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GLOBAL SOLUTIONS TO THE ONE-DIMENSIONAL COMPRESSIBLE NAVIER–STOKES–POISSON EQUATIONS WITH LARGE DATA*

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Abstract. In this paper, we study the global solutions with large data away from vacuum to the Cauchy problem of the one-dimensional compressible Navier–Stokes–Poisson system with density-dependent viscosity coefficient and density- and temperature-dependent heat-conductivity coefficient. The proof is based on some detailed analysis on the bounds on the density and temperature functions.

Key words. Navier–Stokes–Poisson equations, global solutions with large data, viscosity and heat-conductivity coefficients

AMS subject classifications. 35Q35, 35D35, 76D05

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1. Introduction.

The compressible Navier–Stokes–Poisson (NSP) system consisting of the compressible Navier–Stokes equations coupled with the Poisson equation models the viscous fluid under the influence of the self-induced electric force:

$$\begin{cases} \rho_\tau + \nabla_\xi \cdot (\rho u) = 0, \\ (\rho u)_\tau + \nabla_\xi \cdot (\rho u \otimes u) + \nabla_\xi p = \rho \nabla_\xi \Phi + \nabla_\xi \cdot \mathbf{T}, \\ (\rho \mathbf{E})_\tau + \nabla_\xi \cdot (\rho u \mathbf{E} + up) = \rho u \cdot \nabla_\xi \Phi + \nabla_\xi \cdot (u\mathbf{T}) + \nabla_\xi \cdot (\kappa(v,\theta)\nabla_\xi \theta), \\ \Delta_\xi \Phi = \rho - \overline{\rho}(\xi), \qquad \lim\limits_{|\xi|\to+\infty} \Phi(\tau,\xi) = 0. \end{cases}$$

Here, $\rho > 0$, $u = (u^1, u^2, u^3)$, $\theta > 0$, $p = p(\rho,\theta)$, e, and Φ denote the density, velocity, absolute temperature, pressure, internal energy, and the electrostatic potential function, respectively. Also the total energy $\mathbf{E} = \frac{1}{2}|u|^2 + e$ and the stress tensor $\mathbf{T} = \mu(\rho,\theta)\left(\nabla_\xi u + (\nabla_\xi u)^t\right) + \nu(\rho,\theta)(\nabla_\xi \cdot u)\mathbf{I}$ with $\mathbf{I}$ being the identity matrix. The viscosity coefficients $\mu(\rho,\theta) > 0$ and $\nu(\rho,\theta)$ satisfy $\mu(\rho,\theta) + \frac{2}{3}\nu(\rho,\theta) > 0$. The thermodynamic variables p, ρ, and e are related by Gibbs equation $de = \theta ds - pd\rho^{-1}$ with s being the specific entropy. $\kappa(\rho,\theta) > 0$ denotes the heat-conductivity coefficient and $\overline{\rho}(\xi) > 0$ is the background doping profile; see [30].

To explain the purpose of this paper, we first give the following remarks on the viscosity and heat-conductivity coefficients:

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- When the viscosity coefficients $\mu(\rho,\theta) > 0,\ \nu(\rho,\theta)$, and the heat-conductivity coefficient $\kappa(\rho,\theta) > 0$ are constants, (1.1) is used in semiconductor theory to model the transport of charged particles under the influence of self-induced electric field; see [30].
- In the kinetic theory, the time evolution of the particle distribution function for the charged particles in a dilute gas can be modeled by the Vlasov–Poisson–Boltzmann system; see [4], [3], [34]. When we derive the NSP (1.1) from the Vlasov–Poisson–Boltzmann system by using the Chapman–Enskog expansion, see [4], [12], [34], the viscosity coefficients $\mu,\ \nu$, and the heat-conductivity coefficient κ depend on the absolute temperature θ and $\nu = -\frac{2}{3}\mu$ for the monatomic gas. If the intermolecular potential is proportional to $r^{-\alpha}$ with $\alpha > 1$, where r represents the intermolecular distance, then $\mu,\ \nu$, and κ are proportional to the temperature to some power:

$$\mu,\ -\nu,\ \kappa \propto \theta^{\frac{\alpha+4}{2\alpha}}.$$

  In particular, for the Maxwellian molecule ($\alpha = 4$), such dependence is linear, while for the hard sphere model and also the case when $\alpha \to +\infty$, the dependence is in the form of $\sqrt{\theta}$.

This paper is concerned with the global existence of large-data solutions when the viscosity coefficients $\mu,\ \nu$, and the heat-conductivity coefficient κ depend on ρ and θ. Unlike the small perturbation solutions, such dependence has a strong influence on the solution behavior and thus leads to difficulties in analysis but not for the case of constant coefficients. In fact, for the one-dimensional compressible Navier–Stokes equations, there are a lot of recent papers on the construction of nonvacuum solutions to the one-dimensional compressible Navier–Stokes equations with density- and temperature-dependent transportation coefficients in various forms; see [1], [5], [18], [19], [21], [22], [23], [24], [25], and the references therein. However, there is a gap between the physical models and the satisfactory existence theory.

The main purpose of this paper is the construction of globally smooth, nonvacuum solutions to the one-dimensional nonisentropic compressible NSP with density dependent viscous coefficient and density- and temperature-dependent heat-conductivity coefficient for arbitrarily large data. We hope that the analysis here can shed some light on the construction of global classical solutions to the fluid model derived from the Vlasov–Poisson–Boltzmann system with large data.

Let x be the Lagrangian space variable, t be the time variable, and $v = \frac{1}{\rho}$ denote the specific volume. Then the one-dimensional compressible NSP system (1.1) with viscous coefficient $\mu(v)$ and heat-conductivity coefficient $\kappa(v,\theta)$ becomes

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v,\theta)_x = \left(\frac{\mu(v)u_x}{v}\right)_x + \frac{\Phi_x}{v}, \\ e_t + p(v,\theta)u_x = \frac{\mu(v)u_x^2}{v} + \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x, \\ \left(\frac{\Phi_x}{v}\right)_x = 1 - v, \qquad \lim_{|x|\to+\infty} \Phi(t,x) = 0. \end{cases} \tag{1.1}$$

Throughout this paper, we will concentrate on the ideal, polytropic gases:

$$p(v,\theta) = \frac{R\theta}{v} = Av^{-\gamma}\exp\left(\frac{\gamma-1}{R}s\right), \quad e = C_v\theta = \frac{R\theta}{\gamma-1}, \tag{1.2}$$

where the specific gas constant R and the specific heat at constant volume C_v are positive constants and $\gamma > 1$ is the adiabatic constant. Moreover, to simplify the presentation, we will only consider the case when the background doping profile $\overline{\rho}$ is a positive constant which is normalized to 1 as in $(1.1)_4$.

Take the initial data

$$(1.3) \qquad \begin{aligned} &(v(0,x), u(0,x), \theta(0,x)) = (v_0(x), u_0(x), \theta_0(x)),\\ &\lim_{x\to\pm\infty}(v_0(x), u_0(x), \theta_0(x)) = (v_\pm, u_\pm, \theta_\pm), \end{aligned}$$

satisfying $v_- = v_+, u_- = u_+, \theta_- = \theta_+$. Without loss of generality, we assume $v_- = v_+ = 1, u_- = u_+ = 0, \theta_- = \theta_+ = 1$.

The first result is concerned with the case

$$(1.4) \qquad \mu(v) = v^{-a}, \quad \kappa(v,\theta) = \theta^b,$$

which is stated as follows.

THEOREM 1.1. *Suppose*

- $(v_0(x)-1, u_0(x), \theta_0(x)-1, \Phi_{0x}(x)) \in H^1(\mathbf{R})$, *and there exist positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *such that*

$$(1.5) \qquad \underline{V} \le v_0(x) \le \overline{V}, \quad \underline{\Theta} \le \theta_0(x) \le \overline{\Theta};$$

- $\frac{1}{3} < a < \frac{1}{2}$;
- b *satisfies one of the following conditions:*
  (i) $1 \le b < \frac{2a}{1-a}$,
  (ii) $0 < b < 1$, $\frac{2-b}{2} + \frac{(a^2-a+2)(1-b)}{(1-2a)(3a-1)} < 1$, $\frac{(1-b)(3+a-2a^2)}{(3a-1)(1-2a)} < 1$.

Then the Cauchy problem (1.1), (1.3) *with* $\mu(v)$ *and* $\kappa(v,\theta)$ *given by* (1.4) *admits a unique global solution* $(v(t,x), u(t,x), \theta(t,x))$ *satisfying*

$$(1.6) \qquad \begin{gathered} (v(t,x)-1, u(t,x), \theta(t,x)-1) \in C^0\left(0,T; H^1(\mathbf{R})\right),\\ (u_x(t,x), \theta_x(t,x)) \in L^2\left(0,T; H^1(\mathbf{R})\right),\\ \Phi_x(t,x) \in C^0\left(0,T; H^2(\mathbf{R})\right),\\ 0 < V_0^{-1} \le v(t,x) \le V_0, \quad 0 < \Theta_0^{-1} \le \theta(t,x) \le \Theta_0 \quad \forall (t,x) \in [0,T]\times\mathbf{R}. \end{gathered}$$

Here $T > 0$ *is any given positive constant and* V_0, Θ_0 *are some positive constants which may depend on* T.

Note that the assumptions imposed on a and b in Theorem 1.1 exclude the case when the viscous coefficient μ and the heat-conductivity coefficient κ are positive constants. The next result will recover this in another setting. The main idea is to use the smallness of $\gamma - 1$ to deduce uniform lower and upper bounds on the absolute temperature. This can be achieved by showing that $(v_0(x) - 1, u_0(x), s_0(x) - \overline{s}) \in H^1(\mathbf{R})$ are bounded in $H^1(\mathbf{R})$ independent of $\gamma - 1$ so that $\|\theta_0(x) - 1\|_{L^\infty(\mathbf{R})}$ can be chosen to be small when γ is close to 1. Here $\overline{s} = \frac{R}{\gamma-1}\ln\frac{R}{A}$ is the far field of the initial entropy $s_0(x)$, that is,

$$\lim_{|x|\to+\infty} s_0(x) = \lim_{|x|\to+\infty}\frac{R}{\gamma-1}\ln\frac{R\theta_0(x)v_0(x)^{\gamma-1}}{A} = \overline{s}.$$

Taking (v,u,s) as the unknown functions, the second global existence theorem can be stated as follows.

THEOREM 1.2. *Suppose we ahve the following:*

- $\|(v_0(x)-1, u_0(x), s_0(x)-\overline{s}, \Phi_{0x}(x))\|_{H^1(\mathbf{R})}$ *is bounded by some positive constant independent of* $\gamma-1$ *and* (1.5) *holds for some* $(\gamma-1)$*-independent positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$.
- $(\gamma-1)\|s_0(x)\|_{L^\infty(\mathbf{R})}$ *is bounded by some constant independent of* $\gamma-1$.
- *The smooth function* $\mu(v)$ *satisfies* $\mu(v)>0$ *for all* $v>0$ *and*

$$\lim_{v\to 0_+}\Psi(v)=-\infty, \qquad \lim_{v\to+\infty}\Psi(v)=+\infty. \tag{1.7}$$

Here,

$$\Psi(v)=\int_1^v \frac{\sqrt{z-\ln z-1}}{z}\mu(z)dz. \tag{1.8}$$

- *For the heat-conductivity coefficient, there are two cases. If* $\kappa(v,\theta)=\kappa(\theta)$ *depends only on* θ, *we only assume* $\kappa(\theta)>0$ *for* $\theta>0$ *with some smoothness condition. If it depends on both* v *and* θ, *then in addition to* $\kappa(v,\theta)>0$ *for all* $v>0, \theta>0$, *we also assume the following. Set* $\kappa_1(v)=\min_{\underline{\Theta}\le\theta\le\overline{\Theta}}\kappa(v,\theta)$ *and assume*

$$\kappa_{\theta\theta}(v,\theta)<0 \quad \forall v>0,\ \theta>0 \tag{1.9}$$

and

$$\overline{\lim_{v\to 0_+}}\frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2}=\overline{\lim_{v\to+\infty}}\frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2}=0. \tag{1.10}$$

- $\gamma-1$ *is sufficiently small.*

Then the Cauchy problem (1.1), (1.3) *admits a unique global solution* $(v(t,x),u(t,x),\theta(t,x))$ *satisfying* (1.6) *and*

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\Big|(v(t,x)-1,u(t,x),\theta(t,x)-1)\Big|=0. \tag{1.11}$$

Remark 1.1. We give the following remarks on Theorems 1.1 and 1.2.

- From the proof of Theorem 1.2, one will notice that the assumption (1.10) can be replaced by the following weaker assumption:

$$\overline{\lim_{v\to 0_+}}\frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2}\le\varepsilon_0, \quad \overline{\lim_{v\to+\infty}}\frac{\frac{\mu(v)}{\kappa_1(v)}}{|\Psi(v)|^2}\le\varepsilon_0. \tag{1.12}$$

  Here $\varepsilon_0>0$ is a suitably chosen sufficiently small positive constant.
- Under the assumptions in Theorem 1.2, when $\gamma-1$ is sufficiently small, although $\|\theta_0-1\|_{H^1(\mathbf{R})}$ is small, $\|(v_0-1,u_0,s_0-\overline{s})\|_{H^1(\mathbf{R})}$ can be large.
- When $\mu(v)$ satisfies certain growth conditions when $v\to 0_+$ and $v\to+\infty$, for example, $\mu(v)\sim v^a$ as $v\to 0_+$ and $\mu(v)\sim v^b$ as $v\to+\infty$ with $a<0, b>-\frac{1}{2}$, then a similar result to Theorem 1.2 also holds even when $(v_0-1,u_0,s_0-\overline{s})\|_{H^1(\mathbf{R})}$, $\underline{V}$, and $\overline{V}$ depend on $\frac{1}{\gamma-1}$ with certain growth conditions as $\gamma\to 1_+$.
- The same arguments for Theorems 1.1 and 1.2 can be applied directly to the compressible Navier–Stokes equations which generalize the previous results [18] and [23], where the viscosity coefficient is assumed to be a positive constant.

- It is worth pointing out that since

$$\left(\frac{\mu(v)u_x}{v}\right)_x = \left(\frac{\mu(v)v_t}{v}\right)_x = \left(\frac{\mu(v)v_x}{v}\right)_t \tag{1.13}$$

 plays an important role in the following analysis, we can only treat the case when $\mu(v)$ is a smooth function of v. Hence, it is interesting to study the case when μ depends on θ.

We now review some related results. First, there are some recent results on the construction of nonvacuum, large solutions to the one-dimensional compressible Navier–Stokes equations with constant viscosity coefficient μ and density- and temperature-dependent heat-conductivity coefficient κ; see [18], [23]. A key ingredient in these works is the pointwise a priori estimates on the specific volume which guarantees that no vacuum or concentration of mass occurs. This together with the standard maximum principle deduce a lower-bound estimate on the absolute temperature $\theta(t,x)$ and, consequently, the main effort in [18], [23] is to obtain the upper-bound estimate on $\theta(t,x)$.

The strategy to prove Theoerem 1.1 can be stated as follows. We note that, however, for the compressible NSP system (1.1), even when the viscosity coefficient $\mu(v)$ is a positive constant, the argument in [25] does not give bounds on $v(t,x)$ because of the nonlocal term $\frac{\Phi_x}{v}$. For this, to prove Theorem 1.1, we will first apply the maximum principle for second-order parabolic equations to obtain a lower-bound estimate on $\theta(t,x)$ in terms of the lower bound on $v(t,x)$ in Lemma 2.4. And then by combining the arguments used in [21] and [25], we can deduce a lower bound and an upper bound on $v(t,x)$ in terms of $\|\theta^{1-b}\|_{L^\infty([0,T]\times\mathbf{R})}$, that is, the estimates (2.35) and (2.36). These two estimates together with the $L^\infty([0,T]\times\mathbf{R})$ estimate on $\theta(t,x)$ given in Lemma 2.9 then yield the desired lower and upper bound on $v(t,x)$ and $\theta(t,x)$ provided that the parameters a and b satisfy certain conditions.

To prove Theorem 1.2, the main idea is to assume the following a priori assumption on the absolute temperature $\theta(t,x)$:

$$\frac{1}{2}\underline{\Theta} \le \theta(t,x) \le 2\overline{\Theta} \tag{1.14}$$

for $(t,x) \in [0,T]\times\mathbf{R}$. Then by some delicate energy-type estimates and by using the argument initiated in [21], we can deduce a uniform (with respect to the time variable t) lower and upper bound on $v(t,x)$ and some uniform-energy estimates on $\left\|\left(v-1,u,(\theta-1)/\sqrt{\gamma-1}\right)(t)\right\|_{H^1(\mathbf{R})}$ in terms of $\left\|\left(v_0-1,u_0,(\theta_0-1)/\sqrt{\gamma-1}\right)\right\|_{H^1(\mathbf{R})}$, $\inf_{x\in\mathbf{R}} v_0(x)$, and $\sup_{x\in\mathbf{R}} v_0(x)$. At the end, to extend the solution globally in time, we only need to close the a priori assumption (1.14) where we need the smallness of $\gamma-1$.

Before concluding the introduction, we point out that there are many results on the construction of global solutions to the NSP system (1.1). In particular, the global existence of smooth small perturbative solutions away from vacuum with the optimal time-decay estimates was recently obtained in [26] for the isentropic flow, and in [37], [16] for the nonisentropic flow. There, it is observed that the electric field affects the large time behavior of the solution so that the momentum decays at the rate $(1+t)^{-\frac{1}{4}}$ which is slower than the rate $(1+t)^{-\frac{3}{4}}$ for the compressible Navier–Stokes system, while the density tends to its asymptotic state at the rate $(1+t)^{-\frac{3}{4}}$ just like the compressible Navier–Stokes system. Moreover, the global existence of a strong solution in Besov-type space was obtained in [15]. On the other hand, it is quite different for the compressible Euler–Poisson system. In fact, it was shown in [14] that the long time convergence rate of the global irrotational solution is enhanced by the dispersion effect due to the coupling of the electric field, namely, both density and velocity tend to the equilibrium constant state at the rate $(1+t)^{-p}$ for any $p \in (1, \frac{3}{2})$.

Note that even though most of the results for the small perturbative solutions are considered for the case when μ, ν, and κ are constants, it is straightforward to show that they hold when μ, ν, and κ are smooth functions of density and temperature.

Finally, for the results with large initial data, the existence of renormalized solutions to the NSP system are obtained in [6], [33], [38]. Note that for the compressible NSP system related to the dynamics of self-gravitating gases in stars, some existence results on the weak solution (renormalized solution) were given in [8], [9], [38]. Since the analysis in these works is based on the weak-convergence argument, only isentropic polytropic gas was studied with a special requirement on the range of adiabatic exponent, i.e., $\gamma > \frac{3}{2}$ with constant viscosity coefficient. For the nonisentropic case, even for the compressible Navier–Stokes system, the only available global existence theory for large data is the construction of the so called "variational solution"; see [11].

The rest of the paper is organized as follows. The proofs of Theorems 1.1 and 1.2 will be given in sections 2 and 3, respectively.

Notations. O(1) or $C_i (i \in \mathbf{N})$ stands for a generic positive constant which is independent of t and x, while $C_i(\cdot, \cdots, \cdot)$ $(i \in \mathbf{N})$ is used to denote some positive constant depending on the arguments listed in the parenthesis. Note that all these constants may vary from line to line. $\|\cdot\|_s$ represents the norm in $H^s(\mathbf{R})$ with $\|\cdot\| = \|\cdot\|_0$ and for $1 \le p \le +\infty$, $L^p(\mathbf{R})$ denotes the standard Lebesgue space.

2. The proof of Theorem 1.1.

To prove Theorem 1.1, we first define the following function space for the solution to the Cauchy problem (1.1), (1.3):

$$
\begin{aligned}
&X(0,T;M_0,M_1;N_0,N_1) \\
&\quad = \left\{ (v,u,\theta,\Phi)(t,x) \;\middle|\; \begin{array}{c} (v-1,u,\theta-1)(t,x) \in C^0\left(0,T;H^1(\mathbf{R})\right) \\ (u_x,\theta_x)(t,x) \in L^2\left(0,T;H^1(\mathbf{R})\right) \\ \Phi_x(t,x) \in C^0\left(0,T;H^2(\mathbf{R})\right) \\ M_0 \le v(t,x) \le M_1, \quad N_0 \le \theta(t,x) \le N_1 \end{array} \right\}.
\end{aligned} \tag{2.1}
$$

Here $T > 0, M_1 \ge M_0 > 0, N_1 \ge N_0 > 0$ are some positive constants.

Under the assumptions given in either Theorems 1.1 or 1.2, we can get the following local existence result.

Lemma 2.1 (local existence). *Under the assumptions in either Theorems* 1.1 *or* 1.2*, there exists a sufficiently small positive constant t_1, which depends only on $\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}$, and $\|(v_0-1, u_0, \theta_0-1)\|_1$, such that the Cauchy problem* (1.1), (1.3) *admits a unique smooth solution $(v(t,x), u(t,x), \theta(t,x), \Phi(t,x)) \in X\left(0, t_1; \frac{1}{2}\underline{V}, 2\overline{V}; \frac{1}{2}\underline{\Theta}, 2\overline{\Theta}\right)$ and $(v(t,x), u(t,x), \theta(t,x), \Phi(t,x))$ satisfy*

$$
\begin{cases} 0 < \frac{\underline{V}}{2} \le v(t,x) \le 2\overline{V}, \\ 0 < \frac{\underline{\Theta}}{2} \le \theta(t,x) \le 2\overline{\Theta}, \end{cases} \tag{2.2}
$$

$$
\sup_{[0,t_1]} \left(\|(v-1, u, \theta-1, \Phi_x)(t)\|_1 \right) \le 2\|(v_0-1, u_0, \theta_0-1, \Phi_0)\|_1, \tag{2.3}
$$

and

$$\lim_{|x|\to\infty}(v(t,x)-1,u(t,x),\theta(t,x)-1,\Phi_x(t,x))=(0,0,0,0). \tag{2.4}$$

Lemma 2.1 can be proved by the standard iteration argument as in [32] for the one-dimensional compressible Navier–Stokes system; we thus omit the details for brevity.

Now we give some properties on the local solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ constructed above. Noticing that

$$\left(u+\left(\frac{\Phi_x}{v}\right)_t\right)_x=u_x+\left[\left(\frac{\Phi_x}{v}\right)_x\right]_t=u_x+(1-v)_t=u_x-v_t=0,$$

we have the following lemma from (2.4).

Lemma 2.2. *Under the conditions in Lemma* 2.1*, we have*

$$u(t,x)=-\left(\frac{\Phi_x(t,x)}{v(t,x)}\right)_t. \tag{2.5}$$

Now we turn to prove Theorem 1.1. Recall that $\mu(v)=v^{-a}$, $\kappa(v,\theta)=\theta^b$, and the constitutive equations (1.2), and thus the Cauchy problem (1.1), (1.3) can be rewritten as

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v,\theta)_x=\left(\frac{u_x}{v^{1+a}}\right)_x+\frac{\Phi_x}{v},\\ C_v\theta_t+p(v,\theta)u_x=\frac{u_x^2}{v^{1+a}}+\left(\frac{\theta^b\theta_x}{v}\right)_x,\\ \left(\frac{\Phi_x}{v}\right)_x=1-v,\qquad \lim_{|x|\to+\infty}\Phi(t,x)=0, \end{cases} \tag{2.6}$$

$$\begin{aligned} &(v(0,x),u(0,x),\theta(0,x))=(v_0(x),u_0(x),\theta_0(x)),\\ &\lim_{|x|\to+\infty}(v_0(x),u_0(x),\theta_0(x))=(1,0,1). \end{aligned} \tag{2.7}$$

Suppose that the local solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ constructed in Lemma 2.1 has been extended to $t=T\geq t_1$ and satisfies the a priori assumption

$$\overline{V}_0\leq v(t,x)\leq\overline{V}_1,\quad \overline{\Theta}_0\leq\theta(t,x)\leq\overline{\Theta}_1 \tag{H_1}$$

for all $x\in\mathbf{R}$, $0\leq t\leq T$, and some positive constants $0<\overline{\Theta}_0\leq\overline{\Theta}_1, 0<\overline{V}_0\leq\overline{V}_1$; we now deduce certain a priori estimates on $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ which are independent of $\overline{\Theta}_0,\overline{\Theta}_1,\overline{V}_0,\overline{V}_1$ but may depend on T.

The first one is concerned with the basic energy estimate. For this, note that

$$\eta(v,u,\theta)=R\phi(v)+\frac{u^2}{2}+\frac{R\phi(\theta)}{\gamma-1}\quad\text{with}\quad\phi(x)=x-\ln x-1$$

is a convex entropy to (2.6) which satisfies

$$\eta(v,u,\theta)_t+\left\{\left(\frac{R\theta}{v}-R\right)u\right\}_x-\left\{\frac{uu_x}{v^{1+a}}+\frac{(\theta-1)\theta_x}{v\theta^{1-b}}\right\}_x+\left\{\frac{u_x^2}{v^{1+a}\theta}+\frac{\theta_x^2}{v\theta^{2-b}}\right\}=\frac{u\Phi_x}{v}. \tag{2.8}$$

With (2.8), since

$$\frac{u\Phi_x}{v}=\left(\frac{u\Phi}{v}+\frac{\Phi}{v}\left(\frac{\Phi_x}{v}\right)_t\right)_x-\frac{1}{2}\left[\left(\frac{\Phi_x}{v}\right)^2\right]_t+\frac{\Phi v_x}{v^2}\left[u+\left(\frac{\Phi_x}{v}\right)_t\right],$$

we can deduce the following lemma by integrating (2.8) with respect to t and x over $[0,T]\times\mathbf{R}$ and from (2.5).

LEMMA 2.3 (basic energy estimates). *Let the conditions in Lemma 2.1 hold and suppose that the local solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ constructed in Lemma 2.1 satisfies the a priori assumption (H_1), then we have for $0\le t\le T$ that*

$$\begin{aligned}&\int_{\mathbf{R}}\left(\eta(v,u,\theta)+\frac{1}{2}\left(\frac{\Phi_x}{v}\right)^2\right)(t,x)dx+\int_0^t\int_{\mathbf{R}}\left(\frac{u_x^2}{v^{1+a}\theta}+\frac{\theta_x^2}{v\theta^{2-b}}\right)(\tau,x)dxd\tau\\ &\quad=\int_{\mathbf{R}}\left(\eta(v_0,u_0,\theta_0)+\frac{1}{2}\left(\frac{\Phi_{0x}}{v_0}\right)^2\right)(x)dx.\end{aligned}\tag{2.9}$$

The next estimate is concerned with a lower-bound estimate on $\theta(t,x)$ in terms of the lower bound on $v(t,x)$.

LEMMA 2.4. *Under the assumptions in Lemma 2.3, we have for $a<1$ that*

$$\frac{1}{\theta(t,x)}\le O(1)+O(1)\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a},\quad x\in\mathbf{R},\ 0\le t\le T.\tag{2.10}$$

Proof. First of all, $(2.6)_3$ implies

$$\begin{aligned}C_v\left(\frac{1}{\theta}\right)_t&=-\frac{u_x^2}{\theta^2v^{1+a}}+\frac{Ru_x}{v\theta}-\frac{2\theta^{1+b}}{v}\left[\left(\frac{1}{\theta}\right)_x\right]^2+\left[\left(\frac{\theta^b}{v}\right)\left(\frac{1}{\theta}\right)_x\right]_x\\ &=\left[\left(\frac{\theta^b}{v}\right)\left(\frac{1}{\theta}\right)_x\right]_x-\left\{\frac{2\theta^{1+b}}{v}\left[\left(\frac{1}{\theta}\right)_x\right]^2+\frac{1}{v^{1+a}\theta^2}\left(u_x-\frac{R\theta v^a}{2}\right)^2\right\}\\ &\quad+\frac{R^2}{4v^{1-a}}.\end{aligned}\tag{2.11}$$

Set

$$h(t,x)=\frac{1}{\theta}-\frac{R^2t}{4C_v}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}$$

and we can deduce that $h(t,x)$ satisfies

$$\begin{cases}C_vh_t\le\left(\frac{\theta^b}{v}h_x\right)_x, & x\in\mathbf{R},\ 0\le t\le T,\\ h(0,x)=\frac{1}{\theta_0(x)}\le\frac{1}{\underline{\Theta}},\end{cases}\tag{2.12}$$

and the standard maximum principle for parabolic equations implies that $h(t,x)\le\frac{1}{\underline{\Theta}}$ holds for all $(t,x)\in[0,T]\times\mathbf{R}$. That is, for $x\in\mathbf{R},0\le t\le T$,

$$\frac{1}{\theta}-\frac{R^2t}{4C_v}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}\le\frac{1}{\underline{\Theta}}.\tag{2.13}$$

This is (2.10) and the proof of Lemma 2.4 is completed. ∎

To use Kanel's method to deduce a lower bound and an upper bound on $v(t,x)$, we need to deduce an estimate on $\left\|\frac{v_x}{v^{1+a}}\right\|$, which is the main concern of our next lemma.

It is worth pointing out that it is in this step that we ask the viscous coefficient μ depend only on v.

LEMMA 2.5. *Under the assumptions in Lemma* 2.3, *we have*

$$\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\theta v_x^2}{v^{3+a}}+g(v)(v-1)\right)dxds \tag{2.14}$$
$$\le \left(\|v_{0x}\|^2+\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\|^2\right)$$
$$+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds+O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1+a}\theta}dxds$$

and

$$g(v)=\int_1^v\frac{dz}{z^{1+a}}=\frac{1-v^{-a}}{a}.$$

Proof. Notice that

$$\left(\frac{v_x}{v^{1+a}}\right)_t=\left(\frac{v_t}{v^{1+a}}\right)_x=\left(\frac{u_x}{v^{1+a}}\right)_x=u_t+p(v,\theta)_x-\frac{\Phi_x}{v}.$$

We have by multiplying the above identity by $\frac{v_x}{v^{1+a}}$ and integrating the resulting equation with respect to t and x over $[0,T]\times\mathbf{R}$ that

$$\frac{1}{2}\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{R\theta v_x^2}{v^{3+a}}dxds \tag{2.15}$$
$$\le O(1)\|v_{0x}\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{R\theta_x v_x}{v^{2+a}}dxds}_{I_1}+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{u_t v_x}{v^{1+a}}dxds}_{I_2}-\underbrace{\int_0^t\int_{\mathbf{R}}\frac{v_x}{v^{1+a}}\frac{\Phi_x}{v}dxds}_{I_3}.$$

Now we estimate I_1, I_2, and I_3 term by term. First, we have from $(2.6)_4$ and the Cauchy–Schwarz inequality that

$$I_3=\int_0^t\int_{\mathbf{R}}g(v)_x\left(\frac{\Phi_x}{v}\right)dxds=-\int_0^t\int_{\mathbf{R}}g(v)\left(\frac{\Phi_x}{v}\right)_x dxds \tag{2.16}$$
$$=-\int_0^t\int_{\mathbf{R}}g(v)(1-v)dxds\ge 0$$

and

$$I_1\le\frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{R\theta v_x^2}{v^{3+a}}dxds+O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1+a}\theta}dxds. \tag{2.17}$$

As to I_2, we have from (2.9) that

$$I_2=\int_{\mathbf{R}}\frac{uv_x}{v^{1+a}}dx\Big|_0^t-\int_0^t\int_{\mathbf{R}}u\left(\frac{v_x}{v^{1+a}}\right)_t dxds \tag{2.18}$$
$$\le\int_{\mathbf{R}}\frac{uv_x}{v^{1+a}}dx+O(1)\left\|(u_0,v_{0x})\right\|^2-\int_0^t\int_{\mathbf{R}}u\left(\frac{u_x}{v^{1+a}}\right)_x dxds$$
$$\le\frac{1}{2}\left\|\frac{v_x}{v^{1+a}}\right\|^2+O(1)\left\|(v_0-1,v_{0x},u_0,\theta_0-1,\Phi_{0x})\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds.$$

Inserting (2.16)–(2.18) into (2.15), we can deduce (2.14) immediately. This completes the proof of Lemma 2.5. ◻

To bound the two terms on the right-hand side of (2.14), we now estimate $\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds$ in the following lemma.

LEMMA 2.6. *Under the assumptions in Lemma 2.3, we have*

(2.19)
$$\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds\le O(1)\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\|^2+O(1)\int_0^t\int_{\mathbf{R}}\frac{(\theta-1)^2}{v^{1-a}}dxds.$$

Proof. Multiplying $(2.6)_2$ by u, we have by integrating the resulting equation with respect to t and x over $[0,T]\times\mathbf{R}$ that

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds &\\ \le O(1)\|u_0\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{R(\theta-1)u_x}{v}dxds}_{I_4}&\\ +\underbrace{\int_0^t\int_{\mathbf{R}}R\left(1-\frac{1}{v}\right)u_xdxds}_{I_5}+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{u\Phi_x}{v}dxds}_{I_6}.&\end{aligned}\tag{2.20}$$

From the basic energy estimate (2.9) and the Cauchy–Schwarz inequality, we can bound I_j $(j=4,5,6)$ as follows:

$$\begin{aligned}I_6&\le\int_0^t\|u(s)\|\left\|\left(\frac{\Phi_x}{v}\right)(s)\right\|ds\le C(T)\|(u_0,v_0-1,\theta_0-1,\Phi_{0x})\|^2,\\ I_5&=\int_0^t\int_{\mathbf{R}}R\left(1-\frac{1}{v}\right)v_tdxds=R\int_{\mathbf{R}}\phi(v)dx\Big|_0^t\\ &=R\left(\int_{\mathbf{R}}\phi(v)dx-\int_{\mathbf{R}}\phi(v_0)dx\right)\\ &\le O(1)\|(u_0,v_0-1,\theta_0-1,\Phi_{0x})\|^2,\\ I_4&\le\frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxds+O(1)\int_0^t\int_{\mathbf{R}}\frac{(\theta-1)^2}{v^{1-a}}dxds.\end{aligned}$$

Substituting the above estimates into (2.20), we can deduce (2.19) and complete the proof of the lemma. ◻

To bound the terms appearing on the right-hand side of (2.19) and (2.14), we need the following lemma.

LEMMA 2.7. *Under the assumptions in Lemma 2.3, we have for $b\neq 0,-1$ that*

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^bds\le C(T),\tag{2.21}$$

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^{b+1}ds\le C(T)\left(1+\|\theta\|_{L^\infty_{T,x}}\right),\tag{2.22}$$

and

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^{b+1}ds\le C(T)\left(1+\|v\|_{L^\infty_{T,x}}\right).\tag{2.23}$$

Proof. We only prove (2.22) because (2.21) and (2.23) can be proved similarly.

From the argument used in [25] we have, from the basic energy estimate (2.9), the Jenssen inequality that for each $i \in \mathbf{Z}$, there are positive constants $A_0 > 0, A_1 > 0$ which are independent of i such that

$$A_0 \leq \int_i^{i+1} v(t,x)dx, \ \int_i^{i+1} \theta(t,x)dx \leq A_1 \quad \forall t \in [0,T]. \tag{2.24}$$

Hence, there exist $a_i(t) \in [i, i+1]$, $b_i(t) \in [i, i+1]$ such that

$$A_0 \leq v(t, a_i(t)), \ \theta(t, b_i(t)) \leq A_1. \tag{2.25}$$

Define

$$g_1(\theta) = \int_1^\theta s^{\frac{b-1}{2}} ds = \frac{2}{b+1}\left(\theta^{\frac{b+1}{2}} - 1\right);$$

for each $x \in \mathbf{R}$, there exists an integer $i \in \mathbf{Z}$ such that $x \in [i, i+1]$ and we can assume without loss of generality that $x \geq b_i(t)$. Thus

$$\begin{aligned}
g_1(\theta(t,x)) &= g_1(\theta(t,b_i(t))) + \int_{b_i(t)}^x g_1(\theta(t,y))_y dy \\
&\leq O(1) + \int_i^{i+1} \left|\theta^{\frac{b-1}{2}}\theta_x\right| dx \\
&\leq O(1) + \left(\int_{\mathbf{R}} \frac{\theta_x^2}{v\theta^{2-b}} dx\right)^{\frac{1}{2}} \left(\int_i^{i+1} v\theta dx\right)^{\frac{1}{2}} \\
&\leq O(1) + \|\theta\|_{L^\infty_{T,x}}^{\frac{1}{2}} \left(\int_{\mathbf{R}} \frac{\theta_x^2}{v\theta^{2-b}} dx\right)^{\frac{1}{2}}.
\end{aligned}$$

The above estimate and (2.9) give (2.22) and thus completes the proof of the lemma. ∎

As a direct corollary of (2.21)–(2.23), we have the following corollary.

Corollary 2.1. *Under the conditions in Lemma* 2.3, *we have*

$$\int_0^t \int_{\mathbf{R}} (\theta - 1)^2 dxds \leq O(1) \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}. \tag{2.26}$$

Proof. In fact, (2.9) together with (2.21) imply that

$$\begin{aligned}
\int_0^t \int_{\mathbf{R}} (\theta-1)^2 dxd\tau &\leq O(1) \int_0^t \int_{\mathbf{R}} (\theta+1)\phi(\theta) dxd\tau \\
&\leq O(1) \int_0^t \max_{x\in\mathbf{R}} \theta(\tau,x) d\tau + O(1) \\
&= O(1) \int_0^t \max_{x\in\mathbf{R}} \left(\theta^{1-b}\theta^b\right) d\tau + O(1) \\
&\leq O(1) \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}} \int_0^t \max_{x\in\mathbf{R}} \theta^b(\tau,x) d\tau + O(1) \\
&\leq O(1) \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}} + O(1),
\end{aligned}$$

and this completes the proof of the corollary. ∎

Having obtained (2.26), we can deduce that

$$\int_0^t\int_{\mathbf{R}}\frac{(\theta-1)^2}{v^{1-a}}dxd\tau\le\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}\int_0^t\int_{\mathbf{R}}(\theta-1)^2dxd\tau\le O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}. \tag{2.27}$$

On the other hand, from (2.9), we have

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{\theta v^{1+a}}dxd\tau&=\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}\frac{1}{v^a\theta^{b-1}}dxd\tau\\&\le\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}dxd\tau\\&\le O(1)\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\|^2\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}.\end{aligned} \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.19) and (2.14), we have the following corollary.

Corollary 2.2. *Under the assumptions in Lemma* 2.3, *we have*

$$\begin{aligned}&\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxd\tau\\&\quad\le O(1)\left\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\right\|^2+O(1)\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}},\end{aligned} \tag{2.29}$$

$$\begin{aligned}&\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\theta v_x^2}{v^{3+a}}+g(v)(v-1)\right)dxd\tau\\&\quad\le O(1)\left\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\right\|^2+O(1)\left(\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}\right)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}.\end{aligned} \tag{2.30}$$

Now we apply Kanel's approach to deduce a lower bound and an upper bound on $v(t,x)$ in terms of $\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}$. To this end, set

$$\Psi(v)=\int_1^v\frac{\sqrt{\phi(z)}}{z^{1+a}}dz. \tag{2.31}$$

Note that there exist positive constants A_2, A_3 such that

$$|\Psi(v)|\ge A_2\left(v^{-a}+v^{\frac{1}{2}-a}\right)-A_3. \tag{2.32}$$

Since

$$\begin{aligned}|\Psi(v)|&=\left|\int_{-\infty}^x\Psi(v(t,y))_y dy\right|\\&\le\int_{\mathbf{R}}\left|\frac{\sqrt{\phi(v)}}{v^{1+a}}v_x\right|dx\\&\le\left\|\sqrt{\phi(v)}\right\|\left\|\frac{v_x}{v^{1+a}}\right\|\\&\le O(1)\left(1+\left(\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{\frac{a}{2}}+\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{\frac{1-a}{2}}\right)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{1}{2}}\right),\end{aligned} \tag{2.33}$$

we have from (2.32) and (2.33) that

$$\left\|\frac{1}{v}\right\|^{a}_{L^\infty_{T,x}}+\|v\|^{\frac12-a}_{L^\infty_{T,x}}\leq O(1)\left(1+\left(\left\|\frac{1}{v}\right\|^{\frac{a}{2}}_{L^\infty_{T,x}}+\left\|\frac{1}{v}\right\|^{\frac{1-a}{2}}_{L^\infty_{T,x}}\right)\left\|\theta^{1-b}\right\|^{\frac12}_{L^\infty_{T,x}}\right). \tag{2.34}$$

Thus if $\frac13<a<\frac12$, we can deduce from (2.34) the following corollary.

COROLLARY 2.3. *Under the conditions in Lemma* 2.3*, if we assume further that* $\frac13<a<\frac12$*, then we have*

$$\frac{1}{v(t,x)}\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{1}{3a-1}}_{L^\infty_{T,x}}\right) \tag{2.35}$$

and

$$v(t,x)\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{2a}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right) \tag{2.36}$$

hold for any $(t,x)\in[0,T]\times\mathbf{R}$.

Consequently, (2.29) *and* (2.30) *can be rewritten as*

$$\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}dxd\tau\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{2a}{3a-1}}_{L^\infty_{T,x}}\right), \tag{2.37}$$

$$\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\theta v_x^2}{v^{3+a}}+g(v)(v-1)\right)dxd\tau\leq O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{2a}{3a-1}}_{L^\infty_{T,x}}\right). \tag{2.38}$$

To get an upper bound on $\theta(t,x)$, we need also the estimate on $\|u_x(t)\|$ which is given in the following lemma.

LEMMA 2.8. *Under the conditions listed in Lemma* 2.3*, we have for* $0\leq t\leq T$ *that*

$$\begin{aligned}&\|u_x(t)\|^2+\|v(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau\\&\quad\leq O(1)\left\|(v_0-1,u_0,\theta_0-1,\Phi_{0x})\right\|^2+O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\&\qquad+O(1)\left(1+\left\|\theta^{1-b}\right\|^{\frac{2(2a-2a^2+1)}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right).\end{aligned} \tag{2.39}$$

Proof. By differentiating $(2.6)_2$ with respect to x, multiplying the resulting identity by u_x, and integrating the result with respect to t and x over $[0,T]\times\mathbf{R}$, we have

$$\begin{aligned}&\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau+\|v-1\|^2\\&\quad\leq O(1)\|u_{0x}\|^2+\underbrace{2\int_0^t\int_{\mathbf{R}}u_{xx}p(v,\theta)_x dxd\tau}_{I_7}+\underbrace{2(1+a)\int_0^t\int_{\mathbf{R}}\frac{u_xv_xu_{xx}}{v^{2+a}}dxd\tau}_{I_8}.\end{aligned} \tag{2.40}$$

For I_7, we have from (2.9) that

$$
\begin{aligned}
I_7 &= 2R\int_0^t\int_{\mathbf{R}} u_{xx}\left(\frac{\theta_x}{v}-\frac{\theta v_x}{v^2}\right)dxd\tau \\
&\le \frac{1}{4}\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1-a}}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta^2 v_x^2}{v^{3-a}}dxd\tau \\
&\le \frac{1}{4}\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau + O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a^2}{(3a-1)(1-2a)}}\right) \\
&\quad + O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{7a-4a^2-1}{(3a-1)(1-2a)}}\right).
\end{aligned} \tag{2.41}
$$

Here we have used the fact that

$$
\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1-a}}dxd\tau &= \int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}v^a\theta^{2-b}dxd\tau \\
&\le O(1)\|v\|_{L^\infty_{T,x}}^a\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}dxd\tau \\
&\le O(1)\|v\|_{L^\infty_{T,x}}^a\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}} \\
&\le O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a^2}{(3a-1)(1-2a)}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{\theta^2 v_x^2}{v^{3-a}}dxd\tau &= \int_0^t\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}\frac{\theta^2}{v^{1-3a}}dxd\tau \\
&\le \int_0^t\int_{\mathbf{R}}\left(\max_{x\in\mathbf{R}}\theta^2(s,x)ds\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\left(\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}dx\right)d\tau \\
&\le O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^2(s,x)ds\right) \\
&\le O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^{1-b}\theta^{1+b}(s,x)ds\right) \\
&\le O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{5a-1}{3a-1}}\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^{1+b}(s,x)ds\right) \\
&\le O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{5a-1}{3a-1}}\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\left(1+\|v\|_{L^\infty_{T,x}}\right) \\
&\le O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{7a-4a^2-1}{(3a-1)(1-2a)}}\right),
\end{aligned}
$$

where (2.9), (2.21)–(2.23), and (2.38) are used.

As for I_8, since (2.36), (2.37), together with the Sobolev inequality, imply

$$
\begin{aligned}
\int_0^t\|u_x(\tau)\|_{L^\infty_x}^2d\tau &\le \int_0^t\|u_x(\tau)\|\|u_{xx}(\tau)\|d\tau \\
&\le \left(\int_0^t\|u_x(\tau)\|^2d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|u_{xx}(\tau)\|^2d\tau\right)^{\frac{1}{2}}
\end{aligned} \tag{2.42}
$$

$$
\begin{aligned}
&\leq \|v\|_{L^\infty_{T,x}}^{1+a}\left(\int_0^t\left\|\frac{u_x}{v^{\frac{1+a}{2}}}(\tau)\right\|^2 d\tau\right)^{\frac12}\left(\int_0^t\left\|\frac{u_{xx}}{v^{\frac{1+a}{2}}}(\tau)\right\|^2 d\tau\right)^{\frac12}\\
&\leq O(1)\left(1+\|\theta^{1-b}\|_{L^\infty_{T,x}}^{\frac{3a}{(3a-1)(1-2a)}}\right)\left(\int_0^t\left\|\frac{u_{xx}}{v^{\frac{1+a}{2}}}(\tau)\right\|^2 d\tau\right)^{\frac12},
\end{aligned}
$$

we can deduce from (2.35)–(2.38) that

(2.43)

$$
\begin{aligned}
I_8 &\leq \frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{u_x^2v_x^2}{v^{3+a}}dxd\tau\\
&\leq \frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau + O(1)\int_0^t\left\|\frac{u_x^2}{v^{1-a}}\right\|_{L^\infty_x}\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}dxd\tau\\
&\leq \frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau + O(1)\left\|\frac1v\right\|_{L^\infty_{T,x}}^{1-a}\left(1+\|\theta^{1-b}\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right)\int_0^t\|u_x(\tau)\|_{L^\infty_x}^2 d\tau\\
&\leq \frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau + O(1)\left(1+\|\theta^{1-b}\|_{L^\infty_{T,x}}^{\frac{2a-2a^2+1}{(3a-1)(1-2a)}}\right)\left(\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau\right)^{\frac12}\\
&\leq \frac12\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau + O(1)\left(1+\|\theta^{1-b}\|_{L^\infty_{T,x}}^{\frac{2(2a-2a^2+1)}{(3a-1)(1-2a)}}\right).
\end{aligned}
$$

Putting (2.40), (2.41), and (2.43) together and noticing that $2(2a-2a^2+1) > 7a-4a^2-1$ imply (2.39), and this completes the proof of Lemma 2.8. ∎

Now we turn to deduce the upper bound on $\theta(t,x)$.

LEMMA 2.9. *Under the conditions in Lemma* 2.3*, we have*

$$
\|\theta\|_{L^\infty_{T,x}} \leq O(1)\left\{1+\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty_x}+\left\|\frac{u_x^2}{v^2}\right\|_{L^\infty_x}+\|\theta\|_{L^\infty_x}^2\right)d\tau\right\}. \tag{2.44}
$$

Proof. From $(2.6)_3$, it is easy to see that for each $p>1$,

$$
\begin{aligned}
&C_v\left[(\theta-1)^{2p}\right]_t + 2p(2p-1)(\theta-1)^{2(p-1)}\frac{\theta^b\theta_x^2}{v}\\
&\quad=\left\{\frac{2p(\theta-1)^{2p-1}\theta^b\theta_x}{v}\right\}_x+\frac{2p(\theta-1)^{2p-1}}{v^{1+a}}u_x^2-\frac{2pR\theta}{v}u_x(\theta-1)^{2p-1}.
\end{aligned}
\tag{2.45}
$$

Integrating (2.45) with respect to x over $\mathbf{R}$, we have

$$
C_v\left(\|\theta-1\|_{L^{2p}}^{2p}\right)_t \leq \underbrace{2p\int_{\mathbf{R}}\frac{u_x^2(\theta-1)^{2p-1}}{v^{1+a}}dx}_{I_9}-\underbrace{2pR\int_{\mathbf{R}}\frac{\theta u_x(\theta-1)^{2p-1}}{v}dx}_{I_{10}}. \tag{2.46}
$$

Since

$$
\begin{aligned}
I_9 &\leq 2pO(1)\|\theta-1\|_{L^{2p}}^{2p-1}\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^{2p}},\\
I_{10} &\leq 2pO(1)\|\theta-1\|_{L^{2p}}^{2p-1}\left\|\frac{\theta u_x}{v}\right\|_{L^{2p}}
\end{aligned}
$$

hold for some positive constant $O(1)$ independent of p, we have

$$\|\theta-1\|_{L^{2p}} \le O(1) + O(1)\int_0^t \left(\left\| \frac{u_x^2}{v^{1+a}} \right\|_{L^{2p}} + \left\| \frac{\theta u_x}{v} \right\|_{L^{2p}} \right) d\tau. \tag{2.47}$$

Letting $p \to \infty$ in (2.47) and exploiting the Cauchy inequality, we can deduce (2.44) immediately and the proof of Lemma 2.9 is complete. ◻

We are now ready to use (2.35), (2.36), and (2.44) to deduce a lower bound and an upper bound on $\theta(t,x)$. First, we have from (2.42) and (2.39) that

$$\begin{aligned}
\int_0^t \|u_x(s)\|^2_{L^\infty_x} ds &\le O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{3a}{(3a-1)(1-2a)}}\right) \\
&\quad \times \left[\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}^{\frac12} \left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{a^2}{(3a-1)(1-2a)}}\right) + 1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a-2a^2+1}{(3a-1)(1-2a)}} \right] \\
&\le O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}^{\frac12} \left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{3a+a^2}{(3a-1)(1-2a)}}\right) \\
&\quad + O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{5a-2a^2+1}{(3a-1)(1-2a)}} + O(1).
\end{aligned} \tag{2.48}$$

Thus, we have from (2.35)–(2.36), (2.48) that

$$\begin{aligned}
&\int_0^t \left(\left\| \frac{u_x^2}{v^{1+a}} \right\|_{L^\infty_x} + \left\| \frac{u_x^2}{v^2} \right\|_{L^\infty_x} \right) d\tau \\
&\quad \le O(1)\left(\left\|\frac1v\right\|_{L^\infty_{T,x}}^{1+a} + \left\|\frac1v\right\|_{L^\infty_{T,x}}^{2} \right) \int_0^t \|u_x(\tau)\|^2_{L^\infty_x} d\tau \\
&\quad \le O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2}{3a-1}}\right) \int_0^t \|u_x(\tau)\|^2_{L^\infty_x} d\tau \\
&\quad \le O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}^{\frac12} \left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{a^2-a+2}{(3a-1)(1-2a)}}\right) + O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{3+a-2a^2}{(3a-1)(1-2a)}} + O(1)
\end{aligned} \tag{2.49}$$

and

$$\begin{aligned}
\int_0^t \max_{x\in\mathbf{R}} \theta^2(s,x) ds &\le \int_0^t \max_{x\in\mathbf{R}} \left(\theta^{1-b}(s,x)\theta^{b+1}(s,x)\right) ds \\
&\le \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}} \int_0^t \max_{x\in\mathbf{R}} \theta^{1+b}(s,x) ds \\
&\le O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}} \left(1 + \|v\|_{L^\infty_{T,x}}\right) \\
&\le O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{7a-6a^2-1}{(3a-1)(1-2a)}}\right).
\end{aligned} \tag{2.50}$$

Inserting (2.49) and (2.50) into (2.44) yields

$$\begin{aligned}
\|\theta\|_{L^\infty_{T,x}} &\le O(1) + O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}^{\frac12} \left(1 + \left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{a^2-a+2}{(3a-1)(1-2a)}}\right) \\
&\quad + O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{3+a-2a^2}{(3a-1)(1-2a)}} + O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{7a-6a^2-1}{(3a-1)(1-2a)}}
\end{aligned} \tag{2.51}$$

$$
\begin{aligned}
&\leq O(1)+O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}^{\frac12}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{a^2-a+2}{(3a-1)(1-2a)}}\right)\\
&\quad+O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{3+a-2a^2}{(3a-1)(1-2a)}}.
\end{aligned}
$$

Based on the estimate (2.10), (2.35), (2.36), and (2.51), we have the following corollary.

COROLLARY 2.4. *Under the assumptions in Lemma* 2.3*, we further assume that* $\frac13 < a < \frac12$ *and one of the following conditions holds:*

(i) $1 \leq b < \frac{2a}{1-a} < 2$*;*

(ii) $0 < b < 1, \frac{2-b}{2} + \frac{(a^2-a+2)(1-b)}{(3a-1)(1-2a)} < 1, \frac{(1-b)(3+a-2a^2)}{(3a-1)(1-2a)} < 1$.

Then there exist positive constants $V_1 > 0, \Theta_1 > 0$*, such that*

$$
\begin{cases} V_1^{-1} \leq v(t,x) \leq V_1, \\ \Theta_1^{-1} \leq \theta(t,x) \leq \Theta_1. \end{cases} \tag{2.52}
$$

Proof. We first consider the case $b \geq 1$. In this case, we have from (2.10), (2.35), and (2.36) that

$$
\left\|\frac{1}{\theta}\right\|_{L^\infty_{T,x}} \leq O(1)+O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{1-a}{3a-1}} \leq O(1)+O(1)\left\|\frac{1}{\theta}\right\|_{L^\infty_{T,x}}^{\frac{(1-a)(b-1)}{3a-1}}
$$

which, together with the assumption $b < \frac{2a}{1-a}$, implies that there exists a positive constant $\Theta_1 > 0$ such that

$$
\theta(t,x) \geq \Theta_1^{-1} > 0 \quad \forall (t,x) \in [0,T]\times \mathbf{R}. \tag{2.53}
$$

Moreover, (2.35), (2.36), (2.53) together with the fact that $b \geq 1$ imply that there exists a positive constant $V_1 > 0$, which may depend on T, such that

$$
V_1^{-1} \leq v(t,x) \leq V_1 \quad \forall (t,x) \in [0,T]\times \mathbf{R}. \tag{2.54}
$$

Thus to prove (2.52), we only need to deduce the upper bound on $\theta(t,x)$. For this purpose, we have from the fact $1 \leq b < \frac{2a}{1-a} < 2$, (2.53), and (2.51) that

$$
\begin{aligned}
\|\theta\|_{L^\infty_{T,x}} &\leq O(1)+O(1)\|\theta\|_{L^\infty_{T,x}}^{\frac{2-b}{2}}\left(1+\left\|\tfrac{1}{\theta}\right\|_{L^\infty_{T,x}}^{\frac{(a^2-a+2)(b-1)}{(3a-1)(1-2a)}}\right)+O(1)\left\|\tfrac{1}{\theta}\right\|_{L^\infty_{T,x}}^{\frac{(3+a-2a^2)(b-1)}{(3a-1)(1-2a)}}\\
&\leq O(1)\left(1+\|\theta\|_{L^\infty_{T,x}}^{\frac{2-b}{2}}\right).
\end{aligned} \tag{2.55}
$$

From (2.55) and the fact that $0 < \frac{2-b}{2} < 1$, one can easily deduce an upper bound on $\theta(t,x)$. This completes the proof of (2.52) for the case $1 \leq b < \frac{2a}{1-a}$.

When $b < 1$, we have from (2.51) that

$$
\|\theta\|_{L^\infty_{T,x}} \leq O(1)+O(1)\|\theta\|_{L^\infty_{T,x}}^{\frac{2-b}{2}+\frac{(a^2-a+2)(1-b)}{(3a-1)(1-2a)}}+O(1)\|\theta\|_{L^\infty_{T,x}}^{\frac{(3+a-2a^2)(1-b)}{(3a-1)(1-2a)}}. \tag{2.56}
$$

From (2.56) and the assumption (ii) of Corollary 2.4, we can deduce an upper bound on $\theta(t,x)$. With this, the lower and upper bound on $v(t,x)$ can be deduced from (2.35) and (2.36). And then (2.10) implies the lower bound on $\theta(t,x)$. This completes the proof of the corollary. ∎

With Corollary 2.4, Theorem 1.1 follows from the standard continuation argument.

3. The proof of Theorem 1.2. First of all, the local solvability of the Cauchy problem (1.1),(1.3) in the function space $X\left(0, t_1; \frac{1}{2}\underline{V}, 2\overline{V}; \frac{1}{2}\underline{\Theta}, 2\overline{\Theta}\right)$ with t_1 depending on $\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}, \|(v_0 - 1, v_0, \theta_0 - 1, \Phi_{0x})\|_1$ can be proved as in Lemma 3.1. Suppose this solution $(v(t,x), u(t,x), \theta(t,x), \Phi(t,x))$ is extended to $t = T \geq t_1$. To apply the continuation argument for global existence, we first set the following a priori estimate:

$$(H_2) \qquad \frac{1}{2}\underline{\Theta} \leq \theta(t,x) \leq 2\overline{\Theta}, \quad (t,x) \in [0,T] \times \mathbf{R}.$$

Here without loss of generality, we can assume that $0 < \underline{\Theta} < 1 < \overline{\Theta}$.

Note that the smallness of $\gamma - 1$ is needed to close the a priori estimate, the generic constants used later are independent of $\gamma - 1$, and, the dependence on this factor will be clearly stated in the estimates when needed.

Similar to Lemma 2.3 we have the following basic energy estimate.

Lemma 3.1. *Under the conditions in Theorem* 1.2, *we have for* $0 \leq t \leq T$ *that*

$$\begin{aligned} &\int_{\mathbf{R}} \left\{ R\phi(v) + \frac{u^2}{2} + \frac{R}{\gamma-1}\phi(\theta) + \frac{\Phi_x^2}{2v^2} \right\}(t,x)dx + \int_0^t \int_{\mathbf{R}} \left(\frac{\mu(v)u_x^2}{v\theta} + \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} \right) dxd\tau \\ &\quad = \int_{\mathbf{R}} \left(R\phi(v_0) + \frac{u_0^2}{2} + \frac{R}{\gamma-1}\phi(\theta_0) + \frac{\Phi_{0x}^2}{2v_0^2} \right)(x)dx. \end{aligned} \tag{3.1}$$

Here, as in section 2, $\phi(x) = x - \ln x - 1$.

Now, we deduce an estimate on $\|\frac{\mu(v)v_x}{v}\|$. For this, similar to Lemma 2.5, we can deduce

$$\begin{aligned} &\left\| \frac{\mu(v)v_x}{v} \right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\mu(v)\theta v_x^2}{v^3} dxd\tau + \int_0^t \int_{\mathbf{R}} g(v)(1-v)dxd\tau \\ &\quad \leq O(1)\|v_{0x}\|^2 + O(1) \underbrace{\int_0^t \int_{\mathbf{R}} \frac{\mu(v)u_x^2}{v} dxd\tau}_{J_1} + O(1) \underbrace{\int_0^t \int_{\mathbf{R}} \frac{\mu(v)\theta_x^2}{v\theta} dxd\tau}_{J_2}. \end{aligned} \tag{3.2}$$

If the a priori estimate (H_2) holds, we have from (3.1) and the assumptions imposed on $\kappa(v,\theta)$ in Theorem 1.2 that

$$J_1 \leq O(1) \int_0^t \int_{\mathbf{R}} \frac{\mu(v)u_x^2}{v\theta} dxd\tau \leq O(1) \left\| \left(v_0 - 1, u_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}}, \Phi_{0x} \right) \right\|^2 \tag{3.3}$$

and

$$\begin{aligned} J_2 &\leq \int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} \frac{\theta\mu(v)}{\kappa(v,\theta)} dxd\tau \\ &\leq O(1) \left\| \frac{\mu(v)}{\kappa_1(v)} \right\|_{L^\infty} \int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} dxd\tau \\ &\leq O(1) \left\| \left(v_0 - 1, u_0, \frac{\theta_0 - 1}{\sqrt{\gamma-1}}, \Phi_{0x} \right) \right\|^2 \left\| \frac{\mu(v)}{\kappa_1(v)} \right\|_{L^\infty}. \end{aligned} \tag{3.4}$$

Putting (3.2), (3.3), and (3.4) together, we obtain the following lemma.

Lemma 3.2. *Under the assumptions in Lemma* 3.1 *and the a priori assumption* (H_2), *we have*

$$\left\|\frac{\mu(v)v_x}{v}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)v_x^2}{v^3}dxd\tau \le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|^2\left(1+\left\|\frac{\mu(v)}{\kappa_1(v)}\right\|_{L^\infty_{T,x}}\right). \tag{3.5}$$

Having obtained (3.1) and (3.5), we can use Kanel's argument, see [21], to deduce the lower and upper bounds on $v(t,x)$ as follows.

LEMMA 3.3. *Under the assumptions in Theorem* 1.2 *and Lemma* 3.2*, there exists a positive constant* $V_2\geq 1$*, which depends only on* $\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|$, $\underline{V},\overline{V},\underline{\Theta}$*, and* $\overline{\Theta}$*, but is independent of* T*, such that*

$$V_2^{-1}\leq v(t,x)\leq V_2,\quad (t,x)\in[0,T]\times\mathbf{R}. \tag{3.6}$$

Proof. Define

$$\Psi(v)=\int_1^v\frac{\sqrt{\phi(z)}}{z}\mu(z)dz,\quad \phi(z)=z-\ln z-1$$

and notice that

$$|\Psi(v)|=\left|\int_{-\infty}^x\Psi(v(t,y))_ydy\right|\leq\int_{\mathbf{R}}\left|\sqrt{\phi(v)}\frac{\mu(v)v_x}{v}\right|dx\leq\|\phi(v)\|_{L^1}^{\frac12}\left\|\frac{\mu(v)v_x}{v}\right\| \le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|^2\left(1+\left\|\frac{\mu(v)}{\kappa_1(v)}\right\|_{L^\infty_{T,x}}\right)^{\frac12}.$$

It is straightforward to deduce (3.6) from the assumptions in Theorem 1.2. This completes the proof of the lemma. ∎

The next lemma is about the estimate on $\|u_x(t)\|$.

LEMMA 3.4. *Under the assumptions in Lemma* 3.3*, we have for each* $0\leq t\leq T$ *that*

$$\|u_x(t)\|^2+\|v(t)-1\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}dxd\tau \le O(1)\left\|\left(v_0-1,v_{0x},u_{0x},u_{0x},\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|^6. \tag{3.7}$$

Since $v(t,x)$ satisfies (3.6) and $\theta(t,x)$ is assumed to satisfy the a priori estimate (H_2), (3.7) can be proved by applying the argument used in the proof of Lemma 2.8. Thus, we omit the detail for brevity.

To close the a priori estimate (H_2), we need to deduce an estimate on $\|\theta_x(t)\|$. For the case when $\kappa(v,\theta)\equiv\kappa(\theta)$, we have the following lemma.

LEMMA 3.5. *Under the assumptions in Theorem* 1.2 *and Lemma* 3.3*, we have*

$$\int_{\mathbf{R}}\frac{|K(\theta)_x|^2}{\gamma-1}dx+\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau \le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10}. \tag{3.8}$$

Here

$$K(\theta)=\int_1^\theta\kappa(z)dz. \tag{3.9}$$

Proof. Multiplying $(1.1)_3$ by $\kappa(\theta)$ and differentiating the resulting equation with respect to x, we get

$$C_vK(\theta)_{tx}+(\kappa(\theta)p(v,\theta)u_x)_x=\left(\frac{\kappa(\theta)\mu(v)u_x^2}{v}\right)_x+\left[\kappa(\theta)\left(\frac{K(\theta)_x}{v}\right)_x\right]_x. \tag{3.10}$$

Multiplying (3.10) by $K(\theta)_x$ and integrating with respect to t and x over $[0,t]\times\mathbf{R}$ give

$$\begin{aligned}&\int_{\mathbf{R}}\frac{C_v}{2}|K(\theta)_x|^2dx+\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau\\ &\quad\le O(1)\left\|\frac{\theta_{0x}}{\sqrt{\gamma-1}}\right\|^2+O(1)\underbrace{\int_0^t\int_{\mathbf{R}}\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|\left|\frac{K(\theta)_x}{v}v_x\right|dxd\tau}_{J_3}\\ &\qquad+\underbrace{\int_0^t\int_{\mathbf{R}}K(\theta)_x\left(\frac{\kappa(\theta)\mu(v)u_x^2}{v}\right)_xdxd\tau}_{J_4}-\underbrace{\int_0^t\int_{\mathbf{R}}K(\theta)_x\left(\kappa(\theta)p(v,\theta)u_x\right)_xdxd\tau}_{J_5}.\end{aligned}\tag{3.11}$$

Notice that

$$\begin{aligned}\|\theta_x\|_{L_x^\infty}&\le O(1)\left\|\frac{K(\theta)_x}{v}\right\|_{L_x^\infty}\le O(1)\left\|\frac{K(\theta)_x}{v}\right\|^{\frac12}\left\|\left(\frac{K(\theta)_x}{v}\right)_x\right\|^{\frac12}\\ &\le O(1)\|\theta_x\|^{\frac12}\left\|\left(\frac{K(\theta)_x}{v}\right)_x\right\|^{\frac12};\end{aligned}\tag{3.12}$$

we have from (3.1), (3.5), (3.6), and the a priori estimate (H_2) that

$$\begin{aligned}J_3&\le\frac{1}{12}\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}v_x^2\theta_x^2dxd\tau\\ &\le\frac{1}{12}\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau\\ &\quad+O(1)\int_0^t\|v_x(\tau)\|^2\|\theta_x(\tau)\|\left\|\sqrt{v\kappa(\theta)}\left(\frac{K(\theta)_x}{v}\right)_x\right\|d\tau\\ &\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau+O(1)\int_0^t\|v_x(\tau)\|^4\|\theta_x(\tau)\|^2d\tau\\ &\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau+O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^6,\end{aligned}\tag{3.13}$$

$$\begin{aligned}J_4&=-\int_0^t\int_{\mathbf{R}}\kappa(\theta)\mu(v)u_x^2\left(\frac{K(\theta)_x}{v}\right)_xdxd\tau-\int_0^t\int_{\mathbf{R}}\frac{K(\theta)_xv_x\kappa(\theta)\mu(v)u_x^2}{v^2}dxd\tau\\ &\le\frac{1}{12}\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}u_x^4dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\theta_x^2v_x^2dxd\tau\\ &\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2dxd\tau+O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10},\end{aligned}\tag{3.14}$$

and

(3.15)
$$\begin{aligned}J_5 &= \int_0^t\int_{\mathbf{R}}\left(\frac{K(\theta)_x}{v}\right)_x v\kappa(\theta)p(v,\theta)u_x dxd\tau + \int_0^t\int_{\mathbf{R}}\frac{K(\theta)_x}{v}v_x\kappa(\theta)p(v,\theta)u_x dxd\tau\\ &\le \frac{1}{12}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\left(u_x^2+\theta_x^2 v_x^2\right)dxd\tau\\ &\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} v\kappa(\theta)\left|\left(\frac{K(\theta)_x}{v}\right)_x\right|^2 dxd\tau + O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^6.\end{aligned}$$

Here we have used the fact that

$$\begin{aligned}\int_0^t\int_{\mathbf{R}} u_x^4 dxd\tau &\le O(1)\int_0^t \|u_x(\tau)\|^2\|u_x(\tau)\|^2_{L_x^\infty}d\tau\\ &\le O(1)\int_0^t \|u_x(\tau)\|^3\|u_{xx}(\tau)\|d\tau\\ &\le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10}.\end{aligned} \tag{3.16}$$

Inserting (3.13)–(3.15) into (3.11), we deduce (3.8) and complete the proof of the lemma. □

Now we turn to the case when $\kappa(v,\theta)$ depends on both v and θ. For this, we have the following lemma.

LEMMA 3.6. *Under the assumptions in Lemma* 3.5*, if* $\kappa_{\theta\theta}(v,\theta) < 0$ *holds for* $v > 0, \theta > 0$*, then we have*

$$\begin{aligned}&\left\|\frac{\theta_x(t)}{\sqrt{\gamma-1}}\right\|^2 + \int_0^t \frac{\kappa(v,\theta)}{v}\theta_{xx}^2 dxd\tau - \int_0^t\int_{\mathbf{R}}\frac{\kappa_{\theta\theta}(v,\theta)}{v}\theta_x^4 dxd\tau\\ &\quad\le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^6.\end{aligned} \tag{3.17}$$

Proof. Differentiating $(1.1)_3$ with respect to x and multiplying the resulting equation by θ_x, we have by integrating over $[0,t]\times\mathbf{R}$ that

$$\begin{aligned}&\frac{C_v}{2}\|\theta_x(t)\|^2 + \int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2 dxd\tau\\ &\quad = \frac{C_v}{2}\|\theta_{0x}\|^2 + \underbrace{\int_0^t\int_{\mathbf{R}}\theta_x\left(\frac{\mu(v)}{v}u_x^2\right)_x dxd\tau}_{J_6} - \underbrace{\int_0^t\int_{\mathbf{R}}\theta_x\left(\frac{\kappa(v,\theta)}{v}\right)_x\theta_{xx}dxd\tau}_{J_7}\\ &\qquad + \underbrace{\int_0^t\int_{\mathbf{R}}\theta_{xx}p(v,\theta)u_x dxd\tau}_{J_8}.\end{aligned} \tag{3.18}$$

For J_6, J_7, and J_8, we have from Lemmas 3.1–3.4 and the a priori estimate (H_2) that

$$J_6 = -\int_0^t \int_{\mathbf{R}} \frac{\mu(v)}{v} u_x^2 \theta_{xx} dx d\tau \tag{3.19}$$
$$\leq \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau + O(1)\int_0^t \int_{\mathbf{R}} u_x^4 dx d\tau$$
$$\leq \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau + O(1)\int_0^t \|u_x(\tau)\|^3 \|u_{xx}(\tau)\| d\tau$$
$$\leq \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau + O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^{10},$$

$$J_8 \leq \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau + O(1)\int_0^t \int_{\mathbf{R}} u_x^2 dx d\tau \tag{3.20}$$
$$\leq \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau + O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^{2},$$

and

(3.21)
$$J_7 = -\int_0^t \int_{\mathbf{R}} \theta_x^2 \left(\frac{\kappa(v,\theta)}{v}\right)_\theta \theta_{xx} dx d\tau - \int_0^t \int_{\mathbf{R}} \theta_x v_x \left(\frac{\kappa(v,\theta)}{v}\right)_v \theta_{xx} dx d\tau$$
$$= \frac{1}{3}\int_0^t \int_{\mathbf{R}} \theta_x^4 \frac{\kappa_{\theta\theta}(v,\theta)}{v} dx d\tau + \frac{1}{3}\int_0^t \int_{\mathbf{R}} \theta_x^3 v_x \left(\frac{\kappa_\theta(v,\theta)}{v}\right)_v dx d\tau$$
$$- \int_0^t \int_{\mathbf{R}} \theta_x v_x \left(\frac{\kappa(v,\theta)}{v}\right)_v \theta_{xx} dx d\tau$$
$$\leq \frac{1}{6}\int_0^t \int_{\mathbf{R}} \theta_x^4 \frac{\kappa_{\theta\theta}(v,\theta)}{v} dx d\tau + \frac{1}{12}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau + O(1)\int_0^t \int_{\mathbf{R}} \theta_x^2 v_x^2 dx d\tau$$
$$\leq \frac{1}{6}\int_0^t \int_{\mathbf{R}} \theta_x^4 \frac{\kappa_{\theta\theta}(v,\theta)}{v} dx d\tau + \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau$$
$$+ O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^{6}.$$

Here we have used the fact that

$$\int_0^t \int_{\mathbf{R}} \theta_x^2 v_x^2 dx d\tau$$
$$\leq \int_0^t \|\theta_x(\tau)\|_{L_x^\infty}^2 \|v_x(\tau)\|^2 d\tau \leq O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^{2} \int_0^t \|\theta_x(\tau)\|_{L_x^\infty}^2 d\tau$$
$$\leq O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^{2} \int_0^t \|\theta_x(\tau)\| \|\theta_{xx}(\tau)\| d\tau$$
$$\leq \frac{1}{12}\int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v} \theta_{xx}^2 dx d\tau + O(1)\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}, \Phi_{0x}\right)\right\|_1^{4} \int_0^t \int_{\mathbf{R}} \theta_x^2 dx d\tau.$$

Inserting (3.19)–(3.21) into (3.18), we obtain

$$\begin{aligned}(3.22)\qquad &\frac{C_v}{2}\|\theta_x(t)\|^2+\frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2dxd\tau-\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa_{\theta\theta}(v,\theta)}{v}\theta_x^4dxd\tau\\ &\quad\leq O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10}.\end{aligned}$$

This is (3.17) and the proof of Lemma 3.6 is completed. ∎

Lemmas 3.1–3.6 imply that under the a priori estimate (H_2), there exist two positive constants $V_2\geq 1$ and $C_1\geq 1$ with V_2 depending only on $\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x},v_{0x}\right)\right\|$, $\underline{V},\overline{V},\underline{\Theta}$, and $\overline{\Theta}$ but independent of T and $\gamma-1$, and C_1 depending only on V_2 but independent of $T>0$ and $\gamma-1$, such that

$$\begin{aligned}(3.23)\qquad &V_2^{-1}\leq v(t,x)\leq V_2,\quad (t,x)\in[0,T]\times\mathbf{R},\\ &\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}},\Phi_x\right)(t)\right\|^2\\ &\quad+\int_0^t\int_{\mathbf{R}}\left(u_x^2+\theta_x^2\right)(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|^2,\\ &\|v_x(t)\|^2+\int_0^t\int_{\mathbf{R}}v_x^2(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^2,\\ &\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}u_{xx}^2(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^6,\\ &\left\|\frac{\theta_x(\tau)}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\theta_{xx}^2(\tau,x)dxd\tau\leq C_1\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^{10}\end{aligned}$$

hold for $0\leq t\leq T$.

To obtain the global existence of solutions, we only need to close the a priori estimate (H_2). For this, we need the smallness of $\gamma-1>0$. In fact, we have from $(3.23)_2$, $(3.23)_5$, and Sobolev's inequality that
(3.24)

$$\|\theta(t)-1\|_{L^\infty_{T,x}}\leq\|\theta(t)-1\|^{\frac{1}{2}}\|\theta_x(t)\|^{\frac{1}{2}}\leq C_1(\gamma-1)^{\frac{1}{2}}\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},\Phi_{0x}\right)\right\|_1^4.$$

On the other hand, since $\theta=\frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right)$, if we set $\overline{s}=\frac{R}{\gamma-1}\ln\frac{R}{A}$, we have

$$\begin{aligned}\theta-1&=\frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right)-1\\ &=\frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right)-\frac{A}{R}\exp\left(\frac{\gamma-1}{R}\overline{s}\right)\\ &=\frac{A}{R}\left(v^{1-\gamma}-1\right)\exp\left(\frac{\gamma-1}{R}s\right)+\frac{A}{R}\left(\exp\left(\frac{\gamma-1}{R}s\right)-\exp\left(\frac{\gamma-1}{R}\overline{s}\right)\right).\end{aligned}$$

Consequently,

$$\|\theta_0 - 1\| \le O(1)\frac{A(\gamma-1)}{R}\exp\left(\frac{\gamma-1}{R}\|s_0\|_{L_x^\infty}\right)\left[\left\|v_0^{-\gamma}\right\|_{L_x^\infty}\|v_0-1\| + \frac{1}{R}\|s_0(x)-\overline{s}\|\right],$$
$$\|\theta_{0x}\| \le O(1)\frac{A(\gamma-1)}{R}\exp\left(\frac{\gamma-1}{R}\|s_0\|_{L_x^\infty}\right)$$
$$\times\left[(\inf v_0(x))^{-\gamma}\|v_{0x}\| + \frac{1}{R}\left(\inf_x v_0(x)\right)^{1-\gamma}\|s_{0x}\|\right]. \tag{3.25}$$

Since $\|v_0(x)\|_{L_x^\infty}, \inf v_0(x), \frac{\gamma-1}{A}\|s_0(x)\|_{L_x^\infty}$ are assumed to be independent of $\gamma-1$, we have from (3.24) and (3.25) that

$$\|\theta(t)-1\|_{L_x^\infty} \le C_2(\gamma-1)^{\frac{1}{2}}\|(v_0-1,u_0,\Phi_{0x})\|_1^3 + C_3(\gamma-1)^2\|(v_0-1,s_0-\overline{s})\|_1^3 \tag{3.26}$$

holds for $0 \le t \le T$.

Thus if $\gamma - 1 > 0$ is chosen to be sufficiently small such that

$$C_2(\gamma-1)^{\frac{1}{2}}\|(v_0-1,u_0,\Phi_{0x})\|_1^3 + C_3(\gamma-1)^2\|(v_0-1,u_0,s_0-\overline{s})\|_1^3 \le \min\left\{\overline{\Theta}-1, 1-\underline{\Theta}\right\}, \tag{3.27}$$

we have from (3.26) and (3.27) that for any $0 \le t \le T,\ x \in \mathbf{R}$,

$$\theta(t,x) \le \|\theta(t,x)-1\|_{L_{T,x}^\infty} + 1 \le 1 + \min\left\{\overline{\Theta}-1, 1-\underline{\Theta}\right\} \le \overline{\Theta}, \tag{3.28}$$

and

$$\theta(t,x) \ge 1 - \|\theta(t,x)-1\|_{L_{T,x}^\infty} \ge 1 - \min\left\{\overline{\Theta}-1, 1-\underline{\Theta}\right\} \ge 1-(1-\underline{\Theta}) = \underline{\Theta}. \tag{3.29}$$

That is

$$\underline{\Theta} \le \theta(t,x) \le \overline{\Theta}, \quad x \in \mathbf{R}, \quad 0 \le t \le T. \tag{3.30}$$

This closes the a priori estimate (H_2) and then Theorem 1.2 follows from the standard continuation argument.

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