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GLOBAL WEAK SOLUTION FOR A HEAT AND SWEAT TRANSPORT SYSTEM IN THREE-DIMENSIONAL FIBROUS POROUS MEDIA WITH CONDENSATION/EVAPORATION AND ABSORPTION*

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Abstract. This paper is concerned with heat and moisture transport in three-dimensional porous textile media with complex phase change, condensation/evaporation, and absorption. The physical process is described as multiphase and multicomponent (air, vapor, water, and heat) flow governed by a system of nonlinear, degenerate, and strongly coupled parabolic equations. In this paper, we prove global existence of a positive weak solution under the physical assumption that the initial and background temperatures are not higher than about 1000 K.

Key words. porous media, multicomponent, condensation/evaporation, absorption, existence

AMS subject classifications. 76S05, 76T30, 35K61, 35M33

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1. Introduction. Simultaneous heat and moisture transport in fibrous porous media is of growing interest in a wide range of industrial and engineering domains. Here we focus our attention on sweat transport in porous textile materials (functional clothing). In this application, the physical process can be viewed as a multiphase and multicomponent fluid flow with complex phase changes, which occur in the form of evaporation/condensation and fiber absorption due to its chemical/physical nature. Some earlier works on mathematical modeling can be found in [7, 14, 28] with relative simple models. More precise descriptions were presented recently for single-component (vapor) models [11, 13, 21, 24, 31] and multicomponent (vapor and air) models [19, 17, 30, 32].

Both the single-component and the multicomponent models are governed mathematically by a system of nonlinear, degenerate, and strongly coupled parabolic system, respectively. One of the important features in these models is its complex phase changes, condensation/evaporation, and fiber absorption, which determine the water content distribution. Due to the strong nonlinearity, coupling, and complex phase change, mathematical analysis for these models is restricted to some simplified cases, though numerous work on numerical simulations with various applications has been done [5, 11, 13, 17, 20, 21, 30]. A nonlinear heat and moisture model was studied in [29] for a pure diffusion problem (without convection and condensation) with a nonsymmetric parabolic part. Existence of weak solutions for a single-component (vapor-heat) system was obtained in [23] in which air motion was neglected and the vapor motion was induced only by convection. Extension to a multicomponent (air-vapor-heat) model was made in [22]. In those works [23, 22], only a simple phase change in a wet zone ($\Gamma_{ce} = H$) of one-dimensional space was considered, and also

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both the water equation and fiber absorption were neglected. More recently, existence of a local (short time) strong solution for a different single-component system was given in [3].

In this paper, we analyze the multicomponent model in the three-dimensional space with complex phase changes, condensation/evaporation, and fiber absorption. The system was described in the previous papers [17, 19, 30] for a wet case. The complex phase changes used in this system follow our recent work [32], which is applicable for the general wet-dry case and results in more precise conservation of mass and energy. We prove the global existence of a positive weak solution for the system with a class of commonly used flux-type boundary conditions under more general assumptions for the saturation pressure function and the absorption rate. The main difficulty of the current paper is that, due to the incorporation of the liquid phase and the high dimensional effects, the approaches used in previous works [23, 22] cannot be extended to the present system. In particular, the Sobolev embedding inequalities used in previous papers are not valid in three-dimensional space. The key to our proof is the physical observation of the unique structure of the system with the conservative phase changes. In particular, the structure of the temperature equation allows us to obtain a maximal estimate by assuming that the initial and background temperatures are less than about 1000 K, which is absolutely true for the clothing assembly problems. Under this assumption and the structure of the system, the upper and lower boundedness of temperature follows from the maximum principle of parabolic equations. Moreover, the maximum estimates of vapor and air concentrations also follow from the maximum principle.

Applications of multicomponent and multiphase flow in porous media can be found in many other areas [4, 6, 9, 17, 18]. One related problem is the fluid flow in petroleum engineering and the groundwater hydrology. Theoretical analysis can be found in [2, 15, 27]. However, in all those theoretical works, temperature was ignored and phase changes do not occur due to the nature of those applications, while both temperature and phase changes play important roles in the textile model. Nonisothermal flows in porous media were considered for certain applications in petroleum engineering and studied numerically by several authors; e.g., see [9] and the references therein. Mathematical analysis for nonisothermal flows in porous media was presented in [10] for some special cases, in which the fluid was assumed to be incompressible but dilatable with the linear density-temperature relation (Oberbeck–Boussinesq approximation) $\rho = \rho_0(1 - \beta(T - T_0))$. In building engineering, a similar mathematical model of two-component flow was presented in [17] to study the heat, air, and vapor transport in porous building materials. Numerical simulations with a finite volume method were done without any analysis. Two multiphase models for the heat and moisture transport process during bread baking were introduced in [18]. During baking, heat transfer is a combination of conduction/radiation and evaporation/condensation. In this process, only a single-component vapor flow is involved due to the absence of air. Again no theoretical analysis was provided. The analysis presented in this paper can be applied directly to those multicomponent heat and mass transport models in those areas.

The rest of the paper is organized as follows. In section 2 we introduce the multicomponent (air-vapor-water-heat) system with phase changes in the form of condensation/evaporation and fiber absorption as presented in [19, 32]. With some general assumptions for the saturation pressure function and the absorption rate, we present the main results. A priori estimates are presented in section 3 under a physical assumption that the initial and background temperatures are lower than about 1000 K. In section 4, a class of approximating solutions are introduced in terms of a regularized quasi-linear parabolic system. Existence of solutions to the regularized system is proved by the Leray–Schauder fixed-point theorem. The convergence of approximating solutions is studied in section 5.

2. The mathematical models and equations. Mathematical models for heat and moisture transport in clothing assemblies were studied by many authors. Here we consider a general multiphase and multicomponent flow model mainly based on the work in [19, 32], which can be viewed as a generalization of models developed earlier in [6, 11, 13, 20, 24].

2.1. The conservation equations. With the generalized Fick's law for binary multicomponent gas mixture (vapor and air) and conservation of mass and energy, the physical process can be described by

$$\partial_t(\epsilon C_{\mathrm{v}}) + \nabla\cdot(\mathbf{u}\epsilon C_{\mathrm{v}}) - \nabla\cdot[D_{\mathrm{g}}\epsilon C\nabla(C_{\mathrm{v}}/C)] = -(1-\epsilon)\partial_t C_f - \Gamma_{\mathrm{ce}}, \tag{2.1}$$

$$\partial_t(\epsilon C_{\mathrm{a}}) + \nabla\cdot(\mathbf{u}\epsilon C_{\mathrm{a}}) - \nabla\cdot[D_{\mathrm{g}}\epsilon C\nabla(C_{\mathrm{a}}/C)] = 0, \tag{2.2}$$

$$\begin{aligned}&\partial_t\big(\bar{C}_{\mathrm{g}}\epsilon CT + \bar{C}_{\mathrm{w}}WT + \bar{C}_{\mathrm{s}}(1-\epsilon)T\big) + \nabla\cdot\big(\mathbf{u}\bar{C}_{\mathrm{g}}\epsilon CT\big)\\ &\qquad - \nabla\cdot(\kappa\nabla T) = \lambda(1-\epsilon)\partial_t C_f + \lambda\Gamma_{\mathrm{ce}},\end{aligned} \tag{2.3}$$

$$\partial_t W = (1-\epsilon)\partial_t C_f + \Gamma_{\mathrm{ce}}, \tag{2.4}$$

where ϵ denotes the porosity of the textile material, C_{v}, C_{a}, T, and W are the vapor density, air density, absolute temperature, and liquid water, respectively, $C = C_{\mathrm{v}} + C_{\mathrm{a}}$ is the density of the gas mixture, $\bar{C}_{\mathrm{g}}$, $\bar{C}_{\mathrm{w}}$, and $\bar{C}_{\mathrm{s}}$ are the specific heat capacities of the gas mixture, liquid water, and fiber, respectively, λ is the latent heat of condensation/evaporation, D_{g} is the molecular diffusion coefficient for gas mixture, and κ is the thermal conductivity of the gas and fiber mixture which is a function of the gas constitution in general.

The group velocity $\mathbf{u}$ of the gas mixture is given by Darcy's law,

$$\mathbf{u} = -\frac{k}{\mu}\nabla P, \tag{2.5}$$

where k and μ denote the permeability and the viscosity of the gas, respectively. For a compressible flow, the viscosity is concentration-dependent with different forms in different applications. Here, we assume that $\mu = \mu_0 C$ with μ_0 being a constant. The gas pressure P is given by the ideal gas law $P = RCT$, with R the universal gas constant.

The phase-change rate consists of two parts: Γ_{ce} and $\partial_t C_f$, where Γ_{ce} is the condensation/evaporation rate by applying local thermal equilibrium and $\partial_t C_f$ is the rate due to absorption by fiber. The condensation/evaporation is generally defined by a truncated Hertz–Knudsen equation [13, 32]:

$$\Gamma_{\mathrm{ce}} = H^+ - a(w)H^-, \qquad H := \beta_{\mathrm{ce}}(1-\epsilon)\frac{P_{\mathrm{v}} - P_{\mathrm{sat}}(T)}{\sqrt{T}}, \tag{2.6}$$

where $a(w)$ is a monotonically increasing function with $a(0) = 0$ and $0 \le a(w) \le 1$, and β_{ce} is a positive constant. The partial vapor pressure is given by $P_{\mathrm{v}} = RC_{\mathrm{v}}T$, and the saturation pressure $P_{\mathrm{sat}}(T)$ at temperature T is determined from experimental measurements [12, 13].

The mount of liquid water absorbed by the fiber with the radius R_f is given in [13, 19, 32] by

$$C_f = \frac{1}{\pi R_f^2} \int_0^{2\pi} \int_0^{R_f} C_f'(r, \vartheta) r dr d\vartheta, \tag{2.7}$$

where $C_f'(r, \vartheta)$, the density (mol/m^3) of liquid water inside the cylindrical fiber, satisfies the evolution equation

$$\frac{\partial C_f'}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(D_f r \frac{\partial C_f'}{\partial r} \right), \qquad 0 < r < R_f, \tag{2.8}$$

with the boundary conditions

$$\left. \frac{\partial C_f'}{\partial r} \right|_{r=0} = 0, \quad D_f \left. \frac{\partial C_f'}{\partial r} \right|_{r=R_f} = \beta_{\rm ab}(T)(W_f'(R_H) - C_f'), \tag{2.9}$$

and a certain initial condition, where D_f and $\beta_{\rm ab}(T)$ denote the diffusion coefficient and resistance coefficient, respectively. $W_f'(R_H)$, the saturation absorption rate via experimental measurements [30], is a function of the relative humidity $R_H = P_{\rm v}/P_{\rm sat}$.

Clearly, the initial-boundary value problem (2.8)–(2.9) has a unique solution. Let $\mathbf{G}$ denote the linear operator from $W_f'(R_H)$ to $C_f'(R_f, t)$, i.e.,

$$C_f'(R_f, t) = \mathbf{G}[g(R_H)]. \tag{2.10}$$

Then from (2.7) we know

$$(1 - \epsilon)\partial_t C_f = \frac{2\beta_{\rm ab}(T)}{R_f}(1 - \epsilon)(W_f'(R_H) - \mathbf{G}[W_f'(R_H)]).$$

By the fundamental estimates of parabolic equation, we have the following lemma.

LEMMA 2.1. *Let $f : (0, \tau) \to \mathbb{R}$ be a nonnegative function. Then*

$$0 \le \mathbf{G}[f](t) \le \max_{0 \le t' \le \tau} f(t') \qquad \forall\, t \in (0, \tau), \quad \forall\, f \in L^\infty(0, \tau), \tag{2.11}$$

$$\int_0^\tau |\mathbf{G}[f](t)|^p dt \le C \int_0^\tau |f(t)|^p dt \quad \forall\, f \in L^p(0, \tau). \tag{2.12}$$

2.2. Mathematical formulation and assumptions. With proper nondimensionalization, the system (2.1)–(2.5) reduces to

$$\partial_t(\rho c) + \nabla \cdot (\rho c \mathbf{u}) - \nabla \cdot (d_{\rm g} \rho \nabla c) = -\Gamma, \tag{2.13}$$

$$\partial_t(\rho \tilde{c}) + \nabla \cdot (\rho \tilde{c} \mathbf{u}) - \nabla \cdot (d_{\rm g} \rho \nabla \tilde{c}) = 0, \tag{2.14}$$

$$\partial_t(\rho\theta + \varsigma_1 w\theta + \varsigma_2\theta) + \nabla \cdot (\theta \rho \mathbf{u}) - \nabla \cdot (\kappa(\rho, c)\nabla\theta) = \lambda\Gamma, \tag{2.15}$$

$$\partial_t w = \Gamma, \tag{2.16}$$

$$\rho \mathbf{u} = -k\nabla(\rho\theta) \tag{2.17}$$

for $x \in \Omega$ and $t > 0$, where ρ is the density of the gas mixture, c and $\tilde{c}$ are the density proportion of vapor and air, respectively, with $c + \tilde{c} = 1$, θ is temperature, and $\kappa(\rho, c)$ is the thermal conductivity. The diffusion coefficient $d_{\rm g}$, the Darcy coefficient k, and the heat capacities $\varsigma_1 = \bar{C}_{\rm w}/\bar{C}_{\rm g}$, ς_2 are assumed to be positive constants.

The phase change rate is defined by $\Gamma = \Gamma_{\rm ce} + \Gamma_{\rm ab}$, where

$$\Gamma_{\rm ce} = H + (1 - a(w))H^-, \tag{2.18}$$

$$\Gamma_{\rm ab} = \beta_1(\theta)(g(r_H) - \mathbf{G}[g(r_H)]), \tag{2.19}$$

where

$$H = \beta_2(\rho c\sqrt{\theta} - p_s(\theta)),$$

$p_s(\theta) = P_{\rm sat}(\theta)/\sqrt{\theta}$, and $g(r_H)$ is a smooth and monotonically increasing function of the relative humidity

$$r_H = \frac{\rho c\sqrt{\theta}}{p_s(\theta)}. \tag{2.20}$$

The linear operator $\mathbf{G}$ defined by (2.10) satisfies the estimates (2.11)–(2.12).

We study the above problem with a class of Robin-type boundary conditions [13, 24, 30]

$$(\mathbf{u}\rho c - d_{\rm g}\rho\nabla c)\cdot \mathbf{n} = \alpha_1(x)(\rho c - v^o(x)), \tag{2.21}$$

$$(\mathbf{u}\rho \tilde{c} - d_{\rm g}\rho\nabla \tilde{c})\cdot \mathbf{n} = \alpha_2(x)(\rho \tilde{c} - a^o(x)), \tag{2.22}$$

$$-\kappa\nabla\theta\cdot\mathbf{n} = \alpha_3(x)(\theta - \theta^o(x)) \tag{2.23}$$

for $x \in \partial\Omega$, where $\alpha_1(x), \alpha_2(x)$, and $\alpha_3(x)$ denote the mass and heat transfer coefficients, and $v^o(x), a^o(x)$, and $\theta^o(x)$ denote the densities and temperature of outer backgrounds.

The initial conditions are given by

$$\rho(x,0) = \rho_0(x), \quad \theta(x,0) = \theta_0(x), \quad c(x,0) = c_0(x), \quad w(x,0) = 0 \quad \text{for } x \in \Omega. \tag{2.24}$$

Before presenting theoretical analysis for the above problem, we make some further assumptions below. Based on the experimental data [12, 13], we assume that $p_s(\theta)$ is a smooth, increasing, and nonnegative function defined on $\mathbb{R}^+$ which satisfies

$$\lim_{\theta\to 0}\frac{p_s(\theta)}{\theta} = 0, \qquad \lim_{\theta\to\infty}\frac{p_s(\theta)}{\theta} = \infty. \tag{2.25}$$

For completeness, we set $p_s(\theta) = 0$ for $\theta < 0$.

Physically, experimental measurement for $g(r_H)$ is available only for $0 \le r_H \le 1$; therefore we define $g(r_H) = 0$ for $r_H \le 0$ and $g(r_H) = g(1)$ for $r_H \ge 1$. Similarly, we define $a(w) = 0$ for $w < 0$. In this way, $p_s(\theta)$, $g(r_H)$, and $a(w)$ are continuous, nondecreasing, piecewise smooth functions defined on $\mathbb{R}$. In particular,

$$0 \le a(s) \le 1 \;\forall\; s \in \mathbb{R}\,. \tag{2.26}$$

It follows from (2.11) and (2.19) that

$$0 \le \mathbf{G}[g(r_H)] \le g(1), \qquad |\Gamma_{\rm ab}| \le \beta_1(\theta)g(1), \tag{2.27}$$

where the resistance coefficient $\beta_1(\theta)$ is assumed to be a smooth increasing function defined for $\theta \in [0,\infty)$ with $\beta_1(0) = 0$ and $\beta_1(\theta) > 0$ for $\theta > 0$.

Moreover, we assume that κ satisfies the following estimates:

$$\kappa_0 + \kappa_1 \rho \le \kappa(\rho, c) \le \kappa_0' + \kappa_1' \rho, \quad \text{provided} \quad \rho \ge 0, \quad 0 \le c \le 1, \tag{2.28}$$

where $\kappa_0, \kappa_1, \kappa_0', \kappa_1'$ are positive constants. Physically, the constant $\nu := \varsigma_1 - 1 = \bar{C}_{\mathrm{w}}/\bar{C}_{\mathrm{g}} - 1$ is positive. Here we make a mild physical assumption for the maximum background and initial temperatures

$$\max \Big(\max_{x\in\Omega} \theta_0(x), \ \max_{x\in\partial\Omega} \theta^o(x) \Big) \le \lambda/\nu. \tag{2.29}$$

We have noted that λ/ν depends solely upon the latent heat and the specific heat capacities of water ($\bar{C}_{\mathrm{w}}$) and the gas mixture ($\bar{C}_{\mathrm{g}}$). With the physical data in [13, 11, 19], we find that $\lambda/\nu \approx 1000\,\mathrm{K}$. Therefore, the above inequality holds for any fiber materials when the initial and background temperatures are lower than about 1000 K.

2.3. Main results. In the rest part of this paper, we denote by T a fixed positive number, instead of temperature. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and

$$I = (0, T), \quad Q_t = \Omega \times (0, t), \quad Q_T = \Omega \times I\,.$$

We denote by $W^{1,q}(\Omega)$ and $\widetilde{W}^{-1,q'}(\Omega)$ the usual Sobolev space [1] and its dual space, respectively, where $1 < q, q' < \infty$ and $1/q + 1/q' = 1$. We denote by $L^q(I; B)$ the usual Bochner space for any Banach space B [8]; therefore $L^{q'}(I; \widetilde{W}^{-1,q'}(\Omega)) = L^q(I; W^{1,q}(\Omega))'$.

DEFINITION 2.2 (weak solution). *We say that the measurable functions* ρ, c, w, θ *form a weak solution to the system* (2.13)–(2.24) *in* Q_T *if*

$$\rho > 0, \quad 0 \le c \le 1, \quad w \ge 0, \quad \theta > 0 \quad \textit{a.e. in} \ \ Q_T,$$

$$\theta \in L^\infty(Q_T), \quad w \in L^2(Q_T),$$
$$\rho, \ \nabla\rho, \ \kappa\nabla\theta, \ \rho\nabla c \in L^{5/4}(Q_T),$$

which, together with

$$\mathbf{u} = -k\rho^{-1}\nabla(\rho\theta),$$

satisfy the variational equations

$$\begin{aligned} &- \int_0^T \int_\Omega \rho c \partial_t \phi_1 \, dxdt - \int_0^T \int_\Omega \rho c \mathbf{u} \cdot \nabla\phi_1 \, dxdt \\ &\quad + \int_0^T \int_\Omega d_{\mathrm{g}} \rho \nabla c \cdot \nabla \phi_1 \, dxdt + \int_0^T \int_{\partial\Omega} \alpha_1 (\rho c - v^o) \phi_1 \, d\sigma_x dt \\ &= \int_\Omega \rho_0(x) c_0(x) \phi_1(x, 0) \, dxdt - \int_0^T \int_\Omega \Gamma \, \phi_1 \, dxdt, \end{aligned}$$

$$\begin{aligned} &- \int_0^T \int_\Omega \rho \tilde{c} \partial_t \phi_2 \, dxdt - \int_0^T \int_\Omega \rho \tilde{c} \mathbf{u} \cdot \nabla\phi_2 \, dxdt \\ &\quad + \int_0^T \int_\Omega d_{\mathrm{g}} \rho \nabla \tilde{c} \cdot \nabla \phi_2 \, dxdt + \int_0^T \int_{\partial\Omega} \alpha_2 (\rho \tilde{c} - a^o) \phi_2 \, d\sigma_x dt \\ &= \int_\Omega \rho_0(x) \tilde{c}_0(x) \phi_2(x, 0) \, dxdt, \end{aligned}$$

$$-\int_0^T\int_\Omega w\partial_t\phi_3\,dxdt = \int_0^T\int_\Omega \Gamma\,\phi_3\,dxdt,$$

and

$$\begin{aligned}&-\int_0^T\int_\Omega(\rho\theta+\varsigma_1 w\theta+\varsigma_2\theta)\partial_t\phi_4 dxdt+\int_0^T\int_\Omega\kappa(\rho,c)\nabla\theta\cdot\nabla\phi_4\,dxdt\\&-\int_0^T\int_\Omega\theta\rho\mathbf{u}\cdot\nabla\phi_4\,dxdt+\int_0^T\int_{\partial\Omega}\big[\alpha_3(\theta-\theta^o)\phi_4\,d\sigma_x dt\\&+\int_0^T\int_{\partial\Omega}\big[\alpha_1(\rho c-v^o)+\alpha_2(\rho\tilde{c}-a^o)\big]\theta\phi_4\,d\sigma_x dt=\lambda\int_0^T\int_\Omega\Gamma\,\phi_4 dxdt\end{aligned}$$

for any $\phi_1,\phi_2,\phi_3,\phi_4\in C^1(\bar{Q}_T)$ which vanish at $t=T$.

Now we give the main result of this paper.

THEOREM 2.3. *Let the assumptions* (2.25)–(2.29) *be satisfied. If the initial data $\rho_0, c_0, w_0, \theta_0$ satisfy*

$$\ln\rho_0,\ \rho_0\ln\rho_0\in L^1(\Omega),\quad \theta_0\in L^\infty(\Omega),$$

$$\rho_0\geq 0,\quad 0\leq c_0\leq 1,\quad w_0\geq 0,\quad \theta_0\geq\underline{\theta},$$

for some positive constant $\underline{\theta}$, then the initial-boundary value problem (2.13)–(2.24) *admits a global weak solution in the sense of Definition* 2.2*, satisfying*

$$\ln\rho,\ \rho\ln\rho\in L^\infty(I;L^1(\Omega)),\quad \theta\in L^\infty(Q_T),$$

$$\rho\geq 0,\quad 0\leq c\leq 1,\quad w\geq 0,\quad \theta\geq\underline{\theta}_T.$$

In the following sections, we denote by $C_{p_1,p_2,\ldots,p_k}$ a generic positive constant, which depends solely upon the quantities $p_1,p_2,\ldots,p_k$ and the physical parameters involved in the equation as well as that in the initial and boundary conditions. In addition, unless specified, we denote by $C(p_1,p_2,\ldots,p_k)$ a generic positive function which is bounded when $p_1,p_2,\ldots,p_k$ are bounded.

3. A priori estimates.

We rewrite the system (2.13)–(2.17) as

$$\partial_t\rho-\nabla\cdot(k\nabla(\rho\theta))=-\Gamma_{\rm ce}-\Gamma_{\rm ab}, \tag{3.1}$$

$$\partial_t(\rho c)-\nabla\cdot(kc\nabla(\rho\theta))-\nabla\cdot(d_{\rm g}\rho\nabla c)=-\Gamma_{\rm ce}-\Gamma_{\rm ab}, \tag{3.2}$$

$$\partial_t w=\Gamma_{\rm ce}+\Gamma_{\rm ab}, \tag{3.3}$$

$$(\rho+\varsigma_1 w+\varsigma_2)\partial_t\theta-\nabla\cdot(\kappa(\rho,c)\nabla\theta)-k\nabla(\rho\theta)\cdot\nabla\theta=(\lambda-\nu\theta)(\Gamma_{\rm ce}+\Gamma_{\rm ab}), \tag{3.4}$$

where (3.1) is obtained by summing up (2.13) and (2.14) and using (2.17), and (3.4) is obtained by subtracting (2.16) times ς_1 and (2.13) times θ from (2.15).

The corresponding boundary conditions for (3.1)–(3.4) are given by

$$-k\nabla(\rho\theta)\cdot\mathbf{n}=\alpha_1(\rho c-v^o)+\alpha_2(\rho\tilde{c}-a^o), \tag{3.5}$$

$$-kc\nabla(\rho\theta)\cdot\mathbf{n}-d_{\rm g}\rho\nabla c\cdot\mathbf{n}=\alpha_1(\rho c-v^o), \tag{3.6}$$

$$-\kappa(\rho,c)\nabla\theta\cdot\mathbf{n}=\alpha_3(\theta-\theta^o) \tag{3.7}$$

for $x\in\partial\Omega$, with the initial conditions (2.24).

In this section, we present a priori estimates for the system of equations (3.1)–(3.7), under the qualitative assumption that the solution satisfies

$$\rho, \theta \geq 1/C_1^*, \quad 0 \leq c \leq 1, \quad w \geq 0 \tag{3.8}$$

for some positive constant C_1^* and $\int_0^T C_2^*(t)dt < \infty$, where

$$C_2^*(t) = \max_{x\in\Omega} \frac{\kappa(\rho, c)|\nabla\rho + \varsigma_1 \nabla w|^2}{(\rho + \varsigma_1 w + \varsigma_2)^3} + \max_{x\in\Omega} \frac{k^2|\nabla(\rho\theta)|^2}{\kappa(\rho, c)(\rho + \varsigma_1 w + \varsigma_2)}. \tag{3.9}$$

Note that the a priori estimates obtained in the following do not depend on C_1^* or $\int_0^T C_2^*(t)dt$.

We assume that the solution is regular enough for the manipulations below. In the next section, we shall construct approximating solutions which will be proved to satisfy the assumption (3.8)–(3.9) and preserve the essential structure used to obtain the a priori estimates in this section. Therefore, the a priori estimates in this section can be applied to the approximating solutions constructed in the next section.

3.1. A priori estimates for temperature. First, we present a priori estimates for the temperature θ by considering the following modified equation:

$$(\rho + \varsigma_1 w + \varsigma_2)\partial_t\theta - \nabla\cdot(\kappa(\rho, c)\nabla\theta) - k\nabla(\rho\theta)\cdot\nabla\theta = (\lambda - \nu\chi_1(\theta))(\Gamma_{\mathrm{ce}} + \Gamma_{\mathrm{ab}}), \tag{3.10}$$

where χ_1 is a Lipschitz continuous truncation function defined by

$$\chi_1(\theta) = \begin{cases} \theta & \text{if} \quad \theta \leq \lambda/\nu, \\ \lambda/\nu & \text{if} \quad \theta \geq \lambda/\nu. \end{cases}$$

Let $\tilde\theta = \theta - \lambda/\nu$ and $\tilde\theta^+ = (|\tilde\theta| + \tilde\theta)/2$. Then $(\lambda - \nu\chi_1(\theta))\tilde\theta^+ \equiv 0$ and $\tilde\theta^+\nabla\theta = \tilde\theta^+\nabla\tilde\theta^+$ a.e. in Q_T. Multiplying (3.10) by $\tilde\theta^+/(\rho + \varsigma_1 w + \varsigma_2)$ and integrating the result over Q_t, we get

$$\begin{aligned}
&\int_\Omega |\tilde\theta^+(x,\tau)|^2 dx + \int_0^\tau\int_\Omega \frac{\kappa(\rho,c)}{\rho+\varsigma_1 w+\varsigma_2}|\nabla\tilde\theta^+|^2 dxdt + \int_0^\tau\int_{\partial\Omega} \frac{\alpha_3(\theta-\theta^o)\tilde\theta^+}{\rho+\varsigma_1 w+\varsigma_2} d\sigma_x dt \\
&\quad\leq \int_\Omega |\tilde\theta^+(x,0)|^2 dx \\
&\qquad + \int_0^\tau\int_\Omega \left(\frac{\kappa(\rho,c)\tilde\theta^+\nabla\tilde\theta^+\cdot(\nabla\rho+\varsigma_1\nabla w)}{(\rho+\varsigma_1 w+\varsigma_2)^2} dxdt + \frac{k\tilde\theta^+\nabla(\rho\theta)\cdot\nabla\tilde\theta^+}{\rho+\varsigma_1 w+\varsigma_2}\right) dxdt \\
&\quad\leq \int_\Omega |\tilde\theta^+(x,0)|^2 dx + \frac{1}{2}\int_0^\tau\int_\Omega \frac{\kappa(\rho,c)}{\rho+\varsigma_1 w+\varsigma_2}|\nabla\tilde\theta^+|^2 dxdt \\
&\qquad + \int_0^\tau\int_\Omega \left(\frac{\kappa(\rho,c)|\nabla\rho+\varsigma_1\nabla w|^2}{(\rho+\varsigma_1 w+\varsigma_2)^3} + \frac{k^2|\nabla(\rho\theta)|^2}{\kappa(\rho,c)(\rho+\varsigma_1 w+\varsigma_2)}\right)|\tilde\theta^+|^2 dxdt \\
&\quad\leq \int_\Omega |\tilde\theta^+(x,0)|^2 dx + \frac{1}{2}\int_0^\tau\int_\Omega \frac{\kappa(\rho,c)}{\rho+\varsigma_1 w+\varsigma_2}|\nabla\tilde\theta^+|^2 dxdt + \int_0^\tau C_2^*(t)\int_\Omega |\tilde\theta^+|^2 dxdt,
\end{aligned} \tag{3.11}$$

where $C_2^*(t)$ is defined in (3.9). By the assumption (2.29), we have

$$\tilde\theta^+(x,0) = 0 \quad \text{for} \quad x \in \Omega,$$

$$\alpha_3(\theta - \theta^o)\tilde{\theta}^+ \geq 0 \quad \text{for} \quad x \in \partial\Omega.$$

Therefore, (3.11) reduces to

$$\int_\Omega |\tilde{\theta}^+(x,\tau)|^2 dx \leq \int_0^\tau C_2^*(t) \int_\Omega |\tilde{\theta}^+|^2 dx dt.$$

By applying Gronwall's inequality, we derive that

$$\int_\Omega |\tilde{\theta}^+(x,t)|^2 dx \leq 0, \quad \text{a.e.} \;\; 0 \leq t \leq T,$$

which implies that $\theta \leq \lambda/\nu$. Hence, $\chi_1(\theta) \equiv \theta$ and the modified equation (3.10) reduces to (3.4).

To provide a uniform lower bound for temperature, we let $\theta = (\underline{\theta} + \underline{\theta}_m)e^{-C_1 t}$, where $\underline{\theta}_m$ and C_1 are positive numbers to be determined. Then $\underline{\theta}$ is the solution of the following equation:

$$\begin{aligned}
&(\rho + \varsigma_1 w + \varsigma_2)\partial_t\underline{\theta} - \nabla\cdot(\kappa(\rho,c)\nabla\underline{\theta}) - k\nabla(\rho\theta)\cdot\nabla\underline{\theta} - C_1(\rho + \varsigma_1 w + \varsigma_2)\underline{\theta} \\
&\quad + \beta_2 e^{C_1 t}(\lambda - \nu\theta)\big(p_s(e^{-C_1 t}\underline{\theta} + e^{-C_1 t}\underline{\theta}_m) - p_s(e^{-C_1 t}\underline{\theta}_m)\big) \\
&\quad - e^{C_1 t}(\lambda - \nu\theta)\big(g(r_H) - \mathbf{G}[g(r_H)]\big)\big(\beta_1(e^{-C_1 t}\underline{\theta} + e^{-C_1 t}\underline{\theta}_m) - \beta_1(e^{-C_1 t}\underline{\theta}_m)\big) \\
&= e^{C_1 t}\left[\frac{C_1}{2}(\rho + \varsigma_1 w + \varsigma_2)e^{-C_1 t}\underline{\theta}_m - (\lambda - \nu\theta)\beta_2 p_s(e^{-C_1 t}\underline{\theta}_m)\right] \\
&\quad + e^{C_1 t}\left[\frac{C_1}{2}(\rho + \varsigma_1 w + \varsigma_2)e^{-C_1 t}\underline{\theta}_m + (\lambda - \nu\theta)\beta_1(e^{-C_1 t}\underline{\theta}_m)(g(r_H) - \mathbf{G}[g(r_H)])\right] \\
&\quad + e^{C_1 t}(\lambda - \nu\theta)\big[\beta_2\rho c\sqrt{\theta} + (1 - a(w))H^-(\rho,c,\theta)\big],
\end{aligned} \tag{3.12}$$

with the boundary condition

$$\kappa(\rho,c)\nabla\underline{\theta}\cdot\mathbf{n} + \alpha_3\underline{\theta} = \alpha_3(e^{C_1 t}\theta^o(x) - \underline{\theta}_m) \tag{3.13}$$

and the initial condition

$$\underline{\theta}(x,0) = \theta_0(x) - \underline{\theta}_m. \tag{3.14}$$

Recall the assumptions (2.25) and that $\beta_1(0) = 0$. By choosing $\underline{\theta}_m$ small enough, we have

$$(\rho + \varsigma_1 w + \varsigma_2)e^{-t}\underline{\theta}_m > (\lambda - \nu\theta)\beta_2 p_s(e^{-t}\underline{\theta}_m)$$

and

$$\left|(\lambda - \nu\theta)\beta_1(e^{-C_1 t}\underline{\theta}_m)(g(r_H) - \mathbf{G}[g(r_H)])\right| \leq C_0 g(1)e^{-C_1 t}\underline{\theta}_m$$

for some constant C_0.

Let

$$C_1 = \max\{2, 4C_0 g(1)/\varsigma_2\}. \tag{3.15}$$

Then the right-hand side of (3.12) is nonnegative.

Moreover, by choosing $\underline{\theta}_m$ to be small enough (dependent only on the initial and background temperatures), the right-hand sides of (3.13)–(3.14) are nonnegative. By multiplying (3.12) by $\underline{\theta}^-/(\rho+\varsigma_1 w+\varsigma_2)$ and integrating the result over $\Omega\times(0,\tau)$, we obtain

$$\int_\Omega |\underline{\theta}^-(x,\tau)|^2 dx \le \int_0^\tau C_2^*(t)\int_\Omega |\underline{\theta}^-(x,\tau)|^2 dxdt, \quad \tau\in(0,T).$$

Using Gronwall's inequality, we derive $\underline{\theta}^- = 0$ a.e. in Q_T. It follows that $\underline{\theta}\ge 0$ and hence

$$\theta \ge \underline{\theta}_m e^{-C_1 T}. \tag{3.16}$$

In the following, we shall denote

$$\theta_{\min} = \underline{\theta}_m e^{-C_1 T}, \qquad \theta_{\max} = \lambda/\nu, \tag{3.17}$$

which are the lower and upper bounds of temperature, respectively.

By integrating (3.1), we get

$$\begin{aligned}\int_\Omega \rho dx &+ \int_0^\tau\int_{\partial\Omega}[\alpha_1(\rho c - v^o)+\alpha_2(\rho\tilde{c}-a^o)]d\sigma_x dt\\ &\le \int_\Omega \rho_0 dx + \int_0^\tau\int_\Omega(\beta_2 p_s(\theta)+\mathbf{G}[g(r_H)])dxdt.\end{aligned}$$

Note that the boundary integral in the above inequality has a lower bound $-C$. Using the boundedness of θ and the boundedness of $\mathbf{G}[g(r_H)]$, we derive that

$$\sup_{0\le t\le T}\int_\Omega \rho(x,t)dx \le C. \tag{3.18}$$

Finally, multiplying (3.1) and (3.4) by $\theta^2/2$ and θ, respectively, and integrating the sum over Q_t, we arrive at

$$\begin{aligned}&\int_\Omega \frac{1}{2}(\rho(x,\tau)+\varsigma_1 w(x,\tau)+\varsigma_2)|\theta(x,\tau)|^2 dx + \int_0^\tau\int_\Omega \kappa(\rho,c)|\nabla\theta|^2 dxdt\\ &\quad+\int_0^\tau\int_{\partial\Omega}\frac{\theta^2}{2}[\alpha_1(\rho c-v^o)+\alpha_2(\rho\tilde{c}-a^o)]d\sigma_x dt + \int_0^\tau\int_{\partial\Omega}\alpha_3(\theta-\theta^o)\theta\, d\sigma_x dt\\ &= \int_\Omega \frac{1}{2}(\rho_0+\varsigma_2)|\theta_0|^2 dx + \int_0^\tau\int_\Omega(\lambda-\nu\theta/2)\theta(\Gamma_{\rm ce}+\Gamma_{\rm ab})dxdt\\ &\le C + \int_0^\tau\int_\Omega \rho dxdt\\ &\le C,\end{aligned}$$

which implies that

$$\begin{aligned}\int_\Omega \frac{\varsigma_2}{2}|\theta(x,\tau)|^2 dx &+ \int_0^\tau\int_\Omega \kappa(\rho,c)|\nabla\theta|^2 dxdt \le C + C\int_0^\tau\int_{\partial\Omega}\theta^2 d\sigma_x dt\\ &\le C + \frac{C}{\kappa_0}\int_0^\tau\int_\Omega \theta^2 dxdt + \frac{\kappa_0}{2}\int_0^\tau\int_\Omega |\nabla\theta^2| dxdt.\end{aligned}$$

By using Gronwall's inequality we can derive that $\max_{0\le t\le T}\int_\Omega \frac{\varsigma_2}{2}|\theta(x,\tau)|^2 dx \le C$ and so

$$\int_0^T\int_\Omega \kappa(\rho,c)|\nabla\theta|^2 dxdt \le C. \tag{3.19}$$

3.2. A priori estimates for the densities. Integrating (3.1) times $\ln\rho$ over Q_T, we obtain

$$
\begin{aligned}
&\int_\Omega \big[\rho(x,\tau)\ln\rho(x,\tau)-\rho(x,\tau)\big]dx + k\int_0^\tau\int_\Omega \frac{\theta}{\rho}|\nabla\rho|^2 dxdt + k\int_0^\tau\int_\Omega \nabla\theta\cdot\nabla\rho dxdt\\
&\quad+\int_0^\tau\int_{\partial\Omega}[\alpha_1(\rho c-v^o)+\alpha_2(\rho\tilde{c}-a^o)]\ln\rho\, d\sigma_x dt + \int_0^\tau\int_\Omega \beta_1(\theta)g(r_H)\ln\rho\big|_{\{\rho>1\}}dxdt\\
&\quad+\int_0^\tau\int_\Omega \beta_2 c\sqrt{\theta}\,\rho\ln\rho dxdt + \int_0^\tau\int_\Omega \beta_1(\theta)\mathbf{G}[g(r_H)]|\ln\rho|\big|_{\{\rho<1\}}dxdt\\
&=\int_0^\tau\int_\Omega \beta_1(\theta)g(r_H)|\ln\rho|\big|_{\{\rho<1\}}dxdt + \int_0^\tau\int_\Omega \beta_1(\theta)\mathbf{G}[g(r_H)]\ln\rho\big|_{\{\rho>1\}}dxdt\\
&\quad+\int_0^\tau\int_\Omega \beta_2\big(p_s(\theta)-(1-a(w))\big[p_s(\theta)-\rho c\sqrt{\theta}\big]^+\big)\ln\rho dxdt + \int_\Omega\big(\rho_0\ln\rho_0-\rho_0\big)dx,
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
&\int_0^\tau\int_{\partial\Omega}[\alpha_1(\rho c-v^o)+\alpha_2(\rho\tilde{c}-a^o)]\ln\rho d\sigma_x dt\\
&\quad\geq -C+\int_0^\tau\int_{\partial\Omega}[\alpha_1(\rho c-v^o)+\alpha_2(\rho\tilde{c}-a^o)]\ln\rho\big|_{\{\rho\geq1\}}d\sigma_x dt\\
&\quad\geq -C+\int_0^\tau\int_{\partial\Omega}\min\{\alpha_1,\alpha_2\}\left(\rho-\frac{\alpha_1 v^o+\alpha_2 a^o}{\min\{\alpha_1,\alpha_2\}}\right)\ln\rho\big|_{\{\rho\geq1\}}d\sigma_x dt\\
&\quad\geq -C.
\end{aligned}
$$

By (2.28) we have

$$\int_0^\tau\int_\Omega \rho|\nabla\theta|^2 dxdt \leq C\int_0^\tau\int_\Omega \kappa(\rho,c)|\nabla\theta|^2 dxdt,$$

and the term $k\int_0^\tau\int_\Omega|\nabla\theta\cdot\nabla\rho|dxdt$ in (3.20) is estimated by

$$k\int_0^\tau\int_\Omega|\nabla\theta\cdot\nabla\rho|dxdt \leq \frac{1}{2}\int_0^\tau\int_\Omega\frac{\theta_{\min}|\nabla\rho|^2}{\rho}dxdt + C\int_0^\tau\int_\Omega\rho|\nabla\theta|^2 dxdt.$$

Since $g(r_H)$ is smooth for $r_H\geq 0$,

$$\beta_1(\theta)g(r_H)|\ln\rho|\big|_{\{\rho<1\}} = \beta_1(\theta)g\left(\frac{c\sqrt{\theta}}{p_s(\theta)}\rho\right)|\ln\rho|\big|_{\{\rho<1\}} \leq C\rho|\ln\rho|\big|_{\{\rho<1\}} \leq C.$$

Substituting the above inequalities into (3.20) gives

$$\int_\Omega \rho(x,t)\ln\rho(x,t)\big|_{\{\rho\geq1\}}dx + \int_0^\tau\int_\Omega\frac{|\nabla\rho|^2}{\rho}dxdt \leq C,$$

or, equivalently,

$$\|\rho\ln(1+\rho)\|_{L^\infty(0,T;L^1(\Omega))}\leq C, \qquad \|\nabla\rho/\sqrt{\rho}\|_{L^2(Q_T)}\leq C. \tag{3.21}$$

By interpolation inequalities of inhomogeneous Sobolev spaces, we have

$$\int_0^T\int_\Omega \rho^{5/3}dxdt \leq C, \tag{3.22}$$

and by (2.18)–(2.19) we have

$$\int_0^T \int_\Omega |\Gamma|^{5/3} dxdt \le C. \tag{3.23}$$

Similarly, we multiply (3.1) and (3.2) by $-c^2$ and $2c$, respectively, and integrate their sum to obtain

$$\begin{aligned}
&\int_\Omega \rho(x,T)c^2(x,T)dx - \int_\Omega \rho_0(x)|c_0(x)|^2dx + 2\int_0^T \int_\Omega d_g\rho|\nabla c|^2 dxdt \\
&\quad + \int_0^T \int_\Omega (2c-c^2)\Gamma\, dxdt + \int_0^T \int_{\partial\Omega} [(2-c)c\alpha_1(\rho c - v^o) - \alpha_2(\rho\tilde{c} - a^o)c^2] d\sigma_x dt = 0.
\end{aligned}$$

By noting the assumptions $0 \le c \le 1$ in (3.8) and the fact that

$$\begin{aligned}
\int_0^T \int_{\partial\Omega} \rho d\sigma_x dt &\le C\int_0^T \int_\Omega \rho dxdt + C\int_0^T \int_\Omega |\nabla\rho| dxdt \\
&\le C\int_0^T \int_\Omega \rho dxdt + C\int_0^T \int_\Omega \frac{1}{\rho}|\nabla\rho|^2 dxdt \le C,
\end{aligned}$$

where we have used (3.21) in the last step, we derive that

$$\int_0^T \int_\Omega \rho|\nabla c|^2 dxdt \le C. \tag{3.24}$$

We leave the estimates for the liquid water w to section 5.

4. Construction of approximating solutions. In this section we construct a class of approximating solutions to the system (3.1)–(3.7) in terms of a regularized system of quasi-linear parabolic equations. The approximating solutions are nonnegative and preserve the essential structure such that the a priori estimates in the last section hold for the approximating solutions. By applying the Aubin–Lions lemma, we can derive the existence of approximating solutions which will be proved to converge to the exact solution of the system (3.1)–(3.7) in the next section.

Throughout this section, we let ε be a fixed small positive number. For any given function f defined on Q_T, we denote by $f_\varepsilon := \mathrm{Ext}_1(f) * \eta_\varepsilon$ the regularization of f, where η_ε is the standard mollifier in $\mathbb{R}^4$ and $\mathrm{Ext}_1(\cdot)$ is the extension operator defined by

$$\mathrm{Ext}_1(f)(x,t) = \begin{cases} f(x,t), & (x,t) \in Q_T, \\ 1, & (x,t) \in \mathbb{R}^4 \backslash Q_T. \end{cases} \tag{4.1}$$

It is easy to see that

$$\begin{aligned}
\|f_\varepsilon\|_{L^p(Q_T)} &\le C + C\|f\|_{L^p(Q_T)}, \\
\|f_\varepsilon\|_{C^1(\bar{Q}_T)} &\le C + C_\varepsilon\|f\|_{L^1(Q_T)},
\end{aligned}$$

$$\min\Big\{1, \inf_{(x,t)\in Q_T} f(x,t)\Big\} \le f_\varepsilon \le \max\Big\{1, \sup_{(x,t)\in Q_T} f(x,t)\Big\}.$$

4.1. Construction of the approximating system. In the following, we construct an approximating system. We expect that this approximating system admits a solution, say an approximating solution, and the approximating solutions converge to a weak solution to the original system as $\varepsilon \to 0$. For this purpose, we say that the measurable function quadruple (ρ, c, w, θ) is an approximating solution of the initial-boundary value problem (3.1)–(3.7) if it is a weak solution of the regularized system

$$\rho_t - \nabla \cdot (k\theta_\varepsilon \nabla\rho) - \nabla \cdot (k(\nabla\theta)_\varepsilon \rho) = -\Gamma, \tag{4.2}$$

$$(\rho c)_t - \nabla \cdot (d_{\mathrm{g}}\rho\nabla c) - \nabla \cdot (k\theta_\varepsilon c\nabla\rho) - \nabla \cdot (k(\nabla\theta)_\varepsilon \rho c) = -\Gamma, \tag{4.3}$$

$$w_t - \varepsilon\Delta w = \Gamma, \tag{4.4}$$

$$\begin{aligned}&(\rho + \varsigma_1 w + \varsigma_2)\partial_t\theta - \nabla \cdot (\kappa(\rho_\varepsilon, c_\varepsilon)\nabla\theta) - \big(k\theta_\varepsilon\nabla\rho + k(\nabla\theta)_\varepsilon\rho\big) \cdot \nabla\theta \\ &- \varepsilon\nabla w \cdot \nabla\theta = (\lambda - \nu\theta)\Gamma,\end{aligned} \tag{4.5}$$

with the boundary conditions

$$[-k\theta_\varepsilon\nabla\rho - k(\nabla\theta)_\varepsilon\rho] \cdot \mathbf{n} = \alpha_1(\rho c - v^o) + \alpha_2(\rho\tilde{c} - a^o), \tag{4.6}$$

$$[-d_{\mathrm{g}}\rho\nabla c - k\theta_\varepsilon\nabla\rho\, c - k(\nabla\theta)_\varepsilon\rho c] \cdot \mathbf{n} = \alpha_1(\rho c - v^o), \tag{4.7}$$

$$\nabla w \cdot \mathbf{n} = 0, \tag{4.8}$$

$$-\kappa(\rho_\varepsilon, c_\varepsilon)\nabla\theta \cdot \mathbf{n} = \alpha_3(\theta - \theta^o) \tag{4.9}$$

and the initial conditions

$$\rho(x,0) = \rho_{0\varepsilon}(x), \quad c(x,0) = c_{0\varepsilon}(x), \quad w(x,0) = 0, \quad \theta(x,0) = \theta_{0\varepsilon}(x) \quad \text{for} \quad x \in \Omega. \tag{4.10}$$

Note that as discussed in section 3, (4.5) is equivalent to

$$\partial_t(\rho\theta + \varsigma_1 w\theta + \varsigma_2\theta) - \nabla \cdot (\kappa(\rho_\varepsilon, c_\varepsilon)\nabla\theta) - \nabla \cdot \big[(k\theta_\varepsilon\nabla\rho + k(\nabla\theta)_\varepsilon\rho + \varepsilon\nabla w)\theta\big] = \lambda\Gamma. \tag{4.11}$$

The rest part of this section sketches a proof for the existence of solution to the above regularized system of equations. We present the Leray–Schauder-type fixed-point theorem below, which is a slight variation of that given in [16]. The proof is straightforward, following the outline in [16].

LEMMA 4.1 (Leray–Schauder). *Let B be a Banach space, and U a closed convex cone of B centered at the origin; i.e., U is closed, convex and $\{ty \,|\, t \geq 0\} \subset U$ for any $y \in U$. Let $M : U \times [0,1] \to U$ be a continuous and compact map such that there exists a positive constant C such that $\|M(z,0)\| \leq \|z\|$ for all $\|z\| \geq C$, $z \in U$. Suppose the map M has the property that the subset*

$$\{z \in U \,|\, z = M(z,s) \ \ \text{for some} \ \ s \in [0,1]\}$$

is bounded in B. Then the map $M(\cdot, 1)$ has a fixed point $z \in U$, i.e., $M(z,1) = z$.

The following lemma is about compactness in Bochner spaces [25].

LEMMA 4.2 (Aubin–Lions). *Let $B_1 \hookrightarrow\hookrightarrow B_2 \hookrightarrow B_3$ be reflexive and separable Banach spaces. Then*

$$\{u \in L^p(I; B_1) \,|\, u_t \in L^q(I; B_3)\} \hookrightarrow\hookrightarrow L^p(I; B_2), \quad 1 < p, q < \infty.$$

For $\phi \in C^\infty(\bar{Q}_T)$ and $1 < q < \infty$, we define the following norms and seminorms:

$$\|\phi\|_{V_q^0(Q_T)} = \left(\int_0^T \|\phi\|^q_{L^q(\Omega)} dt + \int_0^T \|\phi\|^q_{L^q(\partial\Omega)} dt\right)^{1/q},$$

$$\|\phi\|_{V_q^1(Q_T)} = \left(\int_0^T \|\phi\|^q_{W^{1,q}(\Omega)} dt + \|\partial_t \phi\|^q_{\widetilde{W}^{-1,q}(\Omega)} dt\right)^{1/q}, \tag{4.12}$$

$$\|\phi\|_{W_q^{2,1}(Q_T)} = \left[\int_0^T \int_\Omega \left(|\phi|^q + \sum_{j=1}^3 |\partial_j \phi|^q + |\partial_t \phi|^q dt + \sum_{i,j=1}^3 |\partial_{ij}\phi|^q\right) dx dt\right]^{1/q}.$$

Let $V_q^0(Q_T)$, $V_q^1(Q_T)$, and $W_q^{2,1}(Q_T)$ be the completion of $C^\infty(\bar{Q}_T)$ with respect to the above norms, respectively. By the Aubin–Lions lemma, we have

$$V_q^1(Q_T) \hookrightarrow\hookrightarrow V_q^0(Q_T) \quad \text{and} \quad W_q^{2,1}(Q_T) \hookrightarrow\hookrightarrow L^q(I; W^{1,q}(\Omega)).$$

In the following, we shall denote

$$X = V_2^0(Q_T) \times V_2^0(Q_T) \times L^2(I; H^1(\Omega)) \times L^2(I; H^1(\Omega)), \tag{4.13}$$

$$X^+ = \{(\rho, c, w, \theta) \in X |\ \rho, c, w, \theta \geq 0\}. \tag{4.14}$$

Clearly, X^+ is a closed convex cone in the Banach space X in the sense of Lemma 4.1. To apply the fixed-point theorem to the regularized system , we need to construct a continuous and compact map whose fixed point is the solution of the regularized system (4.2)–(4.10).

4.2. Construction of the map *M*. Let $\chi(\cdot)$ and $\bar{\chi}(\cdot)$ be truncated operators defined as follows:

$$\chi(c) = \begin{cases} 0, & c \leq 0, \\ c, & 0 \leq c \leq 1, \\ 1, & c \geq 1, \end{cases} \qquad \bar{\chi}(\theta) = \begin{cases} \theta_{\min}, & 0 \leq \theta \leq \theta_{\min}, \\ \theta, & \theta_{\min} \leq \theta \leq \theta_{\max}, \\ \theta_{\max}, & \theta \geq \theta_{\max}, \end{cases}$$

where $\theta_{\min}$ and $\theta_{\max}$ are given in (3.17).

For any given $(\rho^0, c^0, w^0, \theta^0) \in X^+$ and $s \in [0,1]$, we define $\tilde{c}^0 = 1 - c^0$. Then we define ρ to be the solution of the quasi-linear parabolic equation

$$\begin{aligned} &\partial_t \rho - k\nabla \cdot ([\bar{\chi}(\theta^0)]_\varepsilon \nabla \rho) - sk\nabla \cdot (\rho (\nabla \theta^0)_\varepsilon) \\ &\quad + s\beta_1(\bar{\chi}(\theta^0))(g(r_H^0) - \mathbf{G}[g(r_H^0)]) + s\beta_2 \rho \chi(c^0)\sqrt{\bar{\chi}(\theta^0)} \\ &= \varepsilon + s\beta_2 p_s(\bar{\chi}(\theta^0)) - s\beta_2(1 - a(w^0))\big[p_s(\bar{\chi}(\theta^0)) - \rho^0 \chi(c^0)\sqrt{\bar{\chi}(\theta^0)}\big]^+, \end{aligned} \tag{4.15}$$

with the initial and boundary conditions

$$\begin{cases} \big[-k[\bar{\chi}(\theta^0)]_\varepsilon \nabla \rho - sk\rho(\nabla\theta^0)_\varepsilon\big] \cdot \mathbf{n} \\ \quad = \alpha_1(\rho\chi(c^0) - v^o) + \alpha_2(\rho\chi(\tilde{c}^0) - a^o) & \text{for } x \in \partial\Omega, \\ \rho(x, 0) = \rho_{0\varepsilon}(x) + \varepsilon & \text{for } x \in \Omega, \end{cases}$$

where

$$r_H^0 = \frac{\sqrt{\bar{\chi}(\theta^0)}}{p_s(\bar{\chi}(\theta^0))} \chi(c^0)\rho. \tag{4.16}$$

With the ρ in hand, we define c to be the solution of the linear parabolic equation

$$\begin{aligned}
&\partial_t(\rho c) - \nabla\cdot(d_{\rm g}\rho\nabla c) - k\nabla\cdot([\bar\chi(\theta^0)]_\varepsilon\nabla\rho\, c) - sk\nabla\cdot(\rho c(\nabla\theta^0)_\varepsilon)\\
&\quad + s\beta_1(\bar\chi(\theta^0))(g(\tilde r_H^0) - \mathbf{G}[g(r_H^0)]) + s\beta_2\rho c\sqrt{\bar\chi(\theta^0)}\\
&= \varepsilon c + s\beta_2 p_s(\bar\chi(\theta^0)) - s\beta_2(1-a(w^0))\big[p_s(\bar\chi(\theta^0)) - \rho^0\chi(c^0)\sqrt{\bar\chi(\theta^0)}\big]^+,
\end{aligned} \tag{4.17}$$

with the initial and boundary conditions

$$\begin{cases} [-d_{\rm g}\rho\nabla c - k[\bar\chi(\theta^0)]_\varepsilon\nabla\rho\, c - sk\rho c(\nabla\theta^0)_\varepsilon]\cdot\mathbf{n} = \alpha_1(\rho c - v^o) & \text{for } x\in\partial\Omega,\\ c(x,0) = c_{0\varepsilon}(x) & \text{for } x\in\Omega, \end{cases}$$

where

$$\tilde r_H^0 = \frac{\sqrt{\bar\chi(\theta^0)}}{p_s(\bar\chi(\theta^0))}\rho c.$$

Moreover, we define w to be the solution of the quasi-linear parabolic equation

$$\begin{aligned}
&w_t - \varepsilon\Delta w + sH^-(\rho,\chi(c^0),\bar\chi(\theta^0))a(w)\\
&\quad = sH^+(\rho,\chi(c^0),\bar\chi(\theta^0)) + s\beta_1(\bar\chi(\theta^0))\big[g(\tilde r_H^0(x,\tau)) - \mathbf{G}[g(\tilde r_H^0)](x,\tau)\big],
\end{aligned} \tag{4.18}$$

with the initial and boundary conditions

$$\begin{cases} \varepsilon\nabla w\cdot\mathbf{n} = 0 & \text{for } x\in\partial\Omega,\\ w(x,0) = 0 & \text{for } x\in\Omega. \end{cases}$$

Finally, we define θ to be the solution of the quasi-linear parabolic equation

$$\begin{aligned}
&(\rho + \varsigma_1 w + \varsigma_2)\partial_t\theta - \nabla\cdot(\kappa(\rho_\varepsilon,c_\varepsilon)\nabla\theta) - \big(k[\bar\chi(\theta^0)]_\varepsilon\nabla\rho + sk(\nabla\theta^0)_\varepsilon\rho\big)\cdot\nabla\theta\\
&\quad - s\varepsilon\nabla w\cdot\nabla\theta + s(\lambda - \nu\bar\chi(\theta^0))p_s(\theta) - s(\lambda-\nu\bar\chi(\theta^0))\beta_1(\theta)(g(r_H^0) - \mathbf{G}[g(r_H^0)])\\
&= s\beta_2(\lambda-\nu\bar\chi(\theta^0))\rho\chi(c^0)\sqrt{\bar\chi(\theta^0)} + \beta_2(1-a(w^0))\big[p_s(\bar\chi(\theta^0)) - \rho^0\chi(c^0)\sqrt{\bar\chi(\theta^0)}\big]^+,
\end{aligned} \tag{4.19}$$

with the initial and boundary conditions

$$\begin{cases} -\kappa(\rho_\varepsilon,c_\varepsilon)\nabla\theta\cdot\mathbf{n} = \alpha_3(\theta-\theta^o) & \text{for } x\in\partial\Omega\\ \theta(x,0) = \theta_{0\varepsilon}(x) & \text{for } x\in\Omega. \end{cases}$$

Let M denote the map from $(\rho^0,c^0,w^0,\theta^0,s)$ to (ρ,c,w,θ). We claim in the following lemma that the map $M : X\times[0,1]\to X$ is well defined, continuous, and compact. The proof can be obtained following the lines given in [22] and so we omit it.

Lemma 4.3. *The mapping $M : X\times[0,1]\to X$ is well defined, continuous, and compact. For any $(\rho^0,c^0,w^0,\theta^0)\in X$ and $s\in[0,1]$, the image $(\rho,c,w,\theta) = M(\rho^0,c^0,w^0,\theta^0)$ satisfies*

$$\|\rho\|_{V_p^1(Q_T)} \le C(\varepsilon^{-1},\|\theta^0\|_{L^2(I;H^1(\Omega))},p,T),\quad 2\le p<\infty, \tag{4.20}$$

$$\|\rho c\|_{V_p^1(Q_T)} \le C(\varepsilon^{-1},\|\theta^0\|_{L^2(I;H^1(\Omega))},p,T),\quad 2\le p<\infty, \tag{4.21}$$

$$\|w\|_{W_p^{2,1}(Q_T)}, \le C(\varepsilon^{-1},\|\theta^0\|_{L^2(I;H^1(\Omega))},p,T),\quad 2\le p<\infty, \tag{4.22}$$

$$\|\theta\|_{W_2^{2,1}(Q_T)} \le C(\varepsilon^{-1},\|\theta^0\|_{L^2(I;H^1(\Omega))},T) \tag{4.23}$$

and

$$\rho \le C(\varepsilon^{-1}, \|\theta^0\|_{L^2(I;H^1(\Omega))}, T), \tag{4.24}$$

$$\rho \ge C^{-1}(\varepsilon^{-1}, \|\theta^0\|_{L^2(I;H^1(\Omega))}, T), \tag{4.25}$$

$$0 \le c \le C(\varepsilon^{-1}, \|\theta^0\|_{L^2(I;H^1(\Omega))}, T), \tag{4.26}$$

$$0 \le w \le C(\varepsilon^{-1}, \|\theta^0\|_{L^2(I;H^1(\Omega))}, T), \tag{4.27}$$

$$0 \le \theta \le C(\varepsilon^{-1}, \|\theta^0\|_{L^2(I;H^1(\Omega))}, T). \tag{4.28}$$

Remark 4.4. To prove the existence of an approximating solution by the Leray–Schauder fixed-point theory, we have introduced the map M in terms of two truncated operators χ and $\bar{\chi}$. These truncated operators will reduce to the identity operator after we prove the boundedness of c and θ. Moreover, when $s = 1$, the fixed point of the map $M(\cdot, 1)$ is a solution of the approximating system (4.2)–(4.11). When $s = 0$, $M(\rho^0, c^0, w^0, \theta^0, 0)$ is the solution of the system

$$\begin{aligned}
&\partial_t \rho - k\nabla \cdot ([\bar{\chi}(\theta^0)]_\varepsilon \nabla\rho) = \varepsilon,\\
&\partial_t(\rho c) - \nabla \cdot (d_{\mathrm{g}}\rho\nabla c) - k\nabla \cdot ([\bar{\chi}(\theta^0)]_\varepsilon \nabla\rho\, c) = \varepsilon c,\\
&w_t - \varepsilon\Delta w = 0,\\
&(\rho + \varsigma_1 w + \varsigma_2)\partial_t\theta - \nabla \cdot (\kappa(\rho_\varepsilon, c_\varepsilon)\nabla\theta) - k[\bar{\chi}(\theta^0)]_\varepsilon \nabla\rho \cdot \nabla\theta\\
&\ = \beta_2(1 - a(w^0))\big[p_s(\bar{\chi}(\theta^0)) - \rho^0\chi(c^0)\sqrt{\bar{\chi}(\theta^0)}\big]^+,
\end{aligned}$$

with the corresponding initial and boundary conditions. One can see that the solution (ρ, c, w, θ) of the above system is uniformly bounded in the space X^+, i.e., $\|M(\rho^0, c^0, w^0, \theta^0, 0)\|_X \le C_\varepsilon$, with a constant C_ε independent of $\rho^0, c^0, w^0, \theta^0$, which implies that $\|M(\rho^0, c^0, w^0, \theta^0, 0)\|_X \le \|(\rho^0, c^0, w^0, \theta^0)\|_X$ for all $\|\rho^0, c^0, w^0, \theta^0\|_X \ge C_\varepsilon$.

4.3. Existence of approximating solutions. By Lemma 4.1, there exists a fixed point for the map $M(\cdot, 1) : X \to X$ if

$$M(\rho^0, c^0, w^0, \theta^0, 0) \le \|(\rho^0, c^0, w^0, \theta^0)\|_X \quad \forall\ \|(\rho^0, c^0, w^0, \theta^0)\|_X \ge C \tag{4.29}$$

for some positive constant C, and all the functions $(\rho, c, w, \theta) \in X$ satisfying

$$(\rho, c, w, \theta) = M(\rho, c, w, \theta, s) \tag{4.30}$$

for some $s \in [0, 1]$ are uniformly bounded in X. It is not difficult to observe that (4.29) holds because for $s = 0$ the solution of (4.15)–(4.19) is bounded, independent of $(\rho^0, c^0, w^0, \theta^0)$. On the other hand, if (ρ, c, w, θ) satisfies (4.30), then it is a solution of the system

$$\begin{aligned}
&\partial_t\rho - k\nabla \cdot ([\bar{\chi}(\theta)]_\varepsilon \nabla\rho) - sk\nabla \cdot (\rho(\nabla\theta)_\varepsilon)\\
&\quad + s\beta_1(\bar{\chi}(\theta))(g(r_H) - \mathbf{G}[g(r_H)]) + s\beta_2\rho\chi(c)\sqrt{\bar{\chi}(\theta)}\\
&\quad = \varepsilon + s\beta_2 p_s(\bar{\chi}(\theta)) - s\beta_2(1 - a(w))\big[p_s(\bar{\chi}(\theta)) - \rho\chi(c)\sqrt{\bar{\chi}(\theta)}\big]^+,
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
&\partial_t(\rho c) - \nabla \cdot (d_{\mathrm{g}}\rho\nabla c) - k\nabla \cdot ([\bar{\chi}(\theta)]_\varepsilon \nabla\rho\, c)\\
&\quad - sk\nabla \cdot (\rho c(\nabla\theta)_\varepsilon) + s\beta_1(\bar{\chi}(\theta))(g(\widetilde{r}_H) - \mathbf{G}[g(r_H)]) + s\beta_2\rho c\sqrt{\bar{\chi}(\theta)}\\
&= \varepsilon c + s\beta_2 p_s(\bar{\chi}(\theta)) - s\beta_2(1 - a(w))\big[p_s(\bar{\chi}(\theta)) - \rho\chi(c)\sqrt{\bar{\chi}(\theta)}\big]^+,
\end{aligned} \tag{4.32}$$

$$
\begin{aligned}
& w_t - \varepsilon\Delta w + sH^-(\rho,\chi(c),\bar\chi(\theta))a(w) \\
& \quad = sH^+(\rho,\chi(c),\bar\chi(\theta)) + s\beta_1(\bar\chi(\theta^0))\big[g(\widetilde r_H) - \mathbf{G}[g(\widetilde r_H)]\big],
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
& (\rho+\varsigma_1 w+\varsigma_2)\partial_t\theta - \nabla\cdot(\kappa(\rho_\varepsilon,c_\varepsilon)\nabla\theta) - \big(k[\bar\chi(\theta)]_\varepsilon\nabla\rho + sk(\nabla\theta)_\varepsilon\rho\big)\cdot\nabla\theta \\
& - s\varepsilon\nabla w\cdot\nabla\theta + s\beta_2(\lambda-\nu\bar\chi(\theta))p_s(\theta) - s\beta_1(\theta)(g(r_H) - \mathbf{G}[g(r_H)]) \\
& = s(\lambda-\nu\bar\chi(\theta))\big[\beta_2\rho\chi(c)\sqrt{\bar\chi(\theta)} + (1-a(w))H^-(\rho,\chi(c),\bar\chi(\theta))\big],
\end{aligned} \tag{4.34}
$$

with the boundary conditions

$$\big[-k[\bar\chi(\theta)]_\varepsilon\nabla\rho - sk(\nabla\theta)_\varepsilon\rho\big]\cdot\mathbf{n} = \alpha_1(\rho\chi(c)-v^o) + \alpha_2(\rho\chi(\tilde c)-a^o), \tag{4.35}$$

$$\big[-d_{\rm g}\rho\nabla c - k[\bar\chi(\theta)]_\varepsilon\nabla\rho\, c - sk(\nabla\theta)_\varepsilon\rho c\big]\cdot\mathbf{n} = \alpha_1(\rho c - v^o), \tag{4.36}$$

$$-\varepsilon\nabla w\cdot\mathbf{n} = 0, \tag{4.37}$$

$$-\kappa(\rho_\varepsilon,c_\varepsilon)\nabla\theta\cdot\mathbf{n} = \alpha_3(\theta-\theta^o) \tag{4.38}$$

and initial conditions

$$\rho(x,0) = \rho_{0\varepsilon}(x)+\varepsilon, \quad \theta(x,0) = \theta_{0\varepsilon}(x), \quad w(x,0) = 0, \quad c(x,0) = c_{0\varepsilon}(x) \quad \text{for} \ \ x\in\Omega. \tag{4.39}$$

In the following, we present uniform estimates for the solution of (4.30), i.e., solution of (4.31)–(4.39). We shall see that the truncation functions χ and $\bar\chi$ can be removed from the above equations.

To prove $0\le c\le 1$, we consider $\tilde c = 1-c$. Equation (4.31) minus (4.32) shows that $\tilde c$ is the solution of the equation

$$
\begin{aligned}
& \partial_t(\rho\tilde c) - \nabla\cdot(d_g\rho\nabla\tilde c) - \nabla\cdot(k[\bar\chi(\theta)]_\varepsilon\nabla\rho\,\tilde c) - s\nabla\cdot(k\rho(\nabla\theta)_\varepsilon\tilde c) - \varepsilon\tilde c \\
& \quad = s\beta_2\rho(c-\chi(c))\sqrt{\bar\chi(\theta)} + s\beta_1(\bar\chi(\theta))\left[g\left(\frac{\sqrt{\bar\chi(\theta)}}{p_s(\bar\chi(\theta))}\rho c\right) - g\left(\frac{\sqrt{\bar\chi(\theta)}}{p_s(\bar\chi(\theta))}\rho\chi(c)\right)\right],
\end{aligned} \tag{4.40}
$$

with the initial and boundary conditions

$$
\begin{cases}
\big[-d_g\rho\nabla\tilde c - k[\bar\chi(\theta)]_\varepsilon\nabla\rho\,\tilde c - sk\rho(\nabla\theta)_\varepsilon\tilde c\big]\cdot\mathbf{n} = \alpha_2\rho\chi(\tilde c) - \vartheta^o & \text{for} \ \ x\in\partial\Omega, \\
\tilde c(x,0) = 1 - c_{0\varepsilon}(x) & \text{for} \ \ x\in\Omega,
\end{cases}
$$

where

$$\vartheta^o = \alpha_2 a^o + \alpha_1\rho(c-\chi(c)) \ge 0.$$

Since the right-hand side of (4.40) is nonnegative and the system has the same structure as (4.17), with the same argument as for $c\ge 0$, we can easily show that $\tilde c\ge 0$ (note that $\rho\chi(\tilde c)\rho\tilde c^- \equiv 0$ on $\partial\Omega$), which implies

$$0\le c\le 1. \tag{4.41}$$

By Lemma 4.3, the conditions (3.8)–(3.9) are satisfied. Therefore, the a priori estimates presented in section 3.1 hold for the solution of (4.31)–(4.39). In particular,

$$\theta_{\min}\le\theta\le\theta_{\max}. \tag{4.42}$$

It follows immediately from the above two inequalities that

$$\bar{\chi}(\theta) = \theta \quad \text{and} \quad \chi(c) = c, \tag{4.43}$$

and the truncated operators $\bar{\chi}$ and χ can be removed from (4.31)–(4.39). Thus the a priori estimates also hold for the approximating solutions, from which we have

$$\int_0^T \int_\Omega \kappa(\rho_\varepsilon, c_\varepsilon)|\nabla\theta|^2 dxdt \le C, \tag{4.44}$$

$$\sup_{0\le t\le T} \int_\Omega \rho(x,t)dx + \int_0^T \int_\Omega \rho^{5/3} dxdt + \int_0^T \int_\Omega \frac{|\nabla\rho|^2}{\rho} dxdt \le C, \tag{4.45}$$

$$\int_0^T \int_\Omega \rho|\nabla c|^2 dxdt \le C, \qquad \int_0^T \int_\Omega |\Gamma|^{5/3} dxdt \le C. \tag{4.46}$$

From (4.42), (4.44), (4.24), and (4.41), we have further

$$\|\theta\|_{L^2(I;H^1(\Omega))} \le C$$

and

$$\|\rho\|_{V_2^0(Q_T)},\ \|c\|_{V_2^0(Q_T)},\ \|w\|_{L^2(I;H^1(\Omega))}\ \le C.$$

In conclusion, (ρ, c, w, θ) are uniformly bounded in X, provided it is solution of (4.30). By Lemma 4.1, there exists a fixed point (ρ, θ, w, c) for the map $M(\cdot, 1)$, which is a solution of the system (4.2)–(4.10). With the regularity of the approximating solution, i.e., $\rho, c \in V_p^1(Q_T)$ and $w, \theta \in W_2^{2,1}(Q_T)$, equation (4.11) holds

5. Global existence of solution. From the last section, we know that there exists an approximating solution $(\rho^\varepsilon, c^\varepsilon, w^\varepsilon, \theta^\varepsilon)$ for the system (4.2)–(4.10). In this section, we prove that there exists a sequence $\varepsilon_j \to 0$ such that the corresponding approximating solutions (with $\varepsilon = \varepsilon_j$) converge to a solution of the system (2.17)–(2.24).

5.1. Compactness of the approximating solutions. We first prove the compactness of the approximating solutions. For the convenience of notation, we neglect the superscripts ε of the approximating solutions. By the estimates in Lemma 4.3, we have $\rho \ge C^{-1}(\varepsilon^{-1}, \|\theta\|_{L^2(I;H^1(\Omega))}, T)$. With this estimate, by multiplying $1/\rho$ on both sides of (4.2) and integrating the result over $\Omega \times (0, \tau)$, we obtain

$$\begin{aligned} &-\int_\Omega \ln\rho(x,t)dx + \int_0^t \int_\Omega \frac{k\theta_\varepsilon}{\rho^2}|\nabla\rho|^2 dxd\tau + \int_0^t \int_{\partial\Omega} \left[\alpha_1\Big(\frac{v^o}{\rho} - c\Big) + \alpha_2\Big(\frac{a^o}{\rho} - \tilde{c}\Big)\right] d\sigma_x d\tau \\ &= -\int_\Omega \ln\rho_{0\varepsilon}(x) - \int_0^t \int_\Omega \frac{k(\nabla\theta)_\varepsilon}{\rho}\cdot\nabla\rho\, dxd\tau + \int_0^t \int_\Omega \frac{\Gamma}{\rho}\, dxd\tau, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \int_0^\tau \int_\Omega \frac{\Gamma}{\rho}\, dxdt &= \beta_2 \int_0^\tau \int_\Omega \left(\frac{[\rho c\sqrt{\theta} - p_s(\theta)]^+}{\rho} - a(w)\frac{[p_s(\theta) - \rho c\sqrt{\theta}]^+}{\rho}\right) dxdt \\ &\quad + \int_0^\tau \int_\Omega \frac{g(r_H) - \mathbf{G}[g(r_H)]}{\rho}\beta_1(\theta)\, dxdt \\ &\le \beta_2 \int_0^\tau \int_\Omega c\sqrt{\theta} dxdt + \beta_1(\theta_{\max}) \int_0^\tau \int_\Omega \frac{g(r_H)}{\rho} dxdt. \end{aligned}$$

Since

$$\frac{g(r_H)}{\rho} = g\left(\frac{c\sqrt{\theta}}{p_s(\theta)}\rho\right)\frac{1}{\rho} \le C\left|\frac{c\sqrt{\theta}}{p_s(\theta)}\right| \le C,$$

we have

$$\int_0^\tau \int_\Omega \frac{\Gamma}{\rho}\, dxdt \le C.$$

From (5.1) we obtain

$$\begin{aligned}\int_{\Omega\cap\{\rho<1\}} |\ln\rho(x,\tau)|dx + \int_0^\tau \int_\Omega \frac{k\theta_\varepsilon}{\rho^2}|\nabla\rho|^2 dxdt \\ \le C + \frac{1}{2}\int_0^t \int_\Omega \frac{k\theta_\varepsilon}{\rho^2}|\nabla\rho|^2 dxd\tau + \int_0^t \int_\Omega \frac{k}{2\theta_\varepsilon}|(\nabla\theta)_\varepsilon|^2 dxd\tau,\end{aligned}$$

which together with the estimates (3.18) implies that

$$\int_0^T \int_\Omega |\ln\rho(x,\tau)|\, dxdt \le C. \tag{5.2}$$

This estimate is useful because it implies that ρ is a.e. positive.

By interpolation inequality in $\mathbb{R}^3$,

$$\begin{aligned}\int_0^T \left(\int_\Omega \rho^2 dx\right)^{2/3} dt \le C\int_0^T \left(\int_\Omega \sqrt{\rho} dx\right)^{8/3} dt \\ + C\int_0^T \left[\left(\int_\Omega \rho dx\right)^{1/3} \int_\Omega \frac{|\nabla\rho|^2}{\rho} dx\right] dt \le C.\end{aligned} \tag{5.3}$$

With Hölder's inequality and (3.22), we have

$$\int_0^T \int_\Omega |\rho\nabla\theta|^{5/4} dxdt \le C\int_0^T \int_\Omega \rho^{5/3} dxdt + C\int_0^T \int_\Omega \rho|\nabla\theta|^2 dxdt \le C, \tag{5.4}$$

$$\int_0^T \int_\Omega |\rho\nabla c|^{5/4} dxdt \le C\int_0^T \int_\Omega \rho^{5/3} dxdt + C\int_0^T \int_\Omega \rho|\nabla c|^2 dxdt \le C, \tag{5.5}$$

$$\int_0^T \int_\Omega |\nabla\rho|^{5/4} dxdt \le C\int_0^T \int_\Omega \rho^{5/3} dxdt + C\int_0^T \int_\Omega \frac{|\nabla\rho|^2}{\rho} dxdt \le C. \tag{5.6}$$

Let $u = \rho c$. Then

$$\|\nabla u\|_{L^{5/4}(Q_T)} \le \|c\nabla\rho\|_{L^{5/4}(Q_T)} + \|\rho\nabla c\|_{L^{5/4}(Q_T)} \le C, \tag{5.7}$$

$$\int_0^T \int_\Omega u^{5/3} dxdt \le \int_0^T \int_\Omega \rho^{5/3} dxdt \le C. \tag{5.8}$$

Note that u is the solution of the variational equation

$$\begin{aligned}\int_{Q_T} \partial_t u\, \psi dxdt = -\int_{Q_T} \left[d_g\nabla u - d_g\nabla\rho\, c + k\theta_\varepsilon\nabla\rho\, c + k(\nabla\theta)_\varepsilon \rho c\right]\cdot\nabla\psi dxdt \\ + \int_{Q_T} \Gamma\psi dxdt - \int_0^T \int_{\partial\Omega} \alpha_1(u - v^o)\psi d\sigma_x dt \qquad \forall\, \psi \in L^2(I; H^1(\Omega)).\end{aligned} \tag{5.9}$$

We observe that

$$\int_{Q_T} |(\nabla\theta)_\varepsilon \rho c|^{5/4} \le \int_{Q_T} \rho^{5/3} dxdt + \int_{Q_T} \rho |(\nabla\theta)_\varepsilon|^2 dxdt \le C,$$

$$\int_{Q_T} |\kappa(\rho_\varepsilon, c_\varepsilon)\nabla\theta|^{5/4} \le \int_{Q_T} \kappa(\rho_\varepsilon, c_\varepsilon)|\nabla\theta|^2 dxdt + \int_{Q_T} |\kappa(\rho_\varepsilon, c_\varepsilon)|^{5/3} dxdt \le C.$$

Plugging the above inequalities into (5.9) gives

$$\left| \int_{Q_T} \partial_t u \, \psi dxdt \right| \le C \|\psi\|_{L^5(I;W^{1,5}(\Omega))},$$

which implies

$$\|\partial_t u\|_{L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))} \le C_T. \tag{5.10}$$

With the same approach we can obtain

$$\|\partial_t \rho\|_{L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))} \le C, \qquad \|\partial_t(\rho\theta + \varsigma_1 w\theta + \varsigma_2\theta)\|_{L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))} \le C. \tag{5.11}$$

Finally, we prove compactness of w in $L^p(Q_T)$. Multiplying (4.4) by w and integrating the result, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |w|^2 dx + \int_\Omega \varepsilon |\nabla w|^2 dx &= \int_\Omega \Gamma w dx \le \int_\Omega (C + C\rho) w dx \\
&\le C + C\int_\Omega |w|^2 dx + C\left(\int_\Omega |\rho|^2 dx\right)^{1/2} \left(\int_\Omega |w|^2 dx\right)^{1/2} \\
&\le C + C\left(\int_\Omega |\rho|^2 dx\right)^{1/3} + C\left[1 + \left(\int_\Omega |\rho|^2 dx\right)^{2/3}\right]\left(\int_\Omega |w|^2 dx\right).
\end{aligned}$$

Using Gronwall's inequality and (5.3), we obtain

$$\max_{0\le t\le T} \int_\Omega |w(x,t)|^2 dx + \int_0^T \int_\Omega \varepsilon |\nabla w|^2 dx \le C, \tag{5.12}$$

$$\int_0^T \int_\Omega |\varepsilon \nabla w|^{5/4} dxdt \le C \int_0^T \int_\Omega \varepsilon^{5/3} dxdt + C\int_0^T \int_\Omega \varepsilon |\nabla w|^2 dxdt \le C. \tag{5.13}$$

Combining (4.4) and the estimate (5.13) we can derive that $\partial_t w$ is uniformly bounded in $L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))$. In the following we shall prove that $\partial_j w$ is uniformly bounded in $L^\infty(I; L^1_{\text{loc}}(\Omega))$ so that w is bounded in $L^\infty(I; W^{1,1}_{\text{loc}}(\Omega))$. Since $W^{1,1}_{\text{loc}}(\Omega) \hookrightarrow\hookrightarrow L^{5/4}_{\text{loc}}(\Omega)$, by the Aubin–Lions lemma we derive that w is compact in $L^{5/4}(I; L^{5/4}(B_R(x_0)))$, where $B_R(x_0)$ is any ball contained in Ω.

We denote by $B_R(x_0)$ a ball centered at $x_0 \in \Omega$ such that $\bar{B}_{2R}(x_0) \subset \Omega$. Let ζ be a smooth function such that $0 \le \zeta \le 1$, $\zeta = 1$ on $B_R(x_0)$, and $\zeta = 0$ outside $B_{2R}(x_0)$. Let h be a small positive number so that $h < \text{dist}(B_{2R}(x_0), \partial\Omega)$. Let $\tau \in (0,T)$ be fixed and let q_h be the solution of the following backward parabolic equation:

$$\begin{aligned}
&\partial_t q_h + \varepsilon \Delta q_h = 0 \qquad \text{in } \ \Omega \times (0,\tau), \\
&\varepsilon \nabla q_h \cdot \nabla \mathbf{n} = 0 \qquad \text{on } \ \partial\Omega, \\
&q_h(x,\tau) = \varphi_h * \eta_h,
\end{aligned} \tag{5.14}$$

where

$$\varphi_h(x) = \begin{cases} \text{sign}[\partial_j w(x,\tau)], & |x - x_0| \le R - h, \\ 0, & |x - x_0| > R - h, \end{cases}$$

and η_h is the standard mollifier in $\mathbb{R}^3$. Then by [26, Lemma 4.42, Chapter 2], we have

$$\int_{B_R} \partial_j w\, q_h(x,\tau)\zeta(x)dx \to \int_{B_R} |\partial_j w|\zeta(x)dx \quad \text{as} \quad h \to 0 \tag{5.15}$$

for any smooth function $\zeta(x)$. By multiplying q_h on both sides of (5.14) and integrating the result, we obtain

$$\frac{1}{2}\int_\Omega |q_h(x,\tau')|^2 dx + \int_{\tau'}^{\tau} \varepsilon |\nabla q_h|^2 dxdt = \frac{1}{2}\int_\Omega |\varphi_h * \eta_h|^2 dx \le C. \tag{5.16}$$

By using the weak maximum principle, we have

$$|q_h| \le 1 \quad \text{in} \;\; \Omega \times (0,\tau). \tag{5.17}$$

Differentiating w with respect to x_j and denoting $\partial_j f = \partial f/\partial x_j$ for any function f gives

$$\partial_t \partial_j w - \varepsilon \Delta \partial_j w = \partial_j \Gamma, \tag{5.18}$$

where

$$\begin{aligned} \partial_j \Gamma = \partial_j H^+ &- a(w)\partial_j H^- - a'(w)\partial_j w H^- \\ &+ \beta_1'(\theta)\partial_j\theta(g(r_H) - \mathbf{G}[g(r_H)]) \\ &+ \beta_1(\theta)(g'(r_H) - \mathbf{G}[g'(r_H)])\partial_j r_H \end{aligned}$$

and

$$\partial_j H = \beta_2 \Big(\partial_j \rho c\sqrt{\theta} + \rho \partial_j c\sqrt{\theta} + \frac{\rho c}{2\sqrt{\theta}}\partial_j\theta - p_s'(\theta)\partial_j\theta\Big),$$

$$\partial_j r_H = \frac{\partial_j \rho c \sqrt{\theta}}{p_s(\theta)} + \frac{\rho \partial_j c\sqrt{\theta}}{p_s(\theta)} + \rho c \partial_j \theta \frac{d}{d\theta}\bigg(\frac{\sqrt{\theta}}{p_s(\theta)}\bigg).$$

With $\|H^-\|_{L^\infty(Q_T)} \le C$, from (4.44), (5.4), (5.5), (5.6), and (5.13), we can see that

$$\begin{aligned} &\|\zeta \partial_j \Gamma\|_{L^1(Q_T)} \\ &\quad \le C(\|\partial_j H\|_{L^1(Q_T)} + \|\zeta\partial_j w\|_{L^1(Q_T)} + \|\partial_j\theta\|_{L^1(Q_T)} + \|\partial_j r_H\|_{L^1(Q_T)}) \\ &\quad \le C(\|\partial_j\rho\|_{L^1(Q_T)} + \|\rho\partial_j c\|_{L^1(Q_T)} + \|\rho\partial_j\theta\|_{L^1(Q_T)} + \|\partial_j\theta\|_{L^1(Q_T)} + \|\zeta\partial_j w\|_{L^1(Q_T)}) \\ &\quad \le C + C\|\zeta\partial_j w\|_{L^1(Q_T)}. \end{aligned} \tag{5.19}$$

By multiplying ζq_h on both sides of (5.18) and integrating the result over $\Omega\times(0,\tau)$, we obtain

$$\int_\Omega \partial_j w(x,\tau) q_h(x,\tau)\zeta(x)dx = \int_0^\tau \int_\Omega \big(\varepsilon \partial_j w \Delta\zeta q_h + 2\varepsilon\partial_j w \nabla\zeta\cdot\nabla q_h + \partial_j\Gamma q_h\zeta\big)dxdt.$$

Let $h \to 0$. By using (5.16), (5.17), and (5.19), we derive that

$$\max_{0\le\tau\le T}\int_\Omega |\partial_j w(x,\tau)|\zeta(x)dx \le C\int_0^\tau\int_\Omega \big(\varepsilon|\partial_j w||q_h| + \varepsilon|\partial_j w||\nabla q_h|\big)dxdt$$
$$+ C + C\int_0^T\int_\Omega |\partial_j w(x,\tau)|\zeta(x)dxdt$$
$$\le C + C\int_0^T\int_\Omega |\partial_j w(x,\tau)|\zeta(x)dxdt.$$

By applying Gronwall's inequality to the above estimate, we get

$$\max_{0\le\tau\le T}\int_\Omega |\partial_j w(x,\tau)|\zeta(x)dx \le C.$$

Therefore, w is uniformly bounded in $L^\infty(I; W^{1,1}(B_R(x_0)))$. By the previous analysis, we know that w is compact in $L^{5/4}(I; L^{5/4}(B_R(x_0)))$, where $B_R(x_0)$ is any ball inside Ω. By using a diagonal process argument, one can prove that there exists a sequence $\varepsilon_j \to 0$ such that the corresponding approximating solutions w^{ε_j} converge a.e. to some function $w \in L^{5/4}(I; L^{5/4}_{\text{loc}}(\Omega))$. By (5.12) we know that w^{ε_j} is uniformly bounded in $L^2(Q_T)$. Hence,

$$w^{\varepsilon_j} \to w \quad \text{in} \quad L^p(Q_T) \quad \text{as} \quad \varepsilon_j \to 0 \quad \text{for} \quad 1 \le p < 2. \tag{5.20}$$

5.2. Convergence of the approximating solutions. Now we prove the convergence of a sequence $(\rho^{\varepsilon_j}, c^{\varepsilon_j}, w^{\varepsilon_j}, \theta^{\varepsilon_j})$ to a solution of the system (2.17)–(2.24). The following lemma is useful for the reader's convenience [22].

LEMMA 5.1. *Let M be a σ-finite measure space and let $f_n, g_n \in L^1(M)$ be two sequences of functions and $f, g, h \in L^1(M)$ such that*

$$f_n \to f \quad \text{a.e. (or in measure)} \quad \text{as} \quad n \to \infty,$$
$$g_n \rightharpoonup g \quad \text{weakly in} \quad L^1(M) \quad \text{as} \quad n \to \infty,$$
$$f_n g_n \rightharpoonup h \quad \text{weakly in} \quad L^1(M) \quad \text{as} \quad n \to \infty.$$

Then $h = fg$ a.e. in M.

We summarize the previous estimates as follows:

(5.21)
$$\theta_{\min} \le \theta(x,t) \le \theta_{\max}, \quad \int_0^t\int_\Omega (\kappa_0 + \kappa_1\rho_\varepsilon)|\nabla\theta|^2 dxd\tau \le C,$$

(5.22)
$$\|\rho\|_{L^{5/4}(I:W^{1,5/4}(\Omega))} \le C, \quad \|\rho\|_{L^{5/3}(Q_T)} \le C_T, \quad \|\rho^{-1}\|_{L^1(Q_T)} \le C,$$

(5.23)
$$0 \le c \le 1, \quad \int_0^T\int_\Omega \rho|\nabla c|^2 dxdt \le C,$$

(5.24)
$$\|u\|_{L^{5/3}(Q_T)} \le C, \quad \|u\|_{L^{5/4}(I;W^{1,5/4}(\Omega)} \le C, \quad \|\partial_t u\|_{L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))} \le C,$$

(5.25)
$$\|\partial_t\rho\|_{L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))} \le C, \qquad \|\partial_t(\rho\theta + \varsigma_1 w + \varsigma_2\theta)\|_{L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))} \le C,$$

(5.26)
$$\|\partial_t w\|_{L^{5/4}(I;\widetilde{W}^{-1,5/4}(\Omega))} \le C, \quad \|(\nabla\theta)_\varepsilon\rho c\|_{L^{5/4}(Q_T)} \le C, \quad \|\theta_\varepsilon\nabla\rho\, c\|_{L^{5/4}(Q_T)} \le C.$$

By the Aubin–Lions lemma, there exists a sequence $\varepsilon_j \to 0$ such that ρ^{ε_j} converges to some ρ in $L^{5/4}(Q_T)$. Since ρ is uniformly bounded in $L^{5/3}(Q_T)$, we further deduce that ρ^{ε_j} converges to ρ in $L^q(Q_T)$ for any $1 \le q < 5/3$. By choosing a proper subsequence, we have

$$
\begin{aligned}
&\rho^{\varepsilon_j} \to \rho \quad \text{strongly} \quad \text{in} \quad L^q(Q_T), \quad 1 \le q < 5/3,\\
&\rho^{\varepsilon_j} \to \rho \quad \text{a.e. in} \quad Q_T,\\
&\rho^{\varepsilon_j} \rightharpoonup \rho \quad \text{weakly} \quad \text{in} \quad L^{5/4}(I, W^{1,5/4}(\Omega)),\\
&\rho^{\varepsilon_j} \to \rho \quad \text{strongly} \quad \text{in} \quad L^{5/4}(I; L^{5/4}(\partial\Omega)),\\
&\partial_t\rho^{\varepsilon_j} \rightharpoonup \partial_t\rho \quad \text{weakly} \quad \text{in} \quad L^{5/4}(I; \widetilde{W}^{-1,5/4}(\Omega)).
\end{aligned} \tag{5.27}
$$

Similarly, there exists a subsequence of $u^{\varepsilon_j} = \rho^{\varepsilon_j} c^{\varepsilon_j}$ (also denoted by u^{ε_j}) such that

$$
\begin{aligned}
&u^{\varepsilon_j} \to u \quad \text{strongly} \quad \text{in} \quad L^q(Q_T), \quad 1 \le q < 5/3,\\
&u^{\varepsilon_j} \to u \quad \text{a.e. in} \quad Q_T,\\
&u^{\varepsilon_j} \to u \quad \text{strongly} \quad \text{in} \quad L^{5/4}(I; L^{5/4}(\partial\Omega)),\\
&u^{\varepsilon_j} \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad L^{5/4}(I, W^{1,5/4}(\Omega)),\\
&\partial_t u^{\varepsilon_j} \rightharpoonup \partial_t u \quad \text{weakly} \quad \text{in} \quad L^{5/4}(I; \widetilde{W}^{-1,5/4}(\Omega))
\end{aligned} \tag{5.28}
$$

and a subsequence of w^{ε_j} (also denoted by w^{ε_j}) such that

$$
\begin{aligned}
&w^{\varepsilon_j} \to w \quad \text{strongly} \quad \text{in} \quad L^q(Q_T)\ (1 \le q < 2),\\
&w^{\varepsilon_j} \to w \quad \text{a.e. in} \quad Q_T,\\
&\varepsilon w^{\varepsilon_j} \to 0 \quad \text{strongly} \quad \text{in} \quad L^2(Q_T),\\
&\partial_t w^{\varepsilon_j} \rightharpoonup \partial_t w \quad \text{weakly} \quad \text{in} \quad L^{5/4}(I; \widetilde{W}^{-1,5/4}(\Omega)).
\end{aligned} \tag{5.29}
$$

Finally, there exists a subsequence of θ^{ε_j} (also denoted by θ^{ε_j}) such that

$$
\begin{aligned}
&\theta^{\varepsilon_j} \to \theta \quad \text{strongly} \quad \text{in} \quad L^q(Q_T)\ (1 \le q < \infty),\\
&\theta^{\varepsilon_j} \to \theta \quad \text{a.e. in} \quad Q_T,\\
&\theta^{\varepsilon_j} \to \theta \quad \text{strongly} \quad \text{in} \quad L^2(I; L^2(\partial\Omega)),\\
&\theta^{\varepsilon_j} \rightharpoonup \theta \quad \text{weakly} \quad \text{in} \quad L^2(I, H^1(\Omega)),\\
&\partial_t(\rho^{\varepsilon_j}\theta^{\varepsilon_j} + \varsigma_1 w^{\varepsilon_j}\theta^{\varepsilon_j} + \varsigma_2\theta^{\varepsilon_j}) \rightharpoonup \partial_t(\rho\theta + \varsigma_1 w\theta + \varsigma_2\theta)\\
&\quad \text{weakly} \quad \text{in} \quad L^{5/4}(I; \widetilde{W}^{-1,5/4}(\Omega)).
\end{aligned} \tag{5.30}
$$

Note that (5.22) contains

$$\iint_{Q_T} |\ln\rho^{\varepsilon_j}|\, dxdt \le C.$$

Since $\rho^{\varepsilon_j} \to \rho \ge 0$ a.e. in Q_T, it follows that

$$\iint_{Q_T} |\ln\rho|\, dxdt \le \liminf_{j\to 0} \iint_{Q_T} |\ln\rho^{\varepsilon_j}|\, dxdt \le C.$$

Hence, ρ must be positive a.e. in Q_T. If we define $c = u/\rho$, then $c^{\varepsilon_j} \to c$ a.e. in Q_T because ρ is a.e. positive. It is not difficult to prove that

$$\nabla\rho^{\varepsilon_j} c^{\varepsilon_j} \rightharpoonup \nabla\rho\, c \quad \text{weakly in } L^{5/4}(Q_T), \tag{5.31}$$

$$\sqrt{\rho^{\varepsilon_j}}\nabla c^{\varepsilon_j} \rightharpoonup \sqrt{\rho}\nabla c \quad \text{weakly in } L^2(Q_T), \tag{5.32}$$

and

$$\int_0^T\int_\Omega \rho|\nabla c|^2 dxdt \le \liminf_{j\to\infty}\int_0^T\int_\Omega \rho^{\varepsilon_j}|\nabla c^{\varepsilon_j}|^2 dxdt \le C. \tag{5.33}$$

With the help of Lemma 5.1, the following convergence can be verified:

$$\kappa(\rho^{\varepsilon_j}_{\varepsilon_j}, c^{\varepsilon_j}_{\varepsilon_j})\nabla\theta^{\varepsilon_j} \rightharpoonup \kappa(\rho,c)\nabla\theta \quad \text{weakly in } L^{5/4}(Q_T), \tag{5.34}$$

$$\theta^{\varepsilon_j}_{\varepsilon_j}\nabla\rho^{\varepsilon_j} \rightharpoonup \theta\nabla\rho \quad \text{weakly in } L^{5/4}(Q_T), \tag{5.35}$$

$$(\nabla\theta^{\varepsilon_j})_{\varepsilon_j}\rho^{\varepsilon_j} \rightharpoonup \nabla\theta\,\rho \quad \text{weakly in } L^{5/4}(Q_T). \tag{5.36}$$

Since $(\rho^{\varepsilon_j}, \theta^{\varepsilon_j}, w^{\varepsilon_j}, c^{\varepsilon_j})$ is a solution of the system (4.2)–(4.11), it satisfies

$$\begin{aligned}
&-\int_0^T\int_\Omega \rho^{\varepsilon_j}\partial_t\phi_1\, dxdt + \int_0^T \left(k\theta^{\varepsilon_j}_{\varepsilon_j}\nabla\rho^{\varepsilon_j} + k\rho^{\varepsilon_j}\nabla(\theta^{\varepsilon_j})_{\varepsilon_j}\right)\cdot\nabla\phi_1\, dxdt\\
&\quad + \int_0^T\int_{\partial\Omega}\left[\alpha_1(\rho^{\varepsilon_j}c^{\varepsilon_j} - v^o) + \alpha_2(\rho^{\varepsilon_j}\tilde{c}^{\varepsilon_j} - a^o)\right]\phi_1\, d\sigma_x dt\\
&\quad = \int_\Omega(\rho_{0\varepsilon_j}(x) + \varepsilon_j)\phi_1(x,0)\, dxdt - \int_0^T\int_\Omega \Gamma^{\varepsilon_j}\phi_1\, dxdt,\\
&-\int_0^T\int_\Omega \rho^{\varepsilon_j}c^{\varepsilon_j}\partial_t\phi_2\, dxdt + \int_0^T \left(k\theta^{\varepsilon_j}_{\varepsilon_j}\nabla\rho^{\varepsilon_j}c^{\varepsilon_j} + k\rho^{\varepsilon_j}\nabla(\theta^{\varepsilon_j})_{\varepsilon_j}c^{\varepsilon_j}\right)\cdot\nabla\phi_2\, dxdt\\
&\quad + \int_0^T\int_\Omega d_g\rho^{\varepsilon_j}\nabla c^{\varepsilon_j}\cdot\nabla\phi_2\, dxdt + \int_0^T\int_{\partial\Omega}\alpha_1(\rho^{\varepsilon_j}c^{\varepsilon_j} - v^o)\phi_2\, d\sigma_x dt\\
&\quad = \int_\Omega(\rho_{0\varepsilon_j}(x) + \varepsilon_j)c_{0\varepsilon_j}(x)\phi_2(x,0)\, dxdt - \int_0^T\int_\Omega \Gamma^{\varepsilon_j}\phi_2\, dxdt,\\
&-\int_0^T\int_\Omega w^{\varepsilon_j}\partial_t\phi_3\, dxdt + \int_0^T \varepsilon_j\nabla w^{\varepsilon_j}\cdot\nabla\phi_3\, dxdt = \int_0^T\int_\Omega \Gamma^{\varepsilon_j}\phi_3\, dxdt,
\end{aligned}$$

and

$$\begin{aligned}
&-\int_0^T\int_\Omega(\rho^{\varepsilon_j}\theta^{\varepsilon_j} + \varsigma_1 w^{\varepsilon_j}\theta^{\varepsilon_j} + \varsigma_2\theta^{\varepsilon_j})\partial_t\phi_4 dxdt + \int_0^T\int_\Omega \kappa(\rho^{\varepsilon_j}_{\varepsilon_j}, c^{\varepsilon_j}_{\varepsilon_j})\nabla\theta^{\varepsilon_j}\cdot\nabla\phi_4\, dxdt\\
&+\int_0^T\int_\Omega \theta^{\varepsilon_j}\left(k\theta^{\varepsilon_j}_{\varepsilon_j}\nabla\rho^{\varepsilon_j} + k\rho^{\varepsilon_j}(\nabla\theta^{\varepsilon_j})_{\varepsilon_j} + \varepsilon_j\nabla w\right)\cdot\nabla\phi_4\, dxdt\\
&+\int_0^T\int_{\partial\Omega}\left[\alpha_1(\rho^{\varepsilon_j}c^{\varepsilon_j} - v^o) + \alpha_2(\rho^{\varepsilon_j}\tilde{c}^{\varepsilon_j} - a^o)\right]\theta\phi_4\, d\sigma_x dt\\
&+\int_0^T\int_{\partial\Omega}\alpha_3(\theta^{\varepsilon_j} - \theta^o)\phi_4\, d\sigma_x dt = \lambda\int_0^T\int_\Omega \Gamma^{\varepsilon_j}\phi_4 dxdt
\end{aligned}$$

for any $\phi_1, \phi_2, \phi_3, \phi_4 \in C^1(\bar{Q}_T)$ which vanish at $t = T$.

By taking the limit $j \to \infty$ and using (5.27)–(5.36), we find that the limit $(\rho, c, \widetilde{w}, \theta)$ satisfies the variational equations,

$$
\begin{aligned}
&-\int_0^T \int_\Omega \rho \partial_t \phi_1 \, dxdt + \int_0^T \big(k\theta\nabla\rho + k\rho\nabla\theta\big) \cdot \nabla\phi_1 \, dxdt \\
&\qquad + \int_0^T \int_{\partial\Omega} \big[\alpha_1(\rho c - v^o) + \alpha_2(\rho\tilde{c} - a^o)\big] \phi_1 \, d\sigma_x dt \\
&\quad = \int_\Omega \rho_0(x)\phi_1(x,0)\, dxdt - \int_0^T \int_\Omega \Gamma\, \phi_1 \, dxdt, \\
&-\int_0^T \int_\Omega \rho c \partial_t \phi_2 \, dxdt + \int_0^T \big(k\theta\nabla\rho\, c + k\rho\nabla\theta\, c\big) \cdot \nabla\phi_2 \, dxdt \\
&\qquad + \int_0^T \int_\Omega d_g \rho \nabla c \cdot \nabla\phi_2 \, dxdt + \int_0^T \int_{\partial\Omega} \alpha_1(\rho c - v^o)\phi_2 \, d\sigma_x dt \\
&\quad = \int_\Omega \rho_0(x) c_0(x) \phi_2(x,0)\, dxdt - \int_0^T \int_\Omega \Gamma\, \phi_2 \, dxdt, \\
&-\int_0^T \int_\Omega w \partial_t \phi_3 \, dxdt = \int_0^T \int_\Omega \Gamma\, \phi_3 \, dxdt,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_0^T \int_\Omega (\rho\theta + \varsigma_1 w\theta + \varsigma_2\theta)\partial_t\phi_4 dxdt + \int_0^T \int_\Omega \kappa(\rho, c)\nabla\theta \cdot \nabla\phi_4 \, dxdt \\
&+ \int_0^T \int_\Omega \theta\big(k\theta\nabla\rho + k\rho\nabla\theta\big) \cdot \nabla\phi_4 \, dxdt \\
&+ \int_0^T \int_{\partial\Omega} \big[\alpha_1(\rho c - v^o) + \alpha_1(\rho\tilde{c} - a^o)\big] \theta\phi_4 \, d\sigma_x dt \\
&+ \int_0^T \int_{\partial\Omega} \big[\alpha_3(\theta - \theta^o)\phi_4 \, d\sigma_x dt = \lambda \int_0^T \int_\Omega \Gamma\, \phi_4 dxdt
\end{aligned}
$$

for any $\phi_1, \phi_2, \phi_3, \phi_4 \in C^1(\bar{Q}_T)$ which vanish at $t = T$. In other words, (ρ, c, w, θ) is a solution to (2.13)–(2.24) in the sense of Definition 2.2.

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