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A CONDITION NUMBER FOR MULTIFOLD CONIC SYSTEMS*

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Abstract. Let $A : Y \to X$ be a linear map and $K \subseteq X$ be a regular closed convex cone. Consider the problem of finding a nontrivial solution to the conic feasibility problem $Ay \in K$. Condition numbers for this problem (as well as for related ones) are studied to quantify various issues concerning properties of the conic feasibility problem. Some issues especially relevant are the behavior of the problem under data perturbations, the geometry of the set of solutions, and the complexity analyses of algorithms that solve the problem. In this paper we define and characterize a condition number that exploits the possible factorization of K as a product of simpler cones. This condition number extends both Renegar's condition number and the one we defined in [*Math. Program.*, 91 (2001), pp. 163–174] for polyhedral conic systems. We see these results as a step in developing a theory of conditioning that takes into account the structure of the problem.

Key words. condition numbers, conic systems

AMS subject classifications. Primary, 90C25; Secondary, 90C31

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1. Introduction.

1.1. Multifold conic systems and condition. Let X, Y be real finite-dimensional vector spaces (not necessarily of the same dimension) endowed with norms $\| \|_X$ and $\| \|_Y$, and let $K \subseteq X$ be a regular closed convex cone (a precise definition is in section 2 below). Denote by $L(Y, X)$ the space of linear maps from Y to X endowed with the operator norm. Given $A \in L(Y, X)$, consider the feasibility problem: decide whether there exists a nontrivial $y \in Y$ satisfying

$$Ay \in K. \tag{1.1}$$

This format encompasses, after homogenization, a large class of feasibility problems. For example, the linear programming feasibility problem corresponds to $K = \mathbb{R}^n_+$, the nonnegative orthant in $\mathbb{R}^n$, and semidefinite programming corresponds to $K = \mathbf{S}^n_+$, the set of $n \times n$ positive semidefinite matrices. Consider also the *alternative* feasibility problem

$$A^* x^* = 0,\ x^* \in K^*, \tag{1.2}$$

where X^*, Y^* are the dual spaces of X, Y, respectively, $A^* \in L(X^*, Y^*)$ is the adjoint of A, and $K^* \subseteq X^*$ is the dual cone of K.

The problem (1.1) is strictly feasible if there exists $y \in Y$ such that $Ay \in \mathsf{int}(K)$. Let $\mathcal{D}$ denote the set of instances $A \in L(Y, X)$ for which (1.1) is strictly feasible.

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Observe that $A \in \mathcal{D}$ if and only if $AY - K = X$, i.e., if and only if the conic system

$$Ay - c \in K, \tag{1.3}$$

is feasible for every $c \in X$.

Likewise, the problem (1.2) is strictly feasible if there exists $x^* \in \mathsf{int}(K^*)$ such that $A^*x^* = 0$. Let $\mathcal{P}$ denote the set of instances $A \in L(Y,X)$ such that A^* is surjective and (1.2) is strictly feasible. Observe that $A \in \mathcal{P}$ if and only if $A^*K^* = Y^*$, i.e., if and only if the conic system

$$A^*x^* = b^*,\ x^* \in K^*, \tag{1.4}$$

is feasible for every $b^* \in Y^*$.

It is easy to see that the sets $\mathcal{D}$ and $\mathcal{P}$ are open, and $\mathcal{P} = \overline{\mathcal{D}}^c$. The sets $\mathcal{D}$ and $\mathcal{P}$ are the set of "well-posed" feasible instances for problems (1.1) and (1.2), respectively. The boundary $\Sigma := \partial\mathcal{D} = \partial\mathcal{P}$ is the set of "ill-posed" instances. Given $A \in \Sigma$, arbitrarily small perturbations of A may yield instances in both $\mathcal{D}$ and $\mathcal{P}$.

The feasibility problems (1.1) and (1.2) can be solved via algorithms (such as interior-point or ellipsoid methods). The theoretical running time of such algorithms grows as A approaches Σ. Consequently, a complexity analysis of these algorithms has been carried out in terms of a measure capturing this distance. Similar remarks hold as well, with natural modifications, for linear conic optimization problems with constraints of the form (1.3) or (1.4). A general analysis of such a type for interior-point methods is due to Renegar [30, 31], who introduced the condition number

$$C(A) = \frac{\|A\|}{\mathsf{dist}(A,\Sigma)} = \frac{\|A\|}{\min_{\widetilde{A}\in\Sigma}\|\widetilde{A} - A\|}. \tag{1.5}$$

Renegar's condition number is thus the normalized inverse of the distance to ill-posedness. The condition number $C(A)$ can also be used in the complexity analysis of the ellipsoid method [17] and in the round-off analysis of interior-point algorithms [10]. For the linear programming feasibility problem ($K = \mathbb{R}^n_+$), the quantities $C(A)$ and $\ln C(A)$ have also been analyzed as random variables when A is random [7, 13]. Bounds for the expected value of $C(A)$ (or for that of $\ln C(A)$) yield average case bounds for the algorithms mentioned above.

It is often the case that a feasibility problem of the form (1.1) is actually the coupling of a number of similar feasibility problems. More precisely, if $X = X_1 \times \cdots \times X_r$ and $K = K_1 \times \cdots \times K_r$, where each $K_j \subseteq X_j$ is a regular closed convex cone, then (1.1) can be written as

$$\begin{aligned} A_1 y &\in K_1 \\ &\vdots \\ A_r y &\in K_r, \end{aligned} \tag{1.6}$$

where each $A_j \in L(Y, X_j)$ is the projection of $A \in L(Y,X)$ onto $L(Y,X_j)$. In this *multifold* case, it may well be the case that $C(A)$ is large, but a natural preconditioning, such as component normalization, could remove the seemingly bad conditioning. This limitation of $C(A)$ may yield a nonessential overestimate on the conditioning of the problem (1.6). The latter in turn often leads to results concerning the geometry of the set of feasible solutions, as well as complexity estimates of algorithms that are overly conservative. Consequently, some condition-based analyses such as those in [10, 27] are presented in terms of the condition number of a problem after performing some appropriate preprocessing steps on the data. In the case of linear programming (i.e., when $X = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, $r = n$, and $K_j = \mathbb{R}_+$) another condition number $\mathscr{C}(A)$ was introduced in [6] (extending ideas in [19]) exploiting the multifold structure of $\mathbb{R}^n_+$. This condition number is close in spirit to $C(A)$ but is invariant under row scaling and is defined (in the feasible case) in terms of a *best conditioned* solution to (1.6). This condition number can also be used in the analysis of algorithms (e.g., the analysis in [10] carries over to $\mathscr{C}(A)$) and has also been studied as a random variable [8, 11, 20].

In this paper we show that the definition and key characterization of $\mathscr{C}(A)$ extend to the general multifold conic system (1.6) for a particular class of norms in X. An immediate consequence of our results is a maxmin characterization of $C(A)$, which emphasizes the close relationship between $C(A)$ and $\mathscr{C}(A)$. In the special case when the multifold structure of (1.6) is ignored, our work is closely related to previous characterizations of the distance to ill-posedness of the system $Ay \in K$ by Freund and Vera [18] and by Cánovas et al. [5]. More precisely, Theorem 1.1 without scaling on the components of (1.6) follows from [18, Thms. 7 and 10]. Furthermore, Theorem 1.1 without scaling also holds for an infinite family of linear inequalities as was shown in [5, Thm. 7]. On the other hand, an extension of the condition number $\mathscr{C}(A)$ to general conic systems was proposed by Lara and Tunçel [22]. However, that condition number conveys only information about the geometry of the set of feasible solutions of (1.6) and does not have a direct relationship to the distance to ill-posedness of $Ay \in K$.

The central results in our paper, namely, Theorems 1.1 and 1.2, can be seen as steps in the development of a theory of *structured condition numbers* in the spirit introduced by Peña [28, 29] and Lewis [23].

1.2. Statement of the main results. Given a triple (X, K, e), with X a finite-dimensional normed space, $K \subseteq X$ a regular closed convex cone, and $e \in \mathsf{int}(K)$ a given point, define $\lambda_{\min} : X \to \mathbb{R}$ as

$$x \mapsto \max\{t \in \mathbb{R} : x - te \in K\}. \tag{1.7}$$

Notice that $\lambda_{\min}$ is *positively homogeneous*, i.e., it satisfies

$$\lambda_{\min}(sx) = s\lambda_{\min}(x) \quad \text{for all} \;\; s \geq 0 \;\; \text{and} \;\; x \in X,$$

and *superlinear*, i.e., it satisfies

$$\lambda_{\min}(x + u) \geq \lambda_{\min}(x) + \lambda_{\min}(u) \quad \text{for all} \;\; x, u \in X.$$

Notice also that $x \in K \Leftrightarrow \lambda_{\min}(x) \geq 0$ and $x \in \mathsf{int}(K) \Leftrightarrow \lambda_{\min}(x) > 0$.

We next proceed with conditioning.

For $j = 1, \ldots, r$, let $e_j \in \mathsf{int}(K_j)$ be fixed and $\lambda^j_{\min} : X_j \to \mathbb{R}$ denote the function (1.7) corresponding to the triple (X_j, K_j, e_j).

Let $\mathbb{R}_{++} = (0, +\infty)$ and $\alpha \in \mathbb{R}^r_{++}$ be given. Define the *condition value* of a point $y \in Y \setminus \{0\}$ to be

$$v_{A,\alpha}(y) := \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j \|y\|}. \tag{1.8}$$

Observe that y is a strict solution to (1.6) if and only if $v_{A,\alpha}(y) > 0$. Define the best conditioned value $\overline{v}_{A,\alpha}$ to be

$$\overline{v}_{A,\alpha} := \max_{y \neq 0} \; v_{A,\alpha}(y).$$

Notice that $A \in \mathcal{D}$ if and only if $\overline{v}_{A,\alpha} > 0$, $A \in \Sigma$ if and only if $\overline{v}_{A,\alpha} = 0$, and $A \in \mathcal{P}$ if and only if $\overline{v}_{A,\alpha} < 0$. Notice also that this is valid for all $\alpha \in \mathbb{R}^r_{++}$.

For $a \in X$ and $\delta > 0$ let

$$\mathbb{B}_X(a, \delta) := \{x \in X : \|x - a\| \le \delta\}.$$

We shall say that the triple (X, K, e) satisfies the *norm compatibility condition* if $\|e\| = 1$ and the following condition holds:

$$\text{(NC)} \qquad \mathbb{B}_X(e, 1) \subseteq K.$$

We will see in section 2 below that a natural, canonical norm can be associated to any triple (X, K, e) such that the triple satisfies the norm compatibility condition for this canonical norm.

We are now ready to state our main results (we delay their proofs to section 3).

THEOREM 1.1. *If each one of the triples* (X_j, K_j, e_j), $j = 1, \ldots, r$, *satisfies the norm compatibility condition* (NC), *then*

$$|\overline{v}_{A,\alpha}| = \min_{\widetilde{A} \in \Sigma} \max_{j=1,\ldots,r} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j}, \tag{1.9}$$

where the norms in the right-hand side are the operator norms induced by the norms in Y *and* X_j.

It is customary to define condition numbers either as a relativized distance to ill-posedness or as the condition of a best conditioned solution. Theorem 1.1 shows that both choices lead to the same notion by taking the *condition number with respect to* α to be

$$C_\alpha(A) := \frac{1}{|\overline{v}_{A,\alpha}|}$$

and requiring the norm in X to satisfy $\|(x_1, \ldots, x_r)\| = \max_{j=1,\ldots,r} \|x_j\|$. Note that the distance to ill-posedness in the right-hand side in Theorem 1.1 is relativized by the vector α.

In the previous development we have assumed that $\alpha_j > 0$ for $j = 1, \ldots, r$. From a perturbation theory viewpoint this corresponds to assuming that all of the A_j can be perturbed and that the magnitude of these perturbations are weighted (or relativized) by the α_j.

We next consider the case where some blocks are rigid, i.e., cannot be perturbed. This amounts to setting the corresponding α_j to zero. To that end, assume that $B \cup N = \{1, \ldots, r\}$ is a partition of $\{1, \ldots, r\}$, with $B \neq \emptyset$. Let $X_N = \prod_{j \in N} X_j$, $K_N = \prod_{j \in N} K_j$, and $A_N = \prod_{j \in N} A_j$. Write also $\alpha_N = (\alpha_j)_{j \in N}$ and $\alpha_B = (\alpha_j)_{j \in B}$. If we allow only perturbations in the blocks A_j for $j \in B$, then the following extension of Theorem 1.1 holds.

THEOREM 1.2. *Assume that each one of the triples* (X_j, K_j, e_j), $j = 1, \ldots, r$, *satisfies the norm compatibility condition. Then, for* $A \notin \Sigma$,

$$\left| \max_{\substack{A_N y \in K_N \\ y \neq 0}} \min_{j \in B} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j \|y\|} \right| = \min_{\substack{\widetilde{A} \in \Sigma \\ \widetilde{A}_N = A_N}} \max_{j \in B} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j} \tag{1.10}$$

with the convention that the left-hand side above is $+\infty$ *if*

$$\{y : A_N y \in K_N,\ y \neq 0\} = \emptyset$$

and the right-hand side is $+\infty$ *if*

$$\{\widetilde{A} \in \Sigma : \widetilde{A}_N = A_N\} = \emptyset.$$

Furthermore, (1.10) *can be seen as a limit case of* (1.9) *in Theorem* 1.1. *More precisely, for* $A \notin \Sigma$,

$$\left| \max_{\substack{A_N y \in K_N \\ y \neq 0}} \min_{j \in B} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j \|y\|} \right| = \lim_{\substack{\alpha_B \text{ fixed} \\ \alpha_N \downarrow 0}} |\overline{v}_{A,\alpha}|$$

and

$$\min_{\substack{\widetilde{A} \in \Sigma \\ \widetilde{A}_N = A_N}} \max_{j \in B} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j} = \lim_{\substack{\alpha_B \text{ fixed} \\ \alpha_N \downarrow 0}} \min_{\widetilde{A} \in \Sigma} \max_{j=1,\ldots,r} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j}.$$

Remark 1.3. The identity (1.10) in Theorem 1.2 does not necessarily hold if $A \in \Sigma$. For instance, consider the example $r = 2$, $B = \{1\}$, $\alpha_1 = 1$, $X_1 = \mathbb{R}$, $X_2 = Y = \mathbb{R}^2$, $K_1 = \mathbb{R}_+$, $K_2 = \mathbb{R}^2_+$, $A_1 = [1\ 0]$, and $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. In this example $A \in \Sigma$, and thus the right-hand side in (1.10) is zero, but a simple calculation shows that the left-hand side is one.

Nevertheless, when $A \in \Sigma$, a modified version of (1.10) holds if the set of ill-posed instances Σ is redefined by taking into account the relationship between the rigid part $A_N Y$ and the cone K_N.

1.3. Geometric interpretation of $\overline{v}_{A,\alpha}$. When $A \in \mathcal{D}$, any point $\overline{y} \in Y$ that satisfies

$$v_{A,\alpha}(\overline{y}) = \overline{v}_{A,\alpha}$$

can be interpreted as a best conditioned point for (1.1). Notice that in this case the best condition value $\overline{v}_{A,\alpha}$ satisfies

$$\overline{v}_{A,\alpha} = \max\{\delta > 0 : \exists y \in Y, \|y\| = 1, \text{ such that (s.t.) } \|x_i\| \leq \delta\alpha_i \Rightarrow Ay + x \in K\}.$$

In particular, $\mathbb{B}_Y(\overline{y}, \rho)$ is contained in the feasible solution set of (1.1) for $\rho = \min_{j=1,\ldots,r} \frac{\overline{v}_{A,\alpha}\alpha_j}{\|A_j\|}$. Furthermore, from Theorem 1.1 and [28, Thm. 2.11], it follows that, for $A \in \mathcal{D}$, the best condition value $\overline{v}_{A,\alpha}$ satisfies

$$\overline{v}_{A,\alpha} = \max\left\{\delta > 0 : \|x_i\| \leq \delta\alpha_i \Rightarrow x \in \{Ay - K : \|y\| \leq 1\}\right\}.$$

In other words, $\overline{v}_{A,\alpha} \prod \mathbb{B}_{X_i}(0, \alpha_i)$ is the largest multiple of $\prod \mathbb{B}_{X_i}(0, \alpha_i)$ contained in the set

$$\{Ay + K : \|y\| \leq 1\}.$$

Likewise, from Theorem 1.1 and [28, Thm. 2.8], it follows that, for $A \in \mathcal{P}$, the best condition value $\overline{v}_{A,\alpha}$ satisfies the following geometric property:

(1.11)
$$|\overline{v}_{A,\alpha}| = \max\left\{\delta > 0 : \|y^*\|^* \leq \delta \Rightarrow y^* \in \left\{A^* x^* : x^* \in K^*, \sum \alpha_i \|x_i^*\|^* \leq 1\right\}\right\}.$$

In other words, $\mathbb{B}_{Y^*}(0, |\overline{v}_{A,\alpha}|)$ is the largest (dual) ball centered at 0 that is contained in the set

$$\left\{ A^* x^* : x^* \in K^*, \sum \alpha_i \|x_i^*\|^* \leq 1 \right\}.$$

The identity (1.11) yields the existence of *well-conditioned* solutions to (1.2), as is stated more precisely in section 2.4 below.

We also note that the above geometric interpretations of $\overline{v}_{A,\alpha}$ can be extended to the case when some of the components of α are zero. Specifically, if $B \cup N = \{1, \dots, r\}$ is a partition of $\{1, \dots, r\}$ such that $\alpha_B > 0$ and $\alpha_N = 0$, then the statements above hold as long as $\overline{v}_{A,\alpha}$ is replaced with

$$\max_{\substack{A_N y \in K_N \\ y \neq 0}} \min_{j \in B} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j \|y\|}.$$

2. Canonical norm and examples.

In this section we recall some basic notions, describe various cones, exhibit explicit descriptions of their corresponding functions $\lambda_{\min}$ and canonical norms, and show how Theorems 1.1 and 1.2 apply in a number of situations.

2.1. Cones and norms.

A *pointed cone* in $\mathbb{R}^n$ is a set $K \subseteq \mathbb{R}^n$ satisfying
(i) for every $x \in \mathbb{R}^n$, if $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$, and
(ii) $K \cap -K = \{0\}$.
A cone is *regular* if it is pointed and has a nonempty interior. In what follows we assume that all cones are closed, convex, and regular.

We first show that the norm compatibility condition for a given triple (X, K, e) can be alternatively stated in terms of other geometric objects. To do so, recall [19] that the *width* of K is given by $\tau_K = \max\{r \in \mathbb{R} \mid \mathbb{B}(x, r) \subseteq K, \|x\| = 1\}$ and the *center* of K is the point $f \in K$ where τ_K is attained. It follows from the regularity of K that $0 < \tau_K \leq 1$.

PROPOSITION 2.1. *Assume that* $\|e\| = 1$. *The following conditions are equivalent:*
(NC) $\mathbb{B}(e, 1) \subseteq K$.
(NC') $|\lambda_{\min}(x) - \lambda_{\min}(u)| \leq \|x - u\|$ *for all* $x, u \in X$.
(W) $\tau_K = 1$ *and* e *is the center of* K.
(L) $\|s\|^* = \langle e, s \rangle$ *for all* $s \in K^*$.

Proof. To prove the equivalence between (NC) and (NC'), first observe that (NC') can be equivalently phrased as

(NC") $\quad |\lambda_{\min}(x + d) - \lambda_{\min}(x)| \leq 1 \quad$ for all $d \in \mathbb{B}(e, 1)$ and $x \in X$.

Assume that (NC") holds. Then in particular for all $d \in \mathbb{B}(e, 1)$ we have

$$\lambda_{\min}(e + d) - \lambda_{\min}(e) \geq -1.$$

So

$$\lambda_{\min}(e + d) \geq -1 + \lambda_{\min}(e) = 0,$$

and consequently $e + d \in K$. Since this holds for all $d \in \mathbb{B}(e, 1)$, we get (NC).

Conversely, assume that (NC) holds. Let $d \in \mathbb{B}(e, 1)$ and $x \in X$ be given. By the construction of $\lambda_{\min}$ we have

$$x - \lambda_{\min}(x) e \in K,$$

and by (NC) we have

$$d + e \in K.$$

Hence $x + d - (\lambda_{\min}(x) - 1)e \in K$. Consequently $\lambda_{\min}(x + d) \geq \lambda_{\min}(x) - 1$ by the construction of $\lambda_{\min}$. Thus

$$\lambda_{\min}(x + d) - \lambda_{\min}(x) \geq -1. \tag{2.1}$$

On the other hand, again by the construction of $\lambda_{\min}$ and by (NC), we have

$$x + d - \lambda_{\min}(x + d)e \in K$$

and

$$-d + e \in K.$$

Hence $x - (\lambda_{\min}(x + d) - 1)e \in K$. Consequently $\lambda_{\min}(x) \geq \lambda_{\min}(x + d) - 1$, i.e.,

$$\lambda_{\min}(x + d) - \lambda_{\min}(x) \leq 1. \tag{2.2}$$

We thus get (NC") from (2.1) and (2.2).

Condition (NC) amounts to saying that $\tau_K = 1$ and that e is the center of K. Hence the equivalence of (NC) and (W). Finally, the equivalence between (W) and (L) is shown in [17, Proposition 2.1]. □

Remark 2.2. (i) Any triple (X, K, e) can be endowed with the following *canonical norm* $\| \ \|_{\mathbf{c}}$ so that (X, K, e) satisfies the norm compatibility condition:

$$\|x\|_{\mathbf{c}} := \min\{\alpha \geq 0 : x + \alpha e \in K, \ -x + \alpha e \in K\}.$$

This canonical norm plays a central role in primal-dual interior-point methods for self-scaled cones. In such a context, it is generally denoted as $| \ |_e$. See, e.g., [25, 26, 32].

(ii) In case the norms of some X_j do not satisfy (NC), one may extend Theorem 1.1 to obtain inequalities involving the widths τ_{K_j} of the respective cones.

As some of the examples below illustrate, the canonical norm $\|x\|_{\mathbf{c}}$ above coincides with commonly used norms in a number of cases.

Example 1 (cone of squares in Euclidean Jordan algebras). Consider $(X, K, e) = (\mathcal{E}, \mathcal{K}, e)$, where $\mathcal{E}$ is an Euclidean Jordan algebra, $\mathcal{K}$ is the closure of the cone of squares in $\mathcal{E}$, and $e \in \mathcal{K}$ is the identity element [14]. In this case

$$\lambda_{\min}(x) = \min_{j=1,\ldots,q} \lambda_j(x),$$

and the canonical norm is the spectral norm

$$\|x\|_{\mathbf{c}} = \max_{j=1,\ldots,q} |\lambda_j(x)|,$$

where the $\lambda_j(x)$, $j = 1, \ldots, q$, are the Jordan algebra eigenvalues of x, i.e., the eigenvalues of the characteristic polynomial $\det(\lambda e - x)$ for a suitable homogeneous polynomial det [14, Chap. 3].

Examples 2–7 specialize Example 1 above. They provide explicit expressions for $\lambda_{\min}(\ )$ and $\| \ \|_{\mathbf{c}}$ for some specific Jordan algebras. It should be noted that the explicit expressions in Examples 2–4 have been known in optimization for some time (see [16, sect. 2] and [25, sect. 3]).

Example 2 (nonnegative orthant). Consider $(X, K, e) = (\mathbb{R}^n, \mathbb{R}^n_+, (1, \dots, 1))$. In this case $q = n,\ \lambda_j(x) = x_j,\ j = 1, \dots, n$. Consequently,

$$\lambda_{\min}(x) = \min_j x_j, \quad \|x\|_{\mathbf{c}} = \|x\|_\infty = \max_j |x_j|.$$

Example 3 (second-order cone). Consider $(X, K, e) = (\mathbb{R}^{n+1}, \mathcal{Q}^n, (1, 0, \dots, 0))$, where $\mathcal{Q}^n$ is the *second-order cone* defined to be

$$\mathcal{Q}^n := \{x = (x_0, \overline{x}), \overline{x} \in \mathbb{R}^n : x_0 \geq \|\overline{x}\|_2\}.$$

In this case $q = 2,\ \lambda_1(x) = x_0 - \|\overline{x}\|_2,\ \lambda_2(x) = x_0 + \|\overline{x}\|_2$. Consequently,

$$\lambda_{\min}(x) = x_0 - \|\overline{x}\|_2, \quad \|x\|_{\mathbf{c}} = |x_0| + \|\overline{x}\|_2.$$

Example 4 (semidefinite cone). Consider $(X, K, e) = (\mathbf{S}^n, \mathbf{S}^n_+, \mathrm{I})$, where $\mathbf{S}^n$ is the set of $n \times n$ symmetric matrices, $\mathbf{S}^n_+$ is the subset of those which are positive semidefinite, and I is the identity matrix. In this case

$$\lambda_{\min}(x) = \min_{j=1,\dots,n} \lambda_j(x) \qquad \text{and} \qquad \|x\|_{\mathbf{c}} = \max_{j=1,\dots,n} |\lambda_j(x)|,$$

where $\lambda_j(x),\ j = 1, \dots, n$, are the usual eigenvalues of x, i.e., the roots of $p(\lambda) := \det(\lambda \mathrm{I} - x)$.

Example 5 (cones of positive semidefinite Hermitian matrices). Consider (X, K, e), where X is the real vector space $\mathrm{Herm}(n, \mathbb{C})$ of $n \times n$ Hermitian matrices with complex entries, K is the cone of positive semidefinite Hermitian matrices in X, and e is the $n \times n$ identity matrix. In this case $q = n,\ \lambda_j(x),\ j = 1, \dots, n$, are the usual eigenvalues of x, i.e., the roots of $p(\lambda) := \det(\lambda \mathrm{I} - x)$ which, it is well known, are real.

Example 6 (cones of positive semidefinite Hermitian matrices with quaternions entries). Consider (X, K, e), where X is the real vector space $\mathrm{Herm}(n, \mathbb{H})$ of $n \times n$ Hermitian matrices with quaternion entries, K is the cone of positive semidefinite Hermitian matrices in X, and e is the $n \times n$ identity matrix. In this case $q = n$ and $\lambda_j(x),\ j = 1, \dots, n$, are the roots (as a univariate polynomial in λ) of the "characteristic polynomial" $\det(\lambda e - x)$ of X. This polynomial is defined as follows [15]. Let J be the $2n \times 2n$ matrix $\left[\begin{smallmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{smallmatrix}\right]$. Then $\mathrm{Herm}(n, \mathbb{H})$ can be written as $\{z \in \mathrm{Herm}(2n, \mathbb{C}) : \overline{z}J = Jz\}$, and, for $z \in \mathrm{Herm}(n, \mathbb{H})$,

$$\det(z) = \mathrm{Pf}(Jz),$$

where $\mathrm{Pf}(Jz)$ is the *Pfaffian* of Jz, i.e., the unique polynomial satisfying $\mathrm{Pf}(J) = 1$ and $\mathrm{Pf}(Jz)^2 = \det(Jz)$. Again, it is well known that the $\lambda_j(x),\ j = 1, \dots, n$, are real. (For a more detailed discussion on Pfaffians, see, e.g., [21].)

Example 7 (cones of squares in the Albert algebra). Consider (X, K, e), where X is the real vector space of 3×3 Hermitian matrices with octonion entries [2, 9], K is the cone of squares in X, i.e., $K = \{x^2 : x \in X\}$, and e is the 3×3 identity matrix. In this case $q = 3$ and $\lambda_j(x),\ j = 1, 2, 3$, are the roots of the characteristic polynomial

$$p(\lambda) = \det(\lambda e - x) = \lambda^3 - \mathrm{trace}(x)\lambda^2 + \sigma(x)\lambda - \det(x),$$

where $\mathrm{trace}(x), \sigma(x), \det(x)$ are defined as follows [12, 15]. For a, b, c octonions and $p, m, n \in \mathbb{R}$,

$$x = \begin{bmatrix} p & a & \overline{b} \\ \overline{a} & m & c \\ b & \overline{c} & n \end{bmatrix},$$

$$\begin{aligned}\mathrm{trace}(x) &= p+m+n,\\ \sigma(x) &= pm+mn+pn-|a|^2-|b|^2-|c|^2,\\ \det(x) &= pmn+b(ac)+\overline{b(ac)}-n|a|^2-m|b|^2-p|c|^2.\end{aligned}$$

Note that we write $b(ac)$ to emphasize the order of the multiplications in the nonassociative ring of octonions. Just as in the previous examples, λ_1, λ_2, and λ_3 are real.

Example 8 (hyperbolicity cones). Let $e \in \mathbb{R}^n$ and $p \in \mathbb{R}[X_1, \ldots, X_n]$ be a *complete hyperbolic polynomial* in the direction e, i.e., a homogeneous polynomial satisfying that, for all $x \in \mathbb{R}^n$, the univariate polynomial $\lambda \mapsto p(\lambda e - x)$ has only real roots, and at least one of them is nonzero for $x \neq 0$. (For a detailed exposition on hyperbolic polynomials, see, e.g., [3, 33].) Consider $(X, K, e) = (\mathbb{R}^n, C(p, e), e)$, where $C(p, e)$ is the closure of the *hyperbolicity cone* for p in the direction e; i.e., $C(p, e)$ is the closure of the connected component of the set $\{x : p(x) > 0\}$ that contains e. In this case

$$\lambda_{\min}(x) = \min_{j=1,\ldots,d} \lambda_j(x),$$

where $\lambda_j(x)$, $j = 1, \ldots, d = \deg(p)$, are the roots of the polynomial $\lambda \mapsto p(\lambda e - x)$, and

$$\|x\|_{\mathbf{c}} = \max_{j=1,\ldots,d} |\lambda_j(x)|.$$

Example 9 (nonnegative, finitely spanned, functions on a compact domain). Assume that $d \in \mathbb{N}$, $D \subseteq \mathbb{R}^n$ is a nonempty compact set, and $f_0, \ldots, f_d$ are continuous, real-valued functions defined on D, with $f_0(x) = 1$, for all $x \in D$. Consider the triple (X, K, e), where

$$X = \mathrm{span}\{f_0, \ldots, f_d\},$$

$$K = \{f \in X : f(x) \geq 0 \text{ for all } x \in D\},$$

and $e \in X$ is the constant function f_0. In this case, for $f \in X$,

$$\lambda_{\min}(f) = \min_{x \in D} f(x)$$

and

$$\|f\|_{\mathbf{c}} = \max_{x \in D} |f(x)|.$$

2.2. Cone reducibility. Assume that X is a finite-dimensional inner product space. Then the map $u \mapsto \langle u, \cdot \rangle$ defines an isomorphism between X and its dual space X^*. The *dual* of a cone $K \subseteq X$ is identified via this isomorphism with the cone

$$K^* = \{u \in X : \langle u, x \rangle \geq 0 \text{ for all } x \in K\}.$$

We say that K is *self-dual* if $K^* = K$. We say that K is *homogeneous* if for all $x, u \in \mathsf{int}(K)$ there exists $g \in \mathrm{Aut}(\mathsf{int}(K))$ such that $gx = u$, where $\mathrm{Aut}(\mathsf{int}(K)) = \{g \in \mathrm{GL}(X) : g(\mathsf{int}(K)) = \mathsf{int}(K)\}$. Here $\mathrm{GL}(X)$ is the general linear group over X. A cone is *symmetric* if it is self-dual and homogeneous [14].

Symmetric cones coincide with *self-scaled cones,* a class of cones that plays a central role in interior-point methods [25]. Nesterov and Todd identified the properties of self-scaled cones as the fundamental building blocks for the development of symmetric primal-dual interior-point algorithms [26, 32] for conic programs over these cones. Symmetric cones have been extensively studied in other areas of mathematics. It can also be shown that they coincide with the cones of squares of Euclidean Jordan algebras (cf. Example 1). Furthermore, they satisfy a unique factorization property; namely, they can be written in a unique way (up to ordering) as a product of cones in the classes described in Examples 3–7 above [14].

Because second-order conic feasibility over a single second-order cone can be solved in closed form, all interesting examples of second-order conic feasiblity problems are written in terms of a nontrivial product of second-order cones (see, e.g., [1, 24]).

Given X and K, one may wonder how the canonical norm and the minimum-eigenvalue constructs depend on different factorizations of (X, K). The following proposition settles this question.

PROPOSITION 2.3. *Let* $X = X_1 \times \cdots \times X_r$, $K = K_1 \times \cdots \times K_r$, *and* $e = (e_1, \ldots, e_r)$, *with* $e_j \in \mathsf{int}(K_j)$. *Then, for all* $x = (x_1, \ldots, x_r) \in X$,

(i)

$$\lambda_{\min}(x) = \min\{\lambda^1_{\min}(x_1), \ldots, \lambda^r_{\min}(x_r)\}.$$

where $\lambda^j_{\min}$ *is the minimum eigenvalue associated to* (X_j, K_j, e_j) *and* $\lambda_{\min}$ *that associated to* (X, K, e).

(ii)

$$\|x\|_{\mathbf{c}} = \max_{j=1,\ldots,r} \|x_j\|_{\mathbf{c},j},$$

where $\| \ \|_{\mathbf{c},j}$ *is the canonical norm associated to* (X_j, K_j, e_j) *and* $\| \ \|_{\mathbf{c}}$ *that associated to* (X, K, e). *In particular, the restriction of* $\| \ \|_{\mathbf{c}}$ *to* X_j *is* $\| \ \|_{\mathbf{c},j}$.

Proof. From (1.7) it follows that, for $x = (x_1, \ldots, x_r) \in X$,

$$\begin{aligned}
\lambda_{\min}(x) &= \max\{t \in \mathbb{R} : x - te \in K\} \\
&= \max\{t \in \mathbb{R} : x_j - te_j \in K_j \ \text{ for } \ j = 1, \ldots, r\} \\
&= \min_{j=1,\ldots,r} \max\{t \in \mathbb{R} : x_j - te_j \in K_j\} \\
&= \min\{\lambda^1_{\min}(x_1), \ldots, \lambda^r_{\min}(x_r)\}.
\end{aligned}$$

This shows part (i). For part (ii) we first claim that

$$B = B_1 \times \cdots \times B_r. \tag{2.3}$$

Indeed, given $d = (d_1, \ldots, d_r) \in X$,

$$\begin{aligned}
d \in B &\Leftrightarrow e + d, e - d \in K \\
&\Leftrightarrow e_j + d_j, e_j - d_j \in K_j \ \text{ for } \ j = 1, \ldots, r \\
&\Leftrightarrow d_j \in B_j \ \text{ for } \ j = 1, \ldots, r \\
&\Leftrightarrow d \in B_1 \times \cdots \times B_r.
\end{aligned}$$

From (2.3) it follows that, for $x = (x_1, \dots, x_r) \in X$,

$$\begin{aligned} \|x\|_{\mathbf{c}} &= \inf\left\{t : \frac{1}{t}x \in B\right\} \\ &= \inf\left\{t : \frac{1}{t}x_j \in B_j \quad \text{for} \quad j = 1, \dots, r\right\} \\ &= \max_{j=1,\dots,r} \inf\left\{t : \frac{1}{t}x_j \in B_j\right\} \\ &= \max_{j=1,\dots,r} \|x_j\|_{\mathbf{c},j}. \qquad \square \end{aligned}$$

We have already mentioned that we endow $L(Y, X)$ with the operator norm with respect to the norms in Y and X. Therefore, the canonical norm in X induces a *canonical norm in* $L(Y, X)$. In particular, in the case where $X = X_1 \times \cdots \times X_r$, Proposition 2.3(ii) yields the relation

$$\|A\|_{\mathbf{c}} = \max_{j=1,\dots,r} \|A_j\|_{\mathbf{c},j}$$

for the canonical norms in $L(Y, X)$ and those in $L(Y, X_j)$, $j = 1, \dots, r$.

Remark 2.4. Note that the factorization mentioned above together with Proposition 2.3(ii) and Examples 3–7 yield expressions for the canonical norm for every symmetric cone. If the factorization is explicit, then the expressions for the canonical norm are explicit as well.

2.3. Condition numbers and the choice of $\boldsymbol{\alpha}$. We mentioned in section 1.1 the role of Renegar's condition number in the analysis of algorithms for conic feasibility problems. We also mentioned there that, in the case of polyhedral cones, the condition number $\mathscr{C}(A)$ exploited the reducibility of the cone $\mathbb{R}^n_+$. We next show how these condition numbers are obtained by appropriately selecting α.

Assume a factorization $X = X_1 \times \cdots \times X_r$ and $K = K_1 \times \cdots \times K_r$. Basic choices for α are

(1) $\alpha_j = \|A\|$ for $j = 1, \dots, r$;

(2) $\alpha_j = \|A_j\|$ for $j = 1, \dots, r$.

The first choice leads to Renegar's condition number $C(A)$ for the norm in $L(Y, X)$ defined by

$$\|A\| = \max_{j=1,\dots,r} \|A_j\| \tag{2.4}$$

because in this case

$$C_\alpha(A) = \frac{\|A\|}{\min_{\widetilde{A}\in\Sigma} \max_{j=1,\dots,r} \|A_j - \widetilde{A}_j\|} = \frac{\|A\|}{\min_{\widetilde{A}\in\Sigma} \|A - \widetilde{A}\|} = C(A).$$

Theorem 1.1 then takes the form of a minmax characterization of the distance to ill-posedness (and therefore of $C(A)$). We note that this can also be obtained from [18, Thms. 7 and 10].

The second choice of α above leads to (extensions of) the condition number $\mathscr{C}(A)$ introduced in [7].

The discusion above assumes that $\alpha_j > 0$ for all $j = 1, \dots, r$. If some cones are rigid, say, $K = K_1 \times \cdots \times K_r \times K_N$ with $\alpha_j > 0$ for $j = 1, \dots, r$ and $\alpha_N = 0$, then, by letting $B = \{1, \dots, r\}$, one defines

$$C(A) = \frac{\|A_B\|}{\displaystyle\min_{\substack{\widetilde{A}\in\Sigma \\ \widetilde{A}_N = A_N}} \max_{j\in B} \|A_j - \widetilde{A}_j\|}$$

and

$$\mathscr{C}(A) = \frac{1}{\left|\displaystyle\max_{\substack{A_N y \in K_N \\ y\neq 0}} \min_{j\in B} \frac{\lambda^j_{\min}(A_j y)}{\|A_j\|\,\|y\|}\right|}.$$

The proof of the following proposition is an immediate consequence of the fact that $\|A_j\| \leq \|A\|$ for all $j = 1, \dots, r$.

Proposition 2.5. *For all $A \in L(Y, X)$, $\mathscr{C}(A) \leq C(A)$. This holds as well if some factors of A are rigid.*

We next see how the choices of α above materialize in the case of polyhedral conic systems.

Example 2 (revisited). Recall that $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $K = \mathbb{R}^n_+$.

(i) Consider the case $r = 1$. We do not decompose K. In this case we take $e = (1, \dots, 1)$ and, as we have seen, $\|x\| = \|x\|_\infty$. This induces the canonical norm in $L(Y, X)$ given by

$$\|A\| = \|A\|_{Y\infty} = \max_{\|y\|=1} \|Ay\|_\infty = \max_{\|y\|=1} \max_{j=1,\dots,n} |A_j y|.$$

We now take $\alpha = \|A\|$. Theorem 1.1 and the fact that $\lambda_{\min}(x) = \min_{j=1,\dots,n} x_j$ give then the following characterization of Renegar's condition number:

$$C(A) = \frac{\|A\|}{\mathsf{dist}(A, \Sigma)} = \frac{\|A\|}{\left|\displaystyle\max_{\|y\|=1} \min_{j=1,\dots,n} Ay\right|}. \tag{2.5}$$

(ii) Consider now the case where $r = n$. Here we take $K_j = [0, +\infty)$ and $e_j = 1$ for $j = 1, \dots, r$. We obtain the canonical norm $\|x\| = |x|$ and the minimum-eigenvalue $\lambda_{\min}(x) = x$. The former induces the canonical norm in $L(Y, X)$ given by

$$\|A\| = \max_{j=1,\dots,n} \|A_j\| = \max_{j=1,\dots,n} \max_{\|y\|=1} |A_j y|,$$

that is, as in case (i) above. Again, take $\alpha_j = \|A\|$ for all j. Then, not surprisingly, Theorem 1.1 characterizes $C(A)$ by (2.5) as well.

(iii) We now take r, K_j, and e_j as in (ii) but choose instead $\alpha_j = \|A_j\| = \max_{\|y\|=1} |A_j y|$. In this case Theorem 1.1 gives us the well-known [7] characterization of $\mathscr{C}(A)$:

$$\mathscr{C}(A) := \frac{1}{|\overline{v}_{A,\alpha}|} = \frac{1}{\displaystyle\min_{\widetilde{A}\in\Sigma} \max_{j=1,\dots,n} \frac{\|A_j - \widetilde{A}_j\|}{\|A_j\|}}.$$

(iv) For $M \in \mathbb{R}^{n\times m}$ consider the system

$$\begin{aligned} My &\geq 0, \\ y &\geq 0. \end{aligned}$$

This system can be thought of as a special case of the above with $A = (M, \mathrm{I})$. The identity matrix I, however, should be considered to be rigid (not subject to perturbations) and its corresponding α_{I} then be set to 0.

By taking $r = 2$ (two blocks, corresponding to M and I) and $\alpha_M = \|M\| = \max_{j=1,\dots,n} \max_{\|y\|=1} |M_j y|$, we obtain (in the right-hand side of (1.10)) Renegar's condition number, and Theorem 1.2 shows that

$$C(M) := \frac{\|M\|}{\min\limits_{\widetilde{M}\in\Sigma} \|M - \widetilde{M}\|} = \frac{\|M\|}{\min\limits_{\widetilde{M}\in\Sigma} \max\limits_{j=1,\dots,n} \|M_j - \widetilde{M}_j\|} = \frac{\|M\|}{\left|\max\limits_{y\geq 0} \min\limits_{j=1,\dots,n} \frac{M_j y}{\|y\|}\right|}.$$

Finally, by taking $r = n$ and $\alpha_j = \|M_j\| = \max_{\|y\|=1} |M_j y|$, $j = 1, \dots, n$, we obtain $\mathscr{C}(M)$ in the left-hand side of (1.10), and now Theorem 1.2 shows that

$$\mathscr{C}(M) := \frac{1}{\left|\max\limits_{y\geq 0} \min\limits_{j=1,\dots,n} \frac{M_j y}{\|M_j\|\|y\|}\right|} = \frac{1}{\min\limits_{\widetilde{M}\in\Sigma} \max\limits_{j=1,\dots,n} \frac{\|M_j - \widetilde{M}_j\|}{\|M_j\|}}.$$

We have revisited Example 2 to see how Theorems 1.1 and 1.2, together with appropriate choices of α, yield characterizations of $C(A)$ and $\mathscr{C}(A)$ in the case of polyhedral conic systems, possibly with rigid components. The other examples in section 2.1, and arbitrary products of them, may be similarly dealt with. We will not do so to avoid being repetitious.

2.4. Well-conditioned solutions. As was noted in section 1.3, for $A \in \mathcal{D}$ any point $\overline{y} \in Y$ that satisfies

$$v_{A,\alpha}(\overline{y}) = \overline{v}_{A,\alpha} \tag{2.6}$$

can be interpreted as a best conditioned solution for (1.1). Indeed, from (2.6) it follows that $A\overline{y} \in \mathsf{int}(K)$ and for each $i = 1, \dots, r$

$$\frac{\mathsf{dist}(A_i\overline{y}, \partial K_i)}{\|\overline{y}\|} \geq \overline{v}_{A,\alpha}\, \alpha_i.$$

The following proposition provides an analogous statement for $A \in \mathcal{P}$.

PROPOSITION 2.6. *Assume that each one of the triples* (X_j, K_j, e_j), $j = 1, \dots, r$, *satisfies the norm compatibility condition* (NC) *and* $A \in \mathcal{P}$. *Then there exists* $\overline{x} \in \mathsf{int}(K^*)$ *such that* $A^*\overline{x} = 0$ *and for each* $i = 1, \dots, r$

$$\frac{\mathsf{dist}(\overline{x}_i, \partial K_i^*)}{\|\overline{x}_i\|} \geq \frac{|\overline{v}_{A,\alpha}|\, \alpha_i\, \tau_{K_i^*}}{r\|A_i\| + |\overline{v}_{A,\alpha}|\alpha_i}. \tag{2.7}$$

In particular, if $\alpha_i = \|A_i\|$, $i = 1, \dots, r$, *then there exists* $\overline{x} \in K^*$ *such that* $A^*\overline{x} = 0$ *and for each* $i = 1, \dots, r$

$$\frac{\mathsf{dist}(\overline{x}_i, \partial K_i^*)}{\|\overline{x}_i\|} \geq \frac{|\overline{v}_{A,\alpha}|\tau_{K_i^*}}{r + |\overline{v}_{A,\alpha}|} \geq \frac{|\overline{v}_{A,\alpha}|\tau_{K_i^*}}{r + 1}.$$

Proof. For each $i = 1, \dots, r$, let $f_i^* \in K_i^*$ be the center of K_i^*, i.e., $\|f_i^*\|^* = 1$ and $\mathsf{dist}(f_i^*, \partial K_i^*) = \tau_{K_i^*}$. Define $\widetilde{x} \in K^*$ and $y^* \in Y^*$ as follows:

$$\widetilde{x}_i := \frac{|\overline{v}_{A,\alpha}|}{r\|A_i\|} f_i^*, \; y^* := -A^*\widetilde{x}.$$

It is immediate that $\|y^*\|^* \leq |\overline{v}_{A,\alpha}|$, so by (1.11) there exists $x^* \in K^*$ such that $A^*x^* = y^* = -A^*\widetilde{x}$ and $\sum_{i=1}^r \alpha_i \|x_i^*\|^* \leq 1$. Thus the point $\overline{x} := x^* + \widetilde{x} \in K^*$ satisfies $A^*\overline{x} = 0$. To finish, we next show that $\overline{x}$ satisfies (2.7). Since $\sum_{i=1}^r \alpha_i \|x_i^*\|^* \leq 1$, it follows that $\|x_i^*\|^* \leq 1/\alpha_i$ for each $i = 1, \ldots, r$. Hence

$$\|\overline{x}_i\| \leq \|x_i^*\| + \|\widetilde{x}_i\| \leq \frac{1}{\alpha_i} + \frac{|\overline{v}_{A,\alpha}|}{r\|A_i\|} = \frac{r\|A_i\| + |\overline{v}_{A,\alpha}|\alpha_i}{r\|A_i\|\alpha_i}. \tag{2.8}$$

On the other hand, since $\mathsf{dist}(f_i^*, \partial K_i^*) = \tau_{K_i^*}$, it follows that

$$\mathsf{dist}(\overline{x}_i, \partial K_i^*) \geq \frac{|\overline{v}_{A,\alpha}|\tau_{K_i^*}}{r\|A_i\|}. \tag{2.9}$$

Inequality (2.7) then follows from (2.8) and (2.9). ∎

3. Proof of the main results. The result is trivial when $\overline{v}_{A,\alpha} = 0$. Therefore, we will assume that $\overline{v}_{A,\alpha} \neq 0$. For ease of exposition, we split the proof of Theorem 1.1 into two parts, namely, Propositions 3.1 and 3.2.

PROPOSITION 3.1.

$$|\overline{v}_{A,\alpha}| \leq \min_{\widetilde{A} \in \Sigma} \max_{j=1,\ldots,r} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j}.$$

Proof. Assume that $\widetilde{A}$ is such that

$$\max_{j=1,\ldots,r} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j} < |\overline{v}_{A,\alpha}|.$$

We need to prove that $\widetilde{A} \notin \Sigma$, i.e., $\widetilde{A} \in \mathcal{P} \cup \mathcal{D}$.

Let $\overline{y}_A \in Y$ be such that $v_{A,\alpha}(\overline{y}_A) = \overline{v}_{A,\alpha}$. Assume without loss of generality that $\|\overline{y}_A\| = 1$. Because each (X_j, K_j, e_j) satisfies the norm compatibility condition, it follows from Proposition 2.1 that, for all $\widetilde{A}$ and $y \in Y \setminus \{0\}$,

$$\begin{aligned} \frac{|\lambda_{\min}^j(A_j y) - \lambda_{\min}^j(\widetilde{A}_j y)|}{\alpha_j} &\leq \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j} \|y\| \\ &\leq \max_{j=1,\ldots,r} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j} \|y\| \\ &< |\overline{v}_{A,\alpha}| \|y\|. \end{aligned} \tag{3.1}$$

In particular

$$\frac{|\lambda_{\min}^j(A_j \overline{y}_A) - \lambda_{\min}^j(\widetilde{A}_j \overline{y}_A)|}{\alpha_j} < |\overline{v}_{A,\alpha}|. \tag{3.2}$$

We now consider the cases $\overline{v}_{A,\alpha} < 0$ and $\overline{v}_{A,\alpha} > 0$ separately.

Case 1: $\overline{v}_{A,\alpha} > 0$. In this case $A \in \mathcal{D}$. From (1.8), (3.2), and the equality $v_{A,\alpha}(\overline{y}_A) = \overline{v}_{A,\alpha}$, we get, for $j = 1, \ldots, r$,

$$\begin{aligned} \frac{\lambda_{\min}^j(\widetilde{A}_j \overline{y}_A)}{\alpha_j} &\geq \frac{\lambda_{\min}^j(A_j \overline{y}_A)}{\alpha_j} - \frac{|\lambda_{\min}^j(A_j \overline{y}_A) - \lambda_{\min}^j(\widetilde{A}_j \overline{y}_A)|}{\alpha_j} \\ &> \overline{v}_{A,\alpha} - \overline{v}_{A,\alpha} = 0. \end{aligned}$$

Therefore

$$v_{\widetilde{A}}(\overline{y}_A) = \min_{j=1,\dots,r} \frac{\lambda^j_{\min}(\widetilde{A}_j\overline{y}_A)}{\alpha_j} > 0,$$

which shows that $\overline{y}_A$ is a strict solution for $\widetilde{A}$ and, consequently, that $\widetilde{A} \in \mathcal{D}$.

Case 2: $\overline{v}_{A,\alpha} < 0$. In this case $A \in \mathcal{P}$. Let y be any point in $Y \setminus \{0\}$. Since $\overline{v}_{A,\alpha} < 0$, we must have $v_{A,\alpha}(y) \le \overline{v}_{A,\alpha} < 0$. Let $\overline{j} = \overline{j}(y)$ be such that $\frac{\lambda^{\overline{j}}_{\min}(A_{\overline{j}}y)}{\alpha_{\overline{j}}\|y\|} = v_{A,\alpha}(y)$. We claim that $\lambda^{\overline{j}}_{\min}(\widetilde{A}_{\overline{j}}y) < 0$. Indeed, by (3.1),

$$\begin{aligned}
&\left|\lambda^{\overline{j}}_{\min}(A_{\overline{j}}y) - \lambda^{\overline{j}}_{\min}(\widetilde{A}_{\overline{j}}y)\right| < -\overline{v}_{A,\alpha}\alpha_{\overline{j}}\|y\| \le -v_{A,\alpha}(y)\alpha_{\overline{j}}\|y\| \\
&\Rightarrow \lambda^{\overline{j}}_{\min}(\widetilde{A}_{\overline{j}}y) - \lambda^{\overline{j}}_{\min}(A_{\overline{j}}y) < -v_{A,\alpha}(y)\alpha_{\overline{j}}\|y\| \\
&\Rightarrow \lambda^{\overline{j}}_{\min}(\widetilde{A}_{\overline{j}}y) - v_{A,\alpha}(y)\alpha_{\overline{j}}\|y\| < -v_{A,\alpha}(y)\alpha_{\overline{j}}\|y\| \\
&\Rightarrow \lambda^{\overline{j}}_{\min}(\widetilde{A}_{\overline{j}}y) < 0.
\end{aligned}$$

Hence, for all $y \in Y \setminus \{0\}$ there exists j such that $\lambda^j_{\min}(\widetilde{A}_jy) < 0$. It follows that

$$\overline{v}_{\widetilde{A},\alpha} = \max_{y\neq 0} v_{\widetilde{A},\alpha}(y) = \max_{y\neq 0}\min_{j=1,\dots,r} \frac{\lambda^j_{\min}(\widetilde{A}_jy)}{\alpha_j\|y\|} < 0,$$

that is, $\widetilde{A} \in \mathcal{P}$. ∎

Recall that, given vector spaces X and Y and a linear mapping $A \in L(Y,X)$, its *adjoint* $A^* \in L(X^*,Y^*)$ is the unique linear mapping that satisfies

$$\langle v, Ay\rangle = \langle A^*v, y\rangle \quad \text{for all } v \in X^*,\ y \in Y.$$

PROPOSITION 3.2.

$$|\overline{v}_{A,\alpha}| \ge \min_{\widetilde{A}\in\Sigma}\max_{j=1,\dots,r} \frac{\|A_j - \widetilde{A}_j\|}{\alpha_j}.$$

Proof. We consider the cases $\overline{v}_{A,\alpha} < 0$ and $\overline{v}_{A,\alpha} > 0$ separately.

Case 1: $\overline{v}_{A,\alpha} < 0$. In this case $A \notin \mathcal{D}$, so it suffices to show that for all $\delta > 0$ there exists $\widetilde{A} \in \mathcal{D}$ such that, for all $j = 1,\dots,r$, $\|A_j - \widetilde{A}_j\| \le \alpha_j(|\overline{v}_{A,\alpha}| + \delta)$. Let $\overline{y}_A \in Y$ be such that $v_A(\overline{y}_A) = \overline{v}_{A,\alpha}$. Assume without loss of generality that $\|\overline{y}_A\| = 1$. By the Hahn–Banach theorem [34, Thm. 5.20], there exists $v \in Y^*$ such that $\langle v, \overline{y}_A\rangle = \|\overline{y}_A\| = 1$ and $\|v\|^* = 1$. For $j = 1,\dots,r$, consider $\widetilde{A}_j \in L(Y,X_j)$ given by

$$\widetilde{A}_j = A_j - \alpha_j(\overline{v}_{A,\alpha} - \delta)\langle v, \cdot\rangle e_j.$$

We claim that $\widetilde{A} \in \mathcal{D}$. To see this, first notice that, for all $j = 1,\dots,r$, $A_j\overline{y}_A - \overline{v}_{A,\alpha}e_j \in K_j$ because

$$\overline{v}_{A,\alpha} = v_{A,\alpha}(\overline{y}_A) \le \frac{\lambda^j_{\min}(A_j\overline{y}_A)}{\alpha_j} = \max\{t \mid A_j\overline{y}_A - \alpha_j te_j \in K_j\}.$$

Therefore,

$$\widetilde{A}_j\overline{y}_A = A_j\overline{y}_A - \alpha_j(\overline{v}_{A,\alpha} - \delta)\langle v, \overline{y}_A\rangle e_j = (A_j\overline{y}_A - \alpha_j\overline{v}_{A,\alpha}e_j) + \alpha_j\delta e_j \in \mathsf{int}(K_j)$$

since K_j is convex and $e_j \in \mathsf{int}(K_j)$. This shows that $\overline{y}_A$ is a strict solution for $\widetilde{A}$. To finish, just observe that $\|A_j - \widetilde{A}_j\| = \alpha_j \|(\overline{v}_{A,\alpha} - \delta)\langle v, \cdot\rangle e_j\| \le \alpha_j |\overline{v}_{A,\alpha} - \delta| \|v\|^* \|e_j\| = \alpha_j(|\overline{v}_{A,\alpha}| + \delta)$.

Case 2: $\overline{v}_{A,\alpha} > 0$. In this case $A \in \mathcal{D}$, so it suffices to show that there exists $\widetilde{A} \notin \mathcal{D}$ such that, for all $j = 1, \dots, r$, $\|A_j - \widetilde{A}_j\| \le \alpha_j \overline{v}_{A,\alpha}$. Let $e = (e_1, \dots, e_r) \in K = K_1 \times \cdots \times K_r$. Let $B_j = \frac{1}{\alpha_j} A_j$ and $B = [B_1, \dots, B_r] \in L(Y, X)$. From Proposition 2.3(i) and the positive homogeneity of $\lambda_{\min}$ it follows that for $y \in Y \setminus \{0\}$

$$
\begin{aligned}
v_{A,\alpha}(y) &= \min_{j=1,\dots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j \|y\|} = \frac{1}{\|y\|} \min_{j=1,\dots,r} \lambda^j_{\min}(B_j y) \\
&= \frac{1}{\|y\|} \lambda_{\min}(By) = \frac{1}{\|y\|} \max\{t : By - te \in K\}.
\end{aligned}
$$

Then, by taking maxima on both sides above,

$$
\overline{v}_{A,\alpha} = \max_{y \ne 0} v_{A,\alpha}(y) = \max_{\|y\|=1} \max\{t : By - te \in K\}.
$$

Since $\overline{v}_{A,\alpha} > 0$ we may rewrite the above as a maximum over a convex set

$$
\overline{v}_{A,\alpha} = \max_{\substack{By - te \in K \\ \|y\| \le 1}} t. \tag{3.3}
$$

Consider the Lagrangian dual (see [4]) of the right-hand side of (3.3):

$$
\begin{aligned}
\min_{x \in K^*} \max_{\substack{\|y\| \le 1 \\ t \in \mathbb{R}}} t + \langle x, By - te\rangle &= \min_{x \in K^*} \max_{\substack{\|y\| \le 1 \\ t \in \mathbb{R}}} t(1 - \langle x, e\rangle) + \langle x, By\rangle \\
&= \min_{\substack{x \in K^* \\ \langle x, e\rangle = 1}} \max_{\|y\| \le 1} \langle B^* x, y\rangle \\
&= \min_{\substack{x \in K^* \\ \langle x, e\rangle = 1}} \|B^* x\|^*. \qquad (3.4)
\end{aligned}
$$

Since both (3.3) and (3.4) are convex programs and satisfy the Slater condition, by [4, Thm. 4.3.7], they attain the same optimal value $\overline{v}_{A,\alpha}$. Hence there exists $\overline{x} \in K^*$ such that $\|B^* \overline{x}\|^* = \overline{v}_{A,\alpha}$ and $\langle \overline{x}, e\rangle = 1$. Let $\widetilde{A}_j = A_j - \alpha_j \langle B^*_j \overline{x}_j, \cdot\rangle e_j = A_j - \langle A^*_j \overline{x}_j, \cdot\rangle e_j$. We claim that $\widetilde{A} \notin \mathcal{D}$. Indeed, otherwise, there would exist $y \in Y$ and $\epsilon > 0$ such that $\widetilde{A} y - \epsilon e \in K$ and, therefore,

$$
\begin{aligned}
0 &\le \langle \overline{x}, \widetilde{A} y - \epsilon e\rangle \qquad (\text{because } \overline{x} \in K^*) \\
&= \langle \overline{x}, Ay - (\langle A^* \overline{x}, y\rangle + \epsilon) e\rangle \\
&= -\epsilon \qquad (\text{because } \langle \overline{x}, e\rangle = 1) \\
&< 0,
\end{aligned}
$$

which is a contradiction. Hence $\widetilde{A} \notin \mathcal{D}$. To finish, observe that

$$
\frac{\|\widetilde{A}_j - A_j\|}{\alpha_j} = \|\langle B^*_j \overline{x}_j, \cdot\rangle e_j\| = \|B^*_j \overline{x}_j\|^* \le \|B^* \overline{x}\|^* = \overline{v}_{A,\alpha}. \qquad \square
$$

We next prove Theorem 1.2. We will need the following result.

LEMMA 3.3. *Assume that $A \notin \Sigma$. If the system*

$$A_N y \in \partial K_N, \quad A_B y \in \mathsf{int}(K_B)$$

has a nontrivial solution, then so does the system

$$A_N y \in \mathsf{int}(K_N), \quad A_B y \in \mathsf{int}(K_B).$$

Proof. Let $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$. By hypothesis, $v_{A,\mathbf{1}}(y) = 0$ and so $\overline{v}_{A,\mathbf{1}} \geq 0$. Since $A \notin \Sigma$, $\overline{v}_{A,\mathbf{1}} > 0$, and so there is $y' \neq 0$ such that $v_{A,\mathbf{1}}(y') > 0$. But this implies that $A_N y' \in \mathsf{int}(K_N)$ and $A_B y' \in \mathsf{int}(K_B)$). ◻

Proof of Theorem 1.2. We first show that

$$\{y \neq 0 \mid A_N y \in K_N\} = \emptyset \iff \{\widetilde{A} \in \Sigma \mid \widetilde{A}_N = A_N\} = \emptyset.$$

This will show that the left-hand side in (1.10) is $+\infty$ if and only if so is the right-hand side.

For the only if direction, assume that there exists $\widetilde{A}_B$ such that $\mathscr{A} = (\widetilde{A}_B, A_N) \in \Sigma$. The latter implies that $\overline{v}_{\mathscr{A},\mathbf{1}} = 0$. Hence, there exists $\overline{y} \in S_Y := \{y \in Y \mid \|y\| = 1\}$ such that $\min_{j=1,\ldots,r} \lambda^j_{\min}(\mathscr{A}_j \overline{y}) = 0$ and, therefore, such that $\lambda^j_{\min}(\mathscr{A}_j \overline{y}) \geq 0$ for $j = 1, \ldots, r$. But this implies that $A_N \overline{y} \in K_N$.

For the if direction, assume that there exists $\overline{y} \neq 0$ such that $A_N \overline{y} \in K_N$. Let $\mathcal{A} = (0, A_N)$. Then, for all $y \neq 0$, and since $0y = 0 \in \partial K_B$,

$$v_{\mathcal{A},\mathbf{1}}(y) = \min_{j=1,\ldots,r} \lambda^j_{\min}(\mathcal{A}_j y) \leq 0.$$

This implies that $\overline{v}_{\mathcal{A},\mathbf{1}} \leq 0$. But $v_{\mathcal{A},\mathbf{1}}(\overline{y}) = 0$ since $A_N \overline{y} \in K_N$. Therefore $\overline{v}_{\mathcal{A},\mathbf{1}} = 0$, which implies that $\mathcal{A} \in \Sigma$.

We now assume that the sets above are nonempty and take limits when $\alpha_N \to 0$. We will show that both the left- and right-hand sides of (1.9) tend to the corresponding sides in (1.10) when $\alpha_N \to 0$. Equation (1.10) will therefore hold since Theorem 1.1 does.

Recall that the left-hand side in the equality of Theorem 1.1 is

$$\left| \max_{y \in S_Y} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} \right|.$$

For any $y \in S_Y$ such that $A_N y \in \mathsf{int}(K_N)$, we have

$$\begin{aligned} &\forall j \in N \ \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} > 0 \\ &\Longrightarrow \forall j \in N \ \lim_{\alpha_j \to 0} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} = +\infty \\ &\Longrightarrow \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} = \min_{j \in B} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j}. \end{aligned} \tag{3.5}$$

On the other hand, for any $y \in S_Y$ such that $A_N y \notin K_N$,

$$\lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} = -\infty. \tag{3.6}$$

Finally, for any $y \in S_Y$ such that $A_N y \in \partial K_N$, we have

$$\lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} = \begin{cases} 0 & \text{if } A_B y \in \mathsf{int}(K_B), \\ \min_{j\in B} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} & \text{otherwise.} \end{cases} \tag{3.7}$$

By taking the maximum over $y \in S_Y$ on the equalities (3.5), (3.6), and (3.7) and using Lemma 3.3, it follows that

$$\max_{y\in S_Y} \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} = \max_{\substack{y\in S_Y \\ A_N y \in K_N}} \min_{j\in B} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j}.$$

Hence to show that the left-hand side in (1.9) tends to the left-hand side in (1.10) when $\alpha_N \to 0$, we need to show that

$$\max_{y\in S_Y} \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} = \lim_{\alpha_N \to 0} \max_{y\in S_Y} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j}. \tag{3.8}$$

If $\{y \neq 0 \mid A_N y \in K_N\} = \emptyset$, then from (3.6) if follows that both sides of (3.8) are $-\infty$. Assume that $\{y \neq 0 \mid A_N y \in K_N\} \neq \emptyset$. From Lemma 3.3 and (3.5) it follows that both sides of (3.8) are finite. Let $\epsilon > 0$ be given. By Lemma 3.3, (3.5), (3.6), and (3.7) there exists $y_\epsilon \in S_Y$ such that $A_N y_\epsilon \in \mathsf{int}(K_N)$ and

$$\max_{y\in S_Y} \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} - \epsilon < \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y_\epsilon)}{\alpha_j}.$$

Thus

$$\begin{aligned} \max_{y\in S_Y} \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} - \epsilon &< \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y_\epsilon)}{\alpha_j} \\ &\leq \lim_{\alpha_N \to 0} \max_{y\in S_Y} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j}. \end{aligned}$$

This shows that the left-hand side in (3.8) is smaller than or equal to the right-hand side. For the reverse inequality let $\epsilon > 0$ be given. By Lemma 3.3, (3.5), (3.6), and (3.7) there exists $y_\epsilon \in S_Y$ such that $A_N y_\epsilon \in \mathsf{int}(K_N)$ and

$$\lim_{\alpha_N \to 0} \max_{y\in S_Y} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} - \epsilon < \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y_\epsilon)}{\alpha_j}.$$

Hence (3.5) yields

$$\begin{aligned} \lim_{\alpha_N \to 0} \max_{y\in S_Y} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j} - \epsilon &< \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y_\epsilon)}{\alpha_j} \\ &= \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y_\epsilon)}{\alpha_j} \\ &\leq \max_{y\in S_Y} \lim_{\alpha_N \to 0} \min_{j=1,\ldots,r} \frac{\lambda^j_{\min}(A_j y)}{\alpha_j}. \end{aligned}$$

Therefore the right-hand side of (3.8) is also smaller than or equal to the left-hand side.

Next, we show that the right-hand side of (1.9), namely,

$$\min_{\widetilde{A}\in\Sigma}\max_{j=1,\ldots,r}\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j},$$

tends to the right-hand side of (1.10) when $\alpha_N\to 0$. Take $\widetilde{A}\in\Sigma$. Then

$$\begin{aligned}\widetilde{A}_N\neq A_N &\Longrightarrow \exists j\in N\ \|A_j-\widetilde{A}_j\|\neq 0\\ &\Longrightarrow \exists j\in N\ \lim_{\alpha_j\to 0}\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j}=+\infty\\ &\Longrightarrow \lim_{\alpha_N\to 0}\max_{j=1,\ldots,r}\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j}=+\infty.\end{aligned}$$

This implies that

$$\lim_{\alpha_N\to 0}\min_{\widetilde{A}\in\Sigma}\max_{j=1,\ldots,r}\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j}=\lim_{\alpha_N\to 0}\min_{\substack{\widetilde{A}\in\Sigma\\ \widetilde{A}_N=A_N}}\max_{j=1,\ldots,r}\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j}. \tag{3.9}$$

But if $\widetilde{A}_N=A_N$, then $\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j}=0$ for all $j\in N$. Therefore,

$$\lim_{\alpha_N\to 0}\min_{\substack{\widetilde{A}\in\Sigma\\ \widetilde{A}_N=A_N}}\max_{j=1,\ldots,r}\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j}=\min_{\substack{\widetilde{A}\in\Sigma\\ \widetilde{A}_N=A_N}}\max_{j\in B}\frac{\|A_j-\widetilde{A}_j\|}{\alpha_j},$$

and the claimed limit follows. ∎

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