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## A NEW ERROR ANALYSIS OF CHARACTERISTICS-MIXED FEMs FOR MISCIBLE DISPLACEMENT IN POROUS MEDIA\*

JILU WANG<sup>†</sup>, ZHIYONG SI<sup>‡</sup>, AND WEIWEI SUN<sup>†</sup>

**Abstract.** The method of characteristics type is especially effective for convection-dominated diffusion problems. Due to the nature of characteristic temporal discretization, the method allows one to use a large time step in many practical computations, while all previous theoretical analyses always required certain restrictions on the time stepsize. Here, we present a new analysis to establish unconditionally optimal error estimates for a modified method of characteristics with a mixed finite element approximation to the miscible displacement problem in  $\mathbb{R}^d$  ( $d = 2, 3$ ). For this purpose, we introduce a new characteristic time-discrete system. We prove that the  $L^2$  error bound the characteristic time-discrete system of the fully discrete method of characteristics to the time-discrete system is  $\tau$ -independent and the numerical solution is bounded in  $W^{1,\infty}$ -norm unconditionally. With the boundedness, optimal error estimates are established in a traditional manner. Numerical results confirm our theoretical analysis and clearly show the unconditional stability.

**Key words.** unconditionally optimal error estimates, modified method of characteristics, mixed finite element method, incompressible miscible flow

**AMS subject classifications.** 65N12, 65N30, 35K61

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**1. Introduction.** We consider the following miscible displacement system modeling an incompressible flow in porous media  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ):

$$(1.1) \quad \Phi \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u})\nabla c) + \mathbf{u} \cdot \nabla c = c_1 q^I - cq^I,$$

$$(1.2) \quad \nabla \cdot \mathbf{u} = q^I - q^P,$$

$$(1.3) \quad \mathbf{u} = -\frac{k(x)}{\mu(c)} \nabla p,$$

for  $t \in [0, T]$ , with the initial condition

$$(1.4) \quad c(x, 0) = c_0(x) \text{ for } x \in \Omega,$$

where we assume that the domain  $\Omega$  is bounded and the condition  $\int_{\Omega} p dx = 0$  is enforced for the uniqueness of the solution.

In an enhanced oil-recovery process, a solvent is injected into an oil reservoir at one well and mixed with oil, and the mixture fluid flows to a neighboring well. The mathematical model (1.1)–(1.3) for describing such a process is established by conservation of mass for the fluids, with incompressibility condition and Darcy’s law

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that the mixture fluid velocity  $\mathbf{u}$  is proportional to the gradient of the fluid pressure  $p$ . Here,  $c$  represents the concentration of one of the fluids.  $\Phi$  denotes the porosity of the medium,  $q^I$  and  $q^P$  are given injection and production sources,  $c_1$  is the concentration of the injection source,  $D(\mathbf{u}) = [D_{ij}(\mathbf{u})]_{d \times d}$  is the diffusion-dispersion tensor (see [5] for details),  $k(x)$  is the permeability of the medium, and  $\mu(c)$  is the concentration-dependent viscosity of the fluid mixture. Numerical methods and analyses for the miscible displacement system (1.1)–(1.3) have been investigated extensively in the last several decades, including finite difference methods [11, 35, 39], finite element methods (FEMs) [4, 7, 12, 21, 25, 28, 30], and characteristic methods [9, 14, 17, 18, 38]. Numerical simulations have been done for various engineering models, e.g., see [22, 35, 45, 46, 47], and a review article was given by Ewing and Wang [20]. In particular, Ewing and Wheeler [21] proposed a fully discrete Galerkin-Galerkin FEM for the miscible displacement problem in two-dimensional space. Douglas, Ewing, and Wheeler [12] introduced a Galerkin-mixed FEM to solve the system (1.1)–(1.3). In both [12] and [21], the linearized semi-implicit Euler scheme was applied for the time discretization, and a time-step condition  $\tau = o(h)$  was required to obtain optimal error estimates.

Since the concentration equation (1.1) is often convection-dominated, i.e., the diffusion coefficient  $D$  is very small, in practical cases, the method of characteristics is more effective for solving such a coupled system. A modified method of characteristics (MMOC) with both finite difference and finite element approximations was proposed by Douglas and Russell [13] for linear convection-dominated diffusion problems. The method is the first order in the characteristic time direction. Later, the method was extended to the nonlinear miscible displacement equations in both two- and three-dimensional spaces [38]. Furthermore, characteristics-mixed FEMs were studied by several authors [9, 14, 18] for the nonlinear miscible displacement problem, in which the MMOC with a mixed finite element approximation was used. The optimal  $L^2$  error estimate in two-dimensional space was established in [14]. The analysis was done under a time-step condition  $\tau = o(h^{2/3} |\log h|^{-1/6})$  for the two-dimensional model. As the author stated, the method and analysis can be extended to the problem in three-dimensional space with a restriction  $\tau = o(h)$ . Moreover, second-order time-discrete schemes were also investigated by several authors [1, 19] for both linear and nonlinear convection-diffusion problems. To maintain the conservation of the mass, an Eulerian-Lagrangian localized adjoint method was studied in [6, 43] for advective-diffusive equations. Further error analysis was presented in [42, 44]. Some other methods of characteristics can be found in [10, 26, 33, 40]. In addition, the characteristics-type methods have been applied and analyzed for many other linear and nonlinear parabolic PDEs from various engineering applications [1, 2, 23, 24, 32, 47]. Numerical simulations show that the time-truncation errors of the MMOC are much smaller than those of standard methods in many convection-dominated models.

The MMOC is based on characteristic tracking, in which the hyperbolic part is approximated by

$$(1.5) \quad \left( \Phi \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c \right) \Big|_{t=t^{n+1}} \approx \Phi \frac{c(x, t^{n+1}) - c(x - \frac{\mathbf{u}(x, t^n)}{\Phi} \tau, t^n)}{\tau}.$$

In this method, the numerical solution at the time step  $t^{n+1}$  depends upon the numerical solution at the previous time step  $t^n$  and at the location which is determined by the numerical velocity of the flow. Error analysis of numerical methods with such

an approximation for the nonlinear system involving an unknown velocity requires the boundedness of numerical concentration in  $W^{1,\infty}$ -norm and the velocity in  $L^\infty$ -norm, e.g., see [14, 18]. A time-step condition immediately arises to bound the numerical solution in  $W^{1,\infty}$ -norm in a traditional manner: mathematical induction/inverse inequality. Clearly, there is a contradiction between the concept of the characteristic temporal discretization and existing theoretical results. On the one hand, due to the temporal scheme of characteristics, the method of characteristics may greatly reduce the temporal error and allow one to use a large time step in practical computations. However, on the other hand, all the previous analyses required certain time-step conditions to obtain optimal error estimates. Such time-step restrictions from theoretical analyses do not support the strength of the MMOC. The restrictions could become more serious for problems in three-dimensional space and/or for nonuniform meshes. The purpose of this paper is to present a new analysis to establish unconditionally optimal error estimates of characteristics-mixed FEMs for the nonlinear and coupled system (1.1)–(1.3). Our results show clearly that the characteristic temporal discrete scheme is effective and the time-step conditions in previous analyses were required mainly due to the weakness of the traditional approach. The analysis is based on a new characteristic time-discrete system, i.e., an iterated sequence of elliptic PDEs. The fully discrete method of characteristics can be viewed as a spatial FE approximation to the time-discrete system. The  $L^2$  error bound of the fully discrete method of characteristics to the time-discrete system is proved to be  $\tau$ -independent, in terms of a temporal-spatial error splitting argument proposed in [27, 28]. We establish the unconditional boundedness of the numerical solution in  $W^{1,\infty}$ -norm by a classical inverse inequality and then the optimal error estimate in a traditional manner. Our results show clearly that the characteristic temporal discrete scheme is effective and the time-step conditions in previous analyses were required mainly due to the weakness of the traditional approach. Moreover, the approach used in this paper can be extended to many other characteristics-type methods.

The paper is organized as follows. In section 2, we present our notation, several useful lemmas, and our main results, and a new characteristic time-discrete system is introduced. In section 3, by making use of the splitting argument, we analyze the temporal and spatial errors, respectively, and the boundedness of numerical solutions is proved without such time-step restrictions. Then, we present optimal error estimates and unconditional stability of the numerical scheme in section 4. In section 5, numerical results are given to confirm our theoretical analysis.

**2. The main results.** We at first define some notation used in this paper. For any integer  $m \geq 0$  and  $1 \leq p \leq \infty$ , let  $W^{m,p}(\Omega)$  be the Sobolev space of functions with the norm

$$\|f\|_{W^{m,p}} = \begin{cases} \left( \sum_{|\beta| \leq m} \int_{\Omega} |D^\beta f|^p dx \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \sum_{|\beta| \leq m} \operatorname{ess\,sup}_{\Omega} |D^\beta f| & \text{for } p = \infty, \end{cases}$$

where

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}},$$

for the multi-index  $\beta = (\beta_1, \dots, \beta_d)$ ,  $\beta_1 \geq 0, \dots, \beta_d \geq 0$ , and  $|\beta| = \beta_1 + \dots + \beta_d$ . When  $p = 2$ , we denote  $W^{m,2}(\Omega)$  by  $H^m(\Omega)$ . For any Banach space  $X$ , let  $L^p(I; X)$  be the space of all measurable functions  $g : I \rightarrow X$  with the norm

$$\|g\|_{L^p(I; X)} = \begin{cases} \left( \int_0^T \|g(t)\|_X^p dt \right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \text{ess sup}_{t \in I} \|g(t)\|_X & \text{for } p = \infty, \end{cases}$$

where  $I = (0, T)$ . Also, we define  $L_0^p(\Omega) = \{f \in L^p(\Omega) : \int_{\Omega} f dx = 0\}$  and  $H(\text{div}; \Omega) = \{\mathbf{f} = (f_1, \dots, f_d) : f_i, \nabla \cdot \mathbf{f} \in L^2(\Omega), 1 \leq i \leq d\}$ .

To avoid technical boundary difficulties, we assume that  $\Omega$  is a rectangle in  $\mathbb{R}^2$  (or cuboid in  $\mathbb{R}^3$ ) and the problem (1.1)–(1.3) and the corresponding FE spaces are  $\Omega$ -periodic as usual [14, 18, 33, 38]. Let  $\pi_h$  be a quasi-uniform decomposition of  $\Omega$  into triangles  $T_j$ ,  $j = 1, \dots, M$ , in  $\mathbb{R}^2$  (or tetrahedra in  $\mathbb{R}^3$ ) of diameter less than  $h$ . For any integer  $r, k \geq 1$ , we define the finite element space [29]:

$$V_h^r = \{\chi \in C^0(\bar{\Omega}) : \chi|_{T_j} \in P_r \text{ for all } T_j \in \pi_h\},$$

where  $P_r$  is the space of polynomials of degree  $r$ . Let  $S_h^k \subset L_0^2(\Omega)$  and  $H_h^k \subset H(\text{div}; \Omega)$  be the Raviart–Thomas [37] spaces of index  $k$  such that  $\text{div } v_h \in S_h^k$  for  $v_h \in H_h^k$ .

Let  $\{t_n | t_n = n\tau; 0 \leq n \leq N\}$  be a uniform partition of  $[0, T]$  with the time step  $\tau = T/N$ , and we denote

$$c^n(x) = c(x, t_n), \quad \mathbf{u}^n(x) = \mathbf{u}(x, t_n), \quad p^n(x) = p(x, t_n).$$

For a sequence of functions  $\{\omega^n\}_{n=0}^N$ , we define

$$D_{\tau} \omega^{n+1} = \frac{\omega^{n+1} - \omega^n}{\tau}.$$

In the rest of the paper, we assume that the permeability  $k(x)$  and the concentration-dependent viscosity  $\mu(s)$  are in the space  $W^{2,\infty}(\Omega)$  and  $W^{2,\infty}(\mathbb{R})$ , respectively. There exist positive constants  $k_0$  and  $\mu_0$  such that

$$(2.1) \quad k_0^{-1} \leq k(x) \leq k_0 \quad \text{for } x \in \Omega,$$

$$(2.2) \quad \mu_0^{-1} \leq \mu(s) \leq \mu_0 \quad \text{for } s \in \mathbb{R}.$$

Moreover, the injection and production sources satisfy

$$(2.3) \quad \|q^I\|_{W^{1,4}}, \|q^P\|_{W^{1,4}} \leq C.$$

The diffusion-dispersion tensor  $D(\mathbf{u}) = \Phi(d_m I + D^*(\mathbf{u}))$  is a  $d \times d$  matrix, where  $d_m > 0$ ,  $D^*(\mathbf{u})$  is symmetric and positive definite and  $\partial_{u_i} D, \partial_{u_i u_j}^2 D \in L^\infty(\Omega)$ . For the system (1.1)–(1.3) being well-posed, we add

$$(2.4) \quad \int_{\Omega} q^I dx = \int_{\Omega} q^P dx.$$

With the above notation, the MMOC with a mixed FEM is to find  $C_h^n \in V_h^r$ ,  $U_h^n \in H_h^k$ , and  $P_h^n \in S_h^k$  such that

$$(2.5) \quad \left( \frac{\mu(C_h^n)}{k(x)} U_h^n, v_h \right) = \left( P_h^n, \nabla \cdot v_h \right),$$

$$(2.6) \quad (\nabla \cdot U_h^n, \varphi_h) = (q^I - q^P, \varphi_h),$$

$$(2.7) \quad \left( \Phi \frac{C_h^{n+1} - \widehat{C}_h^n}{\tau}, \phi_h \right) + (D(U_h^n) \nabla C_h^{n+1}, \nabla \phi_h) = (c_1 q^I - C_h^{n+1} q^I, \phi_h)$$

for all  $\phi_h \in V_h^r$ ,  $v_h \in H_h^k$ , and  $\varphi_h \in S_h^k$ , where

$$\widehat{\omega}^n := \omega^n(\widehat{x}) = \omega^n \left( x - \frac{U_h^n}{\Phi} \tau \right)$$

and  $C_h^0 = I_h c_0$  with  $I_h$  being the Lagrangian interpolation operator.

In this paper, we assume that the system (1.1)–(1.3) admits a unique solution satisfying

$$(2.8) \quad \begin{aligned} & \|c_0\|_{H^{r+1}} + \|c\|_{L^\infty(I; H^{r+1})} + \|c_t\|_{L^\infty(I; H^{r+1})} + \|c\|_{L^\infty(I; W^{2,4})} + \|c_{tt}\|_{L^2(I; L^2)} \\ & + \|\mathbf{u}\|_{L^\infty(I; H^{k+1})} + \|\mathbf{u}_t\|_{L^\infty(I; L^2)} + \|\mathbf{u}_t\|_{L^2(I; H^1)} + \|p\|_{L^\infty(I; H^{k+1})} \leq K. \end{aligned}$$

Next we present our main results in the following theorem.

**THEOREM 2.1.** *Suppose that the system (1.1)–(1.3) has a unique solution  $(c, \mathbf{u}, p)$  satisfying (2.8). Then, there exist positive constants  $\tau_0$  and  $h_0$  such that when  $\tau < \tau_0$  and  $h < h_0$ , the finite element system (2.5)–(2.7) admits a unique solution  $(C_h^m, U_h^m, P_h^m)$ ,  $m = 0, 1, \dots, N$ , which satisfies that*

$$(2.9) \quad \max_{0 \leq m \leq N} (\|C_h^m - c^m\|_{L^2} + \|U_h^m - \mathbf{u}^m\|_{L^2} + \|P_h^m - p^m\|_{L^2}) \leq C_0(\tau + h^{r+1} + h^{k+1}),$$

where  $C_0$  is a positive constant independent of  $\tau$  and  $h$ .

**Remark 2.2.** The scheme (2.5)–(2.7) was studied by several authors, such as [14, 18], in which optimal error estimates were obtained with certain time-step conditions. Here, we establish the estimate (2.9) unconditionally and show that such time-step restrictions are unnecessary.

To prove Theorem 2.1, we introduce a characteristic time-discrete system:

$$(2.10) \quad U^n = -\frac{k(x)}{\mu(C^n)} \nabla P^n,$$

$$(2.11) \quad \nabla \cdot U^n = q^I - q^P,$$

$$(2.12) \quad \Phi \frac{C^{n+1} - \widetilde{C}^n}{\tau} - \nabla \cdot (D(U^n) \nabla C^{n+1}) = c_1 q^I - C^{n+1} q^I,$$

with periodic boundary conditions and the following initial condition:

$$(2.13) \quad C^0(x) = c_0(x),$$

where  $x \in \Omega$ ,  $t \in [0, T]$ , and

$$\widetilde{\omega}^n := \omega^n(\widetilde{x}) = \omega^n \left( x - \frac{U^n}{\Phi} \tau \right).$$

The condition  $\int_\Omega P^n dx = 0$  is enforced for the uniqueness of the solution. The above system can be viewed as an iterated sequence of elliptic PDEs.

The key to the proof of Theorem 2.1 is a temporal-spatial error splitting:

$$\begin{aligned} \|\mathcal{C}_h^n - c^n\|_{L^2} &\leq \|e_c^n\|_{L^2} + \|\mathcal{C}^n - \mathcal{C}_h^n\|_{L^2}, \\ \|U_h^n - \mathbf{u}^n\|_{L^2} &\leq \|e_u^n\|_{L^2} + \|U^n - U_h^n\|_{L^2}, \\ \|P_h^n - p^n\|_{L^2} &\leq \|e_p^n\|_{L^2} + \|P^n - P_h^n\|_{L^2}, \end{aligned}$$

where

$$e_c^n = \mathcal{C}^n - c^n, \quad e_u^n = U^n - \mathbf{u}^n, \quad e_p^n = P^n - p^n.$$

Following the splitting, we shall establish the error estimates of  $(e_c^n, e_u^n, e_p^n)$  and regularity analysis of solutions to the characteristic time-discrete system. Then, we analyze errors in the spatial direction to get a priori estimates of numerical solutions, with which Theorem 2.1 can be proved in a routine way.

In our work, the following lemmas are useful.

LEMMA 2.3 (Gagliardo–Nirenberg inequality [34]). *Let  $u$  be a function defined on  $\Omega$  and  $\partial^s u$  be any partial derivative of  $u$  of order  $s$ . Then*

$$\|\partial^j u\|_{L^p} \leq C \|\partial^m u\|_{L^r}^\alpha \|u\|_{L^q}^{1-\alpha} + C \|u\|_{L^q}$$

for  $0 \leq j < m$  and  $\frac{j}{m} \leq \alpha \leq 1$  with

$$\frac{1}{p} = \frac{j}{n} + \alpha \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - \alpha) \frac{1}{q},$$

except  $1 < r < \infty$  and  $m - j - \frac{n}{r}$  is a nonnegative integer, in which case the above estimate holds only for  $\frac{j}{m} \leq \alpha < 1$ .

LEMMA 2.4. (a) *If  $f \in W^{1,\infty}(\Omega)$  and  $g_1, g_2 \in L^p(\Omega)$ , then*

$$\|f(x - g_1\tau) - f(x - g_2\tau)\|_{L^p} \leq C\tau \|f\|_{W^{1,\infty}} \|g_1 - g_2\|_{L^p}$$

for  $1 \leq p \leq \infty$ .

(b) *If  $g_1, g_2 \in L^\infty(\Omega)$ , and  $\tau(\|g_1\|_{W^{1,\infty}} + \|g_2\|_{W^{1,\infty}}) \leq \frac{1}{2}$ , then*

$$\begin{aligned} \|f(x - g_1\tau) - f(x - g_2\tau)\|_{L^p} &\leq C\tau \|f\|_{W^{1,q_1}} \|g_1 - g_2\|_{L^{q_2}}, \\ \|f(x) - f(x - g_2\tau)\|_{H^{-1}} &\leq C\tau \|f\|_{L^2} \|g_2\|_{W^{1,4}} \end{aligned}$$

for  $1/q_1 + 1/q_2 = 1/p$  and  $1 < p < \infty$ .

LEMMA 2.5 ( $W^{k,p}$ -estimates of elliptic problems [8, 16]). *Suppose that  $v$  is a solution of the following boundary value problem:*

$$-\Delta v = f \quad \text{in } \Omega,$$

with periodic boundary conditions and  $\int_\Omega v dx = 0$ . Then,

$$\|v\|_{W^{k,p}} \leq C \|f\|_{W^{k-2,p}}, \quad 2 \leq p < \infty.$$

The proof of Lemma 2.4 can be found in the literature, such as [13, 14].

In this paper, we denote by  $C$  a generic positive constant and by  $\epsilon$  a generic small positive constant, which are independent of  $n, h, \tau$ , and  $C_0$ .



**3. A priori estimates of numerical solutions.** First, we introduce several types of projections. Let  $\mathcal{Q}_h^n : H^1(\Omega) \rightarrow V_h^r$  denote an elliptic projection which satisfies

$$(3.1) \quad (D(U^{n-1})\nabla(v - \mathcal{Q}_h^n v), \nabla\phi_h) = 0, \quad v \in H^1 \text{ for all } \phi_h \in V_h^r,$$

with  $\int_{\Omega}(v - \mathcal{Q}_h^n v)dx = 0$ , for  $n = 1, 2, \dots, N$ , and  $\mathcal{Q}_h^0 := I_h$ . Let  $\Pi_h : L^2(\Omega) \rightarrow S_h^k$  and  $R_h : H(\text{div}; \Omega) \rightarrow H_h^k$  be two projections satisfying

$$(3.2) \quad (v - \Pi_h v, \varphi_h) = 0, \quad v \in L^2 \text{ for all } \varphi_h \in S_h^k,$$

$$(3.3) \quad (\nabla \cdot (w - R_h w), \varphi_h) = 0, \quad w \in H(\text{div}) \text{ for all } \varphi_h \in S_h^k.$$

By classical theories of Galerkin and mixed FEMs for linear elliptic problems [14, 15, 36, 41], we have

$$(3.4) \quad \|\mathcal{Q}_h^n v\|_{W^{1,\infty}} \leq C\|v\|_{W^{1,\infty}},$$

$$(3.5) \quad \|v - \mathcal{Q}_h^n v\|_{L^2} + h\|\nabla(v - \mathcal{Q}_h^n v)\|_{L^2} \leq Ch^2\|v\|_{H^2},$$

for  $n = 0, 1, \dots, N$ , and

$$(3.6) \quad \|v - \Pi_h v\|_{L^2} \leq Ch^m\|v\|_{H^m}, \quad 2 \leq m \leq k + 1,$$

$$(3.7) \quad \|\omega - R_h \omega\|_{L^2} \leq Ch^m\|\omega\|_{H^m}, \quad 2 \leq m \leq k + 1.$$

The following inverse inequality will always be used in this paper:

$$(3.8) \quad \|v_h\|_{W^{m,p}} \leq Ch^{(d/p-d/q)}\|v_h\|_{W^{m,q}}, \quad 1 \leq q \leq p \leq \infty, \quad m = 0, 1,$$

for any  $v_h$  in finite element spaces.

Here, we define

$$\xi_c^n = \mathcal{C}_h^n - \mathcal{Q}_h^n \mathcal{C}^n, \quad n = 0, 1, \dots, N.$$

In this section, we shall prove the following theorem.

**THEOREM 3.1.** *Suppose that the system (1.1)–(1.3) has a unique solution  $(c, \mathbf{u}, p)$  satisfying (2.8). Then, there exist positive constants  $\tau^*$  and  $h^*$  such that when  $\tau < \tau^*$  and  $h < h^*$ , the finite element system (2.5)–(2.7) admits a unique solution  $(\mathcal{C}_h^m, U_h^m, P_h^m)$ ,  $m = 0, 1, \dots, N$ , such that*

$$(3.9) \quad \max_{0 \leq m \leq N} \|U_h^m\|_{L^\infty} \leq C_0^*,$$

$$(3.10) \quad \max_{0 \leq m \leq N} \|\xi_c^m\|_{L^2}^2 + \sum_{m=0}^N \tau \|\xi_c^m\|_{H^1}^2 \leq C_0^* h^4,$$

where  $C_0^*$  is a positive constant, independent of  $n, \tau, h$ , and  $C_0$ .

**3.1. Analysis of the characteristic time-discrete system.** From the miscible displacement system (1.1)–(1.3) and the characteristic time-discrete system (2.10)–(2.12), we observe that the error function  $(e_p^n, e_u^n, e_c^n)$  satisfies the following equations:

$$(3.11) \quad -\nabla \cdot \left( \frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^n \right) = \nabla \cdot \left[ \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n \right],$$

$$(3.12) \quad e_u^n = -\frac{k(x)}{\mu(\mathcal{C}^n)} \nabla e_p^n - \left( \frac{k(x)}{\mu(\mathcal{C}^n)} - \frac{k(x)}{\mu(c^n)} \right) \nabla p^n,$$

(3.13)

$$\begin{aligned} & \Phi D_\tau e_c^{n+1} - \nabla \cdot (D(U^n) \nabla e_c^{n+1}) \\ &= \Phi \frac{c^n - \bar{c}^n - (C^n - \tilde{C}^n)}{\tau} + \nabla \cdot \left( (D(U^n) - D(\mathbf{u}^{n+1})) \nabla c^{n+1} \right) - e_c^{n+1} q^I + R_{tr}^{n+1}, \end{aligned}$$

for  $x \in \Omega$  and  $t \in [0, T]$ , with  $e_c^0(x) = 0$  and periodic boundary conditions, where

$$\bar{\omega}^n := \omega^n(\bar{x}) = \omega^n \left( x - \frac{\mathbf{u}^n}{\Phi} \tau \right),$$

and

$$R_{tr}^{n+1} = \Phi c_t^{n+1} + \mathbf{u}^{n+1} \cdot \nabla c^{n+1} - \Phi \frac{c^{n+1} - \bar{c}^n}{\tau}$$

defines a truncation error. By the regularity assumptions (2.8), it is easy to see [13]

$$(3.14) \quad \left( \sum_{n=0}^{N-1} \tau \|R_{tr}^{n+1}\|_{L^2}^2 \right)^{\frac{1}{2}} \leq C\tau.$$

**THEOREM 3.2.** *Suppose that the system (1.1)–(1.3) has a unique solution  $(c, \mathbf{u}, p)$  satisfying (2.8). Then there exists  $\tau' > 0$  such that when  $\tau < \tau'$ , the time-discrete system (2.10)–(2.12) admits a unique solution  $(C^m, U^m, P^m)$ ,  $m = 0, 1, \dots, N$ , which satisfies*

$$(3.15) \quad \max_{0 \leq m \leq N} (\|e_c^m\|_{H^1} + \|e_u^m\|_{H^1} + \|e_p^m\|_{H^1}) \leq C_0' \tau,$$

$$(3.16) \quad \max_{0 \leq m \leq N} \tau \|\nabla U^m\|_{L^\infty} \leq 1/2,$$

$$(3.17) \quad \max_{0 \leq m \leq N} (\|U^m\|_{H^2} + \|C^m\|_{W^{1,\infty}}) + \left( \sum_{m=1}^N \tau \|D_\tau C^m\|_{H^2}^2 \right)^{1/2} \leq C_0'.$$

*Proof.* Since at each time step the characteristic time-discrete system (2.10)–(2.12) is a linear system of elliptic PDEs, the existence and uniqueness of solutions to the system follow the boundedness of the solutions. So, we only need to prove the above estimates.

Before we study (3.15)–(3.17), we prove a primary estimate

$$(3.18) \quad \tau^{1/2} \|C^m\|_{W^{2,4}} + \|e_c^m\|_{H^2} \leq 1$$

by mathematical induction for  $m = 0, 1, \dots, N$ .

Since  $C^0(x) = c_0(x)$ , we see from (2.10)–(2.11) that  $U^0 = \mathbf{u}^0$  and  $P^0 = p^0$ . Then, (3.15) and (3.18) hold for  $m = 0$ .

We assume that (3.18) holds for  $m \leq n$  for some integer  $n \geq 0$ , which implies

$$(3.19) \quad \|C^m\|_{H^2} \leq \|e_c^m\|_{H^2} + \|c^m\|_{H^2} \leq 1 + K.$$

Combining (2.10) with (2.11), we obtain the equation for  $P^m$ :

$$(3.20) \quad -\Delta P^m = \frac{\mu(C^m)}{k(x)} \left( \nabla \left( \frac{k(x)}{\mu(C^m)} \right) \cdot \nabla P^m + (q^I - q^P) \right),$$

which shows that

$$(3.21) \quad \|P^m\|_{H^2} + \|U^m\|_{H^1} \leq C.$$

In this subsection, we further assume that the generic constant  $C$  is independent of  $C'_0$ . By Lemma 2.5, we have

$$\begin{aligned} \|P^m\|_{H^3} &\leq C \left\| \frac{\mu(\mathcal{C}^m)}{k(x)} \left( \nabla \left( \frac{k(x)}{\mu(\mathcal{C}^m)} \right) \cdot \nabla P^m + (q^I - q^P) \right) \right\|_{H^1} \\ &\leq C \left\| \frac{k(x)}{\mu(\mathcal{C}^m)} \right\|_{H^2} \|\nabla P^m\|_{L^\infty} + C \left\| \nabla \left( \frac{k(x)}{\mu(\mathcal{C}^m)} \right) \right\|_{L^4} \|P^m\|_{W^{2,4}} + C \|q^I - q^P\|_{H^1}. \end{aligned}$$

By Lemma 2.3,  $\|P^m\|_{W^{1,\infty}} \leq C \|P^m\|_{W^{2,4}} \leq \epsilon \|P^m\|_{H^3} + C_\epsilon \|P^m\|_{L^2}$ . It follows immediately that

$$\|P^m\|_{H^3} \leq C,$$

which with (2.10) further implies

$$(3.22) \quad \|U^m\|_{L^\infty} \leq C \|U^m\|_{H^2} \leq C.$$

Similarly, from (3.20),

$$\|P^m\|_{W^{3,4}} \leq C (\|\mathcal{C}^m\|_{W^{2,4}} \|P^m\|_{W^{1,\infty}} + \|\mathcal{C}^m\|_{W^{1,\infty}} \|P^m\|_{W^{2,4}} + \|q^I - q^P\|_{W^{1,4}}).$$

Since  $\tau^{1/2} \|\mathcal{C}^m\|_{W^{1,\infty}} \leq C \tau^{1/2} \|\mathcal{C}^m\|_{W^{2,4}} \leq C$ , from (2.10), we see that

$$\begin{aligned} \tau \|U^m\|_{W^{1,\infty}} &\leq C \tau \|U^m\|_{W^{2,4}} \\ &\leq C \tau \|\mathcal{C}^m\|_{W^{2,4}} \|P^m\|_{W^{1,\infty}} + C \tau \|\mathcal{C}^m\|_{W^{1,\infty}} \|P^m\|_{W^{2,4}} + C \tau \|P^m\|_{W^{3,4}} \\ &\leq C \tau \|\mathcal{C}^m\|_{W^{2,4}} \|P^m\|_{W^{2,4}} + C \tau \|q^I - q^P\|_{W^{1,4}} \\ (3.23) \quad &\leq \frac{1}{2} \end{aligned}$$

when  $\tau \leq \tau_1$  for some  $\tau_1 > 0$ . Furthermore, we multiply (3.11) by  $e_p^m$  to get

$$(3.24) \quad \|\nabla e_p^m\|_{L^2} \leq C \left\| \left( \frac{k(x)}{\mu(\mathcal{C}^m)} - \frac{k(x)}{\mu(c^m)} \right) \nabla p^m \right\|_{L^2} \leq C \|e_c^m\|_{L^2} \|\nabla p^m\|_{L^\infty} \leq C \|e_c^m\|_{L^2}.$$

From (3.11) and (3.12), we further have

$$(3.25) \quad \|\Delta e_p^m\|_{L^2} \leq C (\|\nabla e_p^m\|_{L^2} + \|e_c^m\|_{H^1}) \leq C \|e_c^m\|_{H^1},$$

$$(3.26) \quad \|e_u^m\|_{L^2} \leq C \|\nabla e_p^m\|_{L^2} + C \|e_c^m\|_{L^2} \|\nabla p^m\|_{L^\infty} \leq C \|e_c^m\|_{L^2},$$

and

$$(3.27) \quad \|e_u^m\|_{H^1} \leq C (\|e_p^m\|_{H^2} + \|e_c^m\|_{H^1}) \leq C \|e_c^m\|_{H^1}$$

for  $m \leq n$ .

To prove (3.18) for  $m = n + 1$ , we multiply (3.13) by  $e_c^{n+1}$  and integrate it over  $\Omega$  to get

$$\begin{aligned}
 (3.28) \quad & \frac{\Phi}{2} D_\tau \|e_c^{n+1}\|_{L^2}^2 + \|\sqrt{D(U^n)} \nabla e_c^{n+1}\|_{L^2}^2 \\
 & \leq \frac{C}{\tau} (c^n - \bar{c}^n - (C^n - \tilde{C}^n), e_c^{n+1}) + C(\|e_u^n\|_{L^2} + \|\mathbf{u}^n - \mathbf{u}^{n+1}\|_{L^2}) \|c^{n+1}\|_{W^{1,\infty}} \|\nabla e_c^{n+1}\|_{L^2} \\
 & \quad + C\|e_c^{n+1}\|_{L^4}^2 \|q^I\|_{L^2} + C\|R_{tr}^{n+1}\|_{L^2} \|e_c^{n+1}\|_{L^2}.
 \end{aligned}$$

By (3.22)–(3.23), Lemma 2.4, and (3.26), we obtain

$$\begin{aligned}
 (3.29) \quad \frac{C}{\tau} (c^n - \bar{c}^n - (C^n - \tilde{C}^n), e_c^{n+1}) & \leq \frac{C}{\tau} \|-(e_c^n - \tilde{e}_c^n) - (\bar{c}^n - \tilde{c}^n)\|_{L^2} \|e_c^{n+1}\|_{L^2} \\
 & \leq C\|e_c^n\|_{H^1} \|U^n\|_{L^\infty} \|e_c^{n+1}\|_{L^2} + C\|c^n\|_{W^{1,\infty}} \|e_u^n\|_{L^2} \|e_c^{n+1}\|_{L^2} \\
 & \leq \epsilon \|e_c^n\|_{H^1}^2 + C_\epsilon \|e_c^{n+1}\|_{L^2}^2.
 \end{aligned}$$

By noting the fact  $\|e_c^{n+1}\|_{L^4}^2 \leq \epsilon \|\nabla e_c^{n+1}\|_{L^2}^2 + C_\epsilon \|e_c^{n+1}\|_{L^2}^2$ , summing (3.28) up with (3.29) leads to

$$\|e_c^{n+1}\|_{L^2}^2 + \sum_{m=0}^n \tau \|\nabla e_c^{m+1}\|_{L^2}^2 \leq C_\epsilon \sum_{m=0}^n \tau \|e_c^{m+1}\|_{L^2}^2 + C \sum_{m=0}^n \tau \|R_{tr}^{m+1}\|_{L^2}^2 + C\tau^2.$$

By Gronwall’s inequality and (3.14), there exists  $\tau_2 > 0$  such that

$$(3.30) \quad \max_{0 \leq m \leq n} \|e_c^{m+1}\|_{L^2}^2 + \sum_{m=0}^n \tau \|\nabla e_c^{m+1}\|_{L^2}^2 \leq C_1 \tau^2$$

when  $\tau \leq \tau_2$ . By (3.24) and (3.26), we see that

$$(3.31) \quad \max_{0 \leq m \leq n} (\|e_u^{m+1}\|_{L^2} + \|\nabla e_p^{m+1}\|_{L^2}) \leq C_2 \tau.$$

The inequalities (3.30)–(3.31) further imply

$$(3.32) \quad \max_{0 \leq m \leq n} \|D_\tau C^{m+1}\|_{L^2} \leq \max_{0 \leq m \leq n} (\|D_\tau e_c^{m+1}\|_{L^2} + \|D_\tau c^{m+1}\|_{L^2}) \leq C_1 + K,$$

$$(3.33) \quad \max_{0 \leq m \leq n} \|D_\tau U^{m+1}\|_{L^2} \leq \max_{0 \leq m \leq n} (\|D_\tau e_u^{m+1}\|_{L^2} + \|D_\tau \mathbf{u}^{m+1}\|_{L^2}) \leq C_2 + K.$$

Moreover, we multiply (3.13) by  $-\nabla \cdot (D(U^n) \nabla e_c^{n+1})$  and integrate it over  $\Omega$  to arrive at

$$\begin{aligned}
 (3.34) \quad & D_\tau (D(U^n) \nabla e_c^{n+1}, \nabla e_c^{n+1}) + \|\nabla \cdot (D(U^n) \nabla e_c^{n+1})\|_{L^2}^2 \\
 & \leq C \|D_\tau D(U^n)\|_{L^2} \|\nabla e_c^n\|_{L^3} \|\nabla e_c^n\|_{L^6} + \frac{C}{\tau} \|c^n - \bar{c}^n - (C^n - \tilde{C}^n)\|_{L^2}^2 \\
 & \quad + C \|\nabla \cdot ((D(U^n) - D(\mathbf{u}^{n+1})) \nabla c^{n+1})\|_{L^2}^2 + C \|e_c^{n+1}\|_{L^6}^2 \|q^I\|_{L^3}^2 + C \|R_{tr}^{n+1}\|_{L^2}^2 \\
 & \leq C \|\nabla e_c^n\|_{L^3} \|\nabla e_c^n\|_{L^6} + C \|e_c^n\|_{H^1}^2 \|U^n\|_{L^\infty}^2 + C \|c^n\|_{W^{1,\infty}}^2 \|e_u^n\|_{L^2}^2 \\
 & \quad + C \|\nabla \cdot ((D(U^n) - D(\mathbf{u}^{n+1})) \nabla c^{n+1})\|_{L^2}^2 + C \|e_c^{n+1}\|_{H^1}^2 + C \|R_{tr}^{n+1}\|_{L^2}^2,
 \end{aligned}$$

where we have used (3.29) and (3.33). By (2.8) and (3.27),

$$\begin{aligned} & \|\nabla \cdot ((D(U^n) - D(\mathbf{u}^{n+1}))\nabla c^{n+1})\|_{L^2}^2 \\ & \leq C(\|\nabla e_c^n\|_{L^2}^2 + \|\mathbf{u}^n - \mathbf{u}^{n+1}\|_{H^1}^2)\|\nabla c^{n+1}\|_{L^\infty}^2 + C(\|e_u^n\|_{L^6}^2 + \|\mathbf{u}^n - \mathbf{u}^{n+1}\|_{L^6}^2)\|c^{n+1}\|_{W^{2,3}}^2 \\ & \leq C\|e_c^n\|_{H^1}^2 + C\|\mathbf{u}^n - \mathbf{u}^{n+1}\|_{H^1}^2. \end{aligned}$$

With (3.22), (3.30), and the above inequality and by noting

$$\begin{aligned} \|\nabla e_c^n\|_{L^3}\|\nabla e_c^n\|_{L^6} & \leq C_\epsilon\|\nabla e_c^n\|_{L^2}^2 + \epsilon\|e_c^n\|_{H^2}^2, \\ \|e_c^{n+1}\|_{H^2} & \leq C\|\nabla \cdot (D(U^n)\nabla e_c^{n+1})\|_{L^2} + C\|e_c^{n+1}\|_{H^1}, \end{aligned}$$

we sum (3.34) to get

$$\begin{aligned} & \|\nabla e_c^{n+1}\|_{L^2}^2 + \sum_{m=0}^n \tau\|e_c^{m+1}\|_{H^2}^2 \\ & \leq C \sum_{m=0}^n \tau\|\nabla e_c^{m+1}\|_{L^2}^2 + C \sum_{m=0}^n \tau(\|R_{tr}^{m+1}\|_{L^2}^2 + \|\mathbf{u}^m - \mathbf{u}^{m+1}\|_{H^1}^2) + C\tau^2. \end{aligned}$$

By applying Gronwall's inequality, (2.8), and (3.14), we get

$$(3.35) \quad \max_{0 \leq m \leq n} \|\nabla e_c^{m+1}\|_{L^2}^2 + \sum_{m=0}^n \tau\|e_c^{m+1}\|_{H^2}^2 \leq C_3\tau^2$$

and therefore,

$$(3.36) \quad \|e_c^{n+1}\|_{H^2} \leq \frac{1}{2}$$

when  $\tau \leq \tau_3$  for some  $\tau_3 > 0$ . The last two estimates further show that

$$(3.37) \quad \max_{0 \leq m \leq n} \|c^{m+1}\|_{H^2} \leq \max_{0 \leq m \leq n} (\|e_c^{m+1}\|_{H^2} + \|c^{m+1}\|_{H^2}) \leq 1 + K,$$

$$(3.38) \quad \max_{0 \leq m \leq n} \|D_\tau c^{m+1}\|_{L^4} \leq \max_{0 \leq m \leq n} (\|D_\tau e_c^{m+1}\|_{L^4} + \|D_\tau c^{m+1}\|_{L^4}) \leq C_3 + K,$$

$$(3.39) \quad \sum_{m=0}^n \tau\|D_\tau c^{m+1}\|_{H^2}^2 \leq 2 \sum_{m=0}^n \tau\|D_\tau e_c^{m+1}\|_{H^2}^2 + 2 \sum_{m=0}^n \tau\|D_\tau c^{m+1}\|_{H^2}^2 \leq 2C_3 + K.$$

To derive an estimate for  $\|c^{n+1}\|_{W^{2,4}}$ , we rewrite (2.12) by

$$-\nabla \cdot (D(U^n)\nabla c^{n+1}) = -\Phi \frac{c^{n+1} - \tilde{c}^n}{\tau} + c_1 q^I - c^{n+1} q^I.$$

By Lemma 2.4, (3.22), and (3.37),

$$\begin{aligned} \|c^{n+1}\|_{W^{2,4}} & \leq C\|U^n\|_{W^{1,4}}\|c^{n+1}\|_{W^{1,\infty}} + \frac{C}{\tau}\|c^{n+1} - c^n\|_{L^4} + \frac{C}{\tau}\|c^n - \tilde{c}^n\|_{L^4} \\ & \quad + C\|c_1 q^I - c^{n+1} q^I\|_{L^4} + C\|c^{n+1}\|_{L^2} \\ & \leq C\|U^n\|_{W^{1,4}}(\epsilon\|c^{n+1}\|_{W^{2,4}} + C\|c^{n+1}\|_{L^2}) + C\|D_\tau c^{n+1}\|_{L^4} \\ & \quad + C\|c^n\|_{W^{1,4}}\|U^n\|_{L^\infty} + C\|c^{n+1}\|_{L^4} + C \\ & \leq \epsilon\|c^{n+1}\|_{W^{2,4}} + C, \end{aligned}$$

which in turn produces

$$(3.40) \quad \|\mathcal{C}^{n+1}\|_{W^{2,4}} \leq C.$$

Thus, (3.18) holds for  $m = n + 1$  if  $\tau' \leq \min\{\tau_1, \tau_2, \tau_3, \frac{1}{4C^2}\}$ . The induction is closed.

Clearly, (3.15) follows from (3.27), (3.30)–(3.31), and (3.35), and (3.16) can be obtained immediately from (3.23). By the Sobolev embedding theorem and (3.40), we have

$$\|\mathcal{C}^{n+1}\|_{W^{1,\infty}} \leq C\|\mathcal{C}^{n+1}\|_{W^{2,4}} \leq C.$$

Thus (3.17) follows from (3.22), (3.39), and the above inequality. The proof of Theorem 3.2 is complete.  $\square$

*Remark 3.3.* Based on the results in Theorem 3.2, we can prove

$$(3.41) \quad \sum_{n=0}^{N-1} \tau \|D_\tau(\mathcal{C}^{n+1} - \mathcal{Q}_h^{n+1}\mathcal{C}^{n+1})\|_{H^{-1}}^2 \leq Ch^4$$

by a similar method as used in [28, 31]. With Theorem 3.2 and the above inequality, we will prove Theorem 3.1 in the following subsection.

**3.2. The proof of Theorem 3.1.** Since we have proved Theorem 3.2, in this subsection we assume that the generic constant  $C$  is independent of  $C_0^*$  and may be dependent upon  $C_0'$ .

From the fully discrete scheme (2.5)–(2.7) and the characteristic time-discrete system (2.10)–(2.12), we obtain the following error equations:

$$(3.42) \quad \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^n - \frac{\mu(\mathcal{C}^n)}{k(x)} U^n, v_h \right) = (P_h^n - \Pi_h P^n, \nabla \cdot v_h),$$

$$(3.43) \quad (\nabla \cdot (U_h^n - U^n), \varphi_h) = 0, \\ (\Phi D_\tau \xi_c^{n+1}, \phi_h) + (D(U_h^n) \nabla \xi_c^{n+1}, \nabla \phi_h) \\ = \frac{\Phi}{\tau} \left( (\mathcal{C}^{n+1} - \mathcal{Q}_h^{n+1} \mathcal{C}^{n+1}) - (\tilde{\mathcal{C}}^n - \mathcal{Q}_h^n \tilde{\mathcal{C}}^n), \phi_h \right) \\ + \frac{\Phi}{\tau} \left( \mathcal{Q}_h^n \hat{\mathcal{C}}^n - \mathcal{Q}_h^n \tilde{\mathcal{C}}^n, \phi_h \right) + \frac{\Phi}{\tau} \left( \hat{\xi}_c^n - \xi_c^n, \phi_h \right) \\ + ((D(U^n) - D(U_h^n)) \nabla \mathcal{Q}_h^{n+1} \mathcal{C}^{n+1}, \nabla \phi_h) - ((\mathcal{C}_h^{n+1} - \mathcal{C}^{n+1}) q^I, \phi_h)$$

$$(3.44) \quad := \sum_{i=1}^5 J_i$$

for all  $v_h \in H_h^k$ ,  $\varphi_h \in S_h^k$ , and  $\phi_h \in V_h^r$ .

Let

$$\xi_u^n = U_h^n - R_h U^n, \quad n = 0, 1, \dots, N.$$

By (3.3) and (3.43), we can see that  $(\nabla \cdot \xi_u^n, \varphi_h) = 0$  for any  $\varphi_h \in S_h^k$ , which implies that  $(P_h^n - \Pi_h P^n, \nabla \cdot \xi_u^n) = 0$ . Taking  $v_h = \xi_u^n$  in (3.42) gives

$$\left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} \xi_u^n + \frac{\mu(\mathcal{C}_h^n)}{k(x)} (R_h U^n - U^n) + \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} - \frac{\mu(\mathcal{C}^n)}{k(x)} \right) U^n, \xi_u^n \right) = 0,$$

which with (3.5), (3.7), and (3.17) shows

$$(3.45) \quad \|\xi_u^n\|_{L^2} \leq C\|R_h U^n - U^n\|_{L^2} + C\|C_h^n - C^n\|_{L^2} \leq Ch^2 + C\|\xi_c^n\|_{L^2}.$$

Now, we prove a primary estimate

$$(3.46) \quad \|\xi_c^m\|_{L^2} \leq h^{7/4}, \quad m = 0, 1, \dots, N,$$

by mathematical induction.

Since  $\xi_c^0 = C_h^0 - I_h C^0 = 0$ , the estimate (3.46) holds for  $m = 0$ .

For some integer  $n \geq 0$ , we assume that (3.46) holds for  $m \leq n$ , which with (3.17), (3.45), and the inverse inequality (3.8) implies

$$(3.47) \quad \begin{aligned} \|U_h^m\|_{L^\infty} &\leq \|R_h U^m\|_{L^\infty} + \|\xi_u^m\|_{L^\infty} \\ &\leq \|U^m\|_{H^2} + Ch^{-d/2}\|\xi_u^m\|_{L^2} \\ &\leq \|U^m\|_{H^2} + Ch^{-d/2}h^{7/4} \\ &\leq C'_0 + 1, \quad m = 0, 1, \dots, n, \end{aligned}$$

when  $h \leq h_1 = \frac{1}{C'_4}$  and  $\tau \leq \tau'$ .

To prove (3.46) for  $m = n + 1$ , we let  $\phi_h = \xi_c^{n+1}$  in (3.44). By Lemma 2.2, (3.7), and (3.45), we have

$$\begin{aligned} J_1 &= \Phi(D_\tau(C^{n+1} - Q_h^{n+1}C^{n+1}), \xi_c^{n+1}) + \frac{\Phi}{\tau} \left( (C^n - Q_h^n C^n) - (\tilde{C}^n - Q_h^n \tilde{C}^n), \xi_c^{n+1} \right) \\ &\leq C\|D_\tau(C^{n+1} - Q_h^{n+1}C^{n+1})\|_{H^{-1}}\|\xi_c^{n+1}\|_{H^1} + C\|C^n - Q_h^n C^n\|_{L^2}\|U^n\|_{W^{1,4}}\|\xi_c^{n+1}\|_{H^1}, \\ J_2 &\leq C\|Q_h^n C^n\|_{W^{1,\infty}}(\|U^n - R_h U^n\|_{L^2} + \|\xi_u^n\|_{L^2})\|\xi_c^{n+1}\|_{L^2}, \\ J_3 &= \frac{\Phi}{\tau}(\tilde{\xi}_c^n - \tilde{\xi}_c^n, \xi_c^{n+1}) + \frac{\Phi}{\tau}(\tilde{\xi}_c^n - \xi_c^n, \xi_c^{n+1}) \\ &\leq C\|\xi_c^n\|_{W^{1,\infty}}(\|U^n - R_h U^n\|_{L^2} + \|\xi_u^n\|_{L^2})\|\xi_c^{n+1}\|_{L^2} + C\|\xi_c^n\|_{L^2}\|U^n\|_{W^{1,4}}\|\xi_c^{n+1}\|_{H^1} \\ &\leq Ch^{-d/2}\|\xi_c^n\|_{H^1}h^{7/4}\|\xi_c^{n+1}\|_{L^2} + C\|\xi_c^n\|_{L^2}\|U^n\|_{W^{1,4}}\|\xi_c^{n+1}\|_{H^1}, \\ J_4 &\leq C(\|U^n - R_h U^n\|_{L^2} + \|\xi_u^n\|_{L^2})\|Q_h^{n+1}C^{n+1}\|_{W^{1,\infty}}\|\nabla \xi_c^{n+1}\|_{L^2}, \\ J_5 &\leq C\|q^I\|_{L^3}(\|\xi_c^{n+1}\|_{L^3}^2 + \|C^{n+1} - Q_h^{n+1}C^{n+1}\|_{L^2}\|\xi_c^{n+1}\|_{L^6}) \\ &\leq C_\epsilon\|\xi_c^{n+1}\|_{L^2}^2 + \epsilon\|\nabla \xi_c^{n+1}\|_{L^2}^2 + C_\epsilon h^4. \end{aligned}$$

With these estimates and by (3.4)–(3.7) and (3.17), (3.44) reduces to

$$\begin{aligned} &\Phi \frac{\|\xi_c^{n+1}\|_{L^2}^2 - \|\xi_c^n\|_{L^2}^2}{2\tau} + \left\| \sqrt{D(U_h^n)} \nabla \xi_c^{n+1} \right\|_{L^2}^2 \\ &\leq C_\epsilon(\|\xi_c^{n+1}\|_{L^2}^2 + \|\xi_c^n\|_{L^2}^2) + \epsilon(\|\nabla \xi_c^{n+1}\|_{L^2}^2 + \|\nabla \xi_c^n\|_{L^2}^2) + C_\epsilon h^4 \\ &\quad + C\|D_\tau(C^{n+1} - Q_h^{n+1}C^{n+1})\|_{H^{-1}}^2. \end{aligned}$$

By (3.41), summing the above inequality leads to

$$(3.48) \quad \|\xi_c^{n+1}\|_{L^2}^2 + \sum_{m=0}^n \tau \|\nabla \xi_c^{m+1}\|_{L^2}^2 \leq C \sum_{m=0}^n \tau \|\xi_c^{m+1}\|_{L^2}^2 + Ch^4.$$

By Gronwall's inequality, there exists  $\tau_4 > 0$  such that when  $\tau < \tau_4$ ,

$$(3.49) \quad \max_{0 \leq m \leq n} \|\xi_c^{m+1}\|_{L^2}^2 + \sum_{m=0}^n \tau \|\nabla \xi_c^{m+1}\|_{L^2}^2 \leq C_4 h^4,$$

and therefore, when  $h \leq h_2 = \frac{1}{C_4^2}$ ,

$$(3.50) \quad \|\xi_c^{n+1}\|_{L^2} \leq h^{7/4}.$$

The induction is closed. Thus, (3.9)–(3.10) can be obtained from (3.47) and (3.49) when  $\tau^* = \min\{\tau', \tau_4\}$ ,  $h^* = \min\{h_1, h_2\}$ , and  $C_0^* \geq \max\{C_0' + 1, C_4\}$ . The proof of Theorem 3.1 is complete.  $\square$

*Remark 3.4.* We have proved that (3.9)–(3.10) hold for any order FEMs with  $r, k \geq 1$ . Moreover, by an inverse inequality, we have

$$(3.51) \quad \max_{0 \leq m \leq N} \|\xi_c^m\|_{H^1} \leq Ch.$$

With the projection error estimates (3.5) and Theorem 3.2, it is easy to see that

$$(3.52) \quad \max_{0 \leq m \leq N} \|C_h^m - c^m\|_{L^2} \leq C(\tau + h^2),$$

$$(3.53) \quad \max_{0 \leq m \leq N} \|C_h^m - c^m\|_{H^1} \leq C(\tau + h).$$

The above error estimates are optimal for the MMOC with a linear FEM, i.e.,  $r = k = 1$ . Obviously, the inequalities (3.52)–(3.53) are not optimal for higher-order approximations. We will prove unconditionally optimal  $L^2$  error estimates of high-order FE approximation by making use of the boundedness of numerical solutions (3.9)–(3.10) in the next section.

**4. The proof of Theorem 2.1.** In this section, we establish optimal error estimates for the fully discrete scheme (2.5)–(2.7) with results given in Theorem 3.1 for a general mixed FEM. Since we have proved Theorems 3.1 and 3.2, hereafter the generic constant  $C$  may be dependent upon  $C_0'$  and  $C_0^*$  and independent of  $C_0$ .

Let

$$\theta_c^n = C_h^n - Q_h^n c^n, \quad n = 0, 1, \dots, N,$$

where  $Q_h^n$  denotes an elliptic projection operator defined by

$$(D(\mathbf{u}^n) \nabla(v - Q_h^n v), \nabla \phi_h) = 0 \quad \text{for all } \phi_h \in V_h^n, \quad v \in H^1(\Omega),$$

with  $\int_{\Omega} (v - Q_h^n v) dx = 0$ . By the classical theory of Galerkin method for linear elliptic problems [48], we have

$$(4.1) \quad \|Q_h^n v\|_{W^{1,\infty}} \leq C \|v\|_{W^{1,\infty}},$$

$$(4.2) \quad \|v - Q_h^n v\|_{L^2} + h \|v - Q_h^n v\|_{H^1} \leq Ch^{r+1} \|v\|_{H^{r+1}},$$

$$(4.3) \quad \left( \sum_{m=0}^{N-1} \tau \|D_{\tau}(v^{m+1} - Q_h^{m+1} v^{m+1})\|_{L^2}^2 \right)^{\frac{1}{2}} \leq Ch^{r+1} (\|v\|_{L^2(I; H^{r+1})} + \|v_t\|_{L^2(I; H^{r+1})}).$$

Now, we start to prove Theorem 2.1. From the finite element system (2.5)–(2.7) and the miscible displacement system (1.1)–(1.3), we obtain the following error equations:

$$(4.4) \quad \left( \frac{\mu(C_h^n)}{k(x)} U_h^n - \frac{\mu(c^n)}{k(x)} \mathbf{u}^n, v_h \right) = (P_h^n - \Pi_h p^n, \nabla \cdot v_h),$$



$$\begin{aligned}
(4.5) \quad & (\nabla \cdot (U_h^n - \mathbf{u}^n), \varphi_h) = 0, \\
& (\Phi D_\tau \theta_c^{n+1}, \phi_h) + (D(U_h^n) \nabla \theta_c^{n+1}, \nabla \phi_h) \\
& = \frac{\Phi}{\tau} \left( (c^{n+1} - Q_h^{n+1} c^{n+1}) - (\bar{c}^n - Q_h^n \bar{c}^n), \phi_h \right) + \frac{\Phi}{\tau} (Q_h^n \bar{c}^n - Q_h^n \bar{c}^n, \phi_h) \\
& \quad + \frac{\Phi}{\tau} (\widehat{\theta}_c^n - \theta_c^n, \phi_h) + ((D(\mathbf{u}^{n+1}) - D(U_h^n)) \nabla Q_h^{n+1} c^{n+1}, \nabla \phi_h) \\
& \quad - ((c_h^{n+1} - c^{n+1}) q^I, \phi_h) + (R_{tr}^{n+1}, \phi_h) \\
(4.6) \quad & := \sum_{i=1}^6 L_i
\end{aligned}$$

for all  $v_h \in H_h^k$ ,  $\varphi_h \in S_h^k$ , and  $\phi_h \in V_h^r$ .

Let

$$\theta_u^n = U_h^n - R_h \mathbf{u}^n, \quad n = 0, 1, \dots, N.$$

By (3.3) and (4.5), we have  $(\nabla \cdot \theta_u^n, \varphi_h) = 0$  for any  $\varphi_h \in S_h^k$ , which shows that  $(P_h^n - \Pi_h p^n, \nabla \cdot \theta_u^n) = 0$ . By (4.4) and (4.5) with  $v_h = \theta_u^n$ , we get

$$\left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} \theta_u^n + \frac{\mu(\mathcal{C}_h^n)}{k(x)} (R_h \mathbf{u}^n - \mathbf{u}^n) + \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} - \frac{\mu(c^n)}{k(x)} \right) \mathbf{u}^n, \theta_u^n \right) = 0.$$

Applying the same approach as used for proving the estimate (3.45), we can easily see

$$\begin{aligned}
(4.7) \quad & \|U_h^n - \mathbf{u}^n\|_{L^2} \leq \|\theta_u^n\|_{L^2} + \|R_h \mathbf{u}^n - \mathbf{u}^n\|_{L^2} \\
& \leq C \|R_h \mathbf{u}^n - \mathbf{u}^n\|_{L^2} + C \|\mathcal{C}_h^n - c^n\|_{L^2} \\
& \leq C h^{k+1} + C h^{r+1} + C \|\theta_c^n\|_{L^2}, \quad n = 0, 1, \dots, N.
\end{aligned}$$

To prove the estimate of  $\|\theta_c^{n+1}\|_{L^2}$ , we take  $\phi_h = \theta_c^{n+1}$  in (4.6). By Lemma 2.2, (4.2), and (4.7),

$$\begin{aligned}
L_1 & = \Phi(D_\tau(c^{n+1} - Q_h^{n+1} c^{n+1}), \theta_c^{n+1}) + \frac{\Phi}{\tau} \left( (c^n - Q_h^n c^n) - (\bar{c}^n - Q_h^n \bar{c}^n), \theta_c^{n+1} \right) \\
& \leq C \|D_\tau(c^{n+1} - Q_h^{n+1} c^{n+1})\|_{L^2} \|\theta_c^{n+1}\|_{L^2} + C \|c^n - Q_h^n c^n\|_{L^2} \|\mathbf{u}^n\|_{W^{1,4}} \|\theta_c^{n+1}\|_{H^1}, \\
L_2 & \leq C \|Q_h^n c^n\|_{W^{1,\infty}} \|U_h^n - \mathbf{u}^n\|_{L^2} \|\theta_c^{n+1}\|_{L^2}, \\
L_3 & = \frac{\Phi}{\tau} (\widehat{\theta}_c^n - \bar{\theta}_c^n, \theta_c^{n+1}) + \frac{\Phi}{\tau} (\bar{\theta}_c^n - \theta_c^n, \theta_c^{n+1}) \\
& \leq C \|\theta_c^n\|_{W^{1,\infty}} \|U_h^n - \mathbf{u}^n\|_{L^2} \|\theta_c^{n+1}\|_{L^2} + C \|\theta_c^n\|_{L^2} \|\mathbf{u}^n\|_{W^{1,4}} \|\theta_c^{n+1}\|_{H^1} \\
& \leq C (\|\xi_c^n\|_{W^{1,\infty}} + \|Q_h^n c^n\|_{W^{1,\infty}} + \|Q_h^n c^n\|_{W^{1,\infty}}) (h^{k+1} + h^{r+1} + \|\theta_c^n\|_{L^2}) \|\theta_c^{n+1}\|_{L^2} \\
& \quad + C \|\theta_c^n\|_{L^2} \|\theta_c^{n+1}\|_{H^1}, \\
L_4 & \leq C (\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{L^2} + \|\mathbf{u}^n - U_h^n\|_{L^2}) \|\nabla Q_h^{n+1} c^{n+1}\|_{L^\infty} \|\nabla \theta_c^{n+1}\|_{L^2}, \\
L_5 & \leq C \|\theta_c^{n+1} - (c^{n+1} - Q_h^{n+1} c^{n+1})\|_{L^2} \|q^I\|_{L^6} \|\theta_c^{n+1}\|_{L^3} \\
& \leq C_\epsilon \|\theta_c^{n+1}\|_{L^2}^2 + \epsilon \|\nabla \theta_c^{n+1}\|_{L^2}^2 + C h^{2(r+1)}, \\
L_6 & \leq C \|R_{tr}^{n+1}\|_{L^2} \|\theta_c^{n+1}\|_{L^2}.
\end{aligned}$$

With (3.17), (4.1), and the above estimates, (4.6) reduces to

$$\begin{aligned}
 & \Phi \frac{\|\theta_c^{n+1}\|_{L^2}^2 - \|\theta_c^n\|_{L^2}^2}{2\tau} + \left\| \sqrt{D(U_h^n)} \nabla \theta_c^{n+1} \right\|_{L^2}^2 \\
 & \leq C \|D_\tau(c^{n+1} - Q_h^{n+1}c^{n+1})\|_{L^2}^2 + C_\epsilon (\|\theta_c^{n+1}\|_{L^2}^2 + \|\theta_c^n\|_{L^2}^2) \\
 & \quad + \epsilon \|\nabla \theta_c^{n+1}\|_{L^2}^2 + C \|R_{tr}^{n+1}\|_{L^2}^2 \\
 (4.8) \quad & + Ch^{-d} \|\xi_c^n\|_{H^1}^2 (h^{2(r+1)} + h^{2(k+1)} + \|\theta_c^n\|_{L^2}^2) + C_\epsilon (\tau^2 + h^{2(r+1)} + h^{2(k+1)}).
 \end{aligned}$$

By (3.14), (4.3), and Theorem 3.1, summing (4.8) leads to

$$\begin{aligned}
 & \|\theta_c^{n+1}\|_{L^2}^2 + \tau \sum_{m=0}^n \|\nabla \theta_c^{m+1}\|_{L^2}^2 \\
 & \leq C_\epsilon \sum_{m=0}^{n+1} \tau \|\theta_c^m\|_{L^2}^2 + C_5 h^{-d} \left( \max_{0 \leq m \leq n} \|\theta_c^m\|_{L^2}^2 + h^{2(r+1)} + h^{2(k+1)} \right) \sum_{m=0}^n \tau \|\xi_c^m\|_{H^1}^2 \\
 & \quad + C \sum_{m=0}^n \tau \|D_\tau(c^{m+1} - Q_h^{m+1}c^{m+1})\|_{L^2}^2 + C \sum_{m=0}^n \tau \|R_{tr}^{m+1}\|_{L^2}^2 \\
 & \quad + C_5 (\tau^2 + h^{2(r+1)} + h^{2(k+1)}) \\
 & \leq C_\epsilon \sum_{m=0}^{n+1} \tau \|\theta_c^m\|_{L^2}^2 + \frac{1}{2} \|\theta_c^{n+1}\|_{L^2}^2 + C (\tau^2 + h^{2(r+1)} + h^{2(k+1)}),
 \end{aligned}$$

when  $h \leq h_3 = \min\{h^*, \frac{1}{2C_5C_0^*}\}$  and  $\tau \leq \tau^*$ . By Gronwall's inequality, there exists  $\tau_5 > 0$  such that

$$(4.9) \quad \max_{0 \leq m \leq N-1} \|\theta_c^{m+1}\|_{L^2} \leq C(\tau + h^{r+1} + h^{k+1})$$

when  $\tau < \tau_5$ . Since  $C_h^0 = I_h c_0$ , with (4.2) we get

$$\begin{aligned}
 & \max_{0 \leq m \leq N} \|C_h^m - c^m\|_{L^2} \leq \max_{0 \leq m \leq N} (\|\theta_c^m\|_{L^2} + \|Q_h^m c^m - c^m\|_{L^2}) \\
 (4.10) \quad & \leq C(\tau + h^{r+1} + h^{k+1}).
 \end{aligned}$$

By (4.7), we further have

$$(4.11) \quad \max_{0 \leq m \leq N} \|U_h^m - \mathbf{u}^m\|_{L^2} \leq C(\tau + h^{r+1} + h^{k+1}).$$

To derive the estimate of  $\|P_h^n - p^n\|_{L^2}$ , we let  $g$  be the solution to the equation

$$(4.12) \quad -\nabla \cdot \left( \frac{k(x)}{\mu(c^n)} \nabla g \right) = P_h^n - \Pi_h p^n$$

with periodic boundary conditions and  $\int_\Omega g dx = 0$ , where we have noted from (3.2) that  $\int_\Omega (P_h^n - \Pi_h p^n) dx = 0$ . By Lemma 2.5, it is easy to get

$$\|g\|_{H^2} \leq C \|P_h^n - \Pi_h p^n\|_{L^2}.$$

Let  $v_h = R_h(\frac{k(x)}{\mu(c^n)} \nabla g)$ . From (3.3) and (4.12), we see that

$$(\varphi_h, \nabla \cdot v_h) = -(\varphi_h, P_h^n - \Pi_h p^n)$$

for any  $\varphi_h \in S_h^k$ . Taking  $\varphi_h = P_h^n - \Pi_h p^n$  in the above equality, with (4.4), we derive that

$$\begin{aligned} \|P_h^n - \Pi_h p^n\|_{L^2}^2 &= - \left( \frac{\mu(\mathcal{C}_h^n)}{k(x)} U_h^n - \frac{\mu(c^n)}{k(x)} \mathbf{u}^n, R_h \left( \frac{k(x)}{\mu(c^n)} \nabla g \right) \right) \\ &\leq C(\|C_h^n - c^n\|_{L^2} + \|U_h^n - \mathbf{u}^n\|_{L^2}) \left\| \frac{k(x)}{\mu(c^n)} \nabla g \right\|_{H^1} \\ &\leq C(\tau + h^{r+1} + h^{k+1}) \|P_h^n - \Pi_h p^n\|_{L^2}, \end{aligned}$$

and therefore, by (3.6)

$$\|P_h^n - p^n\|_{L^2} \leq C(\tau + h^{r+1} + h^{k+1}), \quad n = 0, 1, \dots, N.$$

Taking  $\tau_0 = \min\{\tau^*, \tau_5\}$ ,  $h_0 = h_3$  and  $C_0 \geq C$  in (4.10), (4.11), and the last inequality, the proof of Theorem 2.1 is complete.  $\square$

**5. Numerical results.** In this section, we present some numerical results to confirm our theoretical analysis. All computations are performed by FreeFem++ in two-dimensional space.

We consider the system

$$(5.1) \quad \frac{\partial c}{\partial t} - \nabla \cdot (D(\mathbf{u}) \nabla c) + \mathbf{u} \cdot \nabla c = f,$$

$$(5.2) \quad \nabla \cdot \mathbf{u} = g,$$

$$(5.3) \quad \mathbf{u} = - \frac{k(x)}{\mu(c)} \nabla p,$$

in  $\Omega = (0, 1) \times (0, 1)$ , with  $D(\mathbf{u}) = \frac{1}{40}(1 + \frac{|\mathbf{u}|^2}{1+|\mathbf{u}|})$  and  $\mu(c) = 1 + c^2$ , where  $f$  and  $g$  are chosen corresponding to the exact solution

$$(5.4) \quad c = 1 + 20e^t(1 + t^2) \sin(x^2) \sin(y^2)(1 - x)^2(1 - y)^2,$$

$$(5.5) \quad p = 3 + 400e^t(1 + t^3)x^2y^2(1 - x)^3(1 - y)^3,$$

$$(5.6) \quad \mathbf{u} = - \frac{1}{\mu(c)} \nabla p.$$

Here, we solve the system (5.1)–(5.3) by the numerical scheme (2.5)–(2.7) with  $k = r = 1$ , in which a uniform triangular partition with  $M + 1$  nodes in each direction is used to generate the FEM mesh, where  $h = \sqrt{2}/M$ . To show the optimal error rates, we choose  $\tau = 10h^2$  and present in Table 1 numerical results at  $t = 0.5, 1.0$ . Clearly, we see that the errors in  $L^2$ -norm are proportional to  $O(\tau + h^2) = O(h^2)$ .

To test the stability of the scheme, we solve the problem (5.1)–(5.3) with several different  $M$  for each  $\tau = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$ . Numerical results are presented in Figure 1, which shows that  $L^2$  errors converge to  $O(\tau)$  as  $\tau/h \rightarrow \infty$ . This shows that the time-step restrictions are not necessary and the scheme is unconditionally stable.

TABLE 1  
 $L^2$  errors of the scheme (2.5)–(2.7).

$t$	$\ c(\cdot, t_n) - C_h^n\ _{L^2}$			
	$M = 10$	$M = 20$	$M = 40$	Order( $\alpha$ )
$t = 0.5$	1.7335e-02	4.3286e-03	1.0178e-03	2.0451
$t = 1.0$	1.6830e-01	4.2809e-02	9.9244e-03	2.0420
$t$	$\ \mathbf{u}(\cdot, t_n) - U_h^n\ _{L^2}$			
	$M = 10$	$M = 20$	$M = 40$	Order( $\alpha$ )
$t = 0.5$	1.8371e-02	4.7699e-03	1.2109e-03	1.9617
$t = 1.0$	6.5716e-02	1.6273e-02	4.0702e-03	2.0065
$t$	$\ p(\cdot, t_n) - P_h^n\ _{L^2}$			
	$M = 10$	$M = 20$	$M = 40$	Order( $\alpha$ )
$t = 0.5$	6.9454e-03	1.7462e-03	4.1837e-04	2.0266
$t = 1.0$	1.3719e-01	3.7177e-02	8.5811e-03	1.9994

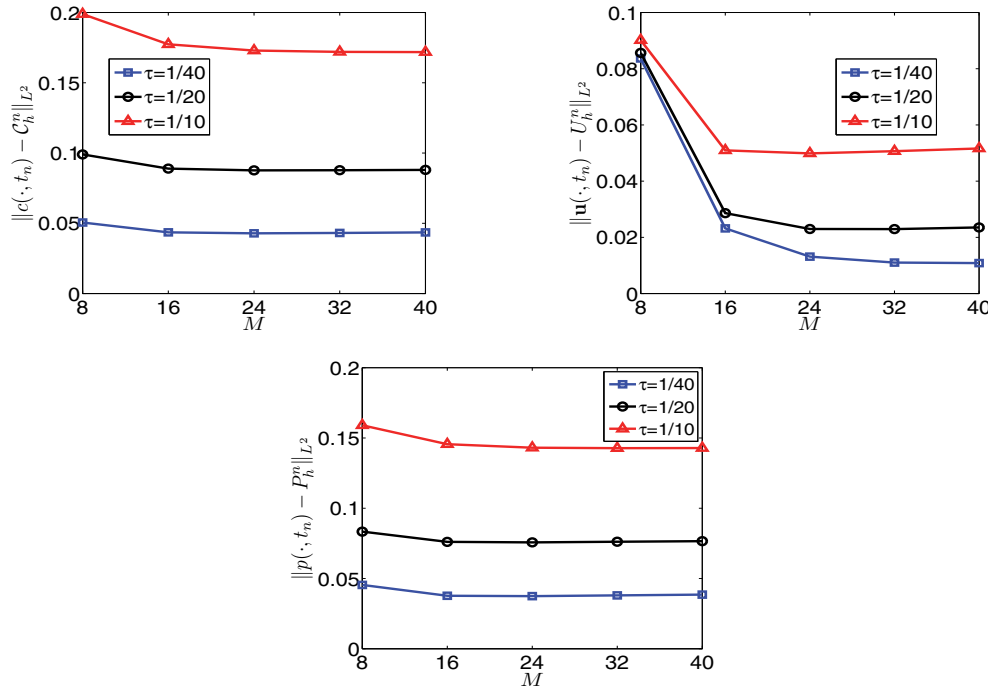


FIG. 1. Stability of the scheme (2.5)–(2.7).

**6. Conclusion.** We have established unconditionally optimal error estimates of characteristics-mixed FEMs ( $k \geq 1$ ) with linearized Euler scheme for miscible displacement problems in two- and three-dimensional spaces. Our theoretical analysis resolves the contradiction between the numerical observation and previous analyses and shows clearly that the MMOC is (almost) unconditionally stable. For the lowest-order Raviart–Thomas FEM, one may not be able to obtain the boundedness of numerical solution in  $W^{1,\infty}$ -norm in three-dimensional space unconditionally from the optimal error estimate. The stability analysis of the lowest-order FEM in a stronger norm is under investigation.

In addition, here we assume that the diffusion-dispersion tensor is positive definite, i.e., a nondegenerate porous media problem. It has been noted that analyses for some degenerate cases were also studied by several authors [4, 9]. In [9], Dawson, Russell, and Wheeler investigated characteristics-mixed FEMs for miscible displacement system in two-dimensional space, where no lower bound was assumed for the diffusion-dispersion coefficient. A suboptimal  $L^2$ -error estimate was obtained under the conditions  $\tau = o(h_c^2)$  and  $h_p^{k+1} = o(h_c^2)$ . Arbogast and Wang [3] considered a volume corrected characteristics-mixed method for a purely transport problem. A lower-order  $L^1$ -error estimate  $O(h/\sqrt{\tau} + h + \tau^m)$  was presented where  $m$  was related to the accuracy of the characteristic tracing. Extension of our splitting approach to the degenerate case will be studied in the future.

Moreover, theoretical analysis presented in this paper is based on the  $\Omega$ -periodic model as usual [9, 14, 18, 38] to avoid the technical difficulties on the boundary. This periodic assumption is physically reasonable. For the problem with Neumann boundary conditions, some further approximation to  $C_h^n(\hat{x})$  was mentioned in [38].

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