-equations* in [30], whose solutions furnish the regularizing transformation.

3.1. Derivation of the RT-equations

We first give the main steps in the derivation of the RT-equations, and then explain how they furnish optimal regularity. For this, assuming there exists a coordinate transformation with Jacobian J mapping Γ_x to Γ_y (the connection of optimal regularity), we write the connection transformation law as

$$\tilde{\Gamma} = \Gamma - J^{-1}dJ, \tag{3.4}$$

where $\Gamma \equiv \Gamma_x$ and $\tilde{\Gamma}^k_{ij} = (J^{-1})^k_\gamma J^\alpha_i J^\beta_j (\Gamma_y)^\gamma_{\alpha\beta}$ is the connection Γ_y transformed as a tensor to x-coordinates. Then differentiating (3.4) by the exterior derivative d and co-derivative δ, (3.4) implies after careful organization the following two equations

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta\left(dJ^{-1}\wedge dJ\right) + d\delta\tilde{\Gamma}, \tag{3.5}$$

$$\Delta J = \delta(J\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - J\delta\tilde{\Gamma}, \tag{3.6}$$

where $\Delta \equiv \delta d + d\delta = \partial^2_{x^0} + \ldots + \partial^2_{x^3}$ is the Euclidean Laplacian, $\langle\cdot\,;\cdot\rangle$ is a matrix-valued inner product and $\wedge$ the wedge product on matrix valued differential forms, (see [30, chapter 3] or [33, chapter 5] for detailed definitions). At the current stage, equations (3.5) and (3.6) neither appear solvable, nor would it be clear whether a solution J would be a true Jacobian integrable to coordinates, i.e. satisfying $\mathrm{Curl}(J) = 0$. To overcome this obstacle, we view $A \equiv \delta\tilde{\Gamma}$ as a free matrix valued parameter function—a choice which appears plausible by the Riemann-flat condition for optimal regularity in [29] which only involves $d\tilde{\Gamma}$, but not $\delta\tilde{\Gamma}$. Substituting $A \equiv \delta\tilde{\Gamma}$ in (3.5) and (3.6), and viewing A as a new unknown matrix valued function, we next impose on equation (3.6) the condition $\mathrm{Curl}(J) = 0$ for integrability, in its equivalent form $d\vec{J} = 0$ on the vectorization $\vec{J}^\mu = J^\mu_\nu \mathrm{d}x^\nu$ of J so that $\mathrm{Curl}(J) \equiv d\vec{J}$. After careful organization, involving a fortuitous cancellation by which the regularity of terms in the equations match up, the computations in [30] lead to the *RT-equations*, the following solvable systems of PDE's which is elliptic regardless of metric signature:

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta\left(dJ^{-1}\wedge dJ\right) + d\left(J^{-1}A\right), \tag{3.7}$$

$$\Delta J = \delta(J\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \tag{3.8}$$

$$d\vec{A} = \overrightarrow{\mathrm{div}}\,(dJ\wedge\Gamma) + \overrightarrow{\mathrm{div}}\,(J\,d\Gamma) - d\left(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\right), \tag{3.9}$$

$$\delta\vec{A} = v. \tag{3.10}$$

Equation (3.9) on the auxiliary field A results from imposing $d\vec{J} = 0$ on (3.8), and one can prove integrablity of J to follow from the coupled equations (3.8) and (3.9). The unknowns $(\tilde{\Gamma}, J, A)$ in the RT-equations, together with the given non-optimal connection components Γ, are viewed as matrix valued differential forms. Arrows denote 'vectorization', mapping matrix valued 0-forms to vector valued 0-forms, (e.g. $\vec{A}^\mu = A^\mu_i\,\mathrm{d}x^i$) and $\overrightarrow{\mathrm{div}}$ is a divergence operation which maps matrix valued k-forms to vector valued k-forms. The vector v in (3.10) is free to impose, representing a 'gauge'-type freedom in the equations, reflecting the fact that smooth transformations preserve optimal connection regularity. The operations on the right hand side are formulated in terms of the Cartan Algebra of matrix valued differential forms based on the Euclidean metric in x-coordinates, (see [30] for detailed definitions and proofs).

3.2. How the RT-equations yield optimal regularity

We now explain how solutions of the RT-equations furnish the coordinate transformations to optimal regularity, on which the proof of theorem 3.1 in [33] is based. It was because our earlier iteration scheme, for solving the RT-equations at higher regularity in [31, 32], did not close at the low regularity of L^p connections, due to the non-linear term $dJ^{-1} \wedge dJ$ in (3.7), that we eventually discovered internal 'gauge'-type transformations on solutions of the RT-equations (3.7)–(3.10), by which one can separate off (3.7) from the remaining equations. This latter system, which turned out to be linear in the unkowns (J, B), is what we refer to as the *reduced* RT-equations [33],

$$\Delta J = \delta(J \cdot \Gamma) - B, \tag{3.11}$$

$$d\vec{B} = \overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J\, d\Gamma), \tag{3.12}$$

$$\delta\vec{B} = v'. \tag{3.13}$$

Our iteration scheme (based on Poisson equations) applies to the reduced RT-equations (3.11)–(3.13) at the low regularity of L^p connections with $d\Gamma \in L^p$, and establishes existence of solutions (J, B) locally, (i.e, in neighborhoods Ω' of points), such that J is point-wise an invertable matrix, [33, theorem 6.3].

It is a built-in property of (3.11)–(3.13) that any solution J is a Jacobian integrable to coordinates, as long that the integrability condition $d\vec{J} \equiv \text{Curl}(J) = 0$ holds on the boundary $\partial\Omega'$, as accomplished in [33, theorem 6.3]. That is, combining (3.11) with (3.12), a computation shows that $\omega \equiv d\vec{J}$ is a solution of the Laplace equation $\Delta\omega = 0$, which together with our boundary data implies that $\omega = 0$ throughout Ω'. This implies that J is integrable to coordinates.

Given now a solution (J, B) of the reduced RT-equations (3.11)–(3.13) with J an integrable and invertable Jacobian, one recovers a solution $(J, \tilde{\Gamma}, A)$ of the full RT-equations (3.7)–(3.10) by introducing[8]

$$\tilde{\Gamma} \equiv \Gamma - J^{-1}dJ, \qquad A \equiv B - \langle dJ; \tilde{\Gamma}\rangle, \qquad \text{and} \qquad v \equiv v' - \delta\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}, \tag{3.14}$$

as can be verified by direct computation using (3.11) to eliminate uncontrolled terms involving $\delta\Gamma$. From interior elliptic estimates, applied to the first RT-equations (3.7), one can then prove that $\tilde{\Gamma}$ is in $W^{1,p}$ on any compactly contained set Ω'_c of Ω', a gain of one derivative over Γ. Defining

$$(\Gamma_y)^{\gamma}_{\alpha\beta} \equiv J^{\gamma}_{k}\left(J^{-1}\right)^{i}_{\alpha}\left(J^{-1}\right)^{j}_{\beta}\tilde{\Gamma}^{k}_{ij}, \tag{3.15}$$

substitution into the first equation in (3.14) yields the connection transformation law (3.4), which implies that Γ_y is the transformed connection of optimal regularity, $\Gamma_y \in W^{1,p}(\Omega'_c)$. These are the basic ideas underlying the proof of theorem 3.1, which is worked out in full detail at the level of weak (distributional) solutions in [33].

[8] The second and third equation in (3.14) define the 'gauge' transformations of the RT-equations, while the first equation defines a projection onto the space of solution of the Riemann-flat condition.

4. Proof of theorems 2.1 and 2.2

4.1. Proof of theorem 2.2

We prove theorem 2.2 by contradiction, using the optimal regularity result of theorem 3.1. So assume there exist a bubble extension $\left(\mathcal{M}^{\mathrm{ext}}_\theta\right)_{\theta\in\mathcal{O}}$ with $C^{0,1}$ Lorentzian metrics and metric connections $\Gamma_\theta \in L^\infty(\Omega)$ subject to the uniform bound (2.2), where (Ω, x) is the single coordinate chart assumed to cover $\mathcal{M}^{\mathrm{ext}}_\theta \setminus \mathring{\mathcal{M}}_\theta$ after suitable identification; we denote here with $\Omega_x \equiv x(\Omega) \subset \mathbb{R}^4$ the image of Ω under the coordinate map following the notation in section 3. In fact, to apply theorem 3.1 it suffices to assume the bound

$$\|\Gamma_\theta\|_{L^\infty(\Omega_x)} + \|\mathrm{Riem}(\Gamma_\theta)\|_{L^p(\Omega_x)} \leqslant M, \tag{4.1}$$

for some constant $M > 0$ independent of θ, some $p > 4$. Equation (4.1) is equivalent to the uniform bound (3.1) of theorem 3.1, as can be shown directly by using Hölder's inequality in combination with the expression for the Riemann curvature tensor as a matrix valued 2-form, $\mathrm{Riem}(\Gamma_x) = \mathrm{d}\Gamma_x + \Gamma_x \wedge \Gamma_x$.

So assume (4.1) and let $q \in \Omega$ be a point on the boundary of $\mathcal{M}_\theta$; (note that this $q \in \Omega$ exists, since Ω is an open set of $\mathcal{M}^{\mathrm{ext}}_\theta$ covering $\mathcal{M}^{\mathrm{ext}}_\theta \setminus \mathring{\mathcal{M}}_\theta$, where $\mathring{\mathcal{M}}_\theta$ is the interior of the maximal Cauchy development $\mathcal{M}_\theta$). Theorem 3.1 implies that there exists a neighborhood $\Omega' \subset \Omega$ of q, (depending on M, but independent of θ), such that for each connection Γ_θ there exists a coordinate transformation $x \to y_\theta$ with Jacobian $(J_\theta)^\mu_\nu = \frac{\partial y^\mu_\theta}{\partial x^\nu} \in W^{1,2p}(\Omega'_x)$ such that the connections Γ_{y_θ} in y_θ-coordinates exhibit optimal regularity, $\Gamma_{y_\theta} \in W^{1,p}(\Omega'_{y_\theta})$, where $\Omega'_{y_\theta} \equiv y_\theta(\Omega')$, subject to the uniform bound

$$\|\Gamma_{y_\theta}\|_{W^{1,p}\left(\Omega'_{y_\theta}\right)} + \|J_\theta\|_{W^{1,2p}(\Omega'_x)} + \|J_\theta^{-1}\|_{W^{1,2p}(\Omega'_x)} < C(M), \tag{4.2}$$

for some constant $C(M) > 0$ independent of θ. The norm of the second and third term in (4.2) are measured in x-coordinates, but the first one is measured in y_θ-coordinates, the coordinates of optimal regularity for each Γ_θ respectively.

In order to express (4.2) as a uniform bound suitably measuring the connections Γ_{y_θ} in a single fixed coordinate system, we first write each Γ_{y_θ} in x-coordinates by transforming connection components as a scalar function, $\Gamma_{y_\theta}(x) \equiv \Gamma_{y_\theta}(y_\theta(x))$. Now, changing integration from x to y_θ-coordinates, we have

$$\int_{\Omega'_x} \Gamma_{y_\theta}(x)\,\mathrm{d}x = \int_{\Omega'_{y_\theta}} \Gamma_{y_\theta}(y_\theta)\ \left|\det\left(J_\theta^{-1}\right)\right| \mathrm{d}y_\theta, \tag{4.3}$$

which, using that the L^∞-norm is identical in x- and y_θ-coordinates under scalar transformations, implies

$$\begin{aligned} \|\Gamma_{y_\theta}(x)\|_{L^p(\Omega'_x)} &\leqslant \|J_\theta^{-1}\|_{L^\infty(\Omega'_x)}\ \|\Gamma_{y_\theta}\|_{L^p\left(\Omega'_{y_\theta}\right)} \\ &\leqslant C_M \|J_\theta^{-1}\|_{W^{1,2p}(\Omega'_x)}\ \|\Gamma_{y_\theta}\|_{L^p\left(\Omega'_{y_\theta}\right)} \end{aligned} \tag{4.4}$$

where we used Morrey's inequality in the last line, and where $C_M > 0$ is a constant depending only on p and Ω', independent of θ[9]. Similarly, the chain rule implies for differentiation in x- versus y_θ-coordinates that

$$\partial_{x^\nu}\Gamma_{y_\theta}(x) = \sum_{\mu=0,\dots,3} (J_\theta)^\mu_\nu \, \partial_{y^\mu_\theta}\Gamma_{y_\theta}, \tag{4.5}$$

which allows us to bound each connection derivative by

$$\begin{aligned}\|\partial_x\Gamma_{y_\theta}(x)\|_{L^p(\Omega'_x)} &\leqslant C_p\,\|J_\theta\|_{L^\infty(\Omega'_x)}\,\|\partial_{y_\theta}\Gamma_{y_\theta}\|_{L^p(\Omega'_x)}\\ &\leqslant C_M\,C_p\,\|J_\theta\|_{W^{1,2p}(\Omega'_x)}\,\|\partial_{y_\theta}\Gamma_{y_\theta}\|_{L^p(\Omega'_x)},\end{aligned} \tag{4.6}$$

where $C_p > 1$ is a combinatorial constant independent of θ to account for the summation in (4.5), ∂_x and ∂_{y_θ} denotes the collection of partial derivatives in respective coordinates (which norms are summed over, cf (2.3)), and we applied again Morrey's inequality. Changing integration to y_θ coordinates, (4.6) implies

$$\|\partial_x\Gamma_{y_\theta}(x)\|_{L^p(\Omega'_x)} \leqslant C_M^2\,C_p\,\|J_\theta\|_{W^{1,2p}(\Omega'_x)}\|J_\theta^{-1}\|_{W^{1,2p}(\Omega'_x)}\,\|\partial_{y_\theta}\Gamma_{y_\theta}\|_{L^p\left(\Omega'_{y_\theta}\right)}, \tag{4.7}$$

following the reasoning in (4.5). Combining (4.4) and (4.7), we can now bound the $W^{1,p}$-norm of $\Gamma_{y_\theta}(x)$ in x-coordinates by

$$\begin{aligned}\|\Gamma_{y_\theta}(x)\|_{W^{1,p}(\Omega'_x)} &\equiv \|\Gamma_{y_\theta}(x)\|_{L^p(\Omega'_x)} + \sum_{\nu=0,\dots,3}\|\partial_{x^\nu}\Gamma_{y_\theta}(x)\|_{L^p(\Omega'_x)}\\ &\leqslant C_M\,C_p\,\|J_\theta^{-1}\|_{W^{1,2p}(\Omega'_x)}\left(1 + C_M\|J_\theta\|_{W^{1,2p}(\Omega'_x)}\right)\|\Gamma_{y_\theta}\|_{W^{1,p}\left(\Omega'_{y_\theta}\right)}.\end{aligned} \tag{4.8}$$

Using finally (4.2) to bound $\|J_\theta^{-1}\|_{W^{1,2p}(\Omega'_x)}$ and $\|\Gamma_{y_\theta}\|_{W^{1,p}(\Omega'_{y_\theta})}$ in (4.8), we obtain the sought-after uniform bound

$$\|\Gamma_{y_\theta}(x)\|_{W^{1,p}(\Omega'_x)} \leqslant C_M\,C_p\,(1 + C_M C(M))\,C(M)^2 \equiv C, \tag{4.9}$$

where both $C > 0$ and Ω' are independent of θ.

Estimate (4.9) contradicts the assumption of theorem 2.2. Namely, the connections Γ_{y_θ} form a set of bubble extensions with $\mathcal{M}^{\text{ext}}_\theta \equiv \mathcal{M}_\theta \cup \Omega'$, because $\mathcal{M}^{\text{ext}}_\theta \setminus \mathring{\mathcal{M}}_\theta$ can be covered by Ω', and since one can identify the coordinates y_θ with x and Ω'_{y_θ} with Ω'_x. Thus, by (4.9), we constructed a set of bubble extensions subject to the uniform bound (2.1), in contradiction to our assumptions in theorem 2.2. We conclude no $C^{0,1}$ Lorentzian metric extension with L^p bounded Riemann curvature, subject to the uniform bound (2.2), exists for $p > 4$. This completes the proof of theorem 2.2. □

4.2. *Proof of theorem 2.1*

Let $(\mathcal{M}_\theta, g_\theta)$ be the maximal Cauchy development of some datum $\theta \in \mathcal{O}$, such that no bubble extension of $(\mathcal{M}_\theta, g_\theta)$ exists with connection Γ_θ in $W^{1,p}(\Omega)$, for some $p > 2$.

[9] Morrey's inequality bounds the Hölder norms of a function f as $\|f\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant C_M\|f\|_{W^{1,p}(\Omega)}$, where $\alpha \equiv 1 - \frac{n}{p}$ and $C_M > 0$ is a constant depending only on n, p and $\Omega \subset \mathbb{R}^n$, see [12].

Assume for contradiction there exists a bubble extension $\mathcal{M}_\theta^{\mathrm{ext}}$ with Lorentzian metric $g_\theta \in C^{0,\alpha}(\Omega)$, metric connection $\Gamma_\theta \in L^{2p}(\Omega)$ and Riemann curvature $\mathrm{Riem}(\Gamma_\theta) \in L^p(\Omega)$, where (Ω, x) is the coordinate chart covering $\mathcal{M}_\theta^{\mathrm{ext}} \setminus \mathring{\mathcal{M}}_\theta$. Now, by theorem 3.1, for any point $q \in \Omega$ on the boundary of $\mathcal{M}_\theta$, there exists a neighborhood Ω' of q, on which a coordinate transformation $x \to y_\theta$ is defined such that Γ_{y_θ} has optimal regularity, $\Gamma_{y_\theta} \in W^{1,p}(\Omega'_{y_\theta})$. This is a contradiction to our incoming assumption that no such bubble extension exists. This completes the proof of theorem 2.1. □

5. Conclusion

The results of this paper, under the assumption that no extensions exist with higher connection regularity $W^{1,p}_{\mathrm{loc}}$, verify the *Strong Cosmic Censorship conjecture with bounded curvature* in the affirmative. This step towards confirming the conjecture uses elliptic PDE theory, (applied to the RT-equations), to lift the problem from the lowest regularity, (of Lipschitz continuous metrics with bounded curvature), by one level, to regularities more accessible to hyperbolic PDE methods.

Data availability statement

No new data were created or analysed in this study.

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Conflict of interest

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