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UNCONDITIONALLY OPTIMAL ERROR ESTIMATES OF A CRANK–NICOLSON GALERKIN METHOD FOR THE NONLINEAR THERMISTOR EQUATIONS*

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Abstract. This paper focuses on unconditionally optimal error analysis of an uncoupled and linearized Crank–Nicolson Galerkin finite element method for the time-dependent nonlinear thermistor equations in d-dimensional space, $d = 2, 3$. In our analysis, we split the error function into two parts, one from the spatial discretization and one from the temporal discretization, by introducing a corresponding time-discrete (elliptic) system. We present a rigorous analysis for the regularity of the solution of the time-discrete system and error estimates of the time discretization. With these estimates and the proved regularity, optimal error estimates of the fully discrete Crank–Nicolson Galerkin method are obtained unconditionally. Numerical results confirm our analysis and show the efficiency of the method.

Key words. unconditional optimal error analysis, linearized Crank–Nicolson scheme, Galerkin FEM, nonlinear thermistor equations

AMS subject classifications. 65N12, 65N30, 35K61

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1. Introduction.

We consider the time-dependent nonlinear thermistor system

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2, \tag{1.1}$$

$$-\nabla\cdot(\sigma(u)\nabla\phi) = 0, \tag{1.2}$$

for $x \in \Omega$ and $t \in [0, T]$, where Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. The initial and boundary conditions are given by

$$\begin{aligned} &u(x,t) = 0, \quad \phi(x,t) = g(x,t) && \text{for } x \in \partial\Omega, \quad t \in [0,T], \\ &u(x,0) = u_0(x) && \text{for } x \in \Omega. \end{aligned} \tag{1.3}$$

The nonlinear system above describes the model of electric heating of a conducting body, where u is the temperature, ϕ is the electric potential, and σ is the temperature-dependent electric conductivity. Following the previous works [15, 41], we assume that $\sigma \in W^{1,\infty}(\mathbb{R})$ and

$$\sigma_1 \le \sigma(s) \le \sigma_2 \tag{1.4}$$

for some positive constants σ_1 and σ_2.

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In the last several decades, theoretical analysis for the time-dependent thermistor equations has been done extensively; e.g., see [4, 6, 11, 39, 40]. Among these works, Yuan and Liu [40] proved the existence and uniqueness of a C^α solution in three-dimensional space, from which further regularity of the solution can be derived with suitable assumptions on the initial and boundary conditions. Numerical methods and analysis for the thermistor system have also been investigated by many authors [3, 5, 15, 38, 41, 42]. For the system in two-dimensional space, the optimal L^2 error estimate of a mixed finite element method with a linearized semi-implicit Euler scheme was obtained in [41] under a weak time-step condition. Error analysis for the three-dimensional model was given in [15], in which a linearized semi-implicit Euler scheme with a linear Galerkin FEM was used. An optimal L^2 error estimate was obtained under the condition $\tau = O(h^{d/6})$. A more general time discretization with higher-order finite element approximations was studied in [3] and an optimal L^2 norm error estimate was given under the conditions $\tau = O(h^{\frac{d}{2p}})$ and $r \geq 2$, where p is the order of the time discretization and r is the degree of piecewise polynomials of the finite element space. A natural question is whether such time-step restrictions are necessary for the numerical solution to achieve optimal convergence rate, although some restrictions are not so severe in practical computation and the previous error estimates are optimal from the accuracy perspective.

There are several different time discretizations for nonlinear parabolic systems: explicit, semiexplicit (or semi-implicit), and implicit. The most popular and widely used one is a linearized (semi)-implicit scheme. At each time step, the scheme only requires the solution of a linear system. However, the time-step condition is always a key issue for such a scheme. To study the error estimate of linearized (semi)-implicit schemes, the boundedness of the numerical solution (or error function) in the L^∞ norm or a stronger norm is often required. If an a priori estimate for numerical solution in such a norm cannot be provided, one may employ mathematical induction with an inverse inequality to bound the numerical solution, such as

$$\|U_h^n - R_h u^n\|_{L^\infty} \leq Ch^{-d/2}\|U_h^n - R_h u^n\|_{L^2} \leq Ch^{-d/2}(\tau^p + h^{r+1}), \tag{1.5}$$

where U_h^n is the finite element solution, u^n is the exact solution, and R_h is a certain projection operator. A time-step condition $\tau = O(h^{\frac{d}{2p}})$ arises immediately from the above inequality. This approach has been widely used in error analysis for many different nonlinear parabolic PDEs; e.g., see [3, 7, 9, 10, 13, 17, 18, 20, 19, **?**, 25, 26, 27, 28, 31, 33, 34, 35, 37]. In all these works, error estimates were established under certain time-step restrictions, which depend upon the dimension, the scheme, and the nonlinearity of the equations in general. The time-step restrictions arising from theoretical analysis may result in the use of a very small time step and are extremely time-consuming in practical computations. A new approach was introduced in our recent works [22, 23] (also see [24]), in which the error estimates of a linearized backward Euler–Galerkin method for a porous media flow and the thermistor system were obtained, respectively, under the condition of h and τ being smaller than a positive constant. In this paper, we propose an uncoupled and linearized Crank–Nicolson Galerkin finite element method for the nonlinear thermistor system and present optimal error estimates in both L^2 and H^1 norms without any stepsize restrictions. In this method, the standard Crank–Nicolson scheme is applied for the linear term in the temperature equation and an extrapolation approximation is used for the nonlinear electric conductivity. At each time step, one only needs to solve two uncoupled linear systems. The main idea of our approach is to split the error function into two parts, the spatially discrete error and the temporally discrete error, by introducing a corresponding time-discrete (elliptic) system. The former arises from the Galerkin FEM discretization for the time-discrete equations and depends solely upon the spatial mesh size h (independent of the time-step size τ). If a suitable regularity of the solution to the time-discrete equations can be proved, the numerical solution is bounded by

$$\|U_h^n - R_h U^n\|_{L^\infty} \leq Ch^{-d/2}\|U_h^n - R_h U^n\|_{L^2} \leq Ch^{-d/2}h^{r+1} \tag{1.6}$$

without any time-step condition, where U^n is the solution of the time-discrete equations. This approach can also be applied to other nonlinear parabolic PDEs and other time discretizations to obtain unconditional convergence and optimal error estimates.

The rest of the paper is organized as follows. In section 2, we present the uncoupled and linearized Crank–Nicolson scheme with a linear Galerkin finite element approximation in the spatial direction and state our main results. After introducing the corresponding time-discrete system, we provide in section 3 a priori estimates and optimal error estimates for the time-discrete solution, which imply a certain regularity of the time-discrete solution. With the regularity obtained, in section 4 we present optimal error estimates of the fully discrete Galerkin finite element solution in both the L^2 norm and the H^1 norm unconditionally. Numerical results are presented in section 5 to confirm our theoretical analysis.

2. The main result. Let Ω be a bounded, smooth, and convex domain in $\mathbb{R}^d$ ($d = 2, 3$). Let π_h be a regular division of Ω into triangles T_j, $j = 1, \ldots, M$, in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$, and denote by $h = \max_{1\leq j\leq M}\{\operatorname{diam} T_j\}$ the mesh size. For a triangle (or tetrahedra) T_j at the boundary, we define $\tilde{T}_j$ to be a triangle with one curved side (or a tetrahedra with one curved face in $\mathbb{R}^3$) with the same vertices as T_j, and set $D_j = \widetilde{T}_j \backslash T_j$. For an interior triangle, we set $\widetilde{T}_j = T_j$ and $D_j = \emptyset$. For a given triangular (or tetrahedral) division of Ω, we define the finite element spaces [32]:

$$V_h = \{v_h \in C(\overline{\Omega}) : v_h|_{T_j} \text{ is linear at each element and } v_h = 0 \text{ on } D_j\},$$
$$S_h = \{v_h \in C(\overline{\Omega}) : v_h|_{\widetilde{T}_j} \text{ is linear at each element}\}.$$

For any $v \in S_h$, we define $\Lambda_h v$ to be a function satisfying $\Lambda_h v = 0$ on D_j and $\Lambda_h v = v$ on T_j. We further define $\widetilde{\Pi}_h : C_0(\overline{\Omega}) \to S_h$ to be the Lagrangian interpolation operator and set $\Pi_h = \Lambda_h \widetilde{\Pi}_h$. Clearly, Π_h is a projection operator from $C_0(\overline{\Omega})$ onto V_h.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with $t_n = n\tau$ and let

$$u^n = u(x, t_n), \quad \phi^n = \phi(x, t_n) \qquad \text{for} \quad n = 0, 1, \ldots, N. \tag{2.1}$$

For any sequence of functions $\{f^n\}_{n=0}^N$, we define

$$D_\tau f^{n+1} = \frac{f^{n+1} - f^n}{\tau}, \qquad \widehat{f}^{n+1/2} = (3f^n - f^{n-1})/2, \tag{2.2}$$

$$\overline{f}^{n+1/2} = \frac{1}{2}(f^n + f^{n+1}) \tag{2.3}$$

for $n = 1, 2, \ldots, N - 1$.

For simplicity, we denote by C a generic positive constant and by ϵ a generic small positive constant, which depend solely upon physical parameters of the problem and independent of τ, h, and n. We assume that $g(\cdot,t) \in H^1(\Omega)$ is given for each fixed $t \geq 0$.

We propose an uncoupled and linearized Crank–Nicolson Galerkin finite element method for the system (1.1)–(1.3), which seeks $U_h^{n+1} \in V_h$ and $\Phi_h^{n+1/2} \in g^{n+1/2} + V_h$, $n = 0, 1, \ldots, N-1$, such that

$$\big(\sigma(\widehat{U}_h^{n+1/2})\nabla\Phi_h^{n+1/2}, \nabla\varphi\big) = 0 \quad \forall\ \varphi \in V_h, \tag{2.4}$$

$$\big(D_\tau U_h^{n+1}, v\big) + \big(\nabla\overline{U}_h^{n+1/2}, \nabla v\big) = \big(\sigma(\widehat{U}_h^{n+1/2})|\nabla\Phi_h^{n+1/2}|^2, v\big) \quad \forall\ v \in V_h. \tag{2.5}$$

At the initial time steps, we choose $U_h^0 = \Pi_h u_0$ and let Φ_h^0 be the Galerkin solution to the potential equation

$$\big(\sigma(u_0)\nabla\Phi_h^0, \nabla\varphi\big) = 0 \quad \forall\ \varphi \in V_h \tag{2.6}$$

and $\widehat{U}_h^{1/2}$ can be calculated by a semi-implict Euler scheme

$$\left(\frac{\widehat{U}_h^{1/2} - u^0}{\tau/2}, v\right) + \big(\nabla\widehat{U}_h^{1/2}, \nabla v\big) = \big(\sigma(u_0)|\nabla\Phi_h^0|^2, v\big) \quad \forall\ v \in V_h. \tag{2.7}$$

By the classical finite element theory for elliptic equations and for interpolation, we have

$$\|U_h^0 - u_0\|_{L^2} + \|\Phi_h^0 - \phi^0\|_{L^{12/5}} \leq Ch^2. \tag{2.8}$$

Here we assume that the solution of the initial/boundary value problem (1.1)–(1.3) exists and satisfies

$$\begin{aligned} &\|u_0\|_{H^2} + \|u\|_{L^\infty((0,T);H^2)} + \|u_t\|_{L^\infty((0,T);H^2)} \\ &\quad + \|u_{tt}\|_{L^\infty((0,T);H^1)} + \|u_{tt}\|_{L^2((0,T);H^2)} + \|u_{ttt}\|_{L^2((0,T);L^2)} \leq C \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\|\phi\|_{L^\infty((0,T);W^{2,12/5})} + \|\phi_t\|_{L^\infty((0,T);W^{1,6})} + \|\nabla\phi\|_{L^\infty((0,T);L^\infty)} \\ &\quad + \|g\|_{L^\infty((0,T);W^{2,12/5})} + \|\nabla g\|_{L^\infty((0,T);L^\infty)} \leq C. \end{aligned} \tag{2.10}$$

The emphasis of this paper is on unconditionally optimal error estimates. The above regularity assumptions may possibly be weakened for the analysis below. We present our main result in the following theorem. The proof will be given in sections 3 and 4.

THEOREM 2.1. *Suppose that the system* (1.1)–(1.2) *with the initial and boundary conditions* (1.3) *has a unique solution* (u,ϕ) *satisfying* (2.9)–(2.10). *Then the finite element system* (2.4)–(2.7) *admits a unique solution* $(U_h^n,\ \Phi_h^{n-1/2})$, $n = 1, \ldots, N$, *such that*

$$\begin{aligned} &\max_{1\leq n\leq N} \|U_h^n - u(\cdot,t_n)\|_{L^2} + \max_{1\leq n\leq N} \|\Phi_h^{n-1/2} - \phi(\cdot,t_{n-1/2})\|_{L^{12/5}} \\ &\quad \leq C(\tau^2 + h^2), \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\max_{1\leq n\leq N} \|U_h^n - u(\cdot,t_n)\|_{H^1} + \max_{1\leq n\leq N} \|\Phi_h^{n-1/2} - \phi(\cdot,t_{n-1/2})\|_{W^{1,12/5}} \\ &\quad \leq C(\tau^2 + h). \end{aligned} \tag{2.12}$$

To prove Theorem 2.1, we introduce a time-discrete system:

$$\nabla\cdot\big(\sigma(\widehat{U}^{n+1/2})\nabla\Phi^{n+1/2}\big)=0 \qquad \text{for } n=0,1,\dots,N-1, \tag{2.13}$$

$$D_\tau U^{n+1}-\Delta\overline{U}^{n+1/2}=\sigma(\widehat{U}^{n+1/2})|\nabla\Phi^{n+1/2}|^2 \quad \text{for } n=0,1,\dots,N-1, \tag{2.14}$$

subject to the boundary/initial conditions

$$\begin{aligned} &U^n(x)=0,\quad \Phi^{n+1/2}(x)=g(x,t_{n+1/2}) && \text{for}\ \ x\in\partial\Omega,\quad 1\le n\le N,\\ &U^0(x)=u_0(x) && \text{for}\ \ x\in\Omega,\\ &\frac{\widehat{U}^{1/2}-U^0}{\tau/2}-\Delta\widehat{U}^{1/2}=\sigma(u_0)|\nabla\phi^0|^2 && \text{for}\ \ x\in\Omega,\\ &\widehat{U}^{1/2}=0, && \text{for}\ \ x\in\partial\Omega, \end{aligned} \tag{2.15}$$

where ϕ^0 is the solution to the elliptic equation

$$\begin{cases}\nabla\cdot(\sigma(u_0)\nabla\phi^0)=0 & \text{in}\ \ \Omega,\\ \phi^0(x)=g(x,0) & \text{for}\ \ x\in\partial\Omega.\end{cases} \tag{2.16}$$

With the solution of the time-discrete system $(U^n,\Phi^{n-1/2})$, we have the following error splitting:

$$\begin{aligned} \|U_h^n-u^n\|&\le\|e^n\|+\|U^n-U_h^n\|,\\ \|\Phi_h^{n-1/2}-\phi^{n-1/2}\|&\le\|\eta^{n-1/2}\|+\|\Phi^{n-1/2}-\Phi_h^{n-1/2}\|, \end{aligned}$$

where

$$e^n=U^n-u^n,\qquad \eta^{n-1/2}=\Phi^{n-1/2}-\phi^{n-1/2}.$$

Note that the fully discrete system (2.4)–(2.5) can be viewed as the spatial discretization of the elliptic system (2.13)–(2.14). The key issue is to prove the regularity of the solution to the time-discrete equations (2.13)–(2.14) required in the error estimates of the Galerkin finite element method. We present estimates of the error functions $(e^n,\eta^{n-1/2})$ and $(U^n-U_h^n,\ \Phi^{n-1/2}-\Phi_h^{n-1/2})$ in sections 3 and 4, respectively.

Moreover, error estimates given in Theorem 2.1 for Φ are defined in the time level $t_{n-1/2}$. To get the solution at the time level t_n, we define

$$\Phi_h^n:=\frac{1}{2}(\Phi_h^{n+1/2}+\Phi_h^{n-1/2}).$$

From Theorem 2.1 we see that

$$\max_{1\le n\le N}\|\Phi_h^n-\phi(\cdot,t_n)\|_{L^{12/5}}\le C(\tau^2+h^2), \tag{2.17}$$

$$\max_{1\le n\le N}\|\Phi_h^n-\phi(\cdot,t_n)\|_{W^{1,12/5}}\le C(\tau^2+h). \tag{2.18}$$

The following lemma can be proved by noting definition (2.3) and using a triangular inequality.

LEMMA 2.2. *Let $\{v^n\}_{n=0}^N$ be a sequence of functions on Ω. Then for any norm $\|\cdot\|$,*

$$\tau\|v^n\| \le 2\tau\sum_{m=1}^n \|\overline{v}^{m-1/2}\| + \tau\|v^0\| \le 2\sqrt{T}\sqrt{\sum_{m=1}^n \tau\|\overline{v}^{m-1/2}\|^2} + \tau\|v^0\|. \tag{2.19}$$

The following lemma is concerned with the $W^{1,p}$ and $W^{2,p}$ estimates of two typical elliptic equations, which can be found in [1, 2, 8, 12, 30].

LEMMA 2.3. *Suppose that $\kappa \in C^\alpha(\overline{\Omega})$ and $\kappa_0^{-1} \le \kappa \le \kappa_0$ for some positive constant κ_0, $c \in L^\infty(\Omega)$ and $c \ge 0$. Then the solution of the elliptic equation*

$$\begin{cases} -\nabla\cdot(\kappa\nabla u) + cu = \nabla\cdot\mathbf{f} & in \;\; \Omega, \\ u = g & on \;\; \partial\Omega \end{cases}$$

satisfies the $W^{1,p}$ estimate

$$\|u\|_{W^{1,p}(\Omega)} \le C_p(\|\mathbf{f}\|_{L^p(\Omega)} + \|g\|_{W^{1,p}(\Omega)}), \quad 2 \le p < \infty,$$

and the solution of the Poisson equation

$$\begin{cases} -\Delta u + cu = f & in \;\; \Omega, \\ u = g & on \;\; \partial\Omega, \end{cases}$$

satisfies the $W^{2,p}$ estimate

$$\|u\|_{W^{2,p}(\Omega)} \le C_p(\|f\|_{L^p(\Omega)} + \|g\|_{W^{2,p}(\Omega)}), \quad 2 \le p < \infty.$$

3. Temporal error analysis.

In this section, we prove the existence and uniqueness of the solution of the time-discrete system (2.13)–(2.15) and establish error bounds for $(e^n, \eta^{n-1/2})$.

THEOREM 3.1. *Suppose that the system* (1.1)–(1.3) *has a unique solution (u, ϕ) satisfying* (2.9)–(2.10)*. Then the time-discrete system* (2.13)–(2.15) *admits a unique solution $(U^n, \Phi^{n-1/2})$ such that*

$$\max_{1\le n\le N}\|U^n\|_{H^2} + \max_{1\le n\le N}\|D_\tau U^n\|_{H^2} + \|D_\tau \widehat{U}^{1/2}\|_{H^2} \le C, \tag{3.1}$$

$$\max_{1\le n\le N}\|\Phi^{n-1/2}\|_{W^{2,12/5}} + \max_{1\le n\le N}\|\nabla\Phi^{n-1/2}\|_{L^p} \le C \quad \forall\ 1 \le p < \infty, \tag{3.2}$$

and

$$\max_{1\le n\le N}\|e^n\|_{H^1} + \max_{1\le n\le N}\|\eta^{n-1/2}\|_{W^{1,12/5}} \le C\tau^2. \tag{3.3}$$

Proof. The existence and uniqueness of the solution to the linear elliptic equations (2.13)–(2.15) are straightforward. In the following, we only prove the estimates (3.1)–(3.3).

Since $U^0 = u^0$, the error functions e^{n+1} and $\eta^{n+1/2}$, $0 \le n \le N-1$, satisfy

$$\begin{aligned} &-\nabla\cdot\left[\sigma(\widehat{U}^{n+1/2})\nabla\eta^{n+1/2}\right] \\ &\quad = \nabla\cdot\left[(\sigma(\widehat{u}^{n+1/2}) - \sigma(\widehat{U}^{n+1/2}))\nabla\phi^{n+1/2}\right] + \nabla\cdot R_\phi^{n+1/2}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &D_\tau e^{n+1} - \Delta\overline{e}^{n+1/2} \\ &\quad = (\sigma(\widehat{U}^{n+1/2}) - \sigma(\widehat{u}^{n+1/2}))|\nabla\phi^{n+1/2}|^2 \\ &\qquad + \sigma(\widehat{U}^{n+1/2})(\nabla\phi^{n+1/2} + \nabla\Phi^{n+1/2})\cdot\nabla\eta^{n+1/2} + R_u^{n+1}, \end{aligned} \tag{3.5}$$

and

$$\frac{2}{\tau}\widehat{e}^{1/2} - \Delta\widehat{e}^{1/2} = R_u^{1/2}, \tag{3.6}$$

where

$$\begin{aligned}
R_u^{1/2} &= u_t|_{t=0} - \frac{u^{1/2}-u^0}{\tau/2} + \sigma(u^0)|\nabla\phi^0|^2 - \sigma(u^{1/2})|\nabla\phi^{1/2}|^2\\
R_u^{n+1} &= \frac{\partial u}{\partial t}\bigg|_{t=t_{n+1/2}} - D_\tau u^{n+1} + \Delta(\overline{u}^{n+1/2} - u^{n+1/2})\\
&\quad + (\sigma(u^{n+1/2}) - \sigma(\widehat{u}^{n+1/2}))|\nabla\phi^{n+1/2}|^2,\\
R_\phi^{n+1/2} &= (\sigma(u^{n+1/2}) - \sigma(\widehat{u}^{n+1/2}))\nabla\phi^{n+1/2}
\end{aligned}$$

are truncation errors. With the regularity given in (2.9)–(2.10), we have the following estimates for truncation errors:

$$\max_{0\le n\le N-1}\|R_\phi^{n+1/2}\|_{L^6} + \left(\sum_{n=0}^{N-1}\tau\|R_u^{n+1}\|_{L^2}^2\right)^{\frac{1}{2}} + \tau\|R_u^{1/2}\|_{L^6} \le C\tau^2. \tag{3.7}$$

To prove (3.1)–(3.3), we first study errors at the initial steps. Multiplying (3.6) by $|\widehat{e}^{1/2}|^4\widehat{e}^{1/2}$ and integrating it over Ω, we get

$$\|\widehat{e}^{1/2}\|_{L^6} \le C\tau^2,$$

which further shows that $\tau\|\widehat{e}^{1/2}\|_{H^2} \le C\tau^2$. Since $H^2 \hookrightarrow C^\alpha$, $\widehat{U}^{1/2} \in C^\alpha(\overline{\Omega})$. By applying the $W^{1,p}$ estimate in Lemma 2.3 to (3.4) with $n = 0$, we derive that

$$\|\eta^{1/2}\|_{W^{1,6}} \le C\|\widehat{e}^{1/2}\|_{L^6} + C\|R_\phi^{1/2}\|_{L^6} \le C\tau^2.$$

By noting the fact $e^0 = 0$, from (3.5) with $n = 0$, we get

$$\|\nabla e^1\|_{L^2}^2 + \tau\|\Delta e^1\|_{L^2}^2 \le \tau(\|\widehat{e}^{1/2}\|_{L^2}^2 + \|(\nabla\phi^{1/2} + \nabla\Phi^{1/2})\nabla\eta^{1/2}\|_{L^2}^2 + \|R_u^1\|_{L^2}^2) \le C\tau^4.$$

To conclude, we have

$$\|e^1\|_{H^1} + \|\eta^{1/2}\|_{W^{1,6}} + \tau^{1/2}\|e^1\|_{H^2} \le C_0\tau^2. \tag{3.8}$$

With the above regularity, we can rewrite (2.13) as

$$-\Delta\Phi^{1/2} = \frac{\sigma'(\widehat{U}^{1/2})}{\sigma(\widehat{U}^{1/2})}\nabla\widehat{U}^{1/2}\cdot\nabla\Phi^{1/2}$$

and apply the $W^{2,p}$ estimate in Lemma 2.3 (with $p = 12/5$) to obtain

$$\|\Phi^{1/2}\|_{W^{2,12/5}} \le C\|\nabla\widehat{U}^{1/2}\|_{L^4}\|\nabla\Phi^{1/2}\|_{L^6} + C\|g^{1/2}\|_{W^{2,12/5}} \le C. \tag{3.9}$$

Therefore, (3.1)–(3.3) hold for $n = 1$.

Second, we present L^2 error estimates for the solution of (3.4)–(3.5). Multiplying (3.4) by $\eta^{n+1/2}$ and integrating the result over Ω, we obtain

$$\|\eta^{n+1/2}\|_{H^1} \le C\|\widehat{e}^{n+1/2}\|_{L^2} + C\tau^2. \tag{3.10}$$

Again, multiplying (3.5) by $\overline{e}^{n+1/2}$ and integrating it over Ω give

$$
\begin{aligned}
&\frac{1}{2}D_\tau\big(\|e^{n+1}\|_{L^2}^2\big)+\|\nabla\overline{e}^{n+1/2}\|_{L^2}^2\\
&\quad\le C\|\widehat{e}^{n+1/2}\|_{L^2}\|\overline{e}^{n+1/2}\|_{L^6}\|\nabla\phi^{n+1/2}\|_{L^6}^2+\|R_u^{n+1}\|_{L^2}\|\overline{e}^{n+1/2}\|_{L^2}\\
&\qquad+\big(\sigma(\widehat{U}^{n+1/2})(\nabla\phi^{n+1/2}+\nabla\Phi^{n+1/2})\overline{e}^{n+1/2},\,\nabla\eta^{n+1/2}\big).
\end{aligned}
$$

Using (2.14) and integrating by parts, the last term in the preceding inequality is bounded by

$$
\begin{aligned}
&|\big(\sigma(\widehat{U}^{n+1/2})(\nabla\phi^{n+1/2}+\nabla\Phi^{n+1/2})\overline{e}^{n+1/2},\,\nabla\eta^{n+1/2}\big)|\\
&\quad\le|\big(\sigma(\widehat{U}^{n+1/2})\overline{e}^{n+1/2}\nabla\phi^{n+1/2},\nabla\eta^{n+1/2}\big)|+|\big(\nabla\cdot(\sigma(\widehat{U}^{n+1/2})\nabla\Phi^{n+1/2})\overline{e}^{n+1/2}\\
&\qquad+\sigma(\widehat{U}^{n+1/2})\nabla\Phi^{n+1/2}\cdot\nabla\overline{e}^{n+1/2},\eta^{n+1/2}\big)|\\
&\quad=|\big(\sigma(\widehat{U}^{n+1/2})\overline{e}^{n+1/2}\nabla\phi^{n+1/2},\nabla\eta^{n+1/2}\big)|+|\big(\sigma(\widehat{U}^{n+1/2})\nabla\eta^{n+1/2}\cdot\nabla\overline{e}^{n+1/2},\eta^{n+1/2}\big)\\
&\qquad+\big(\sigma(\widehat{U}^{n+1/2})\nabla\phi^{n+1/2}\cdot\nabla\overline{e}^{n+1/2},\eta^{n+1/2}\big)|\\
&\quad\le C\|\overline{e}^{n+1/2}\|_{L^2}\|\nabla\eta^{n+1/2}\|_{L^2}\\
&\qquad+C\|\nabla\overline{e}^{n+1/2}\|_{L^2}\big(\|\nabla\eta^{n+1/2}\|_{L^2}\|\eta^{n+1/2}\|_{L^\infty}+\|\eta^{n+1/2}\|_{L^2}\big).
\end{aligned}
$$

Applying the maximum principle to the elliptic equation (2.13), we obtain $\|\Phi^{n+1/2}\|_{L^\infty}\le C$ and so $\|\eta^{n+1/2}\|_{L^\infty}\le C$ for $1\le n\le N-1$. It follows that

$$
\begin{aligned}
&\frac{1}{2}D_\tau\big(\|e^{n+1}\|_{L^2}^2\big)+\frac{1}{2}\|\nabla\overline{e}^{n+1/2}\|_{L^2}^2\\
&\quad\le C\|\widehat{e}^{n+1/2}\|_{L^2}^2+C\|\overline{e}^{n+1/2}\|_{L^2}^2+C\|\eta^{n+1/2}\|_{H^1}^2+C\|R_u^{n+1}\|_{L^2}^2,\\
&\quad\le C\left(\|e^{n+1}\|_{L^2}^2+\|e^n\|_{L^2}^2+\|e^{n-1}\|_{L^2}^2\right)+C\tau^4,
\end{aligned}
$$

where we have noted (3.10) and used the inequality $\|\cdot\|_{L^6}\le C\|\cdot\|_{H^1}$. Applying Gronwall's inequality to the above inequality, with (3.10) we see that there exists a positive constant $\tau_1>0$ such that when $\tau<\tau_1$,

$$
\max_{1\le n\le N}\|e^{n+1}\|_{L^2}^2+\max_{1\le n\le N-1}\|\eta^{n+1/2}\|_{H^1}^2+\sum_{n=0}^{N-1}\|\overline{e}^{n+1/2}\|_{H^1}^2\tau\le C\tau^4. \tag{3.11}
$$

Finally, we study the regularity in (3.1)–(3.2) and the estimate for $\|\eta^{n+1}\|_{W^{1,12/5}}$. Note that the above estimate implies that

$$
\|D_\tau e^{n+1}\|_{L^2}^2+\sum_{m=0}^{n}\tau\|D_\tau\overline{e}^{m+1/2}\|_{H^1}^2\le C\tau^2, \tag{3.12}
$$

$$
\|D_\tau U^{n+1}\|_{L^2}\le\|D_\tau u^{n+1}\|_{L^2}+\|D_\tau e^{n+1}\|_{L^2}\le C
$$

for $0\le n\le N-1$. Regarding (3.5) as an elliptic equation and applying the H^2 estimate in Lemma 2.3, with (3.12) we obtain

(3.13)

$$
\begin{aligned}
&\|\overline{e}^{n+1/2}\|_{H^2}\\
&\quad\le C\left(\|D_\tau e^{n+1}\|_{L^2}+\|\widehat{e}^{n+1/2}\|_{L^2}+\|\nabla\eta^{n+1/2}\|_{L^2}+\|\nabla\eta^{n+1/2}\|_{L^4}^2+\|R_u^{n+1}\|_{L^2}\right)\\
&\quad\le C\tau+C\|\nabla\eta^{n+1/2}\|_{L^2}^{1/2}\|\nabla\eta^{n+1/2}\|_{L^6}^{3/2}\\
&\quad\le C_1\tau(1+\|\nabla\eta^{n+1/2}\|_{L^6}^{3/2})
\end{aligned}
$$

for $n=1,\dots,N-1$.

Now we prove a primary estimate

$$\|\nabla \eta^{n+1/2}\|_{L^6} \leq 1 \tag{3.14}$$

for $0 \leq n \leq N-1$ by mathematical induction. It is easy to see from (3.8) that (3.14) holds for $n = 0$ if $\tau < 1/\sqrt{C_0}$. We assume that (3.14) holds for $0 \leq n \leq k$. Then from (3.13) and (3.8) we have

$$\|\overline{e}^{n+1/2}\|_{H^2} \leq (2C_1 + C_0\tau)\tau \quad \text{for} \quad 0 \leq n \leq k,$$

and by Lemma 2.1,

$$\|e^{n+1}\|_{H^2} \leq 2\sum_{m=0}^{n} \|\overline{e}^{m+1/2}\|_{H^2} + \|e^0\|_{H^2} \leq (2C_1 + C_0\tau)T \quad \text{for} \quad 1 \leq n \leq k.$$

Hence,

$$\|U^{n+1}\|_{H^2} \leq \|u^{n+1}\|_{H^2} + (2C_1 + C_0\tau)T \leq C_3 \quad \text{for} \quad 1 \leq n \leq k.$$

Since $H^2 \hookrightarrow C^\alpha$ in $\mathbb{R}^d$ $(d = 2, 3)$, we have further

$$\|\widehat{U}^{k+3/2}\|_{C^\alpha} \leq C\|U^{k+1}\|_{H^2} + C\|U^k\|_{H^2} \leq C_4.$$

With the Hölder regularity of $\sigma(\widehat{U}^{k+3/2})$, by applying the $W^{1,p}$ estimate in Lemma 2.3 to (3.4) for $n = k+1$, we obtain

$$\begin{aligned} \|\nabla\eta^{k+3/2}\|_{L^6} &\leq C_5\|(\sigma(\widehat{u}^{k+3/2}) - \sigma(\widehat{U}^{k+3/2}))\nabla\phi^{k+3/2}\|_{L^6} + C_5\|R_\phi^{k+1/2}\|_{L^6} \\ &\leq C_6\|\widehat{e}^{k+3/2}\|_{L^6} + C_6\tau^2 \\ &\leq C_7\|\widehat{e}^{k+3/2}\|_{H^1} + C_6\tau^2 \\ &\leq C_8\left(\|e^{k+1}\|_{H^1} + \|e^k\|_{H^1}\right) + C_6\tau^2 \\ &\leq C_9\sum_{m=0}^{k+1}\|\overline{e}^{m+1/2}\|_{H^1} + C_6\tau^2 \leq C_{10}\tau, \end{aligned} \tag{3.15}$$

where we have used Lemma 2.1 and (3.11).

By choosing $\tau < 1/\max\{C_0, C_{10}\}$, we get $\|\nabla\eta^{k+3/2}\|_{L^6} \leq 1$ and we complete the induction. Thus, we have proved that (3.14) holds for $0 \leq n \leq N-1$, which together with (3.13) implies that $\max_{1\leq n\leq N-1}\|\overline{e}^{n+1/2}\|_{H^2} \leq C\tau$. By using Lemma 2.1 again, we obtain

$$\max_{1\leq n\leq N}\|U^n\|_{H^2} \leq C. \tag{3.16}$$

Since $H^2 \hookrightarrow C^\alpha$ in $\mathbb{R}^d$ $(d = 2, 3)$, with the Hölder continuity of $\widehat{U}^{n+1/2}$, we apply the $W^{1,p}$ estimate in Lemma 2.3 to (2.13) and derive that

$$\max_{1\leq n\leq N-1}\|\nabla\Phi^{n+1/2}\|_{L^p} \leq C_p \quad \forall\ 1 \leq p < \infty, \tag{3.17}$$

where we have noted $\|g\|_{W^{1,p}} \leq C$. With the estimates (3.16)–(3.17), we can perform the $W^{2,p}$ estimate (with $p = 12/5$) for the elliptic equation (2.13) to obtain

$$\max_{1\leq n\leq N-1}\|\Phi^{n+1/2}\|_{W^{2,12/5}} \leq C, \tag{3.18}$$

where we have also used the assumption $\|g\|_{W^{2,12/5}} \leq C$.

With the above estimates, multiplying (3.5) by $-\Delta\overline{e}^{n+1/2}$ and using the inequality $\|\overline{e}^{n+1/2}\|_{H^2} \le C\|\Delta\overline{e}^{n+1/2}\|_{L^2}$, we obtain

$$\begin{aligned} D_\tau\|e^{n+1}\|_{H^1}^2 + \|\overline{e}^{n+1/2}\|_{H^2}^2 &\le C(\|\nabla\phi^{n+1/2}\|_{L^\infty}^2 + \|\nabla\Phi^{n+1/2}\|_{L^3}^2)\|\nabla\eta^{n+1/2}\|_{L^6}^2 \\ &\quad + C\|\widehat{e}^{n+1}\|_{L^2}^2 + C\|R_u^{n+1}\|_{L^2}^2 \\ &\le C\|\widehat{e}^{n+1/2}\|_{H^1}^2 + C\tau^4, \end{aligned}$$

where we have used (3.15). By Gronwall's inequality, we see that

$$\max_{0\le n\le N-1} \|e^{n+1}\|_{H^1}^2 + \sum_{n=0}^{N-1}\tau\|\overline{e}^{n+1/2}\|_{H^2}^2 \le C\tau^4, \tag{3.19}$$

and by using (3.15) again, we have

$$\|\nabla\eta^{n+1/2}\|_{L^{12/5}} \le C\tau^2. \tag{3.20}$$

Moreover, by (3.19) and Lemma 2.2, we have further

$$\|e^n\|_{H^2} \le C\tau^{-1}\left(\sum_{k=0}^{n-1}\tau\|\overline{e}^{k+1/2}\|_{H^2}^2\right)^{\frac{1}{2}} \le C\tau,$$

which implies that $\|D_\tau e^n\|_{H^2} \le C\tau^{-1}\|e^n\|_{H^2} \le C$. With the regularity assumption (2.9), we arrive at

$$\|D_\tau U^n\|_{H^2} \le \|D_\tau e^n\|_{H^2} + \|D_\tau u^n\|_{H^2} \le C. \tag{3.21}$$

So far we have proved that there exists a positive constant τ_0 such that (3.1)–(3.3) hold for $\tau < \tau_0$. Also we have proved that for any $0 < \tau \le T$, (3.1)–(3.3) hold for $n = 1$.

For $\tau \ge \tau_0$, we choose a fixed $4 \le p < \infty$ and define

$$\rho_k := \max_{1\le n\le k}(\|\nabla\Phi^{n-1/2}\|_{L^p} + \|U^n\|_{H^2}) \quad \text{for} \quad 1 \le k \le N.$$

Then, by writing (2.13) as

$$\left(1 - \frac{\tau}{2}\Delta\right)U^{k+1} = \left(1 + \frac{\tau}{2}\Delta\right)U^k + \sigma(\widehat{U}^{k+1/2})|\nabla\Phi^{k+1/2}|^2\tau, \tag{3.22}$$

and applying the $W^{1,p}$ and $W^{2,p}$ estimates to (2.13) and (3.22) (since $\tau_0 \le \tau \le T$), we get

$$\|\nabla\Phi^{k+1/2}\|_{L^p} + \|U^{k+1}\|_{H^2} \le f(\rho_k) \quad \text{for} \quad 1 \le k \le N-1,$$

where $f(\rho_k)$ is a positive constant dependent of ρ_k. Let $\overline{f}(s) = f(s) + s$ and let $\overline{f}^{(0)}(s) = \overline{f}(s)$. For any positive integer n we define $\overline{f}^{(n)}(s) = \overline{f}(\overline{f}^{(n-1)}(s))$. Then the last inequality implies that

$$\rho_{k+1} \le \overline{f}(\rho_k) \quad \text{for} \quad 1 \le k \le N-1,$$

and by iterating the above inequality we derive that

$$\max_{1\le n\le N}(\|\nabla\Phi^{n-1/2}\|_{L^p} + \|U^n\|_{H^2}) \le \overline{f}^{([T/\tau_0]+1)}(\|\nabla\Phi^{1/2}\|_{L^p} + \|U^1\|_{H^2}) \le C, \tag{3.23}$$

where we have noted that the boundedness of $\|\nabla\Phi^{1/2}\|_{L^p} + \|U^1\|_{H^2}$ was proved in (3.8) and (3.9) without any restriction on the time step τ. Since $\tau_0 \le \tau \le T$, (3.1)–(3.3) follow immediately.

The proof of Theorem 3.1 is completed. $\square$

4. Spatial error analysis. In this section, we present error estimates of the Galerkin finite element method for the time-discrete system (2.13)–(2.14). Let $P_h^0\phi^0 = g^0 + \Pi_h(\phi^0 - g^0)$ and $P_h^{n-1/2}\Phi^{n-1/2} = g^{n-1/2} + \Pi_h(\Phi^{n-1/2} - g^{n-1/2})$ for $n = 1, 2, \ldots, N$, and define $R_h : H_0^1(\Omega) \to V_h$ to be a Riesz projection operator defined by

$$\big(\nabla(v - R_h v), \nabla w\big) = 0 \quad \forall\ v \in H_0^1(\Omega) \ \text{ and } \ w \in V_h.$$

We summarize some basic inequalities below. The proof follows the classical finite element theory for elliptic equations; see [14, 36].

$$\|w\|_{W^{m,p}} \le Ch^{(d/p-d/q)}\|w\|_{W^{m,q}}, \quad w \in V_h, \quad 1 \le q \le p \le \infty,\ m = 0, 1, \tag{4.1}$$

$$\|w\|_{W^{1,p}} \le Ch^{-1}\|w\|_{L^p}, \qquad 1 \le p \le \infty, \tag{4.2}$$

$$\|v - \Pi_h v\|_{L^p} + h\|v - \Pi_h v\|_{W^{1,p}} \le Ch^2\|v\|_{W^{2,p}}, \quad p > d/2, \tag{4.3}$$

$$\begin{aligned}&\|\Phi^{n-1/2} - P_h^{n-1/2}\Phi^{n-1/2}\|_{L^p} + h\|\Phi^{n-1/2} - P_h^{n-1/2}\Phi^{n-1/2}\|_{W^{1,p}}\\ &\quad \le Ch^2\|\Phi^{n-1/2} - g^{n-1/2}\|_{W^{2,p}}, \quad p > d/2,\end{aligned} \tag{4.4}$$

$$\|\nabla\Pi_h v\|_{L^p} \le C\|v\|_{W^{1,p}} \quad \forall\ v \in W^{1,p}(\Omega) \ \text{ with } \ p > d, \tag{4.5}$$

and

$$\|R_h v\|_{W^{1,p}} \le C\|v\|_{W^{1,p}} \quad \forall\ \ v \in W^{1,p}, \quad 1 < p \le \infty, \tag{4.6}$$

$$\|v - R_h v\|_{L^p} + h\|v - R_h v\|_{W^{1,p}} \le Ch^2\|v\|_{W^{2,p}} \quad \forall\ \ v \in W^{2,p}, \quad 1 < p < \infty, \tag{4.7}$$

$$\|v - R_h v\|_{L^p} \le Ch^{[(d+2p)q-dp]/(2p)}\|v\|_{W^{2,q}}, \quad dp/(d + 2p) \le q \le p. \tag{4.8}$$

Let $\eta^0 = \Phi_h^0 - P_h^0\phi^0$ and

$$e_h^n = U_h^n - R_h U^n, \quad \eta_h^{n-1/2} = \Phi_h^{n-1/2} - P_h^{n-1/2}\Phi^{n-1/2}, \quad \text{for} \ \ 1 \le n \le N.$$

We present error estimates of the spatial discretization in the following theorem.

THEOREM 4.1. *Suppose that the system* (1.1)–(1.3) *has a unique solution* (u, ϕ) *satisfying* (2.9)–(2.10)*. Then the fully discrete finite element system* (2.4)–(2.7) *admits a unique solution* $(U_h^n, \Phi_h^{n-1/2})$*,* $n = 1, 2, \ldots, N$*, such that*

$$\max_{1\le n\le N}\|e_h^n\|_{L^2} + \max_{1\le n\le N}\|\eta_h^{n-1/2}\|_{L^{12/5}} \le Ch^2, \tag{4.9}$$

$$\max_{1\le n\le N}\|\nabla e_h^n\|_{L^2} + \max_{1\le n\le N}\|\nabla\eta_h^{n-1/2}\|_{L^{12/5}} \le Ch. \tag{4.10}$$

Proof. At each time step of the scheme, one only needs to solve two uncoupled linear discrete elliptic systems. It is easy to see that coefficient matrices in both systems are symmetric and positive definite. The existence and uniqueness of the Galerkin finite element solution follow immediately. Since the inequality (4.10) follows from (4.9) via the inverse inequality (4.2), it suffices to prove (4.9).

Let $\Phi^0 = \phi^0$. The solution of the time-discrete equations (2.13)–(2.14) satisfies

$$\big(D_\tau U^{n+1}, v\big) + \big(\nabla\overline{U}^{n+1/2}, \nabla v\big) = \big(\sigma(\widehat{U}^{n+1/2})|\nabla\Phi^{n+1/2}|^2, v\big), \tag{4.11}$$

$$\big(\sigma(\widehat{U}^{n+1/2})\nabla\Phi^{n+1/2}, \nabla\varphi\big) = 0, \qquad n = 0, 1, \ldots, N, \tag{4.12}$$

for any $v, \varphi \in V_h$, and

$$\left(\frac{\widehat{U}^{1/2} - u_0}{\tau/2}, \, v\right) + \big(\nabla \widehat{U}^{1/2}, \, \nabla v\big) = \big(\sigma(u_0)|\nabla \phi^0|^2, \, v\big), \quad \forall \, v \in V_h. \tag{4.13}$$

From the above equations and the corresponding finite element system (2.4)–(2.7), we find that the error functions $e_h^{n+1}, \eta_h^{n+1/2} \in V_h$, $0 \le n \le N$, satisfy

$$\begin{aligned}
&\big(D_\tau e_h^{n+1}, \, v\big) + \big(\nabla \overline{e}_h^{n+1/2}, \, \nabla v\big) \\
&\quad = \big(D_\tau(U^{n+1} - R_h U^{n+1}), \, v\big) + \big((\sigma(\widehat{U}_h^{n+1/2}) - \sigma(\widehat{U}^{n+1/2}))|\nabla \Phi^{n+1/2}|^2, \, v\big) \\
&\qquad + 2\big((\sigma(\widehat{U}_h^{n+1/2}) - \sigma(\widehat{U}^{n+1/2}))\nabla \Phi^{n+1/2} \cdot \nabla(\Phi_h^{n+1/2} - \Phi^{n+1/2}), \, v\big) \\
&\qquad + \big(\sigma(\widehat{U}_h^{n+1/2})|\nabla(\Phi_h^{n+1/2} - \Phi^{n+1/2})|^2, \, v\big) \\
&\qquad + 2\big(\sigma(\widehat{U}^{n+1/2})\nabla \Phi^{n+1/2} \cdot \nabla(\Phi_h^{n+1/2} - \Phi^{n+1/2}), \, v\big) \\
&\quad := \sum_{i=1}^{5} I_i^{n+1/2}(v),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\big(\sigma(\widehat{U}^{n+1/2})\nabla \eta_h^{n+1/2}, \, \nabla \varphi\big) \\
&\quad = -\big((\sigma(\widehat{U}_h^{n+1/2}) - \sigma(\widehat{U}^{n+1/2}))\nabla \Phi_h^{n+1/2}, \, \nabla \varphi\big) \\
&\qquad + \big(\sigma(\widehat{U}^{n+1/2})\nabla(\Phi^{n+1/2} - P_h^{n+1/2}\Phi^{n+1/2}), \, \nabla \varphi\big),
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\left(\frac{\widehat{e}_h^{1/2} - e_h^0}{\tau/2}, v\right) - \big(\nabla \widehat{e}_h^{1/2}, \nabla v\big) &= \big(\sigma(u_0)(|\nabla \phi^0|^2 - |\nabla \Phi_h^0|^2), v\big) \\
&\quad + \big(D_{\tau/2}\widehat{U}^{1/2} - R_h D_{\tau/2}\widehat{U}^{1/2}, v\big)
\end{aligned} \tag{4.16}$$

for all $v, \varphi \in V_h$, where $D_{\tau/2}\widehat{U}^{1/2} = 2(\widehat{U}^{1/2} - u^0)/\tau$.

To prove (4.9)–(4.10), we first estimate error functions at the initial step. Since $\nabla \cdot (\sigma(u^0)\nabla \phi^0) = 0$ in Ω, by setting $v = \widehat{e}_h^{1/2}$ in (4.16) we have

$$\begin{aligned}
\big|\big(D_{\tau/2}\widehat{U}^{1/2} - R_h D_{\tau/2}\widehat{U}^{1/2}, \widehat{e}_h^{1/2}\big)\big| &\le C\|D_{\tau/2}\widehat{U}^{1/2}\|_{H^2} h^2 \|\widehat{e}_h^{1/2}\|_{L^2} \\
&\le \epsilon \|\widehat{e}_h^{1/2}\|_{H^1} + C\epsilon^{-1}h^4
\end{aligned}$$

and

$$\begin{aligned}
&|\big(\sigma(u_0)(|\nabla \phi^0|^2 - |\nabla \Phi_h^0|^2), \widehat{e}_h^{1/2}\big)| \\
&\quad \le |\big(\sigma(u_0)|\nabla(\Phi_h^0 - \phi^0)|^2, \, \widehat{e}_h^{1/2}\big)| + 2|\big(\sigma(u_0)\nabla \Phi^0 \cdot \nabla(\Phi_h^0 - \phi^0), \, \widehat{e}_h^{1/2}\big)| \\
&\quad = |\big(\sigma(u_0)|\nabla(\Phi_h^0 - \phi^0)|^2, \, \widehat{e}_h^{1/2}\big)| + 2|\big(\sigma(u_0)\nabla \Phi^0(\Phi_h^0 - \phi^0), \, \nabla \widehat{e}_h^{1/2}\big)| \\
&\quad \le C\|\widehat{e}_h^{1/2}\|_{L^6} \|\nabla(\phi^0 - \Phi_h^0)\|_{L^{12/5}}^2 + C\|\Phi_h^0 - \phi^0\|_{L^2} \|\nabla \widehat{e}_h^{1/2}\|_{L^2} \\
&\quad \le \epsilon \|\widehat{e}_h^{1/2}\|_{H^1} + C\epsilon^{-1}h^4,
\end{aligned}$$

where we have used integration by parts and (2.8). With the above estimates, (4.16) reduces to

$$\|\widehat{e}_h^{1/2}\|_{L^2}^2 \le \|e_h^0\|_{L^2}^2 + C\tau h^4 \le Ch^4. \tag{4.17}$$

Second, we present estimates for $\|e_h^{n+1}\|_{L^2}$ and $\|\eta_h^{n+1/2}\|_{L^{12/5}}$ for $0 \le n \le N-1$. For this purpose, we take $v = \overline{e}_h^{n+1/2}$ in (4.14) and we have

$$
\begin{aligned}
I_1^{n+1/2}(\overline{e}_h^{n+1/2}) &\le \|\overline{e}_h^{n+1/2}\|_{L^2}\|D_\tau U^{n+1} - R_h D_\tau U^{n+1}\|_{L^2} \\
&\le C\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}\|D_\tau U^{n+1} - R_h D_\tau U^{n+1}\|_{L^2} \\
&\le \epsilon\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}^2 + C\epsilon^{-1}\|D_\tau U^{n+1}\|_{H^2}^2 h^4, \\
I_2^{n+1/2}(\overline{e}_h^{n+1/2}) &\le C(\|\widehat{e}_h^{n+1/2}\|_{L^2} + \|\widehat{U}^{n+1/2} - R_h\widehat{U}^{n+1/2}\|_{L^2})\|\nabla\Phi^{n+1/2}\|_{L^6}^2\|\overline{e}_h^{n+1/2}\|_{L^6} \\
&\le C(\|\widehat{e}_h^{n+1/2}\|_{L^2} + h^2)\|\nabla\overline{e}_h^{n+1/2}\|_{L^2} \\
&\le \epsilon\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}^2 + C\epsilon^{-1}(\|\widehat{e}_h^{n+1/2}\|_{L^2}^2 + h^4), \\
I_3^{n+1/2}(\overline{e}_h^{n+1/2}) &\le C\|\overline{e}_h^{n+1/2}\|_{L^6}(\|\widehat{e}_h^{n+1/2}\|_{L^2} + \|\widehat{U}^{n+1/2} - R_h\widehat{U}^{n+1/2}\|_{L^2}) \\
&\quad\cdot(\|\nabla\eta_h^{n+1/2}\|_{L^6} + \|\nabla(\Phi^{n+1/2} - P_h^{n+1/2}\Phi^{n+1/2})\|_{L^6})\|\nabla\Phi^{n+1/2}\|_{L^6} \\
&\le C\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}(\|\widehat{e}_h^{n+1/2}\|_{L^2} + h^2)(h^{-d/4}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}} \\
&\quad + \|\Phi^{n+1/2} - g^{n+1/2}\|_{H^2}) \\
&\le \epsilon\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}^2 + C\epsilon^{-1}(\|\widehat{e}_h^{n+1/2}\|_{L^2}^2 + h^4)(h^{-d/2}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}}^2 + C), \\
I_4^{n+1/2}(\overline{e}_h^{n+1/2}) &\le C\|\overline{e}_h^{n+1/2}\|_{L^6}\|\nabla(\Phi^{n+1/2} - \Phi_h^{n+1/2})\|_{L^{12/5}}^2 \\
&\le C\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}(\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}}^2 + h^2) \\
&\le \epsilon\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}^2 + C\epsilon^{-1}(\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}}^4 + h^4) \\
I_5^{n+1/2}(\overline{e}_h^{n+1/2}) &\le \|\overline{e}_h^{n+1/2}\|_{L^6}\|\Phi_h^{n+1/2} - \Phi^{n+1/2}\|_{L^{12/5}}\|\nabla\Phi^{n+1/2}\|_{L^{12/5}} \\
&\le \epsilon\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}^2 + C\epsilon^{-1}\|\Phi_h^{n+1/2} - \Phi^{n+1/2}\|_{L^{12/5}}^2,
\end{aligned}
\tag{4.18}
$$

where we have used (4.5) in the estimate of $I_3^{n+1/2}(\overline{e}_h^{n+1/2})$.

By applying the $W^{1,p}$ estimate [29] to (4.15) and using (4.4), we obtain

$$
\begin{aligned}
&\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}} \\
&\quad\le C\big(\|(\sigma(\widehat{U}_h^{n+1/2}) - \sigma(\widehat{U}^{n+1/2}))\nabla\Phi_h^{n+1/2}\|_{L^{12/5}} \\
&\qquad + \|\sigma(\widehat{U}^{n+1/2})\nabla(\Phi^{n+1/2} - P_h^{n+1/2}\Phi^{n+1/2})\|_{L^{12/5}}\big) \\
&\quad\le C\|\widehat{U}_h^{n+1/2} - \widehat{U}^{n+1/2}\|_{L^4}(\|\nabla\eta_h^{n+1/2}\|_{L^6} + \|\nabla P_h^{n+1/2}\Phi^{n+1/2}\|_{L^6}) + Ch \\
&\quad\le C(\|\widehat{e}_h^{n+1/2}\|_{L^4} + \|\widehat{U}^{n+1/2} - R_h\widehat{U}^{n+1/2}\|_{L^4})(\|\nabla\eta_h^{n+1/2}\|_{L^6} \\
&\qquad + \|\nabla P_h^{n+1/2}\Phi^{n+1/2}\|_{L^6}) + Ch \\
&\quad\le Ch^{-d/4}(\|\widehat{e}_h^{n+1/2}\|_{L^2} + h^2)(h^{-d/4}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}} + C) + Ch \\
&\quad\le C\big(h^{-d/2}\|\widehat{e}_h^{n+1/2}\|_{L^2}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}} + h^{-d/4}\|\widehat{e}_h^{n+1/2}\|_{L^2} \\
&\qquad + h^{2-d/2}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}} + h\big),
\end{aligned}
\tag{4.19}
$$

where we have used the inverse inequality (4.1) and (4.8) with $q = 2$ and $p = 4$. To estimate $\|\eta^{n+1/2}\|_{L^{12/5}}$, we rewrite (4.15) as

$$
\begin{aligned}
&\big(\sigma(\widehat{U}^{n+1/2})\nabla(\Phi^{n+1/2} - \Phi_h^{n+1/2}), \nabla\varphi\big) \\
&\qquad + \big((\sigma(\widehat{U}^{n+1/2}) - \sigma(\widehat{U}_h^{n+1/2}))\nabla\Phi_h^{n+1/2}, \nabla\varphi\big) = 0.
\end{aligned}
$$

Following the Nitsche technique, we define ψ as the solution of the elliptic equation

$$-\nabla\cdot\big(\sigma(\widehat{U}^{n+1/2})\nabla\psi\big) = |\Phi^{n+1/2}-\Phi_h^{n+1/2}|^{2/5}(\Phi^{n+1/2}-\Phi_h^{n+1/2})$$

with the boundary condition $\psi = 0$ on $\partial\Omega$. The solution ψ to the above elliptic equation satisfies that

$$\|\psi\|_{W^{2,12/7}} \le C\|\Phi^{n+1/2}-\Phi_h^{n+1/2}\|_{L^{12/5}}^{7/5}.$$

Since

$$\begin{aligned}
&\|\Phi^{n+1/2}-\Phi_h^{n+1/2}\|_{L^{12/5}}^{12/5}\\
&\quad= \big(\sigma(\widehat{U}^{n+1/2})\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2}),\ \nabla\psi\big)\\
&\quad= \big(\sigma(\widehat{U}^{n+1/2})\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2}),\ \nabla(\psi_0-\Pi_h\psi)\big)\\
&\qquad+ \big(\sigma(\widehat{U}^{n+1/2})\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2}),\ \nabla\Pi_h\psi\big)\\
&\quad= \big(\sigma(\widehat{U}^{n+1/2})\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2}),\ \nabla(\psi-\Pi_h\psi)\big)\\
&\qquad+ \big((\sigma(\widehat{U}_h^{n+1/2})-\sigma(\widehat{U}^{n+1/2}))\nabla\Phi_h^{n+1/2},\ \nabla\Pi_h\psi\big)\\
&\quad\le C\|\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2})\|_{L^{12/5}}\|\nabla(\psi-\Pi_h\psi)\|_{L^{12/7}}\\
&\qquad+ C\|\widehat{U}^{n+1/2}-\widehat{U}_h^{n+1/2}\|_{L^2}\|\nabla\Phi_h^{n+1/2}\|_{L^4}\|\nabla\Pi_h\psi\|_{L^4}\\
&\quad\le Ch\|\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2})\|_{L^{12/5}}\|\psi\|_{W^{2,12/7}}\\
&\qquad+ \|\widehat{U}^{1/2}-\widehat{U}_h^{n+1/2}\|_{L^2}\big(\|\nabla\eta_h^{n+1/2}\|_{L^4}+\|\nabla P_h^{n+1/2}\Phi^{1/2}\|_{L^4}\big)\|\psi\|_{W^{2,12/7}}\\
&\quad\le C\|\psi\|_{W^{2,12/7}}\big[h\|\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2})\|_{L^{12/5}}\|\widehat{U}^{n+1/2}-\widehat{U}_h^{n+1/2}\|_{L^2}\\
&\qquad+ \big(h^{-\frac{d}{6}}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}}+\|\Phi^{n+1/2}\|_{W^{2,12/5}}+\|g^{n+1/2}\|_{W^{2,12/5}}\big)\big],
\end{aligned}$$

we derive that

$$\begin{aligned}
\|\Phi^{n+1/2}-\Phi_h^{n+1/2}\|_{L^{12/5}} &\le Ch\|\nabla(\Phi^{n+1/2}-\Phi_h^{n+1/2})\|_{L^{12/5}}\\
&\quad+ C\|\widehat{U}^{n+1/2}-\widehat{U}_h^{n+1/2}\|_{L^2}\big(h^{-d/6}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}}+C\big).
\end{aligned}\tag{4.20}$$

By (4.17) and (4.18)–(4.20), (4.14) reduces to

$$\begin{aligned}
&D_\tau\big(\|e_h^{n+1}\|_{L^2}^2\big)+\|\nabla\overline{e}_h^{n+1/2}\|_{L^2}^2\\
&\quad\le C\epsilon^{-1}\|D_\tau U^{n+1}\|_{H^1}^2h^4 + C\left(\|e_h^n\|_{L^2}^2+\|e_h^{n-1}\|_{L^2}^2\right)\\
&\qquad+ C\big(h^{-d/3}\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}}^2\|\widehat{e}_h^{n+1/2}\|_{L^2}^2+\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}}^4+h^4\big)
\end{aligned}\tag{4.21}$$

for $n = 1, \ldots, N-1$.

Now we prove a primary estimate

$$\|e_h^n\|_{L^2} \le h^{7/4}, \quad 0 \le n \le N,\tag{4.22}$$

by using mathematical induction.

By (2.8) and (4.7), this inequality holds for $n = 0$ if $h < h_1$ for some given positive constant h_1. If we assume that (4.22) holds for $0 \le n \le k$, then from (4.17) we know that $\|\widehat{e}_h^{n+1/2}\|_{L^2} \le 2h^{7/4} + Ch^2$ for $0 \le n \le k$, and from the inequalities (4.19) we see that there exists a positive constant h_2 such that when $h < h_2$,

$$\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}} \le Ch \quad \text{for} \ \ 0 \le n \le k.\tag{4.23}$$

With the above inequalities, (4.21) reduces to

$$D_\tau\left(\|e_h^{n+1}\|_{L^2}^2\right) + \|\nabla e_h^{n+1/2}\|_{L^2}^2 \le C\left(\|e_h^n\|_{L^2}^2 + \|e_h^{n-1}\|_{L^2}^2\right) + Ch^4$$

for $0 \le n \le k$, which implies that there exists a positive constant C_{11} satisfying

$$\|e_h^{n+1}\|_{L^2} \le C_{11}h^2 \tag{4.24}$$

for $0 \le n \le k$. Therefore, $\|e_h^{k+1}\|_{L^2} < h^{7/4}$ if $h < \min(h_1, h_2, 1/C_{11}^4)$, which completes the induction. Thus, (4.23)–(4.24) hold for all $0 \le n \le N-1$. From (4.20) we derive that $\|\Phi^{n+1/2} - \Phi_h^{n+1/2}\|_{L^{12/5}} \le Ch^2$ and therefore,

$$\|\eta_h^{n+1/2}\|_{L^{12/5}} \le \|\Phi^{n+1/2} - \Phi_h^{n+1/2}\|_{L^{12/5}} + \|P_h^{n+1/2}\Phi^{n+1/2} - \Phi^{n+1/2}\|_{L^{12/5}} \le Ch^2 \tag{4.25}$$

for $0 \le n \le N-1$. Then (4.9)–(4.10) follow immediately.

So far we have proved that the estimate (4.9) holds if $h < h_0$ for some positive constant h_0. It remains to show that

$$\max_{0\le n\le N} \|e_h^n\|_{L^2} + \max_{0\le n\le N-1} \|\eta_h^{n+1/2}\|_{L^{12/5}} \le C \tag{4.26}$$

for $h \ge h_0$. In fact, substituting $\varphi = \Phi_h^{n+1/2}$ in (2.4) we get

$$\|\nabla\Phi_h^{n+1/2}\|_{L^2} \le C, \quad n = 1, 2, \dots, N.$$

By the inverse inequality (4.1), we have

$$\|\nabla\Phi_h^{n+1/2}\|_{L^\infty} \le Ch_0^{-2}, \quad n = 1, 2, \dots, N,$$

and therefore,

$$\|\nabla\eta_h^{n+1/2}\|_{L^{12/5}} \le C_{h_0}, \quad n = 1, 2, \dots, N,$$

where C_{h_0} is a positive constant dependent upon h_0. From (2.4) with $v = \overline{U}^{n+1/2}$ we get

$$\begin{aligned} D_\tau\left(\|U_h^{n+1}\|_{L^2}^2\right) + \|\nabla\overline{U}_h^{n+1/2}\|_{L^2}^2 &\le C\|\nabla\Phi_h^{n+1/2}\|_{L^\infty}^2\|\overline{U}_h^{n+1/2}\|_{L^2} \\ &\le C\|\nabla\Phi_h^{n+1/2}\|_{L^\infty}^2\|\nabla\overline{U}_h^{n+1/2}\|_{L^2} \\ &\le C\|\nabla\Phi_h^{n+1/2}\|_{L^\infty}^4 + \frac{1}{2}\|\nabla\overline{U}_h^{n+1/2}\|_{L^2}^2, \end{aligned}$$

which implies that

$$\|U_h^{n+1}\|_{L^2}^2 \le \|U_h^n\|_{L^2}^2 + C_{h_0}\tau \le \cdots \le C_{h_0}T.$$

This completes the proof of Theorem 4.1. ∎

Theorem 2.1 follows from Theorem 3.1 and Theorem 4.1, together with (4.4) and (4.7). ∎

5. Numerical results. In this section, we present two numerical examples to illustrate our theoretical results. The computations are performed with the software FEniCS.

Example 5.1. We rewrite the system (1.1)–(1.2) by

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2 + f_1, \tag{5.1}$$

$$-\nabla \cdot (\sigma(u)\nabla\phi) = f_2, \tag{5.2}$$

where $\Omega = (0,1) \times (0,1)$ and

$$\sigma(u) = \frac{1}{1+u^2} + 1.$$

The functions f_1, f_2, and the Dirichlet boundary conditions are chosen corresponding to the exact solution

$$u(x,y,t) = \exp(x+y-t), \quad \phi(x,y,t) = 1 + \sin(x+y+t).$$

A uniform triangular partition with $M+1$ nodes in each direction is used in our computation. We solve the system by the linearized Crank–Nicolson Galerkin method with linear elements and quadratic elements, respectively, while our analysis was given only for the linear case. To confirm our error estimates in the L^2 norm, we choose $\tau = h$ for the linear FEM and $\tau = h^{3/2}$ for the quadratic FEM. We present the numerical results in Tables 1 and 2. We can see clearly from Table 1 that the L^2 errors of the linear FEM are proportional to h^2 and from Table 2 that the L^2 errors of the quadratic FEM are proportional to h^3. To see the errors in the H^1 norm, we take $\tau = h^{1/2}$ for the linear FEM and $\tau = h^{3/2}$ for the quadratic FEM, and we present numerical results in Tables 3 and 4. All these results are in good agreement with our theoretical analysis.

To show the unconditional stability, we test the linearized Crank–Nicolson Galerkin method with linear elements, $h = 1/80$, and the large time steps $\tau = h, 5h, 10h, 20h$. We present numerical results in Table 5. The results show that the scheme is stable for large time steps, although the numerical results with $\tau = 20h$ seem not very accurate.

TABLE 1
L^2 errors of linear FEM with $h = \tau = 1/M$ (Example 5.1).

	$\|U_h^n - u(\cdot,t_n)\|_{L^2}$			
t	$M=20$	$M=40$	$M=80$	Order
1.0	7.8063e-05	1.9587e-05	4.9042e-06	2.00
2.0	9.9117e-05	2.4975e-05	6.2605e-06	1.99
3.0	8.7998e-05	2.2134e-05	5.5422e-06	1.99
4.0	5.6591e-05	1.4242e-05	3.5666e-06	1.99
	$\|\Phi_h^n - \phi(\cdot,t_n)\|_{L^2}$			
t	$M=20$	$M=40$	$M=80$	Order
1.0	7.2691e-05	1.7791e-05	4.3746e-06	2.03
2.0	9.7524e-05	2.3836e-05	5.8610e-06	2.03
3.0	1.3954e-04	3.4376e-05	8.4930e-06	2.02
4.0	1.4342e-04	3.5393e-05	8.7511e-06	2.02

Table 2
L^2 errors of quadratic FEM with $h = 1/M$ and $\tau = h^{3/2}$ (Example 5.1).

	$\|U_h^n - u(\cdot, t_n)\|_{L^2}$			
t	$M = 10$	$M = 20$	$M = 40$	Order
1.0	8.8214e-06	1.2113e-06	1.5496e-07	2.92
2.0	2.0196e-05	2.5321e-06	3.1437e-07	3.00
3.0	1.7212e-05	2.1149e-06	2.5866e-07	3.03
4.0	4.7866e-06	5.5294e-07	6.3392e-08	3.12
	$\|\Phi_h^n - \phi(\cdot, t_n)\|_{L^2}$			
t	$M = 10$	$M = 20$	$M = 40$	Order
1.0	3.4542e-05	4.0535e-06	5.0059e-07	3.05
2.0	3.9065e-05	4.7335e-06	4.3953e-07	3.24
3.0	4.1164e-05	3.2571e-06	6.2613e-07	3.02
4.0	3.0560e-06	2.5956e-05	3.7303e-07	3.06

Table 3
H^1 errors of linear FEM with $h = 1/M$ and $\tau = h^{1/2}$ (Example 5.1).

	$\|U_h^n - u(\cdot, t_n)\|_{H^1}$			
t	$M = 40$	$M = 80$	$M = 160$	Order
1.0	5.6024e-03	2.3706e-03	9.9037e-04	1.25
2.0	3.6159e-03	1.8179e-03	7.7195e-04	1.11
3.0	2.8675e-03	1.3254e-03	6.4766e-04	1.07
4.0	1.7923e-03	8.2924e-04	3.5460e-04	1.17
	$\|\Phi_h^n - \phi(\cdot, t_n)\|_{H^1}$			
t	$M = 40$	$M = 80$	$M = 160$	Order
1.0	4.1118e-03	2.2510e-03	1.1312e-03	0.93
2.0	4.8396e-03	2.0593e-03	1.3143e-03	0.94
3.0	5.0531e-03	2.7701e-03	1.0180e-03	1.16
4.0	5.1689e-03	1.9872e-03	1.4263e-03	0.93

Table 4
H^1 errors of quadratic FEM with $h = \tau = 1/M$ (Example 5.1).

	$\|U_h^n - u(\cdot, t_n)\|_{H^1}$			
t	$M = 10$	$M = 20$	$M = 40$	Order
1.0	1.6700e-03	3.1162e-04	5.9398e-05	2.41
2.0	1.3279e-03	2.8247e-04	6.4458e-05	2.18
3.0	1.0156e-03	2.2534e-04	5.2879e-05	2.13
4.0	5.0056e-04	9.5814e-05	1.9466e-05	2.34
	$\|\Phi_h^n - \phi(\cdot, t_n)\|_{H^1}$			
t	$M = 10$	$M = 20$	$M = 40$	Order
1.0	1.8020e-03	4.4199e-04	1.0614e-04	2.04
2.0	1.6539e-03	4.0090e-04	9.4222e-05	2.07
3.0	1.6141e-03	3.8446e-04	8.8813e-05	2.09
4.0	1.5725e-03	3.7110e-04	8.5031e-05	2.10

Example 5.2. In the second example, we consider the system (5.1)–(5.2) in the three-dimensional space with the exact solution

$$u(x, y, z, t) = \exp(2x + y - z)(2t + \sin(t)), \tag{5.3}$$

$$\phi(x, y, z, t) = \sin(x - 2y)\cos(z)\exp(t). \tag{5.4}$$

We use a uniform tetrahedral partition with $M + 1$ nodes in each direction (see Figure 1). The total number of tetrahedra is $6M^3$ and the total number of vertices

Table 5
L^2 *errors of linear FEM with* $h = 1/M$ *and* $\tau = kh$ *(Example* 5.1*).*

	$\|U_h^n - u(\cdot, t_n)\|_{L^2}$			
t	$k = 1$	$k = 5$	$k = 10$	$k = 20$
1.0	4.9042e-06	3.6772e-05	2.2480e-04	1.6693e-03
2.0	6.2605e-06	8.0524e-05	3.1973e-04	1.5965e-03
3.0	5.5422e-06	6.6692e-05	2.6145e-04	1.1952e-03
4.0	3.5666e-06	1.7407e-05	6.6093e-05	3.9786e-04
	$\|\Phi_h^n - \phi(\cdot, t_n)\|_{L^2}$			
t	$k = 1$	$k = 5$	$k = 10$	$k = 20$
1.0	4.3746e-06	1.4126e-04	5.5525e-04	1.3714e-03
2.0	5.8610e-06	1.2561e-04	4.9591e-04	1.0407e-03
3.0	8.4930e-06	1.1959e-04	4.6941e-04	8.5232e-04
4.0	8.7511e-06	1.1356e-04	4.4541e-04	7.2633e-04

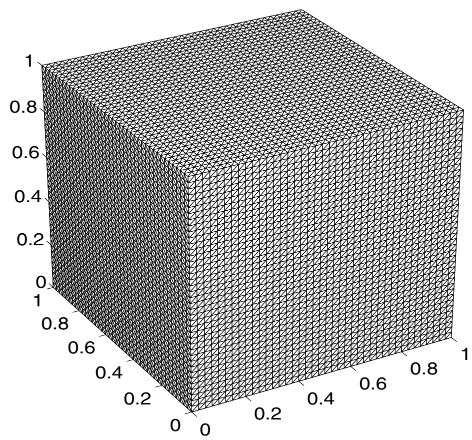

Fig. 1. *The three-dimensional mesh (Example* 5.2*).*

Table 6
L^2 *errors of linear FEM with* $h = \tau = 1/M$ *(Example* 5.2*).*

	$\|U_h^n - u(\cdot, t_n)\|_{L^2}$			
t	$M = 10$	$M = 20$	$M = 40$	Order
1.0	1.1089e-03	2.8319e-04	7.1320e-05	1.9793
2.0	8.6316e-04	2.2523e-04	5.6987e-05	1.9605
3.0	4.0520e-04	1.0626e-04	2.6895e-05	1.9566
4.0	3.6125e-04	9.4243e-05	2.3822e-05	1.9613
	$\|\Phi_h^n - \phi(\cdot, t_n)\|_{L^2}$			
t	$M = 10$	$M = 20$	$M = 40$	Order
1.0	4.0129e-04	1.1562e-04	2.9779e-05	1.8761
2.0	7.8577e-04	2.1723e-04	5.5611e-05	1.9103
3.0	7.7231e-04	2.1482e-04	5.5087e-05	1.9047
4.0	5.0533e-04	1.4801e-04	3.8404e-05	1.8589

is $(M+1)^3$. We solve the system by the proposed Crank–Nicolson Galerkin method with linear elements. Table 6 contains the L^2 errors of the numerical solution with $\tau = h$ and $h = 1/10, 1/20, 1/40$. Similarly, we can see that the L^2 errors for both u and ϕ are proportional to h^2.

TABLE 7
L^2 errors of linear FEM with $h = 1/M$ and $\tau = kh$ (Example 5.2).

	$\|U_h^n - u(\cdot, t_n)\|_{L^2}$		
t	$k = 1$	$k = 5$	$k = 10$
1.0	7.1320e-05	2.5286e-04	2.3962e-03
2.0	5.6987e-05	2.4133e-04	1.6463e-03
3.0	2.6895e-05	1.0359e-04	8.5356e-04
4.0	2.3822e-05	2.6147e-04	1.1001e-03
	$\|\Phi_h^n - \phi(\cdot, t_n)\|_{L^2}$		
t	$k = 1$	$k = 5$	$k = 10$
1.0	2.9779e-05	2.7311e-04	4.6730e-04
2.0	5.5611e-05	1.2094e-04	1.3614e-03
3.0	5.5087e-05	1.0514e-04	1.7040e-03
4.0	3.8404e-05	1.0333e-04	1.6532e-03

Previous analyses for three-dimensional problems often required stronger time-step conditions than those for two-dimensional problems. Here we test the linear Galerkin method for the problem in the three-dimensional space with $h = 1/80$ and large time steps $\tau = h, 5h, 10h$ and present the results in Table 7. Numerical results show that the scheme is unconditionally stable.

6. Conclusions. We have presented an uncoupled and linearized Crank–Nicolson Galerkin finite element method for the nonlinear time-dependent thermistor equations in the d-dimensional space ($d = 2, 3$) and provided unconditionally optimal error estimates in L^2 and H^1 norms for the temperature and $L^{12/5}$ and $W^{1,12/5}$ norms for the potential, while existing analysis requires certain time-step restrictions. Our numerical results confirm our analysis and show that no time-step conditions are needed. The $L^{12/5}$ norm estimate is done here due to the nature of the nonlinearity of the equations. In the error analysis of the finite element method, we need to estimate the nonlinear term (see section 4) by

$$\begin{aligned}&\big(\sigma(\widehat{U}_h^{n+1/2})|\nabla(\Phi_h^{n+1/2} - \Phi^{n+1/2})|^2, \, \overline{e}_h^{n+1/2}\big)\\ &\quad\le C\|\overline{e}_h^{n+1/2}\|_{L^6}\|\nabla(\Phi^{n+1/2} - \Phi_h^{n+1/2})\|_{L^{12/5}}^2\\ &\quad\le C\|\overline{e}_h^{n+1/2}\|_{H^1}\|\nabla(\Phi^{n+1/2} - \Phi_h^{n+1/2})\|_{L^{12/5}}^2.\end{aligned}$$

Thus, the estimates of $\Phi^{n-1/2} - \Phi_h^{n-1/2}$ in the space $W^{1,12/5}$ are needed. Moreover, error estimates and regularity of the solution of the time-discrete system (2.13)–(2.15) in the corresponding spaces are also required.

The approach used in this paper consists of two parts: (i) regularity and error analysis for the time discrete system and (ii) τ-independent optimal error estimate of the FEM for the time discrete PDEs. We believe that such an approach can be extended to many other nonlinear equations and other numerical schemes to obtain unconditionally optimal error estimates, although this paper only focuses on the electric heating model with a linear finite element method.

As an example, we sketch an outline of the analysis of the Crank–Nicolson scheme for the general semilinear parabolic equation:

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{6.1}$$

The analysis of the semilinear equation can be found in the literature; e.g., see [16] and references therein. Here we assume that the function f is locally twice differentiable with respect to u and the solution of the above equation exists and satisfies the regularity condition

(6.2)
$$u,\, u_t \in L^\infty((0,T);H^{r+1}(\Omega)), \quad u_{tt} \in\in L^\infty((0,T);H^2(\Omega)), \quad u_{ttt} \in L^\infty((0,T);L^2(\Omega)).$$

Let S_h^r be the finite element subspace of $H_0^1(\Omega)$ consisting of piecewise polynomials of degree $r \geq 1$. The Crank–Nicolson Galerkin FE system is defined by

$$\begin{aligned} \big(D_\tau U_h^{n+1}, v\big) + \big(\nabla \overline{U}_h^{n+1/2}, \nabla v\big) &= \big(f(\widehat{U}_h^{n+1/2}), v\big) \quad \forall\, v \in S_h^r \ \text{ for } \ n = 1,2,\ldots,N-1,\\ \big(D_\tau U_h^1, v\big) + \big(\nabla U_h^1, \nabla v\big) &= \big(f(u_0), v\big) \quad \forall\, v \in S_h^r \end{aligned}$$

with the initial condition $U_h^0 = \Pi_h u^0$. The corresponding time-discrete PDEs are defined by

$$\begin{cases} D_\tau U^{n+1} - \Delta \overline{U}^{n+1/2} = f(\widehat{U}^{n+1/2}) & \text{in } \ \Omega,\\ U^{n+1} = 0 & \text{on } \ \partial\Omega \end{cases} \tag{6.3}$$

for $n = 1,2,\ldots,N-1$, where U^1 is solved by the backward Euler scheme

$$\begin{cases} D_\tau U^1 - \Delta U^1 = f(u_0) & \text{in } \ \Omega,\\ U^1 = 0 & \text{on } \ \partial\Omega \end{cases} \tag{6.4}$$

with $U^0 = u_0$. Therefore, the error function $e^n = u^n - U^n$ satisfies the equation

$$D_\tau e^{n+1} - \Delta \overline{e}^{n+1/2} = f(\widehat{U}^{n+1/2}) - f(\widehat{u}^{n+1/2}) + R_{tr}^{n+1/2} \tag{6.5}$$

with the boundary condition $e^{n+1}|_{\partial\Omega} = 0$, where $R_{tr}^{n+1/2}$ denotes the truncation error.

By applying the approach used in section 3 for the time-discrete system (6.3)–(6.5) and noting the regularity condition (6.2), one can prove the regularity of the solution

$$\|U^{n+1}\|_{H^2} + \|D_\tau U^{n+1}\|_{H^2} \leq C \tag{6.6}$$

and the error estimate

$$\|U^{n+1} - u^{n+1}\|_{H^1} \leq C\tau^2. \tag{6.7}$$

The former is required by FEM analysis.

With the proved regularity in (6.6), we can establish the error estimate

$$\|R_h U^{k+1} - U_h^{k+1}\|_{L^2} \leq Ch^2 \tag{6.8}$$

under the induction assumption $\|U_h^n\|_{L^\infty} \leq \|R_h U^n\|_{L^\infty} + 1$ for $0 \leq n \leq k$, where the constant C above does not depend on k. By using (6.8) and the inverse inequality, we see that there exists $h_0 > 0$, independent of τ, h, and k, such that

$$\begin{aligned} \|U_h^{k+1}\|_{L^\infty} &\leq \|R_h U^{k+1}\|_{L^\infty} + \|R_h U^{k+1} - U_h^{k+1}\|_{L^\infty}\\ &\leq \|R_h U^{k+1}\|_{L^\infty} + Ch^{-d/2}\|R_h U^{k+1} - U_h^{k+1}\|_{L^2}\\ &\leq \|R_h U^{k+1}\|_{L^\infty} + 1 \end{aligned}$$

when $h < h_0$. Thus the mathematical induction is completed. With the boundedness of numerical solution in L^∞, the optimal error estimate

$$\max_{0\le n\le N} \|R_h u^n - U_h^n\|_{L^2} \le C(\tau^2 + h^{r+1}) \tag{6.9}$$

can be proved in the traditional way.

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