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## THE LIMIT OF THE BOLTZMANN EQUATION TO THE EULER EQUATIONS FOR RIEMANN PROBLEMS\*

FEIMIN HUANG<sup>†</sup>, YI WANG<sup>†</sup>, YONG WANG<sup>†</sup>, AND TONG YANG<sup>‡</sup>

**Abstract.** The convergence of the Boltzmann equation to the compressible Euler equations when the Knudsen number tends to zero has been a long-standing open problem in kinetic theory. In the setting of a Riemann solution that contains the generic superposition of shock, rarefaction wave, and contact discontinuity to the Euler equations, we succeed in justifying this limit by introducing hyperbolic waves with different solution backgrounds to capture the extra masses carried by the hyperbolic approximation of the rarefaction wave and the diffusion approximation of contact discontinuity.

**Key words.** hydrodynamic limit, Boltzmann equation, Euler equations, Riemann solution

**AMS subject classifications.** 76P05, 35Q20, 74J40, 82B40, 82C40

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**1. Introduction.** As the fundamental equation in statistical mechanics, the Boltzmann equation takes the form of

$$f_t + \xi \cdot \nabla_X f = \frac{1}{\varepsilon} Q(f, f),$$

where  $f(t, X, \xi)$  is the density distribution function of particles at time  $t$  with location  $X$  and velocity  $\xi$ . In this equation, the physical parameter  $\varepsilon > 0$  called the Knudsen number is proportional to the mean free path of the interacting particles.

It was known since its derivation that the Boltzmann equation is closely related to the systems of fluid dynamics, in particular, the systems of Euler and Navier–Stokes equations. In fact, the first derivation of the fluid dynamical components and systems from the kinetic equations can be traced back to Maxwell and Boltzmann. Their early derivations rest on some arguments about how the various terms in a kinetic equation balance each other. These balance arguments seem arbitrary to some extent. For this, Hilbert proposed a systematic expansion in 1912, and Enskog and Chapman independently proposed another expansion in 1916 and 1917, respectively.

Either the Hilbert expansion or the Chapman–Enskog expansion yields the compressible Euler equations in the leading order with respect to the Knudsen number  $\varepsilon$ , and the compressible Navier–Stokes equations and Burnett equations in the subsequent orders. To justify these formal approximations in rigorous mathematics, that

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is, hydrodynamic limits, has been proved to be extremely challenging and mostly remains open, in part because the basic well-posedness and regularity questions are still mostly unsolved for these fluid equations.

The justification of the fluid limits of the Boltzmann equation is also related to the Hilbert's sixth problem, "Mathematical treatment of the axioms of physics." which is delivered at lectures in the International Congress of Mathematicians at Paris in 1900 and says that, "The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. . . . Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua. . . . Further, the mathematician has the duty to test exactly in each instance whether the new axioms are compatible with the previous ones."

The goal of this paper is to justify the limiting process of the Boltzmann equation to the system of the compressible Euler equations in the setting of Riemann solutions. The Riemann problem was first formulated and studied by Riemann in the 1860s when he studied one space dimensional isentropic gas dynamics with initial data being two constant states. The solution to this problem turns out to be fundamental in the theory of hyperbolic conservation laws because it not only captures the local and global behavior of solutions but also fully represents the effect of nonlinearity in the structure of the solutions. It is now well known that for the system of Euler equations, there are three basic wave patterns, that is, shock wave, rarefaction wave, and contact discontinuity. These three types of waves have essential differences: shock is compressive, rarefaction is expansive, and contact discontinuity has some diffusive structure. Therefore, how to study the hydrodynamic limit of the Boltzmann equation for the full Riemann solution that consists of the superposition of these three typical waves is still very challenging in mathematics.

In this paper, by introducing two types of hyperbolic waves that carry the extra masses in the hyperbolic approximation of the rarefaction wave profile and the diffusive approximation of the contact discontinuity, we succeed in proving rigorously that there exists a family of solutions to the Boltzmann equation that converges to a Maxwellian determined by a Riemann solution consisting of three basic wave patterns when the Knudsen number tends to zero. Furthermore, a convergence rate is obtained in terms of the Knudsen number.

By coping with the essential properties of individual wave pattern, the hydrodynamic limit for a single wave was justified in the previous works separately. More precisely, by using the compressibility of shock wave profile, Yu [41] showed that when the solution of the Euler equations (1.2) contains only noninteracting shocks, there exists a sequence of solutions to the Boltzmann equation that converge to a local Maxwellian defined by the solution of the Euler equations (1.2) uniformly away from the shock in any fixed time interval. In this work, a generalized Hilbert expansion was introduced, and the analytic technique of matching the inner and outer expansions developed by Goodman and Xin [17] for conservation laws was used. On the other hand, by using the time decay properties of the rarefaction wave, a similar problem was studied by Xin and Zeng [40]. Moreover, by using the diffusive structure in the contact discontinuity as for the Navier–Stokes equations, the hydrodynamic limit to the contact discontinuity was proved by Huang, Wang, and Yang in [20].

However, up to now, how to deal with the general Riemann solution that consists of all three basic waves is still a challenging open problem. This is mainly due to the

difficulty in handling the wave interactions and also unifying the different approaches in the analysis used for each single wave pattern. In order to overcome these difficulties in justifying the limit, our main idea in this paper is to introduce two families of hyperbolic waves, called hyperbolic wave I and II, that capture the propagation of the extra mass created by the approximate hyperbolic rarefaction wave profile in the viscous setting and the diffusion approximation of contact discontinuity.

We now briefly explain why the two families of hyperbolic waves we introduced are essential for the proof. As in the previous works on the rarefaction wave in the setting of either Navier–Stokes equations or the Boltzmann equation, the approximate rarefaction wave is constructed as a hyperbolic wave profile. Therefore, we need to precisely capture the error in the second order of the approximation for the Boltzmann equation in term of the Knudsen number, that is, in the Navier–Stokes level. And this requires studying the propagation of the extra mass induced by the viscosity and heat conductivity. For this, we introduce hyperbolic wave I as a solution to the linearized system around the approximate rarefaction wave profile with source terms given by the viscosity and heat conductivity induced by the rarefaction wave profile to recover the viscous terms. We can show that hyperbolic wave I decays like the first order derivative of the rarefaction wave profile so that the decay properties given in Lemma 2.2 are good enough to carry out the analysis.

The main difficulty comes from the approximation of the contact discontinuity. First, such an approximation, that is, a 2-viscous contact wave, behaves like a diffusion wave profile as for the Navier–Stokes equations. Due to the lack of sufficient decay in  $\varepsilon$  and the nonconservative error terms when taking the antiderivative of the perturbation, we need to remove the leading error terms and nonconservative terms in such approximation before taking the antiderivative. Hyperbolic wave II is constructed to remove these error terms due to the viscous contact wave approximation. Note that the construction of hyperbolic wave II cannot be done simply around the 2-viscous contact wave as hyperbolic wave I for the rarefaction wave profile. Otherwise, the wave interaction terms thus induced will lead to insufficient decay in  $\varepsilon$  due to the 2-viscous contact wave and it seems that these terms are essential in wave interactions. Instead, it is constructed around the superposition of the approximate 1-rarefaction wave, hyperbolic wave I, the 2-viscous contact wave, and the 3-shock profile as a whole. Thus some wave interaction terms can be absorbed in hyperbolic wave II and the other wave interaction terms can be handled by some subtle and careful calculations. Due to the nonconservative terms and insufficient decay rates of  $\varepsilon$  of error terms induced by the 2-contact wave, we cannot use the antiderivative technique to analyze hyperbolic wave II. Since the derivative of the 3-shock profile is negative and tends to infinity as the Knudsen number  $\varepsilon \rightarrow 0+$ , we have to impose the condition at the time  $t = T$  (see (2.74) below) for the linearized hyperbolic system of hyperbolic wave II so that the monotonicity of the 3-shock wave is fully utilized. This idea is different from the previous stability analysis on the shock profile which is based on the antiderivative technique.

With the help of these two hyperbolic waves and the corresponding new estimates, we can justify the limiting process from the Boltzmann equation to compressible Euler equations for the generic Riemann problems by elaborate analysis after a hyperbolic scaling.

We now formulate the problem. Consider the Boltzmann equation with slab symmetry (see [6])

$$(1.1) \quad f_t + \xi_1 f_x = \frac{1}{\varepsilon} Q(f, f),$$

where  $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$  and  $x \in \mathbf{R}^1$ . Here, the collision operator takes the form of

$$Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}_+^2} (f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi') - f(\xi)g(\xi_*) - f(\xi_*)g(\xi))B(|\xi - \xi_*|, \hat{\theta}) d\xi_* d\Omega,$$

where  $\xi', \xi'_*$  are the velocities after an elastic collision of two particles with velocities  $\xi, \xi_*$  before the collision. Here,  $\hat{\theta}$  is the angle between the relative velocity  $\xi - \xi_*$  and the unit vector  $\Omega$  in  $\mathbf{S}_+^2 = \{\Omega \in \mathbf{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0\}$ . The conservations of momentum and energy yield the following relations between the velocities before and after collision:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \Omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega] \Omega.$$

We will concentrate on the hard sphere model where the cross section takes the form of

$$B(|\xi - \xi_*|, \hat{\theta}) = |(\xi - \xi_*, \Omega)| = |\xi - \xi_*| \cos \hat{\theta}.$$

On the other hand, it is noted that the analysis can be applied to at least hard potential if we can assume Lemma 2.3 on the shock wave profile holds true.

As we mentioned earlier, formally, when the Knudsen number  $\varepsilon$  tends to zero, the limit of the Boltzmann equation (1.1) is the system of compressible Euler equations that consists of conservations of mass, momentum, and energy:

$$(1.2) \quad \begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = 0, \quad i = 2, 3, \\ \left[ \rho \left( e + \frac{|u|^2}{2} \right) \right]_t + \left[ \rho u_1 \left( E + \frac{|u|^2}{2} \right) + p u_1 \right]_x = 0, \end{cases}$$

where

$$(1.3) \quad \begin{cases} \rho(t, x) = \int_{\mathbf{R}^3} \varphi_0(\xi) f(t, x, \xi) d\xi, \\ \rho u_i(t, x) = \int_{\mathbf{R}^3} \varphi_i(\xi) f(t, x, \xi) d\xi, \quad i = 1, 2, 3, \\ \rho \left( e + \frac{|u|^2}{2} \right) (t, x) = \int_{\mathbf{R}^3} \varphi_4(\xi) f(t, x, \xi) d\xi. \end{cases}$$

Here,  $\rho$  is the density,  $u = (u_1, u_2, u_3)$  is the macroscopic velocity,  $e$  is the internal energy, and  $p = R\rho\theta$  with  $R$  being the gas constant is the pressure. Note that the temperature  $\theta$  is related to the internal energy by  $e = \frac{3}{2}R\theta$ , and  $\varphi_i(\xi) (i = 0, 1, 2, 3, 4)$  are the collision invariants given by

$$(1.4) \quad \varphi_0(\xi) = 1, \quad \varphi_i(\xi) = \xi_i \quad (i = 1, 2, 3), \quad \varphi_4(\xi) = \frac{1}{2}|\xi|^2$$

that satisfy

$$\int_{\mathbf{R}^3} \varphi_i(\xi) Q(g_1, g_2) d\xi = 0 \quad \text{for } i = 0, 1, 2, 3, 4.$$

Instead of using either Hilbert expansion or Chapman–Enskog expansion, we will apply the macro-micro decomposition introduced in [28]. For a solution  $f(t, x, \xi)$  of (1.1), set

$$f(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi),$$

where the local Maxwellian  $\mathbf{M}(t, x, \xi) = \mathbf{M}_{[\rho, u, \theta]}(\xi)$  represents the macroscopic component of the solution defined by the five conserved quantities, i.e., the mass density  $\rho(t, x)$ , the momentum  $\rho u(t, x)$ , and the total energy  $\rho(e + \frac{1}{2}|u|^2)(t, x)$  given in (1.3), through

$$(1.5) \quad \mathbf{M} = \mathbf{M}_{[\rho, u, \theta]}(t, x, \xi) = \frac{\rho(t, x)}{\sqrt{(2\pi R\theta(t, x))^3}} e^{-\frac{|\xi - u(t, x)|^2}{2R\theta(t, x)}}$$

and  $\mathbf{G}(t, x, \xi)$  represents the microscopic component.

From now on, the inner product of  $g_1$  and  $g_2$  in  $L^2_\xi(\mathbf{R}^3)$  with respect to a given Maxwellian  $\tilde{\mathbf{M}}$  is denoted by

$$(1.6) \quad \langle g_1, g_2 \rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{\mathbf{M}}} g_1(\xi) g_2(\xi) d\xi.$$

If  $\tilde{\mathbf{M}}$  is the local Maxwellian  $\mathbf{M}$  defined in (1.5), the macroscopic space is spanned by the following five pairwise orthogonal base:

$$(1.7) \quad \begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}} \mathbf{M}, \\ \chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}} \mathbf{M} \text{ for } i = 1, 2, 3, \\ \chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}} \left( \frac{|\xi - u|^2}{R\theta} - 3 \right) \mathbf{M}, \\ \langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad i, j = 0, 1, 2, 3, 4. \end{cases}$$

For brevity, if  $\tilde{\mathbf{M}}$  is the local Maxwellian  $\mathbf{M}$ , we will simply use  $\langle \cdot, \cdot \rangle$  to denote  $\langle \cdot, \cdot \rangle_{\mathbf{M}}$ . By using the above base, the macroscopic projection  $\mathbf{P}_0$  and microscopic projection  $\mathbf{P}_1$  can be defined as

$$\mathbf{P}_0 g = \sum_{j=0}^4 \langle g, \chi_j \rangle \chi_j, \quad \mathbf{P}_1 g = g - \mathbf{P}_0 g.$$

Note that a function  $g(\xi)$  is called microscopic if

$$\int g(\xi) \varphi_i(\xi) d\xi = 0, \quad i = 0, 1, 2, 3, 4,$$

where again  $\varphi_i(\xi)$  represents the collision invariants.

Notice that the solution  $f(t, x, \xi)$  to the Boltzmann equation (1.1) satisfies

$$\mathbf{P}_0 f = \mathbf{M}, \quad \mathbf{P}_1 f = \mathbf{G},$$

and the Boltzmann equation (1.1) becomes

$$(1.8) \quad (\mathbf{M} + \mathbf{G})_t + \xi_1 (\mathbf{M} + \mathbf{G})_x = \frac{1}{\varepsilon} [2Q(\mathbf{M}, \mathbf{G}) + Q(\mathbf{G}, \mathbf{G})].$$

By integrating the product of (1.8) and the collision invariants  $\varphi_i(\xi)$  ( $i = 0, 1, 2, 3, 4$ ) with respect to  $\xi$  over  $\mathbf{R}^3$ , one has the following system for the fluid variables  $(\rho, u, \theta)$ :

$$(1.9) \quad \begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = - \int \xi_1^2 \mathbf{G}_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = - \int \xi_1 \xi_i \mathbf{G}_x d\xi, \quad i = 2, 3, \\ \left[ \rho \left( e + \frac{|u|^2}{2} \right) \right]_t + \left[ \rho u_1 \left( e + \frac{|u|^2}{2} \right) + p u_1 \right]_x = - \int \frac{1}{2} \xi_1 |\xi|^2 \mathbf{G}_x d\xi. \end{cases}$$

Note that the above fluid-type system is not self-contained and one more equation for the microscopic component  $\mathbf{G}$  is needed, which can be obtained by applying the projection operator  $\mathbf{P}_1$  to (1.8):

$$(1.10) \quad \mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \frac{1}{\varepsilon} [\mathbf{L}_M \mathbf{G} + Q(\mathbf{G}, \mathbf{G})].$$

Here  $\mathbf{L}_M$  is the linearized collision operator of  $Q(f, f)$  with respect to the local Maxwellian  $\mathbf{M}$  given by

$$\mathbf{L}_M g = 2Q(\mathbf{M}, g) = Q(\mathbf{M}, g) + Q(g, \mathbf{M}).$$

Note that the null space  $\mathfrak{N}$  of  $\mathbf{L}_M$  is spanned by the macroscopic variables:

$$\chi_j(\xi), \quad j = 0, 1, 2, 3, 4.$$

Furthermore, there exists a positive constant  $\tilde{\sigma} > 0$  such that for any function  $g(\xi) \in \mathfrak{N}^\perp$  (cf. [18]),

$$\langle g, \mathbf{L}_M g \rangle \leq -\tilde{\sigma} \langle \nu(|\xi|) g, g \rangle,$$

where  $\nu(|\xi|) = O(1)(1 + |\xi|)$  is the collision frequency for the hard sphere model.

Consequently, the linearized collision operator  $\mathbf{L}_M$  is a dissipative operator on  $L^2(\mathbf{R}^3)$ , and its inverse  $\mathbf{L}_M^{-1}$  is a bounded operator on  $\mathfrak{N}^\perp$ . It follows from (1.10) that

$$(1.11) \quad \mathbf{G} = \varepsilon \mathbf{L}_M^{-1} [\mathbf{P}_1(\xi_1 \mathbf{M}_x)] + \Pi$$

with

$$(1.12) \quad \Pi = \mathbf{L}_M^{-1} [\varepsilon(\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{G}_x)) - Q(\mathbf{G}, \mathbf{G})].$$

Plugging (1.11) into (1.9) gives

$$(1.13) \quad \begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4\varepsilon}{3} (\mu(\theta) u_{1x})_x - \int \xi_1^2 \Pi_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = \varepsilon (\mu(\theta) u_{ix})_x - \int \xi_1 \xi_i \Pi_x d\xi, \quad i = 2, 3, \\ \left[ \rho \left( \theta + \frac{|u|^2}{2} \right) \right]_t + \left[ \rho u_1 \left( \theta + \frac{|u|^2}{2} \right) + p u_1 \right]_x = \varepsilon (\kappa(\theta) \theta_x)_x + \frac{4\varepsilon}{3} (\mu(\theta) u_1 u_{1x})_x \\ + \varepsilon \sum_{i=2}^3 (\mu(\theta) u_i u_{ix})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_x d\xi, \end{cases}$$

where the viscosity coefficient  $\mu(\theta) > 0$  and the heat conductivity coefficient  $\kappa(\theta) > 0$  are smooth functions of the temperature  $\theta$ . Here, we normalize the gas constant  $R$  to be  $\frac{2}{3}$  so that  $e = \theta$  and  $p = \frac{2}{3} \rho \theta$ .



Since the problem considered in this paper is one dimensional in the space variable  $x \in \mathbf{R}$ , in the macroscopic level, it is more convenient to rewrite (1.1) and system (1.2) in the *Lagrangian* coordinates. For this, set the coordinate transformation

$$(1.14) \quad (t, x) \Rightarrow \left( t, \int_{(0,0)}^{(t,x)} \rho(\tau, y) dy - (\rho u_1)(\tau, y) d\tau \right),$$

where  $\int_A^B f dy + g d\tau$  represents a line integraton from point  $A$  to point  $B$  on  $\mathbf{R}^+ \times \mathbf{R}$ . Here, the value of the integration is unique because of the conservation of mass.

We will still denote the *Lagrangian* coordinates by  $(t, x)$  for simplicity of notation. Then (1.1) and (1.2) in the Lagrangian coordinates become, respectively,

$$(1.15) \quad f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = \frac{1}{\varepsilon} Q(f, f)$$

and

$$(1.16) \quad \begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ u_{it} = 0, \quad i = 2, 3, \\ \left( \theta + \frac{|u|^2}{2} \right)_t + (p u_1)_x = 0. \end{cases}$$

Moreover, (1.9)–(1.13) take the form of

$$(1.17) \quad \begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = - \int \xi_1^2 \mathbf{G}_x d\xi, \\ u_{it} = - \int \xi_1 \xi_i \mathbf{G}_x d\xi, \quad i = 2, 3, \\ \left( \theta + \frac{|u|^2}{2} \right)_t + (p u_1)_x = - \int \frac{1}{2} \xi_1 |\xi|^2 \mathbf{G}_x d\xi, \end{cases}$$

$$(1.18) \quad \mathbf{G}_t - \frac{u_1}{v} \mathbf{G}_x + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \frac{1}{\varepsilon} (\mathbf{L}_M \mathbf{G} + Q(\mathbf{G}, \mathbf{G}))$$

with

$$(1.19) \quad \mathbf{G} = \varepsilon \mathbf{L}_M^{-1} \left( \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_x) \right) + \Pi_1,$$

$$(1.20) \quad \Pi_1 = \mathbf{L}_M^{-1} \left[ \varepsilon \left( \mathbf{G}_t - \frac{u_1}{v} \mathbf{G}_x + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x) \right) - Q(\mathbf{G}, \mathbf{G}) \right],$$

and

$$(1.21) \quad \begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4\varepsilon}{3} \left( \frac{\mu(\theta)}{v} u_{1x} \right)_x - \int \xi_1^2 \Pi_{1x} d\xi, \\ u_{it} = \varepsilon \left( \frac{\mu(\theta)}{v} u_{ix} \right)_x - \int \xi_1 \xi_i \Pi_{1x} d\xi, \quad i = 2, 3, \\ \left( \theta + \frac{|u|^2}{2} \right)_t + (p u_1)_x = \varepsilon \left( \frac{\kappa(\theta)}{v} \theta_x \right)_x + \frac{4\varepsilon}{3} \left( \frac{\mu(\theta)}{v} u_1 u_{1x} \right)_x \\ + \varepsilon \sum_{i=2}^3 \left( \frac{\mu(\theta)}{v} u_i u_{ix} \right)_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_{1x} d\xi. \end{cases}$$

The Riemann problem for the Euler system (1.16) is an initial value problem with initial data

$$(v, u, \theta)(t = 0, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases}$$

where,  $u = (u_1, u_2, u_3)$ ,  $u_{\pm} = (u_{1\pm}, 0, 0)$ , and  $v_{\pm} > 0, u_{1\pm}, \theta_{\pm} > 0$  are constants. It is known that the generic solution to the Riemann problem consists of three waves that propagate at different speeds, that is, shock, rarefaction wave, and contact discontinuity (cf. [11, 27]). We denote this solution by  $(\tilde{V}, \tilde{U}, \tilde{\Theta})(t, x)$ . Note that  $\tilde{U} = (\tilde{U}_1, 0, 0)$ .

Given the right end state  $(v_+, u_{1+}, \theta_+)$ , the following wave curves for the left end state  $(v, u_1, \theta)$  in the phase space are defined with  $v > 0$  and  $\theta > 0$  for the Euler equations (1.16):

- contact discontinuity curve,

$$(1.22) \quad CD(v_+, u_{1+}, \theta_+) = \{(v, u_1, \theta) | u_1 = u_{1+}, p = p_+, v \neq v_+\};$$

- $i$ -rarefaction wave curve ( $i = 1, 3$ ),

$$(1.23) \quad R_i(v_+, u_{1+}, \theta_+) := \left\{ (v, u_1, \theta) \left| v < v_+, u_1 = u_{1+} - \int_{v_+}^v \lambda_i(\eta, s_+) d\eta, s(v, \theta) = s_+ \right. \right\},$$

where  $s_+ = s(v_+, \theta_+)$  and  $\lambda_i = \lambda_i(v, s)$  is the  $i$ th characteristic speed of (1.16);

- $i$ -shock wave curve ( $i = 1, 3$ ),

$$(1.24) \quad S_i(v_+, u_{1+}, \theta_+) := \left\{ (v, u_1, \theta) \left| \begin{aligned} -s_i(v_+ - v) - (u_{1+} - u_1) &= 0, \\ -s_i(u_{1+} - u_1) + (p_+ - p) &= 0, \quad \text{and } \lambda_{i+} < s_i < \lambda_{i-}, \\ -s_i(E_+ - E) + (p_+ u_{1+} - p u_1) &= 0, \end{aligned} \right. \right\},$$

where  $E = \theta + \frac{|u|^2}{2}, p = \frac{2\theta}{3v}, E_+ = \theta_+ + \frac{|u_+|^2}{2}, p_+ = \frac{2\theta_+}{3v_+}$ ,  $\lambda_{i\pm} = \lambda_i(v_{\pm}, \theta_{\pm})$ , and  $s_i$  is the  $i$ -shock speed.

For definiteness, we consider the case when the solution to the Riemann problem is a superposition of a 1-rarefaction and a 3-shock wave with a contact discontinuity in between, that is,  $(v_-, u_{1-}, \theta_-) \in R_1\text{-}CD\text{-}S_3(v_+, u_{1+}, \theta_+)$ . Then there exist uniquely two intermediate states  $(v_*, u_{1*}, \theta_*)$  and  $(v^*, u_1^*, \theta^*)$  such that  $(v_-, u_{1-}, \theta_-) \in R_1(v_*, u_{1*}, \theta_*)$ ,  $(v_*, u_{1*}, \theta_*) \in CD(v^*, u_1^*, \theta^*)$ , and  $(v^*, u_1^*, \theta^*) \in S_3(v_+, u_{1+}, \theta_+)$ .

Hence, the wave pattern  $(\tilde{V}, \tilde{U}, \tilde{\mathcal{E}})(t, x)$  can be written as

$$(1.25) \quad \begin{pmatrix} \tilde{V} \\ \tilde{U}_1 \\ \tilde{\mathcal{E}} \end{pmatrix} (t, x) = \begin{pmatrix} v^{r_1} + v^{cd} + v^{s_3} \\ u_1^{r_1} + u_1^{cd} + u_1^{s_3} \\ E^{r_1} + E^{cd} + E^{s_3} \end{pmatrix} (t, x) - \begin{pmatrix} v_* + v^* \\ u_{1*} + u_1^* \\ E_* + E^* \end{pmatrix}, \quad \tilde{U}_i = 0, (i = 2, 3),$$

where  $(v^{r_1}, u_1^{r_1}, \theta^{r_1})(t, x)$  is the 1-rarefaction wave defined in (1.23) with the right state  $(v_+, u_{1+}, \theta_+)$  given by  $(v_*, u_{1*}, \theta_*)$ ,  $(v^{cd}, u_1^{cd}, \theta^{cd})(t, x)$  is the contact discontinuity defined in (1.22) with the states  $(v_-, u_{1-}, \theta_-)$  and  $(v_+, u_{1+}, \theta_+)$  given by  $(v_*, u_{1*}, \theta_*)$  and  $(v^*, u_1^*, \theta^*)$ , respectively, and  $(v^{s_3}, u_1^{s_3}, \theta^{s_3})(t, x)$  is the 3-shock wave defined in (1.24) with the left state  $(v_-, u_{1-}, \theta_-)$  given by  $(v^*, u_1^*, \theta^*)$ .

Consequently, we can define

$$(1.26) \quad \tilde{\Theta}(t, x) = \left( \tilde{\mathcal{E}}(t, x) - \frac{\tilde{U}(t, x)^2}{2} \right).$$

Due to the singularity of the rarefaction wave at  $t = 0$ , in this paper we consider the problem in the time interval  $[h, T]$  for any small fixed  $h > 0$  up to any arbitrarily fixed time  $T > 0$ . To investigate the interaction between the waves and the initial layer is another interesting topic that will not be discussed here. With the above preparation, the main result can be stated as follows.

**THEOREM 1.1.** *Let  $(\tilde{V}, \tilde{U}, \tilde{\Theta})(t, x)$  be a Riemann solution to the Euler equations which is a superposition of a 1-rarefaction wave, a 2-contact discontinuity, and a 3-shock wave, and let  $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)|$  be the wave strength. There exist a small positive constant  $\delta_0$  and a global Maxwellian  $\mathbf{M}_* = \mathbf{M}_{[v_*, u_*, \theta_*]}$  such that if the wave strength satisfies  $\delta \leq \delta_0$ , then in any time interval  $[h, T]$  with  $0 < h < T$ , there exists a positive constant  $\varepsilon_0 = \varepsilon_0(\delta, h, T)$  such that if the Knudsen number  $\varepsilon \leq \varepsilon_0$ , then the Boltzmann equation admits a family of smooth solutions  $f^{\varepsilon, h}(t, x, \xi)$  satisfying*

$$\sup_{(t, x) \in \Sigma_{h, T}} \|f^{\varepsilon, h}(t, x, \xi) - \mathbf{M}_{[\tilde{v}, \tilde{u}, \tilde{\theta}]}(t, x, \xi)\|_{L_\xi^2(\frac{1}{\sqrt{\mathbf{M}_*})}} \leq C_{h, T} \varepsilon^{\frac{1}{5}} |\ln \varepsilon|,$$

where  $\Sigma_{h, T} = \{(t, x) | h \leq t \leq T, |x| \geq h, |x - s_3 t| \geq h\}$ , the norm  $\|\cdot\|_{L_\xi^2(\frac{1}{\sqrt{\mathbf{M}_*})}}$  is  $\|\frac{\cdot}{\sqrt{\mathbf{M}_*}}\|_{L_\xi^2(\mathbf{R}^3)}$ , and the positive constant  $C_{h, T}$  depends on  $h$  and  $T$  but is independent of  $\varepsilon$ .

*Remark 1.* Theorem 1.1 shows that away from the initial time  $t = 0$ , the contact discontinuity at  $x = 0$ , and the shock discontinuity at  $x = s_3 t$ , for small total wave strength  $\delta \leq \delta_0$  and Knudsen number  $\varepsilon \leq \varepsilon_0$ , there exists a family of smooth solutions  $f^{\varepsilon, h}(t, x, \xi)$  of the Boltzmann equation which tends to the Maxwellian  $\mathbf{M}_{[\tilde{v}, \tilde{u}, \tilde{\theta}]}(t, x, \xi)$  with  $(\tilde{V}, \tilde{U}, \tilde{\Theta})(t, x)$  being the Riemann solution to the Euler equations as a superposition of a 1-rarefaction wave, a 2-contact discontinuity, and a 3-shock wave when  $\varepsilon \rightarrow 0$  with a convergence rate  $\varepsilon^{\frac{1}{5}} |\ln \varepsilon|$ . Note that this superposition of waves is the most generic case for the Riemann problem. Similar results hold for any other superpositions of waves by using the same analysis.

*Remark 2.* The proof of the above theorem crucially depends on the introduction of two kinds of hyperbolic waves. Hyperbolic wave I was constructed by Huang, Wang, and Yang [22] for the compressible Navier–Stokes equations to recover the viscous terms to the inviscid approximation of rarefaction wave pattern where the rarefaction wave structure plays an important role in the construction.

Hyperbolic wave II is constructed to remove the error terms due to the viscous contact wave approximation. Note that the construction of hyperbolic wave II can not be done simply around the contact wave approximation as hyperbolic wave I for the rarefaction wave. Otherwise, the wave interaction terms thus induced will lead to insufficient decay in terms of the Knudsen number. Instead, it is constructed around the superposition of the approximate 1-rarefaction wave, hyperbolic wave I, the 2-viscous contact wave, and the 3-shock profile as a whole. Moreover, it also takes care of the nonconservative terms in the previous reduced system so that energy estimates can be taken for antiderivative of the perturbation.

*Remark 3.* Note that the analysis can also be applied to the vanishing viscosity limit of the one-dimensional compressible Navier–Stokes equations. In fact, the vanishing viscosity limit of the one-dimensional compressible Navier–Stokes equations in some sense can be viewed as a special case of a hydrodynamic limit of the Boltzmann equation to the Euler equations by neglecting the microscopic effect.

*Remark 4.* If the total wave strength  $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)| \leq \delta_0$ , then from the wave curves defined in (1.22), (1.23), and (1.24), we know that

$\delta^{R_1}, \delta^{CD}, \delta^{S_3} \leq C\delta_0$ , where  $\delta^{R_1}, \delta^{CD}, \delta^{S_3}$  are the wave strengths of rarefaction wave, contact discontinuity, and shock wave, respectively.

Let us now review some previous works on the hydrodynamic limits to the Boltzmann equation. For the case when the Euler equations have smooth solutions, the vanishing Knudsen number limit of the Boltzmann equation has been studied even in the case with an initial layer; cf. Caffisch [7], Lachowicz [26], Nishida [33], and Ukai and Asona [37]. However, as is well known, solutions of the Euler equations in general develop singularities, such as shock waves and contact discontinuities. Therefore, how to verify the hydrodynamic limit from the Boltzmann equation to the Euler equations with basic wave patterns becomes a natural problem in the process to the general setting. In this direction, with slab symmetry, as mentioned earlier, there were studies on each individual wave pattern. For superposition of different types of waves, to our knowledge there is only one result, given in [21], about the superposition of two rarefaction waves and one contact discontinuity.

On the other hand, for the incompressible equations, there are works such as those by Bardos, Golse, and Levermore, Bardos et al., Bardos and Ukai, Golse and Saint-Raymond, Levermore and Masmoudi, and Sone, which studied direct derivations of the incompressible Navier–Stokes equations in the long time scaling, about which more is known; cf. [2, 4, 5, 16, 25, 35, 36] and the references therein. In particular, Golse and Saint-Raymond showed that the limits of suitably rescaled sequences of the DiPerna–Lions renormalized solutions to the Boltzmann equation are the Leray solutions to the incompressible Navier–Stokes equations. However, even in this aspect, the uniqueness and regularity of the solution are still big issues. Since we will concentrate on the compressible Euler limit in this paper, we will not go into details about the incompressible limits. One can read the monographs about the related topics and mathematical theory of the Boltzmann equation, cf. [9, 10, 13, 14].

Furthermore, the Boltzmann equation provides more information than the classical fluid dynamical systems so that it describes some phenomena which cannot be modeled by using the classical systems, such as Euler and Navier–Stokes equations. This kind of interesting phenomena, such as the thermal creep flow in a rarefied gas, was known since the time of Maxwell. Some mathematical formulations and numerical computations on the basis of kinetic equations were studied since 1960s; cf. the works by Sone [35, 36]. However, the justification of this kind of fluid dynamics is almost open with rigorous mathematical theory.

Finally, we briefly outline the proof of the theorem. First, we define the individual wave profile. Then we introduce the first family of hyperbolic wave by linearizing around the approximate rarefaction profile and by adding the viscosity and heat conductivity terms induced by the profile. Then we define the first approximation of the superposition of this hyperbolic wave together with the three basic wave patterns so that it takes care of the hyperbolicity of the rarefaction wave in the viscous setting.

Based on this, we linearize the fluid system around this profile and consider the propagation of the extra error due to the contact discontinuity approximation and then define a second set of hyperbolic waves. By adding these two sets of hyperbolic waves to the superposition of the three basic wave profiles, we will perform the energy estimate on the Boltzmann equation with suitable initial data through the macro-micro decomposition. Precisely, for the macroscopic component, we will consider the antiderivative of the perturbation after applying a hyperbolic scaling. By using the dissipation in the fluid-type system and the linearized Boltzmann operator on the microscopic component, we can close the energy estimate through a suitable chosen a priori assumption. Then the statements in the theorem follow.

The rest of the paper will be arranged as follows. In section 2, we will construct the approximate solutions to the Boltzmann equation corresponding to the basic wave patterns to the Euler system. Then we obtain the detailed information on the difference between the Riemann solution to Euler system and the approximate solution to the Boltzmann equation by the construction. In section 3, we will construct a family of solutions to the Boltzmann equation around the approximate solution by using the energy method to close the a priori estimate. Since the proofs of Propositions 3.6 and 3.7 about the lower and higher order energy estimates, respectively, are very technical and long, we put them in the appendices.

**Notations.** Throughout this paper, the positive generic constants which are independent of  $\varepsilon, T, h$  are denoted by  $c, C, C_i (i = 1, 2, 3, \dots)$ , while  $C_{h,T}$  represents a generic positive constant depending on  $h$  and  $T$  but independent of  $\varepsilon$ . We will use  $\|\cdot\|$  to denote the standard  $L_2(\mathbf{R}; dy)$  norm and  $\|\cdot\|_{H^i} (i = 1, 2, 3, \dots)$  to denote the Sobolev  $H^i(\mathbf{R}; dy)$  norm. Sometimes, we also use  $O(1)$  to denote a uniform bounded constant which is independent of  $\varepsilon, T, h$ .

**2. Approximate wave patterns.** In this section, we will construct the approximate wave profile that consists of three basic wave patterns and two hyperbolic waves. For this, we will first recall the construction of the approximate rarefaction wave for the Boltzmann equation. Then we will introduce hyperbolic wave I to correct the error terms coming from the hyperbolic approximation. Then we will construct the viscous contact wave to Boltzmann equation and study the nonconservative error terms. The viscous shock profile to the Boltzmann equation will then be recalled. With the above wave patterns, we will introduce hyperbolic wave II to take care of the error terms due to the viscous contact wave by avoiding the interaction between the viscous contact wave with the wave patterns defined earlier.

**2.1. Rarefaction wave.** For the rarefaction wave, since there is no exact rarefaction wave profile for either the Navier–Stokes equations or the Boltzmann equation, the following approximate rarefaction wave profile satisfying the Euler equations was introduced in [32, 39]. For completeness of the presentation, we include its definition and the properties obtained in the above two papers as follows.

If  $(v_-, u_{1-}, \theta_-) \in R_1(v_+, u_{1+}, \theta_+)$ , then there exists a 1-rarefaction wave  $(v^{r_1}, u_1^{r_1}, E^{r_1})(x/t)$  which is a global solution to the following Riemann problem:

$$(2.1) \quad \begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ E_t + (pu_1)_x = 0, \\ (v, u_1, \theta)(t = 0, x) = \begin{cases} (v_-, u_{1-}, \theta_-), & x < 0, \\ (v_+, u_{1+}, \theta_+), & x > 0. \end{cases} \end{cases}$$

Consider the following inviscid Burgers equation with Riemann data

$$(2.2) \quad \begin{cases} w_t + ww_x = 0, \\ w(t = 0, x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases}$$

If  $w_- < w_+$ , then the above Riemann problem admits a rarefaction wave solution

$$(2.3) \quad w^r(t, x) = w^r\left(\frac{x}{t}\right) = \begin{cases} w_-, & \frac{x}{t} \leq w_-, \\ \frac{x}{t}, & w_- \leq \frac{x}{t} \leq w_+, \\ w_+, & \frac{x}{t} \geq w_+. \end{cases}$$

As in [39], the approximate rarefaction wave  $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$  to the problem (2.1) can be constructed by the solution of the Burgers equation

$$(2.4) \quad \begin{cases} w_t + ww_x = 0, \\ w(0, x) = w_\sigma(x) = w\left(\frac{x}{\sigma}\right) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\sigma}, \end{cases}$$

where  $\sigma > 0$  is a small parameter to be determined later to be  $\varepsilon^{\frac{1}{2}}$ . Note that the solution  $w_\sigma^r(t, x)$  of the problem (2.4) is given by

$$w_\sigma^r(t, x) = w_\sigma(x_0(t, x)), \quad x = x_0(t, x) + w_\sigma(x_0(t, x))t.$$

The smooth approximate rarefaction wave profile denoted by  $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$  can be defined by

$$(2.5) \quad \begin{cases} S^{R_1}(t, x) = s(V^{R_1}(t, x), \Theta^{R_1}(t, x)) = s_+, \\ w_\pm = \lambda_{1\pm} := \lambda_1(v_\pm, \theta_\pm), \\ w_\sigma^r(t, x) = \lambda_1(V^{R_1}(t, x), s_+), \\ U_1^{R_1}(t, x) = u_{1+} - \int_{v_+}^{V^{R_1}(t, x)} \lambda_1(v, s_+)dv, \\ U_i^{R_1}(t, x) \equiv 0, \quad i = 2, 3. \end{cases}$$

Note that  $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$  defined above satisfies

$$(2.6) \quad \begin{cases} V_t^{R_1} - U_{1x}^{R_1} = 0, \\ U_{1t}^{R_1} + P_x^{R_1} = 0, \\ U_{it}^{R_1} = 0, \quad i = 2, 3, \\ \mathcal{E}_t^{R_1} + (P^{R_1} U_1^{R_1})_x = 0, \end{cases}$$

where  $P^{R_1} = p(V^{R_1}, \Theta^{R_1}) = \frac{2\Theta^{R_1}}{3V^{R_1}}$  and  $\mathcal{E}^{R_1} = \Theta^{R_1} + \frac{|U^{R_1}|^2}{2}$ . The properties of the rarefaction wave profile can be summarized as follows.

LEMMA 2.1 (see [39]). *The approximate rarefaction waves  $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$  constructed in (2.5) have the following properties:*

- (1)  $U_{1x}^{R_1}(t, x) > 0$  for  $x \in \mathbf{R}, t > 0$ .
- (2) For any  $1 \leq p \leq +\infty$ , the following estimates holds:

$$\begin{aligned} \|(V^{R_1}, U_1^{R_1}, \Theta^{R_1})_x\|_{L^p(dx)} &\leq C \min \{ \delta^{R_1} \sigma^{-1+1/p}, (\delta^{R_1})^{1/p} t^{-1+1/p} \}, \\ \|(V^{R_1}, U_1^{R_1}, \Theta^{R_1})_{xx}\|_{L^p(dx)} &\leq C \min \{ \delta^{R_1} \sigma^{-2+1/p}, \sigma^{-1+1/p} t^{-1} \}, \end{aligned}$$

where the positive constant  $C$  depends only on  $p$  and the wave strength.

- (3) If  $x \geq \lambda_{1+}^{R_1} t$ , then

$$\begin{aligned} |(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x) - (v_+, u_+, \theta_+)| &\leq C e^{-\frac{2|x-\lambda_{1+}t|}{\sigma}}, \\ |\partial_x^k (V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)| &\leq \frac{C}{\sigma^k} e^{-\frac{2|x-\lambda_{1+}t|}{\sigma}}, \quad k = 1, 2. \end{aligned}$$

- (4) There exist positive constants  $C$  and  $\sigma_0$  such that for  $\sigma \in (0, \sigma_0)$  and  $t > 0$ ,

$$\sup_{x \in \mathbf{R}} |(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})(t, x) - (v^{r_1}, u^{r_1}, E^{r_1})\left(\frac{x}{t}\right)| \leq \frac{C}{t} [\sigma \ln(1+t) + \sigma |\ln \sigma|].$$

**2.2. Hyperbolic wave I.** Since the whole wave profile consists of a shock wave whose rate of change in the shock region is of the order of  $\varepsilon^{-1}$ , we have to consider the antiderivative of the perturbation in order to cope with the correct sign as in the stability analysis. From (2.6), we know that the approximate rarefaction wave  $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$  satisfies the compressible Euler equations exactly without viscous terms. Thus if we carry out the energy estimates to the antiderivative variables, the error terms due to the viscous terms from the approximate rarefaction wave are not good enough to get the desired estimates. In order to overcome this difficulty, we introduce hyperbolic wave I to recover these viscous terms.

This hyperbolic wave denoted by  $(d_1, d_2, d_3)(t, x)$  can be defined as follows. Consider a linear system

$$(2.7) \quad \begin{cases} d_{1t} - d_{2x} = 0, \\ d_{2t} + (p_v^{R_1} d_1 + p_{u_1}^{R_1} d_2 + p_E^{R_1} d_3)_x = \frac{4}{3} \varepsilon \left( \frac{\mu(\Theta^{R_1}) U_{1x}^{R_1}}{V^{R_1}} \right)_x, \\ d_{3t} + [(pu_1)_v^{R_1} d_1 + (pu_1)_{u_1}^{R_1} d_2 + (pu_1)_E^{R_1} d_3]_x \\ = \varepsilon \left( \frac{\kappa(\Theta^{R_1}) \Theta_x^{R_1}}{V^{R_1}} \right)_x + \frac{4}{3} \varepsilon \left( \frac{\mu(\Theta^{R_1}) U_1^{R_1} U_{1x}^{R_1}}{V^{R_1}} \right)_x, \end{cases}$$

where  $p = \frac{R\theta}{v} = p(v, u, E) = \frac{2E-u^2}{3v}$  and  $p_v^{R_1} = p_v(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})$ , etc. Note that the left-hand side of the above system is the linearization of the Euler equation around the rarefaction wave approximation. We want to solve this linear hyperbolic system (2.7) on the time interval  $[h, T]$ . First, we diagonalize the above system by rewriting it as

$$(2.8) \quad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_t + \left[ A^{R_1} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \right]_x = \begin{pmatrix} 0 \\ H_1^{R_1} \\ H_2^{R_1} \end{pmatrix},$$

where  $H_1^{R_1} = \varepsilon \left( \frac{\mu(\Theta^{R_1}) U_{1x}^{R_1}}{V^{R_1}} \right)_x$ ,  $H_2^{R_1} = \varepsilon \left( \frac{\kappa(\Theta^{R_1}) \Theta_x^{R_1}}{V^{R_1}} \right)_x + \varepsilon \left( \frac{\mu(\Theta^{R_1}) U_1^{R_1} U_{1x}^{R_1}}{V^{R_1}} \right)_x$ . Here, the matrix

$$A^{R_1} = \begin{pmatrix} 0 & -1 & 0 \\ p_v^{R_1} & p_{u_1}^{R_1} & p_E^{R_1} \\ (pu_1)_v^{R_1} & (pu_1)_{u_1}^{R_1} & (pu_1)_E^{R_1} \end{pmatrix}$$

has three distinct eigenvalues  $\lambda_1^{R_1} := \lambda_1(V^{R_1}, s_{\pm}) < 0 \equiv \lambda_2^{R_1} < \lambda_3(V^{R_1}, s_{\pm}) := \lambda_3^{R_1}$ , and the corresponding left and right eigenvectors denoted  $l_j^{R_1}, r_j^{R_1}$  ( $j = 1, 2, 3$ ), respectively, satisfy

$$L^{R_1} A^{R_1} R^{R_1} = \text{diag}(\lambda_1^{R_1}, 0, \lambda_3^{R_1}) \equiv \Lambda^{R_1}, \quad L^{R_1} R^{R_1} = \text{Id.},$$

Here  $L^{R_1} = (l_1^{R_1}, l_2^{R_1}, l_3^{R_1})^t$ ,  $R^{R_1} = (r_1^{R_1}, r_2^{R_1}, r_3^{R_1})$  with  $l_i^{R_1} = l_i(V^{R_1}, U_1^{R_1}, s_+)$  and  $r_i^{R_1} = r_i(V^{R_1}, U_1^{R_1}, s_+)$  ( $i = 1, 2, 3$ ) and  $\text{Id.}$  is the  $3 \times 3$  identity matrix. Now we set

$$(2.9) \quad (D_1, D_2, D_3)^t = L^{R_1} (d_1, d_2, d_3)^t.$$

Then

$$(2.10) \quad (d_1, d_2, d_3)^t = R^{R_1} (D_1, D_2, D_3)^t,$$

and  $(D_1, D_2, D_3)$  satisfies the system

$$\begin{aligned}
 & \left( \begin{array}{c} D_1 \\ D_2 \\ D_3 \end{array} \right)_t + \left[ \Lambda^{R_1} \left( \begin{array}{c} D_1 \\ D_2 \\ D_3 \end{array} \right) \right]_x \\
 (2.11) \quad & = L^{R_1} \left( \begin{array}{c} 0 \\ H_1^{R_1} \\ H_2^{R_1} \end{array} \right) + L_t^{R_1} R^{R_1} \left( \begin{array}{c} D_1 \\ D_2 \\ D_3 \end{array} \right) + L_x^{R_1} R^{R_1} \Lambda^{R_1} \left( \begin{array}{c} D_1 \\ D_2 \\ D_3 \end{array} \right).
 \end{aligned}$$

Because the 1-Riemann invariant is constant along the approximate rarefaction wave curve, we have

$$L_t^{R_1} = -\lambda_1^{R_1} L_x^{R_1}.$$

Substituting the above equation into (2.11), we obtain the diagonalized system

$$(2.12) \quad \begin{cases} D_{1t} + (\lambda_1^{R_1} D_1)_x = b_{12}^{R_1} H_1^{R_1} + b_{13}^{R_1} H_2^{R_1} + a_{12}^{R_1} V_x^{R_1} D_2 + a_{13}^{R_1} V_x^{R_1} D_3, \\ D_{2t} = b_{22}^{R_1} H_1^{R_1} + b_{23}^{R_1} H_2^{R_1} + a_{22}^{R_1} V_x^{R_1} D_2 + a_{23}^{R_1} V_x^{R_1} D_3, \\ D_{3t} + (\lambda_3^{R_1} D_3)_x = b_{32}^{R_1} H_1^{R_1} + b_{33}^{R_1} H_2^{R_1} + a_{32}^{R_1} V_x^{R_1} D_2 + a_{33}^{R_1} V_x^{R_1} D_3, \end{cases}$$

where  $a_{ij}^{R_1}, b_{ij}^{R_1}$  are some given functions of  $V^{R_1}, U_1^{R_1}$ , and  $S^{R_1} = s_{\pm}$ . Note that in the diagonalized system (2.12), the equations of  $D_2, D_3$  are decoupled from  $D_1$  because of the property of the rarefaction wave.

Now we impose the following boundary condition to the above linear hyperbolic system (2.12) in the domain  $(t, x) \in [h, T] \times \mathbf{R}$ :

$$(2.13) \quad D_1(t = h, x) = 0, \quad D_2(t = T, x) = D_3(t = T, x) = 0.$$

With this boundary condition, we can solve the linear diagonalized hyperbolic system (2.12) under the conditions (2.13). Moreover, we have the following estimates on the solution.

LEMMA 2.2. *There exists a positive constant  $C_{h,T}$  independent of  $\varepsilon$  such that*

(1)

$$\left\| \frac{\partial^k}{\partial x^k} d_i(t, \cdot) \right\|_{L^2(dx)}^2 \leq C_{h,T} \frac{\varepsilon^2}{\sigma^{2k+1}}, \quad i = 1, 2, 3, \quad k = 0, 1, 2, 3.$$

(2) *If  $x > \lambda_{1+}t$ , then we have*

$$|d_i(x, t)| \leq C_{h,T} \frac{1}{\sigma} e^{-\frac{|x-\lambda_{1+}t|}{\sigma}}, \quad |d_{ix}(x, t)| \leq C_{h,T} \frac{1}{\sigma^2} e^{-\frac{|x-\lambda_{1+}t|}{\sigma}}, \quad i = 1, 2, 3.$$

The proof of Lemma 2.2 can be done similarly as in [22] for the compressible Navier–Stokes equations.

**2.3. Viscous contact wave.** In this subsection, we construct the contact wave  $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$  for the Boltzmann equation motivated by [24]. Consider the Euler system (1.16) with Riemann initial data

$$(2.14) \quad (v, u, \theta)(t = 0, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases}$$

where  $u_{\pm} = (u_{1\pm}, 0, 0)$  and  $v_{\pm} > 0, \theta_{\pm} > 0, u_{1\pm}$  are given constants. It is known (cf. [34]) that the Riemann problem (1.16), (2.14) admits a contact discontinuity

$$(2.15) \quad (v^{cd}, u^{cd}, \theta^{cd})(t, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases}$$



provided that  $(v_-, u_{1-}, \theta_-) \in CD(v_+, u_{1+}, \theta_+)$ , that is,

$$(2.16) \quad u_{1+} = u_{1-}, \quad p_- := \frac{2\theta_-}{3v_-} = p_+ := \frac{2\theta_+}{3v_+}.$$

Then for the Navier–Stokes equations, by the energy equation  $(1.21)_4$  and the mass equation  $(1.21)_1$  with  $v \approx \frac{2\theta}{3p_+}$  (cf. [21] and [23]), we can obtain the following nonlinear diffusion equation:

$$(2.17) \quad \theta_t = \varepsilon(a(\theta)\theta_x)_x, \quad a(\theta) = \frac{9p_+\kappa(\theta)}{10\theta}.$$

From [1] and [12], we know that the nonlinear diffusion equation (2.17) admits a unique self-similar solution  $\hat{\Theta}(\eta)$ ,  $\eta = \frac{x}{\sqrt{\varepsilon(1+t)}}$  satisfying the boundary conditions  $\hat{\Theta}(\pm\infty, t) = \theta_{\pm}$ . Let  $\delta^{CD} = |\theta_+ - \theta_-|$ , then  $\hat{\Theta}(t, x)$  has the property that

$$(2.18) \quad \hat{\Theta}_x(t, x) = \frac{O(1)\delta^{CD}}{\sqrt{\varepsilon(1+t)}} e^{-\frac{cx^2}{\varepsilon(1+t)}}$$

with some positive constant  $c$  depending only on  $\theta_{\pm}$ .

Correspondingly, we can define the Navier–Stokes profile by

$$(2.19) \quad \begin{aligned} \hat{V} &= \frac{2}{3p_+}\hat{\Theta}, \\ \hat{U}_1 &= u_{1+} + \frac{2\varepsilon a(\hat{\Theta})}{3p_+}\hat{\Theta}_x, \quad \hat{U}_i = 0, i = 2, 3. \end{aligned}$$

For the Boltzmann equation, if we still use the above Navier–Stokes profile  $(\hat{V}, \hat{U}, \hat{\Theta})$ , we cannot get any decay with respect to the Knudsen number  $\varepsilon$  due to the nonfluid component. Hence, we construct a Boltzmann contact wave as follows. Set

$$(2.20) \quad \mathbf{G}^{CD}(t, x, \xi) = \frac{3\varepsilon}{2v\theta} \mathbf{L}_M^{-1} \left\{ \mathbf{P}_1 \left[ \xi_1 \left( \frac{|\xi - u|^2}{2\theta} \Theta_x^{CD} + \xi \cdot U_x^{CD} \right) \mathbf{M} \right] \right\}$$

and

$$(2.21) \quad \Pi_{11}^{CD} = \mathbf{L}_M^{-1} \left[ \varepsilon \left( -\frac{u_1}{v} \mathbf{G}_x^{CD} + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x^{CD}) \right) - Q(\mathbf{G}^{CD}, \mathbf{G}^{CD}) \right],$$

where  $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$  is the viscous contact wave for the Boltzmann equation to be constructed later.

Note that for the Boltzmann equation, the leading terms in the energy equation  $(1.21)_4$  can be written as

$$(2.22) \quad \theta_t + p_+ u_{1x} = \varepsilon \left( \frac{\kappa(\theta)\theta_x}{v} \right)_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_{11x}^{CD} d\xi + u_{1+} \int \frac{1}{2} \xi_1^2 \Pi_{11x}^{CD} d\xi.$$

By the definition of  $\Pi_{11}^{CD}$  in (2.21), we have

$$(2.23) \quad - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_{11}^{CD} d\xi + u_{1+} \int \frac{1}{2} \xi_1^2 \Pi_{11}^{CD} d\xi = \Delta_{11} + \Delta_{12},$$

where

$$(2.24) \quad \Delta_{11} = \varepsilon^2 \left[ g_{11} \theta_x \Theta_x^{CD} + g_{12} v_x \Theta_x^{CD} + g_{13} (\Theta_x^{CD})^2 + g_{14} \Theta_{xx}^{CD} \right]$$

with  $g_{1i} = g_{1i}(v, u, \theta)$ , ( $i = 1, 2, 3, 4$ ) being smooth functions of  $(v, u, \theta)$ , and

$$(2.25) \quad \Delta_{12} = O(1)\varepsilon^2 \left[ (|v_x| + |u_x| + |\theta_x| + |\Theta_x^{CD}| + |U_x^{CD}|) |U_x^{CD}| + |u_x| |\Theta_x^{CD}| + |U_{xx}^{CD}| \right].$$

Thus, by choosing the leading term and dropping the higher order term  $\Delta_{12x}$  in (2.22), we have

$$(2.26) \quad \theta_t = \varepsilon(a(\theta)\theta_x)_x + \frac{3}{5}\Delta_{11x},$$

where  $a(\theta)$  is defined in (2.17) and  $\Delta_{11}$  is defined in (2.24). To represent the microscopic effect on the wave profile, we want to define  $\Theta^{CD}$  to be close to  $\hat{\Theta}(\frac{x}{\sqrt{\varepsilon(1+t)}}) + \hat{\Theta}^{nf}(t, x)$  with  $\hat{\Theta}$  being determined by (2.17), (2.18) and where  $\hat{\Theta}^{nf}$  represents the part of the nonlinear diffusion wave coming from the nonfluid component not appearing in the Navier–Stokes level. Moreover, the term  $\hat{\Theta}^{nf}$  decays faster than  $\hat{\Theta}$  so that it can be viewed as the perturbation around the Navier–Stokes profile  $\hat{\Theta}$ . To construct  $\hat{\Theta}^{nf}$ , we linearize (2.26) around the Navier–Stokes profile  $\hat{\Theta}$  and drop all the higher order terms. This leads to a linear diffusion equation for  $\hat{\Theta}^{nf}$ ,

$$(2.27) \quad \hat{\Theta}_t^{nf} = \varepsilon(a(\hat{\Theta})\hat{\Theta}_x^{nf})_x + \varepsilon(a'(\hat{\Theta})\hat{\Theta}_x\hat{\Theta}^{nf})_x + \frac{3}{5}\tilde{\Delta}_{11x},$$

where  $\tilde{\Delta}_{11} = \varepsilon^2(\tilde{g}_{11} + \frac{2}{3p_+}\tilde{g}_{12} + \tilde{g}_{13})(\hat{\Theta}_x)^2 + \varepsilon^2\tilde{g}_{14}\hat{\Theta}_{xx}$  with  $\tilde{g}_{1i} = \tilde{g}_{1i}(\hat{V}, \hat{U}, \hat{\Theta})$  ( $i = 1, 2, 3, 4$ ). Integrating (2.27) with respect to  $x$  yields that

$$(2.28) \quad \Xi_{1t} = \varepsilon a(\hat{\Theta})\Xi_{1xx} + \varepsilon a'(\hat{\Theta})\hat{\Theta}_x\Xi_{1x} + \frac{3}{5}\tilde{\Delta}_{11},$$

where

$$(2.29) \quad \Xi_1(t, x) = \int_{-\infty}^x \hat{\Theta}^{nf}(t, x) dx.$$

Note that  $\tilde{\Delta}_{11}$  takes the form of  $\frac{\varepsilon}{1+t}A^1(\frac{x}{\sqrt{\varepsilon(1+t)}})$  and satisfies that

$$|\tilde{\Delta}_{11}| = O(\delta^{CD})\varepsilon(1+t)^{-1}e^{-\frac{x^2}{4\alpha(\theta_{\pm})\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty.$$

We can check that there exists a self-similar solution  $\Xi_1(\frac{x}{\sqrt{\varepsilon(1+t)}})$  for (2.27) with the boundary conditions  $\Xi_1(-\infty) = 0, \Xi_1(+\infty) = \Xi_{1+}$ . Here  $\Xi_{1+}$  can be any given constant satisfying  $|\Xi_{1+}| < \delta^{CD}$ . It is worth pointing out that even though the function  $\Xi_1(t, x)$  depends on the constant  $\Xi_{1+}$ ,  $\hat{\Theta}^{nf}(t, x) = \Xi_{1x}(t, x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ . That is, the choice of the constant  $\Xi_{1+}$  has no influence on the ansatz as long as  $|\Xi_{1+}| < \delta^{CD}$ . From now on, we fix  $\Xi_{1+}$  so that the function  $\Xi_1(t, x)$  is uniquely determined and its derivative  $\Xi_{1x} = \hat{\Theta}^{nf}$  has the property

$$(2.30) \quad |\hat{\Theta}^{nf}| = |\Xi_{1x}| = O(\delta^{CD})\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}e^{-\frac{x^2}{4\alpha(\theta_{\pm})\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty.$$

Then we apply a similar procedure to construct the second and third components of the velocity of the contact wave denoted by  $U_i^{CD}$  ( $i = 2, 3$ ) as follows. The leading part of the equation for  $u_i$  in (1.21) <sub>$i$</sub>  ( $i = 2, 3$ ) is

$$(2.31) \quad u_{it} = \varepsilon \left( \frac{3p_+\mu(\theta)}{2\theta} u_{ix} \right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi.$$

First, we have

$$(2.32) \quad - \int \xi_1 \xi_i \Pi_{11}^{CD} d\xi = \Delta_{i1} + \Delta_{i2},$$

where

$$(2.33) \quad \Delta_{i1} = \varepsilon^2 \left[ g_{i1} \theta_x \Theta_x^{CD} + g_{i2} v_x \Theta_x^{CD} + g_{i3} (\Theta_x^{CD})^2 + g_{i4} \Theta_{xx}^{CD} \right]$$

with  $g_{ij}$ , ( $i = 2, 3$ ,  $j = 1, 2, 3, 4$ ) being the smooth functions of  $(v, u, \theta)$  and

$$(2.34) \quad \Delta_{i2} = O(1)\varepsilon^2 \left[ (|v_x| + |u_x| + |\theta_x| + |\Theta_x^{CD}| + |U_x^{CD}|) |U_x^{CD}| + |u_x| |\Theta_x^{CD}| + |U_{xx}^{CD}| \right].$$

Thus we expect the viscous contact wave  $U_i^{CD}$  ( $i = 2, 3$ ) to satisfy the linear equation

$$(2.35) \quad U_{it}^{CD} = \varepsilon \left( \frac{3p_+ \mu(\hat{\Theta})}{2\hat{\Theta}} U_{ix}^{CD} \right)_x + \tilde{\Delta}_{i1x}, \quad i = 2, 3,$$

where  $\tilde{\Delta}_{i1} = \varepsilon^2 (\tilde{g}_{i1} + \frac{2}{3p_+} \tilde{g}_{i2} + \tilde{g}_{i3}) (\hat{\Theta}_x)^2 + \varepsilon^2 \tilde{g}_{i4} \hat{\Theta}_{xx}$  with  $\tilde{g}_{ij} = \tilde{g}_{ij}(\hat{V}, \hat{U}, \hat{\Theta})$  ( $i = 2, 3$ ,  $j = 1, 2, 3, 4$ ). Integrating (2.35) with respect to  $x$  yields that

$$(2.36) \quad \Xi_{it} = \varepsilon \frac{3p_+ \mu(\hat{\Theta})}{2\hat{\Theta}} \Xi_{ixx} + \tilde{\Delta}_{i1},$$

where

$$(2.37) \quad \Xi_i(t, x) = \int_{-\infty}^x U_i^{CD}(t, x) dx.$$

Note that  $\tilde{\Delta}_{i1}$  takes the form  $\frac{\varepsilon}{1+t} A^i \left( \frac{x}{\sqrt{\varepsilon(1+t)}} \right)$ ,  $i = 2, 3$ , and satisfies that

$$|\tilde{\Delta}_{i1}| = O(\delta^{CD}) \varepsilon (1+t)^{-1} e^{-\frac{x^2}{4a(\theta_{\pm})\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty.$$

We can check that there exists a self-similar solution  $\Xi_i \left( \frac{x}{\sqrt{\varepsilon(1+t)}} \right)$  for (2.35) with the boundary conditions  $\Xi_i(-\infty) = 0$ ,  $\Xi_i(+\infty) = \Xi_{i+}$ , ( $i = 2, 3$ ). Again, here  $\Xi_{i+}$  can be any given constant satisfying  $|\Xi_{i+}| < \delta^{CD}$ . As we explained before, the choice of the constant  $\Xi_{i+}$  has no influence on the ansatz as long as  $|\Xi_{i+}| < \delta^{CD}$ . We fix  $\Xi_{i+}$  so that the function  $\Xi_i(t, x)$  is uniquely determined and the derivative  $\Xi_{ix} = U_i^{CD}$  has the property

$$(2.38) \quad |U_i^{CD}| = |\Xi_{ix}| = O(\delta^{CD}) \varepsilon^{\frac{1}{2}} (1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{4b(\theta_{\pm})\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty$$

with  $b(\theta_{\pm}) = \max\{a(\theta_{\pm}), \frac{3p_+ \mu(\theta_{\pm})}{2\theta_{\pm}}\}$ .

In summary, the viscous contact wave  $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$  can be defined by

$$(2.39) \quad \begin{aligned} V^{CD} &= \frac{2}{3p_+} (\hat{\Theta} + \hat{\Theta}^{nf}), \\ U_1^{CD} &= u_{1+} + \frac{2}{3p_+} \left[ \varepsilon a(\hat{\Theta}) \hat{\Theta}_x + \varepsilon (a(\hat{\Theta}) \hat{\Theta}^{nf})_x + \frac{3}{5} \tilde{\Delta}_{11} \right], \\ U_i^{CD} &= \Xi_{ix} \quad (i = 2, 3), \\ \Theta^{CD} &= \hat{\Theta} + \hat{\Theta}^{nf} + H, \end{aligned}$$

where

$$(2.40) \quad H = -\frac{\varepsilon V^{CD}}{p_+} \left[ a(\hat{\Theta})\hat{\Theta}_t + (a(\hat{\Theta})\hat{\Theta}^{nf})_t \right] + \frac{4\varepsilon^2\mu(\hat{\Theta})}{3p_+} \left[ (a(\hat{\Theta})\hat{\Theta}_x)_x + (a(\hat{\Theta})\hat{\Theta}^{nf})_{xx} \right] - \frac{3}{2}V^{CD} \int \xi_1^2 \tilde{\Pi}_{11} d\xi$$

is chosen such that the momentum equation that the viscous contact wave satisfies has an error term with sufficient decay in  $\varepsilon$ . Without  $H$ , the error term in the momentum equation decay like  $\varepsilon^{\frac{1}{2}}$ . In order to get  $\varepsilon$  order decay in the error term, we should introduce the higher order approximate term  $H$  in the definition of  $\Theta^{CD}$  in the contact wave. Here,  $\int \xi_1^2 \tilde{\Pi}_{11} d\xi$  in (2.40) is the corresponding function defined in (2.21) by replacing both the variables  $(v, u, \theta)$  and  $(V^{CD}, U^{CD}, \Theta^{CD})$  by  $(\hat{V}, \hat{U}, \hat{\Theta})$ , and it satisfies

$$(2.41) \quad \int \xi_1^2 \tilde{\Pi}_{11} d\xi = O(1)\varepsilon^2 |(\hat{\Theta}_x^2, \hat{\Theta}_{xx})|.$$

Hence, by (2.18), (2.30), and (2.41), we have

$$(2.42) \quad H = O(\delta^{CD})\varepsilon(1+t)^{-2}e^{-\frac{\varepsilon x^2}{\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty.$$

Now the contact wave  $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$  defined in (2.39) satisfies the system

$$(2.43) \quad \begin{cases} V_t^{CD} - U_{1x}^{CD} = 0, \\ U_{1t}^{CD} + P_x^{CD} = \frac{4\varepsilon}{3} \left( \frac{\mu(\Theta^{CD})}{V^{CD}} U_{1x}^{CD} \right)_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi + Q_1^{CD}, \\ U_{it}^{CD} = \varepsilon \left( \frac{\mu(\Theta^{CD})}{V^{CD}} U_{ix}^{CD} \right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi + Q_i^{CD}, i = 2, 3, \\ \mathcal{E}_t^{CD} + (P^{CD} U_1^{CD})_x = \varepsilon \left( \frac{\kappa(\Theta^{CD})}{V^{CD}} \Theta_x^{CD} \right)_x + \frac{4\varepsilon}{3} \left( \frac{\mu(\Theta^{CD}) U_1^{CD} U_{1x}^{CD}}{V^{CD}} \right)_x \\ + \sum_{i=2}^3 \varepsilon \left( \frac{\mu(\Theta^{CD}) U_i^{CD} U_{ix}^{CD}}{V^{CD}} \right)_x - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi + Q_4^{CD}, \end{cases}$$

where  $P^{CD} = \frac{2\Theta^{CD}}{3V^{CD}}$ ,  $\mathcal{E}^{CD} = \Theta^{CD} + \frac{|U^{CD}|^2}{2}$ ,  $-\int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi$  ( $i = 1, 2, 3$ ), and  $-\int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi$  are the corresponding functions defined in (2.21) by replacing the variables  $(v, u, \theta)$  by  $(V^{CD}, U^{CD}, \Theta^{CD})$ , respectively. Moreover,

$$(2.44) \quad \begin{aligned} Q_1^{CD} &= \frac{2\varepsilon}{5p_+} \tilde{\Delta}_{11t} - \frac{8\varepsilon^2}{15p_+} \left( \frac{\mu(\hat{\Theta})\tilde{\Delta}_{11x}}{V^{CD}} \right)_x \\ &\quad - \frac{4\varepsilon}{3} \left( \frac{\mu(\Theta^{CD}) - \mu(\hat{\Theta})}{V^{CD}} U_{1x}^{CD} \right)_x + \int \xi_1^2 (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi \\ &= O(1)\delta^{CD}\varepsilon(1+t)^{-2}e^{-\frac{\varepsilon x^2}{\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty, \end{aligned}$$

$$(2.45) \quad \begin{aligned} Q_i^{CD} &= -\varepsilon \left( \frac{\mu(\Theta^{CD}) - \mu(\hat{\Theta})}{V^{CD}} U_{ix}^{CD} \right)_x + \int \xi_1 \xi_i (\Pi_{i1}^{CD} - \tilde{\Pi}_{i1})_x d\xi + \tilde{\Delta}_{i2x} \\ &= O(1)\delta^{CD}\varepsilon(1+t)^{-2}e^{-\frac{\varepsilon x^2}{\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty, \quad i = 2, 3, \end{aligned}$$

and

$$\begin{aligned}
 (2.46) \quad Q_4^{CD} &= -\frac{5\varepsilon}{3} \left[ (a(\Theta^{CD}) - a(\hat{\Theta}) - a'(\hat{\Theta})(\hat{\Theta}^{CD} - \hat{\Theta}))(\hat{\Theta}_x + \hat{\Theta}_x^{nf}) + a(\Theta^{CD})H_x \right. \\
 &\quad \left. + a'(\hat{\Theta})(\hat{\Theta}^{CD} - \hat{\Theta})\hat{\Theta}_x^{nf} \right]_x + \frac{3p+\varepsilon}{2} \left( \frac{\kappa(\Theta^{CD})\Theta_x^{CD}H}{\Theta^{CD}(\Theta^{CD} - H)} \right)_x + H_t \\
 &\quad + \frac{2U_{1x}^{CD}H}{3V^{CD}} - \frac{4\varepsilon\mu(\Theta^{CD})}{3V^{CD}}(U_{1x}^{CD})^2 - \sum_{i=2}^3 \frac{\varepsilon\mu(\Theta^{CD})}{V^{CD}}(U_{ix}^{CD})^2 \\
 &\quad + \int \xi_1 \frac{|\xi|^2}{2} (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi + (U_1^{CD} - u_{1+}) \int \xi_1^2 (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi \\
 &\quad + \sum_{i=2}^3 U_i^{CD} \int \xi_1 \xi_i (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi + \sum_{i=1}^3 U_i^{CD} Q_i^{CD} + \tilde{\Delta}_{12x} \\
 &= O(1)\delta^{CD}\varepsilon(1+t)^{-2}e^{-\frac{cx^2}{\varepsilon(1+t)}} \quad \text{as } x \rightarrow \pm\infty
 \end{aligned}$$

with some positive constant  $c > 0$  depending only on  $\theta_{\pm}$  and  $\tilde{\Delta}_{i2x}$  ( $i = 1, 2, 3$ ) being the corresponding functions defined in (2.25) and (2.34) by replacing both  $(v, u, \theta)$  and  $(V^{CD}, U^{CD}, \Theta^{CD})$  by  $(\hat{V}, \hat{U}, \hat{\Theta})$ .

Note that from (2.18), we have

$$(2.47) \quad |(V^{CD}, U^{CD}, \Theta^{CD})(t, x) - (v^{cd}, u^{cd}, \theta^{cd})(t, x)| = O(1)\delta^{CD}e^{-\frac{cx^2}{2\varepsilon(1+t)}}.$$

**2.4. Shock profile.** In this subsection, we will first recall the shock profile  $F^{S_3}(x - \bar{s}_3 t, \xi)$  of the Boltzmann equation (1.1) in Eulerian coordinates with its existence and properties given in the papers by Caffisch and Nicolaenko [8] and Liu and Yu [30], [31]. Then we will state the corresponding properties in the Lagrangian coordinates used in this paper.

First,  $F^{S_3}(x - \bar{s}_3 t, \xi)$  satisfies

$$(2.48) \quad \begin{cases} -\bar{s}_3(F^{S_3})' + \xi_1(F^{S_3})' = \frac{1}{\varepsilon}Q(F^{S_3}, F^{S_3}), \\ F^{S_3}(\pm\infty, \xi) = \mathbf{M}_{\pm}(\xi) := \mathbf{M}_{[\rho_{\pm}, u_{\pm}, \theta_{\pm}]}(\xi), \end{cases}$$

where  $' = \frac{d}{d\vartheta}$ ,  $\vartheta = x - \bar{s}_3 t$ ,  $u_{\pm} = (u_{1\pm}, 0, 0)$ , and  $(\rho_{\pm}, u_{\pm}, \theta_{\pm})$  satisfy the Rankine-Hugoniot condition

$$(2.49) \quad \begin{cases} -\bar{s}_3(\rho_+ - \rho_-) + (\rho_+ u_{1+} - \rho_- u_{1-}) = 0, \\ -\bar{s}_3(\rho_+ u_{1+} - \rho_- u_{1-}) + (\rho_+ u_{1+}^2 + p_+ - \rho_- u_{1-}^2 - p_-) = 0, \\ -\bar{s}_3(\rho_+ E_+ - \rho_- E_-) + (\rho_+ u_{1+} E_+ + p_+ u_{1+} - \rho_- u_{1-} E_- - p_- u_{1-}) = 0 \end{cases}$$

and the Lax entropy condition

$$(2.50) \quad \lambda_{3+}^E < \bar{s}_3 < \lambda_{3-}^E$$

with  $\bar{s}_3$  being 3-shock wave speed and  $\lambda_3^E = u_1 + \frac{\sqrt{10\theta}}{3}$  being the third characteristic eigenvalue of the Euler equations in the Eulerian coordinate and  $\lambda_{3\pm}^E = u_{1\pm} + \frac{\sqrt{10\theta_{\pm}}}{3}$ .

By the macro-micro decomposition around the local Maxwellian  $\mathbf{M}^{S_3}$ , set

$$F^{S_3}(x, t, \xi) = \mathbf{M}^{S_3}(x, t, \xi) + \mathbf{G}^{S_3}(x, t, \xi),$$

where

$$\mathbf{M}^{S_3}(x, t, \xi) = \mathbf{M}_{[\rho^{S_3}, u^{S_3}, \theta^{S_3}]}(x, t, \xi) = \frac{\rho^{S_3}(x, t)}{\sqrt{(2\pi R\theta^{S_3}(x, t))^3}} e^{-\frac{|\xi - u^{S_3}(x, t)|^2}{2R\theta^{S_3}(x, t)}}$$

with

$$(2.51) \quad \begin{pmatrix} \rho^{S_3} \\ \rho^{S_3} u_i^{S_3} \\ \rho^{S_3} (\theta^{S_3} + \frac{|u^{S_3}|^2}{2}) \end{pmatrix} = \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ \xi_i \\ \frac{|\xi|^2}{2} \end{pmatrix} F^{S_3}(x, t, \xi) d\xi, \quad i = 1, 2, 3.$$

With respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathbf{M}^{S_3}}$  defined in (1.6), we can now define the macroscopic projection  $\mathbf{P}_0^{S_3}$  and microscopic projection  $\mathbf{P}_1^{S_3}$  by

$$(2.52) \quad \mathbf{P}_0^{S_3} g = \sum_{j=0}^4 \langle g, \chi_j^{S_3} \rangle_{\mathbf{M}^{S_3}} \chi_j^{S_3}, \quad \mathbf{P}_1^{S_3} g = g - \mathbf{P}_0^{S_3} g,$$

where  $\chi_j^{S_3}$  ( $j = 0, 1, 2, 3, 4$ ) are the corresponding pairwise orthogonal base defined in (1.7) by replacing  $(\rho, u, \theta, \mathbf{M})$  by  $(\rho^{S_3}, u^{S_3}, \theta^{S_3}, \mathbf{M}^{S_3})$ .

Under the above macro-micro decomposition, the solution  $F^{S_3} = F^{S_3}(x - \bar{s}_3 t, \xi)$  satisfies

$$\mathbf{P}_0^{S_3} F^{S_3} = \mathbf{M}^{S_3}, \quad \mathbf{P}_1^{S_3} F^{S_3} = \mathbf{G}^{S_3},$$

and the Boltzmann equation (2.48) becomes

$$(2.53) \quad (\mathbf{M}^{S_3} + \mathbf{G}^{S_3})_t + \xi_1 (\mathbf{M}^{S_3} + \mathbf{G}^{S_3})_x = \frac{1}{\varepsilon} [2Q(\mathbf{M}^{S_3}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})].$$

Correspondingly, we have the following fluid-type system for the fluid components of shock profile:

$$(2.54) \quad \begin{cases} \rho_t^{S_3} + (\rho^{S_3} u_1^{S_3})_x = 0, \\ (\rho^{S_3} u_1^{S_3})_t + [\rho^{S_3} (u_1^{S_3})^2 + p^{S_3}]_x = \frac{4\varepsilon}{3} (\mu(\theta^{S_3}) u_{1x}^{S_3})_x - \int \xi_1^2 \Pi_x^{S_3} d\xi, \\ (\rho^{S_3} u_i^{S_3})_t + (\rho^{S_3} u_1^{S_3} u_i^{S_3})_x = \varepsilon (\mu(\theta^{S_3}) u_{ix}^{S_3})_x - \int \xi_1 \xi_i \Pi_x^{S_3} d\xi, \quad i = 2, 3, \\ \left[ \rho^{S_3} \left( \theta^{S_3} + \frac{|u^{S_3}|^2}{2} \right) \right]_t + \left[ \rho^{S_3} u_1^{S_3} \left( \theta^{S_3} + \frac{|u^{S_3}|^2}{2} \right) + p^{S_3} u_1^{S_3} \right]_x = \varepsilon (\kappa(\theta^{S_3}) \theta_x^{S_3})_x \\ + \frac{4\varepsilon}{3} (\mu(\theta^{S_3}) u_1^{S_3} u_{1x}^{S_3})_x + \varepsilon \sum_{i=2}^3 (\mu(\theta^{S_3}) u_i^{S_3} u_{ix}^{S_3})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_x^{S_3} d\xi. \end{cases}$$

In fact, from the invariance of (2.48) by changing  $\xi_i$  with  $-\xi_i$  and the fact that  $u_{i\pm} = 0$ , we have  $u_i^{S_3} = \int \xi_1 \xi_i \Pi_x^{S_3} d\xi \equiv 0$  for  $i = 2, 3$ .

And the equation for the nonfluid component  $\mathbf{G}^{S_3}$  is

$$\mathbf{G}_t^{S_3} + \mathbf{P}_1^{S_3} (\xi_1 \mathbf{M}_x^{S_3}) + \mathbf{P}_1^{S_3} (\xi_1 \mathbf{G}_x^{S_3}) = \frac{1}{\varepsilon} [\mathbf{L}_{\mathbf{M}^{S_3}} \mathbf{G}^{S_3} + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})].$$

Here  $\mathbf{L}_{\mathbf{M}^{S_3}}$  is the linearized collision operator of  $Q(F^{S_3}, F^{S_3})$  with respect to the local Maxwellian  $\mathbf{M}^{S_3}$ :

$$\mathbf{L}_{\mathbf{M}^{S_3}} g = 2Q(\mathbf{M}^{S_3}, g) = Q(\mathbf{M}^{S_3}, g) + Q(g, \mathbf{M}^{S_3}).$$

Thus

$$\begin{aligned} \mathbf{G}^{S_3} &= \varepsilon \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} [\mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_x^{S_3})] + \Pi^{S_3}, \\ \Pi^{S_3} &= \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} [\varepsilon(\mathbf{G}_t^{S_3} + \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_x^{S_3})) - Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})]. \end{aligned}$$

Now we recall the properties of the shock profile  $F^{S_3}(x - \bar{s}_3 t, \xi)$  that are given or can be induced by [31, Theorem 6.8].

LEMMA 2.3 (see [31]). *If the shock wave strength  $\delta^{S_3}$  is small enough, then the Boltzmann equation (1.1) admits a 3-shock profile solution  $F^{S_3}(x - \bar{s}_3 t, \xi)$  uniquely up to a shift satisfying the following properties:*

- (1) *The shock profile converges to its far fields exponentially fast with an exponent proportional to the magnitude of the shock wave strength, that is,*

$$\begin{cases} |(\rho^{S_3} - \rho_{\pm}, u_1^{S_3} - u_{1\pm}, \theta^{S_3} - \theta_{\pm})| \leq C \delta^{S_3} e^{-c_{\pm} \frac{\delta^{S_3} |\vartheta|}{\varepsilon}} & \text{as } \vartheta \rightarrow \pm\infty, \\ \left( \int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} \leq C (\delta^{S_3})^2 e^{-c_{\pm} \frac{\delta^{S_3} |\vartheta|}{\varepsilon}} & \text{as } \vartheta \rightarrow \pm\infty, \end{cases}$$

with  $\delta^{S_3}$  being the 3-shock strength and  $\mathbf{M}_0$  being the global Maxwellian which is close to the shock profile with its precise definition given in [31, Theorem 6.8].

- (2) *The compressibility of the 3-shock profile is*

$$(\lambda_3^E)_{\vartheta} < 0, \quad \lambda_3^E = u_1^{S_3} + \frac{\sqrt{10\theta^{S_3}}}{3}.$$

- (3) *The following properties hold:*

$$\rho_{\vartheta}^{S_3} \sim u_{1\vartheta}^{S_3} \sim \theta_{\vartheta}^{S_3} \sim (\lambda_3^E)_{\vartheta} \sim \frac{1}{\varepsilon} \left( \int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}},$$

where  $A \sim B$  denotes the equivalence of the quantities  $A$  and  $B$ , and

$$\begin{cases} u_i^{S_3} \equiv 0, \quad \int \xi_1 \xi_i \Pi^{S_3} d\xi \equiv 0, \quad i = 2, 3, \\ |\partial_{\vartheta}^k(\rho^{S_3}, u_1^{S_3}, \theta^{S_3})| \leq C \frac{(\delta^{S_3})^{k-1}}{\varepsilon^{k-1}} |(\rho_{\vartheta}^{S_3}, u_{1\vartheta}^{S_3}, \theta_{\vartheta}^{S_3})|, \quad k \geq 2, \\ \left( \int \frac{\nu(|\xi|) |\partial_{\vartheta}^k \mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} \leq C \frac{(\delta^{S_3})^k}{\varepsilon^k} \left( \int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}}, \quad k \geq 1, \\ \left| \int \xi_1 \varphi_i(\xi) \Pi_{\vartheta}^{S_3} d\xi \right| \leq C \delta^{S_3} |u_{1\vartheta}^{S_3}|, \quad i = 1, 2, 3, 4, \end{cases}$$

where  $\varphi_i(\xi)$  ( $i = 1, 2, 3, 4$ ) are the collision invariants defined in (1.4).

Now we rewrite this shock profile in Lagrangian coordinate by using the transformation (1.14) and use  $(\tilde{t}, \tilde{x})$  for the Lagrangian coordinate to distinguish it from the Eulerian coordinate  $(t, x)$  at this moment. Then the shock profile in Lagrangian coordinate can be written as  $\tilde{F}^{S_3}(\tilde{x} - s_3 \tilde{t}, \xi)$  with  $s_3$  determined by the 3-shock wave curve given in (1.24). First, from the Rankine–Hugoniot condition in Eulerian and Lagrangian coordinates, we have

$$-\bar{s}_3(\rho_+ - \rho_-) + (\rho_+ u_{1+} - \rho_- u_{1-}) = 0$$

and

$$-s_3(v_+ - v_-) - (u_{1+} - u_{1-}) = 0$$

with  $v_{\pm} = \frac{1}{\rho_{\pm}}$ , respectively. Thus we have the following relation between  $\bar{s}_3$  and  $s_3$ :

$$(2.55) \quad s_3 = \rho_{\pm}(\bar{s}_3 - u_{1\pm}).$$

On the other hand, we have from (2.54)<sub>1</sub> that

$$(2.56) \quad \rho^{S_3}(x - s_3 t)[\bar{s}_3 - u_1^{S_3}(x - s_3 t)] = \text{const.} = \rho_{\pm}(\bar{s}_3 - u_{1\pm}).$$

Note that

$$\begin{aligned} \tilde{x} - s_3 \tilde{t} &= - \int_0^t \rho^{S_3} u_1^{S_3}(-\bar{s}_3 \tau) d\tau + \int_0^x \rho^{S_3}(y - \bar{s}_3 \tau) dy - \int_0^t \rho_{\pm}(\bar{s}_3 - u_{1\pm}) d\tau \\ &= - \int_0^t \rho^{S_3} u_1^{S_3}(-\bar{s}_3 \tau) d\tau + \int_0^x \rho^{S_3}(y - \bar{s}_3 \tau) dy \\ &\quad - \int_0^t \rho^{S_3}(-s_3 \tau)[\bar{s}_3 - u_1^{S_3}(-s_3 \tau)] d\tau \\ &= \int_0^x \rho^{S_3}(y - \bar{s}_3 \tau) dy - \bar{s}_3 \int_0^t \rho^{S_3}(-s_3 \tau) d\tau = \int_0^{x - \bar{s}_3 t} \rho^{S_3}(\eta) d\eta, \end{aligned}$$

where in the second equality we have used (2.56).

This shows that under the Lagrangian transformation (1.14), the shock profile  $F^{S_3}(x - \bar{s}_3 t, \xi)$  in the Eulerian coordinate can be exactly transformed to the shock profile  $\tilde{F}^{S_3}(\tilde{x} - s_3 \tilde{t}, \xi)$  in the Lagrangian coordinate. Moreover, we have

$$\tilde{F}_{\eta}^{S_3}(\eta, \xi) = \rho^{S_3} F_{\vartheta}^{S_3}(\vartheta, \xi)$$

with  $\eta = \tilde{x} - s_3 \tilde{t}$ .

For simplicity of notation, from now on, we use  $(t, x)$  to denote the Lagrangian coordinate and  $F^{S_3}(\eta, \xi)$  with  $\eta = x - s_3 t$  to denote the 3-shock profile of Boltzmann equation in Lagrangian coordinate. And in the Lagrangian coordinate, we have the following lemma.

**LEMMA 2.4.** *Assume that  $(v_-, u_-, \theta_-) \in S_3(v_+, u_+, \theta_+)$ ; then there exists a unique shock profile  $F^{S_3}(\eta, \xi)$  with  $\eta = x - s_3 t$  up to a shift, to the Boltzmann equation (1.15) in Lagrangian coordinate. Moreover, there are positive constants  $c_{\pm}$  and  $C$  such that for  $\eta \in \mathbf{R}$ ,*

$$\begin{cases} s_3 V_{\eta}^{S_3} = -U_{1\eta}^{S_3} > 0, \\ U_i^{S_3} \equiv 0, \quad \int \xi_i \xi_i \Pi_1^{S_3} d\xi \equiv 0, \quad i = 2, 3, \\ (|V^{S_3} - v_{\pm}|, |U_1^{S_3} - u_{1\pm}|, |\Theta^{S_3} - \theta_{\pm}|) \leq C \delta^{S_3} e^{-\frac{c_{\pm} \delta^{S_3} |\eta|}{\varepsilon}} \quad \text{as } \eta \rightarrow \pm\infty, \\ \left( \int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} \leq C (\delta^{S_3})^2 e^{-c_{\pm} \frac{\delta^{S_3} |\eta|}{\varepsilon}} \quad \text{as } \eta \rightarrow \pm\infty. \end{cases}$$

Furthermore, we have

$$V_{\eta}^{S_3} \sim U_{1\eta}^{S_3} \sim \Theta_{\eta}^{S_3} \sim \frac{1}{\varepsilon} \left( \int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}},$$



and

$$|\partial_\eta^k(V^{S_3}, U_1^{S_3}, \Theta^{S_3})| \leq C \frac{(\delta^{S_3})^{k-1}}{\varepsilon^{k-1}} |(V_\eta^{S_3}, U_{1\eta}^{S_3}, \Theta_\eta^{S_3})|, \quad k \geq 2,$$

$$\left( \int \frac{\nu(|\xi|)|\partial_\eta^k \mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} \leq C \frac{(\delta^{S_3})^k}{\varepsilon^k} \left( \int \frac{\nu(|\xi|)|\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}}, \quad k \geq 1,$$

and

$$\left| \int \xi_1 \varphi_i(\xi) \Pi_{1\eta}^{S_3} d\xi \right| \leq C \delta^{S_3} |U_{1\eta}^{S_3}|, \quad i = 1, 2, 3, 4,$$

with  $\varphi_i(\xi)$  being the collision invariants.

Furthermore, we have

$$(2.57) \quad \begin{cases} V_t^{S_3} - U_{1x}^{S_3} = 0, \\ U_{1t}^{S_3} + P_x^{S_3} = \frac{4}{3} \varepsilon \left( \frac{\mu(\Theta^{S_3}) U_{1x}^{S_3}}{V^{S_3}} \right)_x - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi, \\ U_{it}^{S_3} = \varepsilon \left( \frac{\mu(\Theta^{S_3}) U_{ix}^{S_3}}{V^{S_3}} \right)_x - \int \xi_1 \xi_i \Pi_{1x}^{S_3} d\xi, \quad i = 2, 3, \\ \mathcal{E}_t^{S_3} + (P^{S_3} U_1^{S_3})_x = \varepsilon \left( \frac{\kappa(\Theta^{S_3}) \Theta_x^{S_3}}{V^{S_3}} \right)_x + \frac{4}{3} \varepsilon \left( \frac{\mu(\Theta^{S_3}) U_1^{S_3} U_{1x}^{S_3}}{V^{S_3}} \right)_x \\ + \varepsilon \sum_{i=2}^3 \left( \frac{\mu(\Theta^{S_3}) U_i^{S_3} U_{ix}^{S_3}}{V^{S_3}} \right)_x - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi, \end{cases}$$

where  $\mathcal{E}^{S_3} = \Theta^{S_3} + \frac{|U^{S_3}|^2}{2}$  and  $(v_\pm, u_\pm, \theta_\pm)$  satisfy the Rankine–Hugoniot condition and Lax entropy condition and  $s_3$  is the 3-shock wave speed.

Correspondingly, we have the following equation for the nonfluid part of the 3-shock profile:

$$\mathbf{G}_t^{S_3} - \frac{U_1^{S_3}}{V^{S_3}} \mathbf{G}_x^{S_3} + \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_x^{S_3}) + \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_x^{S_3}) = \frac{1}{\varepsilon} [\mathbf{L}_{\mathbf{M}^{S_3}} \mathbf{G}^{S_3} + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})].$$

Here,  $\mathbf{L}_{\mathbf{M}^{S_3}}$  is the linearized collision operator of  $Q(F^{S_3}, F^{S_3})$  with respect to the local Maxwellian  $\mathbf{M}^{S_3}$ :

$$\mathbf{L}_{\mathbf{M}^{S_3}} g = 2Q(\mathbf{M}^{S_3}, g) = Q(\mathbf{M}^{S_3}, g) + Q(g, \mathbf{M}^{S_3}).$$

Thus

$$(2.58) \quad \begin{aligned} \mathbf{G}^{S_3} &= \varepsilon \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} \left[ \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_x^{S_3}) \right] + \Pi_1^{S_3}, \\ \Pi_1^{S_3} &= \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} \left[ \varepsilon \left( \mathbf{G}_t^{S_3} - \frac{U_1^{S_3}}{V^{S_3}} \mathbf{G}_x^{S_3} + \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_x^{S_3}) \right) - Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3}) \right]. \end{aligned}$$

**2.5. Hyperbolic wave II.** The purpose of this subsection is to construct the second hyperbolic wave. Up to now, we can define the following approximate composite wave profile  $(\bar{V}, \bar{U}, \bar{\mathcal{E}})(t, x)$ :

$$(2.59) \quad \begin{pmatrix} \bar{V} \\ \bar{U}_1 \\ \bar{\mathcal{E}} \end{pmatrix} (t, x) = \begin{pmatrix} V^{R_1} + d_1 + V^{CD} + V^{S_3} \\ U_1^{R_1} + d_2 + U_1^{CD} + U_1^{S_3} \\ \mathcal{E}^{R_1} + d_3 + \mathcal{E}^{CD} + \mathcal{E}^{S_3} \end{pmatrix} (t, x) - \begin{pmatrix} v_* + v^* \\ u_{1*} + u_1^* \\ E_* + E^* \end{pmatrix},$$

$$\bar{U}_i = U_i^{CD}, \quad i = 2, 3,$$

where  $\bar{\mathcal{E}} = \bar{\Theta} + \frac{|\bar{U}|^2}{2}$ ,  $(V^{R_1}, U_1^{R_1}, \mathcal{E}^{R_1})(t, x)$  is the 1-rarefaction wave defined in (2.5) with the right state  $(v_+, u_{1+}, E_+)$  replaced by  $(v_*, u_{1*}, E_*)$ ,  $(V^{CD}, U_1^{CD}, \mathcal{E}^{CD})(t, x)$  is the viscous contact wave defined in (2.39) with the states  $(v_-, u_{1-}, E_-)$  and  $(v_+, u_{1+}, E_+)$  replaced by  $(v_*, u_{1*}, E_*)$  and  $(v^*, u_1^*, E^*)$ , respectively, and  $(V^{S_3}, U_1^{S_3}, \mathcal{E}^{S_3})(t, x)$  is the fluid part of the 3-shock profile of Boltzmann equation defined in (2.57) with the left state  $(v_-, u_{1-}, E_-)$  replaced by  $(v^*, u_1^*, E^*)$ .

Moreover, we can check that this profile satisfies

$$(2.60) \quad \begin{cases} \bar{V}_t - \bar{U}_{1x} = 0, \\ \bar{U}_{1t} + \bar{P}_x = \frac{4}{3}\varepsilon \left( \frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}} \right)_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi + \bar{Q}_{1x} + Q_1^{CD}, \\ \bar{U}_{it} = \varepsilon \left( \frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}} \right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi - \int \xi_1 \xi_i \Pi_{1x}^{S_3} d\xi + \bar{Q}_{ix} + Q_i^{CD}, \quad i = 2, 3, \\ \bar{\mathcal{E}}_t + (\bar{P}\bar{U}_1)_x = \varepsilon \left( \frac{\kappa(\bar{\Theta})\bar{\Theta}_x}{\bar{V}} \right)_x + \frac{4}{3}\varepsilon \left( \frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1x}}{\bar{V}} \right)_x + \sum_{i=2}^3 \varepsilon \left( \frac{\mu(\bar{\Theta})\bar{U}_i\bar{U}_{ix}}{\bar{V}} \right)_x \\ - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi + \bar{Q}_{4x} + Q_4^{CD}, \end{cases}$$

where  $\bar{P} = p(\bar{V}, \bar{\Theta})$ ,  $Q_i^{CD}$  ( $i = 1, 2, 3, 4$ ) are defined in (2.44), (2.45), and (2.46), respectively, and

$$\begin{aligned} \bar{Q}_1 &= \left[ \bar{P} - (p_v^{R_1} d_1 + p_{u_1}^{R_1} d_2 + p_E^{R_1} d_3) - P^{R_1} - P^{CD} - P^{S_3} + p_* + p^* \right] \\ &\quad - \frac{4}{3}\varepsilon \left[ \frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}} - \frac{\mu(\Theta^{R_1})U_{1x}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1x}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1x}^{S_3}}{V^{S_3}} \right], \\ \bar{Q}_i &= -\varepsilon \left[ \frac{\mu(\bar{\Theta})}{\bar{V}} - \frac{\mu(\Theta^{CD})}{V^{CD}} \right] U_{ix}^{CD}, \quad i = 2, 3, \\ \bar{Q}_4 &= \left[ \bar{P}\bar{U}_1 - ((pu_1)_v^{R_1} d_1 + (pu_1)_{u_1}^{R_1} d_2 + (pu_1)_E^{R_1} d_3) - P^{R_1} U_1^{R_1} \right. \\ &\quad \left. - P^{CD} U_1^{CD} - P^{S_3} U_1^{S_3} + p_* u_{1*} + p^* u_1^* \right] \\ &\quad - \varepsilon \left[ \frac{\kappa(\bar{\Theta})\bar{\Theta}_x}{\bar{V}} - \frac{\kappa(\Theta^{R_1})\Theta_x^{R_1}}{V^{R_1}} - \frac{\kappa(\Theta^{CD})\Theta_x^{CD}}{V^{CD}} - \frac{\kappa(\Theta^{S_3})\Theta_x^{S_3}}{V^{S_3}} \right] \\ &\quad - \frac{4}{3}\varepsilon \left[ \frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1x}}{\bar{V}} - \frac{\mu(\Theta^{R_1})U_1^{R_1}U_{1x}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_1^{CD}U_{1x}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_1^{S_3}U_{1x}^{S_3}}{V^{S_3}} \right] \\ &\quad - \sum_{i=2}^3 \varepsilon \left[ \frac{\mu(\bar{\Theta})}{\bar{V}} - \frac{\mu(\Theta^{CD})}{V^{CD}} \right] U_i^{CD} U_{ix}^{CD}. \end{aligned}$$

From (2.60), we have that

$$(2.61) \quad \begin{aligned} \bar{\Theta} &= \Theta^{R_1} + d_3 + \Theta^{CD} + \Theta^{S_3} - (\theta_* + \theta^*) \\ &\quad - \frac{1}{2} [|\bar{U}|^2 - |U^{R_1}|^2 - |U^{CD}|^2 - |U^{S_3}|^2 + u_{1*}^2 + (u_1^*)^2]. \end{aligned}$$

Direct computation yields

$$\begin{aligned}
 (2.62) \quad \bar{Q}_1 &= O(1) \left[ |(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, U_1^{R_1} - u_{1*}, d_1, d_2, d_3)| \right. \\
 &\quad \times |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \Theta^{CD} - \theta_*, V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| \\
 &\quad + |(V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| |(V^{CD} - v^*, U_1^{CD} - u_{1*}, U_1^{CD} \\
 &\quad - u_1^*, \Theta^{CD} - \theta^*)| + \varepsilon |U_{1x}^{R_1}| |(V^{CD} - v_*, \Theta^{CD} - \theta_*, V^{S_3} - v^*, \Theta^{S_3} - \theta^*)| \\
 &\quad + \varepsilon |U_{1x}^{CD}| |(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, d_1, d_2, d_3, V^{S_3} - v^*, \Theta^{S_3} - \theta^*)| \\
 &\quad + \varepsilon |U_{1x}^{S_3}| |(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, d_1, d_2, d_3, V^{CD} - v^*, \Theta^{CD} - \theta^*)| \left. \right] \\
 &:= \bar{Q}_{11} + \bar{Q}_{12}, \\
 (2.63) \quad \bar{Q}_i &= O(1)\varepsilon [(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, U_1^{R_1} - u_{1*}, d_1, d_2, d_3, V^{S_3} \\
 &\quad - v^*, \Theta^{S_3} - \theta^*, U_1^{S_3} - u_1^*) |U_{ix}^{CD}|], \quad i = 2, 3,
 \end{aligned}$$

and

$$\begin{aligned}
 (2.64) \quad \bar{Q}_4 &= O(1) \left[ |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, d_1, d_2, d_3)| \right. \\
 &\quad \times |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \Theta^{CD} - \theta_*, V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| \\
 &\quad + |(V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| |(V^{CD} - v^*, U_1^{CD} - u_1^*, \Theta^{CD} - \theta^*)| \\
 &\quad + \varepsilon |(U_{1x}^{R_1}, \Theta_x^{R_1})| |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \Theta^{CD} - \theta_*, V^{S_3} - v^*, \Theta^{S_3} - \theta^*)| \\
 &\quad + \varepsilon |(U_{1x}^{CD}, \Theta_x^{CD})| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, d_1, d_2, d_3, V^{S_3} \\
 &\quad - v^*, \Theta^{S_3} - \theta^*)| + \varepsilon |(U_{1x}^{S_3}, \Theta_x^{S_3})| |(V^{R_1} - v_*, \Theta^{R_1} \\
 &\quad - \theta_*, d_1, d_2, d_3, V^{CD} - v^*, U_1^{CD} - u_1^*, \Theta^{CD} - \theta^*)| \left. \right] \\
 &:= \bar{Q}_{41} + \bar{Q}_{42}.
 \end{aligned}$$

Here,  $\bar{Q}_{11}$ ,  $\bar{Q}_i$  ( $i = 2, 3$ ), and  $\bar{Q}_{41}$  represent the interaction of waves in different families, and  $\bar{Q}_{12}$  and  $\bar{Q}_{42}$  represent the error terms coming from the approximate rarefaction wave and the hyperbolic wave I.

First, we estimate the interaction terms  $\bar{Q}_{11}$ ,  $\bar{Q}_i$  ( $i = 2, 3$ ), and  $\bar{Q}_{41}$  by dividing the whole domain  $\Omega = \{(t, x) | (t, x) \in [h, T] \times \mathbf{R}\}$  into three regions:

$$\begin{aligned}
 \Omega_{R_1} &= \{(t, x) \in \Omega | 2x \leq \lambda_{1*}t\}, \\
 \Omega_{CD} &= \{(t, x) \in \Omega | \lambda_{1*}t < 2x < s_3t\}, \\
 \Omega_{S_3} &= \{(t, x) \in \Omega | 2x \geq s_3t\},
 \end{aligned}$$

where  $\lambda_{1*} = \lambda_1(v_*, \theta_*)$  and  $s_3$  is the 3-shock speed.

From Lemma 2.1, we have the following estimates in each region:

- in  $\Omega_{R_1}$ ,
 
$$\begin{aligned}
 |V^{S_3} - v^*| &= O(1)\delta^{S_3} e^{-\frac{c\delta^{S_3}|x-s_3t|}{\varepsilon}} = O(1)e^{-\frac{C|x|}{\varepsilon}} e^{-\frac{Ch}{\varepsilon}}, \\
 |(V^{CD} - v_*, V^{CD} - v^*)| &= O(1)\delta^{CD} e^{-\frac{C|x||\lambda_{1*}|t}{8\varepsilon(1+t)}} e^{-\frac{C(\lambda_{1*}t)^2}{8\varepsilon(1+t)}} = O(1)e^{-\frac{Ch|x|}{\varepsilon}} e^{-\frac{Ch}{\varepsilon}};
 \end{aligned}$$

- in  $\Omega_{CD}$ ,
 
$$|(V^{R_1} - v_*, d_1, d_2, d_3)| = O(1)e^{-\frac{2|x-\lambda_{1*}t|}{\sigma}} = O(1)e^{-\frac{C_h|x|}{\sigma}} e^{-\frac{C_h}{\sigma}},$$

$$|V^{S_3} - v^*| = O(1)e^{-\frac{c\delta^{S_3}|x-s_3t|}{\varepsilon}} = O(1)e^{-\frac{C_h|x|}{\varepsilon}} e^{-\frac{C_h}{\varepsilon}};$$
- in  $\Omega_{S_3}$ ,
 
$$|(V^{R_1} - v_*, d_1, d_2, d_3)| = O(1)e^{-\frac{2|x-\lambda_{1*}t|}{\sigma}} = O(1)e^{-\frac{2|x|}{\sigma}} e^{-\frac{C_h}{\sigma}},$$

$$|(V^{CD} - v_*, V^{CD} - v^*)| = O(1)\delta^{CD} e^{-\frac{C|x||s_3t|}{8\varepsilon(1+t)}} e^{-\frac{C(s_3t)^2}{8\varepsilon(1+t)}} = O(1)e^{-\frac{C_h|x|}{\varepsilon}} e^{-\frac{C_h}{\varepsilon}}.$$

Note that we just give the pointwise estimates of  $V$  component and  $d_i$  ( $i = 1, 2, 3$ ) in each region; similar estimates also hold for the  $U_1$  and  $\Theta$  components. In summary, we have

$$(2.65) \quad |(\bar{Q}_{11}, \bar{Q}_2, \bar{Q}_3, \bar{Q}_{41})| = C_{h,T} e^{-\frac{C_h|x|}{\sigma}} e^{-\frac{C_h}{\sigma}}$$

with  $\sigma = \varepsilon^{\frac{1}{5}}$  and for some positive constants  $C_{h,T}$  and  $C_h$  independent of  $\varepsilon$ .

In order to remove the nonconservative error terms  $Q_i^{CD}$  ( $i = 1, 2, 3, 4$ ) coming from the definition of the viscous contact wave, we now introduce the following hyperbolic wave  $\vec{b} \triangleq (b_1, b_{21}, b_{22}, b_{23}, b_3)$ :

$$(2.66) \quad \begin{cases} b_{1t} - b_{21x} = 0, \\ b_{21t} + [\bar{P}_v b_1 + \bar{P}_{u_1} b_{21} + \bar{P}_{u_2} b_{22} + \bar{P}_{u_3} b_{23} + \bar{P}_E b_3]_x = -Q_1^{CD}, \\ b_{22t} = -Q_2^{CD}, \\ b_{23t} = -Q_3^{CD}, \\ b_{3t} + [(\bar{P}\bar{U}_1)_v b_1 + (\bar{P}\bar{U}_1)_{u_1} b_{21} + (\bar{P}\bar{U}_1)_{u_2} b_{22} + (\bar{P}\bar{U}_1)_{u_3} b_{23} + (\bar{P}\bar{U}_1)_E b_3]_x = -Q_4^{CD}, \end{cases}$$

where  $P = \frac{2\Theta}{3V} = P(V, U, \mathcal{E})$  and  $\bar{P}_v = P_v(\bar{V}, \bar{U}, \bar{\mathcal{E}})$ , etc. For later use, we denote  $b_2 = (b_{21}, b_{22}, b_{23})$ . Now we want to solve this linear hyperbolic system (2.66) on the interval  $[h, T]$ . First, we diagonalize the above system. Rewrite the system (2.66) as

$$(2.67) \quad \begin{pmatrix} b_1 \\ b_{21} \\ b_{22} \\ b_{23} \\ b_3 \end{pmatrix}_t + \left[ \bar{A}(\bar{V}, \bar{U}, \bar{\mathcal{E}}) \begin{pmatrix} b_1 \\ b_{21} \\ b_{22} \\ b_{23} \\ b_3 \end{pmatrix} \right]_x = \begin{pmatrix} 0 \\ -Q_1^{CD} \\ -Q_2^{CD} \\ -Q_3^{CD} \\ -Q_4^{CD} \end{pmatrix},$$

where the matrix

$$\bar{A}(\bar{V}, \bar{U}, \bar{\mathcal{E}}) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ \bar{P}_v & \bar{P}_{u_1} & \bar{P}_{u_2} & \bar{P}_{u_3} & \bar{P}_E \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\bar{P}\bar{U}_1)_v & (\bar{P}\bar{U}_1)_{u_1} & (\bar{P}\bar{U}_1)_{u_2} & (\bar{P}\bar{U}_1)_{u_3} & (\bar{P}\bar{U}_1)_E \end{pmatrix}$$

has three distinct eigenvalues  $\bar{\lambda}_1 := \lambda_1(\bar{V}, \bar{P}) < 0 = \bar{\lambda}_2 < \lambda_3(\bar{V}, \bar{P}) := \bar{\lambda}_3$  (here  $\bar{\lambda}_2$  being 3-repeated eigenvalues) with the corresponding left and right eigenvectors denoted by

$$\bar{l}_1, \bar{l}_{21}, \bar{l}_{22}, \bar{l}_{23}, \bar{l}_3; \quad \bar{r}_1, \bar{r}_{21}, \bar{r}_{22}, \bar{r}_{23}, \bar{r}_3.$$

It holds that

$$\bar{L}\bar{A}\bar{R} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_2, \bar{\lambda}_2, \bar{\lambda}_3) \equiv \bar{\Lambda},$$

$$\bar{L}\bar{R} = \text{Id}.$$

Here  $\bar{L} = (\bar{l}_1, \bar{l}_{21}, \bar{l}_{22}, \bar{l}_{23}, \bar{l}_3)^t, \bar{R} = (\bar{r}_1, \bar{r}_{21}, \bar{r}_{22}, \bar{r}_{23}, \bar{r}_3)$  with  $\bar{L} = \bar{L}(\bar{V}, \bar{U}, \bar{\mathcal{E}})$  and  $\bar{R} = \bar{R}(\bar{V}, \bar{U}, \bar{\mathcal{E}})$  ( $i = 1, 2, 3$ ) and  $\text{Id}$ . is the  $5 \times 5$  identity matrix. Specially, we can choose

$$(2.68) \quad \bar{l}_{21} = (\bar{P}, -\bar{U}_1, 0, 0, 1), \quad \bar{l}_{22} = (0, 0, 1, 0, 0), \quad \bar{l}_{23} = (0, 0, 0, 1, 0).$$

Set

$$(2.69) \quad \vec{B} \triangleq (B_1, B_{21}, B_{22}, B_{23}, B_3)^t = \bar{L} \cdot (b_1, b_{21}, b_{22}, b_{23}, b_3);$$

then

$$(2.70) \quad (b_1, b_{21}, b_{22}, b_{23}, b_3)^t = \bar{R} \cdot (B_1, B_{21}, B_{22}, B_{23}, B_3)^t,$$

and  $\vec{B}$  satisfies the system

$$(2.71) \quad \begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix}_t + \left[ \bar{\Lambda} \begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix} \right]_x = \bar{L} \begin{pmatrix} 0 \\ -Q_1^{CD} \\ -Q_2^{CD} \\ -Q_3^{CD} \\ -Q_4^{CD} \end{pmatrix} + \bar{L}_t \bar{R} \begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix} + \bar{L}_x \bar{R} \bar{\Lambda} \begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix}.$$

For simplicity, denote

$$(2.72) \quad \vec{Q}^{CD} = (0 - Q_1^{CD}, -Q_2^{CD}, -Q_3^{CD}, -Q_4^{CD})^t.$$

We obtain a diagonalized system

$$(2.73) \quad \begin{cases} B_{1t} + (\bar{\lambda}_1 B_1)_x = \bar{l}_1 \cdot \vec{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{1t} + \bar{\lambda}_i \bar{l}_{1x}) \cdot \bar{r}_i B_i + \bar{l}_{1t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}, \\ B_{21t} = \bar{l}_{21} \cdot \vec{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{21t} + \bar{\lambda}_i \bar{l}_{21x}) \cdot \bar{r}_i B_i + \bar{l}_{21t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}, \\ B_{22t} = \bar{l}_{22} \cdot \vec{Q}^{CD}, \\ B_{23t} = \bar{l}_{23} \cdot \vec{Q}^{CD}, \\ B_{3t} + (\bar{\lambda}_3 B_3)_x = \bar{l}_3 \cdot \vec{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{3t} + \bar{\lambda}_i \bar{l}_{3x}) \cdot \bar{r}_i B_i + \bar{l}_{3t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}. \end{cases}$$

Now we impose the following boundary condition to the linear hyperbolic system (2.73) on the domain  $(t, x) \in [h, T] \times \mathbf{R}$ :

$$(2.74) \quad (B_1, B_{21}, B_{22}, B_{23}, B_3)(t = T, x) = 0.$$

We can solve the linear diagonalized hyperbolic system (2.73) under the condition (2.74) to have the following lemma.

LEMMA 2.5. *There exists a positive constant  $\delta_0$  such that if the wave strength  $\delta \leq \delta_0$ , then there exists a positive constant  $C_{h,T}$  which is independent of  $\varepsilon$ , such that*

$$(2.75) \quad \begin{aligned} & \left\| \frac{\partial^k}{\partial x^k} (b_1, b_{21}, b_{22}, b_{23}, b_3)(t, \cdot) \right\|_{L^2(dx)}^2 \\ & + \int_h^T \left\| \sqrt{|U_{1x}^{S_3}|} \frac{\partial^k}{\partial x^k} (b_1, b_{21}, b_{22}, b_{23}, b_3)(t, \cdot) \right\|_{L^2(dx)}^2 dt \\ & \leq C_{h,T} \varepsilon^{\frac{5}{2}-2k}, \quad k = 0, 1, 2, 3. \end{aligned}$$

*Proof.* From the wave curves defined in (1.22)–(1.24), we know  $\delta^{CD} + \delta^{S_3} \leq C\delta_0$ . Let  $N = \frac{1}{\delta_0}$  and  $\delta \leq \delta_0 \ll 1$ ; then we have

$$0 < c_0 \leq \left(\frac{\hat{V}}{v^*}\right)^{\pm N}, \left(\frac{V^{S_3}}{v^*}\right)^{\pm N} \leq C_0 < \infty,$$

where  $\hat{V}$  is defined in (2.19) and  $c_0, C_0$  are independent of  $\delta_0$ .

Without loss of generality, we assume that  $v^* = 1$  and  $\hat{V}_x > 0$ . If  $v^* \neq 1$ , then we just replace  $\hat{V}^{\pm N}$  and  $(V^{S_3})^{\pm N}$  by  $(\hat{V}/v^*)^{\pm N}$  and  $(V^{S_3}/v^*)^{\pm N}$ , respectively.

First, multiplying (2.73)<sub>1</sub> by  $[(\hat{V})^{-N} + (V^{S_3})^{-N}]B_1$ , we obtain

$$\begin{aligned} (2.76) \quad & \left(\frac{1}{2}[\hat{V}^{-N} + (V^{S_3})^{-N}]B_1^2\right) + \frac{1}{2}N(\hat{V}^{-N-1}\hat{V}_t + (V^{S_3})^{-N-1}V_t^{S_3})B_1^2 \\ & + (\bar{\lambda}_1(\bar{V}, \bar{P})B_1)_x[\hat{V}^{-N} + (V^{S_3})^{-N}]B_1 \\ & = [\hat{V}^{-N} + (V^{S_3})^{-N}]B_1 \left( \bar{l}_1 \cdot \vec{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{1t} + \bar{\lambda}_i \bar{l}_{1x}) \cdot \bar{r}_i B_i + \bar{l}_{1t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j} \right). \end{aligned}$$

Now the second and third terms on the left-hand side of (2.76) can be estimated by

$$\begin{aligned} (2.77) \quad & \frac{1}{2}N(\hat{V}^{-N-1}\hat{V}_t + (V^{S_3})^{-N-1}V_t^{S_3})B_1^2 + (\bar{\lambda}_1(\bar{V}, \bar{P})B_1)_x[\hat{V}^{-N} + (V^{S_3})^{-N}]B_1 \\ & \leq C_{h,T}B_1^2 + C(\hat{V}_x + |U_{1x}^{S_3}|)B_1^2 - \frac{1}{2}N|\bar{\lambda}_1(\bar{V}, \bar{P})|\hat{V}^{-N-1}\hat{V}_x B_1^2 \\ & - \frac{1}{2}N \left( 1 + \frac{|\bar{\lambda}_1(\bar{V}, \bar{P})|}{s_3} \right) (V^{S_3})^{-N-1}|U_{1x}^{S_3}|B_1^2 + (\dots)_x. \end{aligned}$$

Then we estimate the right-hand side of (2.76) term by term. On one hand,

$$(2.78) \quad |[\hat{V}^{-N} + (V^{S_3})^{-N}]B_1 \bar{l}_1 \cdot \vec{Q}^{CD}| \leq CB_1^2 + C|\vec{Q}^{CD}|^2.$$

On the other hand, we have

$$(2.79) \quad |[\hat{V}^{-N} + (V^{S_3})^{-N}](\bar{l}_{1t} + \bar{\lambda}_1 \bar{l}_{1x}) \cdot \bar{r}_1 B_1^2| \leq C_{h,T}B_1^2 + C\hat{V}_x B_1^2 + C|U_{1x}^{S_3}|B_1^2,$$

$$\begin{aligned} (2.80) \quad & \left| [\hat{V}^{-N} + (V^{S_3})^{-N}]B_1 \bar{l}_{1t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j} \right| \\ & \leq C_{h,T}|\vec{B}|^2 + C|U_{1x}^{S_3}|(B_1^2 + |B_{21}|^2 + |B_{22}|^2 + |B_{23}|^2) \end{aligned}$$

and

$$\begin{aligned} (2.81) \quad & |[\hat{V}^{-N} + (V^{S_3})^{-N}](\bar{l}_{1t} + \bar{\lambda}_3 \bar{l}_{1x}) \cdot \bar{r}_3 B_1 B_3| \\ & \leq C_{h,T}(B_1^2 + B_3^2) + C\hat{V}_x(B_1^2 + B_3^2) + C|\lambda_3^{S_3} - s_3||U_{1x}^{S_3}|(B_1^2 + B_3^2) \\ & + C_{h,T} \left[ e^{-\frac{c_h}{\sigma}} e^{-\frac{c_h|x|}{\sigma}} + |(d_{1x}, d_{2x}, d_{3x})| \right] (B_1^2 + B_3^2) \\ & \leq C_{h,T}(B_1^2 + B_3^2) + C\hat{V}_x(B_1^2 + B_3^2) + C\delta|U_{1x}^{S_3}|(B_1^2 + B_3^2). \end{aligned}$$

Substituting (2.77)–(2.81) into (2.76) and choosing  $N$  large enough give  
 (2.82)

$$\begin{aligned} & \left( \frac{1}{2} [\hat{V}^{-N} + (V^{S_3})^{-N}] B_1^2 \right)_t \\ & \quad - \frac{1}{4} N |\bar{\lambda}_1(\bar{V}, \bar{P})| \hat{V}^{-N-1} |\hat{V}_x| B_1^2 - \frac{1}{4} N \left( 1 + \frac{|\bar{\lambda}_1(\bar{V}, \bar{P})|}{s_3} \right) (V^{S_3})^{-N-1} |U_{1x}^{S_3}| B_1^2 \\ & \geq -C_{h,T} |\bar{B}|^2 - C_{h,T} |\bar{Q}^{CD}|^2 - C |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2 - C |\hat{V}_x| B_3^2 - C \delta |U_{1x}^{S_3}| B_3^2. \end{aligned}$$

By multiplying (2.73) <sub>$j+1$</sub>  by  $(V^{S_3})^{-N} B_{2j}$  ( $j = 1, 2, 3$ ) and taking the summation of the equations together, we obtain

(2.83)

$$\begin{aligned} & \left( \frac{1}{2} (V^{S_3})^{-N} \sum_{j=1}^3 |B_{2j}|^2 \right)_t - \frac{1}{2} N (V^{S_3})^{-N-1} |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2 \\ & = (V^{S_3})^{-N} \left( \sum_{j=1}^3 B_{2j} \bar{l}_{2j} \cdot \bar{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{21t} + \bar{\lambda}_i \bar{l}_{21x}) \cdot \bar{r}_i B_i B_{21} + \bar{l}_{21t} \sum_{j=1}^3 \bar{r}_{2j} B_{2j} B_{21} \right). \end{aligned}$$

It is easy to check that

(2.84)

$$\left| (V^{S_3})^{-N} \sum_{j=1}^3 B_{2j} \bar{l}_{2j} \cdot \bar{Q}^{CD} \right| \leq C \sum_{j=1}^3 |B_{2j}|^2 + C |\bar{Q}^{CD}|^2.$$

From the construction of viscous contact wave and (2.68), it holds that  $l_{21x}^{CD} = O(1)\delta^{CD}$ . Then the terms on the right-hand side of (2.83) can be estimated by

(2.85)

$$\begin{aligned} & |(V^{S_3})^{-N} (\bar{l}_{21t} + \bar{\lambda}_1 \bar{l}_{21x}) \cdot \bar{r}_1 B_1 B_{21}| \\ & \leq C_{h,T} |\bar{B}|^2 + C |U_{1x}^{S_3}| (B_1^2 + |B_{21}|^2) \\ & \quad + C_{h,T} \left[ e^{-\frac{c_h}{\sigma}} e^{-\frac{c_h |x|}{\sigma}} + |(d_{1x}, d_{2x}, d_{3x})| \right] (B_1^2 + |B_{21}|^2) \\ & \leq C_{h,T} |\bar{B}|^2 + C |U_{1x}^{S_3}| (B_1^2 + |B_{21}|^2). \end{aligned}$$

Similar to (2.81) and (2.85), we have

(2.86)

$$|(V^{S_3})^{-N} (\bar{l}_{21t} + \bar{\lambda}_3 \bar{l}_{21x}) \cdot \bar{r}_3 B_3 B_{21}| \leq C_{h,T} |\bar{B}|^2 + C \delta |U_{1x}^{S_3}| (B_3^2 + |B_{21}|^2)$$

and

(2.87)

$$\left| (V^{S_3})^{-N} \bar{l}_{21t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j} B_{21} \right| \leq C_{h,T} |\bar{B}|^2 + C |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2.$$

Substituting (2.84)–(2.87) into (2.83) and choosing  $N$  large enough give

(2.88)

$$\begin{aligned} & \left( \frac{1}{2} (V^{S_3})^{-N} \sum_{j=1}^3 |B_{2j}|^2 \right)_t - \frac{1}{4} N (V^{S_3})^{-N-1} |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2 \\ & \geq -C_{h,T} |\bar{B}|^2 - C |\bar{Q}^{CD}|^2 - C \delta |U_{1x}^{S_3}| B_3^2 - C |U_{1x}^{S_3}| (B_1^2 + B_{21}^2). \end{aligned}$$

Multiplying (2.73)<sub>5</sub> by  $\hat{V}^N B_3$  yields

$$(2.89) \quad \left(\frac{1}{2}\hat{V}^N B_3^2\right)_t + \frac{1}{2}\bar{\lambda}_{3x}(\bar{V}, \bar{P})\hat{V}^N B_3^2 - \frac{\bar{\lambda}_3}{2}N\hat{V}^{N-1}|\hat{V}_x|B_3^2 \\ = \frac{1}{2}N\hat{V}^{N-1}\hat{V}_t B_3^2 + \hat{V}^N B_3 \left( \bar{l}_3 \cdot \bar{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{3t} + \bar{\lambda}_i \bar{l}_{3x}) \cdot \bar{r}_i B_i + \bar{l}_{3t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j} \right).$$

The wave interaction estimations imply

$$(2.90) \quad \bar{\lambda}_{3x} = \lambda_{3x}^{R_1} + \lambda_{3x}^{CD} + \lambda_{3x}^{S_3} + (\bar{\lambda}_{3x} - \lambda_{3x}^{R_1} - \lambda_{3x}^{CD} - \lambda_{3x}^{S_3}) \leq \lambda_{3x}^{S_3} + C|\hat{V}_x| + C_{h,T},$$

so that

$$(2.91) \quad \frac{1}{2}\bar{\lambda}_{3x}\hat{V}^N B_3^2 \leq \frac{1}{2}\lambda_{3x}^{S_3}\hat{V}^N B_3^2 + C|\hat{V}_x|\hat{V}^N B_3^2 + C_{h,T}B_3^2.$$

The other terms on the right-hand side of (2.89) can be estimated similarly to (2.81)–(2.85). Then we have

$$(2.92) \quad \left|\frac{1}{2}N\hat{V}^{N-1}\hat{V}_t B_3^2 + \hat{V}^N B_3 \bar{l}_3 \cdot \bar{Q}^{CD}\right| \leq CB_3^2 + C|\bar{Q}^{CD}|^2,$$

$$(2.93) \quad |\hat{V}^N(\bar{l}_{3t} + \bar{\lambda}_1 \bar{l}_{3x}) \cdot \bar{r}_1 B_1 B_3| \\ \leq C_{h,T}(B_1^2 + B_3^2) + \beta|U_{1x}^{S_3}|\hat{V}^N B_3^2 + C_\beta|U_{1x}^{S_3}|B_1^2 + C|\hat{V}_x|\hat{V}^N(B_1^2 + B_3^2),$$

$$(2.94) \quad |\hat{V}^N(\bar{l}_{3t} + \bar{\lambda}_3 \bar{l}_{3x}) \cdot \bar{r}_3 B_3^2| \leq C_{h,T}\hat{V}^N B_3^2 + C\delta|U_{1x}^{S_3}|\hat{V}^N B_3^2 + C|\hat{V}_x|\hat{V}^N B_3^2,$$

and

$$(2.95) \quad \left|\hat{V}^N B_3 \bar{l}_{3t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}\right| \leq C_{h,T}|\bar{B}|^2 + \beta|U_{1x}^{S_3}|\hat{V}^N B_3^2 + C_\beta|U_{1x}^{S_3}|\sum_{j=1}^3 |B_{2j}|^2.$$

By choosing  $\beta$  and  $\delta_0$  small enough, substituting (2.91)–(2.94) into (2.89) gives

$$(2.96) \quad \left(\frac{1}{2}\hat{V}^N B_3^2\right)_t - \frac{|\lambda_{3x}^{S_3}|}{4}\hat{V}^N B_3^2 - \frac{\bar{\lambda}_3}{4}N\hat{V}^{N-1}|\hat{V}_x|B_3^2 \\ \geq -C_{h,T}|\bar{B}|^2 - C|\bar{Q}^{CD}|^2 - C|\hat{V}_x|\hat{V}^N B_1^2 - C_\beta|U_{1x}^{S_3}|\left(B_1^2 + \sum_{j=1}^3 |B_{2j}|^2\right).$$

By combining (2.82), (2.88), and (2.96) and noticing that  $\delta_0 \ll 1$  and  $N \gg 1$ , we can get

$$(2.97) \quad \left(\frac{1}{2}[\hat{V}^{-N} + (V^{S_3})^{-N}]B_1^2 + \frac{1}{2}(V^{S_3})^{-N}\sum_{j=1}^3 |B_{2j}|^2 + \frac{1}{2}\hat{V}^N B_3^2\right)_t \\ - \frac{1}{4}N|\bar{\lambda}_1|\hat{V}^{-N-1}|\hat{V}_x|B_1^2 - \frac{1}{4}N\left(1 + \frac{|\bar{\lambda}_1|}{s_3}\right)\hat{V}^{-N-1}|U_{1x}^{S_3}|B_1^2 \\ - \frac{1}{4}N(V^{S_3})^{-N-1}|U_{1x}^{S_3}|\sum_{j=1}^3 |B_{2j}|^2 - \frac{|\lambda_{3x}^{S_3}|}{4}\hat{V}^N B_3^2 - \frac{\bar{\lambda}_3}{4}N\hat{V}^{N-1}|\hat{V}_x|B_3^2 \\ \geq -C_{h,T}|\bar{B}|^2 - C_{h,T}|Q^{CD}|^2.$$



Integrating (2.97) over  $[t, T] \times \mathbf{R}$  with  $t \in (h, T)$  then yields

$$(2.98) \quad \int_{\mathbf{R}} |\vec{B}|^2 dx + \int_t^T \int_{\mathbf{R}} [|\hat{V}_x|(B_1^2 + B_3^2) + |U_x^{S_3}||\vec{B}|^2] dx dt \leq C_{h,T} \int_{\mathbf{R}} |\vec{B}|^2 + C_{h,T} \varepsilon^{\frac{5}{2}} \quad \forall t \in [h, T].$$

Applying the Gronwall inequality to (2.98) gives

$$(2.99) \quad \int_{\mathbf{R}} |\vec{B}|^2 dx + \int_t^T \int_{\mathbf{R}} [|\hat{V}_x|(B_1^2 + B_3^2) + |U_{1,x}^{S_3}||\vec{B}|^2] dx dt \leq C_{h,T} \varepsilon^{\frac{5}{2}} \quad \forall t \in [h, T].$$

This completes the proof for the case when  $k = 0$  in Lemma 2.5. The case  $k = 1, 2, 3$  can be proved similarly to the differentiated system, and we omit the details for brevity.  $\square$

**2.6. Superposition of waves.** With the above preparation, finally, the approximate superposition wave  $(V, U, \mathcal{E})(t, x)$  can be defined by

$$(2.100) \quad \begin{pmatrix} V \\ U_i \\ \mathcal{E} \end{pmatrix} (t, x) = \begin{pmatrix} \bar{V} + b_1 \\ \bar{U}_i + b_{2i} \\ \bar{\mathcal{E}} + b_3 \end{pmatrix} (t, x), \quad i = 1, 2, 3,$$

where  $\mathcal{E} = \Theta + \frac{|U|^2}{2}$ .

Thus, we have

$$(2.101) \quad \Theta = \bar{\Theta} - \sum_{i=1}^3 \bar{U}_i b_{2i} + b_3 - \frac{|b_2|^2}{2},$$

where  $b_2 = (b_{21}, b_{22}, b_{23})^t$  and  $|b_2|^2 = \sum_{i=1}^3 b_{2i}^2$ .

From the construction of the contact wave and Lemma 2.1 and by noting that  $\sigma = \varepsilon^{\frac{1}{5}}$ , we have the following relation between the approximate wave pattern  $(V, U, \mathcal{E}, \Theta)(t, x)$  of the Boltzmann equation and the inviscid superposition wave pattern  $(\bar{V}, \bar{U}, \bar{\mathcal{E}}, \bar{\Theta})(t, x)$  to the Euler equations:

$$(2.102) \quad \begin{aligned} & |(V, U, \mathcal{E}, \Theta)(t, x) - (\bar{V}, \bar{U}, \bar{\mathcal{E}}, \bar{\Theta})(t, x)| \\ & \leq C \left[ |(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1}, \Theta^{R_1})(t, x) - (v^{r_1}, u^{r_1}, E^{r_1}, \theta^{r_1})(t, x)| + |(d_1, d_2, d_3)(t, x)| \right. \\ & \quad + |(V^{CD}, U^{CD}, \mathcal{E}^{CD}, \Theta^{CD})(t, x) - (v^{cd}, u^{cd}, E^{cd}, \theta^{cd})(t, x)| \\ & \quad + |(V^{S_3}, U^{S_3}, \mathcal{E}^{S_3}, \Theta^{S_3})(t, x) - (v^{s_3}, u^{s_3}, E^{s_3}, \theta^{s_3})(t, x)| \\ & \quad \left. + |(b_1, b_{21}, b_{22}, b_{23}, b_3)(t, x)| \right] \\ & \leq C_{h,T} \left[ \frac{1}{t} (\sigma \ln(1+t) + \sigma |\ln \sigma|) + \frac{\varepsilon}{\sigma} + \delta^{CD} e^{-\frac{cx^2}{\varepsilon(1+t)}} + \delta^{S_3} e^{-c \frac{\delta^{S_3} |x-s_3 t|}{\varepsilon}} + \varepsilon^{\frac{3}{4}} \right] \\ & \leq C_{h,T} \left[ \varepsilon^{\frac{1}{5}} |\ln \varepsilon| + \delta^{CD} e^{-\frac{cx^2}{\varepsilon(1+t)}} + \delta^{S_3} e^{-c \frac{\delta^{S_3} |x-s_3 t|}{\varepsilon}} \right]. \end{aligned}$$

Moreover, the approximate wave pattern  $(V, U, \mathcal{E}, \Theta)(t, x)$  satisfies

$$(2.103) \quad \begin{cases} V_t - U_{1x} = 0, \\ U_{1t} + P_x = \frac{4}{3}\varepsilon \left( \frac{\mu(\Theta)U_{1x}}{V} \right)_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi + \bar{Q}_{1x} + Q_{1x}, \\ U_{it} = \varepsilon \left( \frac{\mu(\Theta)U_{ix}}{V} \right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi - \int \xi_1 \xi_i \Pi_{1x}^{S_3} d\xi + \bar{Q}_{ix} + Q_{ix}, \quad i = 2, 3, \\ \mathcal{E}_t + (PU_1)_x = \varepsilon \left( \frac{\kappa(\Theta)\Theta_x}{V} \right)_x + \frac{4}{3}\varepsilon \left( \frac{\mu(\Theta)U_1 U_{1x}}{V} \right)_x + \sum_{i=2}^3 \varepsilon \left( \frac{\mu(\Theta)U_i U_{ix}}{V} \right)_x \\ \quad - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi + \bar{Q}_{4x} + Q_{4x}, \end{cases}$$

where  $P = p(V, \Theta)$  and

$$(2.104) \quad \begin{aligned} Q_1 &= [P - \bar{P} - (\bar{P}_v b_1 + \bar{P}_u \cdot b_2 + \bar{P}_E b_3)] - \frac{4}{3}\varepsilon \left[ \frac{\mu(\Theta)U_{1x}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1x}}{V} \right], \\ &:= Q_{11} + Q_{12}, \\ Q_i &= -\varepsilon \left[ \frac{\mu(\Theta)U_{ix}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{ix}}{V} \right], \quad i = 2, 3, \\ Q_4 &= [PU_1 - \bar{P}\bar{U}_1 - ((\bar{P}\bar{U}_1)_v b_1 + (\bar{P}\bar{U}_1)_u \cdot b_2 + (\bar{P}\bar{U}_1)_E b_3)] \\ &\quad - \varepsilon \left[ \left( \frac{\kappa(\Theta)\Theta_x}{V} - \frac{\kappa(\bar{\Theta})\bar{\Theta}_x}{V} \right) + \frac{4}{3} \left( \frac{\mu(\Theta)U_1 U_{1x}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_1 \bar{U}_{1x}}{V} \right) \right. \\ &\quad \left. + \sum_{i=2}^3 \left( \frac{\mu(\Theta)U_i U_{ix}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_i \bar{U}_{ix}}{V} \right) \right] \\ &:= Q_{41} + Q_{42}. \end{aligned}$$

Straightforward calculation shows that

$$(2.105) \quad (Q_{11}, Q_{41}) = O(1)|\bar{b}|^2.$$

**3. Proof of main result.** With the above preparation, we will give the proof of the main theorem as follows. For this, we will first reformulate the problem in the following subsection. The energy estimates will then be given in the second subsection.

**3.1. Reformulation of the problem.** We now reformulate the system by introducing a scaling for the independent variables. Set

$$(3.1) \quad y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}.$$

In the following, we will also use the notation  $(v, u, \theta)(\tau, y)$ ,  $\mathbf{G}(\tau, y, \xi)$ ,  $\Pi_1(\tau, y, \xi)$ , and  $(V, U, \Theta)(\tau, y)$ , etc., in the scaled independent variables. Set the perturbation around the superposition wave  $(V, U, \Theta)(\tau, y)$  by

$$(3.2) \quad \begin{aligned} (\phi, \psi, \omega, \zeta)(\tau, y) &= (v - V, u - U, E - \mathcal{E}, \theta - \Theta)(\tau, y), \\ \tilde{\mathbf{G}}(\tau, y, \xi) &= \mathbf{G}(\tau, y, \xi) - \mathbf{G}^{S_3}(\tau, y, \xi), \\ \tilde{f}(\tau, y, \xi) &= f(\tau, y, \xi) - F^{S_3}(\tau, y, \xi). \end{aligned}$$

Under this scaling, the hydrodynamic limit problem is reduced to a time asymptotic stability problem for the Boltzmann equation.

In particular, we can choose the initial value as

$$(3.3) \quad (\phi, \psi, \omega) \left( \tau = \frac{h}{\varepsilon}, y \right) = (0, 0, 0), \quad \tilde{\mathbf{G}} \left( \tau = \frac{h}{\varepsilon}, y, \xi \right) = 0.$$

Introduce the antiderivative variables

$$(\Phi, \Psi, \bar{W})(\tau, y) = \int_{-\infty}^y (\phi, \psi, \omega)(\tau, y') dy'.$$

Then  $(\Phi, \Psi, \bar{W})(\tau, y)$  satisfies that

$$(3.4) \quad \begin{cases} \Phi_\tau - \Psi_{1y} = 0, \\ \Psi_{1\tau} + (p - P) = \frac{4}{3} \left( \frac{\mu(\theta)u_{1y}}{v} - \frac{\mu(\Theta)U_{1y}}{V} \right) - \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi - \bar{Q}_1 - Q_1, \\ \Psi_{i\tau} = \left( \frac{\mu(\theta)u_{iy}}{v} - \frac{\mu(\Theta)U_{iy}}{V} \right) - \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi - \bar{Q}_i - Q_i, \quad i = 2, 3, \\ \bar{W}_\tau + (pu_1 - PU_1) = \left( \frac{\kappa(\theta)\theta_y}{v} - \frac{\kappa(\Theta)\Theta_y}{V} \right) + \frac{4}{3} \left( \frac{\mu(\theta)u_1 u_{1y}}{v} - \frac{\mu(\Theta)U_1 U_{1y}}{V} \right) \\ + \sum_{i=2}^3 \left( \frac{\mu(\theta)u_i u_{iy}}{v} - \frac{\mu(\Theta)U_i U_{iy}}{V} \right) - \int \xi_1 \frac{|\xi|^2}{2} (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi - \bar{Q}_4 - Q_4. \end{cases}$$

To precisely capture the dissipation of heat conduction, we introduce another variable related to the absolute temperature,

$$W = \bar{W} - U \cdot \Psi = \bar{W} - \sum_{i=1}^3 U_i \Psi_i,$$

then

$$(3.5) \quad \zeta = W_y - \left( \frac{|\Psi_y|^2}{2} - U_y \cdot \Psi \right).$$

Linearizing the system (3.4) around the approximate wave pattern  $(V, U, \Theta)(\tau, y)$  implies that

$$(3.6) \quad \begin{cases} \Phi_\tau - \Psi_{1y} = 0, \\ \Psi_{1\tau} - \frac{Z}{V} \Phi_y + \frac{2}{3V} W_y + \frac{2}{3V} U_y \cdot \Psi - \frac{4}{3} \frac{\mu'(\Theta)}{V} (W_y + U_y \cdot \Psi) U_{1y} = \frac{4}{3} \frac{\mu(\Theta)}{V} \Psi_{1yy} \\ - \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi + N_1 - \bar{Q}_1 - Q_1, \\ \Psi_{i\tau} + \frac{\mu(\Theta)U_{iy}}{V^2} \Phi_y - \frac{\mu'(\Theta)}{V} (W_y + U_y \cdot \Psi) U_{iy} = \frac{\mu(\Theta)}{V} \Psi_{iyy} \\ - \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi + N_i - \bar{Q}_i - Q_i, \quad i = 2, 3, \\ W_\tau + Z \Psi_{1y} - \sum_{i=2}^3 \frac{\mu(\Theta)U_{iy}}{V} \Psi_{iy} + U_\tau \cdot \Psi - \frac{\kappa(\Theta)}{V} (U_y \cdot \Psi)_y + \frac{\kappa(\Theta)}{V^2} \Theta_y \Phi_y \\ - \frac{\kappa'(\Theta)}{V} (W_y + U_y \cdot \Psi) \Theta_y = \frac{\kappa(\Theta)}{V} W_{yy} - \int \xi_1 \frac{|\xi|^2}{2} (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \\ + \sum_{i=1}^3 U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi + N_4 - \bar{Q}_4 + \sum_{i=1}^3 U_i \bar{Q}_i - Q_4 + \sum_{i=1}^3 U_i Q_i, \end{cases}$$

where

$$(3.7) \quad Z = P - \frac{4}{3} \frac{\mu(\Theta)U_{1y}}{V},$$

$$(3.8) \quad N_1 = \frac{p-P}{V} \Phi_y + \frac{1}{3V} |\Psi_y|^2 + \frac{4}{3} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} \right) \Psi_{1yy} \\ + \frac{4}{3} U_{1y} \left[ \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} + \frac{\mu(\Theta)}{V^2} \Phi_y - \frac{\mu'(\Theta)}{V} \left( \zeta + \frac{|\Psi_y|^2}{2} \right) \right] \\ = O(1) [|\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{1yy}|^2],$$

$$(3.9) \quad N_i = \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} \right) \Psi_{iyy} + U_{iy} \left[ \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} + \frac{\mu(\Theta)}{V^2} \Phi_y - \frac{\mu'(\Theta)}{V} \left( \zeta + \frac{|\Psi_y|^2}{2} \right) \right] \\ = O(1) [|\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{iyy}|^2], \quad i = 2, 3,$$

and

$$(3.10) \quad N_4 = -(p-P)\Psi_{1y} - \frac{\kappa(\Theta)}{V} \Psi_y \cdot \Psi_{yy} + \left( \frac{\kappa(\theta)}{v} - \frac{\kappa(\Theta)}{V} \right) \zeta_y \\ + \frac{4}{3} \left( \frac{\mu(\theta)u_{1y}}{v} - \frac{\mu(\Theta)U_{1y}}{V} \right) \Psi_{1y} + \sum_{i=2}^3 \left( \frac{\mu(\theta)u_{iy}}{v} - \frac{\mu(\Theta)U_{iy}}{V} \right) \Psi_{iy} \\ + \Theta_y \left[ \frac{\kappa(\theta)}{v} - \frac{\kappa(\Theta)}{V} + \frac{\kappa(\Theta)}{V^2} \Phi_y - \frac{\kappa'(\Theta)}{V} \left( \zeta + \frac{|\Psi_y|^2}{2} \right) \right] \\ = O(1) [|\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{yy}|^2 + |\zeta_y|^2].$$

We now derive the equation for the nonfluid component  $\tilde{\mathbf{G}}(\tau, y, \xi)$  in the scaled independent variables. From (1.18), we have

$$(3.11) \quad \tilde{\mathbf{G}}_\tau - \mathbf{L}_M \tilde{\mathbf{G}} = \frac{u_1}{v} \tilde{\mathbf{G}}_y - \frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) - \left[ \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y) - \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_y^{S_3}) \right] \\ + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_1,$$

where

$$(3.12) \quad J_1 = (\mathbf{L}_M - \mathbf{L}_{M^{S_3}}) \mathbf{G}^{S_3} + \left( \frac{u}{v} - \frac{U_1^{S_3}}{V^{S_3}} \right) \mathbf{G}_y^{S_3} - \left[ \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_y^{S_3}) - \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_y^{S_3}) \right].$$

Let

$$(3.13) \quad \mathbf{G}^{R_1}(\tau, y, \xi) = \frac{3}{2v\theta} \mathbf{L}_M^{-1} \left\{ \mathbf{P}_1 \left[ \xi_1 \left( \frac{|\xi - u|^2}{2\theta} \Theta_y^{R_1} + \xi \cdot U_y^{R_1} \right) \mathbf{M} \right] \right\}$$

and

$$(3.14) \quad \tilde{\mathbf{G}}_1(\tau, y, \xi) = \tilde{\mathbf{G}}(\tau, y, \xi) - \mathbf{G}^{R_1}(\tau, y, \xi) - \mathbf{G}^{CD}(\tau, y, \xi),$$

where  $\mathbf{G}^{CD}(\tau, y, \xi)$  is defined in (2.20).

Then  $\mathbf{G}_1(\tau, y, \xi)$  satisfies

(3.15)

$$\tilde{\mathbf{G}}_{1\tau} - \mathbf{L}_M \tilde{\mathbf{G}}_1 = \frac{u_1}{v} \tilde{\mathbf{G}}_y - \frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_1 + J_2 - \mathbf{G}_\tau^{R_1} - \mathbf{G}_\tau^{CD}$$

with

(3.16)

$$\begin{aligned} J_2 &= - \left[ \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y) - \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_y^{S_3}) \right. \\ &\quad \left. - \frac{3}{2v\theta} \mathbf{P}_1 \left( \xi_1 \left( \frac{|\xi - u|^2}{2\theta} (\Theta_y^{R_1} + \Theta_y^{CD}) + \xi \cdot (U_y^{R_1} + U_y^{CD}) \right) \mathbf{M} \right) \right] \\ &= - \left[ \frac{3}{2v\theta} \mathbf{P}_1 \left( \xi_1 \left[ \frac{|\xi - u|^2}{2\theta} (\theta - \Theta^{R_1} - \Theta^{CD})_y + \xi \cdot (u - U^{R_1} - U^{CD})_y \right] \mathbf{M} \right) \right. \\ &\quad \left. - \frac{3}{2V^{S_3} \Theta^{S_3}} \mathbf{P}_1^{S_3} \left( \xi_1 \left[ \frac{|\xi - U^{S_3}|^2}{2\Theta^{S_3}} \Theta_y^{S_3} + \xi \cdot U_y^{S_3} \right] \mathbf{M}^{S_3} \right) \right]. \end{aligned}$$

Notice that in (3.14) and (3.15),  $\mathbf{G}^{R_1}$  and  $\mathbf{G}^{CD}$  are subtracted from  $\tilde{\mathbf{G}}$  when carrying out the lower order energy estimates because  $\int_{\frac{h}{\varepsilon}}^{\frac{T}{\varepsilon}} \|(\Theta_y^{R_1}, U_{1y}^{R_1})\|_{L^2(dy)}^2 d\tau$  is uniformly bounded with respect to  $\varepsilon$ , while  $\int_{\frac{h}{\varepsilon}}^{\frac{T}{\varepsilon}} \|\Theta_y^{CD}\|_{L^2(dy)}^2 d\tau$  is only of the order of  $\varepsilon^{-\frac{1}{2}}$ . Both do not give any decay with respect to Knudsen number  $\varepsilon$  in the above integrals.

From (1.15) and the scaling transformation (3.1), we have

(3.17) 
$$f_\tau - \frac{u_1}{v} f_y + \frac{\xi_1}{v} f_y = Q(f, f).$$

Thus, we have the equation for  $\tilde{f}$  defined in (3.2),

(3.18) 
$$\tilde{f}_\tau - \frac{u_1}{v} \tilde{f}_y + \frac{\xi_1}{v} \tilde{f}_y = \mathbf{L}_M \tilde{\mathbf{G}} + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_F,$$

with

(3.19)

$$J_F = \left( \frac{u_1}{v} - \frac{U_1^{S_3}}{V^{S_3}} \right) F_y^{S_3} - \left( \frac{1}{v} - \frac{1}{V^{S_3}} \right) \xi_1 F_y^{S_3} + 2Q(\mathbf{M} - \mathbf{M}^{S_3}, \mathbf{G}^{S_3}) + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}).$$

The estimation on the fluid and nonfluid components governed by the above equations will be given in the next subsection. In the following, we will state the main estimate we want to obtain and also give the a priori estimate.

Note that to prove the main theorem in this paper, it is sufficient to prove the following theorem on the Boltzmann equation (3.17) in the scaled independent variables based on the construction of the approximate wave pattern.

**THEOREM 3.1.** *There exist small positive constants  $\delta_1$  and a global Maxwellian  $\mathbf{M}_\star = \mathbf{M}_{[v_\star, u_\star, \theta_\star]}$  such that if the wave strength  $\delta$  satisfies  $\delta \leq \delta_1$ , then on the time interval  $[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]$  for any  $0 < h < T$ , there is a positive constant  $\varepsilon_1(\delta, h, T)$ . If the Knudsen number  $\varepsilon \leq \varepsilon_1$ , then the problem (3.17) admits a family of smooth solution  $f^{\varepsilon, h}(\tau, y, \xi)$  satisfying*

(3.20) 
$$\sup_{\tau \in [\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]} \sup_{y \in \mathbf{R}} \|f^{\varepsilon, h}(\tau, y, \xi) - \mathbf{M}_{[v, u, \theta]}(\tau, y, \xi)\|_{L_\xi^2(\frac{1}{\sqrt{\mathbf{M}_\star})}} \leq C\varepsilon^{\frac{1}{5}}.$$

Consider the reformulated system (3.6) and (3.15). Since the local existence of solution to (3.6) and (3.15) is known (cf. [19] and [38]), to prove the existence on the time interval  $[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]$ , we only need to close the following a priori estimate by the continuity argument. Set

$$\begin{aligned}
 (3.21) \quad \mathcal{N}(\tau) &= \sup_{\frac{h}{\varepsilon} \leq \tau' \leq \tau} \left\{ \|(\Phi, \Psi, W)(\tau', \cdot)\|^2 + \|(\phi, \psi, \zeta)(\tau', \cdot)\|_1^2 + \iint \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy \right. \\
 &\quad \left. + \sum_{|\alpha'|=1} \iint \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy + \sum_{|\alpha|=2} \iint \frac{|\partial^\alpha \tilde{f}|^2}{\mathbf{M}_\star} d\xi dy \right\} \\
 &\leq \chi^2 = \varepsilon^{\frac{1}{10}} \quad \forall \tau \in \left[ \frac{h}{\varepsilon}, \frac{T}{\varepsilon} \right],
 \end{aligned}$$

where  $\partial^\alpha, \partial^{\alpha'}$  denote the derivatives with respect to  $y$  and  $\tau$  and  $\mathbf{M}_\star$  is a global Maxwellian to be chosen.

*Remark 5.* In the paper, we simply choose the initial data for the Boltzmann equation (3.17) as

$$(3.22) \quad f^{\varepsilon, h} \left( \tau = \frac{h}{\varepsilon}, y, \xi \right) = \mathbf{M}_{[V, U, \Theta]} \left( \frac{h}{\varepsilon}, y, \xi \right) + \mathbf{G}^{S_3} \left( \frac{h}{\varepsilon}, y, \xi \right)$$

so that

$$(3.23) \quad \mathcal{N}(\tau)|_{\tau=\frac{h}{\varepsilon}} \leq C_{h, T} \varepsilon^{\frac{1}{2}}.$$

In this case, the functional measuring the perturbation  $\mathcal{N}(\tau)$  is smaller at the initial time  $\tau = \frac{h}{\varepsilon}$  than the estimate given in Theorem 3.1 in the whole time interval when  $\varepsilon$  is small.

Note that the a priori assumption (3.21) implies that

$$(3.24) \quad \|(\Phi, \Psi, W)\|_{L^\infty}^2 + \|(\phi, \psi, \zeta, \omega)\|_{L^\infty}^2 \leq C\chi^2$$

and

$$\begin{aligned}
 (3.25) \quad \left\| \int \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi \right\|_{L^\infty_y} &\leq C \left( \iint \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left( \iint \frac{|\tilde{\mathbf{G}}_{1y}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \\
 &\leq C \left( \iint \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left( \iint \frac{|\tilde{\mathbf{G}}_y|^2 + |\mathbf{G}_y^{R_1}|^2 + |\mathbf{G}_y^{CD}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \\
 &\leq C\chi [\chi + \|(v_y, u_y, \theta_y)(\Theta_y^{R_1}, U_{1y}^{R_1}, \Theta_y^{CD}, U_y^{CD})\|_{L^2(dy)} \\
 &\quad + \|(\Theta_{yy}^{R_1}, U_{1yy}^{R_1}, \Theta_{yy}^{CD}, U_{yy}^{CD})\|_{L^2(dy)}] \leq C_{h, T} \chi^2.
 \end{aligned}$$

Furthermore, for  $|\alpha'| = 1$ ,

$$\begin{aligned}
 (3.26) \quad \left\| \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi \right\|_{L^\infty_y} &\leq C \left( \iint \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left( \iint \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}_y|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \\
 &\leq C \left( \iint \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left( \iint \frac{|\partial^{\alpha'} \tilde{f}_y|^2 + |\partial^{\alpha'} (\mathbf{M} - \mathbf{M}^{S_3})_y|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned} &\leq C\chi \left[ \chi + \sum_{|\alpha|=2} \|\partial^\alpha(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})\| \right. \\ &\quad + \|(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3}) \left( \sum_{|\alpha|=2} \partial^\alpha + \sum_{|\alpha'|=1} \partial^{\alpha'} \right) (V^{S_3}, U^{S_3}, \Theta^{S_3})\| \\ &\quad + \sum_{|\alpha'|=1} \|\partial^{\alpha'}(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3}) \cdot \partial^{\alpha'}(V^{S_3}, U^{S_3}, \Theta^{S_3})\| \\ &\quad \left. + \sum_{|\alpha'|=1} \|\partial^{\alpha'}(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})\|_{L^4(dy)}^2 \right] \leq C_{h,T}\chi^2. \end{aligned}$$

From (1.17) and (2.103), we have

$$(3.27) \quad \begin{cases} \phi_\tau - \psi_{1y} = 0, \\ \psi_{1\tau} + (p - P)_y = -\frac{4}{3} \left[ \left( \frac{\mu(\Theta)}{V} U_{1y} \right)_y - \left( \frac{\mu(\Theta^{S_3})}{V^{S_3}} U_{1y}^{S_3} \right)_y \right] \\ \quad + \int \xi_1^2 \Pi_{11y}^{CD} d\xi - \bar{Q}_{1y} - Q_{1y} - \int \xi_1^2 \tilde{\mathbf{G}}_y d\xi, \\ \psi_{i\tau} = - \left( \frac{\mu(\Theta)}{V} U_{iy} \right)_y + \int \xi_1 \xi_i \Pi_{11y}^{CD} d\xi - \bar{Q}_{iy} - Q_{iy} - \int \xi_1 \xi_i \tilde{\mathbf{G}}_y d\xi, \quad i = 2, 3, \\ \omega_\tau + (pu_1 - PU_1)_y = - \left[ \left( \frac{\kappa(\Theta)}{V} \Theta_y \right)_y - \left( \frac{\kappa(\Theta^{S_3})}{V^{S_3}} \Theta_y^{S_3} \right)_y \right] - \frac{4}{3} \left[ \left( \frac{\mu(\Theta)}{V} U_1 U_{1y} \right)_y \right. \\ \quad \left. - \left( \frac{\mu(\Theta^{S_3})}{V^{S_3}} U_1^{S_3} U_{1y}^{S_3} \right)_y \right] - \sum_{i=2}^3 \left( \frac{\mu(\Theta)}{V} U_i U_{iy} \right)_y + \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11y}^{CD} d\xi \\ \quad - \bar{Q}_{4y} - Q_{4y} - \frac{1}{2} \int \xi_1 |\xi|^2 \tilde{\mathbf{G}}_y d\xi. \end{cases}$$

And from (2.62), we get

$$(3.28) \quad \begin{aligned} \bar{Q}_{1y} &= \varepsilon [\bar{P} - P^{R_1} - P^{CD} - P^{S_3} - (p_v^{R_1} d_1 + p_{u_1}^{R_1} d_2 + p_E^{R_1} d_3)]_x \\ &\quad + \varepsilon^2 \left( \frac{\mu(\bar{\Theta}) \bar{U}_{1x}}{V} - \frac{\mu(\Theta^{R_1}) U_{1x}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD}) U_{1x}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3}) U_{1x}^{S_3}}{V^{S_3}} \right)_x \\ &= O(1)\varepsilon \left[ |(V_x^{R_1}, U_x^{R_1}, \mathcal{E}_x^{R_1}, d_{ix})(|d_i|^2, V^{S_3} - v_*, U^{S_3} - u_*, \mathcal{E}^{S_3} - E_*)| + |d_i d_{ix}| \right] \\ &\quad + O(1)\varepsilon^2 \left[ |(d_{2xx}, d_{1x} d_{2x}, V_x^{R_1} d_{2x}, U_x^{R_1} d_{1x})| + |(U_{xx}^{R_1}, U_x^{R_1} V_x^{R_1})(d_1, V^{S_3} - v_*)| \right. \\ &\quad \left. + |(U_{xx}^{S_3}, U_x^{S_3} V_x^{S_3})(V^{R_1} - v_*, d_1)| + |(U_x^{S_3}, V_x^{S_3})(U_x^{R_1}, V_x^{R_1}, d_{1x}, d_{2x})| \right] \\ &:= \bar{Q}_{13} + \bar{Q}_{14}, \end{aligned}$$

where  $\bar{Q}_{13}$  represents the wave interaction satisfying

$$(3.29) \quad \begin{aligned} \bar{Q}_{13} &= O(1)\varepsilon \left[ |(V_x^{R_1}, U_x^{R_1}, \mathcal{E}_x^{R_1}, d_{ix})(V^{S_3} - v_*, U^{S_3} - u_*, \mathcal{E}^{S_3} - E_*)| \right] \\ &\quad + O(1)\varepsilon^2 \left[ |(U_{xx}^{R_1}, U_x^{R_1} V_x^{R_1})(d_1, V^{S_3} - v_*)| + |(U_{xx}^{S_3}, U_x^{S_3} V_x^{S_3})(V^{R_1} - v_*, d_1)| \right. \\ &\quad \left. + |(U_x^{S_3}, V_x^{S_3})(U_x^{R_1}, V_x^{R_1}, d_{1x}, d_{2x})| \right] \leq C_{h,T} e^{-\frac{c|x|}{\sigma}} e^{-\frac{c}{\sigma} t}, \end{aligned}$$

and  $\bar{Q}_{14}$  represents the terms related to the hyperbolic waves  $d_i$  ( $i = 1, 2, 3$ ) satisfying

$$(3.30) \quad \begin{aligned} \bar{Q}_{14} &= O(1)\varepsilon \left[ |(V_x^{R_1}, U_x^{R_1}, \mathcal{E}_x^{R_1}, d_{ix})| |d_i|^2 + |d_i d_{ix}| \right] \\ &\quad + O(1)\varepsilon^2 \left[ |(d_{2xx}, d_{1x}d_{2x}, V_x^{R_1}d_{2x}, U_x^{R_1}d_{1x})| \right]. \end{aligned}$$

Then we have

$$(3.31) \quad \int_{\frac{h}{\varepsilon}}^\tau \int |\bar{Q}_{1y}|^2 dy d\tau \leq C \int_{\frac{h}{\varepsilon}}^\tau \int (|\bar{Q}_{13}|^2 + |\bar{Q}_{14}|^2) dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}.$$

Similar estimates hold for  $\bar{Q}_{iy}$  ( $i = 2, 3, 4$ ).

By (2.104), straightforward calculation gives

$$(3.32) \quad \begin{aligned} Q_{1y} &= \varepsilon [P - \bar{P} - (\bar{p}_v b_1 + \bar{p}_u \cdot b_2 + \bar{p}_E b_3)]_x - \frac{4}{3} \varepsilon^2 \left( \frac{\mu(\Theta)U_{1x}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1x}}{V} \right)_x \\ &= O(1)\varepsilon \sum_{i=1}^3 \left[ |b_i|^2 |(\bar{V}_x, \bar{U}_{1x}, \bar{\mathcal{E}}_x)| + |b_i| |b_{ix}| \right] \\ &\quad + O(1)\varepsilon^2 \sum_{i=1}^3 \left[ |b_{ixx}| + |b_{ix}|^2 + |b_i| |(\bar{U}_{xx}, \bar{V}_x \bar{U}_x, \bar{V}_x \bar{\Theta}_x)| + |b_{ix}| |(\bar{V}_x, \bar{U}_x, \bar{\Theta}_x)| \right]. \end{aligned}$$

Hence,

$$(3.33) \quad \int_{\frac{h}{\varepsilon}}^\tau \int |Q_{1y}|^2 dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}.$$

Similar estimates hold for  $Q_{iy}$  ( $i = 2, 3, 4$ ).

Thus from the system (3.27), we have

$$(3.34) \quad \|(\phi_\tau, \psi_\tau, \omega_\tau)\|^2 \leq C_{h,T} \chi^2$$

and

$$(3.35) \quad \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 \leq C \|(\phi_\tau, \psi_\tau, \omega_\tau, U_\tau \cdot \psi)\|^2 \leq C_{h,T} \chi^2.$$

Now we want to obtain the estimates on  $\|\partial^\alpha(\phi, \psi, \zeta)\|^2$  for  $|\alpha| = 2$ . For brevity, we only calculate  $|\partial_{yy}(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})|$ ; the others can be estimated similarly. From (1.3) and (2.51), we have

$$(3.36) \quad \begin{aligned} &|\partial_{yy}(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})| \\ &= O(1) |(V_{yy}^{S_3}, U_{yy}^{S_3}, \Theta_{yy}^{S_3}, (V_y^{S_3})^2, (U_y^{S_3})^2, (\Theta_y^{S_3})^2)| \cdot |(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})| \\ &\quad + O(1) |(V_y^{S_3}, U_y^{S_3}, \Theta_y^{S_3})| \cdot |(v_y - V_y^{S_3}, u_y - U_y^{S_3}, \theta_y - \Theta_y^{S_3})| \\ &\quad + O(1) |(v_y - V_y^{S_3}, u_y - U_y^{S_3}, \theta_y - \Theta_y^{S_3})|^2 + O(1) \sum_{i=0}^4 \int |\varphi_i(\xi)| \cdot |\partial_{yy} \tilde{f}| d\xi. \end{aligned}$$

Therefore, we have

$$(3.37) \quad \begin{aligned} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 &\leq C \left[ \|\partial^\alpha(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})\|^2 + \|\partial^\alpha(V^{R_1}, U^{R_1}, \Theta^{R_1})\|^2 \right. \\ &\quad \left. + \|\partial^\alpha(V^{CD}, U^{CD}, \Theta^{CD})\|^2 + \|\partial^\alpha(d_1, d_2, d_3)\|^2 \right. \\ &\quad \left. + \|\partial^\alpha(b_1, b_{21}, b_{22}, b_{23}, b_3)\|^2 \right] + C \int \int \frac{|\partial^\alpha \tilde{f}|^2}{\mathbf{M}_*} d\xi dy \leq C_{h,T} \chi^2. \end{aligned}$$



Finally, by noticing the fact that  $f = \mathbf{M} + \mathbf{G}$  and  $F^{S_3} = \mathbf{M}^{S_3} + \mathbf{G}^{S_3}$ , (3.37) with  $|\alpha| = 2$  yields

$$(3.38) \quad \int \int \frac{|\partial^\alpha \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy \leq C \int \int \frac{|\partial^\alpha \tilde{f}|^2}{\mathbf{M}_\star} d\xi dy + C \int \int \frac{|\partial^\alpha (\mathbf{M} - \mathbf{M}^{S_3})|^2}{\mathbf{M}_\star} d\xi dy \leq C_{h,T} \chi^2,$$

where in the last inequality we have used a similar argument used for (3.26).

Before closing the a priori estimate (3.21), we list some basic lemmas based on the celebrated H-theorem for later use. The first lemma is from [15].

LEMMA 3.2. *There exists a positive constant C such that*

$$\int \frac{\nu(|\xi|)^{-1} Q(f, g)^2}{\tilde{\mathbf{M}}} d\xi \leq C \left\{ \int \frac{\nu(|\xi|) f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{g^2}{\tilde{\mathbf{M}}} d\xi + \int \frac{f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{\nu(|\xi|) g^2}{\tilde{\mathbf{M}}} d\xi \right\},$$

where  $\tilde{\mathbf{M}}$  can be any Maxwellian so that the above integrals are well-defined.

Based on Lemma 3.2, the following three lemmas are taken from [29]. Their proofs are straightforward by using the Cauchy inequality.

LEMMA 3.3. *If  $\theta/2 < \theta_\star < \theta$ , then there exist two positive constants  $\tilde{\sigma} = \tilde{\sigma}(v, u, \theta; v_\star, u_\star, \theta_\star)$  and  $\eta_0 = \eta_0(v, u, \theta; v_\star, u_\star, \theta_\star)$  such that if  $|v - v_\star| + |u - u_\star| + |\theta - \theta_\star| < \eta_0$ , we have for  $g(\xi) \in \mathfrak{N}^\perp$ ,*

$$- \int \frac{g \mathbf{L}_\mathbf{M} g}{\mathbf{M}_\star} d\xi \geq \tilde{\sigma} \int \frac{\nu(|\xi|) g^2}{\mathbf{M}_\star} d\xi.$$

LEMMA 3.4. *Under the assumptions in Lemma 3.3, we have for each  $g(\xi) \in \mathfrak{N}^\perp$ ,*

$$\begin{aligned} \int \frac{\nu(|\xi|)}{\mathbf{M}} |\mathbf{L}_\mathbf{M}^{-1} g|^2 d\xi &\leq \tilde{\sigma}^{-2} \int \frac{\nu(|\xi|)^{-1} g^2}{\mathbf{M}} d\xi \quad \text{and} \\ \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{L}_\mathbf{M}^{-1} g|^2 d\xi &\leq \tilde{\sigma}^{-2} \int \frac{\nu(|\xi|)^{-1} g^2}{\mathbf{M}_\star} d\xi. \end{aligned}$$

LEMMA 3.5. *Under the conditions in Lemma 3.3, for any positive constants k and  $\lambda$ , it holds that*

$$\left| \int \frac{g_1 \mathbf{P}_1(|\xi|^k g_2)}{\mathbf{M}_\star} d\xi - \int \frac{g_1 |\xi|^k g_2}{\mathbf{M}_\star} d\xi \right| \leq C_{k,\lambda} \int \frac{\lambda |g_1|^2 + \lambda^{-1} |g_2|^2}{\mathbf{M}_\star} d\xi,$$

where the constant  $C_{k,\lambda}$  depends on k and  $\lambda$ .

**3.2. Energy estimates.** To close the a priori estimate (3.21) and to prove Theorem 3.1, we need the following energy estimates given in Propositions 3.6 and 3.7. First, the lower order estimates to the system (3.6) and (3.15) are given in the following proposition.

PROPOSITION 3.6. *Under the assumptions of Theorem 3.1, there exist positive constants C and  $C_{h,T}$  independent of  $\varepsilon$  such that*

$$\begin{aligned} &\sup_{\frac{h}{\varepsilon} \leq \tau_1 \leq \tau} \left[ \|(\Phi, \Psi, W, \Phi_y)(\tau_1, \cdot)\|^2 + \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau_1, y, \xi) d\xi dy \right] \\ &+ \int_{\frac{h}{\varepsilon}}^\tau \left[ \left\| \sqrt{|U_{1y}^{S_3}|} (\Psi, W) \right\|^2 + \|(\Phi_y, \Psi_y, W_y, \zeta, \Psi_\tau, W_\tau)\|^2 \right] d\tau \\ &+ \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \end{aligned}$$

$$\begin{aligned} &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau \\ &\quad + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{2}{5}}. \end{aligned}$$

For brevity of the presentation, we also put the proof of Proposition 3.6 in the appendices.

Then we perform the higher order estimates. First, we apply  $\partial_y$  to system (3.6) to get the following system for  $(\phi, \psi, \zeta)$ :

$$(3.39) \quad \left\{ \begin{aligned} &\phi_{\tau} - \psi_{1y} = 0, \\ &\psi_{1\tau} - \frac{Z}{V} \phi_y + \frac{2}{3V} \zeta_y + H_1 = \left( \frac{4\mu(\Theta)}{3V} \psi_{1y} \right)_y - \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \\ &\quad + N_5 - \bar{Q}_{1y} - Q_{1y}, \\ &\psi_{i\tau} + H_i = \left( \frac{\mu(\Theta)}{V} \psi_{iy} \right)_y - \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi + N_{i+4} \\ &\quad - \bar{Q}_{iy} - Q_{iy}, \quad i = 2, 3, \\ &\zeta_{\tau} + Z\psi_{1y} + H_4 = \left( \frac{\kappa(\Theta)}{V} \zeta_y \right)_y - \int \xi_1 \frac{|\xi|^2}{2} (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \\ &\quad + \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi + \sum_{i=1}^3 U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \\ &\quad + N_8 - \bar{Q}_{4y} - Q_{4y} + \sum_{i=1}^3 U_i (\bar{Q}_{iy} + Q_{iy}), \end{aligned} \right.$$

where  $Z$  is defined in (3.7). Here, the linear terms are

$$(3.40) \quad H_1 = -\frac{4\mu'(\Theta)}{3V} U_{1y} \zeta_y - \left( \frac{Z}{V} \right)_y \phi + \left( \frac{2}{3V} \right)_y \zeta - \left( \frac{4\mu'(\Theta)}{3V} U_{1y} \right)_y \zeta,$$

$$(3.41) \quad H_i = \frac{\mu(\Theta) U_{iy}}{V^2} \phi_y - \frac{\mu'(\Theta)}{V} U_{iy} \zeta_y - \left( \frac{\mu(\Theta) U_{iy}}{V^2} \right)_y \phi - \left( \frac{\mu'(\Theta)}{V} U_{iy} \right)_y \zeta, \quad i = 2, 3,$$

$$(3.42) \quad \begin{aligned} H_4 = &-\frac{U_{1y}}{V} \left( Z\phi - \frac{2}{3}\zeta + \frac{4\mu'(\Theta)U_{1y}}{3}\zeta + \frac{4\mu(\Theta)}{3}\psi_{1y} \right) + \left( \frac{\kappa(\Theta)}{V^2} \Theta_y \phi \right)_y \\ &- \sum_{i=2}^3 \frac{U_{iy}}{V} \left( -\frac{\mu(\Theta)U_{iy}}{V} \phi + \mu'(\Theta)U_{iy}\zeta + 2\mu(\Theta)\psi_{iy} \right) - \left( \frac{\kappa'(\Theta)}{V} \Theta_y \zeta \right)_y, \end{aligned}$$

and the nonlinear terms are

$$(3.43) \quad N_5 = \left[ \frac{p-P}{V} \phi + \frac{4}{3} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} \right) \psi_{1y} + \frac{4}{3} U_{1y} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} + \frac{\mu(\Theta)}{V^2} \phi - \frac{\mu'(\Theta)}{V} \zeta \right) \right]_y$$

$$= O(1)[|(\phi, \psi, \zeta)|^2 + |(\phi_y, \psi_{1y}, \zeta_y)|^2 + |\psi_{1yy}|^2],$$

$$(3.44) \quad N_{i+4} = \left[ \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} \right) \psi_{iy} + U_{iy} \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} + \frac{\mu(\Theta)}{V^2} \phi - \frac{\mu'(\Theta)}{V} \zeta \right) \right]_y$$

$$= O(1)[|(\phi, \psi, \zeta)|^2 + |(\phi_y, \psi_{iy}, \zeta_y)|^2 + |\psi_{iyy}|^2], \quad i = 2, 3,$$

and

$$(3.45) \quad N_8 = -(p-P)\psi_{1y} + \frac{U_{1y}}{V}(p-P)\phi + 2 \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} \right) \left( \frac{4}{3} U_{1y} \psi_{1y} + \sum_{i=2}^3 U_{iy} \psi_{iy} \right) + \left( \frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} + \frac{\mu(\Theta)}{V^2} \phi - \frac{\mu'(\Theta)}{V} \zeta \right) \left( \frac{4}{3} U_{1y}^2 + \sum_{i=2}^3 U_{iy}^2 \right) + \frac{4\mu(\theta)}{3v} \psi_{1y}^2 + \sum_{i=2}^3 \frac{\mu(\theta)}{v} \psi_{iy}^2 + \left[ \left( \frac{\kappa(\theta)}{v} - \frac{\kappa(\Theta)}{V} \right) \zeta_y + \Theta_y \left( \frac{\kappa(\theta)}{v} - \frac{\kappa(\Theta)}{V} + \frac{\kappa(\Theta)}{V^2} \phi - \frac{\kappa'(\Theta)}{V} \zeta \right) \right]_y$$

$$= O(1)[|(\phi, \psi, \zeta)|^2 + |(\phi_y, \psi_y, \zeta_y)|^2 + |\zeta_{yy}|^2].$$

To derive the estimate on the higher order derivatives, applying  $\partial_y$  to the system

(3.39) gives

$$(3.46) \quad \left\{ \begin{aligned} &\phi_{y\tau} - \psi_{1yy} = 0, \\ &\psi_{1y\tau} - \frac{Z}{V} \phi_{yy} + \frac{2}{3V} \zeta_{yy} + H_5 = \left( \frac{4}{3} \frac{\mu(\Theta)}{V} \psi_{1y} \right)_{yy} \\ &\quad - \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_{yy} d\xi + N_{5y} - \bar{Q}_{1yy} - Q_{1yy}, \\ &\psi_{iy\tau} + H_{i+4} = \left( \frac{\mu(\Theta)}{V} \psi_{iy} \right)_{yy} - \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_{yy} d\xi \\ &\quad + (N_{i+4})_y - \bar{Q}_{iyy} - Q_{iyy}, \quad i = 2, 3, \\ &\zeta_{y\tau} + Z\psi_{1yy} + H_8 = \left( \frac{\kappa(\Theta)}{V} \zeta_y \right)_{yy} - \int \xi_1 \frac{|\xi|^2}{2} (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_{yy} d\xi \\ &\quad + \sum_{i=1}^3 \left[ \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi \right]_y + \sum_{i=1}^3 \left[ U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right]_y \\ &\quad + N_{8y} - \bar{Q}_{4yy} - Q_{4yy} + \sum_{i=1}^3 \left[ U_i (\bar{Q}_{iy} + Q_{iy}) \right]_y, \end{aligned} \right.$$

where

$$(3.47) \quad H_5 = -\frac{4}{3} \frac{\mu'(\Theta)}{V} U_{1y} \zeta_{yy} - 2 \left( \frac{Z}{V} \right)_y \phi_y - \left( \frac{Z}{V} \right)_{yy} \phi + \left( \frac{4}{3V} \right)_y \zeta_y + \left( \frac{2}{3V} \right)_{yy} \zeta - \left( \frac{8\mu'(\Theta)}{3V} U_{1y} \right)_y \zeta_y - \left( \frac{4\mu'(\Theta)}{3V} U_{1y} \right)_{yy} \zeta,$$

(3.48)

$$H_{i+4} = \frac{\mu(\Theta)U_{iy}}{V^2}\phi_{yy} - \frac{\mu'(\Theta)}{V}U_{iy}\zeta_{yy} + 2\left(\frac{\mu(\Theta)U_{iy}}{V^2}\right)_y\phi_y + \left(\frac{\mu(\Theta)U_{iy}}{V^2}\right)_{yy}\phi - 2\left(\frac{\mu'(\Theta)}{V}U_{iy}\right)_y\zeta_y - \left(\frac{\mu'(\Theta)}{V}U_{iy}\right)_{yy}\zeta, \quad i = 2, 3,$$

and

(3.49)

$$H_8 = Z_y\psi_{1y} - \left[\frac{U_{1y}}{V}\left(Z\phi - \frac{2}{3}\zeta + \frac{4\mu'(\Theta)U_{1y}}{3}\zeta + \frac{4\mu(\Theta)}{3}\psi_{1y}\right)\right]_y + \left(\frac{\kappa(\Theta)}{V^2}\Theta_y\phi\right)_{yy} - \sum_{i=2}^3 \left[\frac{U_{iy}}{V}\left(-\frac{\mu(\Theta)U_{iy}}{V}\phi + \mu'(\Theta)U_{iy}\zeta + 2\mu(\Theta)\psi_{iy}\right)\right]_y - \left(\frac{\kappa'(\Theta)}{V}\Theta_y\zeta\right)_{yy}.$$

By using the above two systems and the equation for the nonfluid component, we can reach the following proposition for the higher order energy estimates.

**PROPOSITION 3.7.** *Under the assumptions of Theorem 3.1, there exist positive constants  $C$  and  $C_{h,T}$  independent of  $\varepsilon$  such that*

$$\begin{aligned} & \sup_{\frac{h}{\varepsilon} \leq \tau_1 \leq \tau} \left[ \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)(\tau_1, \cdot)\|^2 \right. \\ & \quad \left. + \sum_{|\alpha|=1} \int \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star}(\tau_1, y, \xi) d\xi dy + \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha \tilde{f}|^2}{2\mathbf{M}_\star}(\tau_1, y, \xi) d\xi dy \right] \\ & \quad + \int_{\frac{h}{\varepsilon}}^{\tau} \sum_{1 \leq |\alpha| \leq 2} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 d\tau + \sum_{1 \leq |\alpha| \leq 2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\ & \leq C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\ & \quad + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Again, the proof of Proposition 3.7 will be given in the appendices.

By combining the above lower and higher order estimates given in Propositions 3.6 and 3.7 and choosing the wave strength  $\delta$ , the bound on the a priori estimate  $\chi$ , and the Knudsen number  $\varepsilon$  to be suitably small, we obtain

$$\begin{aligned} \mathcal{N}(\tau) & + \int_{\frac{h}{\varepsilon}}^{\tau} \left[ \sum_{0 \leq |\alpha| \leq 2} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 + \|\sqrt{|U_{1y}^{S_3}|}(\Psi, W)\|^2 \right] d\tau \\ & \quad + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy d\tau \\ & \quad + \sum_{1 \leq |\alpha| \leq 2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\partial^\alpha \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy d\tau \leq C_{h,T} \varepsilon^{\frac{2}{5}}. \end{aligned}$$

Therefore, we close the a priori assumption (3.21) and then complete the proof of Theorem 3.1.

**4. Appendices.** As mentioned, since the proofs of Propositions 3.1 and 3.2 are technical and long, we put them in the following three subsections.

**4.1. Proof of Proposition 3.6.** Proof of Proposition 3.1. Firstly, from the fact that

$$v_- < V^{R_1} < v_*, \quad V^{CD} \in (\min\{v_*, v^*\}, \max\{v_*, v^*\}) \quad \text{and} \quad v^* < V^{S_3} < v_+,$$

we have

$$v_- - \delta^{CD} - \|(d_1, b_1)\|_{L^\infty} \leq V \leq v_+ + \delta^{CD} + \|(d_1, b_1)\|_{L^\infty}.$$

Thus

$$(4.1) \quad \frac{v_-}{2} \leq V \leq 2v_+ \quad \text{if} \quad \varepsilon \ll 1 \quad \text{and} \quad \delta^{CD} \ll 1.$$

From (2.57)<sub>1</sub> and (2.57)<sub>2</sub>, we can obtain

$$(4.2) \quad Z^{S_3} := P^{S_3} - \frac{4}{3} \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}}{V^{S_3}} = (a_1 - s_3^2 V^{S_3}) - \int \xi_1^2 \Pi_1^{S_3} d\xi,$$

where  $a_1 = p_+ + s_3^2 v_+ = p^* + s_3^2 v^*$ .

Since

$$(4.3) \quad 0 < p_+ \leq (a_1 - s_3^2 V^{S_3}) \leq p^*, \quad \text{and} \quad \left| \int \xi_1^2 \Pi_1^{S_3} d\xi \right| \leq C\delta^{S_3},$$

we have

$$p_+ - C\delta^{S_3} \leq Z^{S_3} \leq p^* + C\delta^{S_3},$$

and then

$$(4.4) \quad \frac{p_+}{2} \leq Z^{S_3} \leq 2p^* \quad \text{if} \quad \delta^{S_3} \ll 1.$$

Now we estimate  $Z$  defined in (3.7) as follows:

$$(4.5) \quad \begin{aligned} Z &= Z^{S_3} + (P^{R_1} - p_*) - \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} + (P^{CD} - p^*) - \frac{4}{3} \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} \\ &\quad + (P - P^{R_1} - P^{CD} - P^{S_3} + p_* + p^*) \\ &\quad - \frac{4}{3} \left[ \frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}}{V^{S_3}} \right] \\ &:= Z^{S_3} + (P^{R_1} - p_*) - \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} + (P^{CD} - p^*) - \frac{4}{3} \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} + Q_5, \end{aligned}$$

where  $Q_5$  is related to the hyperbolic waves and the wave interaction terms given by

$$(4.6) \quad \begin{aligned} Q_5 &= O(1) \sum_{i=1}^3 |(d_i, b_i)| + O(1)|(d_{2y}, b_{2y})| + O(1)e^{-\frac{C_h}{\sigma}} \\ &= O(1) \left( \frac{\varepsilon}{\sigma} + \varepsilon^{\frac{3}{4}} + \frac{\varepsilon^2}{\sigma^2} + e^{-\frac{C_h}{\sigma}} \right) \end{aligned}$$

with  $\sigma = \varepsilon^{\frac{1}{3}}$ .

From the properties of the approximate rarefaction wave, we have

$$(4.7) \quad 0 \leq P^{R_1} - p_* \leq p_- - p_* \quad \text{and} \quad \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} \leq C_{h,T} \varepsilon.$$

The properties of the viscous contact wave imply that

$$(4.8) \quad P^{CD} - p^* \leq C_{h,T} \varepsilon \quad \text{and} \quad \frac{4\mu(\Theta^{CD})U_{1y}^{CD}}{3V^{CD}} \leq C_{h,T} \varepsilon^{\frac{3}{2}}.$$

Substituting (4.2), (4.6), (4.7), and (4.8) into (4.5), we know that there exist positive constants  $c$  and  $C$  such that

$$(4.9) \quad 0 < c \leq Z \leq C,$$

provided  $\varepsilon \ll 1$  and the wave strength  $\delta \ll 1$ .

*Step 1. Estimation on  $\|(\Phi, \Psi, W)(\tau, \cdot)\|^2$ .* By multiplying (3.6)<sub>1</sub> by  $\Phi$ , (3.6)<sub>2</sub> by  $\frac{V}{Z}\Psi_1$ , (3.6)<sub>3</sub> by  $V^{S_3}\Psi_i$ , (3.6)<sub>4</sub> by  $\frac{2}{3Z^2}W$ , respectively, and adding all of them together, we have

$$(4.10) \quad \begin{aligned} I_1(\Phi, \Psi, W)_\tau + I_2(\Psi, W) + I_3(\Psi_y, W_y) &= I_4(\Psi, W, \Phi_y, \Psi_y, W_y) \\ &+ \frac{V\Psi_1}{Z}(N_1 - \bar{Q}_1 - Q_1) + V^{S_3} \sum_{i=2}^3 (N_i - \bar{Q}_i - Q_i)\Psi_i \\ &+ \frac{2W}{3Z^2} \left( N_4 - \bar{Q}_4 + \sum_{i=1}^3 U_i \bar{Q}_i - Q_4 + \sum_{i=1}^3 U_i Q_i \right) + K_1 + (\cdots)_y, \end{aligned}$$

where

$$(4.11) \quad \begin{aligned} I_1(\Phi, \Psi, W) &= \frac{\Phi^2}{2} + \frac{V}{2Z}\Psi_1^2 + \frac{V^{S_3}}{2} \sum_{i=2}^3 \Psi_i^2 + \frac{W^2}{3Z^2}, \\ I_2(\Psi, W) &= \left[ \frac{2}{3Z}U_{1y} - \frac{1}{2}\left(\frac{V}{Z}\right)_\tau \right] \Psi_1^2 - V_\tau^{S_3} \sum_{i=2}^3 \frac{\Psi_i^2}{2} - \left(\frac{1}{3Z^2}\right)_\tau W^2, \\ I_3(\Psi_y, W_y) &= \frac{4\mu(\Theta)}{3Z}\Psi_{1y}^2 + \frac{V^{S_3}\mu(\Theta)}{V} \sum_{i=2}^3 \Psi_{iy}^2 + \frac{2\kappa(\Theta)}{3Z^2V}W_y^2, \end{aligned}$$

(4.12)

$$\begin{aligned} I_4(\Psi, W, \Phi_y, \Psi_y, W_y) &= -\frac{2}{3Z^2}W\Psi_1(U_{1\tau} + Z_y) - \frac{2W}{3Z^2} \sum_{i=2}^3 U_{i\tau}\Psi_i \\ &- \left[ \frac{2\Psi_1}{3Z} + \frac{\mu(\Theta)V^{S_3}}{V^2}\Phi_y - \frac{\mu'(\Theta)V^{S_3}}{V}(W_y + U_y \cdot \Psi) \right] \sum_{i=2}^3 U_{iy}\Psi_i \\ &- \frac{4}{3} \left( \frac{\mu(\Theta)}{Z} \right)_y \Psi_1\Psi_{1y} - \sum_{i=2}^3 \left( \frac{V^{S_3}\mu(\Theta)}{V} \right)_y \Psi_i\Psi_{iy} - \frac{2}{3} \left( \frac{\kappa(\Theta)}{VZ^2} \right)_y WW_y \\ &+ \frac{4}{3} \frac{\mu'(\Theta)}{Z}U_{1y}\Psi_1(W_y + U_y \cdot \Psi) - \frac{2\kappa(\Theta)}{3V^2Z^2}\Theta_y\Phi_yW + \frac{2\kappa(\Theta)}{3VZ^2}W(U_y \cdot \Psi)_y \\ &+ \frac{2\kappa'(\Theta)}{3VZ^2}\Theta_yW(W_y + U_y \cdot \Psi) - \sum_{i=2}^3 \frac{2\mu(\Theta)U_{iy}}{3Z^2V}W\Psi_{iy} := \sum_{i=1}^{11} I_4^i, \end{aligned}$$

and  $K_1$  denotes the nonfluid parts given by

$$(4.13) \quad \begin{aligned} K_1 &= -\frac{V}{Z}\Psi_1 \int \xi_1^2(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi - \sum_{i=2}^3 V^{S_3}\Psi_i \int \xi_1\xi_i(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi \\ &+ \frac{2W}{3Z^2} \left[ -\int \frac{1}{2}\xi_1|\xi|^2(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi + \sum_{i=1}^3 U_i \int \xi_1\xi_i(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi \right]. \end{aligned}$$

From (4.9), we have

$$(4.14) \quad c(\Phi, \Psi, W)^2 \leq I_1 \leq C(\Phi, \Psi, W)^2$$

for some positive constants  $c$  and  $C$ .

Now we estimate  $I_i$ , ( $i = 2, 3, 4$ ) term by term. Note that

$$(4.15) \quad Z_\tau^{S_3} = -s_3^2 U_{1y}^{S_3} + s_3 \int \xi_1^2 \Pi_{1y}^{S_3} d\xi$$

and

$$(4.16) \quad Z_\tau = -s_3^2 U_{1y}^{S_3} + s_3 \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + Z_\tau^{CD} + Z_\tau^{R_1} + Q_6,$$

where  $Z^{R_1} = P^{R_1} - \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}}$ ,  $Z^{CD} = P^{CD} - \frac{4}{3} \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}}$ , and

$$(4.17) \quad \begin{aligned} Q_6 &= (P - P^{R_1} - P^{CD} - P^{S_3})_\tau \\ &\quad - \frac{4}{3} \left[ \frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}}{V^{S_3}} \right]_\tau \\ &= \frac{2}{3} \left( \frac{\Theta_\tau}{V} - \frac{\Theta_\tau^{R_1}}{V^{R_1}} - \frac{\Theta_\tau^{CD}}{V^{CD}} - \frac{\Theta_\tau^{S_3}}{V^{S_3}} \right) \\ &\quad - \frac{2}{3} \left( \frac{\Theta V_\tau}{V^2} - \frac{\Theta^{R_1} V_\tau^{R_1}}{(V^{R_1})^2} - \frac{\Theta^{CD} V_\tau^{CD}}{(V^{CD})^2} - \frac{\Theta^{S_3} V_\tau^{S_3}}{(V^{S_3})^2} \right) \\ &\quad - \frac{4}{3} \left( \frac{\mu(\Theta)U_{1y\tau}}{V} - \frac{\mu(\Theta^{R_1})U_{1y\tau}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1y\tau}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1y\tau}^{S_3}}{V^{S_3}} \right) \\ &\quad + \frac{4}{3} \left[ \frac{\mu(\Theta)U_{1y}V_\tau}{V^2} - \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}V_\tau^{R_1}}{(V^{R_1})^2} - \frac{\mu(\Theta^{CD})U_{1y}^{CD}V_\tau^{CD}}{(V^{CD})^2} - \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}V_\tau^{S_3}}{(V^{S_3})^2} \right] \\ &\quad - \frac{4}{3} \left[ \frac{\mu'(\Theta)U_{1y}\Theta_\tau}{V} - \frac{\mu'(\Theta^{R_1})U_{1y}^{R_1}\Theta_\tau^{R_1}}{V^{R_1}} \right. \\ &\quad \left. - \frac{\mu'(\Theta^{CD})U_{1y}^{CD}\Theta_\tau^{CD}}{V^{CD}} - \frac{\mu'(\Theta^{S_3})U_{1y}^{S_3}\Theta_\tau^{S_3}}{V^{S_3}} \right] := \sum_{i=1}^5 Q_{6i}. \end{aligned}$$

Then we get for the first term of  $I_2$  that

$$(4.18) \quad \begin{aligned} \frac{2}{3Z} U_{1y} - \frac{1}{2} \left( \frac{V}{Z} \right)_\tau &= \frac{1}{6Z} U_{1y} + \frac{V}{2Z^2} Z_\tau \\ &= \frac{U_{1y}^{S_3}}{6Z^2} (Z^{S_3} - 3s_3^2 V^{S_3}) + \frac{U_{1y}^{S_3}}{6Z^2} [(Z - Z^{S_3}) - 3s_3^2 (V - V^{S_3})] + \frac{s_3 V}{2Z^2} \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \\ &\quad + \frac{1}{6Z} (U_{1y}^{R_1} + U_{1y}^{CD} + d_{2y} + b_{21y}) + \frac{V}{2Z^2} \left( Z_\tau^{CD} + Z_\tau^{R_1} + \sum_{i=1}^5 Q_{6i} \right) \\ &= \frac{|U_{1y}^{S_3}|}{6Z^2} \left[ -(a_1 - s_3^2 V^{S_3}) + 3s_3^2 V^{S_3} + \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \right] + \frac{s_3 V}{2Z^2} \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + Q_7, \end{aligned}$$

where

$$(4.19) \quad Q_7 = \frac{U_{1y}^{S_3}}{6Z^2} [(Z - Z^{S_3}) - 3s_3^2(V - V^{S_3})] + \frac{1}{6Z} (U_{1y}^{R_1} + U_{1y}^{CD} + d_{2y} + b_{21y}) \\ + \frac{V}{2Z^2} \left( Z_\tau^{CD} + Z_\tau^{R_1} + \sum_{i=1}^5 Q_{6i} \right).$$

First, straightforward calculation gives

$$-(a_1 - s_3^2 V^{S_3}) + 3s_3^2 V^{S_3} \geq 4p_* - C\delta^{S_3} \geq 3p_* \quad \text{if } \delta^{S_3} \ll 1,$$

$$\left| \int \xi_1^2 \Pi_1^{S_3} d\xi \right| \leq C\delta^{S_3} \quad \text{and} \quad \left| \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \right| \leq C\delta^{S_3} |U_{1y}^{S_3}|.$$

And (4.18) implies that

$$(4.20) \quad \left[ \frac{2}{3Z} U_{1y} - \frac{1}{2} \left( \frac{V}{Z} \right)_\tau \right] \Psi_1^2 \geq C^{-1} |U_{1y}^{S_3}| \Psi_1^2 + Q_7 \Psi_1^2,$$

provided  $\delta \ll 1$ .

By the definition of the approximate wave pattern defined in (2.100) and (2.101), we have

$$(4.21) \quad Q_{61} = O(1)\varepsilon [ |(\Theta_t^{R_1}, U_{1t}^{R_1})| |(V^{CD} - v_*, U_1^{CD} - u_{1*}, V^{S_3} - v^*, U_1^{S_3} - u_1^*)| \\ + |(\Theta_t^{R_1} d_1, U_{1t}^{R_1} d_2)| \\ + |(\Theta_t^{CD}, U_{1t}^{CD})| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, V^{S_3} - v^*, U_1^{S_3} - u_1^*, d_1, d_2)| \\ + |(\Theta_t^{S_3}, U_{1t}^{S_3})| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, V^{CD} - v^*, U_1^{CD} - u_1^*, d_1, d_2)| \\ + |(d_{2t}, d_{3t})| + |b_1| |(\Theta_t^{R_1}, \Theta_t^{CD}, \Theta_t^{S_3})| ] + \frac{2}{3V} \left( b_3 - \bar{U} \cdot b_2 - \frac{|b_2|^2}{2} \right)_\tau \\ \leq C_{h,T} \varepsilon + C|(b_2, b_{2\tau}, b_{3\tau})|.$$

On the other hand, direct calculation gives

$$(4.22) \quad Q_{63} = O(1)\varepsilon^2 [ |U_{1xt}^{R_1}| |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \Theta^{CD} \\ - \theta_*, V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| \\ + |U_{1xt}^{CD}| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, V^{S_3} \\ - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*, d_1, d_2, d_3)| \\ + |U_{1xt}^{S_3}| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, V^{CD} \\ - v^*, U_1^{CD} - u_1^*, \Theta^{CD} - \theta^*, d_1, d_2, d_3)| \\ + |U_{1xt}^{R_1}| |(d_1, d_2, d_3)| + |(b_1, b_2, b_3)| |(U_{1xt}^{R_1}, U_{1xt}^{CD}, U_{1xt}^{S_3})| \\ + |d_{2xt}| ] - \frac{4\mu(\Theta)}{3V} b_{21y\tau} \\ \leq C_{h,T} \varepsilon + C|b_{21y\tau}|.$$

Similar estimates hold for  $Q_{62}$ ,  $Q_{64}$ , and  $Q_{65}$ .



Moreover, we have

$$(4.23) \quad |Z_\tau^{R_1}| = \left| \left( P^{R_1} - \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} \right)_\tau \right| \leq C_{h,T} \varepsilon$$

and

$$(4.24) \quad |Z_\tau^{CD}| = \left| \left( P^{CD} - \frac{4}{3} \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} \right)_\tau \right| \leq C_{h,T} \varepsilon^2.$$

Now from (4.5), (4.19), and (4.21)–(4.24), we have

$$(4.25) \quad \begin{aligned} \int_{\frac{h}{\varepsilon}}^\tau \int |Q_\tau| \Psi_1^2 dy d\tau &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int |(b, b_y, b_\tau, b_{y\tau})| |\Psi_1|^2 dy d\tau \\ &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \|(b, b_y, b_\tau, b_{y\tau})\| \|\Psi_1\| \|\Psi_1\|_{L^\infty} d\tau \\ &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau + C_{h,T} \varepsilon^{\frac{3}{4}} \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_1\|^{\frac{3}{2}} \|\Psi_{1y}\|^{\frac{1}{2}} d\tau \\ &\leq \beta \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_{1y}\|^2 d\tau + C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau. \end{aligned}$$

Substituting (4.25) into (4.20) gives that

$$(4.26) \quad \begin{aligned} \int_{\frac{h}{\varepsilon}}^\tau \int \left[ \frac{2}{3Z} U_{1y} - \frac{1}{2} \left( \frac{V}{Z} \right)_\tau \right] \Psi_1^2 dy d\tau \\ \geq C^{-1} \int_{\frac{h}{\varepsilon}}^\tau \int |U_{1y}^{S_3}| \Psi_1^2 dy d\tau - \beta \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_{1y}\|^2 d\tau - C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau, \end{aligned}$$

provided  $\delta^{S_3} \ll 1$ .

On the other hand, we have

$$-V_\tau^{S_3} \sum_{i=2}^3 \frac{\Psi_i^2}{2} = \sum_{i=2}^3 |U_{1y}^{S_3}| \frac{\Psi_i^2}{2}.$$

Similar to (4.26), we have

$$\begin{aligned} \int_{\frac{h}{\varepsilon}}^\tau \int \left[ - \left( \frac{1}{3Z^2} \right)_\tau W^2 \right] dy d\tau &= \int_{\frac{h}{\varepsilon}}^\tau \int \frac{2}{3Z^3} Z_\tau W^2 dy d\tau \\ &= \int_{\frac{h}{\varepsilon}}^\tau \int \frac{2}{3Z^3} \left( -s_3^2 U_{1y}^{S_3} + s_3 \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + Z_\tau^{CD} + Z_\tau^{R_1} + Q_6 \right) W^2 dy d\tau \\ &\geq C^{-1} \int_{\frac{h}{\varepsilon}}^\tau \int |U_{1y}^{S_3}| W^2 dy d\tau - \beta \int_{\frac{h}{\varepsilon}}^\tau \|W_y\|^2 d\tau - C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int W^2 dy d\tau. \end{aligned}$$

Thus we complete the estimation for  $I_2$  as

$$(4.27) \quad \begin{aligned} \int_{\frac{h}{\varepsilon}}^\tau \int I_2(\Psi, W) dy d\tau &\geq C^{-1} \int_{\frac{h}{\varepsilon}}^\tau \int |U_{1y}^{S_3}| |(\Psi, W)|^2 dy d\tau \\ &\quad - \beta \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_{1y}, W_y)\|^2 d\tau - C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int |(\Psi, W)|^2 dy d\tau. \end{aligned}$$

Direct computation yields that

$$(4.28) \quad I_3(\Psi_y, W_y) \geq C^{-1} |(\Psi_y, W_y)|^2.$$

Now we estimate  $I_4$  in (4.12). Note that

$$U_{1\tau} + Z_y = - \int \xi_1^2 \Pi_{11y}^{CD} d\xi - \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + \bar{Q}_{1y} + Q_{1y}.$$

Thus

$$\begin{aligned} I_4^1 &= \frac{2}{3Z^2} W \Psi_1 \left[ \int \xi_1^2 \Pi_{11y}^{CD} d\xi + \int \xi_1^2 \Pi_{1y}^{S_3} d\xi - \bar{Q}_{1y} - Q_{1y} \right] \\ &= \frac{2}{3Z^2} W \Psi_1 \left( \int \xi_1^2 \Pi_{11y}^{CD} d\xi + \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \right) + \left( \frac{2W\Psi_1}{3Z^2} \right)_y (\bar{Q}_1 + Q_1) + (\cdots)_y \\ &\leq \beta |(\Psi_{1y}, W_y)|^2 + C_\beta \delta^{S_3} |U_{1y}^{S_3}| |(\Psi_1, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi_1, W)|^2 + (\cdots)_y, \end{aligned}$$

where from now on,  $\beta$  is a small positive constant to be determined and  $C_\beta$  is some positive constant depending on  $\beta$  but independent of  $h, T$ , and  $\varepsilon$ , while  $C_{h,T,\beta}$  depends on  $h, T, \beta$  but is independent of  $\varepsilon$ .

Similarly, we have

$$\begin{aligned} I_4^2 &= \sum_{i=2}^3 \frac{2W\Psi_i}{3Z^2} \left[ - \left( \frac{\mu(\Theta)U_{iy}}{V} \right)_y + \int \xi_1 \xi_i \Pi_{11y}^{CD} d\xi + \int \xi_1 \xi_i \Pi_{1y}^{S_3} d\xi - \bar{Q}_{iy} - Q_{iy} \right] \\ &= \sum_{i=2}^3 \frac{2W\Psi_i}{3Z^2} \left( \int \xi_1 \xi_i \Pi_{11y}^{CD} d\xi + \int \xi_1 \xi_i \Pi_{1y}^{S_3} d\xi \right) \\ &\quad + \sum_{i=2}^3 \left( \frac{2W\Psi_i}{3Z^2} \right)_y \left( \frac{\mu(\Theta)U_{iy}}{V} + \bar{Q}_i + Q_i \right) + (\cdots)_y \\ &\leq \beta \sum_{i=2}^3 |(\Psi_{iy}, W_y)|^2 + C_\beta \delta^{S_3} \sum_{i=2}^3 |U_{1y}^{S_3}| |(\Psi_i, W)|^2 \\ &\quad + C_{h,T,\beta} \varepsilon \sum_{i=2}^3 |(\Psi_i, W)|^2 + (\cdots)_y. \end{aligned}$$

Next, we can show that

$$(4.29) \quad \begin{aligned} I_4^3 &= \sum_{i=2}^3 U_i \left( \frac{2\Psi_1\Psi_i}{3Z} \right)_y - \left[ \frac{\mu(\Theta)V^{S_3}}{V^2} \Phi_y - \frac{\mu'(\Theta)V^{S_3}}{V} (W_y + U_y \cdot \Psi) \right] \sum_{i=2}^3 U_{iy} \Psi_i + (\cdots)_y \\ &\leq \beta |(\Phi_y, \Psi_y, W_y)|^2 + C_\beta \delta^{S_3} |U_{1y}^{S_3}| |(\Psi, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi, W)|^2 + (\cdots)_y, \end{aligned}$$

where we have used the fact that  $U_i = \bar{U}_i + b_{i+1} = U_i^{CD} + b_{i+1} = O(1)\varepsilon^{\frac{1}{2}}$ , ( $i = 2, 3$ ) and that  $U_{iy} = U_{iy}^{CD} + (b_{i+1})_y = O(1)\varepsilon^{\frac{3}{4}}$ ,  $i = 2, 3$ .

By the Hölder inequality, we have

$$(4.30) \quad \sum_{i=4}^{11} I_4^i \leq \beta |(\Phi_y, \Psi_y, W_y)|^2 + C_\beta \delta^{S_3} |U_{1y}^{S_3}| |(\Psi, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi, W)|^2.$$

Hence, we have

$$(4.31) \quad \begin{aligned} I_4(\Psi, W, \Phi_y, \Psi_y, W_y) &\leq \beta |(\Phi_y, \Psi_y, W_y)|^2 + (C_\beta \delta^{S_3} \\ &\quad + C_{h,T} \varepsilon) |U_{1y}^{S_3}| |(\Psi, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi, W)|^2 + (\cdots)_y. \end{aligned}$$

Now we estimate the nonlinear terms  $\frac{V}{Z}N_1\Psi_1$  by

$$(4.32) \quad \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z}N_1\Psi_1 dyd\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left[ |\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{1yy}|^2 \right] dyd\tau \\ \leq C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta, \Psi_{1yy})\|^2 d\tau.$$

Similarly, we have

$$\int_{\frac{h}{\varepsilon}}^{\tau} \int \sum_{i=2,3} V^{S_3}N_i\Psi_i + \frac{2W}{3Z^2}N_4dyd\tau \leq C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta, \Psi_{yy}, \zeta_y)\|^2 d\tau.$$

We now turn to estimate the terms  $-\frac{V}{Z}\Psi_1\bar{Q}_1$ ,  $-\sum_{i=2}^3 V^{S_3}\bar{Q}_i\Psi_i$  and  $-\frac{2W}{3Z^2}(\bar{Q}_4 - \sum_{i=1}^3 U_i\bar{Q}_i)$ . From (2.62) and (2.64), we have

$$(4.33) \quad \left| -\frac{V}{Z}\Psi_1\bar{Q}_1 \right|, \left| -\sum_{i=2}^3 V^{S_3}\bar{Q}_i\Psi_i \right|, \left| -\frac{2W}{3Z^2} \left( \bar{Q}_4 - \sum_{i=1}^3 U_i\bar{Q}_i \right) \right| \\ \leq C|(\Psi, W)|\|(\bar{Q}_{11}, \bar{Q}_2, \bar{Q}_3, \bar{Q}_{41})\| + C|(\Psi, W)|\|(\bar{Q}_{12}, \bar{Q}_{42})\|.$$

From (2.65), we have

$$(4.34) \quad \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)|\|(\bar{Q}_{11}, \bar{Q}_2, \bar{Q}_3, \bar{Q}_{41})\| dyd\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)| e^{-\frac{c|x|}{\sigma}} e^{-\frac{C_h}{\sigma}} dyd\tau \\ \leq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C_{h,T} e^{-\frac{C_h}{2\sigma}}.$$

On the other hand, from (2.62)–(2.64) and Lemma 2.3 and noting that  $\sigma = \varepsilon^{\frac{1}{5}}$ , we have

$$(4.35) \quad \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)|\|(\bar{Q}_{12}, \bar{Q}_{42})\| dyd\tau \\ \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)| \left[ \sum_{i=1}^3 |d_i|^2 + \varepsilon|(d_{2x}, d_{3x})| + \varepsilon|(U_{1x}^{R_1}, \Theta_x^{R_1})|(d_1, d_2, d_3) \right] dyd\tau \\ \leq C_{h,T} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|_{L^2(dy)} \sum_{i=1}^3 [\|d_i\|_{L^\infty} \|d_i\|_{L^2(dy)} + \varepsilon\|d_{ix}\|_{L^2(dy)} + \varepsilon\|d_i\|_{L^2(dy)}] d\tau \\ \leq C_{h,T} \left(\frac{\varepsilon}{\sigma}\right)^{\frac{3}{2}} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|_{L^2(dy)} d\tau \leq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{2}{5}}.$$

Now we estimate the terms  $-\frac{V}{Z}\Psi_1Q_1$ ,  $-\sum_{i=2}^3 V^{S_3}Q_i\Psi_i$ , and  $-\frac{2W}{3Z^2}(Q_4 - \sum_{i=1}^3 U_iQ_i)$ . For this, from (2.104) and (2.105), we have

$$(4.36) \quad \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[ -\frac{V}{Z}\Psi_1Q_{11} - \frac{2W}{3Z^2}(Q_{41} - U_1Q_{11}) \right] dyd\tau \\ \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi_1, W)|\|(Q_{11}, Q_{41})\| dyd\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi_1, W)| \sum_{i=1}^3 |b_i|^2 dyd\tau \\ \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|_{L^2(dy)} \sum_{i=1}^3 \|b_i\|_{L^\infty} \|b_i\|_{L^2(dy)} d\tau \\ \leq C_{h,T} \varepsilon^{\frac{3}{2}} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\| d\tau \leq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|^2 d\tau + C_{h,T} \varepsilon.$$

On the other hand, from Lemma 2.5 with the properties of hyperbolic wave II, we can show that

$$\begin{aligned}
 (4.37) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[ -\frac{V}{Z} \Psi_1 Q_{12} - \sum_{i=2}^3 V^{S_3} Q_i \Psi_i - \frac{2W}{3Z^2} \left( Q_{42} - U_1 Q_{12} - \sum_{i=2}^3 U_i Q_i \right) \right] dy d\tau \\
 &= \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[ \left( \frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left( \frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1y}}{\bar{V}} \right) \right. \\
 &\quad + \sum_{i=2}^3 \left( V^{S_3}\Psi_i - \frac{2U_iW}{3Z^2} \right) \left( \frac{\mu(\Theta)U_{iy}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{iy}}{\bar{V}} \right) \\
 &\quad + \frac{2W}{3Z^2} \left( \frac{\mu(\Theta)\Theta_y}{V} - \frac{\mu(\bar{\Theta})\bar{\Theta}_y}{\bar{V}} \right) + \frac{8W}{9Z^2} \left( \frac{\mu(\Theta)U_1U_{1y}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1y}}{\bar{V}} \right) \\
 &\quad \left. + \frac{2W}{3Z^2} \sum_{i=2}^3 \left( \frac{\mu(\Theta)U_iU_{iy}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_i\bar{U}_{iy}}{\bar{V}} \right) \right] dy d\tau := \sum_{i=1}^5 I_{5i}.
 \end{aligned}$$

For brevity, we only compute  $I_{51}$  in the following because  $I_{5i}$  ( $i = 2, 3, 4, 5$ ) can be estimated similarly:

$$\begin{aligned}
 (4.38) \quad I_{51} &= \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left( \frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left( \frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1y}}{\bar{V}} \right) dy d\tau \\
 &= \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left( \frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left[ \frac{\mu(\Theta)}{V} b_{2y} + \left( \frac{\mu(\Theta)}{V} - \frac{\mu(\bar{\Theta})}{\bar{V}} \right) \bar{U}_{1y} \right] dy d\tau \\
 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[ \left( \frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \frac{\mu(\Theta)}{V} \right]_y b_2 dy d\tau \\
 &\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left( \frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left( \frac{\mu(\Theta)}{V} - \frac{\mu(\bar{\Theta})}{\bar{V}} \right) \bar{U}_{1y} dy d\tau := I_{51}^1 + I_{51}^2.
 \end{aligned}$$

We can further have

$$\begin{aligned}
 (4.39) \quad I_{51}^1 &= O(1) \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |b_2| [ |(\Psi_{1y}, W_y)| + |(V_y, U_{1y}, \Theta_y, Z_y)| |(\Psi_1, W)| ] dy d\tau \\
 &\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} [ \|(\Psi_{1y}, W_y)\|_{L^2(dy)}^2 + \|\sqrt{|U_{1y}^{S_3}|} |(\Psi_1, W)|\|_{L^2(dy)}^2 ] d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|_{L^2(dy)}^2 d\tau + C_{\beta} \int_{\frac{h}{\varepsilon}}^{\tau} \|(b_1, b_2, b_3)\|_{L^2(dy)}^2 d\tau \\
 &\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} [ \|(\Psi_{1y}, W_y)\|_{L^2(dy)}^2 + \|\sqrt{|U_{1y}^{S_3}|} |(\Psi_1, W)|\|_{L^2(dy)}^2 ] d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|_{L^2(dy)}^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 (4.40) \quad I_{51}^2 &= O(1) \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi_1, W)| |\bar{U}_{1y}| |(b_1, b_2, b_3)| dy d\tau \\
 &\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \left\| \sqrt{|U_{1y}^{S_3}|} |(\Psi_1, W)| \right\|^2 d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

By integrating (4.10) with respect to  $y$  and  $\tau$ , then combining all the above estimates and choosing  $\beta, \delta^{S_3}, \varepsilon$ , and  $\chi$  small enough, we have

$$\begin{aligned}
 (4.41) \quad & \|(\Phi, \Psi, W)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \left[ \|\sqrt{|U_{1y}^{S_3}|}(\Psi, W)\|^2 + \|(\Psi_y, W_y)\|^2 \right] d\tau \\
 & \leq C(\beta + \chi) \int_{\frac{h}{\varepsilon}}^\tau \|\Phi_y\|^2 d\tau + C\chi \int_{\frac{h}{\varepsilon}}^\tau \|(\zeta, \Psi_{yy}, \zeta_y)\|^2 d\tau \\
 & \quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi, W)\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int K_1 dy d\tau + C_{h,T,\beta} \varepsilon^{\frac{2}{5}}.
 \end{aligned}$$

Now we turn to estimate the microscopic term  $\int_{\frac{h}{\varepsilon}}^\tau \int K_1 dy d\tau$  in (4.41). Here, we will only estimate  $K_{11} =: -\int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z} \Psi_1 \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi dy d\tau$  because the other terms in  $\int_{\frac{h}{\varepsilon}}^\tau \int K_1 dy d\tau$  can be estimated similarly. Let  $\mathbf{M}_\star$  be a global Maxwellian with the state  $(v_\star, u_\star, \theta_\star)$  satisfying  $\frac{1}{2}\theta < \theta_\star < \theta$  and  $|v - v_\star| + |u - u_\star| + |\theta - \theta_\star| \leq \eta_0$  so that Lemma 3.3 holds, and Lemmas 2.3 and 2.4 hold with  $\mathbf{M}_0$  being replaced by  $\mathbf{M}_\star$ . Note that the above choice of the global Maxwellian  $\mathbf{M}_\star$  can be obtained if the total wave strength  $\delta$  is small enough. By the definitions of  $\Pi_1, \Pi_{11}^{CD}$ , and  $\Pi_1^{S_3}$  given in (1.20), (2.21), and (2.58) respectively, we have

$$\begin{aligned}
 (4.42) \quad & \Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3} = \mathbf{L}_M^{-1} [\tilde{\mathbf{G}}_\tau - \frac{u_1}{v} \tilde{\mathbf{G}}_{1y} + \frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y}) - Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)] \\
 & \quad - \mathbf{L}_M^{-1} [2Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})] + J_3 + J_4 + J_5.
 \end{aligned}$$

Here

$$(4.43) \quad \bar{\mathbf{G}}^{CD}(\tau, y, \xi) = \frac{3}{2V_{CD}\Theta_{CD}} \mathbf{L}_{M^{CD}}^{-1} \left\{ \mathbf{P}_1^{CD} \left[ \xi_1 \left( \frac{|\xi - U^{CD}|^2}{2\Theta_{CD}} \Theta_y^{CD} + \xi \cdot U_y^{CD} \right) \mathbf{M}^{CD} \right] \right\},$$

$$\begin{aligned}
 (4.44) \quad & J_3 = (\mathbf{L}_M^{-1} - \mathbf{L}_{M^{S_3}}^{-1}) [\mathbf{G}_\tau^{S_3} - Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})] - \left( \frac{u_1}{v} \mathbf{L}_M^{-1} - \frac{U_1^{S_3}}{V_{S_3}} \mathbf{L}_{M^{S_3}}^{-1} \right) \mathbf{G}_y^{S_3} \\
 & \quad + \left( \frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1 - \frac{1}{V_{S_3}} \mathbf{L}_{M^S}^{-1} \mathbf{P}_1^{S_3} \right) (\xi_1 \mathbf{G}_y^{S_3}) - \mathbf{L}_M^{-1} [2Q(\mathbf{G}^{S_3}, \mathbf{G}^{R_1} + \mathbf{G}^{CD})],
 \end{aligned}$$

$$\begin{aligned}
 (4.45) \quad & J_4 = - \left( \frac{u_1}{v} \mathbf{L}_M^{-1} \mathbf{G}_y^{CD} - \frac{U_1^{CD}}{V_{CD}} \mathbf{L}_{M^{CD}}^{-1} \bar{\mathbf{G}}_y^{CD} \right) \\
 & \quad - [\mathbf{L}_M^{-1} Q(\mathbf{G}^{CD}, \mathbf{G}^{CD}) - \mathbf{L}_{M^{CD}}^{-1} Q(\bar{\mathbf{G}}^{CD}, \bar{\mathbf{G}}^{CD})] \\
 & \quad + \left[ \frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1(\xi_1 \mathbf{G}_y^{CD}) - \frac{1}{V_{CD}} \mathbf{L}_{M^{CD}}^{-1} \mathbf{P}_1^{CD}(\xi_1 \bar{\mathbf{G}}_y^{CD}) \right] \\
 & \quad - \mathbf{L}_M^{-1} [2Q(\mathbf{G}^{R_1}, \mathbf{G}^{CD})],
 \end{aligned}$$

and

$$(4.46) \quad J_5 = \mathbf{L}_M^{-1} \left[ -\frac{u_1}{v} \mathbf{G}_y^{R_1} + \frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1(\xi_1 \mathbf{G}_y^{R_1}) - Q(\mathbf{G}^{R_1}, \mathbf{G}^{R_1}) \right].$$

By (4.42), we have

$$\begin{aligned}
 (4.47) \quad & K_{11} = - \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_\tau) d\xi dy d\tau \\
 & \quad + \int_{\frac{h}{\varepsilon}}^\tau \int \frac{u_1 V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_{1y}) d\xi dy d\tau
 \end{aligned}$$

$$\begin{aligned}
 & - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{vZ} \int \xi_1^2 \mathbf{L}_M^{-1} [\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y})] d\xi dy d\tau \\
 & + \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \mathbf{L}_M^{-1} [Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)] d\xi dy d\tau \\
 & + \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{vZ} \int \xi_1^2 \mathbf{L}_M^{-1} [2Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} \\
 & + \mathbf{G}^{CD} + \mathbf{G}^{S_3})] d\xi dy d\tau - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 J_3 d\xi dy d\tau \\
 & - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 J_4 d\xi dy d\tau - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 J_5 d\xi dy d\tau := \sum_{i=1}^8 K_{11}^i.
 \end{aligned}$$

For the integral  $K_{11}^1$ , we have

$$\begin{aligned}
 (4.48) \quad K_{11}^1 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_{1\tau}) d\xi dy d\tau \\
 & - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}(\mathbf{G}_\tau^{R_1} + \mathbf{G}_\tau^{CD}) d\xi dy d\tau =: K_{11}^{11} + K_{11}^{12}.
 \end{aligned}$$

Note that the linearized operator  $\mathbf{L}_M^{-1}$  satisfies, for any  $h \in \mathfrak{N}^\perp$ ,

$$(4.49) \quad (\mathbf{L}_M^{-1}g)_\varsigma = \mathbf{L}_M^{-1}(g_\varsigma) - 2\mathbf{L}_M^{-1}\{Q(\mathbf{L}_M^{-1}g, M_\varsigma)\} \quad \text{for } \varsigma = \tau, y.$$

Then we have

$$\begin{aligned}
 (4.50) \quad K_{11}^{11} &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \Psi_1 \int \xi_1^2 (\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1)_\tau d\xi dy d\tau \\
 & - 2 \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}\{Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1, M_\tau)\} d\xi dy d\tau \\
 & = - \int \int \left[ \frac{V}{Z} \Psi_1 \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_1) \right] (\tau, y, \xi) d\xi dy \\
 & + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left( \frac{V}{Z} \Psi_1 \right)_\tau \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_1) d\xi dy d\tau \\
 & - 2 \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}\{Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1, \mathbf{M}_\tau)\} d\xi dy d\tau.
 \end{aligned}$$

The Hölder inequality and Lemma 3.4 yield

$$(4.51) \quad \left| \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_1) d\xi \right|^2 \leq C \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi.$$

Moreover, from Lemmas 3.2–3.4, we have

$$\begin{aligned}
 (4.52) \quad & \int \xi_1^2 \mathbf{L}_M^{-1}\{Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1, \mathbf{M}_\tau)\} d\xi \leq C \left( \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{L}_M^{-1}\{Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1, \mathbf{M}_\tau)\}|^2 d\xi \right)^{\frac{1}{2}} \\
 & \leq C \left( \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}} \cdot \left( \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{M}_\tau|^2 d\xi \right)^{\frac{1}{2}} \\
 & \leq C |v_\tau, u_\tau, \theta_\tau| \left( \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}}.
 \end{aligned}$$

Combining (4.50)–(4.52) gives

$$\begin{aligned}
 (4.53) \quad K_{11}^{11} &\leq \beta \left[ \|\Psi(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_{1\tau}, \sqrt{|U_{1y}^{S_3}|}\Psi_1)\|^2 d\tau \right] \\
 &\quad + C_\beta \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_1\|^2 d\tau + C_{h,T}(\varepsilon^{\frac{1}{4}} + \chi) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 d\tau.
 \end{aligned}$$

On the other hand, by (3.13) and (2.20), we have

$$\begin{aligned}
 (4.54) \quad K_{11}^{12} &= - \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}(\mathbf{G}_\tau^{R_1} + \mathbf{G}_\tau^{CD}) d\xi dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int |\Psi_1| |(\Theta_{y\tau}^{R_1}, U_{1y\tau}^{R_1}, \Theta_{y\tau}^{CD}, U_{y\tau}^{CD})| \\
 &\quad + |(\Theta_y^{R_1}, U_{1y}^{R_1}, \Theta_y^{CD}, U_y^{CD})|(v_\tau, u_\tau, \theta_\tau)| dy d\tau \\
 &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_1\|^2 d\tau + C(C_{h,T} \varepsilon + \delta) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}},
 \end{aligned}$$

which, together with (4.53), implies

$$\begin{aligned}
 (4.55) \quad K_{11}^1 &\leq \beta \left[ \|\Psi(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_{1\tau}, \sqrt{|U_{1y}^{S_3}|}\Psi_1)\|^2 d\tau \right] + C_{h,T} \varepsilon^{\frac{1}{2}} \\
 &\quad + C_\beta \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_1\|^2 d\tau + C[\delta + C_{h,T}(\varepsilon^{\frac{1}{4}} + \chi)] \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 d\tau.
 \end{aligned}$$

We now turn to  $K_{11}^2$ . By (4.49), we have

$$\begin{aligned}
 (4.56) \quad K_{11}^2 &= - \int_{\frac{h}{\varepsilon}}^\tau \int \left( \frac{u_1 V \Psi_1}{v Z} \right)_y \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_1) d\xi dy d\tau \\
 &\quad + 2 \int_{\frac{h}{\varepsilon}}^\tau \int \frac{u_1 V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_M^{-1}\{Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1, \mathbf{M}_y)\} d\xi dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int [|\Psi_{1y}| + |\Psi_1|(V_y, Z_y)] \\
 &\quad + |\Psi_1|(v_y, u_y, \theta_y)| \left( \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}} dy d\tau \\
 &\leq \beta \int_{\frac{h}{\varepsilon}}^\tau \left\| \left( \Psi_{1y}, \sqrt{|U_{1y}^{S_3}|}\Psi_1 \right) \right\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_1\|^2 d\tau + C_{h,T}(\varepsilon^{\frac{1}{4}} + \chi) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau.
 \end{aligned}$$

To estimate  $K_{11}^3$ , notice that

$$(4.57) \quad \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y}) = [\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_1)]_y + \sum_{j=0}^4 \langle \xi_1 \tilde{\mathbf{G}}_1, \chi_j \rangle \mathbf{P}_1(\chi_{jy}).$$

Then from (4.49) and (4.57), we have

$$\begin{aligned}
 (4.58) \quad K_{11}^3 &= \int_{\frac{h}{\varepsilon}}^{\tau} \int \left( \frac{V\Psi_1}{vZ} \right)_y \int \xi_1^2 \mathbf{L}_M^{-1} [\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_1)] d\xi dy d\tau \\
 &\quad - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{vZ} \int \xi_1^2 \mathbf{L}_M^{-1} \left[ \sum_{j=0}^4 \langle \xi_1 \tilde{\mathbf{G}}_1, \chi_j \rangle \mathbf{P}_1(\chi_{jy}) \right] d\xi dy d\tau \\
 &\quad - 2 \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{vZ} \int \xi_1^2 \mathbf{L}_M^{-1} \{ Q(\mathbf{L}_M^{-1} [\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_1)], \mathbf{M}_y) \} d\xi dy d\tau \\
 &\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \int (|U_{1y}^{S_3}| |\Psi_1|^2 + |\Psi_{1y}|^2) dy d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C_{h,T} (\varepsilon^{\frac{1}{4}} + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\phi_y, \psi_y, \zeta_y)|^2 dy d\tau.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (4.59) \quad K_{11}^4 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left( \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{L}_M^{-1} \{ Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1) \}|^2 d\xi \right)^{\frac{1}{2}} dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \leq C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau
 \end{aligned}$$

and

$$\begin{aligned}
 (4.60) \quad K_{11}^5 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left( \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}} \\
 &\quad \times \left[ \left( \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{G}^{S_3}|^2 d\xi \right)^{\frac{1}{2}} + |(\Theta_y^{R_1}, \Theta_y^{CD}, U_{1y}^{R_1}, U_y^{CD})| \right] dy d\tau \\
 &\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \int |U_{1y}^{S_3}| |\Psi_1|^2 dy d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau.
 \end{aligned}$$

By (4.44), we have

$$\begin{aligned}
 (4.61) \quad K_{11}^6 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \left\{ (\mathbf{L}_M^{-1} - \mathbf{L}_{M^{S_3}}^{-1}) \mathbf{G}_\tau^{S_3} - (\mathbf{L}_M^{-1} - \mathbf{L}_{M^{S_3}}^{-1}) Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3}) \right. \\
 &\quad - \left( \frac{u_1}{v} \mathbf{L}_M^{-1} - \frac{U_1^{S_3}}{V^{S_3}} \mathbf{L}_{M^{S_3}}^{-1} \right) \mathbf{G}_y^{S_3} + \left( \frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1 - \frac{1}{V^{S_3}} \mathbf{L}_{M^S}^{-1} \mathbf{P}_1^{S_3} \right) (\xi_1 \mathbf{G}_y^{S_3}) \\
 &\quad \left. - \mathbf{L}_M^{-1} [2Q(\mathbf{G}^{S_3}, \mathbf{G}^{R_1} + \mathbf{G}^{CD})] \right\} d\xi dy d\tau := \sum_{i=1}^5 K_{11}^{6i}.
 \end{aligned}$$

For brevity, we will only consider the term  $K_{11}^{61}$  because similar estimates hold for the terms  $K_{11}^{6i}$  ( $i = 2, 3, 4, 5$ ). Direct calculation yields



(4.62)

$$\begin{aligned}
 K_{11}^{61} &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \mathbf{L}_M^{-1} [2Q(\mathbf{M}^{S_3} - \mathbf{M}, \mathbf{L}_{\mathbf{M}^{S_3}}^{-1}(\mathbf{G}_\tau^{S_3}))] d\xi \\
 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left( \int \frac{\nu(|\xi|)|\mathbf{M}^{S_3} - \mathbf{M}|^2}{\mathbf{M}_*} d\xi \right)^{\frac{1}{2}} \left( \int \frac{\nu(|\xi|)|\mathbf{L}_{\mathbf{M}^{S_3}}^{-1}(\mathbf{G}_\tau^{S_3})|^2}{\mathbf{M}_*} d\xi \right)^{\frac{1}{2}} dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| |(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})| \left( \int \frac{\nu^{-1}(|\xi|)|\mathbf{G}_\tau^{S_3}|^2}{\mathbf{M}_*} d\xi \right)^{\frac{1}{2}} dy d\tau \\
 &\leq C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \int |U_{1y}^{S_3}| |\Psi_1|^2 dy d\tau + C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\Phi_y, \Psi_y, \zeta)|^2 dy d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

By (4.45), we have

$$\begin{aligned}
 K_{11}^7 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \left\{ - \left( \frac{u_1}{v} \mathbf{L}_M^{-1} \mathbf{G}_y^{CD} - \frac{U_1^{CD}}{V^{CD}} \mathbf{L}_{M^{CD}}^{-1} \bar{\mathbf{G}}_y^{CD} \right) \right. \\
 &\quad - [\mathbf{L}_M^{-1} Q(\mathbf{G}^{CD}, \mathbf{G}^{CD}) - \mathbf{L}_{M^{CD}}^{-1} Q(\bar{\mathbf{G}}^{CD}, \bar{\mathbf{G}}^{CD})] \\
 &\quad + \left[ \frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1(\xi_1 \mathbf{G}_y^{CD}) - \frac{1}{V^{CD}} \mathbf{L}_{M^{CD}}^{-1} \mathbf{P}_1^{CD}(\xi_1 \bar{\mathbf{G}}_y^{CD}) \right] \\
 &\quad \left. - \mathbf{L}_M^{-1} [2Q(\mathbf{G}^{R_1}, \mathbf{G}^{CD})] \right\} d\xi dy d\tau := \sum_{i=1}^4 K_{11}^{7i}.
 \end{aligned}$$

For illustration, we only consider  $K_{11}^{71}$  as follows:

(4.63)

$$\begin{aligned}
 K_{11}^{71} &= \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \left[ \left( \frac{u_1}{v} - \frac{U_1^{CD}}{V^{CD}} \right) \mathbf{L}_M^{-1} \mathbf{G}_y^{CD} + \frac{U_1^{CD}}{V^{CD}} (\mathbf{L}_M^{-1} - \mathbf{L}_{M^{CD}}^{-1}) \mathbf{G}_y^{CD} \right. \\
 &\quad \left. + \frac{U_1^{CD}}{V^{CD}} \mathbf{L}_{M^{CD}}^{-1} (\mathbf{G}_y^{CD} - \bar{\mathbf{G}}_y^{CD}) \right] d\xi dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| |(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})| \cdot |(\Theta_{yy}^{CD}, U_{yy}^{CD})| \\
 &\quad + |(\Theta_y^{CD}, U_y^{CD})| |(v_y, u_y, \theta_y)| dy d\tau \\
 &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau \\
 &\quad + C_{h,T} \chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

Then, by (4.46), we have

(4.64)

$$\begin{aligned}
 K_{11}^8 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \mathbf{L}_M^{-1} \left[ - \frac{u_1}{v} \mathbf{G}_y^{R_1} + \frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1(\xi_1 \mathbf{G}_y^{R_1}) + Q(\mathbf{G}^{R_1}, \mathbf{G}^{R_1}) \right] d\xi dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left[ |(v_y, u_y, \theta_y)| |(\Theta_y^{R_1}, U_y^{R_1})| + |(\Theta_y^{R_1}, U_y^{R_1})|^2 + |(\Theta_{yy}^{R_1}, U_{yy}^{R_1})| \right] dy d\tau \\
 &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

By collecting all the above estimates, we have

$$\begin{aligned}
 (4.65) \quad K_{11} &\leq \beta \left[ \|\Psi(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_{1\tau}, \sqrt{|U_{1y}^{S_3}|} \Psi_1)\|^2 d\tau \right] + C_\beta \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*}(\tau, y, \xi) d\xi dy \\
 &\quad + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C(\chi + \beta + \delta) \int_{\frac{h}{\varepsilon}}^\tau \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau \\
 &\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_1\|^2 d\tau + C[\delta + C_{h,T}\chi] \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^\tau \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 (4.66) \quad &\|(\Phi, \Psi, W)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \left[ \|\sqrt{|U_{1y}^{S_3}|}(\Psi, W)\|^2 + \|(\Psi_y, W_y)\|^2 \right] d\tau \\
 &\leq C \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*}(\tau, y, \xi) d\xi dy + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi, W)\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_\tau, W_\tau)\|^2 d\tau \\
 &\quad + C(\chi + \beta + \delta) \int_{\frac{h}{\varepsilon}}^\tau \|(\Phi_y, \zeta)\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
 &\quad + C(\delta + C_{h,T}\chi) \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^\tau \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{2}{5}},
 \end{aligned}$$

where we have used the smallness of  $\delta, \beta, \varepsilon,$  and  $\chi$ .

*Step 2. Estimation on  $\|\Phi_y(\tau, \cdot)\|^2$ .* Note that the dissipation term does not contain the term  $\|\Phi_y\|^2$ . To complete the lower order energy estimate, we have to estimate  $\Phi_y$ . From (3.6)<sub>2</sub>, we have

$$\begin{aligned}
 (4.67) \quad &\frac{4}{3} \frac{\mu(\Theta)}{V} \Phi_{y\tau} - \Psi_{1\tau} + \frac{Z}{V} \Phi_y = \frac{2}{3V} W_y + \frac{2}{3V} U_y \cdot \Psi \\
 &\quad - \frac{4}{3} \frac{\mu'(\Theta)}{V} U_{1y}(W_y + U_y \cdot \Psi) - N_1 + \bar{Q}_1 + Q_1 + \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi.
 \end{aligned}$$

Multiplying (4.67) by  $\Phi_y$  yields

$$\begin{aligned}
 (4.68) \quad &\left( \frac{2\mu(\Theta)}{3V} \Phi_y^2 - \Phi_y \Psi_1 \right)_\tau + \frac{Z}{V} \Phi_y^2 = \left( \frac{2\mu(\Theta)}{3V} \right)_\tau \Phi_y^2 + \Psi_{1y}^2 + \left[ \frac{2}{3V} U_y \cdot \Psi + \frac{2}{3V} W_y \right. \\
 &\quad \left. - \frac{4}{3} \frac{\mu'(\Theta)}{V} U_{1y}(W_y + U_y \cdot \Psi) - N_1 + \bar{Q}_1 + Q_1 \right. \\
 &\quad \left. + \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right] \Phi_y + (\dots)_y,
 \end{aligned}$$

where we have used the fact

$$\Phi_y \Psi_{1\tau} = (\Phi_y \Psi_1)_\tau - (\Phi_\tau \Psi_1)_y + \Psi_{1y}^2.$$

Integrating (4.68) with respect to  $y$  and  $\tau$  and using Hölder inequality give

$$\begin{aligned}
 (4.69) \quad &\int \left[ \frac{2\mu(\Theta)}{3V} \Phi_y^2 - \Phi_y \Psi_1 \right] (\tau, y) dy + \int_{\frac{h}{\varepsilon}}^\tau \int \frac{Z}{2V} \Phi_y^2 dy d\tau \\
 &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|\Psi\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_y, W_y)\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int (N_1^2 + \bar{Q}_1^2 + Q_1^2) dy d\tau \\
 &\quad + C\delta \int_{\frac{h}{\varepsilon}}^\tau \|\sqrt{|U_{1y}^{S_3}|} \Psi\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right|^2 dy d\tau.
 \end{aligned}$$

By (2.62), (3.8), and (2.104) and the Cauchy inequality, one has

$$(4.70) \quad \int_{\frac{h}{\varepsilon}}^{\tau} \int (N_1^2 + \bar{Q}_1^2 + Q_1^2) dy d\tau \leq C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau + C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|\psi_{1y}\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.$$

Now we estimate the last term on the right-hand side of (4.69). By (4.42), we have

$$(4.71) \quad \begin{aligned} K_2 &:= \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right|^2 dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_{\tau}) d\xi \right|^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \left( \frac{u_1}{v} \tilde{\mathbf{G}}_{1y} \right) d\xi \right|^2 dy d\tau \\ &\quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \left[ \frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y}) \right] d\xi \right|^2 dy d\tau \\ &\quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} [Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)] d\xi \right|^2 dy d\tau \\ &\quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} [2Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})] d\xi \right|^2 dy d\tau \\ &\quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \xi_1^2 (|J_3|^2 + |J_4|^2 + |J_5|^2) d\xi dy d\tau := \sum_{i=1}^6 K_2^i. \end{aligned}$$

Then we can obtain

$$(4.72) \quad \begin{aligned} K_2^1 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \frac{\nu(|\xi|) |\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_{\tau}|^2}{\mathbf{M}_{\star}} d\xi \cdot \int \nu^{-1}(|\xi|) \xi_1^4 \mathbf{M}_{\star} d\xi \right| dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_{\tau}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau. \end{aligned}$$

Similarly,

$$(4.73) \quad \begin{aligned} K_2^2 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_{1y}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} (|\mathbf{G}_y^{R_1}|^2 + |\mathbf{G}_y^{CD}|^2) d\xi dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Theta_{yy}^{R_1}, U_{1yy}^{R_1}, \Theta_{yy}^{CD}, U_{yy}^{CD})\|^2 d\tau \\ &\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \int |(v_y, u_y, \theta_y)|^2 \cdot |(\Theta_y^{R_1}, U_{1y}^{R_1}, \Theta_y^{CD}, U_y^{CD})|^2 dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Moreover,

$$\begin{aligned}
 (4.74) \quad K_2^3 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \frac{\nu(|\xi|) |\mathbf{L}_M^{-1}[\frac{1}{\nu} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y})]|^2}{\mathbf{M}_{[2v_*, 2u_*, 2\theta_*]}} d\xi \cdot \int \nu^{-1}(|\xi|) \xi_1^4 \mathbf{M}_{[2v_*, 2u_*, 2\theta_*]} d\xi \right| dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu^{-1}(|\xi|) |\frac{1}{\nu} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y})|^2}{\mathbf{M}_{[2v_*, 2u_*, 2\theta_*]}} d\xi dy d\tau \leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_{1y}|^2}{\mathbf{M}_*} d\xi dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_*} d\xi dy d\tau + C\varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

From Lemma 3.2, we have

$$\begin{aligned}
 (4.75) \quad K_2^4 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu^{-1}(|\xi|) |Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)|^2}{\mathbf{M}_*} d\xi dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi \cdot \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi dy d\tau \\
 &\leq C_{h,T} \chi \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi dy d\tau.
 \end{aligned}$$

Using similar idea as for (4.60) and (4.62), we can obtain

$$\begin{aligned}
 (4.76) \quad K_2^5 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu^{-1}(|\xi|) |Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})|^2}{\mathbf{M}_*} d\xi dy d\tau \\
 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi \int \frac{\nu(|\xi|) (|\mathbf{G}^{R_1}|^2 + |\mathbf{G}^{CD}|^2 + |\mathbf{G}^{S_3}|^2)}{\mathbf{M}_*} d\xi dy d\tau \\
 &\leq C(\delta + \varepsilon) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi dy d\tau
 \end{aligned}$$

and

$$\begin{aligned}
 (4.77) \quad K_2^6 &\leq C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu^{-1}(|\xi|) (|J_3|^2 + |J_4|^2 + |J_5|^2)}{\mathbf{M}_*} d\xi dy d\tau \\
 &\leq C(\delta + \varepsilon) \int_{\frac{h}{\varepsilon}}^\tau \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

Substituting the estimations of  $K_2^i$  ( $i = 1, 2, \dots, 6$ ) into (4.71), we obtain

$$\begin{aligned}
 (4.78) \quad K_2 &= \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right|^2 dy d\tau \\
 &\leq C(\delta + \varepsilon) \int_{\frac{h}{\varepsilon}}^\tau \|(\Phi_y, \Psi_y, \zeta, \phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
 &\quad + C(\delta + \chi) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau.
 \end{aligned}$$

Thus combining (4.69), (4.70), and (4.78) yields

$$(4.79) \quad \begin{aligned} \|\Phi_y(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \|\Phi_y\|^2 d\tau &\leq C\|\Psi_1(\tau, \cdot)\|^2 + C \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_y, W_y, \zeta)\|^2 d\tau \\ &+ C(\delta + \chi) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\ &+ C\delta \int_{\frac{h}{\varepsilon}}^\tau \|\sqrt{|U_{1y}^{S_3}|} \Psi\|^2 d\tau + C[\delta + C_{h,T}\chi] \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau. \end{aligned}$$

*Step 3. Estimation on the nonfluid component.* The microscopic component  $\tilde{\mathbf{G}}_1$  can be estimated by using (3.15). Multiplying (3.15) by  $\frac{v\tilde{\mathbf{G}}_1}{\mathbf{M}_\star}$  gives

$$(4.80) \quad \begin{aligned} \left(\frac{v\tilde{\mathbf{G}}_1^2}{2\mathbf{M}_\star}\right)_{1\tau} - \frac{v\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} \mathbf{L}_M \tilde{\mathbf{G}}_1 &= v_\tau \frac{\tilde{\mathbf{G}}_1^2}{2\mathbf{M}_\star} + \left\{ \frac{u_1}{v} \tilde{\mathbf{G}}_y - \frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) \right. \\ &\quad \left. + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_1 + J_2 - \mathbf{G}_\tau^{R_1} - \mathbf{G}_\tau^{CD} \right\} \frac{v\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} \\ &= \left\{ -\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) + 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1 + vJ_2 \right. \\ &\quad \left. - v\mathbf{G}_\tau^{R_1} - v\mathbf{G}_\tau^{CD} + u_1 \mathbf{G}_y^{R_1} + u_1 \mathbf{G}_y^{CD} \right\} \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} + (\dots)_y. \end{aligned}$$

The Cauchy inequality implies

$$(4.81) \quad \begin{aligned} &\int_{\frac{h}{\varepsilon}}^\tau \int \int (-v\mathbf{G}_\tau^{R_1} - v\mathbf{G}_\tau^{CD} + u_1 \mathbf{G}_y^{R_1} + u_1 \mathbf{G}_y^{CD}) \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} d\xi dy d\tau \\ &\leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int \sum_{|\alpha|=2} |\partial^\alpha(\Theta^{CD}, U^{CD}, \Theta^{R_1}, U^{R_1})|^2 dy d\tau \\ &\quad + C \int_{\frac{h}{\varepsilon}}^\tau \int |(\Theta_y^{CD}, U_y^{CD}, \Theta_y^{R_1}, U_y^{R_1})|^2 [|(v_\tau, u_\tau, \theta_\tau)|^2 + |(v_y, u_y, \theta_y)|^2] dy d\tau \\ &\leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T}\varepsilon^{\frac{1}{2}}. \end{aligned}$$

Notice that  $\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) = \xi_1 \tilde{\mathbf{G}}_y - \sum_{j=0}^4 \langle \xi_1 \tilde{\mathbf{G}}_y, \chi_j \rangle \chi_j$ . Then we have

$$(4.82) \quad \begin{aligned} &\int_{\frac{h}{\varepsilon}}^\tau \int \int \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} d\xi dy d\tau \\ &= \int_{\frac{h}{\varepsilon}}^\tau \int \int \xi_1 (\tilde{\mathbf{G}}_{1y} + \mathbf{G}_y^{R_1} + \mathbf{G}_y^{CD}) \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} - \sum_{j=0}^4 \int_{\frac{h}{\varepsilon}}^\tau \int \int \langle \xi_1 \tilde{\mathbf{G}}_y, \chi_j \rangle \chi_j \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} \\ &\leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau \\ &\quad + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \|\partial_y(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T}\varepsilon^{\frac{1}{2}}, \end{aligned}$$

where we have used the fact that

$$|\langle \xi_1 \tilde{\mathbf{G}}_y, \chi_j \rangle|^2 \leq \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_\star} d\xi.$$

Lemmas 3.2–3.5 and wave interaction estimates imply

$$\begin{aligned} (4.83) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \int \int Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) \frac{\nu \tilde{\mathbf{G}}_1}{\mathbf{M}_\star} d\xi dy d\tau \\ & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left( \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi \right)^{\frac{1}{2}} \left( \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}^2}{\mathbf{M}_\star} d\xi \int \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_\star} d\xi \right)^{\frac{1}{2}} dy d\tau \\ & \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\Theta_y^{CD}, U_y^{CD}, \Theta_y^{R_1}, U_y^{R_1})|^4 dy d\tau \\ & \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Moreover, Lemma 2.5 and Cauchy's inequality and wave interaction estimates imply

$$\begin{aligned} (4.84) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \int \int (J_1 + J_2) \frac{\nu \tilde{\mathbf{G}}_1}{\mathbf{M}_\star} d\xi dy d\tau \leq \frac{\tilde{\sigma}}{16} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau \\ & \quad + C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

Integrating (4.80) with respect to  $\xi, y$ , and  $\tau$  and using (3.23), (4.81)–(4.84), and the smallness of  $\chi, \delta, \varepsilon$  yield that

$$\begin{aligned} (4.85) \quad & \int \int \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy d\tau \\ & \leq C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau \\ & \quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_y|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

On the other hand, from the fluid-type system (3.6), we can get an estimate for  $\|(\Psi_\tau, W_\tau)\|^2$  as follows:

$$\begin{aligned} (4.86) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_\tau, W_\tau)\|^2 d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, W_y, \Psi_{yy}, \zeta_y)\|^2 d\tau + C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \|\sqrt{|U_{1y}^{S_3}}|(\Psi, W)\|^2 d\tau \\ & \quad + C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\ & \quad + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

From (3.5), we have

$$\begin{aligned}
 (4.87) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \|\zeta\|^2 d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|W_y\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_y|^4 dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int |U_y \cdot \Psi|^2 dy d\tau \\
 & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|W_y\|^2 d\tau + C\chi^2 \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi_y\|^2 d\tau \\
 & \quad + C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \|\sqrt{|U_{1y}^{S_3}|} \Psi\|^2 d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi\|^2 d\tau.
 \end{aligned}$$

In summary, collecting the estimates (4.66), (4.79), and (4.85)–(4.87), we complete the proof of Proposition 3.6.

**4.2. Proof of Proposition 3.7.** The proof is divided into the following five steps.

*Step 1. Estimation on  $\|(\phi, \psi, \zeta)(\tau, \cdot)\|^2$ .* Similar to (4.10), we multiply (3.39)<sub>1</sub> by  $\phi$ , (3.39)<sub>2</sub> by  $\frac{V}{Z}\psi_1$ , (3.39)<sub>3</sub> by  $\psi_i$ , and (3.39)<sub>4</sub> by  $\frac{2\zeta}{3Z^2}$ , respectively, and add them together to have

$$\begin{aligned}
 (4.88) \quad & \left( \frac{\phi^2}{2} + \frac{V}{2Z} \psi_1^2 + \sum_{i=2}^3 \frac{\psi_i^2}{2} + \frac{\zeta^2}{3Z^2} \right)_{\tau} + \frac{4\mu(\Theta)}{3Z} \psi_{1y}^2 + \sum_{i=2}^3 \frac{\mu(\Theta)}{V} \psi_{iy}^2 + \frac{2\kappa(\Theta)}{3Z^2 V} \zeta_y^2 \\
 & = I_6(\phi, \psi, \zeta, \psi_y, \zeta_y) + \frac{V}{Z} \psi_1 (N_5 - \bar{Q}_{1y} - Q_{1y}) + \sum_{i=2}^3 \psi_i (N_{i+4} - \bar{Q}_{iy} - Q_{iy}) \\
 & \quad + \frac{2\zeta}{3Z^2} [N_8 - \bar{Q}_{4y} - Q_{4y} + \sum_{i=1}^3 U_i (\bar{Q}_{iy} + Q_{iy})] + K_3 + (\dots)_y,
 \end{aligned}$$

where

$$\begin{aligned}
 (4.89) \quad I_6(\phi, \psi, \zeta, \psi_y, \zeta_y) & = \left( \frac{V}{2Z} \right)_{\tau} \psi_1^2 + \left( \frac{1}{3Z^2} \right)_{\tau} \zeta^2 + \left( \frac{2}{3Z} \right)_y \psi_1 \zeta - \frac{V}{Z} \psi_1 H_1 - \sum_{i=2}^3 \psi_i H_i \\
 & \quad - \frac{2}{3Z^2} \zeta H_4 - \frac{4\mu(\Theta)}{3V} \left( \frac{V}{Z} \right)_y \psi_1 \psi_{1y} - \frac{\kappa(\Theta)}{V} \left( \frac{2}{3Z^2} \right)_y \zeta \zeta_y
 \end{aligned}$$

and

$$\begin{aligned}
 (4.90) \quad K_3 & = -\frac{V}{Z} \psi_1 \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - \sum_{i=2}^3 \psi_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \\
 & \quad + \frac{2\zeta}{3Z^2} \left[ -\frac{1}{2} \int \xi_1 |\xi|^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi + \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi \right. \\
 & \quad \left. + \sum_{i=1}^3 U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right].
 \end{aligned}$$

Direct calculation shows that

$$\begin{aligned}
 (4.91) \quad I_6 & \leq C|(V_{\tau}, Z_{\tau}, V_y, U_y, \Theta_y, Z_y)| [ |(\phi_y, \psi_y, \zeta_y)|^2 + |(\phi, \psi, \zeta)|^2 ] \\
 & \leq C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) |(\phi_y, \psi_y, \zeta_y)|^2 + C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) |(\phi, \psi, \zeta)|^2.
 \end{aligned}$$

Thus, integrating (4.88) with respect to  $\tau$  and  $y$  and using the Cauchy inequality yield that

$$\begin{aligned}
 (4.92) \quad & \|(\phi, \psi, \zeta)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau \leq C(\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi_y\|^2 d\tau \\
 & + C(\beta + \delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C_{\beta} \int_{\frac{h}{\varepsilon}}^{\tau} \int \sum_{i=1}^4 (\bar{Q}_{iy}^2 + Q_{iy}^2) dy d\tau \\
 & + C \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\psi, \zeta)| [ |N_5| + \sum_{i=2}^3 |N_{i+4}| + |N_8| ] dy d\tau + \int_{\frac{h}{\varepsilon}}^{\tau} \int K_3 dy d\tau.
 \end{aligned}$$

By (3.31) and (3.33), we have

$$(4.93) \quad \int_{\frac{h}{\varepsilon}}^{\tau} \int \sum_{i=1}^4 (\bar{Q}_{iy}^2 + Q_{iy}^2) dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}.$$

From (3.43), (3.44), and (3.45), we have

$$\begin{aligned}
 (4.94) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\psi, \zeta)| \left[ |N_5| + \sum_{i=2}^3 |N_{i+4}| + |N_8| \right] dy d\tau \\
 & \leq C_{h,T} \chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \psi_{yy}, \zeta_{yy})\|^2 d\tau.
 \end{aligned}$$

Now we estimate the microscopic term  $\int_{\frac{h}{\varepsilon}}^{\tau} \int K_3 dy d\tau$  in (4.92). We only estimate study  $K_{31} := -\int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \psi_1 \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi dy d\tau$  and  $K_{32} := \frac{\zeta}{3Z^2} \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi$  because the other terms in  $\int_{\frac{h}{\varepsilon}}^{\tau} \int K_3 dy d\tau$  can be estimated similarly.

For  $K_{31}$ , integration by parts with respect to  $y$  and the Cauchy inequality yield

$$\begin{aligned}
 (4.95) \quad & K_{31} = \int_{\frac{h}{\varepsilon}}^{\tau} \int \left( \frac{V}{Z} \psi_1 \right)_y \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi dy d\tau \\
 & \leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|\psi_{1y}\|^2 d\tau + C(\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|\psi_1\|^2 d\tau \\
 & + C_{\beta} \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right|^2 dy d\tau,
 \end{aligned}$$

where the last term has been estimated in (4.78).

For  $K_{32}$ , we have

$$\begin{aligned}
 (4.96) \quad & K_{32} = - \sum_{i=1}^3 \int_{\frac{h}{\varepsilon}}^{\tau} \int \left( \frac{\zeta \psi_i}{3Z^2} \right)_y \int \xi_1 \xi_i \Pi_1 d\xi dy d\tau \\
 & \leq C \sum_{i=1}^3 \int_{\frac{h}{\varepsilon}}^{\tau} \int \left( |\zeta_y| |\psi| + |\psi_y| |\zeta| + |Z_y| |\psi| |\zeta| \right) \\
 & \quad \cdot \left[ \left| \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right| + \left| \int \xi_1 \xi_i (\Pi_{11}^{CD} + \Pi_1^{S_3}) d\xi \right| \right] dy d\tau \\
 & \leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau + C_{\beta} (\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi, \zeta)\|^2 d\tau \\
 & + C_{\beta} \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right|^2 dy d\tau.
 \end{aligned}$$



Substituting (4.93)–(4.96) and (4.78) into (4.92) and choosing  $\beta, \varepsilon, \delta, \chi$  suitably small yield that

$$(4.97) \quad \begin{aligned} \|(\phi, \psi, \zeta)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \|(\psi_y, \zeta_y)\|^2 d\tau &\leq C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \|\phi_y\|^2 d\tau \\ &+ C_\beta(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_y, \psi_{yy}, \zeta_{yy})\|^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{1}{2}} \\ &+ C(\delta + \chi) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau. \end{aligned}$$

*Step 2. Estimation on  $\|\phi_y(\tau, \cdot)\|^2$ .* To estimate the term  $\int_{\frac{h}{\varepsilon}}^\tau \|\phi_y\|^2 d\tau$ , we first rewrite (3.39)<sub>2</sub> as

$$(4.98) \quad \begin{aligned} \frac{4}{3} \frac{\mu(\Theta)}{V} \phi_{y\tau} - \psi_{1\tau} + \frac{Z}{V} \phi_y &= -\frac{4}{3} \left( \frac{\mu(\Theta)}{V} \right)_y \psi_{1y} + \frac{2}{3V} \zeta_y + H_1 \\ &+ \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - N_5 + \bar{Q}_{1y} + Q_{1y} \end{aligned}$$

by using the equation of conservation of the mass (3.39)<sub>1</sub>. Multiplying (4.98) by  $\phi_y$ , we get

$$(4.99) \quad \begin{aligned} \left( \frac{2\mu(\Theta)}{3V} \phi_y^2 - \phi_y \psi_1 \right)_\tau &+ (\phi_\tau \psi_1)_y + \frac{Z}{V} \phi_y^2 \\ &= \psi_{1y}^2 + \left[ -\frac{4}{3} \left( \frac{\mu(\Theta)}{V} \right)_y \psi_{1y} + \frac{2}{3V} \zeta_y + H_1 \right. \\ &\quad \left. + \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - N_5 + \bar{Q}_{1y} + Q_{1y} \right] \phi_y. \end{aligned}$$

Integrating (4.99) with respect to  $\tau, y$  and using (4.93), (4.94), and the Cauchy inequality yield

$$(4.100) \quad \begin{aligned} \|\phi_y(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \|\phi_y\|^2 d\tau &\leq C\|\psi_1(\tau, \cdot)\|^2 + C \int_{\frac{h}{\varepsilon}}^\tau \|(\psi_y, \zeta_y)\|^2 d\tau \\ &+ C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi, \psi, \zeta)\|^2 d\tau + C\chi \int_{\frac{h}{\varepsilon}}^\tau \|\psi_{1yy}\|^2 d\tau \\ &+ C_{h,T} \varepsilon^{\frac{1}{2}} + \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right|^2 dy d\tau. \end{aligned}$$

For the microscopic term  $\int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right|^2 dy d\tau$ , by (4.42), we have

$$(4.101) \quad \begin{aligned} K_4 &:= \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right|^2 dy d\tau \\ &\leq C \left[ \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_\tau)_y d\xi \right|^2 dy d\tau + \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\mathbf{L}_M^{-1} \frac{u_1}{v} \tilde{\mathbf{G}}_{1y})_y d\xi \right|^2 dy d\tau \right] \end{aligned}$$

$$\begin{aligned}
& + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 [\mathbf{L}_M^{-1} \frac{1}{\nu} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y})]_y d\xi \right|^2 dy d\tau \\
& + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 [\mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)]_y d\xi \right|^2 dy d\tau \\
& + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 [\mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})]_y d\xi \right|^2 dy d\tau \\
& + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (J_{3y} + J_{4y} + J_{5y}) d\xi \right|^2 dy d\tau \Big] := \sum_{i=1}^6 K_4^i.
\end{aligned}$$

Then we have

$$\begin{aligned}
(4.102) \quad K_4^1 & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \tilde{\mathbf{G}}_{y\tau} d\xi \right|^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_{\tau}, \mathbf{M}_y)\} d\xi \right|^2 dy d\tau \\
& \leq C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\
& \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_{\tau}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau.
\end{aligned}$$

Similar estimates hold for  $K_4^i$  ( $i = 2, 3$ ). Moreover,

$$\begin{aligned}
(4.103) \quad K_4^4 & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_{1y}) d\xi \right|^2 dy d\tau \\
& \quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1), \mathbf{M}_y)\} d\xi \right|^2 dy d\tau \\
& \leq C_{h,T\chi} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2)}{\mathbf{M}_{\star}} d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}},
\end{aligned}$$

$$(4.104) \quad K_4^5 \leq C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2)}{\mathbf{M}_{\star}} d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}},$$

and

$$\begin{aligned}
(4.105) \quad K_4^6 & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{|J_{3y}|^2 + |J_{4y}|^2 + |J_{5y}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\
& \leq C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \phi_{yy}, \psi_{yy}, \zeta_{yy}\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned}$$

Substituting (4.102)–(4.105) into (4.101) gives

$$\begin{aligned}
(4.106) \quad K_4 & \leq C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
& \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \phi_{yy}, \psi_{yy}, \zeta_{yy}\|^2 d\tau \\
& \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) (|\tilde{\mathbf{G}}_{\tau}|^2 + |\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2)}{\mathbf{M}_{\star}} d\xi dy d\tau.
\end{aligned}$$

Thus we have

$$\begin{aligned}
 (4.107) \quad & \|\phi_y(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^\tau \|\phi_y\|^2 d\tau \leq C\|\psi_1(\tau, \cdot)\|^2 + C \int_{\frac{h}{\varepsilon}}^\tau \|(\psi_y, \zeta_y)\|^2 d\tau \\
 & + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
 & + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi, \psi, \zeta)\|^2 d\tau + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \mathbf{G}|^2 d\xi dy d\tau \\
 & + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)(\sum_{|\alpha'|=1} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 + |\tilde{\mathbf{G}}_1|^2)}{\mathbf{M}_\star} d\xi dy d\tau.
 \end{aligned}$$

We now turn to the time derivatives. To estimate  $\|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2$ , we need to use the system (3.27). By multiplying (3.27)<sub>1</sub> by  $\phi_\tau$ , (3.27)<sub>2</sub> by  $\psi_{1\tau}$ , (3.27)<sub>3</sub> by  $\psi_{i\tau}$  ( $i = 2, 3$ ), and (3.27)<sub>4</sub> by  $\zeta_\tau$ , respectively, and adding them together, after integrating with respect to  $\tau$  and  $y$ , we have

$$\begin{aligned}
 (4.108) \quad & \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_\tau, \psi_\tau, \zeta_\tau)(\tau, \cdot)\|^2 d\tau \\
 & \leq C(\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi, \psi, \zeta)\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau \\
 & + C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_y|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
 \end{aligned}$$

*Step 3. Estimation on  $\|(\phi_y, \psi_y, \zeta_y)(\tau, \cdot)\|^2$ .* Multiplying (3.46)<sub>1</sub> by  $\phi_y$ , (3.46)<sub>2</sub> by  $\frac{V}{Z}\psi_{1y}$ , (3.46)<sub>3</sub> by  $\psi_{iy}$ , and (3.46)<sub>4</sub> by  $\frac{2}{3Z^2}\zeta_y$ , adding them together gives

$$\begin{aligned}
 (4.109) \quad & \left( \frac{\phi_y^2}{2} + \frac{V}{2Z}\psi_{1y}^2 + \sum_{i=2}^3 \frac{\psi_{iy}^2}{2} + \frac{\zeta_y^2}{3Z^2} \right)_\tau + \frac{4\mu(\Theta)}{3Z}\psi_{1yy}^2 + \sum_{i=2}^3 \frac{\mu(\Theta)}{V}\psi_{iyy}^2 + \frac{\kappa(\Theta)}{3Z^2V}\zeta_{yy}^2 \\
 & = I_7(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \psi_{yy}, \zeta_{yy}) - (N_5 - \bar{Q}_{1y} - Q_{1y}) \left( \frac{V\psi_{1y}}{Z} \right)_y \\
 & - \sum_{i=2}^3 (N_{i+4} - \bar{Q}_{iy} - Q_{iy})\psi_{iyy} \\
 & - \left( \frac{2\zeta_y}{3Z^2} \right)_y \left[ N_8 - \bar{Q}_{4y} - Q_{4y} + \sum_{i=1}^3 U_i(\bar{Q}_{iy} + Q_{iy}) \right] + K_5 + (\dots)_y,
 \end{aligned}$$

where

$$\begin{aligned}
 (4.110) \quad & I_7(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \psi_{yy}, \zeta_{yy}) \\
 & = \left( \frac{V}{2Z} \right)_\tau \psi_{1y}^2 + \left( \frac{1}{3Z^2} \right)_\tau \zeta_y^2 + \left( \frac{2}{3Z} \right)_y \psi_{1y}\zeta_y - \frac{V}{Z}\psi_{1y}H_5 - \sum_{i=2}^3 \psi_{iy}H_{i+4} \\
 & - \frac{2}{3Z^2}\zeta_yH_8 - \left( \frac{4\mu(\Theta)\psi_{1y}}{3V} \right)_y \left( \frac{V}{Z} \right)_y \psi_{1y} - \left( \frac{4\mu(\Theta)}{3V} \right)_y \psi_{1y} \frac{V}{Z}\psi_{1yy} \\
 & - \sum_{i=2}^3 \left( \frac{\mu(\Theta)}{V} \right)_y \psi_{iy}\psi_{iyy} - \left( \frac{\kappa(\Theta)\zeta_y}{V} \right)_y \left( \frac{2}{3Z^2} \right)_y \zeta_y - \left( \frac{\kappa(\Theta)}{V} \right)_y \zeta_y \frac{2}{3Z^2}\zeta_{yy} \\
 & \leq \beta|(\phi_{yy}, \psi_{yy}, \zeta_{yy})|^2 + C_\beta(\delta + C_{h,T}\varepsilon^{\frac{1}{2}})|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)|^2
 \end{aligned}$$

and

$$(4.111) \quad \begin{aligned} K_5 = & \left( \frac{V\psi_{1y}}{Z} \right)_y \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi + \sum_{i=2}^3 \psi_{iy} \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \\ & + \left( \frac{2\zeta_y}{3Z^2} \right)_y \left[ \int \xi_1 \frac{|\xi|^2}{2} (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi \right. \\ & \left. - \sum_{i=1}^3 U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right]. \end{aligned}$$

Integrating (4.109) with respect to  $\tau, y$  and substituting (4.93), (4.94), and (4.106) into (4.109) and (4.111), choosing  $\beta, \chi, \varepsilon$  small enough, we have

$$(4.112) \quad \begin{aligned} & \|(\phi_y, \psi_y, \zeta_y)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_{yy}, \zeta_{yy})\|^2 d\tau \\ & \leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi_{yy}\|^2 d\tau + C_\beta (\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\ & \quad + C_{h,T} \chi \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\ & \quad + C_{h,T} \chi \sum_{|\alpha|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau. \end{aligned}$$

Again, to recover  $\|\phi_{yy}\|^2$  in the dissipation rate, applying  $\partial_y$  to (3.27)<sub>2</sub>, we get

$$(4.113) \quad \begin{aligned} & \psi_{1y\tau} + (p - P)_{yy} \\ & = -\frac{4}{3} \left[ \frac{\mu(\Theta)}{V} U_{1y} - \frac{\mu(\Theta^{S_3})}{V^{S_3}} U_{1y}^{S_3} \right]_{yy} - \int \xi_1 \xi_i \Pi_{11yy}^{CD} d\xi - \bar{Q}_{1yy} - Q_{1yy} - \int \xi_1^2 \tilde{\mathbf{G}}_{yy} d\xi. \end{aligned}$$

Note that

$$(4.114) \quad (p - P)_{yy} = -\frac{p}{v} \phi_{yy} + \frac{2}{3v} \zeta_{yy} - \frac{1}{v} (p - P) V_{yy} - \frac{\phi}{v} P_{yy} - \frac{2v_y}{v} (p - P)_y - \frac{2P_y}{v} \phi_y.$$

Multiplying (4.113) by  $-\phi_{yy}$  and using (4.114) imply

$$(4.115) \quad \begin{aligned} & - \int \psi_{1y} \phi_{yy}(\tau, y) dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{p}{2v} \phi_{yy}^2 dy d\tau \\ & \leq C(\delta + C_{h,T} \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)\|^2 d\tau \\ & \quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_{1yy}, \zeta_{yy})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau, \end{aligned}$$

where we have used the fact that

$$(4.116) \quad \int_{\frac{h}{\varepsilon}}^{\tau} \int (|\bar{Q}_{1yy}|^2 + |Q_{1yy}|^2) dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}.$$

To estimate  $\|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2$  and  $\|(\phi_{\tau\tau}, \psi_{\tau\tau}, \zeta_{\tau\tau})\|^2$ , we use the system (3.27) again. Applying  $\partial_y$  to (3.27) and multiplying the four equations of (3.27) by  $\phi_{y\tau}$ ,  $\psi_{1y\tau}$ ,  $\psi_{iy\tau}$  ( $i = 2, 3$ ),  $\zeta_{y\tau}$ , respectively, then adding them together and integrating with respect to  $\tau$  and  $y$ , we have

$$(4.117) \quad \begin{aligned} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2 d\tau &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\ &+ C_{h,T} \chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau \\ &+ C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau. \end{aligned}$$

Similarly, we can obtain

$$(4.118) \quad \begin{aligned} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{\tau\tau}, \psi_{\tau\tau}, \zeta_{\tau\tau})\|^2 d\tau &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\ &+ C_{h,T} \chi \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau \\ &+ C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau. \end{aligned}$$

A suitable linear combination of (4.112)–(4.118) gives

$$(4.119) \quad \begin{aligned} \|(\phi_y, \psi_y, \zeta_y)(\tau, \cdot)\|^2 &- \int \psi_{1y} \phi_{yy} dy + \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 d\tau \\ &\leq C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\ &+ C_{h,T} \chi \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau \\ &+ C_{h,T} \chi \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned}$$

*Step 4: Estimation on the nonfluid component.* To close the a priori estimate, we also need to estimate the derivatives on the nonfluid component  $\tilde{\mathbf{G}}$ , i.e.,  $\partial^\alpha \tilde{\mathbf{G}}$ , ( $|\alpha| = 1, 2$ ). For this, from (3.11) we obtain

$$(4.120) \quad \begin{aligned} v \tilde{\mathbf{G}}_\tau - v \mathbf{L}_M \tilde{\mathbf{G}} &= u_1 \tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) - v \left[ \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y) - \frac{1}{v S_3} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_y^{S_3}) \right] \\ &+ 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1. \end{aligned}$$

Applying  $\partial_y$  on (4.120), we have

$$(4.121) \quad v\tilde{\mathbf{G}}_{y\tau} - (v\mathbf{L}_M\tilde{\mathbf{G}})_y = \left\{ u_1\tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1\tilde{\mathbf{G}}_y) - v \left[ \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_y^{S_3}) \right] \right. \\ \left. + 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1 \right\}_y - v_y\tilde{\mathbf{G}}_\tau.$$

Multiplying (4.121) by  $\frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star}$  gives

$$(4.122) \quad \left( \frac{v\tilde{\mathbf{G}}_y^2}{2\mathbf{M}_\star} \right)_\tau - \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} (v\mathbf{L}_M\tilde{\mathbf{G}})_y \\ = \left\{ u_1\tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1\tilde{\mathbf{G}}_y) - v \left[ \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_y^{S_3}) \right] \right. \\ \left. + 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1 \right\}_y \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} - v_y\tilde{\mathbf{G}}_\tau \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} + v_\tau \frac{\tilde{\mathbf{G}}_y^2}{2\mathbf{M}_\star}.$$

Then Lemmas 3.2–3.5 and the Cauchy inequality give

$$(4.123) \quad - \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} (v\mathbf{L}_M\tilde{\mathbf{G}})_y d\xi dy d\tau \\ = - \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} (v\mathbf{L}_M\tilde{\mathbf{G}}_y + 2v_yQ(\mathbf{M}, \tilde{\mathbf{G}}) + 2vQ(\mathbf{M}_y, \tilde{\mathbf{G}})) d\xi dy d\tau \\ \geq \frac{7\tilde{\sigma}}{8} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau - C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau \\ - C\varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau - C_{h,T} \varepsilon^{\frac{1}{2}}$$

and

$$(4.124) \quad \int_{\frac{h}{\varepsilon}}^\tau \int \int -v_y\tilde{\mathbf{G}}_\tau \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} + v_\tau \frac{\tilde{\mathbf{G}}_y^2}{2\mathbf{M}_\star} + (u_1\tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1\tilde{\mathbf{G}}_y))_y \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\ \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_\tau^2}{\mathbf{M}_\star} d\xi dy d\tau \\ + C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_{yy}^2}{\mathbf{M}_\star} d\xi dy d\tau.$$

Note that

$$(4.125) \quad J_6 \triangleq -v \left[ \frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_y^{S_3}) \right] \\ = -\frac{3}{2\theta}\mathbf{P}_1 \left[ \xi_1\mathbf{M} \left( \frac{|\xi - u|^2}{2\theta}(\theta_y - \Theta_y^{S_3}) + \xi \cdot (u_y - U_y^{S_3}) \right) \right] \\ - v \left[ \frac{3}{2v\theta}\mathbf{P}_1 \left[ \xi_1\mathbf{M} \left( \frac{|\xi - u|^2}{2\theta}\Theta_y^{S_3} + \xi \cdot U_y^{S_3} \right) \right] \right] \\ - \frac{3}{2V^{S_3}\Theta^{S_3}}\mathbf{P}_1^{S_3} \left[ \xi_1\mathbf{M}^{S_3} \left( \frac{|\xi - U^{S_3}|^2}{2\Theta^{S_3}}\Theta_y^{S_3} + \xi \cdot U_y^{S_3} \right) \right].$$

Then Lemmas 3.2–3.5 and wave interaction estimates imply that

$$\begin{aligned}
 (4.126) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \int \int (vJ_1 + J_6)_y \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\
 & \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau \\
 & \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|_{H^1(dy)}^2 d\tau + C_{h,T\varepsilon}^{\frac{1}{2}}
 \end{aligned}$$

and

$$\begin{aligned}
 (4.127) \quad & \int_{\frac{h}{\varepsilon}}^{\tau} \int \int (vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}))_y \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\
 & = \int_{\frac{h}{\varepsilon}}^{\tau} \int \int (v_y Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + 2vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}_y)) \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\
 & \leq C_{h,T\varepsilon}^{\frac{1}{2}} + \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau + C(\chi + \delta) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau.
 \end{aligned}$$

Thus, integrating (4.122) with respect to  $\xi, y$ , and  $\tau$  and using (4.123), (4.124) and (4.126), (4.127), we obtain

$$\begin{aligned}
 (4.128) \quad & \int \int \frac{|\tilde{\mathbf{G}}_y|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_y|^2 d\xi dy d\tau \\
 & \leq C_{h,T\varepsilon}^{\frac{1}{2}} + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_{yy}|^2 d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{yy}, \zeta_{yy})\|^2 d\tau \\
 & \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_\tau|^2) d\xi dy d\tau \\
 & \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} [\|(\phi, \psi, \zeta)\|^2 + \|(\phi, \psi, \zeta)_y\|^2] d\tau.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (4.129) \quad & \int \int \frac{|\tilde{\mathbf{G}}_\tau|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_\tau|^2 d\xi dy d\tau \\
 & \leq C_{h,T\varepsilon}^{\frac{1}{2}} + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_{y\tau}|^2 d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{y\tau}, \zeta_{y\tau})\|^2 d\tau \\
 & \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2) d\xi dy d\tau \\
 & \quad + CC(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} [\|(\phi, \psi, \zeta)\|^2 + \|(\phi, \psi, \zeta)_\tau\|^2] d\tau.
 \end{aligned}$$

*Step 5: Highest order estimates.* Finally, we estimate the highest order derivatives, that is,  $-\int \psi_{1y} \phi_{yy} dy$  and  $\int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\partial^\alpha \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy d\tau$  with  $|\alpha| = 2$  in (4.119). To

do so, it is sufficient to study  $\int \int \frac{|\partial^\alpha \tilde{f}|^2}{2\mathbf{M}_\star} d\xi dy$  ( $|\alpha| = 2$ ) in view of (3.38) and (3.37). Using the same idea in [20], we obtain the estimation for the highest order derivative terms, i.e.,

$$(4.130) \quad \begin{aligned} & \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha \tilde{f}|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\ & \leq C(\eta_0 + \delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} (|\tilde{\mathbf{G}}_y|^2 + |\tilde{\mathbf{G}}_\tau|^2 + |\tilde{\mathbf{G}}_1|^2) d\xi dy d\tau \\ & \quad + C(\eta_0 + \delta + C_{h,T}\chi) \sum_{|\alpha|=0}^2 \int_{\frac{h}{\varepsilon}}^\tau \|\partial^\alpha(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}, \end{aligned}$$

where  $\eta_0$  is defined in Lemma 3.3.

Noting that

$$\begin{aligned} - \int \psi_{1y} \phi_{yy} dy & \leq \beta \|\psi_{1y}\|^2 + C_\beta \|\phi_{yy}\|^2 \\ & \leq \beta \|\psi_{1y}\|^2 + C_\beta \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha \tilde{f}|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + C_{h,T,\beta} \varepsilon^{\frac{1}{2}}, \end{aligned}$$

and combining the estimates (4.119), (4.128), (4.129), and (4.130), we complete the proof of Proposition 3.7.

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