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# Statistical characteristics of queue with fractional Brownian motion input

J. Chen, H.S. Bhatia, R.G. Addie and M. Zukerman

The fractional Brownian motion has attracted significant attention because it accurately represents Internet traffic characteristics and is amenable to analysis. In this letter, we discover a link between the probability density function of the steady-state queue size of a queue with a fractional Brownian input and the Generalised Gamma distribution, and provide the mean, variance, third central moment and skewness. We also provide new simulation results which validate the accuracy of these queueing statistics.

*Introduction:* Since the traffic for core and metropolitan Internet links is an aggregations of flows from many users and many of these users transmit their data independently, it can be assumed to follow a Gaussian process by the central limit theorem [1]. Also, the long range dependence (LRD) characteristics of the Internet traffic has been well established [2, 3, 4]. Thus, a Gaussian LRD process, the fractional Brownian motion (fBm), has been considered as the model of choice for heavily multiplexed Internet traffic and its accuracy has been validated by various publications including [5, 6]. Although a queue fed by fBm input has been considered as a fundamentally important model for Internet queueing performance analysis [5, 7, 8, 9, 10], no explicit accurate results for the mean, variance, third central moment and skewness for the occupancy,  $Q$ , of an fBm queue, are available. This letter, for the first time, provides a link between the fBm queue size distribution and the Generalized Gamma distribution [11] (a special case of the Amoroso distribution [12]) which leads to very accurate closed-form approximations for these important queueing statistics of an fBm queue. The approximations are validated by simulation results. Also, simplified expressions are provided for these results in certain cases when the Hurst parameter takes certain special values.

*The model and related work:* We consider a single server queue with constant service rate,  $C$  [B/s], fed by an fBm input process with the Hurst parameter  $H$ , the mean input rate  $m$  [B/s], and the variance  $\sigma_1^2$ . Introducing the mean net input,  $\mu = m - C$ , we can characterize the fBm queueing model by three parameters:  $H$ ,  $\mu$  and  $\sigma_1^2$ . The probability density function of  $Q$  as provided in [6] is:

$$\frac{P(Q \in (x, x + dx))}{dx} \approx \frac{\nu \alpha^{\frac{\beta}{\nu}}}{\Gamma\left(\frac{\beta}{\nu}\right)} x^{\beta-1} e^{-\alpha x^\nu}, \quad (1)$$

where

$$\alpha = \frac{(1-H)^{2H-2} |\mu|^{2H}}{2H^{2H} \sigma_1^2}, \quad (2)$$

$$\beta = \frac{1}{H} - 1,$$

and

$$\nu = 2(1-H).$$

Observe that this approximation of the fBm queue size distribution is known as the Generalised Gamma distribution. The Generalised Gamma distribution is a special case of the Amoroso distribution [12, 13, 14] used originally to model income rates. The Amoroso distribution has the following density

$$f_X(x; a, d, g, p) = \frac{p(x-g)^{d-1} \exp\left(-\left(\frac{x-g}{a}\right)^p\right)}{a^d \Gamma\left(\frac{d}{p}\right)}, \quad (3)$$

where  $a > 0$ ,  $d > 0$ ,  $p > 0$ ,  $g \in \mathbb{R}$  and  $x \geq g$  and  $\Gamma(\cdot)$  denotes the gamma function. Stacy [11] defined the Generalised Gamma distribution as the special case of the Amoroso distribution where  $g = 0$  and identified the

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cumulative distribution function (CDF) and moment-generating function (MGF), in this case, as

$$F(x; a, d, p) = \frac{\gamma\left(\frac{d}{p}, \left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}, \quad (4)$$

and

$$M(t; a, d, p) = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} \frac{\Gamma\left(\frac{d+k}{p}\right)}{\Gamma\left(\frac{d}{p}\right)}, \quad (5)$$

respectively, where  $\gamma(\cdot)$  in (4) denotes the lower incomplete gamma function. The mean and variance when  $g = 0$  are

$$E[Q] = a \frac{\Gamma((d+1)/p)}{\Gamma(d/p)}, \quad (6)$$

and

$$\text{Var}[Q] = a^2 \left[ \frac{\Gamma((d+2)/p)}{\Gamma(d/p)} - \left( \frac{\Gamma((d+1)/p)}{\Gamma(d/p)} \right)^2 \right], \quad (7)$$

respectively [13]. From the MGF, we have also derived the third central moment of  $Q$  to be given by

$$E[(Q - E[Q])^3] = \frac{a^3 \Gamma\left(\frac{d+3}{p}\right)}{\Gamma(d/p)} + 2 \left( a \frac{\Gamma((d+1)/p)}{\Gamma(d/p)} \right)^3 - \frac{3a^3 \Gamma\left(\frac{d+1}{p}\right) \Gamma\left(\frac{d+2}{p}\right)}{\Gamma^2(d/p)}. \quad (8)$$

Substituting  $g = 0$ ,  $d = \beta$ ,  $p = \nu$  and  $a = \alpha^{-1/\nu}$  in (3) we obtain (1), so it is a Generalised Gamma density.

*The mean, variance, third central moment and Skewness:* Replacing  $d$ ,  $p$  and  $a$  with their expressions in terms of  $\alpha$  and  $H$ , we find:

$$E[Q] \approx \alpha^{\frac{-1}{2H-2}} \frac{\Gamma\left(\frac{(1-H)/H+1}{2-2H}\right)}{\Gamma\left(\frac{1-H}{H(2-2H)}\right)} = \alpha^{\frac{1}{2H-2}} \frac{\Gamma\left(\frac{1}{2H(1-H)}\right)}{\Gamma\left(\frac{1}{2H}\right)}, \quad (9)$$

$$\text{Var}[Q] \approx \frac{\alpha^{\frac{1}{H-1}}}{\Gamma\left(\frac{1}{2H}\right)} \left[ \Gamma\left(\frac{1+H}{2H(1-H)}\right) - \frac{\Gamma^2\left(\frac{1}{2H(1-H)}\right)}{\Gamma\left(\frac{1}{2H}\right)} \right], \quad (10)$$

and

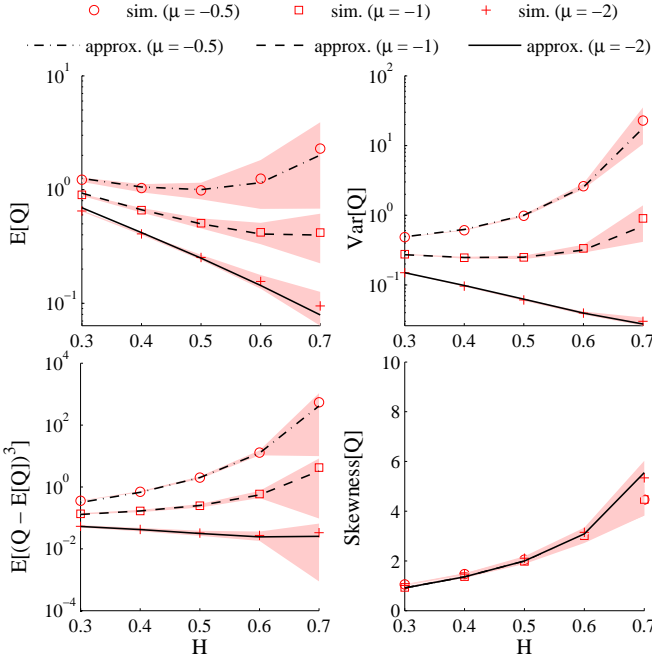
$$E[(Q - E[Q])^3] \approx \alpha^{\frac{3}{2H-2}} \left[ \frac{\Gamma\left(\frac{2H+1}{2H(1-H)}\right)}{\Gamma\left(\frac{1}{2H}\right)} + \frac{2\Gamma^3\left(\frac{1}{2H(1-H)}\right)}{\Gamma^3\left(\frac{1}{2H}\right)} - \frac{3\Gamma\left(\frac{H+1}{2H(1-H)}\right) \Gamma\left(\frac{1}{2H(1-H)}\right)}{\Gamma^2\left(\frac{1}{2H}\right)} \right]. \quad (11)$$

The skewness of  $Q$ , can be expressed as

$$\text{Skewness}[Q] = \frac{E[(Q - E[Q])^3]}{(\text{Var}[Q])^{3/2}}. \quad (12)$$

By (10) and (11),  $\alpha$  is canceled out in (12), so  $\text{Skewness}[Q]$  is a function of only one parameter,  $H$ .

*Validation:* We validate our mean, variance, third central moment and skewness results by simulations of fBm queues with  $\sigma_1 = 1$  and different values of  $H$  and  $\mu$ . We have used the simulation method described in [6]. The simulation results for  $E[Q]$ ,  $\text{Var}[Q]$ ,  $E[(Q - E[Q])^3]$  and  $\text{Skewness}[Q]$  are compared with those from (9) - (12), as shown in Figure 1, where the 95% confidence intervals based on Student's t-distribution are presented as the coloured area. The agreement between our theoretical results and the simulation results can be observed in the figure. The setting  $\sigma_1 = 1$  is without loss of generality because  $\sigma_1$  can be viewed as a scale parameter. For example, for any value of  $\sigma_1$ , using  $\mu$  and  $\sigma_1$  in units of  $\sigma_1$  (i.e., dividing both by  $\sigma_1$ ) will give the scaled mean and variance values as presented in figure 1 (i.e., the mean divided by  $\sigma_1$  and the variance divided by  $\sigma_1^2$ ). Note that  $H$  is not affected by the unit size.



**Fig. 1** Approximations vs. simulation results for  $E[Q]$ ,  $\text{Var}[Q]$ ,  $E[(Q - E[Q])^3]$  and  $\text{Skewness}[Q]$  with  $\sigma_1 = 1$ .

*Simplifications of  $E[Q]$  and  $\text{Var}[Q]$  for certain  $H$  values:* In the Brownian case of  $H = 0.5$ , where  $\sigma_1^2 = m$  as in a Poisson process, (2) reduces to

$$\alpha = \frac{2|\mu|}{\sigma_1^2}.$$

Thus, (9) and (10) can be further simplified as

$$E[Q] = \left(\frac{2|\mu|}{\sigma_1^2}\right)^{-1} = \frac{\sigma_1^2}{2|\mu|}, \quad (13)$$

and

$$\text{Var}[Q] = \left(\frac{2|\mu|}{\sigma_1^2}\right)^{-2} \left(2 - \frac{1^2}{1}\right) = \left(\frac{\sigma_1^2}{2|\mu|}\right)^2 = (E[Q])^2. \quad (14)$$

An alternative way to obtain  $E[Q]$  and  $\text{Var}[Q]$  for the case  $H = 0.5$  is to use the exact solution for  $P(Q > x)$  from [15], which is

$$P(Q > x) = e^{-\frac{2\mu}{\sigma_1^2}x}.$$

Then the mean and variance of  $Q$  are obtained by

$$E[Q] = \int_0^{\infty} P(Q > x) dx,$$

and

$$\text{Var}[Q] = 2 \int_0^{\infty} x P(Q > x) dx - \left(\int_0^{\infty} P(Q > x) dx\right)^2,$$

which are consistent with (13) and (14). Since  $|\mu| = C - m$ , we can express  $E[Q]$  by (13)

$$E[Q] = \frac{\sigma_1^2}{2(C - m)},$$

which is half of that for the equivalent M/M/1 queueing system, where  $\sigma_1^2 = m$  and  $E[Q] = m/(C - m)$ . It is also equal to the mean of an equivalent M/D/1 queue under heavy traffic (where the utilization approaches 1) based on the Pollaczek Khintchine Formula.

When  $H = 1 - \frac{1}{2n}$ , for  $n = 1, 2, \dots$ , we have  $\frac{1}{2H(1-H)} = \frac{1}{2H} + n$ , and

$$\begin{aligned} \Gamma\left(\frac{1}{2H(1-H)}\right) &= \Gamma\left(\frac{1}{2H} + n\right) \\ &= \left(n - 1 + \frac{1}{2H}\right) \left(n - 2 + \frac{1}{2H}\right) \dots \left(1 + \frac{1}{2H}\right) \frac{1}{2H} \Gamma\left(\frac{1}{2H}\right). \end{aligned}$$

Then, (9) can be expressed without the Gamma function as

$$E[Q] = \alpha^{\frac{1}{2H-2}} \left[ \frac{1}{2(1-H)} - 1 + \frac{1}{2H} \right] \dots \left(1 + \frac{1}{2H}\right) \frac{1}{2H}. \quad (15)$$

*Conclusion:* We have established, for the first time, the link between the fBm queue length distribution and the Amoroso distribution (or its special case of the Generalized Gamma distribution). This adds an important application to the long list of applications of the Amoroso distribution [16, 17, 18]. This link has provided new closed-form approximations for the mean, variance, third central moment and skewness of an fBm queue which have been validated by simulations. We have also provided simplified expressions for the mean and variance for a range of cases.

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