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Published in:
SIAM Journal on Mathematical Analysis

Published: 01/01/2014

Published version (DOI):
10.1137/130920617

Publication details:
LIU, H., YANG, T., ZHAO, H., & ZOU, Q. (2014). One-dimensional Compressible Navier-Stokes Equations with Temperature Dependent Transport Coefficients and Large Data. *SIAM Journal on Mathematical Analysis*, *46*(3), 2185-2228. https://doi.org/10.1137/130920617

SIAM J. MATH. ANAL.
Vol. 46, No. 3, pp. 2185–2228

ONE-DIMENSIONAL COMPRESSIBLE NAVIER–STOKES EQUATIONS WITH TEMPERATURE DEPENDENT TRANSPORT COEFFICIENTS AND LARGE DATA*

HONGXIA LIU†, TONG YANG‡, HUIJIANG ZHAO§, AND QINGYANG ZOU¶

Abstract. This paper is concerned with the Cauchy problem of the one-dimensional compressible Navier–Stokes equations with degenerate temperature dependent transport coefficients which satisfy conditions from the consideration in kinetic theory. A result on the existence and uniqueness of a globally smooth nonvacuum solution is obtained provided that the $(\gamma-1)\cdot(H^3(\mathbf{R})$-norm of the initial perturbation$)< C$ for some positive constant C independent of $\gamma-1$. Here $\gamma>1$ is the adiabatic gas constant. This is a Nishida–Smoller type global solvability result with large data.

Key words. compressible Navier–Stokes equations, global solution with large data, temperature dependent transport coefficients, Nishida–Smoller type result

AMS subject classifications. 35Q35, 35D35, 74D10, 76D05, 76N10

DOI. 10.1137/130920617

1. Introduction and main result.

The motion of one-dimensional compressible flow of a viscous ideal fluid can be described by the Navier–Stokes system

$$\begin{cases} \rho_\tau + (\rho u)_y = 0, \\ (\rho u)_\tau + \left(\rho u^2 + p(\rho,\theta)\right)_y = (\mu u_y)_y, \\ (\rho\mathcal{E})_\tau + (\rho u\mathcal{E} + up(\rho,\theta))_y = (\kappa\theta_y + \mu u u_y)_y. \end{cases} \tag{1.1}$$

Here y and τ represent the space variable and the time variable, respectively, and the primary dependent variables are fluid density ρ, fluid velocity u, and temperature θ. $\mathcal{E} = e + \frac{1}{2}u^2$ is the specific total energy with e being the specific internal energy. The pressure p, the internal energy e, and the transport coefficients $\mu>0$ (viscosity) and $\kappa>0$ (heat conductivity) are functions of ρ and θ. The thermodynamic variables ρ, p, e, s, and θ are related by the Gibbs equation $de = ds - pd\rho^{-1}$, where s is the specific entropy.

Motivated by the study in the kinetic theory of gases, we are interested in constructing global smooth nonvacuum solutions to the Cauchy problem of the system

*Received by the editors May 10, 2013; accepted for publication (in revised form) March 19, 2014; published electronically June 26, 2014. This work was supported by a grant from the National Natural Science Foundation of China under contract 11261160485.
http://www.siam.org/journals/sima/46-3/92061.html

†Department of Mathematics, Jinan University, Guangzhou 510632, China (hongxia-liu@163.net). This author was supported by a grant from the National Natural Science Foundation of China under contract 11271160.

‡Corresponding author. Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, China (matyang@cityu.edu.hk). This author was supported by the Joint Research Fund of the National Natural Science Foundation of China and the Research Grants Council of Hong Kong, N-CityU102/12.

§School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China (hhjjzhao@whu.edu.cn). This author was supported by a grant from the National Natural Science Foundation of China under contract 10925103.

¶School of Science, Wuhan University of Science and Technology, Wuhan 430081, China (qyzou@whu.edu.cn). This author was supported by The Fundamental Research Funds for the Central Universities.

(1.1) with large initial data for the case when the transport coefficients $\mu > 0$ and $\kappa > 0$ are functions of temperature.

More precisely, recall that the Boltzmann equation with slab symmetry takes the form

$$f_\tau + \xi_1 f_y = \frac{1}{\varepsilon} Q(f, f), \tag{1.2}$$

where the unknown function $f(\tau, y, \xi) \geq 0$ stands for the distribution density of particles with position $y \in \mathbf{R}$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ at time $\tau \geq 0$, $\varepsilon > 0$ is the Knudsen number proportional to the mean free path and measures the adiabaticity of the gas, and $Q(f, f)$ is the Boltzmann collision operator defined by

$$Q(f, g) = \frac{1}{2} \int_{\mathbf{R}^3 \times \mathbf{S}^2} q(|\xi - \xi_*|, \theta) \Big(f' g'_* + f'_* g' - f g_* - f_* g \Big) d\xi_* d\omega.$$

Here $q(|\xi - \xi_*|, \theta) \geq 0$ is the cross section that is determined by the interaction potential of two colliding particles. Here, $f = f(\tau, y, \xi)$, $f_* = f(\tau, y, \xi_*)$, $f' = f(\tau, y, \xi')$, $f'_* = f(\tau, y, \xi'_*)$, $\cos\theta = (\xi - \xi_*) \cdot \omega / |\xi - \xi_*|$, $\omega \in \mathbf{S}^2$, and

$$\xi' = \xi - ((\xi - \xi_*) \cdot \omega)\omega, \quad \xi'_* = \xi_* + ((\xi - \xi_*) \cdot \omega)\omega$$

is the relation between the velocities ξ', ξ'_* after and the velocities ξ, ξ_* before the collision by the conservation of momentum and energy. For details see [3].

It is well known that by employing the celebrated Chapman–Enskog expansion, the compressible Navier–Stokes equations (1.1) are the first order approximation of the Boltzmann equation (1.2) in terms of ε and the viscosity μ and heat conductivity κ are functions of temperature; cf. [3], [4], [8], [29]. In particular, if the intermolecule potential is propotional to $r^{-\alpha}$ with r being the molecule distance, then

$$\mu, \quad \kappa \propto \theta^{\frac{\alpha+4}{2\alpha}}.$$

Note that for Maxwellian molecules ($\alpha = 4$) the dependence is linear, while for elastic spheres ($\alpha \to +\infty$) the dependence is like $\sqrt{\theta}$.

The above dependence has a strong influence on the solution behavior and leads to difficulty in analysis for global existence with large data. In fact, as pointed out in [11], temperature dependence of the viscosity μ turns out be especially problematic and challenging. Even though there are works about the density dependence in μ and temperature and density dependence in κ (cf. [1] [5], [11], [12], [14], [16], [17], [18], [19], [27], and the references therein), no result has been obtained for the case when μ depends on temperature. Hence, the result obtained in the paper can be viewed a small progress in this direction.

Throughout this paper, we will assume

$$\mu = \mu(\theta) > 0, \quad \kappa = \kappa(\theta) > 0 \quad \forall \theta > 0. \tag{1.3}$$

To state the main result, let x be the Lagrangian space variable, t be the time variable, and $v = \frac{1}{\rho}$ denote the specific volume. Then the system (1.1) with $\mu = \mu(\theta)$, $\kappa = \kappa(\theta)$ becomes

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v, \theta)_x = \left(\frac{\mu(\theta) u_x}{v} \right)_x, \\ e_t + p(v, \theta) u_x = \frac{\mu(\theta) u_x^2}{v} + \left(\frac{\kappa(\theta) \theta_x}{v} \right)_x. \end{cases} \tag{1.4}$$

Assume that the initial data satisfy

$$(v(0,x),u(0,x),\theta(0,x))=(v_0(x),u_0(x),\theta_0(x)),\quad \lim_{x\to\pm\infty}(v_0(x),u_0(x),\theta_0(x))=(1,0,1). \tag{1.5}$$

Throughout this paper, we will concentrate on the ideal, polytropic gases:

$$p(v,\theta)=\frac{R\theta}{v}=Av^{-\gamma}\exp\left(\frac{\gamma-1}{R}s\right),\quad e=C_v\theta=\frac{R\theta}{\gamma-1}, \tag{1.6}$$

where the specific gas constants A, R, and the specific heat at constant volume C_v are positive constants and $\gamma>1$ is the adiabatic constant. Note that for the model of monatomic gas, $\gamma=5/3$, which does not satisfy the condition imposed in the following theorem.

We will present a Nishida–Smoller type result for the above problem. That is, we will be concerned with the global solvability result for the Cauchy problem (1.4), (1.5) under the assumption that $(\gamma-1)\cdot\|(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}})\|_{H^3(\mathbf{R})}<C$ for some positive constant C which does not depend on $\gamma-1$. (For the corresponding original Nishida–Smoller type global existence result for one-dimensional ideal polytropic isentropic compressible Euler system, please refer to [23], whereas the nonisentropic case was analyzed by Liu [21] and also Temple [28]). Note that for the case of constant transport coefficients or for the case when the viscosity coefficient μ depends only on the density while the heat conductivity coefficient κ depends on both density and temperature, such a Nishida–Smoller type result has been obtained in [15] and [16], respectively. The main purpose of our present paper is to show that such a type of result also holds for the case when the transport coefficients depend on the temperature which satisfy conditions from the consideration in kinetic theory.

To state the main result, we will choose the velocity u, the specific volume v, and the entropy s as unknown functions and use $\overline{s}=\frac{R}{\gamma-1}\ln\frac{R}{A}$ to denote the far field of the initial entropy $s_0(x)$, that is,

$$\lim_{|x|\to+\infty}s_0(x)=\lim_{|x|\to+\infty}\frac{R}{\gamma-1}\ln\frac{R\theta_0(x)v_0(x)^{\gamma-1}}{A}=\bar{s}.$$

With the above notation in hand, our main result in this paper can be stated as follows.

THEOREM 1.1. *Suppose that*

- $L_{0i}=\|(v_0(x)-1,u_0(x),s_0(x)-\bar{s})\|_{H^i(\mathbf{R})}$ *($i=1,2,3$) is bounded by some positive constant independent of $\gamma-1$ and there are $(\gamma-1)$-independent positive constants* $0<\underline{V}_0<1$, $\overline{V}_0>1$, $0<\underline{\Theta}_0<1$, $\overline{\Theta}_0>1$ *such that*

$$\underline{V}_0\le v_0(x)\le\overline{V}_0,\quad \underline{\Theta}_0\le\theta_0(x)\le\overline{\Theta}_0\quad \forall x\in\mathbf{R};$$

- *$\mu(\theta)$ and $\kappa(\theta)$ are smooth for $\theta>0$ and satisfy* (1.3) *for $\theta>0$;*
- *$\gamma-1>0$ is sufficiently small such that*

$$(\gamma-1)L_{03}^2\exp\left(\exp\left(C\left(L_{02}^2+1\right)\right)\right)\le\min\left\{\left(1-\underline{\Theta}_0\right)^2,\left(\overline{\Theta}_0-1\right)^2\right\} \tag{1.7}$$

holds for some generic sufficiently large positive constant C depending only on $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, *and* $\overline{\Theta}_0$.

Then the Cauchy problem (1.4), (1.5) *admits a unique global solution* $(v(t,x),u(t,x),\theta(t,x))$ *satisfying*

(1.8)
$$\begin{cases}\underline{V}\le v(t,x)\le \overline{V}, \quad \underline{\Theta}_0\le \theta(t,x)\le \overline{\Theta}_0 \quad \forall (t,x)\in[0,+\infty)\times\mathbf{R},\\ (v(t,x)-1,u(t,x),\theta(t,x)-1)\in C\left([0,+\infty);H^3(\mathbf{R})\right)\cap C^1\left([0,+\infty);H^1(\mathbf{R})\right),\\ v_x(t,x)\in L^2\left([0,+\infty);H^2(\mathbf{R})\right), \quad (u_x(t,x),\theta_x(t,x))\in L^2\left([0,+\infty);H^3(\mathbf{R})\right),\end{cases}$$

and

$$\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}\left|(v(t,x)-1,u(t,x),\theta(t,x)-1)\right|=0. \tag{1.9}$$

Here $\underline{V}$ *and* $\overline{V}$ *are some positive constants depending only on* L_{01}, $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, *and* $\overline{\Theta}_0$.

Remark 1.2. Several remarks concerning Theorem 1.1 follow:

1. Even for the case when the viscosity coefficient μ and the heat conductivity coefficient κ are functions of both v and θ, a similar result still holds if $\mu(v,\theta)$ and $\kappa(v,\theta)$ satisfy
   - $\mu(v,\theta)>0$, $\kappa(v,\theta)>0$ hold for $v>0$, $\theta>0$,
   - there exist constants $a<0$, $b>-\frac{1}{2}$ such that
   $$\mu(v,\theta)\sim\begin{cases}v^a, & v\to 0_+,\\ v^b, & v\to+\infty,\end{cases}$$
   for any $\theta\in\left[\underline{\Theta}_0,\overline{\Theta}_0\right]$.
2. From the proof of Theorem 1.1, it is easy to see that even when $\|(v_0-1,u_0,s_0-\bar{s})\|_{H^3(\mathbf{R})}$, $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, and $\overline{\Theta}_0$ depend on $\gamma-1$, with $\lim_{\gamma\to1_+}\|(v_0-1,u_0,\theta_0-1)\|_{H^3(\mathbf{R})}=+\infty$, $\lim_{\gamma\to1_+}(\underline{V}_0,\underline{\Theta}_0)=(0,0)$, $\lim_{\gamma\to1_+}\left(\overline{V}_0,\overline{\Theta}_0\right)=(+\infty,+\infty)$, a result similar to Theorem 1.1 also holds provided that the above limits satisfy certain growth conditions when $\gamma\to1_+$.

Now we sketch the main ideas used in the proof of Theorem 1.1. As pointed out in [1], [5], [17], [27], the key point to the global solvability of the Cauchy problem (1.4), (1.5) with large data is to obtain the positive lower and upper bounds for the specific volume $v(t,x)$ and the absolute temperature $\theta(t,x)$. For the Cauchy problem (1.4), (1.5), to our knowledge, according to the dependence of the viscosity coefficient μ and the heat conductivity coefficient κ on v and/or θ, there are two existing effective approaches:

(i) The first approach is developed by Kazhikhov and Shelukhin in [19] for the case when the viscosity coefficient μ is a positive constant, i.e., $\mu=\mu_0>0$ for some constant μ_0. In fact, only the case when both μ and κ are positive constants is discussed in [19]. However, the argument developed in [19] can be applied to the case when μ is a positive constant and κ depends on both v and θ. The main idea in [19] is to deduce an explicit formula for $v(t,x)$, cf. [19], as follows:
For each $i\in\mathbf{Z}$ and $x\in[i,i+1]$

$$v(t,x)=\frac{1+\frac{R}{\mu_0}\int_0^t\theta(\tau,x)B_i(\tau,x)Y_i(\tau)d\tau}{B_i(t,x)Y_i(t)}. \tag{1.10}$$

Here

$$B_i(t,x) = \frac{v_0(a_i(t))}{v_0(x)v(t,a_i(t))} \exp\left(\frac{1}{\mu_0}\int_x^{a_i(t)} (u(t,y)-u_0(y))dy\right),$$
$$Y_i(t) = \exp\left(\frac{R}{\mu_0}\int_0^t \left(\frac{\theta}{v}\right)(\tau, a_i(t))d\tau\right),$$

and for each integer $i \in \mathbf{Z}$, $a_i(t) \in [i, i+1]$ satisfies

$$A_1 \le v(t, a_i(t)) \le A_2$$

with $A_i > 0$ being the two positive roots of $x - \ln x + 1 = C$ for some sufficiently large positive constant C depending only on the initial data.
With the expression (1.10), one can first deduce a positive lower bound for $v(t,x)$. Then by employing the standard maximum principle for the parabolic equation

$$\begin{aligned} C_v\left(\frac{1}{\theta}\right)_t &= -\frac{\mu u_x^2}{\theta^2 v} + \frac{Ru_x}{v\theta} - \frac{2\theta\kappa}{v}\left[\left(\frac{1}{\theta}\right)_x\right]^2 + \left[\left(\frac{\kappa}{v}\right)\left(\frac{1}{\theta}\right)_x\right]_x \\ &= \left[\left(\frac{\kappa}{v}\right)\left(\frac{1}{\theta}\right)_x\right]_x - \left\{\frac{2\theta\kappa}{v}\left[\left(\frac{1}{\theta}\right)_x\right]^2 + \frac{\mu}{v\theta^2}\left(u_x - \frac{R\theta}{2\mu}\right)^2\right\} \\ &\quad + \frac{R^2}{4v\mu}, \end{aligned} \tag{1.11}$$

one can deduce a positive lower bound for $\theta(t,x)$.
With the lower bounds on both v and θ, the argument used in [19] leads to the upper bound estimate on v provided that κ satisfies

$$\min_{v\ge V_1>0,\theta\ge\Theta_1>0} \kappa(v,\theta) \ge C(V_1,\Theta_1) > 0$$

for some positive constant $C(V_1,\Theta_1) > 0$. Then the upper bound on θ follows and the global existence of solution is proved.

(ii) The second approach was introduced in [27] to treat the case when μ and κ are degenerate functions of v and/or θ, say, for example, $\mu = v^{-a}$, $\kappa = \theta^b$ for some positive constants $a > 0$, $b > 0$. In such a case, the argument used in [19] cannot be used. And the main idea in [27] is to first derive the lower bound for θ in term of the lower bound of v

$$\left\|\frac{1}{\theta}\right\|_{L^\infty([0,T]\times\mathbf{R})} \le O(1)\left\{1 + \left\|\frac{1}{v}\right\|_{L^\infty([0,T]\times\mathbf{R})}^{1-a}\right\}, \quad 0 \le a < 1, \tag{1.12}$$

by applying the maximum principle to (1.11), and then to deduce the lower and upper bounds for v in terms of θ,

$$\left\|\frac{1}{v}\right\|_{L^\infty([0,T]\times\mathbf{R})} \le O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty([0,T]\times\mathbf{R})}^{\frac{1}{3a-1}}\right), \quad \frac{1}{3} < a < \frac{1}{2}, \tag{1.13}$$

and

$$\|v\|_{L^\infty([0,T]\times\mathbf{R})} \le O(1)\left(1 + \left\|\theta^{1-b}\right\|_{L^\infty([0,T]\times\mathbf{R})}^{\frac{2a}{(3a-1)(1-2a)}}\right), \quad \frac{1}{3} < a < \frac{1}{2}, \tag{1.14}$$

by applying Kanel's argument [14].

From the above estimates, we have

$$\|\theta\|_{L^\infty([0,T]\times\mathbf{R})} \le O(1)\left\{1+\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty([0,T]\times\mathbf{R})} + \left\|\frac{u_x^2}{v^2}\right\|_{L^\infty([0,T]\times\mathbf{R})} + \|\theta\|^2_{L^\infty([0,T]\times\mathbf{R})}\right)d\tau\right\}. \tag{1.15}$$

With this relation, one can deduce the desired lower and upper bounds on v and θ if a and b satisfy certain conditions.

It is worth pointing out that a key point in the second approach is that when the viscosity coefficient μ depends only on v, one can obtain from the continuity equation $(1.4)_1$ and the momentum equation $(1.4)_2$ that

$$\left(\frac{\mu(v)v_x}{v}\right)_t = \left(\frac{\mu(v)v_t}{v}\right)_x = \left(\frac{\mu(v)u_x}{v}\right)_x = u_t + p(v,\theta)_x. \tag{1.16}$$

From this and the fact that the gas under consideration is ideal polytropic, one can then deduce a desired estimate on $\|\frac{\mu(v)v_x}{v}\|$, which is sufficient to deduce the positive uniform lower and upper bounds on $v(t,x)$ by exploiting Kanel's argument [14].

We note, however, that when the viscosity coefficient μ depends on θ, the situation is different because the identity corresponding to (1.16) now becomes

$$\left(\frac{\mu(\theta)v_x}{v}\right)_t = u_t + p(v,\theta)_x + \frac{\mu'(\theta)}{v}\left(\theta_t v_x - u_x\theta_x\right), \quad \mu'(\theta) = \frac{d\mu(\theta)}{d\theta}. \tag{1.17}$$

From $(1.4)_3$, the last term on the right-hand side of (1.17) is highly nonlinear so that to control the possible growth of $(v(t,x),u(t,x),\theta(t,x))$ is very difficult for large initial perturbation.

The key observation in this paper is that the constitutive relations (1.6) gives

$$\theta(t,x) = \frac{A}{R}v(t,x)^{1-\gamma}\exp\left(\frac{\gamma-1}{R}(s(t,x)-\bar{s})\right)\exp\left(\frac{\gamma-1}{R}\bar{s}\right),$$

which implies that when $\gamma-1>0$ is sufficiently small, the absolute temperature $\theta(t,x)$ can be sufficiently close to $\frac{A}{R}\exp\left(\frac{\gamma-1}{R}\bar{s}\right)=1$ provided that $v(t,x)$ is bounded from both below and above by some positive constants independent of $\gamma-1$ and $\lim_{\gamma\to1_+}(\gamma-1)\|s(t,x)-\bar{s}\|_{L^\infty([0,T]\times\mathbf{R})}=0$. Motivated by such an observation, suppose that $(v(t,x),u(t,x),\theta(t,x))$ is a solution of the Cauchy problem (1.4), (1.5) defined on the strip $\Pi_T=[0,T]\times\mathbf{R}$ for some positive constant $T>0$; if we assume a priori that the absolute temperature $\theta(t,x)$ satisfies

$$\|\theta(t,\cdot)-1\|_{H^3(\mathbf{R})} \le \varepsilon \quad \forall x\in\mathbf{R},\ t\in[0,T], \tag{1.18}$$

for some sufficiently small positive constant $\varepsilon>0$, then by some delicate energy type estimates and using the argument initiated in [14], we can use the smallness of ε and $\gamma-1>0$ to control the possible growth of the solution $(v(t,x),u(t,x),\theta(t,x))$ caused by the nonlinearities of the Navier–Stokes equation (1.4) under our consideration to deduce an uniform in time positive lower and upper bound on $v(t,x)$ and some uniform energy estimates on $\|(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}})(t)\|_{H^3(\mathbf{R})}$ in terms of $\|(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}})\|_{H^3(\mathbf{R})}$, $\inf_{x\in\mathbf{R}} v_0(x)$, $\sup_{x\in\mathbf{R}} v_0(x)$, $\inf_{x\in\mathbf{R}}\theta_0(x)$, and $\sup_{x\in\mathbf{R}}\theta_0(x)$. As a by-product of these estimates, if $\inf_{x\in\mathbf{R}} v_0(x)$, $\sup_{x\in\mathbf{R}} v_0(x)$, $\inf_{x\in\mathbf{R}} \theta_0(x)$, and $\sup_{x\in\mathbf{R}} \theta_0(x)$ are independent of $\gamma-1$, one has that

$$\|\theta(t)-1\|_{H^3(\mathbf{R})} \le \sqrt{\gamma-1}\tilde{C}\left(\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_{H^3(\mathbf{R})}\right) \tag{1.19}$$

holds for all $x \in \mathbf{R}$, $t \in [0,T]$. Here $\tilde{C}(x)$ is some function satisfying $\tilde{C}(x) > 0$ for $x > 0$ and $\tilde{C}(0) = 0$. Such an estimate (1.19) implies roughly that if $\|(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}})\|_{H^3(\mathbf{R})}$ is independent of $\gamma-1$, the positive constant ε in the a priori assumption (1.18) can be taken as the order of $\sqrt{\gamma-1}$ and thus can be sufficiently small as $\gamma-1>0$ is sufficiently small.

With the above estimates in hand, one can thus use the continuation argument to extend the local solution step by step to a global one provided that $\gamma-1>0$ is taken sufficiently small such that the conditions listed in Theorem 1.1 hold.

The rest of this paper is organized as follows. In section 2, we will give some identities for later use. For the precise way to use the smallness of ε and $\gamma-1>0$ to control the possible growth of the solution $(v(t,x), u(t,x), \theta(t,x))$ caused by the nonlinearities of the Navier–Stokes equations (1.4) under our consideration to deduce the desired energy estimates, see section 3, and for the extension of the local solution step by step to a global one by combining the a priori estimates obtained in section 3 with the continuation argument, see section 4.

Notation. O(1) or $C_i (i \in \mathbf{N})$ stands for a generic positive constant which is independent of t, x, and $\gamma-1$ but may depend only on $\underline{\Theta}_0$, $\overline{\Theta}_0$, $\underline{V}_0$, and $\overline{V}_0$, while $C(\cdot,\cdots,\cdot)$ is used to denote some positive constant depending only on the arguments listed in the parentheses. Note that all these constants may vary from line to line. $\|\cdot\|_s$ represents the norm in $H^s(\mathbf{R})$ with $\|\cdot\| = \|\cdot\|_0$ and for $1 \le p \le +\infty$, $L^p(\mathbf{R})$ denotes the standard Lebesgue space.

Finally, to simplify the presentation, we can assume without loss of generality that the gas constants $A = R = 1$ and consequently $\bar{s} = 0$.

2. Preliminaries.

The main purpose of this section is to list some identities which will be used in the following sections.

First, notice that $(1.4)_1$ implies

$$\begin{aligned}\left(\frac{\mu(\theta)v_x}{v}\right)_t &= \mu(\theta)\left(\frac{v_x}{v}\right)_t + \frac{\mu'(\theta)\theta_t v_x}{v}\\ &= \mu(\theta)\left(\frac{v_t}{v}\right)_x + \frac{\mu'(\theta)\theta_t v_x}{v}\\ &= \left(\frac{\mu(\theta)u_x}{v}\right)_x + \frac{\mu'(\theta)}{v}\left(\theta_t v_x - u_x\theta_x\right);\end{aligned}$$

we have from $(1.4)_2$ that

$$\left(\frac{\mu(\theta)v_x}{v}\right)_t = u_t + \left(\frac{\theta}{v}\right)_x + \frac{\mu'(\theta)}{v}\left(\theta_t v_x - u_x\theta_x\right). \tag{2.1}$$

Here recall that all the gas constants A and R have been normalized to be 1.

On the other hand, we can get from $(1.4)_3$ that

$$
\begin{aligned}
\frac{\theta_t}{\gamma-1} &= \frac{\mu(\theta)u_x^2}{v} + \left(\frac{\kappa(\theta)\theta_x}{v}\right)_x - \frac{\theta u_x}{v} \\
&= \frac{\mu(\theta)u_x^2 + \kappa'(\theta)\theta_x^2 + \kappa(\theta)\theta_{xx} - \theta u_x}{v} - \frac{\kappa(\theta)\theta_x v_x}{v^2} \\
&\le C(v,\theta)\left(u_x^2 + \theta_x^2 + |u_x| + |\theta_x v_x| + |\theta_{xx}|\right),
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
\frac{\theta_{tx}}{\gamma-1} &= \frac{\kappa(\theta)\theta_{xxx} + 3\kappa'(\theta)\theta_x\theta_{xx} + \kappa''(\theta)\theta_x^3}{v} \\
&\quad + \frac{\mu'(\theta)\theta_x u_x^2 + 2\mu(\theta)u_x u_{xx} - \theta_x u_x - \theta u_{xx}}{v} \\
&\quad - \frac{2\kappa(\theta)\theta_{xx}v_x + 2\kappa'(\theta)\theta_x^2 v_x + \kappa(\theta)\theta_x v_{xx}}{v^2} \\
&\quad - \frac{\mu(\theta)u_x^2 v_x - \theta u_x v_x}{v^2} + \frac{2\kappa(\theta)\theta_x v_x^2}{v^3} \\
&\le C(v,\theta)\Big(|\theta_{xxx}| + |(v_x,\theta_x)|\,|\theta_{xx}| + (1+|u_x|)\,|u_{xx}| + |\theta_x|\,|v_{xx}| \\
&\quad + (1+|(u_x,\theta_x)|)\,|(v_x,u_x,\theta_x)|^2\Big),
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
\frac{\theta_{txx}}{\gamma-1} &= \frac{2\kappa(\theta)\theta_x v_x^3}{v^4} + \frac{6\kappa(\theta)\theta_{xx}v_x^2 + 6\kappa'(\theta)\theta_x^2 v_x^2}{v^3} \\
&\quad + \frac{6\kappa(\theta)\theta_x v_x v_{xx} + 2\mu(\theta)u_x^2 v_x^2 - 2\theta u_x v_x^2}{v^3} \\
&\quad - \frac{2\mu'(\theta)u_x^2 v_x\theta_x + 4\mu(\theta)u_x v_x u_{xx} + \mu(\theta)u_x^2 v_{xx} + 3\kappa''(\theta)\theta_x^3 v_x}{v^2} \\
&\quad - \frac{9\kappa'(\theta)\theta_x\theta_{xx}v_x + 3\kappa'(\theta)\theta_x^2 v_{xx} + 3\kappa(\theta)\theta_{xxx}v_x + 3\kappa(\theta)\theta_{xx}v_{xx}}{v^2} \\
&\quad - \frac{\kappa(\theta)\theta_x v_{xxx} - 2\theta_x v_x u_x - 2\theta u_{xx}v_x - \theta u_x v_{xx}}{v^2} \\
&\quad + \frac{\mu''(\theta)u_x^2\theta_x^2 + \mu'(\theta)u_x^2\theta_{xx} + 4\mu'(\theta)\theta_x u_x u_{xx}}{v} \\
&\quad + \frac{2\mu(\theta)u_{xx}^2 + 2\mu(\theta)u_{xxx}u_x + \kappa'''(\theta)\theta_x^4}{v} \\
&\quad + \frac{6\kappa''(\theta)\theta_x^2\theta_{xx} + 3\kappa'(\theta)\theta_{xx}^2 + 4\kappa'(\theta)\theta_{xxx}\theta_x + \kappa(\theta)\theta_{xxxx}}{v} \\
&\quad - \frac{\theta_{xx}u_x + 2\theta_x u_{xx} + \theta u_{xxx}}{v} \\
&\le C(v,\theta)\Big(|\theta_{xxxx}| + |u_{xxx}| + |(u_x,\theta_x)|\,|(v_{xxx},\theta_{xxx})| \\
&\quad + |(v_{xx},u_{xx},\theta_{xx})|^2 \\
&\quad + |(v_{xx},u_{xx},\theta_{xx})|\,|(v_x,u_x,\theta_x)|\,(1+|(v_x,u_x,\theta_x)|) \\
&\quad + |(v_x,u_x,\theta_x)|^3\,(1+|(v_x,u_x,\theta_x)|)\Big).
\end{aligned} \tag{2.4}
$$

Next, we give some identities related to the pressure p. Recall that for $p = \frac{\theta}{v}$, we have

$$
\left(\frac{\theta}{v}\right)_x = \frac{\theta_x}{v} - \frac{\theta v_x}{v^2} \le C(v,\theta)(|\theta_x| + |v_x|), \tag{2.5}
$$

$$\left(\frac{\theta}{v}\right)_{xx} = \frac{\theta_{xx}}{v} - \frac{2\theta_x v_x + \theta v_{xx}}{v^2} + \frac{2\theta v_x^2}{v^3} \le C(v,\theta)\left(|(v_{xx},\theta_{xx})| + |(v_x,\theta_x)|^2\right), \tag{2.6}$$

$$\left(\frac{\theta}{v}\right)_{xxx} = \frac{\theta_{xxx}}{v} - \frac{3\theta_{xx} v_x + 3\theta_x v_{xx} + \theta v_{xxx}}{v^2} + \frac{6\theta_x v_x^2 + 6\theta v_x v_{xx}}{v^3} - \frac{6\theta v_x^3}{v^4} \le C(v,\theta)\left(|(v_x,\theta_x)|^3 + |(v_{xx},\theta_{xx})|\,|(v_x,\theta_x)| + |(v_{xxx},\theta_{xxx})|\right). \tag{2.7}$$

To deduce the energy type estimates on $v(t,x)$, we need some identities on the derivative of $\frac{\mu(\theta)v_x}{v}$ with respect to x up to the second order, which are listed below:

$$\left(\frac{\mu(\theta)v_x}{v}\right)_x = \frac{\mu'(\theta)\theta_x v_x + \mu(\theta)v_{xx}}{v} - \frac{\mu(\theta)v_x^2}{v^2} \le C(v,\theta)\left(|(v_x,\theta_x)|^2 + |v_{xx}|\right), \tag{2.8}$$

$$\begin{aligned}\left(\frac{\mu(\theta)v_x}{v}\right)_{xx} - \frac{\mu(\theta)v_{xxx}}{v} &= \frac{\mu''(\theta)\theta_x^2 v_x + \mu'(\theta)\theta_{xx}v_x + 2\mu'(\theta)\theta_x v_{xx}}{v} \\ &\quad - \frac{2\mu'(\theta)\theta_x v_x^2 + 3\mu(\theta)v_x v_{xx}}{v^2} + \frac{2\mu(\theta)v_x^3}{v^3} \\ &\le C(v,\theta)\left(|(v_x,\theta_x)|^3 + |(v_x,\theta_x)|\,|(v_{xx},\theta_{xx})|\right).\end{aligned} \tag{2.9}$$

Moreover, for derivatives of both $\frac{\mu(\theta)u_x^2}{v}$ and $\frac{\mu(\theta)u_x}{v}$ with respect to x up to the second order or the third order, respectively, we have

$$\left(\frac{\mu(\theta)u_x^2}{v}\right)_x = \frac{\mu'(\theta)\theta_x u_x^2 + 2\mu(\theta)u_x u_{xx}}{v} - \frac{\mu(\theta)u_x^2 v_x}{v^2} \le C(v,\theta)\left(|u_x u_{xx}| + u_x^2\left(|\theta_x| + |v_x|\right)\right), \tag{2.10}$$

$$\begin{aligned}\left(\frac{\mu(\theta)u_x^2}{v}\right)_{xx} &= \frac{\mu''(\theta)\theta_x^2 u_x^2 + \mu'(\theta)\theta_{xx}u_x^2 + 4\mu'(\theta)\theta_x u_x u_{xx} + 2\mu(\theta)u_{xx}^2 + 2\mu(\theta)u_x u_{xxx}}{v} \\ &\quad - \frac{2\mu'(\theta)\theta_x u_x^2 v_x + 4\mu(\theta)u_x u_{xx} v_x + \mu(\theta)u_x^2 v_{xx}}{v^2} + \frac{2\mu(\theta)u_x^2 v_x^2}{v^3} \\ &\le C(v,\theta)\Big(|(v_x,u_x,\theta_x)|^4 + u_{xx}^2 + \left|(v_x,u_x,\theta_x)\right|^2 |(v_{xx},u_{xx},\theta_{xx})| \\ &\quad + u_x^2|v_{xx}| + |u_x u_{xxx}|\Big)\end{aligned} \tag{2.11}$$

and

$$\left(\frac{\mu(\theta)u_x}{v}\right)_x = \frac{\mu'(\theta)\theta_x u_x + \mu(\theta)u_{xx}}{v} - \frac{\mu(\theta)u_x v_x}{v^2} \le C(\theta,v)\left(|u_{xx}| + |(v_x,u_x,\theta_x)|^2\right), \tag{2.12}$$

$$
\begin{aligned}
(2.13)\qquad &\left(\frac{\mu(\theta)u_x}{v}\right)_{xx}-\frac{\mu(\theta)u_{xxx}}{v}\\
&=\frac{\mu''(\theta)\theta_x^2u_x+\mu'(\theta)\theta_{xx}u_x+2\mu'(\theta)\theta_xu_{xx}}{v}\\
&\quad-\frac{2\mu'(\theta)\theta_xv_xu_x+2\mu(\theta)u_{xx}v_x+\mu(\theta)u_xv_{xx}}{v^2}+\frac{2\mu(\theta)u_xv_x^2}{v^3}\\
&\le C(\theta,v)\left(|(v_x,u_x,\theta_x)|^3+|u_x|\,|v_{xx}|+|(v_x,u_x,\theta_x)|\,|(u_{xx},\theta_{xx})|\right),
\end{aligned}
$$

$$
\begin{aligned}
(2.14)\qquad &\left(\frac{\mu(\theta)u_x}{v}\right)_{xxx}-\frac{\mu(\theta)u_{xxxx}}{v}\\
&=\frac{\mu'''(v)\theta_x^3v_x+3\mu''(\theta)\theta_x^2u_{xx}+3\mu''(\theta)u_x\theta_x\theta_{xx}+3\mu'(\theta)\theta_xu_{xxx}}{v}\\
&\quad+\frac{\mu'(\theta)\theta_{xxx}u_x+3\mu'(\theta)\theta_{xx}u_{xx}}{v}\\
&\quad-\frac{3\mu''(v)\theta_x^2u_xv_x+3\mu'(\theta)\theta_{xx}u_xv_x+6\mu'(\theta)u_{xx}\theta_xv_x}{v^2}\\
&\quad-\frac{3\mu'(\theta)\theta_xu_xv_{xx}+3\mu(\theta)u_{xxx}v_x+3\mu(\theta)v_{xx}u_{xx}+\mu(\theta)u_xv_{xxx}}{v^2}\\
&\quad+\frac{6\mu'(v)\theta_xu_xv_x^2+6\mu(v)u_{xx}v_x^2+6\mu(v)v_{xx}u_xv_x}{v^3}-\frac{6\mu(\theta)u_xv_x^3}{v^4}\\
&\le C(v,\theta)\Big(|(v_x,u_x,\theta_x)|^4+|(v_x,u_x,\theta_x)|^2\,|(v_{xx},u_{xx},\theta_{xx})|\\
&\quad+|(v_x,u_x,\theta_x)|\,|(u_{xxx},\theta_{xxx})|+|u_x|\,|v_{xxx}|+|u_{xx}|\,|(v_{xx},\theta_{xx})|\Big).
\end{aligned}
$$

Finally, for the derivatives of $\frac{\kappa(\theta)\theta_x}{v}$ with respect to x up to the third order, we have

$$
\begin{aligned}
(2.15)\qquad \left(\frac{\kappa(\theta)\theta_x}{v}\right)_x&=\frac{\kappa'(\theta)\theta_x^2+\kappa(\theta)\theta_{xx}}{v}-\frac{\kappa(\theta)\theta_xv_x}{v^2}\\
&\le C(v,\theta)\left(\theta_x^2+v_x^2+|\theta_{xx}|\right),
\end{aligned}
$$

(2.16)

$$
\begin{aligned}
\left(\frac{\kappa(\theta)\theta_x}{v}\right)_{xx}-\frac{\kappa(\theta)\theta_{xxx}}{v}&=\frac{\kappa''(\theta)\theta_x^3+3\kappa'(\theta)\theta_x\theta_{xx}}{v}\\
&\quad-\frac{2\kappa(\theta)\theta_{xx}v_x+2\kappa'(\theta)\theta_x^2v_x+\kappa(\theta)\theta_xv_{xx}}{v^2}+\frac{2\kappa(\theta)\theta_xv_x^2}{v^3},\\
&\le C(v,\theta)\left(|(v_x,\theta_x)|^3+|(v_x,\theta_x)|\,|\theta_{xx}|+|\theta_x|\,|v_{xx}|\right),
\end{aligned}
$$

(2.17)

$$
\begin{aligned}
\left(\frac{\kappa(\theta)\theta_x}{v}\right)_{xxx}-\frac{\kappa(\theta)\theta_{xxxx}}{v}&=\frac{\kappa'''(\theta)\theta_x^4+6\kappa''(\theta)\theta_x^2\theta_{xx}+3\kappa'(\theta)\theta_{xx}^2+4\kappa'(\theta)\theta_x\theta_{xxx}}{v}\\
&\quad-\frac{3\kappa''(\theta)\theta_x^3v_x+9\kappa'(\theta)v_x\theta_x\theta_{xx}+3\kappa'(\theta)\theta_x^2v_{xx}}{v^2}\\
&\quad-\frac{3\kappa(\theta)\theta_{xxx}v_x+3\kappa(\theta)\theta_{xx}v_{xx}+\kappa(\theta)\theta_xv_{xxx}}{v^2}\\
&\quad+\frac{6\kappa(\theta)\theta_{xx}v_x^2+6\kappa'(\theta)\theta_x^2v_x^2+6\kappa(\theta)\theta_xv_xv_{xx}}{v^3}-\frac{6\kappa(\theta)\theta_xv_x^3}{v^4}
\end{aligned}
$$

$$\leq C(v,\theta)\Big(|(v_x,\theta_x)|^4 + |(v_x,\theta_x)|^2 |(v_{xx},\theta_{xx})| + |\theta_{xx}| |(v_{xx},\theta_{xx})| + |(v_x,\theta_x)| |\theta_{xxx}| + |\theta_x| |v_{xxx}| \Big).$$

3. Energy estimates. To prove Theorem 1.1, we first define the following set of functions for which the solutions to the Cauchy problem (1.4), (1.5) will be sought:

$$X^k(0,T;A_0,A_1;B_0,B_1) = \left\{ \Big(v,u,\theta\Big)(t,x) \middle| \begin{array}{c} \big(v-1,u,\theta-1\big)(t,x) \in C^0\left(0,T;H^k(\mathbf{R})\right) \\ \big(u_x,\theta_x\big)(t,x) \in L^2\left(0,T;H^k(\mathbf{R})\right) \\ A_0 \leq v(t,x) \leq A_1, \quad B_0 \leq \theta(t,x) \leq B_1. \end{array} \right\}. \tag{3.1}$$

Here $k \geq 1$ is an integer, $T > 0$ is a given constant, and A_i, B_i $(i = 0,1)$ are some positive constants.

Under the assumptions given in Theorems 1.1, we can get the following local existence result.

Lemma 3.1 (local existence). *Under the assumptions listed in Theorem* 1.1*, there exists a sufficiently small positive constant* t_1*, which depends only on* $\|(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}})\|_3$, $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, *and* $\overline{\Theta}_0$*, such that the Cauchy problem* (1.4), (1.5) *admits a unique smooth solution* $(v(t,x),u(t,x),\theta(t,x)) \in X^3$ $(0,t_1;\,\frac{1}{2}\underline{V}_0, 2\overline{V}_0;\,\frac{1}{2}\underline{\Theta}_0,$ $2\overline{\Theta}_0)$ *which implies that* $(v(t,x),u(t,x),\theta(t,x))$ *satisfies*

$$\begin{cases} 0 < \frac{1}{2}\underline{V}_0 \leq v(t,x) \leq 2\overline{V}_0, \\ 0 < \frac{1}{2}\underline{\Theta}_0 \leq \theta(t,x) \leq 2\overline{\Theta}_0 \end{cases} \tag{3.2}$$

for all $(t,x) \in [0,t_1] \times \mathbf{R}$.

Moreover, one can deduce that

$$\max_{t\in[0,t_1]} \left\{ \left\| \left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t) \right\|_3 \right\} \leq 2 \left\| \left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right) \right\|_3. \tag{3.3}$$

Lemma 3.1 can be proved by employing the standard iteration argument as in [1], [10], [22]; the only difference here is that since $\gamma - 1$ is sufficiently small in our case, we need to pay particular attention to dealing with those terms containing negative powers of $\gamma - 1$. Since the modification is straightforward, we omit the details for brevity.

Remark 3.2. In Lemma 3.1 the time interval on which the local solution is constructed is claimed to depend on $\|(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}})\|_3$; an advantage of such a dependence is that we can deduce the estimate (3.3) by the smallness of t_1. In fact, even if t_1 is assumed to depend on $\|(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}})\|_1$ only, a similar local solvability result of the Cauchy problem (1.4), (1.5) still holds, but in such a case, the local solution $(v(t,x),u(t,x),\theta(t,x))$ constructed in such a way satisfies

$$\max_{t\in[0,t_1]}\left\{\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_1\right\}\le 2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1, \tag{3.4}$$
$$\max_{t\in[0,t_1]}\left\{\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_3\right\}\le C(t_1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_3.$$

Here $C(t_1)$ is some positive constant depending only on t_1. If we combine the continuation argument with the latter local existence result to extend the local solutions step by step to a global one, the presentation will be rather complex, and this is why we use the local existence result stated in Lemma 3.1.

Suppose that the local solution $(v(t,x),u(t,x),\theta(t,x))$ constructed in Lemma 3.1 has been extended to the time step $t=T>0$ and satisfies the a priori assumptions

$$\|\theta(t)-1\|_3\le\varepsilon,\quad 0<M_1^{-1}\le v(t,x)\le M_1,\quad \|(v(t)-1,u(t))\|_3\le N_1 \tag{3.5}$$

for all $x\in\mathbf{R}$, $0\le t\le T$; we now turn to deduce certain energy type estimates on $(v(t,x),u(t,x),\theta(t,x))$ in terms of the initial perturbation. Our main idea here is to use the smallness of both ε and $\gamma-1$ to control the possible growth of the solution $(v(t,x),u(t,x),\theta(t,x))$ constructed in Lemma 3.1 which is caused by the nonlinearities of the system (1.4) under consideration.

In fact, under the assumption that $0<\varepsilon<\min\left\{\overline{\Theta}_0-1,1-\underline{\Theta}_0\right\}$, we have from the a priori assumption (3.5) that

$$0<\underline{\Theta}_0\le\theta(t,x)\le\overline{\Theta}_0,\quad \|\theta_x(t)\|_{W^{1,\infty}(\mathbf{R})}\le\varepsilon,\quad \|(u,v)(t)\|_{W^{2,\infty}(\mathbf{R})}\le N_1+1 \tag{3.6}$$

hold for all $x\in\mathbf{R}$, $0\le t\le T$. Without loss of generality, we may assume in the rest of this manuscript that $M_1\ge1$, $N_1\ge1$.

Now we turn to deducing certain energy type estimates on $(v(t,x),u(t,x),\theta(t,x))$. Before doing so, recall that we will use C or $O(1)$ to denote some generic positive constant independent of $\gamma-1$, M_1, and N_1 but may only depend on $\underline{\Theta}_0$, $\overline{\Theta}_0$, $\underline{V}_0$, and $\overline{V}_0$, and $C(\cdot,\cdot)$ stands for some positive constant which depends only on the quantities listed in the parentheses.

The first one is concerned with the basic energy estimate. For this purpose, recall that $R=1$; then it is well known that

$$\eta(v,u,\theta)=\phi(v)+\frac{u^2}{2}+\frac{\phi(\theta)}{\gamma-1},\quad \phi(x)=x-\ln x-1$$

is a convex entropy to (1.4) which satisfies

$$\eta(v,u,\theta)_t+\left\{\left(\frac{\theta}{v}-1\right)u-\frac{\mu(\theta)uu_x}{v}+\frac{(\theta-1)\kappa(\theta)\theta_x}{v\theta}\right\}_x+\left\{\frac{\mu(\theta)u_x^2}{v\theta}+\frac{\kappa(\theta)\theta_x^2}{v\theta^2}\right\}=0.$$

Integrating the above identity with respect to t, x over $[0,t]\times\mathbf{R}$, we have the next lemma.

LEMMA 3.3 (basic energy estimate). *Under the conditions listed in Lemma 3.1, suppose that the local solution $(v(t,x),u(t,x),\theta(t,x))$ constructed in Lemma 3.1 has been extended to the time step $t=T$; then we have for $0\le t\le T$ that*

$$\int_{\mathbf{R}}\eta(v,u,\theta)dx+\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)u_x^2}{v\theta}+\frac{\kappa(\theta)\theta_x^2}{v\theta^2}\right)dxd\tau=\int_{\mathbf{R}}\eta(v_0,u_0,\theta_0)dx. \tag{3.7}$$

Notice that the a priori assumption (3.5) and (3.6) implies that $0 < \underline{\Theta}_0 \le \theta(t,x) \le \overline{\Theta}_0$ holds for all $x \in \mathbf{R}$, $0 \le t \le T$; we have from the assumption (1.3) and the fact that the upper and lower bounds of $v_0(x)$, $\theta_0(x)$ do not depend on $\gamma - 1$ that there exist some positive constants $C\left(\underline{\Theta}_0, \overline{\Theta}_0\right) > 0$ and $C\left(\underline{V}_0, \overline{V}_0\right) > 0$ such that

$$\begin{cases} \frac{\mu(\theta(t,x))}{\theta(t,x)} \ge C\left(\underline{\Theta}_0, \overline{\Theta}_0\right) > 0, \quad \frac{\kappa(\theta(t,x))}{\theta^2(t,x)} \ge C\left(\underline{\Theta}_0, \overline{\Theta}_0\right) > 0, \\ \phi(\theta(t,x)) \ge C\left(\underline{\Theta}_0, \overline{\Theta}_0\right)(\theta(t,x)-1)^2, \\ \phi(v_0(x)) \le C\left(\underline{V}_0, \overline{V}_0\right)(v_0(x)-1)^2, \quad \phi(\theta_0(x)) \le C\left(\underline{\Theta}_0, \overline{\Theta}_0\right)(\theta_0(x)-1)^2 \end{cases}$$

hold for all $x \in \mathbf{R}$, $0 \le t \le T$. Consequently we have from (3.7) that

$$\begin{aligned} &\left\| \left(\sqrt{\phi(v)}, u, \frac{\theta - 1}{\sqrt{\gamma - 1}} \right)(t) \right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{u_x^2 + \theta_x^2}{v} dx d\tau \\ &\quad \le O(1) \left\| \left(\frac{\theta_0 - 1}{\sqrt{\gamma - 1}}, v_0 - 1, u_0 \right) \right\|^2. \end{aligned} \tag{3.8}$$

Here recall that $O(1)$ denotes some positive constant depending only on $\underline{\Theta}_0$, $\overline{\Theta}_0$, $\underline{V}_0$, and $\overline{V}_0$.

Now we turn to deriving the lower and upper bounds on the specific volume $v(t,x)$. To do so, we need to deduce an estimate on $\|\frac{\mu(\theta)v_x}{v}\|$. For this purpose, we have by multiplying (2.1) by $\frac{\mu(\theta)v_x}{v}$ that

$$\begin{aligned} \frac{1}{2}\left\{ \left(\frac{\mu(\theta)v_x}{v} \right)^2 \right\}_t + \frac{\mu(\theta)\theta v_x^2}{v^3} &= \left(\frac{\mu(\theta)v_x u}{v} \right)_t - \left(\frac{\mu(\theta)u_x u}{v} \right)_x + \frac{\mu(\theta)u_x^2}{v} \\ &\quad + \frac{\mu(\theta)v_x\theta_x}{v^2} + \frac{\mu'(\theta)(u_x\theta_x - v_x\theta_t)(uv - \mu(\theta)v_x)}{v^2}. \end{aligned} \tag{3.9}$$

Integrating the above identity with respect to t and x over $[0,t] \times \mathbf{R}$, it follows that

$$\begin{aligned} &\frac{1}{2}\left\| \frac{\mu(\theta)v_x}{v} \right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\mu(\theta)\theta v_x^2}{v^3} dx d\tau \\ &\quad = \frac{1}{2}\left\| \frac{\mu(\theta_0)v_{0x}}{v_0} \right\|^2 + \underbrace{\int_{\mathbf{R}} \left(\frac{\mu(\theta)uv_x}{v} - \frac{\mu(\theta_0)u_0v_{0x}}{v_0} \right) dx}_{I_1} + \underbrace{\int_0^t \int_{\mathbf{R}} \frac{\mu(\theta)u_x^2}{v} dx d\tau}_{I_2} \\ &\qquad + \underbrace{\int_0^t \int_{\mathbf{R}} \frac{\mu(\theta)v_x\theta_x}{v^2} dx d\tau}_{I_3} + \underbrace{\int_0^t \int_{\mathbf{R}} \frac{\mu'(\theta)(u_x\theta_x - v_x\theta_t)(uv - \mu(\theta)v_x)}{v^2} dx d\tau}_{I_4}. \end{aligned} \tag{3.10}$$

With the basic energy estimate (3.8) in hand, the estimates on the terms I_i $(i = 1,2,3)$ can be treated the same as in [15] for the case of constant transport coefficients. In fact, by making use of the Cauchy inequality, the a priori assumption (3.5) and its consequence (3.6), and the estimate (3.8), it follows that

$$
\begin{aligned}
(3.11)\qquad I_1 &\le \frac{1}{4}\left\|\frac{\mu(\theta)v_x}{v}\right\|^2 + \|u\|^2 + O(1)\left\|(v_{0x},u_0)\right\|^2\\
&\le \frac{1}{4}\left\|\frac{\mu(\theta)v_x}{v}\right\|^2 + O(1)\left\|\left(v_0-1,u_0,v_{0x},\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2,\\
I_2 &\le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2,\\
I_3 &\le \frac{1}{4}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau + \int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_x^2}{v\theta}dxd\tau\\
&\le \frac{1}{4}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau + O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2.
\end{aligned}
$$

As to I_4, $(1.4)_3$ together with the constitution relation $e=\frac{\theta}{\gamma-1}$ implies

$$
u_x\theta_x - v_x\theta_t = u_x\theta_x - (\gamma-1)v_x\left(\frac{\kappa'(\theta)\theta_x^2+\kappa(\theta)\theta_{xx}+\mu(\theta)u_x^2-u_x\theta}{v} - \frac{\kappa(\theta)\theta_x v_x}{v^2}\right),
$$

from which we can get that

$$
\begin{aligned}
(3.12)\qquad I_4 = &\int_0^t\int_{\mathbf{R}}\frac{\mu'(\theta)uu_x\theta_x}{v}dxd\tau - \int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\mu'(\theta)v_xu_x\theta_x}{v^2}dxd\tau\\
&-\int_0^t\int_{\mathbf{R}}\frac{(\gamma-1)\mu'(\theta)}{v^2}\left(\kappa'(\theta)u\theta_x^2v_x+\kappa(\theta)u\theta_{xx}v_x\right)\\
&+\frac{(\gamma-1)\mu'(\theta)}{v^2}\left(\mu(\theta)uv_xu_x^2-u\theta v_xu_x\right)dxd\tau\\
&+\int_0^t\int_{\mathbf{R}}\frac{(\gamma-1)\mu'(\theta)v_x^2}{v^3}\left(\mu(\theta)\kappa'(\theta)\theta_x^2+\mu(\theta)\kappa(\theta)\theta_{xx}\right)\\
&+\frac{(\gamma-1)\mu'(\theta)v_x^2}{v^3}\left(\kappa(\theta)u\theta_x+\mu^2(\theta)u_x^2-\mu(\theta)\theta u_x\right)dxd\tau\\
&-\int_0^t\int_{\mathbf{R}}\frac{(\gamma-1)\mu(\theta)\mu'(\theta)\kappa(\theta)\theta_xv_x^3}{v^4}dxd\tau\\
= &\sum_{i=1}^5 K_i.
\end{aligned}
$$

It is easy to see that if the viscosity coefficient μ does not depend on θ, all the terms $K_i(i=1,2,3,4,5)$ are equal to zero and consequently we can deduce a nice bound on $\|\frac{\mu v_x}{v}\|$, while if μ is a function of θ, all these terms are highly nonlinear so that controling the possible growth of $(v(t,x),u(t,x),\theta(t,x))$ caused by these nonlinearities is very difficult for large initial perturbation. To yield a nice bound on these terms in our case considered in this paper, our main observation is that the terms $K_i(i=3,4,5)$ contain a factor $\gamma-1$, while the terms $K_i(i=1,2)$ have a factor θ_x, and our main idea is to use these two factors, i.e., the smallness of $\gamma-1$ and $\|\theta_x(t)\|_{L^\infty(\mathbf{R})}$, to control $K_i(i=1,2,3,4,5)$ suitably.

More precisely, for $K_i(i=3,4,5)$, we will use the smallness of $\gamma-1$ to control the nonlinearities appeared in these terms. In fact, for K_4 and K_5, under the a priori assumption (3.5), we can get that

(3.13)

$$\begin{aligned}K_4+K_5&=(\gamma-1)\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}\left(\frac{\mu'(\theta)\kappa'(\theta)\theta_x^2+\mu'(\theta)\kappa(\theta)\theta_{xx}+\mu'(\theta)\mu(\theta)u_x^2}{\theta}\right.\\&\quad\left.+\frac{\mu'(\theta)\kappa(\theta)u\theta_x}{\mu(\theta)\theta}-\frac{\mu'(\theta)\kappa(\theta)v_x\theta_x}{v\theta}-\mu'(\theta)u_x\right)dxd\tau\\&\le O(1)(\gamma-1)\left\|\left(\theta_x^2,\theta_{xx},u\theta_x,v_x^2,\frac{\theta_xv_x}{v},u_x^2,u_x\right)\right\|_{L^\infty}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau\\&\le O(1)(\gamma-1)N_1^2M_1\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau,\end{aligned}$$

while for K_3, we have from the Cauchy inequality, the a priori assumption (3.5), (3.6), and the estimate (3.8) that

(3.14)

$$\begin{aligned}|K_3|&\le\frac18\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau+O(1)(\gamma-1)^2\int_0^t\int_{\mathbf{R}}\frac{u^2(\theta_x^4+\theta_{xx}^2+u_x^4+u_x^2)}{v}dxd\tau\\&\le\frac18\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau+O(1)(\gamma-1)^2\|u\|^2_{L^\infty([0,T]\times\mathbf{R})}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_{xx}^2}{v}dxd\tau\\&\quad+O(1)(\gamma-1)^2\left\|\left(u^2\theta_x^2,u^2,u^2u_x^2\right)\right\|_{L^\infty([0,T]\times\mathbf{R})}\int_0^t\int_{\mathbf{R}}\frac{u_x^2+\theta_x^2}{v}dxd\tau\\&\le\frac18\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau+O(1)(\gamma-1)^2N_1^2\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_{xx}^2}{v}dxd\tau\\&\quad+O(1)(\gamma-1)^2\left(\varepsilon^2N_1^2+N_1^4\right)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2.\end{aligned}$$

For K_1 and K_2, we will use the smallness of ε to control the possible growth of the solutions mentioned above. To this end, we can deduce from the a priori assumption (3.7), the basic energy estimate (3.8), and the Cauchy inequality that

$$\begin{aligned}K_2&\le\left(\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau\right)^{\frac12}\left(\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\left(\mu'(\theta)\right)^2\theta_x^2u_x^2}{v\theta}dxd\tau\right)^{\frac12}\\&\le\frac18\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau+4\left\|\frac{\mu(\theta)(\mu'(\theta))^2\theta_x^2}{\theta}\right\|_{L^\infty([0,T]\times\mathbf{R})}\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v}dxd\tau\\&\le\frac18\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau+O(1)\varepsilon^2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}K_1&\le O(1)\int_0^t\|u\|_{L^\infty(\mathbf{R})}\left\|\frac{u_x}{\sqrt v}\right\|\left\|\frac{\theta_x}{\sqrt v}\right\|d\tau\\&\le O(1)\int_0^t\|u\|^{\frac12}\|u_x\|^{\frac12}\left\|\frac{u_x}{\sqrt v}\right\|\left\|\frac{\theta_x}{\sqrt v}\right\|d\tau\\&\le O(1)\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v}dxd\tau\\&\quad+\int_0^t\|\sqrt v\|_{L^\infty(\mathbf{R})}\left\|\frac{1}{\sqrt v}\right\|_{L^\infty(\mathbf{R})}\|u\|\left\|\frac{u_x}{\sqrt v}\right\|\|\theta_x\|\left\|\frac{\theta_x}{\sqrt v}\right\|d\tau\end{aligned}\tag{3.16}$$

$$\leq O(1)\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v}dxd\tau+\varepsilon M_1\max_{\tau\in[0,t]}\{\|u(\tau)\|\}\int_0^t\int_{\mathbf{R}}\frac{u_x^2+\theta_x^2}{v}dxd\tau$$
$$\leq O(1)\left(1+\varepsilon M_1\right)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^3.$$

Here, to deduce the above two estimates, we have used the fact that $\|\theta_x(t)\|_{L^\infty(\mathbf{R})}\leq\varepsilon$, $\|\theta_x(t)\|\leq\varepsilon$, $M_1^{-1}\leq v(t,x)\leq M_1$, which follow from the a priori assumption (3.5) and its consequence (3.6). Moreover, we have assumed without loss of generality that $\|(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}})\|\geq 1$.

Inserting (3.13)–(3.16) into (3.12), we can get that

(3.17)
$$\begin{aligned}I_4\leq&\left(\frac{1}{4}+O(1)(\gamma-1)M_1N_1^2\right)\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau\\&+O(1)(\gamma-1)^2N_1^2\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_{xx}^2}{v}dxd\tau\\&+O(1)\left(1+\varepsilon^2+\varepsilon M_1+(\gamma-1)^2\left(\varepsilon^2N_1^2+N_1^4\right)\right)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^3.\end{aligned}$$

Thus, if we plug (3.11), (3.17) into (3.10), it yields that

$$\begin{aligned}&\left\|\frac{\mu(\theta)v_x}{v}\right\|^2+\left(\frac{1}{2}-C_1(\gamma-1)N_1^2M_1\right)\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau\\&\quad\leq O(1)\|v_{0x}\|^2+O(1)(\gamma-1)^2N_1^2\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_{xx}^2}{v}dxd\tau\\&\qquad+O(1)\left(1+\varepsilon M_1+(\gamma-1)^2\left(\varepsilon^2N_1^2+N_1^4\right)\right)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^3.\end{aligned}\tag{3.18}$$

Here C_1 is some generic positive constant independent of t, x, M_1, N_1, and $\gamma-1$.

Having obtained (3.18), if we assume that $\gamma-1$ and ε are small enough such that

$$\begin{cases}\frac{1}{4}\leq\frac{1}{2}-C_1(\gamma-1)N_1^2M_1,\\ \varepsilon M_1\leq 1,\\ (\gamma-1)^2\left(\varepsilon^2N_1^2+N_1^4\right)\leq 1,\end{cases}\tag{H$_1$}$$

then the above analysis yields the following result

Lemma 3.4. *Under the conditions listed in Lemma* 3.2*, if we further assume that* $\gamma-1$ *and* ε *are chosen sufficiently small such that* (H$_1$) *holds true, then we get*

$$\begin{aligned}\left\|\frac{\mu(\theta)v_x}{v}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_x^2}{v^3}dxd\tau\leq&\ O(1)\left\|\left(v_0-1,v_{0x},u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^3\\&+O(1)(\gamma-1)\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_{xx}^2}{v}dxd\tau.\end{aligned}\tag{3.19}$$

From (3.19), it is easy to see that to deduce an estimate on $\|\frac{\mu(\theta)v_x}{v}\|$, we need to deduce an estimate on $\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_{xx}^2}{v}dxd\tau$ first. To this end, we can get by differentiating $(1.4)_3$ with respect to x once and by multiplying the resulting identity by θ_x that

$$\begin{aligned}(3.20)\qquad &\frac{1}{2(\gamma-1)}\frac{d}{dt}\left(\theta_x^2\right)+\frac{\kappa(\theta)\theta_{xx}^2}{v}\\ &=\left[\left(\frac{\mu(\theta)u_x^2}{v}-u_xp+\left(\frac{\kappa(\theta)\theta_x}{v}\right)_x\theta_x\right)\right]_x\\ &\quad+\frac{\theta u_x\theta_{xx}}{v}-\frac{\mu(\theta)u_x^2\theta_{xx}}{v}-\frac{\kappa'(\theta)\theta_x^2\theta_{xx}}{v}+\frac{\kappa(\theta)\theta_x v_x\theta_{xx}}{v^2}.\end{aligned}$$

Here we have used the fact that $C_v = (\gamma-1)^{-1}$.

Integrating the above equality with respect to t and x over $[0,t]\times\mathbf{R}$, we get

$$\begin{aligned}(3.21)\qquad &\frac{1}{2}\left\|\frac{\theta_x}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau\\ &=\frac{1}{2}\left\|\frac{\theta_{0x}}{\sqrt{\gamma-1}}\right\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\theta u_x\theta_{xx}}{v}dxd\tau}_{I_5}-\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_x^2\theta_{xx}}{v}dxd\tau}_{I_6}\\ &\quad-\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\kappa'(\theta)\theta_x^2\theta_{xx}}{v}dxd\tau}_{I_7}+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_x v_x\theta_{xx}}{v^2}dxd\tau}_{I_8}.\end{aligned}$$

From the basic energy estimate (3.8), the Cauchy inequality, and the a priori assumption (3.5) and (3.6), we can get that

$$(3.22)\qquad |I_5|\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau+O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2,$$

$$(3.23)\qquad |I_6|\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau+O(1)N_1^2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2,$$

$$\begin{aligned}(3.24)\qquad |I_7|&\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\frac{(\kappa'(\theta))^2\theta_x^4}{v\kappa(\theta)}dxd\tau\\ &\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau+O(1)\varepsilon^2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2,\end{aligned}$$

$$\begin{aligned}(3.25)\qquad |I_8|&\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau+O(1)\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_x^2v_x^2}{v^3}dxd\tau\\ &\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau\\ &\quad+O(1)\left\|\frac{v_x}{v}\right\|_{L^\infty([0,T]\times\mathbf{R})}^2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2\\ &\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau\\ &\quad+O(1)M_1^2N_1^2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2.\end{aligned}$$

Inserting (3.22)–(3.25) into (3.21) yields

$$\left\|\frac{\theta_x}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau \le O(1)\left(N_1^2+M_1^2N_1^2\right)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^2. \tag{3.26}$$

As a direct consequence of the estimates (3.19) and (3.26), we can deduce the following.

Lemma 3.5. *Under the same conditions listed in Lemma* 3.3*, if* $\gamma-1$ *is further assumed to be sufficiently small such that*

$$(\gamma-1)M_1^2N_1^2\le 1, \tag{H$_2$}$$

then we arrive at

$$\left\|\frac{v_x}{v}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{v_x^2}{v^3}dxd\tau\le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^4. \tag{3.27}$$

With the estimates (3.8) and (3.27) in hand, we now apply Kanel's approach [14] to deduce a uniform lower bound and a uniform upper bound for $v(t,x)$. To this end, set

$$\Psi(v)=\int_1^v\frac{\sqrt{\phi(z)}}{z}dz,\quad \phi(z)=z-\ln z-1. \tag{3.28}$$

Note that there exist positive constants A_1, A_2 such that

$$|\Psi(v)|\ge A_1\left(v^{\frac12}+|\ln v|^{\frac32}\right)-A_2. \tag{3.29}$$

Since

$$\begin{aligned}|\Psi(v)|&=\left|\int_{-\infty}^x\Psi(v(t,y))_ydy\right|\\ &\le\int_{\mathbf{R}}\left|\frac{\sqrt{\phi(v)}}{v}v_x\right|dx\\ &\le\left\|\sqrt{\phi(v)}\right\|\left\|\frac{v_x}{v}\right\|\\ &\le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^3,\end{aligned}$$

we can deduce from the above inequality, (3.28), and (3.29) that there exists a generic positive constant $C_2>0$, which may depend only on $\underline{\Theta}_0$, $\overline{\Theta}_0$, $\underline{V}_0$, and $\overline{V}_0$, such that

$$\exp\left(-C_2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^2\right)\le v(t,x)\le C_2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^6 \tag{3.30}$$

holds for all $(t,x)\in[0,T]\times\mathbf{R}$.

Moreover, notice that (3.30) and the Taylor expansion for $\phi(v)$ tell us that there exists an $\eta\in[0,1]$ such that

$$
\begin{aligned}
(v(t,x)-1)^2 &= 2(\eta+(1-\eta)v(t,x))^2\phi(v(t,x))\\
&\le 2\left(1+\|v\|_{L^\infty([0,T]\times\mathbf{R})}\right)^2\phi(v(t,x))\\
&\le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{12}\phi(v(t,x));
\end{aligned}
$$

we can get from the above estimate, (3.30), (3.8), and (3.27) that

$$
\begin{aligned}
&\left\|\left(v(t)-1,u(t),v_x(t),\frac{\theta(t)-1}{\sqrt{\gamma-1}}\right)\right\|^2 \le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{14},\\
&\int_0^t \|(u_x(s),v_x(s),\theta_x(s))\|^2\,ds \le O(1)\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{22}.
\end{aligned}
\tag{3.31}
$$

Here and in the rest of this manuscript, we assume without loss of generality that

$$
\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1 \ge 1.
$$

The above analysis yields the following result.

LEMMA 3.6. *Under the conditions listed in Lemma* 3.4*, we can deduce that* (3.30) *and* (3.31) *hold for all* $(t,x)\in[0,T]\times\mathbf{R}$.

To employ the continuation argument to extend the local solutions step by step to a global one, we need to close the a priori assumption (3.5) listed above, and for this purpose, we should derive certain higher order energy type estimates. Before doing so, to simplify the presentation, we will use the notation

$$
N_{0i} = \left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_i, \quad i=1,2,3
$$

in the rest of this paper.

First, based on the a priori assumption (3.6) and the lower and upper bounds of $v(t,x)$ obtained in Lemma 3.5, we now turn to derive certain energy type estimates on $(v_x(t,x),u_x(t,x),\theta_x(t,x))$. To this end, we will first get the estimate on $\|u_x\|$.

LEMMA 3.7. *Under the same conditions of Lemma* 3.5*, we can get that*

$$
\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v}dxd\tau \le O(1)N_{01}^{90}. \tag{3.32}
$$

Proof. Differentiating $(1.4)_2$ with respect to x once, multiplying the result by u_x, and integrating the final identity with respect to t and x over $[0,t]\times\mathbf{R}$, we can get that

$$
\begin{aligned}
&\frac{1}{2}\|u_x\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xx}^2}{v}dxd\tau\\
&\quad=\frac{1}{2}\|u_{0x}\|^2-\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)}{v}\right)_x u_xu_{xx}dxd\tau}_{I_9}+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\theta}{v}\right)_x u_{xx}dxd\tau}_{I_{10}}.
\end{aligned}
\tag{3.33}
$$

The Cauchy inequality together with the estimates (3.8), (3.27), (3.30), and (3.31) yields

(3.34)
$$\begin{aligned} I_{10} &\leq \frac{1}{3}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xx}^2}{v}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\left(\frac{\theta_x^2}{v}+\frac{v_x^2}{v^3}\right)dxd\tau \\ &\leq \frac{1}{3}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xx}^2}{v}dxd\tau + O(1)N_{01}^4, \end{aligned}$$

(3.35)
$$\begin{aligned} I_9 &\leq \frac{1}{3}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xx}^2}{v}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}v^2u_x^2\left(v_x^2+\theta_x^2\right)dxd\tau \\ &\leq \frac{1}{3}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xx}^2}{v}dxd\tau + O(1)N_{01}^{12}\int_0^t\left(\|u_x\|_{L^\infty}^2\|v_x\|^2+\|\theta_x\|_{L^\infty}^2\|u_x\|^2\right)d\tau \\ &\leq \frac{1}{3}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xx}^2}{v}dxd\tau + O(1)N_{01}^{20} + O(1)N_{01}^{34}\int_0^t\|u_x\|\|u_{xx}\|d\tau \\ &\leq \frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xx}^2}{v}dxd\tau + O(1)N_{01}^{90}. \end{aligned}$$

Here we have used the fact that $\|\theta_x\|_{L^\infty([0,T]\times\mathbf{R})}\leq\varepsilon\leq 1$ when dealing with I_9.

Equations (3.34), (3.5) together with (3.33) imply that (3.32) holds. This completes the proof of Lemma 3.6. ∎

Combining the estimates on $\|u_x\|$ and $\|v_x\|$ obtained in (3.27) and (3.32) with (3.21), we can get the following result.

LEMMA 3.8. *Under the same conditions listed in Lemma* 3.6*, we have*

$$\left\|\frac{\theta_x}{\sqrt{\gamma-1}}\right\|^2 + \int_0^t\int_{\mathbf{R}}\frac{\theta_{xx}^2}{v}dxd\tau \leq O(1)N_{01}^{186}\exp\left(C_2N_{01}^2\right). \tag{3.36}$$

Proof. To prove (3.36), we only need to deduce better upper bounds on the terms I_j $(j=6,7,8)$ on the right-hand side of (3.21) in term of N_{01}. Since now we have already obtained the uniform lower and upper bounds on $v(t,x)$ and $\theta(t,x)$, we have from the estimates (3.8), (3.27), (3.30), and (3.32) that

$$\begin{aligned} I_6 &\leq \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{u_x^4}{v}dxd\tau \\ &\leq \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)\exp\left(C_2N_{01}^2\right)\int_0^t\|u_x\|_{L_x^\infty}^2\|u_x\|^2d\tau \\ &\leq \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)\exp\left(C_2N_{01}^2\right)\int_0^t\|u_x\|^3\|u_{xx}\|d\tau \\ &\leq \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)N_{01}^{90}\exp\left(C_2N_{01}^2\right)\int_0^t\|(u_x,u_{xx})\|^2d\tau \\ &\leq \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)N_{01}^{186}\exp\left(C_2N_{01}^2\right). \end{aligned} \tag{3.37}$$

Similarly, noticing that the a priori assumption (3.5) implies

$$\|\theta_x(t)\|_{L^\infty([0,T]\times\mathbf{R})}\leq\varepsilon\leq 1,$$

we can deduce from (3.8), (3.27), and (3.30) that

$$
\begin{aligned}
(3.38)\qquad I_7 &\le \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^4}{v}dxd\tau\\
&\le \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)N_{01}^2,
\end{aligned}
$$

$$
\begin{aligned}
(3.39)\qquad I_8 &\le \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2 v_x^2}{v}dxd\tau\\
&\le \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)N_{01}^{12}\int_0^t\int_{\mathbf{R}}\frac{v_x^2}{v^3}dxd\tau\\
&\le \frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xx}^2}{v}dxd\tau + O(1)N_{01}^{16}.
\end{aligned}
$$

Inserting (3.22), (3.37)–(3.39) into (3.21), we can get (3.36) immediately. This completes the proof of Lemma 3.7. ▯

The energy type estimates obtained in Lemmas 3.3–3.7 imply that under the a priori assumption (3.5) and if we assume that $\gamma-1$ and $\varepsilon>0$ are chosen sufficiently small such that (H_1) and (H_2) hold, then there exists some positive constant O(1) which may depend only on $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, and $\overline{\Theta}_0$ but independent of T, x, and $\gamma-1$ such that $v(t,x)$ satisfies (3.30) for all $(t,x)\in[0,T]\times\mathbf{R}$. Moreover, we have from (3.30), (3.31), (3.32), and (3.36) that

$$
\begin{aligned}
&\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_1^2 + \int_0^t\left(\|v_x(\tau)\|^2+\|(u_x,\theta_x)(\tau)\|_1^2\right)d\tau\\
&\quad\le O(1)N_{01}^{192}\exp\left(C_2N_{01}^2\right).
\end{aligned}
$$

The above inequality together with the fundamental inequality $e^x\ge x^k/k!$ ($x\ge 0$, k is any positive integer) yields

$$
\begin{aligned}
(3.40)\qquad &\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_1^2 + \int_0^t\left(\|v_x\|^2+\|(u_x,\theta_x)\|_1^2\right)(\tau)d\tau\\
&\quad\le O(1)N_{01}^2\exp\left(C_3N_{01}^2\right)
\end{aligned}
$$

with $C_3=C_2+1$.

Now we derive the second order energy estimates on $(v(t,x),u(t,x),\theta(t,x))$. First for the corresponding estimate on $u_{xx}(t,x)$, we have by differentiating $(1.4)_2$ with respect to x twice, multiplying the result by u_{xx}, and integrating the final identity with respect to t and x over $[0,t]\times\mathbf{R}$ that

$$
\begin{aligned}
(3.41)\qquad &\frac{1}{2}\|u_{xx}(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau\\
&\quad=\frac{1}{2}\|u_{0xx}\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\theta}{v}\right)_{xx}u_{xxx}dxd\tau}_{I_{11}}\\
&\qquad-\underbrace{\int_0^t\int_{\mathbf{R}}\left[\left(\frac{\mu(\theta)u_x}{v}\right)_{xx}-\frac{\mu(\theta)u_{xxx}}{v}\right]u_{xxx}dxd\tau}_{I_{12}}.
\end{aligned}
$$

Equation (2.6) together with the estimates (3.30) and (3.40) yields

$$
\begin{aligned}
I_{11} \le & \frac{1}{4}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau \\
& + O(1)\int_0^t\int_{\mathbf{R}} \left(\frac{\theta_{xx}^2}{v} + \frac{\theta_x^2 v_x^2 + v_{xx}^2}{v^3} + \frac{v_x^4}{v^5}\right)dxd\tau \\
\le & \frac{1}{4}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau \\
& + O(1)e^{O(1)N_{01}^2}\int_0^t \left(\|(\theta_{xx}, v_{xx})\|^2 + \|\theta_x\|_{L^\infty}^2\|v_x\|^2 + \|v_x\|^6\right)d\tau \\
\le & \frac{1}{4}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau + O(1)N_{01}^2 e^{O(1)N_{01}^2} \\
& + O(1)e^{O(1)N_{01}^2}\int_0^t \|v_{xx}\|^2 d\tau.
\end{aligned}
\tag{3.42}
$$

Similarly, we have from (2.13) and the estimates (3.30) and (3.40) that

$$
\begin{aligned}
I_{12} \le & \frac{1}{4}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \Big[\left|(u_{xx},\theta_{xx})\right|^2\left|(v_x,u_x,\theta_x)\right|^2 \\
& + v_{xx}^2u_x^2 + \left|(v_x,u_x,\theta_x)\right|^6\Big]dxd\tau \\
\le & \frac{1}{4}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|(v_x,u_x,\theta_x)\|_{L^\infty}^4\|(v_x,u_x,\theta_x)\|^2 d\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left[\left|(u_{xx},\theta_{xx})\right|^2\left|(v_x,u_x,\theta_x)\right|^2 + v_{xx}^2u_x^2\right]dxd\tau \\
\le & \frac{1}{4}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left[\left|(u_{xx},\theta_{xx})\right|^2\left|(v_x,u_x,\theta_x)\right|^2 + v_{xx}^2u_x^2\right]dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|(v_x,u_x,\theta_x)\|^4\|(v_{xx},u_{xx},\theta_{xx})\|^2 d\tau \\
\le & \frac{1}{4}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)u_{xxx}^2}{v}dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left[\left|(u_{xx},\theta_{xx})\right|^2\left|(v_x,u_x,\theta_x)\right|^2 + v_{xx}^2u_x^2\right]dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\left\|\left(v_0-1, u_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10} \\
\times & \int_0^t \|(v_x,u_x,\theta_x)\|^2\|(v_{xx},u_{xx},\theta_{xx})\|^2 d\tau.
\end{aligned}
\tag{3.43}
$$

Inserting (3.42) and (3.43) into (3.41), we can deduce from (3.30) that

$$\begin{aligned}(3.44)\qquad &\|u_{xx}\|^2+\int_0^t\|u_{xxx}\|^2d\tau\\ &\le O(1)N_{02}^2e^{O(1)N_{01}^2}+O(1)e^{O(1)N_{01}^2}\int_0^t\|v_{xx}\|^2d\tau\\ &\quad+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left[|(u_{xx},\theta_{xx})|^2\,|(v_x,u_x,\theta_x)|^2+v_{xx}^2u_x^2\right]dxd\tau\\ &\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\|(v_x,u_x,\theta_x)\|^2\,\|(v_{xx},u_{xx},\theta_{xx})\|^2\,d\tau.\end{aligned}$$

To deduce an estimate on $\|\theta_{xx}\|$, we have by differentiating $(1.4)_3$ with respect to x twice, multiplying the result by θ_{xx}, and then integrating the final resulting identity with respect to t and x over $[0,t]\times\mathbf{R}$ that

$$\begin{aligned}(3.45)\quad &\frac{1}{2}\left\|\frac{\theta_{xx}}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxx}^2}{v}dxd\tau\\ &=\frac{1}{2}\left\|\frac{\theta_{0xx}}{\sqrt{\gamma-1}}\right\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)u_x^2}{v}\right)_x\theta_{xxx}dx\tau}_{I_{13}}+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\theta u_x}{v}\right)_x\theta_{xxx}dxd\tau}_{I_{14}}\\ &\quad\underbrace{-\int_0^t\int_{\mathbf{R}}\left(\left(\frac{\kappa(\theta)\theta_x}{v}\right)_{xx}-\frac{\kappa(\theta)\theta_{xxx}}{v}\right)\theta_{xxx}dxd\tau}_{I_{15}}.\end{aligned}$$

Now we deal with I_j $(j=13,14,15)$ term by term. For I_{13}, we can get from (2.10), (3.30), and (3.40) that

$$\begin{aligned}(3.46)\qquad I_{13}&\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)}{v}\theta_{xxx}^2dxd\tau\\ &\quad+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_x^2u_x^4+u_x^2u_{xx}^2+u_x^4v_x^2\right)dxd\tau\\ &\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)}{v}\theta_{xxx}^2dxd\tau+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}u_x^2u_{xx}^2dxd\tau\\ &\quad+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\|u_x\|^2\|u_{xx}\|^2\left(\|\theta_x\|^2+\|v_x\|^2\right)d\tau\\ &\le\frac{1}{5}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)}{v}\theta_{xxx}^2dxd\tau+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}u_x^2u_{xx}^2dxd\tau\\ &\quad+O(1)N_{01}^2e^{O(1)N_{01}^2}.\end{aligned}$$

Moreover, since (3.40) implies

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}\left(\theta_x^4\left(\theta_x^2+v_x^2\right)+\theta_x^2v_x^4\right)dxd\tau\\ &\quad\le\int_0^t\left(\|\theta_x\|^2\|\theta_{xx}\|^2\left(\|\theta_x\|^2+\|v_x\|^2\right)+\|v_x\|^2\|\theta_x\|^2\|v_{xx}\|^2\right)d\tau\\ &\quad\le O(1)N_{01}^2e^{O(1)N_{01}^2}+\int_0^t\|v_x\|^2\|\theta_x\|^2\|v_{xx}\|^2d\tau,\end{aligned}$$

we have from (3.30), (3.40), and (2.16) that

$$
\begin{aligned}
I_{15} \leq\ & \frac{1}{5}\int_0^t\int_{\mathbf{R}} \frac{\kappa(\theta)}{v}\theta_{xxx}^2 dxd\tau + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\theta_x^2+v_x^2\right)\theta_{xx}^2 dxd\tau \\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\theta_x^4\left(\theta_x^2+v_x^2\right)+\theta_x^2 v_x^4\right) dxd\tau \\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \theta_x^2 v_{xx}^2 dxd\tau \\
\leq\ & \frac{1}{5}\int_0^t\int_{\mathbf{R}} \frac{\kappa(\theta)}{v}\theta_{xxx}^2 dxd\tau + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_x\|^2\|\theta_x\|^2\|v_{xx}\|^2 d\tau \\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2} + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\theta_{xx}^2\left(\theta_x^2+v_x^2\right)+\theta_x^2 v_{xx}^2\right) dxd\tau
\end{aligned}
\tag{3.47}
$$

and

$$
\begin{aligned}
I_{14} \leq\ & \frac{1}{5}\int_0^t\int_{\mathbf{R}} \frac{\kappa(\theta)}{v}\theta_{xxx}^2 dxd\tau \\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\theta_x^2 u_x^2 + u_{xx}^2 + u_x^2 v_x^2\right) dxd\tau \\
\leq\ & \frac{1}{5}\int_0^t\int_{\mathbf{R}} \frac{\kappa(\theta)}{v}\theta_{xxx}^2 dxd\tau \\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \left(\|u_{xx}\|^2 + \|\theta_x\|\|u_x\|^2\|\theta_{xx}\| + \|u_x\|^2\|v_x\|^4\right) d\tau \\
\leq\ & \frac{1}{5}\int_0^t\int_{\mathbf{R}} \frac{\kappa(\theta)}{v}\theta_{xxx}^2 dxd\tau + O(1)N_{01}^2 e^{O(1)N_{01}^2}.
\end{aligned}
\tag{3.48}
$$

Plugging (3.46)–(3.48) into (3.45), we obtain from the a priori assumption (3.5) and the estimate (3.30) that

$$
\begin{aligned}
&\left\|\frac{\theta_{xx}}{\sqrt{\gamma-1}}\right\|^2 + \int_0^t\int_{\mathbf{R}} \frac{\theta_{xxx}^2}{v} dxd\tau \\
&\quad \leq O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\left(u_{xx}^2+\theta_{xx}^2\right)\left(v_x^2+u_x^2+\theta_x^2\right)+\left(u_x^2+\theta_x^2\right)v_{xx}^2\right) dxd\tau \\
&\qquad + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_x\|^2\|\theta_x\|^2\|v_{xx}\|^2 d\tau + O(1)N_{02}^2 e^{O(1)N_{01}^2}.
\end{aligned}
\tag{3.49}
$$

The estimate on $\|v_{xx}\|$ is complex in some sense. To do the estimate, we first differentiate (2.1) with respect to x once, multiply the result by $(\frac{\mu(\theta)v_x}{v})_x$, and then integrate the final result with respect to t and x over $[0,t]\times\mathbf{R}$ to deduce that

$$\frac{1}{2}\left\|\left(\frac{\mu(\theta)v_x}{v}\right)_x\right\|^2 = \frac{1}{2}\left\|\left(\frac{\mu(\theta_0)v_{0x}}{v_0}\right)_x\right\|^2 + \underbrace{\int_0^t\int_{\mathbf{R}} u_{tx}\left(\frac{\mu(\theta)v_x}{v}\right)_x dxd\tau}_{I_{16}} + \underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)v_x}{v}\right)_x\left(\frac{\theta}{v}\right)_{xx} dxd\tau}_{I_{17}} + \underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)v_x}{v}\right)_x\left(\frac{\mu'(\theta)}{v}(\theta_t v_x-\theta_x u_x)\right)_x dxd\tau}_{I_{18}}. \tag{3.50}$$

To estimate the terms appearing on the right-hand side of (3.50) term by term, we first deduce from the identity

$$\begin{aligned} I_{16} &= \int_0^t\int_{\mathbf{R}}\left(u_x\left(\frac{\mu(\theta)v_x}{v}\right)_x\right)_t dxd\tau - \int_0^t\int_{\mathbf{R}} u_x\left(\frac{\mu(\theta)v_x}{v}\right)_{tx} dxd\tau \\ &= \int_{\mathbf{R}} u_x\left(\frac{\mu(\theta)v_x}{v}\right)_x dx - \int_{\mathbf{R}} u_{0x}\left(\frac{\mu(\theta_0)v_{0x}}{v_0}\right)_x dx \\ &\quad + \int_0^t\int_{\mathbf{R}} u_{xx}\left(\frac{\mu(\theta)v_x}{v}\right)_t dxd\tau \\ &= \int_{\mathbf{R}} u_x\left(\frac{\mu(\theta)v_x}{v}\right)_x dx - \int_0^t\int_{\mathbf{R}} u_{0x}\left(\frac{\mu(\theta_0)v_{0x}}{v_0}\right)_x dx \\ &\quad + \int_0^t\int_{\mathbf{R}}\left(\frac{\mu'(\theta)\theta_t v_x u_{xx}}{v} + \frac{\mu(\theta)u_{xx}^2}{v} - \frac{\mu(\theta)v_x u_x u_{xx}}{v^2}\right) dxd\tau \end{aligned} \tag{3.51}$$

and the estimate (3.30) and (3.40) that

$$\begin{aligned} I_{16} &\le \frac{1}{4}\left\|\left(\frac{\mu(\theta)v_x}{v}\right)_x\right\|^2 + O(1)N_{02}^2 e^{O(1)N_{01}^2} \\ &\quad + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|\theta_t v_x| + |v_x u_x|\right)|u_{xx}|dxd\tau. \end{aligned} \tag{3.52}$$

Equation (2.2) together with (3.30) and (3.40) yields

$$\begin{aligned} \int_0^t\int_{\mathbf{R}}|\theta_t v_x||u_{xx}|dxd\tau &\le O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2} \\ &\quad \times \int_0^t\int_{\mathbf{R}}|v_x u_{xx}|\left(u_x^2 + |\theta_{xx}| + \theta_x^2 + |\theta_x v_x| + |u_x|\right)dxd\tau \\ &\le O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2} \\ &\quad \times \int_0^t\int_{\mathbf{R}}\left(u_{xx}^2 + v_x^2\theta_{xx}^2 + v_x^2(u_x^4+\theta_x^4) + v_x^2 u_x^2 + v_x^4\theta_x^2\right)dxd\tau \\ &\le O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\left(\|u_{xx}\|^2 + \|v_x\|^2\|\theta_x\|^2\|\theta_{xx}\|^2\right. \\ &\quad \left. + \|v_x\|^2\|u_x\|^2\|u_{xx}\|^2 + \|v_x\|^2\right)d\tau \end{aligned} \tag{3.53}$$

$$
\begin{aligned}
&+ O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_x\|_{L^\infty}^4\|\theta_x\|^2 d\tau\\
&+ O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} v_x^2\theta_{xx}^2 dxd\tau\\
\le\ & O(1)N_{01}^2 e^{O(1)N_{01}^2} + O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} v_x^2\theta_{xx}^2 dxd\tau\\
&+ O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_{xx}\|^2\|v_x\|^2\|\theta_x\|^2 d\tau\\
\le\ & O(1)N_{01}^2 e^{O(1)N_{01}^2} + O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} v_x^2\theta_{xx}^2 dxd\tau\\
&+ O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_{xx}\|^2 d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t\int_{\mathbf{R}} |u_xv_x||u_{xx}|dxd\tau &\le \int_0^t\int_{\mathbf{R}} u_{xx}^2 dxd\tau + \int_0^t\int_{\mathbf{R}} v_x^2u_x^2 dxd\tau\\
&\le \int_0^t\int_{\mathbf{R}} u_{xx}^2 dxd\tau + \int_0^t \|u_x\|\|u_{xx}\|\|v_x\|^2 dxd\tau\\
&\le O(1)N_{01}^2 e^{O(1)N_{01}^2}.
\end{aligned} \tag{3.54}
$$

Inserting the estimates of (3.53) and (3.54) into (3.52) yields that

$$
\begin{aligned}
I_{16} \le\ & \frac{1}{4}\left\|\left(\frac{\mu(\theta)v_x}{v}\right)_x\right\|^2 + O(1)N_{02}^2 e^{O(1)N_{01}^2}\\
&+ O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\left\{\int_0^t\int_{\mathbf{R}} \theta_{xx}^2v_x^2 dxd\tau + \int_0^t \|v_{xx}\|^2 d\tau\right\}.
\end{aligned} \tag{3.55}
$$

As to I_{17}, we have from (2.6), (2.8), (3.30), and (3.40) that

$$
\begin{aligned}
I_{17} =\ & \int_0^t\int_{\mathbf{R}}\left(\frac{\mu'(\theta)\theta_xv_x+\mu(\theta)v_{xx}}{v}-\frac{\mu(\theta)v_x^2}{v}\right)\\
&\times\left(\frac{\theta_{xx}}{v}-\frac{2\theta_xv_x+\theta v_{xx}}{v^2}+\frac{2\theta v_x^2}{v^3}\right)dxd\tau\\
\le\ & -\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta_{xx}v_{xx}}{v^2}dxd\tau\\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|v_{xx}|+|\theta_{xx}|\right)\left(|\theta_xv_x|+v_x^2\right)dxd\tau\\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|\theta_xv_x|+v_x^2\right)^2 dxd\tau\\
\le\ & -\frac{2}{3}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau\\
&+ O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_{xx}^2+\theta_x^2v_x^2+v_x^4\right)dxd\tau\\
\le\ & -\frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau
\end{aligned} \tag{3.56}
$$

$$
\begin{aligned}
&+O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \left(\|\theta_{xx}\|^2+\|\theta_x\|^2\|v_x\|^4+\|v_x\|^6\right)d\tau\\
&\leq -\frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau+O(1)N_{01}^2 e^{O(1)N_{01}^2}.
\end{aligned}
$$

Finally for I_{18}, noticing the identities (2.2) and (2.3) and due to

$$
\begin{aligned}
I_{18}=\int_0^t\int_{\mathbf{R}}&\left(\frac{\mu'(\theta)\theta_x v_x+\mu(\theta)v_{xx}}{v}-\frac{\mu(\theta)v_x^2}{v^2}\right)\left(\frac{\mu''(\theta)\theta_x}{v}-\frac{\mu'(\theta)v_x}{v^2}\right)(\theta_t v_x-\theta_x u_x)\\
&+\frac{\mu'(\theta)}{v}\left(\frac{\mu'(\theta)\theta_x v_x+\mu(\theta)v_{xx}}{v}-\frac{\mu(\theta)v_x^2}{v^2}\right)(\theta_{tx}v_x+\theta_t v_{xx}-\theta_{xx}u_x-\theta_x u_{xx})dxd\tau,
\end{aligned}
$$

we have from the estimate (3.30) that

(3.57)

$$
\begin{aligned}
I_{18}\leq\ & O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|v_x\theta_x|+|v_x|^2+|v_{xx}|\right)\Big\{(|\theta_x|+|v_x|)\\
&\times\left[|\theta_x u_x|+(\gamma-1)\left(|v_x u_x^2|+|v_x\theta_{xx}|+|v_x\theta_x^2|+|v_x|^3+|v_x u_x|\right)\right]\\
&+(\gamma-1)\left(|v_x u_x u_{xx}|+|v_x\theta_x^3|+|v_x\theta_x\theta_{xx}|+|v_x\theta_{xxx}|+|v_x u_x\theta_x|+|u_x^2\theta_x v_x|\right.\\
&\left.+|v_x u_{xx}|+|v_x u_x|^2+|v_x\theta_x|^2+|v_x^2\theta_{xx}|+|v_x\theta_x v_{xx}|+|v_x^2 u_x|+|v_x^3\theta_x|\right)\\
&+(\gamma-1)\left(|u_x^2 v_{xx}|+|v_{xx}\theta_{xx}|+|\theta_x^2 v_{xx}|+|v_x^2 v_{xx}|+|u_x v_{xx}|\right)\\
&+|u_x\theta_{xx}|+|\theta_x u_{xx}|\Big\}dxd\tau\\
\leq\ & \sum_{j=1}^5 L_j.
\end{aligned}
$$

Here

$$
\begin{aligned}
L_1&=O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_x^2+\theta_x^2+|v_{xx}|\right)\\
&\quad\times\left(\left|\theta_x^2 u_x\right|+|v_x\theta_x u_x|+|\theta_{xx}u_x|+|\theta_x u_{xx}|\right)dxd\tau,\\
L_2&=O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_x^2+\theta_x^2+|v_{xx}|\right)\left(\theta_x^4+v_x^4+u_x^4\right)dxd\tau,\\
L_3&=O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_x^2+\theta_x^2+|v_{xx}|\right)\left(v_x^2+\theta_x^2+u_x^2\right)\\
&\quad\times(|\theta_{xx}|+|v_{xx}|+|u_{xx}|)\,dxd\tau,\\
L_4&=O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_x^2+\theta_x^2+|v_{xx}|\right)(|\theta_{xx}v_{xx}|+|v_x\theta_{xxx}|)\,dxd\tau,\\
L_5&=O(1)(\gamma-1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_x^2+\theta_x^2+|v_{xx}|\right)\\
&\quad\times\left(|v_x u_x\theta_x|+|v_x u_{xx}|+|v_x^2 u_x|+|u_x v_{xx}|\right)dxd\tau.
\end{aligned}
$$

For any $\eta>0$, we have from (3.40) that

$$\int_0^t\int_{\mathbf{R}}|v_{xx}|(|u_x|+|\theta_x|)(|\theta_{xx}|+|u_{xx}|)dxd\tau \tag{3.58}$$
$$\le \eta\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau+C(\eta)\int_0^t\int_{\mathbf{R}}\left(u_x^2+\theta_x^2\right)\left(|\theta_{xx}^2|+|u_{xx}^2|\right)dxd\tau,$$

$$\int_0^t\int_{\mathbf{R}}\left(\theta_x^4+v_x^4+u_x^4\right)dxd\tau \tag{3.59}$$
$$\le \int_0^t\left(\|\theta_x\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2+\|v_x\|_{L^\infty}^2\right)\|(\theta_x,u_x,v_x)\|^2d\tau$$
$$\le O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\left(\|\theta_x\|\|\theta_{xx}\|+\|u_x\|\|u_{xx}\|+\|v_x\|\|v_{xx}\|\right)d\tau$$
$$\le \eta\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\|v_x\|^2d\tau$$
$$+O(1)N_{01}^2e^{O(1)N_{01}^2}\left(\int_0^t\|\theta_x\|^2d\tau\right)^{\frac12}\left(\int_0^t\|\theta_{xx}\|^2d\tau\right)^{\frac12}$$
$$+O(1)N_{01}^2e^{O(1)N_{01}^2}\left(\int_0^t\|u_x\|^2d\tau\right)^{\frac12}\left(\int_0^t\|u_{xx}\|^2d\tau\right)^{\frac12}$$
$$\le \eta\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau+O(1)N_{01}^2e^{O(1)N_{01}^2},$$

and

$$\int_0^t\int_{\mathbf{R}}\left(\theta_x^4u_x^2+u_x^2v_x^2\theta_x^2\right)dxd\tau \tag{3.60}$$
$$\le \int_0^t\left(\|\theta_x\|^2\|u_x\|^2\|\theta_{xx}\|^2+\|v_x\|^2\|u_x\|\|u_{xx}\|\|\theta_x\|\|\theta_{xx}\|\right)d\tau$$
$$\le O(1)N_{01}^2e^{O(1)N_{01}^2}.$$

Consequently we can deduce from (3.58)–(3.60) that

$$L_1\le \frac{1}{10}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau+O(1)N_{01}^2e^{O(1)N_{01}^2} \tag{3.61}$$
$$+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_x^2+v_x^2+u_x^2\right)\left(\theta_{xx}^2+u_{xx}^2\right)dxd\tau.$$

Repeating the above argument and under the assumption that ε and $\gamma-1$ are chosen sufficiently small such that (H_1) and (H_2) hold, we can get from (3.30) and (3.40) that

$$L_2\le \frac{1}{20}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau \tag{3.62}$$
$$+O(1)(\gamma-1)^2\left(1+N_1^2\right)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_x^6+v_x^6+u_x^6\right)dxd\tau$$
$$\le \frac{1}{20}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau+O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\left(\|\theta_{xx}\|^2\|\theta_x\|^4\right.$$
$$\left.+\|v_{xx}\|^2\|v_x\|^4+\|u_{xx}\|^2\|u_x\|^4\right)d\tau$$

$$
\begin{aligned}
&\leq \frac{1}{20}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau + O(1)N_{01}^2e^{O(1)N_{01}^2}\\
&\quad + O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t \|v_x\|^2\|v_{xx}\|^2d\tau,
\end{aligned}
$$

$$
\begin{aligned}
L_3 &\leq (\gamma-1)N_1^2\int_0^t\int_{\mathbf{R}}|v_{xx}|(|v_{xx}|+|u_{xx}|+|\theta_{xx}|)dxd\tau\\
&\quad + \frac{(\gamma-1)N_1^2}{20}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau\\
&\quad + O(1)(\gamma-1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_x^2+v_x^2+u_x^2\right)^2(|u_{xx}|+|\theta_{xx}|)\,dxd\tau\\
&\quad + O(1)(\gamma-1)N_1^2N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_x^4+v_x^4+u_x^4\right)dxd\tau\\
&\leq O(1)(\gamma-1)N_1^2N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau + O(1)N_{01}^2e^{O(1)N_{01}^2},
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
L_4 &\leq \frac{\gamma-1}{20}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau\\
&\quad + O(1)(\gamma-1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}|v_xv_{xx}\theta_{xxx}|dxd\tau\\
&\quad + O(1)(\gamma-1)N_1^2N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}(|\theta_{xx}v_{xx}|+|\theta_{xxx}v_x|)\,dxd\tau\\
&\leq \frac{\gamma-1}{10}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau\\
&\quad + O(1)(\gamma-1)N_1^2N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\theta_{xxx}^2dxd\tau\\
&\quad + O(1)\left(1+(\gamma-1)N_1^2\right)N_{01}^2e^{O(1)N_{01}^2},
\end{aligned}
\tag{3.64}
$$

and

$$
\begin{aligned}
L_5 &\leq O(1)(\gamma-1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_x^4+\theta_x^4+\theta_x^2v_x^2u_x^2+v_x^4u_x^2\right)dxd\tau\\
&\quad + O(1)(\gamma-1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_{xx}^2+v_x^2u_{xx}^2+u_x^2v_{xx}^2\right)dxd\tau\\
&\leq \left(O(1)(\gamma-1)N_1^2N_{01}^2e^{O(1)N_{01}^2}+\eta\right)\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xx}^2}{v^3}dxd\tau\\
&\quad + O(1)N_{01}^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}v_x^2u_{xx}^2dxd\tau + O(1)N_{01}^2e^{O(1)N_{01}^2}.
\end{aligned}
\tag{3.65}
$$

Recall that $\eta > 0$ is any given sufficiently small positive constant.

Inserting the estimates of (3.61)–(3.65) into (3.57), and if we assume further that

$$
(\gamma-1)N_1^4 \leq 1, \quad O(1)(\gamma-1)N_1^2N_{01}^2e^{O(1)N_{01}^2} \leq \frac{1}{20}, \tag{H$_3$}
$$

then we get

$$
\begin{aligned}
I_{18} \le\ & O(1)N_{01}^2 e^{O(1)N_{01}^2} + \frac{1}{5}\int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)\theta v_{xx}^2}{v^3} dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\theta_x^2+v_x^2+u_x^2\right)\left(\theta_{xx}^2+u_{xx}^2\right) dxd\tau \\
& + O(1)(\gamma-1)^{\frac{1}{2}} N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \theta_{xxx}^2 dxd\tau \\
& + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_x\|^2\|v_{xx}\|^2 d\tau.
\end{aligned}
\tag{3.66}
$$

Now, inserting (3.55), (3.56), and (3.66) into (3.50), we can get that

$$
\begin{aligned}
& \left\|\left(\frac{\mu(\theta)v_x}{v}\right)_x\right\|^2 + \int_0^t\int_{\mathbf{R}} \frac{\mu(\theta)\theta v_{xx}^2}{v^3} dxd\tau \\
& \le O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\theta_x^2+v_x^2+u_x^2\right)\left(\theta_{xx}^2+u_{xx}^2\right) dxd\tau \\
& \quad + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_x\|^2\|v_{xx}\|^2 d\tau \\
& \quad + O(1)(\gamma-1)^{\frac{1}{2}} N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \theta_{xxx}^2 dxd\tau \\
& \quad + O(1)N_{02}^2 e^{O(1)N_{01}^2}.
\end{aligned}
\tag{3.67}
$$

Since

$$
\left(\frac{\mu(\theta)v_x}{v}\right)_x = \frac{\mu'(\theta)\theta_x v_x}{v} + \frac{\mu(\theta)v_{xx}}{v} - \frac{\mu(\theta)v_x^2}{v^2},
$$

we can deduce from (3.30) and (3.40) that

$$
\left\|\left(\frac{\mu(\theta)v_x}{v}\right)_x\right\|^2 \ge O(1)N_{01}^{-12}\|v_{xx}\|^2 - O(1)N_{01}^2 e^{O(1)N_{01}^2}.
$$

Based on the above estimate and (3.67), we finally get from (3.30) and (3.40) that

$$
\begin{aligned}
& \|v_{xx}\|^2 + \int_0^t \|v_{xx}\|^2 d\tau \\
& \le O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(\theta_x^2+v_x^2+u_x^2\right)\left(\theta_{xx}^2+u_{xx}^2\right) dxd\tau \\
& \quad + O(1)N_{01}^2 e^{O(1)N_{01}^2}\int_0^t \|v_x\|^2\|v_{xx}\|^2 d\tau \\
& \quad + O(1)(\gamma-1)^{\frac{1}{2}} N_{01}^2 e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \theta_{xxx}^2 dxd\tau \\
& \quad + O(1)N_{02}^2 e^{O(1)N_{01}^2}.
\end{aligned}
\tag{3.68}
$$

A suitable linear combination of (3.44), (3.49), and (3.68) yields the following result.

LEMMA 3.9. *Under the same condition listed in Lemma* 3.7, *if we further assume that* $\gamma - 1$ *is sufficiently small such that the assumption* (H_3) *holds, then we have*

$$\left\| \left(v_{xx}, u_{xx}, \frac{\theta_{xx}}{\sqrt{\gamma-1}} \right) \right\|^2 + \int_0^t \left\| (v_{xx}, u_{xxx}, \theta_{xxx})(\tau) \right\|^2 d\tau \leq O(1) N_{02}^2 \exp\left(N_{01}^2 \exp\left(C_4 N_{01}^2 \right) \right). \tag{3.69}$$

Moreover, (3.40) *and* (3.69) *together with Sobolev's embedding inequality imply*

$$\left\| \left(v-1, u, \frac{\theta-1}{\sqrt{\gamma-1}} \right)(t) \right\|_{W^{1,\infty}(\mathbf{R})} \leq O(1) N_{02} \exp\left(\frac{1}{2} N_{01}^2 \exp\left(C_4 N_{01}^2 \right) \right), \quad 0 \leq t \leq T. \tag{3.70}$$

Here C_4 denotes some positive constant depending only on $\underline{\Theta}_0$, $\overline{\Theta}_0$, $\underline{V}_0$, and $\overline{V}_0$.

Proof. In fact, multiplying (3.68) by a sufficiently large positive number λ and adding the result with (3.44) and (3.49), we can deduce from the fact that $\gamma - 1$ is sufficiently small that

$$\begin{aligned} &\left\| \left(v_{xx}, u_{xx}, \frac{\theta_{xx}}{\sqrt{\gamma-1}} \right) \right\|^2 + \int_0^t \left\| (v_{xx}, u_{xxx}, \theta_{xxx}) \right\|^2 d\tau \\ &\leq O(1) N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \int_{\mathbf{R}} \left(\theta_x^2 + v_x^2 + u_x^2 \right) \left(\theta_{xx}^2 + u_{xx}^2 \right) dx d\tau \\ &\quad + O(1) N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \int_{\mathbf{R}} v_{xx}^2 \left(u_x^2 + \theta_x^2 \right) dx d\tau \\ &\quad + O(1) N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \left(\|v_x\|^2 + \|\theta_x\|^2 \right) \left(\|v_{xx}\|^2 + \|\theta_{xx}\|^2 \right) d\tau \\ &\quad + O(1) N_{02}^2 e^{O(1)N_{01}^2}. \end{aligned} \tag{3.71}$$

Due to

$$\begin{aligned} &\int_0^t \int_{\mathbf{R}} v_{xx}^2 (u_x^2 + \theta_x^2) dx d\tau \\ &\leq \int_0^t \|v_{xx}\|^2 \left(\|u_x\| \|u_{xx}\| + \|\theta_x\| \|\theta_{xx}\| \right) d\tau, \end{aligned} \tag{3.72}$$

$$\begin{aligned} &\int_0^t \int_{\mathbf{R}} \left(\theta_x^2 + v_x^2 + u_x^2 \right) \left(\theta_{xx}^2 + u_{xx}^2 \right) dx d\tau \\ &\leq \int_0^t \left(\|u_{xx}\|^2 + \|\theta_{xx}\|^2 \right) \left(\|u_x\|^2 + \|u_{xx}\|^2 + \|\theta_x\|^2 + \|\theta_{xx}\|^2 \right) d\tau \\ &\quad + \int_0^t \|v_x\|^2 \left(\|u_{xx}\| \|u_{xxx}\| + \|\theta_{xx}\| \|\theta_{xxx}\| \right) d\tau, \end{aligned} \tag{3.73}$$

(3.69) follows immediately by inserting (3.72) and (3.73) into (3.71) and by employing Gronwall's inequality and the estimates (3.30) and (3.40). This completes the proof of Lemma 3.8. ∎

Remark 3.10. Several remarks concerning the second order energy type estimates follow:

- Since the Navier–Stokes system (1.4) is a hyperbolic-parabolic coupled system, the estimates (3.40) contain no information on $\int_0^t \|v_{xx}\|^2 d\tau$. Fortunately, the term $\int_0^t \int_{\mathbf{R}} v_x^2 v_{xx}^2 dx d\tau$ does not appear on the right-hand side of (3.71) and consequently our analysis can be continued.

- Since $\theta_t = (\gamma-1)\big(-\frac{R\theta u_x}{v}+\frac{\mu(\theta)u_x^2}{v}+\big(\frac{\kappa(\theta)\theta_x}{v}\big)_x\big)$, to deduce an estimate on $\|v_{xx}\|$, we need to deal with the term $(\gamma-1)\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\mu'(\theta)\kappa(\theta)}{v^3}\theta_{xx}v_{xx}^2dxd\tau$; cf. (3.64) for details. It is easy to see that to bound such a term, we need to deduce an estimate on $\|\theta_{xx}\|_{L^\infty([0,T]\times\mathbf{R})}$, and as a result, we had to close the energy type estimates in $H^3(\mathbf{R})$.

We now turn to deduce the third order energy type estimates on $(v(t,x),u(t,x),\theta(t,x))$. To this end, we first consider the estimate on θ_{xxx} and obtain by differentiating $(1.4)_3$ with respect to x three times, multiplying the result by θ_{xxx}, and integrating the final result with respect to t and x over $[0,t]\times\mathbf{R}$ that

$$
\begin{aligned}
(3.74)\qquad &\frac{1}{2}\left\|\frac{\theta_{xxx}}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxxx}^2}{v}dxd\tau\\
&=\frac{1}{2}\left\|\frac{\theta_{0xxx}}{\sqrt{\gamma-1}}\right\|^2\underbrace{-\int_0^t\int_{\mathbf{R}}\theta_{xxxx}\left(\frac{\mu(\theta)u_x^2}{v}\right)_{xx}dxd\tau}_{I_{19}}\\
&\quad\underbrace{-\int_0^t\int_{\mathbf{R}}\theta_{xxxx}\left(\left(\frac{\kappa(\theta)\theta_x}{v}\right)_{xxx}-\frac{\kappa(\theta)\theta_{xxxx}}{v}\right)dxd\tau}_{I_{20}}\\
&\quad\underbrace{+\int_0^t\int_{\mathbf{R}}\theta_{xxxx}\left(\frac{\theta u_x}{v}\right)_{xx}dxd\tau}_{I_{21}}.
\end{aligned}
$$

To estimate I_j $(j=19,20,21)$ term by term, we have from (2.11) and the estimates (3.30), (3.40), (3.69), and (3.70) that

$$
\begin{aligned}
(3.75)\qquad I_{19}&\le\frac{1}{10}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxxx}^2}{v}dxd\tau\\
&\quad+O(1)\exp\left(O(1)N_{01}^2\right)\int_0^t\int_{\mathbf{R}}\left(\theta_x^4u_x^4+\theta_{xx}^2u_x^4+\theta_x^2u_x^2u_{xx}^2+u_{xx}^4\right.\\
&\quad\left.+u_{xxx}^2u_x^2+\theta_x^2v_x^2u_x^4+v_{xx}^2u_x^4+v_x^4u_x^4\right)dxd\tau\\
&\le\frac{1}{10}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxxx}^2}{v}dxd\tau+O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right).
\end{aligned}
$$

Here we have used the fact that

$$
\begin{aligned}
\int_0^t\int_{\mathbf{R}}u_{xx}^4dxd\tau&\le\int_0^t\|u_{xxx}\|\|u_{xx}\|^3d\tau\le\int_0^t\|u_{xxx}\|^2+\int_0^t\|u_{xx}\|^6d\tau\\
&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)
\end{aligned}
$$

and

$$
\int_0^t\int_{\mathbf{R}}u_x^8dxd\tau\le\int_0^t\|u_x\|_{L^\infty}^6\|u_x\|^2d\tau\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right).
$$

Similarly, (2.17) and (2.13) together with (3.30), (3.40), (3.69), and (3.70) imply

$$
\begin{aligned}
I_{20} \leq &\frac{1}{10}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxxx}^2}{v}dxd\tau \\
&+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_{xx}^2v_{xx}^2+v_{xxx}^2\right)dxd\tau \\
&+O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)
\end{aligned}
\tag{3.76}
$$

and

$$
I_{21} \leq \frac{1}{10}\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxxx}^2}{v}dxd\tau+O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right). \tag{3.77}
$$

Inserting (3.75)–(3.77) into (3.74) yields

$$
\begin{aligned}
&\left\|\frac{\theta_{xxx}}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\|\theta_{xxxx}(\tau)\|^2d\tau \\
&\leq O(1)N_{03}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right) \\
&\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_{xx}^2v_{xx}^2+v_{xxx}^2\right)dxd\tau \\
&\leq O(1)N_{03}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right) \\
&\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\|\theta_{xx}\|\|\theta_{xxx}\|\|v_{xx}\|^2d\tau \\
&\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}v_{xxx}^2dxd\tau \\
&\leq O(1)N_{03}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right) \\
&\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\left(\|\theta_{xx}\|^2+\|v_{xx}\|^2\right)\|\theta_{xxx}\|^2d\tau \\
&\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}v_{xxx}^2dxd\tau.
\end{aligned}
\tag{3.78}
$$

To deduce an estimate on u_{xxx}, we have by differentiating $(1.4)_2$ with respect to x three times, multiplying the result by u_{xxx}, and then integrating the final result with respect to t and x over $[0,t]\times\mathbf{R}$ that

$$
\begin{aligned}
&\frac{1}{2}\|u_{xxx}\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)u_{xxxx}^2}{v}dxd\tau \\
&=\frac{1}{2}\|u_{0xxx}\|^2\underbrace{-\int_0^t\int_{\mathbf{R}}\left(\left(\frac{\mu(\theta)u_x}{v}\right)_{xxx}-\frac{\mu(\theta)u_{xxxx}}{v}\right)u_{xxxx}dxd\tau}_{I_{22}} \\
&\quad+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\theta}{v}\right)_{xxx}u_{xxxx}dxd\tau}_{I_{23}}.
\end{aligned}
\tag{3.79}
$$

Equations (2.7) and (2.14) together with the estimates (3.30), (3.40), (3.69), and (3.70) imply

$$I_{22} \leq \frac{1}{5}\int_0^t \|u_{xxxx}\|^2 d\tau + O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} u_x^2 v_{xxx}^2 dx d\tau + O(1)N_{02}^2 \exp\left(N_{01}^2 \exp\left(C_4 N_{01}^2\right)\right) \tag{3.80}$$

and

$$I_{23} \leq \frac{1}{5}\int_0^t \|u_{xxxx}\|^2 d\tau + O(1)e^{O(1)N_{01}^2}\int_0^t \|v_{xxx}\|^2 d\tau + O(1)N_{02}^2 \exp\left(N_{01}^2 \exp\left(C_4 N_{01}^2\right)\right). \tag{3.81}$$

Putting (3.80), (3.81), and (3.79) together, we can obtain

$$\begin{aligned} &\|u_{xxx}(t)\|^2 + \int_0^t \|u_{xxxx}(\tau)\|^2 dx d\tau \\ &\leq O(1)N_{03}^2 \exp\left(N_{01}^2 \exp\left(C_4 N_{01}^2\right)\right) \\ &\quad + O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}} \left(u_x^2 v_{xxx}^2 + v_{xxx}^2\right) dx d\tau. \end{aligned} \tag{3.82}$$

Finally, to get an estimate on v_{xxx}, we have from (2.1) that

$$\left(\frac{\mu(\theta)v_x}{v}\right)_{txx} = u_{txx} + \left(\frac{\theta}{v}\right)_{xxx} + \left(\frac{\mu'(\theta)}{v}\left(\theta_t v_x - u_x\theta_x\right)\right)_{xx}. \tag{3.83}$$

Multiplying (3.83) by $(\frac{\mu(\theta)v_x}{v})_{xx}$ and integrating the resulting identity with respect to t and x over $[0,t]\times\mathbf{R}$, we have

$$\begin{aligned} &\frac{1}{2}\left\|\left(\frac{\mu(\theta)v_x}{v}\right)_{xx}\right\|^2 \\ &= \underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)v_x}{v}\right)_{xx} u_{txx} dx d\tau}_{I_{24}} + \underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)v_x}{v}\right)_{xx}\left(\frac{\theta}{v}\right)_{xxx} dx d\tau}_{I_{25}} \\ &\quad + \underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(\theta)v_x}{v}\right)_{xx}\left(\frac{\mu'(\theta)}{v}\left(\theta_t v_x - u_x\theta_x\right)\right)_{xx} dx d\tau}_{I_{26}}. \end{aligned} \tag{3.84}$$

To deal with the terms that appear on the right-hand side of (3.84) term by term, we first have

$$\begin{aligned} I_{24} &= \int_0^t\int_{\mathbf{R}}\left[\left(u_{xx}\left(\frac{\mu(\theta)v_x}{v}\right)_{xx}\right)_t - u_{xx}\left(\frac{\mu(\theta)v_x}{v}\right)_{txx}\right] dx d\tau \\ &= \int_{\mathbf{R}} u_{xx}\left(\frac{\mu(\theta)v_x}{v}\right)_{xx} dx - \int_{\mathbf{R}} u_{0xx}\left(\frac{\mu(\theta_0)v_{0x}}{v_0}\right)_{xx} dx \\ &\quad + \int_0^t\int_{\mathbf{R}} u_{xxx}\left(\frac{\mu(\theta)v_x}{v}\right)_{tx} dx d\tau \\ &= \int_{\mathbf{R}} u_{xx}\left(\frac{\mu(\theta)v_x}{v}\right)_{xx} dx - \int_{\mathbf{R}} u_{0xx}\left(\frac{\mu(\theta_0)v_{0x}}{v_0}\right)_{xx} dx \end{aligned} \tag{3.85}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbf{R}}\left(\frac{\mu'(\theta)\theta_t v_x}{v}+\frac{\mu(\theta)u_{xx}}{v}-\frac{\mu(\theta)v_xu_x}{v^2}\right)_x u_{xxx}dxd\tau\\
&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)+\frac{1}{5}\left\|\left(\frac{\mu(\theta)v_x}{v}\right)_{xx}\right\|^2\\
&+O(1)e^{O(1)N_{01}^2}\underbrace{\int_0^t\int_{\mathbf{R}}|\theta_{tx}v_xu_{xxx}|dxd\tau}_{J_1}\\
&+O(1)e^{O(1)N_{01}^2}\underbrace{\int_0^t\int_{\mathbf{R}}|u_{xxx}\theta_t|\left(|\theta_xv_x|+|v_x^2|+|v_{xx}|\right)dxd\tau}_{J_2}\\
&+O(1)e^{O(1)N_{01}^2}\underbrace{\int_0^t\int_{\mathbf{R}}|u_{xxx}|\left(|u_{xxx}|+|(u_{xx},v_xu_x)(v_x,\theta_x)|++|v_{xx}u_x|\right)dxd\tau}_{J_3}\\
&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)+\frac{1}{5}\left\|\left(\frac{\mu(\theta)v_x}{v}\right)_{xx}\right\|^2.
\end{aligned}
$$

Here we have used the estimates

$$
\begin{aligned}
J_1&\le O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(v_x^2u_{xxx}^2+\theta_{tx}^2\right)dxd\tau\\
&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right),\\
J_2&\le O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(u_{xxx}^2\left(\theta_x^2v_x^2+v_x^4+1\right)+\theta_t^2+\theta_t^2v_{xx}^2\right)dxd\tau\\
&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right),\\
J_3&\le\int_0^t\|u_{xxx}\|^2d\tau\\
&\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(u_{xx}^2v_x^2+\theta_x^2u_{xx}^2+u_x^2v_x^4+u_x^2v_{xx}^2+\theta_x^2v_x^2u_x^2\right)dxd\tau\\
&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right),
\end{aligned}
$$

which follow from (2.2), (2.3), (3.30), (3.40), (3.69), and (3.70).

As to the term I_{25}, we get from (2.7), (2.9), (3.30), (3.40), (3.69), and (3.70) that

(3.86)

$$
\begin{aligned}
I_{25}\le&-\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xxx}^2}{v^3}dxd\tau\\
&+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\Big\{\left(|\theta_{xxx}|+|(\theta_{xx},v_{xx})v_x|+|v_x|^2|(v_x,\theta_x)|+|\theta_xv_{xx}|\right)\\
&\times\left(|\theta_x^2v_x|+|v_x\theta_{xx}|+|\theta_xv_{xx}|+|v_xv_{xx}|+|v_x|^3+|\theta_xv_x^2|\right)\\
&+|v_{xxx}|\left(|\theta_{xxx}|+|(\theta_{xx},v_{xx})v_x|+|v_{xx}\theta_x|+|\theta_x^2v_x|+|v_x^2(v_x,\theta_x)|\right)\Big\}dxd\tau\\
\le&-\frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xxx}^2}{v^3}dxd\tau+O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right).
\end{aligned}
$$

To treat the term I_{26} is much more complex than the other terms on the right-hand side of (3.84). Although this process is similar to the proof of I_{18}, we shall give the proof in detail for the reader's convenience. In fact, notice first from (3.30) that

(3.87)
$$\begin{aligned}&\left(\frac{\mu'(\theta)}{v}(\theta_t v_x - u_x\theta_x)\right)_x\\ &\quad\le O(1)\exp(O(1)N_{01}^2)\\ &\qquad\times\Big(|\theta_t v_x(\theta_x, v_x)| + |\theta_t v_{xx}| + |\theta_{tx}v_x| + |\theta_x u_x(\theta_x, v_x)| + |u_{xx}\theta_x| + |u_x\theta_{xx}|\Big),\end{aligned}$$

(3.88)
$$\begin{aligned}&\left(\frac{\mu'(\theta)}{v}(\theta_t v_x - u_x\theta_x)\right)_{xx}\\ &\quad\le O(1)\exp(O(1)N_{01}^2)\\ &\qquad\times\Big(|\theta_t\theta_x^2 v_x| + |\theta_t\theta_{xx}v_x| + |\theta_t\theta_x v_{xx}| + |\theta_t\theta_x v_x^2| + |\theta_t v_{xxx}|\\ &\qquad + |\theta_t v_{xx}v_x| + |\theta_t v_x^3| + |\theta_{tx}\theta_x v_x| + |\theta_{tx}v_{xx}| + |\theta_{tx}v_x^2| + |\theta_{txx}v_x|\\ &\qquad + |\theta_x^3 u_x| + |\theta_x u_x\theta_{xx}| + |\theta_x^2 u_{xx}| + |\theta_x^2 u_x v_x| + |\theta_x u_{xxx}|\\ &\qquad + |\theta_x u_{xx}v_x| + |u_x\theta_{xxx}| + |\theta_x u_x v_{xx}| + |u_{xx}\theta_{xx}| + |v_x u_x\theta_{xx}| + |\theta_x u_x v_x^2|\Big);\end{aligned}$$

then, combining the estimate of (2.9) and (3.87), (3.88) together, we have

(3.89)
$$\begin{aligned}I_{26}\le{}& O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\Big\{|\theta_t v_{xxx}^2| + |(v_x,u_x,\theta_x)|^6 + |(v_x,u_x,\theta_x)|^2|(v_{xx},u_{xx},\theta_{xx})|^2\\ &+ |\theta_t|^2\Big[|(v_x,u_x,\theta_x)|^6 + |(v_{xx},u_{xx},\theta_{xx})|^2|(v_x,u_x,\theta_x)|^2\Big]\\ &+ |\theta_{tx}|^2\Big[|(v_x,u_x,\theta_x)|^4 + |(v_{xx},u_{xx},\theta_{xx})|^2\Big]\\ &+ |\theta_{txx}v_x|^2 + |(v_x,u_x,\theta_x)|^8 + |(v_x,u_x,\theta_x)|^4|(v_{xx},u_{xx},\theta_{xx})|^2\\ &+ |(u_{xx},\theta_{xx})|^4 + |(u_x,\theta_x)|^2|(\theta_{xxx},u_{xxx})|^2\Big\}dxd\tau.\end{aligned}$$

Due to (2.2), we have from (3.30), (3.40), (3.69), and (3.70) that

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}|\theta_t|^2\Big[|(v_x,u_x,\theta_x)|^6 + |(v_{xx},u_{xx},\theta_{xx})|^2|(v_x,u_x,\theta_x)|^2\Big]dxd\tau\\ &\quad\le O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(u_x^4 + \theta_x^4 + |u_x|^2 + |\theta_x v_x|^2 + |\theta_{xx}|^2\right)\\ &\qquad\times\Big[|(v_x,u_x,\theta_x)|^6 + |(v_{xx},u_{xx},\theta_{xx})|^2|(v_x,u_x,\theta_x)|^2\Big]dxd\tau\\ &\quad\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right).\end{aligned}\tag{3.90}$$

By employing the same argument, we can get from (2.3) and (3.30) that

(3.91)
$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}|\theta_{tx}|^2\Big[|(v_x,u_x,\theta_x)|^4+|(v_{xx},u_{xx},\theta_{xx})|^2\Big]dxd\tau\\ &\quad\le O(1)\exp(O(1)N_{01}^2)\int_0^t\int_{\mathbf{R}}\Big(|\theta_{xxx}|^2+|(v_x,\theta_x)|^2|\theta_{xx}|^2\\ &\qquad+\left(1+|u_x|^2\right)|u_{xx}|^2+|\theta_x|^2|v_{xx}|^2\\ &\qquad+\left(1+|(u_x,\theta_x)|^2\right)|(v_x,u_x,\theta_x)|^4\Big)\Big[|(v_x,u_x,\theta_x)|^4+|(v_{xx},u_{xx},\theta_{xx})|^2\Big]dxd\tau.\end{aligned}$$

Since (3.30), (3.40), (3.69), and (3.70) tell us that

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}\theta_x^{10}dxd\tau&\le\int_0^t\|\theta_x\|_{L^\infty}^6\|\theta_x\|^2\|\theta_{xx}\|^2d\tau\\ &\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right),\end{aligned}\tag{3.92}$$

we can get that

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}|\theta_{tx}|^2\Big[|(v_x,u_x,\theta_x)|^4+|(v_{xx},u_{xx},\theta_{xx})|^2\Big]dxd\tau\\ &\quad\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)\\ &\qquad+\int_0^t\int_{\mathbf{R}}\left(\theta_{xxx}^2+v_{xx}^2\right)|(v_{xx},u_{xx},\theta_{xx})|^2dxd\tau.\end{aligned}\tag{3.93}$$

On the other hand, we have from (2.4), (3.30), (3.40), (3.69), and (3.70) that

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}}|\theta_{txx}v_x|^2dxd\tau\\ &\quad\le O(1)(\gamma-1)^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}|v_x|^2\Big(|\theta_{xxxx}|^2+|u_{xxx}|^2\\ &\qquad+|(u_x,\theta_x)|^2|(v_{xxx},\theta_{xxx})|^2+|(v_{xx},u_{xx},\theta_{xx})|^4\\ &\qquad+|(v_{xx},u_{xx},\theta_{xx})|^2|(v_x,u_x,\theta_x)|^2(1+|(v_x,u_x,\theta_x)|)^2\\ &\qquad+|(v_x,u_x,\theta_x)|^6\left(1+|(v_x,u_x,\theta_x)|^2\right)\Big)dxd\tau\\ &\quad\le O(1)(\gamma-1)^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|v_x|^2|\theta_{xxxx}|^2+|v_x|^2|v_{xx}|^4\right)dxd\tau\\ &\qquad+O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right).\end{aligned}\tag{3.94}$$

Inserting the estimates of (3.90)–(3.94) into (3.89), and by combining the estimate of θ_t, we can get that

$$\begin{aligned}I_{26}&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)\\ &\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_{xxx}^2+v_{xx}^2\right)|(v_{xx},u_{xx},\theta_{xx})|^2dxd\tau\\ &\quad+O(1)(\gamma-1)^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|v_x|^2|\theta_{xxxx}|^2+|v_x|^2|v_{xx}|^4\right)dxd\tau\end{aligned}\tag{3.95}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbf{R}}|\theta_t v_{xxx}^2|dxd\tau\\
&\le O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)\\
&\quad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_{xxx}^2+v_{xx}^2\right)|(v_{xx},u_{xx},\theta_{xx})|^2dxd\tau\\
&\quad+O(1)(\gamma-1)^2e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|v_x|^2|\theta_{xxxx}|^2+|v_x|^2|v_{xx}|^4\right)dxd\tau\\
&\quad+O(1)(\gamma-1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(1+u_x^2+\theta_x^2+v_x^2+\theta_{xx}^2\right)|v_{xxx}|^2dxd\tau.
\end{aligned}
$$

Plugging (3.85), (3.86), and (3.95) into (3.84), and let $\gamma-1$ be small enough such that

$$
(H_4)\qquad O(1)(\gamma-1)^{\frac12}N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)\le 1,
$$

we can deduce from (3.30), (3.40), (3.69), and (3.70) that

$$
\begin{aligned}
&\left\|\left(\frac{\mu(v)v_x}{v}\right)_{xx}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(\theta)\theta v_{xxx}^2}{v^3}dxd\tau\\
&\quad\le O(1)N_{03}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)\\
&\qquad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_{xxx}^2+v_{xx}^2\right)|(v_{xx},u_{xx},\theta_{xx})|^2dxd\tau\\
&\qquad+O(1)(\gamma-1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|\theta_{xxxx}|^2+|v_{xx}|^4\right)dxd\tau\\
&\qquad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}|\theta_{xx}|^2|v_{xxx}|^2dxd\tau.
\end{aligned}\tag{3.96}
$$

Notice that

$$
\begin{aligned}
\left\|\left(\frac{\mu(v)v_x}{v}\right)_{xx}\right\|^2&\ge O(1)N_{01}^{-12}\|v_{xxx}\|^2\\
&\quad-O(1)N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right);
\end{aligned}\tag{3.97}
$$

then, we can get from (3.96) that

$$
\begin{aligned}
&\|v_{xxx}(t)\|^2+\int_0^t\int_{\mathbf{R}}v_{xxx}^2(\tau)dxd\tau\\
&\quad\le O(1)N_{03}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)\\
&\qquad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(\theta_{xxx}^2+v_{xx}^2\right)|(v_{xx},u_{xx},\theta_{xx})|^2dxd\tau\\
&\qquad+O(1)(\gamma-1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}\left(|\theta_{xxxx}|^2+|v_{xx}|^4\right)dxd\tau\\
&\qquad+O(1)e^{O(1)N_{01}^2}\int_0^t\int_{\mathbf{R}}|\theta_{xx}|^2|v_{xxx}|^2dxd\tau.
\end{aligned}\tag{3.98}
$$

Performing $(3.78)+(3.82)+\lambda\times(3.98)$, letting λ be sufficiently large, and making use of Gronwall's inequality, we can get the following lemma.

LEMMA 3.11. *Under the same conditions listed in Lemma* 3.8, *if we let* $\gamma - 1$ *small enough such that* (H$_4$) *holds, then we have*

$$\begin{aligned}(3.99)\qquad &\left\|\left(v_{xxx}, u_{xxx}, \frac{\theta_{xxx}}{\sqrt{\gamma-1}}\right)(t)\right\|^2 \\ &\quad + \int_0^t \left(\|v_{xxx}(\tau)\|^2 + \|(u_{xxxx}, \theta_{xxxx})(\tau)\|^2\right) d\tau \\ &\le O(1)N_{03}^2 \exp\left(O(1)N_{02}^2 \exp\left(O(1)N_{01}^2\right)\right) \\ &\le O(1)N_{03}^2 \exp\left(\exp\left(C_5 N_{02}^2\right)\right).\end{aligned}$$

Here C_5 *denotes some generic positive constant depending only on* $\underline{\Theta}_0$, $\overline{\Theta}_0$, $\underline{V}_0$, *and* $\overline{V}_0$.

As a direct consequence of Lemmas 3.3–3.9, we have the following corollary.

COROLLARY 3.12 (energy type a priori estimates). *Let* $(v(t,x), u(t,x), \theta(t,x))$ *be the local solution constructed in Lemma* 3.1 *which has been extended to the time step* $t = T \ge t_1$ *and assume that* $(v(t,x), u(t,x), \theta(t,x))$ *satisfies the a priori assumption* (3.5); *then if* $\varepsilon > 0$ *and* $\gamma - 1 > 0$ *are chosen sufficiently small such that*

$$\text{(H)}\qquad \begin{cases} \frac{1}{4} \le \frac{1}{2} - C_1(\gamma-1)N_1^2 M_1, \\ \varepsilon M_1 \le 1, \\ (\gamma-1)^2\left(\varepsilon^2 N_1^2 + N_1^4\right) \le 1, \\ (\gamma-1)M_1^2 N_1^2 \le 1, \\ (\gamma-1)N_1^4 \le 1, \\ (\gamma-1)C_6 N_1^2 \exp\left(C_6 N_{01}^2\right) \le \frac{1}{20}, \\ (\gamma-1)^{\frac{1}{2}} C_6 N_{02}^2 \exp\left(N_{01}^2 \exp\left(C_4 N_{01}^2\right)\right) \le 1, \end{cases}$$

there exists a generic positive constant C_7 *which depends only on* $\underline{\Theta}_0$, $\overline{\Theta}_0$, $\underline{V}_0$, *and* $\overline{V}_0$ *such that*

$$\begin{aligned}(3.100)\qquad &\left\|\left(v-1, u, \frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_3^2 + \int_0^t \left(\|v_x(\tau)\|_2^2 + \|(u_x, \theta_x)(\tau)\|_3^2\right) d\tau \\ &\le C_7 N_{03}^2 \exp\left(\exp\left(C_5 N_{02}^2\right)\right)\end{aligned}$$

holds for any $0 \le t \le T$. *Here* C_1 *is the positive constant defined in* (3.18) *while* $C_6 > 0$ *is used to specify the generic positive constants in* (H$_3$) *and* (H$_4$).

Moreover if we assume further that $\gamma - 1 > 0$ *is sufficiently small further such that*

$$(3.101)\qquad \sqrt{\gamma-1}\sqrt{C_7} N_{03} \exp\left(\frac{1}{2}\exp\left(C_5 N_{02}^2\right)\right) \le \min\left\{\left(1 - \underline{\Theta}_0\right), \left(\overline{\Theta}_0 - 1\right)\right\},$$

then we can deduce that

$$(3.102)\qquad \underline{\Theta}_0 \le \theta(t,x) \le \overline{\Theta}_0 \quad \forall (t,x) \in [0,T] \times \mathbf{R}.$$

4. The proof of our main result. This section is devoted to proving our main result, which is based on the continuation argument. Before doing so, noticing that $\theta = \frac{A}{R} v^{1-\gamma} \exp\left(\frac{\gamma-1}{R} s\right)$, $\overline{s} = \frac{R}{\gamma-1} \ln \frac{R}{A}$, and recalling that we have assume that $A = 1$, $R = 1$ which imply that $\overline{s} = 0$, we have

$$\begin{aligned}\theta - 1 &= v^{1-\gamma} \exp\left((\gamma-1)s\right) - 1\\ &= v^{1-\gamma} \exp\left((\gamma-1)s\right) - \exp\left((\gamma-1)\overline{s}\right)\\ &= \left(v^{1-\gamma} - 1\right) \exp\left((\gamma-1)s\right) + \exp\left((\gamma-1)s\right) - \exp\left((\gamma-1)\overline{s}\right).\end{aligned}$$

Consequently

$$\begin{aligned}\|\theta_0 - 1\| &\le O(1)(\gamma-1) \exp\left((\gamma-1)\|s_0\|_{L^\infty_x}\right)\\ &\quad\times \left[\left\|v_0^{-\gamma}\right\|_{L^\infty_x} \|v_0 - 1\| + \|s_0(x) - \overline{s}\|\right],\\ \|\theta_{0x}\| &\le O(1)(\gamma-1) \exp\left((\gamma-1)\|s_0\|_{L^\infty_x}\right)\\ &\quad\times \left[\left(\inf_x v_0(x)\right)^{-\gamma} \|v_{0x}\| + \left(\inf_x v_0(x)\right)^{1-\gamma} \|s_{0x}\|\right],\\ \|\theta_{0xx}\| &\le O(1)(\gamma-1) \exp\left((\gamma-1)\|s_{0x}\|_{L^\infty_x}\right)\\ &\quad\times \left[\left(\inf_x v_0(x)\right)^{-\gamma} \|(v_{0xx}, s_{0xx})\| + \left(\inf_x v_0(x)\right)^{-\gamma-1} \|(v_{0x}^2, s_{0x}^2)\|\right],\\ \|\theta_{0xxx}\| &\le O(1)(\gamma-1) \exp\left((\gamma-1)\|s_{0x}\|_{L^\infty_x}\right)\\ &\quad\times \left[\left(\inf_x v_0(x)\right)^{-\gamma-2} \|(v_{0x}^3, s_{0x}^3)\| + \left(\inf_x v_0(x)\right)^{-\gamma} \|(v_{0xxx}, s_{0xxx})\|\right.\\ &\quad\left. + \left(\inf_x v_0(x)\right)^{-\gamma-1} \|(v_{0x}, s_{0x})(v_{0xx}, s_{0xx})\|\right].\end{aligned} \tag{4.1}$$

Thus, if we assume that $\|v_0(x)\|_{L^\infty_x}$, $\inf_x v_0(x)$, $\frac{\gamma-1}{A}\|s_0(x)\|_{L^\infty_x}$ are independent of $\gamma - 1$, we have from (4.1) and the assumptions listed in Theorem 1.1 that

$$\left\|\frac{\theta_0 - 1}{\sqrt{\gamma-1}}\right\|_i \le C_8 \sqrt{\gamma-1}\, \|(v_0 - 1, u_0, s_0 - \overline{s})\|_i, \quad i = 1, 2, 3,$$

which means that if $\|(v_0 - 1, u_0, s_0 - \overline{s})\|_i$ is chosen independent of $\gamma - 1$, $\|\frac{\theta_0 - 1}{\sqrt{\gamma-1}}\|_i$ can also be chosen independent of $\gamma - 1$ for $i = 1, 2, 3$. Moreover, one can easily deduce that

$$N_{0i} := \left\|\left(v_0 - 1, u_0, \frac{\theta_0 - 1}{\sqrt{\gamma-1}}\right)\right\|_i \le C_9 L_{0i} := C_9 \|(v_0 - 1, u_0, s_0 - \overline{s})\|_i, \quad i = 1, 2, 3. \tag{4.2}$$

Here C_8 and C_9 are some positive constant depending only on $\overline{V}_0$, $\underline{V}_0$, $\overline{\Theta}_0$, and $\underline{\Theta}_0$ but independent of $\gamma - 1$.

Now we prove Theorem 1.1. First under the conditions listed in Theorem 1.1, we have from the local existence result stated in Lemma 3.1 that there exists a sufficiently small positive constant t_1, which depends only on $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, $\overline{\Theta}_0$ and $\|(v_0 - 1, u_0, \theta_0 - 1)\|_3$, such that the Cauchy problem (1.4), (1.5) admits a unique smooth solution $(v(t,x), u(t,x), \theta(t,x) \in X^3\left(0, t_1; \frac{1}{2}\underline{V}_0, 2\overline{V}_0; \frac{1}{2}\underline{\Theta}_0, 2\overline{\Theta}_0\right)$ which satisfies

$$\frac{1}{2}\underline{V}_0 \le v(t,x) \le 2\overline{V}_0, \quad \frac{1}{2}\underline{\Theta}_0 \le \theta(t,x) \le 2\overline{\Theta}_0 \tag{4.3}$$

and

$$\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_3\le 2\left\|\left(v_0-1,u_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_3:=2N_{03} \tag{4.4}$$

for all $0\le t\le t_1$, $x\in\mathbf{R}$.

Equation (4.2) together with (4.4) implies

$$\|(v-1,u)(t)\|_3\le 2N_{03},\quad \|\theta(t)-1\|_3\le 2\sqrt{\gamma-1}N_{03},\quad 0\le t\le t_1. \tag{4.5}$$

Thus if we set $T=t_1$, $M_1=\max\left\{2\overline{V}_0,2\underline{V}_0^{-1}\right\}$, $\varepsilon=2\sqrt{\gamma-1}N_{03}$, $N_1=2N_{03}$, since N_{03}, $\overline{V}_0$, $\underline{V}_0$, $\overline{\Theta}_0$, and $\underline{\Theta}_0$ are assumed to be independent of $\gamma-1$, one can easily deduce that there exists a positive constant $\gamma_1>1$ such that if $1<\gamma\le\gamma_1$ we have that

$$\begin{cases}\frac14\le\frac12-4C_1(\gamma-1)N_{03}^2\max\left\{2\overline{V}_0,2\underline{V}_0^{-1}\right\},\\ 2\sqrt{\gamma-1}N_{03}\max\left\{2\overline{V}_0,2\underline{V}_0^{-1}\right\}\le 1,\\ 16\gamma(\gamma-1)^2N_{03}^4\le 1,\\ 4(\gamma-1)N_{03}^2\left(\max\left\{2\overline{V}_0,2\underline{V}_0^{-1}\right\}\right)^2\le 1,\\ 16(\gamma-1)N_{03}^4\le 1,\\ 4(\gamma-1)C_6N_{03}^2\exp\left(C_6N_{01}^2\right)\le\frac{1}{20},\\ (\gamma-1)^{\frac12}C_6N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right)\le 1\end{cases} \tag{4.6}$$

and

$$2\sqrt{\gamma-1}N_{03}\le\min\left\{\left(1-\underline{\Theta}_0\right),\left(\overline{\Theta}_0-1\right)\right\} \tag{4.7}$$

hold.

The above analysis tells us that all the conditions listed in Corollary 3.12 hold with $T=t_1$ and consequently we have from Corollary 3.12 that the local solution $(v(t,x),u(t,x),\theta(t,x))$ constructed above satisfies

$$\begin{aligned}&\underline{\Theta}_0\le\theta(t,x)\le\overline{\Theta}_0,\\ &\quad\exp\left(-C_2N_{01}^2\right)\le v(t,x)\le C_2N_{01}^6,\\ &\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_3^2+\int_0^t\left(\|v_x(\tau)\|_2^2+\|(u_x,\theta_x)(\tau)\|_3^2\right)d\tau\\ &\quad\le C_7N_{03}^2\exp\left(\exp\left(C_5N_{02}^2\right)\right)\end{aligned} \tag{4.8}$$

for all $0\le t\le t_1$, $x\in\mathbf{R}$. Here C_2 and C_7 are positive constants defined in Corollary 3.12.

Now, we take $(v(t_1,x),u(t_1,x),\theta(t_1,x))$ as initial data. We have from the estimates (4.8) and Lemma 3.1 that the local solution $(v(t,x),u(t,x),\theta(t,x))$ constructed above can be extended to the time step $t=t_1+t_2$ for some suitably small positive constant t_2 depending only on N_{03}, $\underline{\Theta}_0$, and $\overline{\Theta}_0$ and satisfies

$$\frac12\exp\left(-C_2N_{01}^2\right)\le v(t,x)\le 2C_2N_{01}^6,\quad \frac12\underline{\Theta}_0\le\theta(t,x)\le 2\overline{\Theta}_0 \tag{4.9}$$

and

$$\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_3 \le 2\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t_1)\right\|_3 \le 2\sqrt{C_7}N_{03}\exp\left(\frac{1}{2}\exp\left(C_5N_{02}^2\right)\right) \tag{4.10}$$

for all $0 \le t \le t_1+t_2$, $x \in \mathbf{R}$. If we set

$$\begin{cases} T = t_1+t_2, \\ \varepsilon = 2\sqrt{(\gamma-1)C_7}N_{03}\exp\left(\frac{1}{2}\exp\left(C_5N_{02}^2\right)\right), \\ M_1 = 2\exp\left(C_2N_{01}^2\right), \\ N_1 = 2\sqrt{C_7}N_{03}\exp\left(\frac{1}{2}\exp\left(C_5N_{02}^2\right)\right), \end{cases}$$

since the positive constants C_2, C_5, C_7, and N_{03} are independent of $\gamma-1$, it is easy to see that we can find a positive constant $\gamma_2>0$ such that

$$\begin{cases} \frac{1}{2}-8C_1C_7(\gamma-1)N_{03}^2\exp\left(C_2N_{01}^2+\exp\left(C_5N_{02}^2\right)\right) \ge \frac{1}{4}, \\ 4\sqrt{(\gamma-1)C_7}N_{03}\exp\left(C_2N_{01}^2+\frac{1}{2}\exp\left(C_5N_{02}^2\right)\right) \le 1, \\ 16\gamma(\gamma-1)^2C_7^2N_{03}^4\exp\left(2\exp\left(C_5N_{02}^2\right)\right) \le 1, \\ 16(\gamma-1)C_7N_{03}^2\exp\left(2C_2N_{01}^2+\exp\left(C_5N_{02}^2\right)\right) \le 1, \\ 16(\gamma-1)C_7^2N_{03}^4\exp\left(2\exp\left(C_5N_{02}^2\right)\right) \le 1, \\ 4(\gamma-1)C_6C_7N_{03}^2\exp\left(C_6N_{01}^2+\exp\left(C_5N_{02}^2\right)\right) \le \frac{1}{20}, \\ (\gamma-1)^{\frac{1}{2}}C_6N_{02}^2\exp\left(N_{01}^2\exp\left(C_4N_{01}^2\right)\right) \le 1 \end{cases} \tag{4.11}$$

and

$$2\sqrt{(\gamma-1)C_7}N_{03}\exp\left(\frac{1}{2}\exp\left(C_5N_{02}^2\right)\right) \le \min\left\{\left(1-\underline{\Theta}_0\right),\left(\overline{\Theta}_0-1\right)\right\} \tag{4.12}$$

hold for all $1<\gamma\le\gamma_2$. The above analysis tells us that all the conditions listed in Corollary 3.1 hold with $T=t_1+t_2$ and consequently we have from Corollary 3.1 that the solution $(v(t,x),u(t,x),\theta(t,x))$ defined on the time interval $[0,t_1+t_2]$ satisfies (4.8) for all $0\le t\le t_1+t_2$ with the same positive constants C_2, C_5, and C_7.

Now, we take $(v(t_1+t_2,x),u(t_1+t_2,x),\theta(t_1+t_2,x)$ as initial data. Noticing that the constants C_2, C_5, and C_7 in (4.8) are independent of the time variable t, we can then extend $(v(t,x),u(t,x),\theta(t,x)$ to the time step $t=t_1+2t_2$ by exploiting Lemma 3.1 again. Repeating the above procedure, if we take $\gamma_0=\min\{\gamma_1,\gamma_2\}$, we can thus extend the solution $(v(t,x),u(t,x),\theta(t,x))$ step by step to a global one provided that $1<\gamma\le\gamma_0$, and as a by-product of the above analysis, we can also deduce that $(v(t,x),u(t,x),\theta(t,x))$ satisfies

$$\left\|\left(v-1,u,\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|_3^2+\int_0^t\left(\|v_x(\tau)\|_2^2+\|(u_x,\theta_x)(\tau)\|_3^2\right)d\tau \le C_7N_{03}^2\exp\left(\exp\left(C_5N_{02}^2\right)\right), \tag{4.13}$$

from which the time asymptotic behavior (1.9) follows easily.

In the above analysis, we assume that $N_{0i}\ge 1$ for $i=1,2,3$. This assumption is indeed without loss of generality because what we are interested in in this paper is the case of large initial perturbation. On the other hand, even without this assumption, all the analysis holds true provided that we replace $\exp\left(O(1)N_{0i}^2\right)$ by $\exp\left(O(1)\left(N_{0i}^2+1\right)\right)$ $(i = 1, 2)$ in the corresponding estimates. Based on this observation, a sufficient condition to guarantee that (4.6), (4.7), (4.11), and (4.12) hold is

$$(\gamma-1)N_{03}^2 \exp\left(\exp\left(C_{10}\left(N_{02}^2+1\right)\right)\right) \le \min\left\{\left(1-\underline{\Theta}_0\right)^2, \left(\overline{\Theta}_0-1\right)^2\right\} \tag{4.14}$$

holds for some generic sufficiently large positive constant C_{10} which may depend only on $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, and $\overline{\Theta}_0$.

Moreover, we have from (4.2) that a sufficient condition to guarantee that (4.14) holds in term of $L_{0i}(i = 1, 2, 3)$ is

$$(\gamma-1)L_{03}^2 \exp\left(\exp\left(C_{11}\left(L_{02}^2+1\right)\right)\right) \le \min\left\{\left(1-\underline{\Theta}_0\right)^2, \left(\overline{\Theta}_0-1\right)^2\right\} \tag{4.15}$$

holds for some generic sufficiently large positive constant C_{11} which may depend only on $\underline{V}_0$, $\overline{V}_0$, $\underline{\Theta}_0$, and $\overline{\Theta}_0$.

Equation (4.15) is exactly the assumption (1.7) imposed in Theorem 1.1 and this completes the proof of Theorem 1.1.

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