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STABILITY CONDITION FOR DIFFERENCE SCHEMES FOR PARABOLIC SYSTEMS*

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Abstract. Recently, Goldberg studied in [M. Goldberg, *SIAM J. Numer. Anal.*, 35 (1998), pp. 478–493, M. Goldberg, *SIAM J. Numer. Anal.*, 35 (1998), pp. 1995–2003] stability conditions of a well-known family of difference schemes for the following multidimensional parabolic system of second order:

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} = \sum_{1\le p\le q\le s} A_{pq}\frac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial x_p \partial x_q} + \sum_{1\le p\le s} B_p \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_p} + C\mathbf{u}(\mathbf{x},t).$$

Here we present a modified stability condition for the above problem, and prove that this condition is optimal in some sense.

AMS subject classifications. 65M06, 65M12

Key words. parabolic systems, difference schemes, stability

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1. Introduction.

Consider the initial value problem for the parabolic systems of second order of the form

$$\text{(1.1a)} \qquad \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} = \sum_{1\le p\le q\le s} A_{pq}\frac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial x_p \partial x_q} + \sum_{1\le p\le s} B_p \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_p} + C\mathbf{u}(\mathbf{x},t),$$

$$\text{(1.1b)} \qquad \mathbf{u}(\mathbf{x},0) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2, \dots, x_s) \in \mathrm{R}^s, \quad 0 \le t \le T,$$

where A_{pq}, B_p, and C are constant $m\times m$ matrices, A_{pq} being Hermitian, $\mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x},t),\dots,u_m(\mathbf{x},t))'$ (prime denoting the transpose) is the unknown m-vector. Here and below we adopt the same notations and symbols as those in [2, 3].

We assume that (1.1) is well posed in the sense of Petrowski (see [7]), i.e., there exists a constant $\delta > 0$ such that

$$\text{(1.2)} \qquad \sum_{1\le p\le q\le s} \sigma_p\sigma_q A_{pq} \ge \delta \sum_{1\le p\le s} \sigma_p^2 I \quad \text{for all real } \sigma_1,\dots,\sigma_s,$$

where I is the identity matrix. Define the spectral radius of a matrix A as

$$\rho(A) = \max\{|\lambda| \; : \; \lambda \text{ is eigenvalue of } A\}.$$

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Let $\Delta x_p > 0$, $p = 1, \dots, s$, and $\Delta t > 0$ be the space and time stepsizes, respectively. Denote $\lambda_p = \frac{\Delta t}{\Delta x_p^2}$, $p = 1, \dots, s$, $\mathbf{u}_k^n = \mathbf{u}(k_1 \Delta x_1, \dots, k_s \Delta x_s, n\Delta t)$, $\mathbf{k} = (k_1, \dots, k_s)$, and E_p the translations

$$E_p \mathbf{u}_{\mathbf{k}}^n = \mathbf{u}(k_1\Delta x_1, \dots, k_{p-1}\Delta x_{p-1}, (k_p+1)\Delta x_p, k_{p+1}\Delta x_{p+1}, \dots, k_s\Delta x_s),\ p = 1, \dots, s.$$

For $0 \le \theta \le 1$, we introduce the difference schemes

$$\mathbf{v}_k^{n+1} - \theta Q \mathbf{v}_k^{n+1} = \mathbf{v}_k^n + (1-\theta) Q \mathbf{v}_k^n, \tag{1.3}$$

where

$$Q\mathbf{v}_k^n = \sum_{1 \le p \le s} \lambda_p A_{pp} (E_p - 2I + E_p^{-1}) \mathbf{v}_k^n$$

$$+ \sum_{1 \le p < q \le s} \frac{1}{4} \sqrt{\lambda_p \lambda_q} A_{pq} (E_p - E_p^{-1})(E_q - E_q^{-1}) \mathbf{v}_k^n$$

$$+ \sum_{1 \le p \le s} \frac{1}{2} \sqrt{\lambda_p \Delta t} (E_p - E_p^{-1}) \mathbf{v}_k^n + \Delta t C \mathbf{v}_k^n.$$

Following [6] and [7] we call the difference schemes in (1.3) L_2 stable if and only if there exist constants $\tau > 0$, $\kappa > 0$, such that

$$\|G(\Delta t, \boldsymbol{\xi})^n\|_2 \le \kappa \quad \text{for all} \ \ \Delta t \le \tau,\ n\Delta t \le T,\ \boldsymbol{\xi} \in \mathrm{R}^s,$$

where $G(\Delta t, \boldsymbol{\xi})$ is the amplification matrix of the difference schemes in (1.3), and $\|\cdot\|_2$ is the spectral norm on the algebra of complex $m \times m$ matrices $\mathbf{C}^{m\times m}$, i.e.,

$$\|M\|_2 = \max\left\{(y^* M^* M y)^{\frac{1}{2}} : y \in \mathbf{C}^m, y^* y = 1\right\}, \quad M \in \mathbf{C}^{m \times m}$$

($*$ denoting the conjugate transpose).

The system in (1.1) is called strongly well posed in the sense of Petrowski if the conditions (1.2) and

$$\sum_{1 \le p \le s} \sigma_p^2 A_{pp} - \sum_{1 \le p < q \le s} \sigma_p \sigma_q A_{pq} \ge \delta \sum_{1 \le p \le s} \sigma_p^2 I \quad \text{for all real} \ \ \sigma_1, \dots, \sigma_s \tag{1.4}$$

for some $\delta > 0$ are satisfied [2, 3]. In [2] Goldberg has studied stability conditions of (1.3) for the initial value problem of (1.1). He proved that the scheme (1.3) is stable unconditionally for $\frac{1}{2} \le \theta \le 1$, and when the system (1.1) is strongly well posed in the sense of Petrowski, this scheme is stable conditionally for $0 \le \theta < \frac{1}{2}$. Later, in [3], Goldberg presented a stability condition for the problem with only Petrowski condition (1.2). The main theorem in [3] is given below.

THEOREM 1.1 (see [3]). *Let $0 \le \theta < \frac{1}{2}$ and* (1.2) *hold. Then the schemes in* (1.3) *are L_2 stable if*

$$\sum_{1 \le p \le s} \lambda_p \rho(A_{pp}) \le \frac{1}{2\Omega(1 - 2\theta)}, \tag{1.5}$$

where $\Omega = \Omega(s) = s$ *for* s *odd, and* $\Omega = \Omega(s) = s - 1$ *for* s *even.*

Our main result is the following. The proof is along the lines of Goldberg's argument in [3].

THEOREM 1.2. *Let* $0 \le \theta < \frac{1}{2}$ *and* $s \ge 3$. *Assume* (1.2) *holds. Then the schemes in* (1.3) *are* L_2 *stable if*

$$\sum_{1\le p\le s} \lambda_p \rho(A_{pp}) \le \frac{1}{2\omega(1-2\theta)}, \tag{1.6}$$

where $\omega = \omega(s) = \frac{s^2}{4(s-1)}$.

Remark 1.1. Obviously, $\frac{\Omega(3)}{\omega(3)} = \frac{8}{3}$, $\frac{\Omega(4)}{\omega(4)} = \frac{9}{4}$, $\frac{\Omega(6)}{\omega(6)} = \frac{25}{9}$, $\frac{\Omega(s)}{\omega(s)} > 3$ for $s = 5, 7$ and all $s \ge 8$, and $\lim_{s\to\infty} \frac{\Omega(s)}{\omega(s)} = 4$. The condition (1.6) is less restrictive than the one in (1.5). In section 3 we shall give an example to show the condition (1.6) is optimal in some sense.

2. Proof of Theorem 1.2. First we prove the following lemma.

LEMMA 2.1. *Let* $s \ge 3$ *and the condition* (1.2) *hold. Then there holds*

(2.1)
$$C_1 \sum_{1\le p\le s} \sigma_p^2 A_{pp} - \sum_{1\le p<q\le s} \sigma_p\sigma_q A_{pq} \ge (s-1)\delta \sum_{1\le p\le s} \sigma_p^2 I \quad \textit{for all real } \sigma_1, \ldots, \sigma_s,$$

where $C_1 = s - 1$.

Remark 2.1. The estimate of $C_1 = s - 1$ in Lemma 2.1 is optimal. That is, for any $\varepsilon > 0$, (2.1) is not true for $C_1 = s - 1 - \varepsilon$, which can be shown by an example in section 3.

Proof of Lemma 2.1. For any fixed p and q satisfying $1 \le p < q \le s$, by letting all the remaining σ_p vanish, we get

$$\sigma_p^2 A_{pp} + \sigma_p\sigma_q A_{pq} + \sigma_q^2 A_{qq} \ge \delta(\sigma_p^2 + \sigma_q^2) I \quad \text{for all real } \sigma_p, \sigma_q. \tag{2.2}$$

Altering the sign of the σ_p in the above inequality (2.2) will lead to

$$\sigma_p^2 A_{pp} - \sigma_p\sigma_q A_{pq} + \sigma_q^2 A_{qq} \ge \delta(\sigma_p^2 + \sigma_q^2) I \quad \text{for all real } \sigma_p, \sigma_q. \tag{2.3}$$

Summing up the inequality (2.3) over $1 \le p < q \le s$ gives (2.1) immediately. The proof of Lemma 2.1 is completed. □

The amplification matrix of the difference schemes in (1.3) are of the form

$$G(\Delta t, \boldsymbol{\xi}) = H_1(\Delta t, \boldsymbol{\xi})^{-1} H_0(\Delta t, \boldsymbol{\xi}), \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_s) \in \mathrm{R}^s,$$

where $H_0(\Delta t, \boldsymbol{\xi})$ and $H_1(\Delta t, \boldsymbol{\xi})$ are the $m \times m$ complex matrices defined by

$$H_0(\Delta t, \boldsymbol{\xi}) = I + (1-\theta) H(\Delta t, \boldsymbol{\xi}), \quad H_1(\Delta t, \boldsymbol{\xi}) = I - \theta H(\Delta t, \boldsymbol{\xi}),$$

and

$$H(\Delta t, \boldsymbol{\xi}) = \sum_{1\le p\le s} 2\lambda_p(\cos\xi_p - 1) A_{pp} - \sum_{1\le p<q\le s} \sqrt{\lambda_p\lambda_q} \sin\xi_p \sin\xi_q A_{pq}$$

$$+ i \sum_{1\le p\le s} \sqrt{\lambda_p \Delta t} \sin\xi_p B_p + \Delta t C \equiv J + iK + \Delta t C,$$

$$J \equiv J(\boldsymbol{\xi}) = \sum_{1\le p\le s} 2\lambda_p(\cos\xi_p - 1)A_{pp} - \sum_{1\le p<q\le s} \sqrt{\lambda_p\lambda_q}\sin\xi_p\sin\xi_q A_{pq},$$

$$K \equiv K(\Delta t, \boldsymbol{\xi}) = \sum_{1\le p\le s} \sqrt{\lambda_p \Delta t}\sin\xi_p B_p.$$

Since the stability of difference schemes is insensitive to the perturbations of order $O(\Delta t)$ (see [5] and [6, sect. 3.9]), we can let the zero order coefficient matrix C in (1.1a) be 0, and then reducing $G(\Delta t, \boldsymbol{\xi})$ to

$$G = G(\Delta t, \boldsymbol{\xi}) = H_1(\Delta t, \boldsymbol{\xi})^{-1} H_0(\Delta t, \boldsymbol{\xi}),$$

where

$$H_0 = H_0(\Delta t, \boldsymbol{\xi}) = I + (1-\theta)[J(\boldsymbol{\xi}) + iK(\Delta t, \boldsymbol{\xi})],$$

$$H_1 = H_1(\Delta t, \boldsymbol{\xi}) = I - \theta[J(\boldsymbol{\xi}) + iK(\Delta t, \boldsymbol{\xi})].$$

In [2] the following matrix inequalities have been proved:

$$J(\boldsymbol{\xi}) \le 0, \quad P(\boldsymbol{\xi}) = I - \theta J(\boldsymbol{\xi}) \ge I. \tag{2.4}$$

For any $\boldsymbol{\xi} \in \mathrm{R}^s$, define the inner product on $\mathbf{C}^m$

$$(\mathbf{y}, \mathbf{z})_\xi \equiv \mathbf{z}^* P\mathbf{y}, \quad \mathbf{y}, \mathbf{z} \in \mathbf{C}^m,$$

and the numerical radius associated with $(\mathbf{y}, \mathbf{z})_\xi$

$$r_\xi(M) \equiv \max\{|(M\mathbf{y}, \mathbf{y})_\xi| : \mathbf{y} \in \mathbf{C}^m, (\mathbf{y}, \mathbf{y})_\xi = 1\}, \quad M \in \mathbf{C}^{m\times m}.$$

This constitutes a ξ-weighted norm on $\mathbf{C}^{m\times m}$ (see [4] or [2, 3]) which is important in our proof.

Obviously there holds $PH_1^{-1}H_0 = R + iKS$, where $R = I + (1-\theta)J$, $S = (1-\theta)I + \theta[I - \theta J - i\theta K]^{-1}[I + (1-\theta)(J + iK)]$. It follows that

$$r_\xi(G) \equiv r_\xi(G(\Delta t, \boldsymbol{\xi})) \le \max_{\mathbf{y}\ne 0} \frac{|\mathbf{y}^* R\mathbf{y}| + |\mathbf{y}^* KS\mathbf{y}|}{\mathbf{y}^* P\mathbf{y}}. \tag{2.5}$$

Now we have the following lemma.

LEMMA 2.2. *Assume that the conditions of Theorem* 1.2 *hold. Then there exist constants* $\gamma \ge 0$, $\tau > 0$, *such that*

$$r_\xi(G(\Delta t, \boldsymbol{\xi})) \le 1 + \gamma\Delta t \quad \textit{for all} \quad \Delta t \le \gamma, \ \ n\Delta t \le T, \ \ \boldsymbol{\xi} \in R^s.$$

Proof. It is proved in [2] that there is a constant $\tau > 0$ such that $G(\Delta t, \boldsymbol{\xi})$ exists for all $\boldsymbol{\xi} \in \mathrm{R}^s$ and $\Delta t \le \tau$. Note that R is Hermitian, so $\mathbf{y}^* R\mathbf{y}$ is real for all $\mathbf{y} \in \mathbf{C}^m$. According to (2.5) it is sufficient to show that there exists a constant $\gamma \ge 0$ such that

$$\mathbf{y}^*(\pm R - P)\mathbf{y} + |\mathbf{y} * KS\mathbf{y}| \le \gamma\Delta t \mathbf{y}^* P\mathbf{y}, \ \ \mathbf{y} \in \mathbf{C}^m, \ \ 0 < \Delta t \le \tau, \ \ \boldsymbol{\xi} \in \mathrm{R}^s. \tag{2.6}$$

The above inequality with the "+" symbol has been proved in [2]. It remains to prove

$$-\mathbf{y}^*(R + P)\mathbf{y} + |\mathbf{y} * KS\mathbf{y}| \le \gamma\Delta t \mathbf{y}^* P\mathbf{y}, \ \ \mathbf{y} \in \mathbf{C}^m, \ \ 0 < \Delta t \le \tau, \ \ \boldsymbol{\xi} \in \mathrm{R}^s. \tag{2.7}$$

Note that, by letting $\sigma_p = \sqrt{\lambda_p}\sin\xi_p$ in Lemma 2.1,

$$R + P = 2I + (1-2\theta)\left(\sum_{1\le p\le s} 2\lambda_p(\cos\xi_p - 1)A_{pp} - \sum_{1\le p<q\le s}\sqrt{\lambda_p\lambda_q}\sin\xi_p\sin\xi_q A_{pq}\right)$$

$$\ge 2I + (1-2\theta)\left(\sum_{1\le p\le s} 2\lambda_p(\cos\xi_p - 1)A_{pp} - (s-1)\sum_{1\le p\le s}\lambda_p\sin^2\xi_p A_{pp}\right.$$

$$\left. + (s-1)\delta\sum_{1\le p\le s}\lambda_p\sin^2\xi_p I\right)$$

$$\ge 2I + (1-2\theta)\left(-\frac{s^2}{s-1}\sum_{1\le p\le s}\lambda_p A_{pp} + (s-1)\delta\sum_{1\le p\le s}\lambda_p\sin^2\xi_p I\right),$$

where we have used the inequality (2.1) and the fact, for all real ξ_p,

$$2(\cos\xi_p - 1) - (s-1)\sin^2\xi_p = (s-1)\left(\cos\xi_p + \frac{1}{s-1}\right)^2 - \frac{s^2}{s-1} \ge -\frac{s^2}{s-1}.$$

Then, for all $\mathbf{y} \in \mathbf{C}^m$ with $\mathbf{y}^* y = 1$,

$$-\mathbf{y}^*(R+P)\mathbf{y} + |\mathbf{y}^* KS\mathbf{y}|$$

$$\le -2 + (1-2\theta)\left(\frac{s^2}{s-1}\sum_{1\le p\le s}\lambda_p\mathbf{y}^* A_{pp}\mathbf{y} - (s-1)\delta\sum_{1\le p\le s}\lambda_p\sin^2\xi_p\right)$$

$$+ \sum_{1\le p\le s}\sqrt{\lambda_p\Delta t}|\sin\xi_p|\cdot|\mathbf{y}^* B_p S\mathbf{y}|$$

$$\le -2 + (1-2\theta)\left(\frac{s^2}{s-1}\sum_{1\le p\le s}\lambda_p r(A_{pp}) - (s-1)\delta\sum_{1\le p\le s}\lambda_p\sin^2\xi_p\right)$$

$$+ \sum_{1\le p\le s}\sqrt{\lambda_p\Delta t}|\sin\xi_p|\cdot|r(B_p S)|, \tag{2.8}$$

where $r(M) \equiv \max\{|\mathbf{y}^* M\mathbf{y}| : \mathbf{y} \in \mathbf{C}^m, \mathbf{y}^*\mathbf{y} = 1\}$. Since A_{pp} is normal, there holds $r(A_{pp}) = \rho(A_{pp})$.

From (1.6) it follows that

$$(1-2\theta)\frac{s^2}{s-1}\sum_{1\le p\le s}\lambda_p\rho(A_{pp}) \le 2. \tag{2.9}$$

Moreover, using $|ab| \le |a|^2 + \frac{|b|^2}{4}$, we get

(2.10)

$$\sqrt{\lambda_p \Delta t}|\sin \xi_p| \cdot |r(B_p S)| \le (1-2\theta)(s-1)\delta\lambda_p \sin^2 \xi_p + \frac{r^2(B_p S)}{4(1-2\theta)(s-1)\delta}\Delta t.$$

Combining (2.8)–(2.10) gives

$$-\mathbf{y}^*(R+P)\mathbf{y} + |\mathbf{y}^* K S \mathbf{y}| \le \gamma \Delta t, \tag{2.11}$$

where

$$\gamma = \frac{1}{4(1-2\theta)(s-1)\delta} \sup_{\substack{0 < \Delta t \le \tau \\ \boldsymbol{\xi} \in \mathrm{R}^s}} \sum_{1 \le p \le s} r^2(B_p S) < \infty.$$

Since $P \ge I$, (2.7) follows from (2.11) immediately. Lemma 2.2 is proved. □

Finally we turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. As in the proof of Theorem 1.2 in [3], there is a constant μ such that the following inequality holds:

$$\|G(\Delta t, \boldsymbol{\xi})\|_2 \le \mu r_\xi(G(\Delta t, \boldsymbol{\xi})^n) \le \mu r_\xi^n(G(\Delta t, \boldsymbol{\xi}))$$

$$\le \mu(1+\gamma\Delta t)^n \le \mu e^{\gamma n \Delta t} \le \mu e^{\gamma T},$$

for all $\Delta t \le \tau, n\Delta t \le T, \boldsymbol{\xi} \in \mathrm{R}^s$, where τ and γ are the constants in Lemma 2.2. Then (1.4) is true for $\kappa = \mu e^{\gamma T}$, and Theorem 1.1 is proved. □

3. An example. In this section we show the inequality (2.1) and the stability condition (1.6) are optimal by considering the following example.

Let $m = 1$ and

$$A_{pp} = 1, \;\; p = 1, \ldots, s; \quad A_{pq} = 2\left(1 - \frac{\varepsilon}{(s-1)^2}\right), \;\; 1 \le p < q \le s, \; 0 < \varepsilon \le 1,$$

$$B_p = 0, \;\; 1 \le p \le s, \quad C = 0.$$

Denote

$$b_{pp} = 1, \;\; p = 1, \ldots, s; \quad b_{pq} = 1 - \frac{\varepsilon}{(s-1)^2}, \;\; 1 \le p \le s, \; 1 \le q \le s, \; p \ne q. \tag{3.1}$$

Then the matrix $B = (b_{pq})_{s \times s}$ is a circulant symmetric matrix, whose minmum of the eigenvalues is $\frac{\varepsilon}{(s-1)^2}$ and maximum $s - \frac{\varepsilon}{s-1}$ (see [1]). Then there holds the following inequality:

$$\frac{\varepsilon}{(s-1)^2} \sum_{1 \le p \le s} \sigma_p^2 \le \sum_{1 \le p,q \le s} \sigma_p \sigma_q b_{pq} \le \left(s - \frac{\varepsilon}{s-1}\right) \sum_{1 \le p \le s} \sigma_p^2 \;\text{ for all real }\; \sigma_1, \ldots, \sigma_s$$

(3.2)

and the two inequalities above become the equalities when $(\sigma_1, \ldots, \sigma_s)$ is the eigenvectors corresponding to the eigenvalues $s - \frac{\varepsilon}{s-1}$ and $\frac{\varepsilon}{(s-1)^2}$, respectively. From (3.2) it follows that

$$\sum_{1 \le p \le q \le s} \sigma_p \sigma_q A_{pq} \ge \frac{\varepsilon}{(s-1)^2} \sum_{1 \le p \le s} \sigma_p^2$$

and

$$\left(s - 1 - \frac{\varepsilon}{s-1} + \varepsilon_1 \right) \sum_{1 \le p \le s} \sigma_p^2 A_{pp} - \sum_{1 \le p < q \le s} \sigma_p \sigma_q A_{pq} \ge \varepsilon_1 \sum_{1 \le p \le s} \sigma_p^2.$$

Since ε and ε_1 can be arbitrary small positive numbers, the inequality (2.1) is optimal.

In this case, the amplification factor is

$$G(\boldsymbol{\xi}) = \frac{1 + (1-\theta) J(\boldsymbol{\xi})}{1 - \theta J(\boldsymbol{\xi})},$$

where

$$J(\boldsymbol{\xi}) = \sum_{1 \le p \le s} 2\lambda_p (\cos \xi_p - 1) A_{pp} - \sum_{1 \le p < q \le s} \sqrt{\lambda_p \lambda_q} \sin \xi_p \sin \xi_q A_{pq}.$$

We need to prove that for any $\bar{\omega}(s) < \omega(s)$ there exist $\boldsymbol{\xi}$ and λ_p, $p = 1, 2, \ldots, s$ satisfying

$$\frac{1}{2\omega(1-2\theta)} < \sum_{1 \le p \le s} \lambda_p \rho(A_{pp}) < \frac{1}{2\bar{\omega}(1-2\theta)}$$

such that $G(\boldsymbol{\xi}) < -1$, which means the schemes corresponding to (3.1) are unstable.

Set $\Delta x_p = \Delta x$, $\lambda_p = \lambda$, $\xi_p = \xi_0$, and $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_0)$ for $p = 1, \ldots, s$. Then $\sum_{1 \le p \le s} \lambda_p \rho(A_{pp}) = s\lambda$. We choose λ satisfying

$$\frac{1}{2\omega(1-2\theta)} < s\lambda < \frac{1}{2\bar{\omega}(1-2\theta)}. \tag{3.3}$$

In fact, there holds $J(\boldsymbol{\xi}) \le 0$ for all $\boldsymbol{\xi}$. Let $f(\xi_0) = 2 + (1-2\theta) J(\boldsymbol{\xi})$. Then

$$f(\xi_0) = 2 + 2(1-2\theta)\lambda \left(s \cos \xi_0 - s - \frac{s(s-1)}{2} \left(1 - \frac{\varepsilon}{(s-1)^2} \right) \sin^2 \xi_0 \right) \quad \text{for} \quad \xi_0 \in \mathrm{R}.$$

It is easy to show that

$$f(\bar{\xi}_0) = \min_{\xi_0 \in \mathrm{R}} f(\xi_0) = 2 - 4s\lambda(1-2\theta)\omega_0(s), \tag{3.4}$$

where $\omega_0(s) = \frac{(a(s-1)+1)^2}{4a(s-1)}$ and $a \equiv 1 - \frac{\varepsilon}{(s-1)^2}$. Since ε can be an arbitrary small positive number, $\omega_0(s) > \omega(s) > \bar{\omega}(s)$. By (3.3), we have

$$\frac{1}{2\omega_0(1-2\theta)} < s\lambda$$

and by (3.4), $f(\bar{\xi}_0) < 0$, i.e., $G(\bar{\boldsymbol{\xi}}) < -1$ for some $\bar{\boldsymbol{\xi}} = (\bar{\xi}_0, \ldots, \bar{\xi}_0)$. The stability condition given in Theorem 1.2 is optimal.

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REFERENCES

[1] P.J. Davis, *Circulant Matrices*, Wiley, New York, 1979.

[2] M. Goldberg, *Stable difference schemes for parabolic systems—A numerical radius approach*, SIAM J. Numer. Anal., 35 (1998), pp. 478–493.

[3] M. Goldberg, *Stable difference schemes for parabolic systems—A numerical radius approach* II, SIAM J. Numer. Anal., 35 (1998), pp. 1995–2003.

[4] M. Goldberg and E. Tadmor, *On the numerical radius and its applications*, Linear Algebra Appl., 42 (1982), pp. 263–284.

[5] H. O. Kreiss, *Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differentialgleichungen approximieren*, Nordisk Tidskr. Informations-Behandling, 2 (1962), pp. 153–181.

[6] R.D. Richtmyer and K.W. Morton, *Difference Methods for Initial-Value Problems*, 2nd ed., Interscience, New York, 1967.

[7] V. Thomée, *Finite difference methods for linear parabolic equations*, in Handbook of Numerical Analysis, Vol. I, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1990, pp. 77–189.