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## LOCAL WELL-POSEDNESS OF PRANDTL EQUATIONS FOR COMPRESSIBLE FLOW IN TWO SPACE VARIABLES\*

YA-GUANG WANG<sup>†</sup>, FENG XIE<sup>†</sup>, AND TONG YANG<sup>‡</sup>

**Abstract.** In this paper, we consider the local well-posedness of the Prandtl boundary layer equations that describe the behavior of the boundary layer in the small viscosity limit of the compressible isentropic Navier–Stokes equations with nonslip boundary condition. Under the strictly monotonic assumption on the tangential velocity in the normal variable, we apply the Nash–Moser–Hörmander iteration scheme and further develop the energy method introduced in [R. Alexander et al., *J. Amer. Math. Soc.*, DOI:S0894-0347(2014)00813-4] to obtain the well-posedness of the equations locally in time.

**Key words.** compressible Prandtl layer equations, monotonic velocity field, energy method, Nash–Moser–Hörmander iteration, local well-posedness

**AMS subject classifications.** 35A07, 35G61, 35K65

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**1. Introduction.** In this paper, we study the well-posedness of the compressible Prandtl boundary layer equations that are derived in the small viscosity limit from the compressible isentropic Navier–Stokes equations with nonslip boundary condition. Note that the Prandtl equations describe the behavior of the characteristic boundary layer in the leading order. Denote by  $\mathbb{T} \times \mathbb{R}^+ = \{(x, \eta) | x \in \mathbb{R}/\mathbb{Z}, 0 \leq \eta < +\infty\}$  the periodic spatial domain, and let  $u(t, x, \eta)$  and  $v(t, x, \eta)$  be the tangential and normal velocity components in the boundary layer. Consider the following compressible Prandtl equations with  $(x, \eta) \in \mathbb{T} \times \mathbb{R}^+$ ,

$$(1.1) \quad \begin{cases} u_t + uu_x + vu_\eta - \frac{1}{\bar{\rho}(t,x)} \partial_\eta^2 u + P_x = 0, \\ \partial_x(\bar{\rho}u) + \partial_\eta(\bar{\rho}v) = -\bar{\rho}_t \end{cases}$$

with the initial data

$$(1.2) \quad u(t, x, \eta)|_{t=0} = u_0(x, \eta),$$

and the boundary and the far-field conditions

$$(1.3) \quad u(t, x, \eta)|_{\eta=0} = 0, \quad v(t, x, \eta)|_{\eta=0} = 0, \quad \lim_{\eta \rightarrow +\infty} u(t, x, \eta) = U(t, x).$$

Here,  $\bar{\rho}(t, x)$  and  $U(t, x)$  are the traces on the boundary  $\{y = 0\}$  of the density and the tangential velocity of the outer Euler flow that satisfy Bernoulli’s law

$$(1.4) \quad U_t + UU_x + P_x = 0$$

with  $P(t, x)$  being the trace of the enthalpy of the outer Euler flow.

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It is well known that the leading order characteristic boundary layer for the incompressible Navier–Stokes equations with nonslip boundary condition is described by the classical Prandtl equations that were proposed by Prandtl [13] in 1904. Under the monotone assumption on the tangential velocity in the normal direction, Oleinik first obtained the local existence of classical solutions in two spatial dimensions by using the Crocco transformation; cf. [11]. This result together with some other extensions in this direction are presented in Oleinik and Samokhin’s classical book [12]. Recently, this well-posedness result was reproved by using an energy method in the framework of Sobolev spaces in [1] and [9] independently. On the other hand, by imposing an additional favorable condition on the pressure, a global in time weak solution was obtained in [14].

When the monotonicity condition is violated, separation of the boundary layer is well expected and observed. For this, E and Engquist constructed a finite time blowup solution to the Prandtl equations in [3]. Recently, when the background shear flow has a nondegenerate critical point, some interesting ill-posedness (or instability) phenomena of solutions to both the linear and nonlinear Prandtl equations around the shear flow were studied; cf. [4, 5, 6, 7]. All these results show that the monotone assumption on the tangential velocity is very important for well-posedness except in the framework of analytic functions studied in [2] and some other references with generalizations.

This paper aims to obtain the local well-posedness of the problem (1.1)–(1.3) for the compressible Prandtl equations in some weighted Sobolev spaces. To state the main results, we first give the following assumptions on the initial data.

Main assumptions (H) on the initial data.

- (H1) For a fixed integer  $k_0 \geq 9$ , the initial data  $u_0(x, \eta)$  satisfy the compatibility condition of the problem (1.1)–(1.3) up to order  $4k_0 + 2$ ;
- (H2) monotone condition  $\partial_\eta u_0(x, \eta) \geq \frac{\sigma_0}{(1+\eta)^{\gamma+2}} > 0$  holds for all  $x \in \mathbb{T}$  and  $\eta \geq 0$  with some positive constant  $\sigma_0$  and a positive integer  $\gamma \geq 2$ ;
- (H3)  $\|(1+\eta)^{\gamma+\alpha_2} D^\alpha (u_0(x, \eta) - U(0, x))\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \leq C_0$ , where  $D^\alpha = \partial_x^{\alpha_1} \partial_\eta^{\alpha_2}$  with  $\alpha = (\alpha_1, \alpha_2)$  and  $|\alpha| = \alpha_1 + \alpha_2 \leq 4k_0 + 2$ ;
- (H4)  $\|(1+\eta)^{\gamma+2+\alpha_2} D^\alpha \partial_\eta u_0\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} \leq \frac{1}{\sigma_0}$  for  $|\alpha| \leq 3k_0$ .

*Remark 1.1.* The positiveness of the lower bound in the assumption (H2) is required for well-posedness as in the classical work by Oleinik and her collaborators on the incompressible Navier–Stokes equations. However, the algebraic decay in (H2) is only needed at infinity as one can see from the proof of Proposition 4.1.

Denote by  $V(t, x)$  the trace of  $\partial_y u_2^E$  on  $\{y = 0\}$  for the normal velocity  $u_2^E$  of outer Euler flow. From the conservation of mass in the Euler equations, we have

$$\partial_t \bar{\rho}(t, x) + \partial_x (\bar{\rho}(t, x) U(t, x)) + \bar{\rho}(t, x) V(t, x) = 0.$$

Here, we have used the fact that  $u_2^E(t, x, y)|_{y=0} = 0$ . Thus, from the problem (1.1)–(1.3), we know that the normal velocity  $v(t, x, \eta)$  can be represented by

$$(1.5) \quad v(t, x, \eta) = V(t, x) \eta + \frac{1}{\bar{\rho}(t, x)} \int_0^\eta \partial_x (\bar{\rho}(t, x) (U(t, x) - u(t, x, \tilde{\eta}))) d\tilde{\eta}.$$

The main result of this paper can be stated as follows.

**THEOREM 1.1.** *Suppose that the outer Euler flow is smooth for  $0 \leq t \leq T_0$ , the density  $\bar{\rho}(t, x)$  has both positive lower and upper bounds, and the Sobolev norm  $H^s([0, T_0] \times \mathbb{R})$  of  $(\bar{\rho}, U, V)$  is bounded for a suitably large integer  $s$ ; moreover, the*

main assumptions (H) on the initial data  $u_0(x, \eta)$  are satisfied. Then there exists  $0 < T \leq T_0$ , such that the initial boundary-value problem (1.1)–(1.3) has a unique classical solution  $(u, v)$  satisfying

$$(1.6) \quad \sum_{|m_1| + [(m_2+1)/2] \leq k_0} \|\langle \eta \rangle^l \partial_{(t,x)}^{m_1} \partial_\eta^{m_2} (u - U)\|_{L^2([0,T] \times \mathbb{T} \times \mathbb{R}^+)} < +\infty$$

for a fixed  $l > \frac{1}{2}$  depending only on  $\gamma$  given in (H) with  $\langle \eta \rangle = (1 + \eta)$ , and

$$(1.7) \quad \sum_{|m_1| + [(m_2+1)/2] \leq k_0 - 1} \sup_{\eta \in \mathbb{R}^+} \|\partial_{(t,x)}^{m_1} \partial_\eta^{m_2} (v - V\eta)(\cdot, \eta)\|_{L^2([0,T] \times \mathbb{T})} < +\infty.$$

*Remark 1.2.*

- (1) When the outer Euler flow density  $\bar{\rho}(t, x)$  is a positive constant, the system (1.1) is reduced to the classical incompressible Prandtl equations. Thus the analysis in this paper works also for the classical incompressible Prandtl equations with general far-field condition and initial data satisfying the main assumptions (H). Note that the case with a uniform outer flow with a slightly different assumption on the initial data was studied in [1].
- (2) It is straightforward to verify that the set of the initial data satisfying the main assumptions (H) is not empty because it contains the functions with polynomial decay in  $\eta$ .

Now, let us make some comments on the analysis in this paper. In principle, we will apply the approach of [1] to study the problem (1.1)–(1.3). There are several crucial differences between the system (1.1) and classical incompressible Prandtl equations. First, the normal velocity  $v$  contains the linearly increasing part  $V(t, x)\eta$  in  $\eta$ , consequently, in estimating the solution to the linearized problem, we need to study the conormal estimates. Second, the divergence free condition in the classical Prandtl system is now replaced by an inhomogeneous equation in (1.1). Moreover, the far-field state is not uniform so that the shear flow is no longer an exact solution to the compressible Prandtl equations (1.1). Therefore, to apply the Nash–Moser–Hömander iteration scheme used in [1] for the nonlinear problem (1.1)–(1.3), we need to construct a suitable zeroth order approximate solution with suitable error estimate. And the construction is given in subsection 4.1 in three steps.

Finally, the rest of the paper is organized as follows. We will first introduce some weighted Sobolev spaces and give some preliminaries in section 2. The well-posedness of the linearized compressible Prandtl equations is given in section 3. In section 4, we introduce the Nash–Moser–Hömander iteration scheme, and construct the first approximate solution as the starting point of iteration. Then the local existence and uniqueness of a solution to the nonlinear problem of the compressible Prandtl equations are proved.

**2. Preliminaries.** In this section, we will introduce some weighted Sobolev spaces and norms for later use. To simplify the notation, we denote by  $\partial_\tau^m$  the summation of tangential derivatives  $\partial_\tau^m = \partial_t^{m_1} \partial_x^{m_2}$  for all  $m = (m_1, m_2) \in \mathbb{N}^2$ ,  $|m| = m_1 + m_2$ . Denote  $\partial_\tau^\alpha$  by  $Z_1^\alpha$ ,  $(\eta \partial_\eta)^\alpha$  by  $Z_2^\alpha$ , and  $Z^m = Z_1^{m_1} Z_2^{m_2}$  for further simplification. Define

$$\|f\|_{H_{co,l}^{k_1,k_2}} = \left( \sum_{0 \leq m_1 \leq k_1, 0 \leq m_2 \leq k_2} \|\langle \eta \rangle^l Z_1^{m_1} Z_2^{m_2} f\|_{L^2([0,T] \times \mathbb{T} \times \mathbb{R}^+)} \right)^{1/2},$$

and

$$\|f\|_{D_{co,l}^{k_1,k_2}} = \left( \sum_{0 \leq m_1 \leq k_1, 0 \leq m_2 \leq k_2} \sup_{\eta \geq 0} \|\langle \eta \rangle^l Z_1^{m_1} Z_2^{m_2} f(\cdot, \eta)\|_{L^2([0,T] \times \mathbb{T})} \right)^{1/2},$$

$$\|f\|_{C_{co,l}^{k_1,k_2}} = \left( \sum_{0 \leq m_1 \leq k_1, 0 \leq m_2 \leq k_2} \sup_{(t,x) \in [0,T] \times \mathbb{T}} \|\langle \eta \rangle^l Z_1^{m_1} Z_2^{m_2} f(t, x, \cdot)\|_{L^2(\mathbb{R}^+)} \right)^{1/2}.$$

Denote

$$\|f\|_{H_{co,l}^k} = \sum_{k_1+k_2=k} \|f\|_{H_{co,l}^{k_1,k_2}}.$$

The function spaces  $D_{co,l}^k$  and  $C_{co,l}^k$  can be defined similarly. Since the conormal operator  $Z^m$  does not communicate with the normal derivative operator  $\partial_\eta$ , the following estimate of commutator is frequently used,

$$(2.1) \quad \|[Z^m, \partial_\eta]f\|_{L_t^2} \lesssim \|\partial_\eta f\|_{H_{co,l}^{m-1}}.$$

Here and in what follows  $0 < a \lesssim b$  means that there exists a uniform constant  $C > 0$  such that  $a \leq Cb$ .

The following weighted Sobolev spaces are also used frequently. Denote by  $\partial_\eta^k$  the  $k$ th normal derivative, for any given  $k_1, k_2 \in \mathbb{N}, l \in \mathbb{R}^+$ , and  $0 < T < +\infty$ ; set

$$\|f\|_{\mathcal{B}_l^{k_1,k_2}} = \left( \sum_{0 \leq |m| \leq k_1, 0 \leq n \leq k_2} \|\langle \eta \rangle^l Z^m \partial_\eta^n f\|_{L^2([0,T] \times \mathbb{T} \times \mathbb{R}^+)}^2 \right)^{1/2},$$

$$\|f\|_{\tilde{\mathcal{B}}_l^{k_1,k_2}} = \left( \sum_{0 \leq |m| \leq k_1, 0 \leq n \leq k_2} \|\langle \eta \rangle^l Z^m \partial_\eta^n f\|_{L^\infty([0,T]; L^2(\mathbb{T} \times \mathbb{R}^+))}^2 \right)^{1/2},$$

$$\|f\|_{\mathcal{A}_l^m} = \left( \sum_{|m_1| + [(m_2+1)/2] \leq |m|} \|\langle \eta \rangle^l Z^{m_1} \partial_\eta^{m_2} f\|_{L^2([0,T] \times \mathbb{T} \times \mathbb{R}^+)}^2 \right)^{1/2}.$$

It is straightforward to verify that

$$\mathcal{A}_l^m = \bigcap_{m_1 + [(m_2+1)/2] \leq m} \mathcal{B}_l^{m_1, m_2}.$$

We also define

$$\|f\|_{\mathcal{D}_l^m} = \left( \sum_{k_1 + [(k_2+1)/2] \leq m} \|\langle \eta \rangle^l Z^{k_1} \partial_\eta^{k_2} f\|_{L_\eta^\infty(L_{t,x}^2)}^2 \right)^{1/2}$$

and

$$\|f\|_{\mathcal{C}_l^m} = \left( \sum_{k_1 + [(k_2+1)/2] \leq m} \|\langle \eta \rangle^l Z^{k_1} \partial_\eta^{k_2} f\|_{L_{t,x}^\infty(L_\eta^2)}^2 \right)^{1/2}.$$

In addition, the homogeneous norms  $\|\cdot\|_{\dot{\mathcal{A}}_l^m}, \|\cdot\|_{\dot{\mathcal{C}}_l^m}, \|\cdot\|_{\dot{\mathcal{D}}_l^m}$  correspond to the summation over  $1 \leq |m_1| + [(m_2 + 1)/2] \leq |m|$ .

For  $1 \leq p \leq +\infty$ , we will use  $\|f\|_{L_l^p(\mathbb{T} \times \mathbb{R}^+)} = \|\langle \eta \rangle^l f\|_{L^p(\mathbb{T} \times \mathbb{R}^+)}$ . It is direct to show the following Sobolev-type embeddings,

$$(2.2) \quad \|f\|_{\mathcal{C}_l^m} \leq C_s \|f\|_{\mathcal{A}_l^{m+2}}, \quad \|f\|_{\mathcal{D}_l^m} \leq C_s \|f\|_{\mathcal{A}_{l+1}^{m+1}}.$$

Moreover, for any  $l \geq 0$  and  $m \geq 2$ , the space  $\mathcal{A}_l^m$  is continuously embedded into  $\mathcal{C}_b^{m-2}$  which is the space of  $(m - 2)$ th order continuously differentiable functions with bounded derivatives. And the following Morse-type inequalities hold.

LEMMA 2.1. *For any proper functions  $f$  and  $g$ , we have*

$$\|fg\|_{\mathcal{A}_l^m} \leq C_m \left\{ \|f\|_{\mathcal{A}_l^m} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\mathcal{A}_l^m} \right\}$$

and

$$\|fg\|_{\mathcal{A}_l^m} \leq C_m \left\{ \|f\|_{\mathcal{C}_l^m} \|g\|_{\mathcal{D}_0^0} + \|f\|_{\mathcal{C}_l^0} \|g\|_{\dot{\mathcal{D}}_0^0} \right\}.$$

Similar inequalities hold in the norms  $\|\cdot\|_{H_{co,l}^m}, \|\cdot\|_{\mathcal{B}_l^{m_1, m_2}}, \|\cdot\|_{\mathcal{C}_l^m}$ , and  $\|\cdot\|_{\mathcal{D}_l^m}$ . Here,  $C_m > 0$  is a constant depending only on  $m$ .

These results can be obtained similarly as those given in [8].

**3. Well-posedness of linearized system.** The strategy to prove the main result, Theorem 1.1, is to apply an iteration scheme to construct a sequence of approximate solution sequences, and then to show these approximate solutions converge in some suitable weighted Sobolev space. Since there is a loss of regularity, the Nash–Moser–Hömander iteration scheme is used for this purpose. In this section, we study the well-posedness of the linearized equations and obtain the required energy estimates of solutions to the linearized equations for the Nash–Moser–Hömander iteration.

Let  $(\tilde{u}, \tilde{v})$  be a smooth background state satisfying the following conditions:

$$\partial_\eta \tilde{u}(t, x, \eta) > 0, \quad \partial_x(\bar{\rho}\tilde{u}) + \partial_\eta(\bar{\rho}\tilde{v}) = -\bar{\rho}_t, \quad \tilde{u}|_{\eta=0} = \tilde{v}|_{\eta=0} = 0, \quad \lim_{\eta \rightarrow +\infty} \tilde{u} = U(t, x).$$

Here  $\tilde{v}$  is given by

$$\tilde{v} = V(t, x)\eta + \frac{1}{\bar{\rho}(t, x)} \int_0^\eta \partial_x(\bar{\rho}(t, x)(U(t, x) - \tilde{u}))d\tilde{\eta} \triangleq V(t, x)\eta + \bar{v}.$$

It extracts the linear increasing part  $V(t, x)\eta$  by introducing the new function  $\bar{v}$ . The linearized problem of (1.1)–(1.3) around  $(\tilde{u}, \tilde{v})$  can be written as

$$(3.1) \quad \begin{cases} u_t + \tilde{u}u_x + \tilde{v}\partial_\eta u + u\tilde{u}_x + \tilde{u}_\eta v - \frac{1}{\bar{\rho}}u_{\eta\eta} = f, \\ \partial_\eta(\bar{\rho}v) + \partial_x(\bar{\rho}u) = 0, \\ u|_{\eta=0} = v|_{\eta=0} = 0, \quad \lim_{\eta \rightarrow +\infty} u(t, x, \eta) = 0, \\ u|_{t=0} = 0. \end{cases}$$

Similarly to [1], by introducing the transformation

$$\omega(t, x, \eta) = \left( \frac{\bar{\rho}u}{\partial_\eta \tilde{u}} \right)_\eta (t, x, \eta),$$

then for classical solutions, from (3.1) we know that  $w$  satisfies the following problem in  $\{t > 0, x \in \mathbb{T}, \eta > 0\}$ :

$$(3.2) \quad \begin{cases} \omega_t + (\tilde{u}\omega)_x + (\eta V\omega)_\eta + (\bar{v}\omega)_\eta - \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta \\ \quad + \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}} \omega - \frac{1}{\bar{\rho}} \omega_{\eta\eta} = \tilde{f}_\eta, \\ \frac{1}{\bar{\rho}} \left( \omega_\eta + 2\omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right) |_{\eta=0} = -\tilde{f} |_{\eta=0}, \\ \omega|_{t=0} = 0, \end{cases}$$

where

$$\xi = \frac{(\partial_t + \tilde{u}\partial_x + \tilde{v}\partial_\eta - \frac{1}{\bar{\rho}}\partial_\eta^2)\tilde{u}_\eta}{\tilde{u}_\eta} - \frac{\tilde{u}\bar{\rho}_x}{\bar{\rho}} \triangleq \xi_1 - \frac{\tilde{u}\bar{\rho}_x}{\bar{\rho}}, \quad \tilde{f} = \frac{\bar{\rho}f}{\tilde{u}_\eta}.$$

To simplify the presentation, we use the notations

$$(3.3) \quad \lambda_{k_1, k_2} = \|\tilde{u} - U\|_{\mathcal{B}_t^{k_1, k_2}} + \|Z^{k_1} \partial_\eta^{k_2} \bar{v}\|_{L^\infty(L^2_{t,x})} + \|Z^{k_1} \partial_\eta^{k_2} \chi\|_{L^\infty(L^2_{t,x})} + \|\xi_1\|_{\mathcal{B}_t^{k_1, k_2}}$$

with  $\chi = \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}}$ . Set

$$(3.4) \quad \lambda_k = \sum_{k_1 + [(k_2 + 1)/2] \leq k} \lambda_{k_1, k_2}.$$

Similarly to [1], we have the following energy estimates of the solution to the problem (3.2).

**THEOREM 3.1.** *Suppose that the outer Euler flow  $(\bar{\rho}(t, x), U(t, x), V(t, x)) \in H^s(\mathbb{R}_+^2)$ , for  $s$  suitably large, and  $\bar{\rho}(t, x)$  has uniform lower and upper positive bound. Moreover, for a given positive  $k$ , the compatibility condition for the problem (3.2) holds up to order  $k$ . Then for any fixed  $l > 1/2$ , we have*

$$(3.5) \quad \|\omega\|_{\mathcal{A}_t^k} \leq C_1(\lambda_4) \|\tilde{f}\|_{\mathcal{A}_t^k} + C_2(\lambda_4) \lambda_k \|\tilde{f}\|_{\mathcal{A}_t^3},$$

with  $C_1(\cdot)$  and  $C_2(\cdot)$  being two smooth functions in their arguments.

As we mentioned in the introduction, the main difference of the linear problem (3.2) from the one studied in [1] is that there is a linear growth term  $\eta V$  in the equation of (3.2). Hence, we can not obtain the estimates of tangential derivatives directly as in [1]. Similarly to [10], we will study the linearized problem (3.2) in some conormal space. First, we have the following.

**LEMMA 3.1 ( $L^2$ -estimate).** *Under the assumptions in Theorem 3.1, there exists a positive constant  $C$  such that*

$$(3.6) \quad \frac{d}{dt} \|\omega\|_{L^2_t(\mathbb{T} \times \mathbb{R}^+)}^2 + \|\omega_\eta\|_{L^2_t(\mathbb{T} \times \mathbb{R}^+)}^2 \leq C(\lambda_{3,1} + 1) \|\omega\|_{L^2_t(\mathbb{T} \times \mathbb{R}^+)}^2 + C \|\tilde{f}\|_{L^2_t(\mathbb{T} \times \mathbb{R}^+)}^2.$$



*Proof.* Multiplying (3.2) by  $\langle \eta \rangle^{2l} \omega$  and integrating it over  $\mathbb{T} \times \mathbb{R}^+$ , we obtain

$$(3.7) \quad \begin{aligned} & \frac{d}{dt} \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 \\ & + 2 \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} \omega \left\{ (\tilde{u}\omega)_x + (\eta V\omega)_\eta + (\bar{v}\omega)_\eta - \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta \right. \\ & \left. + \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}} \omega - \frac{1}{\bar{\rho}} \omega_{\eta\eta} - \tilde{f}_\eta \right\} dx d\eta = 0. \end{aligned}$$

It is straightforward to obtain

$$\begin{aligned} & \int_{\mathbb{T} \times \mathbb{R}^+} \{(\tilde{u}\omega)_x + (\bar{v}\omega)_\eta\} \langle \eta \rangle^{2l} \omega dx d\eta \\ & = \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} \frac{\omega^2}{2} \left( \frac{-\bar{\rho}_t - \bar{\rho}V - \tilde{u}\bar{\rho}_x}{\bar{\rho}} \right) - l\bar{v}\omega^2 \langle \eta \rangle^{2l-1} dx d\eta \\ & \lesssim (1 + \|\tilde{u}\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} + \|\bar{v}\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}) \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 \end{aligned}$$

and

$$\int_{\mathbb{T} \times \mathbb{R}^+} (\eta V\omega)_\eta \langle \eta \rangle^{2l} \omega dx d\eta = \int_{\mathbb{T} \times \mathbb{R}^+} V\omega^2 \left( \frac{1}{2} \langle \eta \rangle^{2l} - l\eta \langle \eta \rangle^{2l-1} \right) dx d\eta \lesssim \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2.$$

On the other hand, by integration by parts and using the boundary condition given in (3.2) we get

$$\begin{aligned} & \int_{\mathbb{T} \times \mathbb{R}^+} \left\{ -\frac{2}{\bar{\rho}} (\omega\chi)_\eta - \frac{1}{\bar{\rho}} \omega_{\eta\eta} - \tilde{f}_\eta \right\} \langle \eta \rangle^{2l} \omega dx d\eta \\ & = \int_{\mathbb{T} \times \mathbb{R}^+} \left\{ \frac{2}{\bar{\rho}} (\omega\chi) + \frac{1}{\bar{\rho}} \omega_\eta + \tilde{f} \right\} (\langle \eta \rangle^{2l} \omega)_\eta dx d\eta, \end{aligned}$$

where the right-hand side can be estimated as follows:

$$\begin{aligned} & \int_{\mathbb{T} \times \mathbb{R}^+} \frac{2}{\bar{\rho}} \omega\chi (\langle \eta \rangle^{2l} \omega)_\eta dx d\eta \\ & \lesssim \|\chi\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} (\|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)} \|\partial_\eta \omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}), \\ & \int_{\mathbb{T} \times \mathbb{R}^+} \frac{1}{\bar{\rho}} \omega_\eta (2l \langle \eta \rangle^{2l-1} \omega + \langle \eta \rangle^{2l} \omega_\eta) \\ & \gtrsim \|\partial_\eta \omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 - \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2, \end{aligned}$$

and

$$\int_{\mathbb{T} \times \mathbb{R}^+} \tilde{f} (2l \langle \eta \rangle^{2l-1} \omega + \langle \eta \rangle^{2l} \omega_\eta) \lesssim \|\tilde{f}\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)} (\|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)} + \|\partial_\eta \omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}).$$

Denote

$$\int_{\mathbb{T} \times \mathbb{R}^+} \left( \left( \xi_1 - \frac{\tilde{u}\bar{\rho}_x}{\bar{\rho}} \right) \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta \langle \eta \rangle^{2l} \omega dx d\eta \triangleq H^1 + H^2.$$

As  $l > 1/2$ , by integration by parts, it follows

$$|H^1| \lesssim \|\xi_1\|_{L_x^\infty(L_{\eta,t}^2)} (\|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)} \|\partial_\eta \omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)})$$

and

$$\begin{aligned} |H^2| &\leq \left| \int_{\mathbb{T} \times \mathbb{R}^+} \frac{\bar{\rho}_x(\tilde{u} - U)}{\bar{\rho}} \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \langle \eta \rangle^{2l} \omega_\eta dx d\eta \right| \\ &\quad + \left| \int_{\mathbb{T} \times \mathbb{R}^+} \left( \frac{\bar{\rho}_x U}{\bar{\rho}} \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta \langle \eta \rangle^2 \omega dx d\eta \right| \\ &\lesssim \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \|\tilde{u} - U\|_{L_x^\infty(L_{\eta,t}^2)} (\|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)} \|\partial_\eta \omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}). \end{aligned}$$

Thus, from (3.7) we obtain

$$\frac{d}{dt} \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 + \|\omega_\eta\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 \leq C(1 + \lambda_{3,1}) \|\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 + C\|\tilde{f}\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2$$

by noting

$$\|\tilde{u}, \bar{v}, \chi\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)} + \|\xi_1\|_{L_x^\infty(L_{\eta,t}^2)} + \|\tilde{u} - U\|_{L_x^\infty(L_{\eta,t}^2)} \lesssim (1 + \lambda_{3,1}).$$

It completes the proof of the estimate (3.6).  $\square$

LEMMA 3.2 (estimates of conormal derivatives). *Under the assumptions in Theorem 3.1, for any fixed  $T > 0$ , there exists a positive constant  $C > 0$  such that*

$$\begin{aligned} \frac{d}{dt} \|\omega\|_{H_{co,t}^m}^2 + \|\omega_\eta\|_{H_{co,t}^m}^2 &\leq C(1 + \lambda_{3,1}^2) \|\omega\|_{H_{co,t}^m}^2 \\ (3.8) \quad &\quad + C(\|\tilde{f}\|_{H_{co,t}^m}^2 + \|\tilde{f}_\eta\|_{H_{co,t}^{m-1}}^2 + (\lambda_{m-1,1}^2 + \lambda_{m,0}^2 + 1) \|\omega\|_{B_t^{3,1}}^2) \end{aligned}$$

holds for  $t \in [0, T]$ .

*Proof.* The proof is divided into four steps.

- (1) Applying the conormal derivative operator  $Z^m$  on the equation in (3.2), multiplying the resulting equation by  $\langle \eta \rangle^{2l} Z^m \omega$ , and integrating it over  $\mathbb{T} \times \mathbb{R}^+$ , it follows

$$\begin{aligned} \frac{d}{dt} \|Z^m \omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 + 2 \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega Z^m \\ \left\{ (\tilde{u}\omega)_x + (\eta V \omega)_\eta + (\bar{v}\omega)_\eta - \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta \right. \\ (3.9) \quad \left. + \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}} \omega - \frac{1}{\bar{\rho}} \omega_{\eta\eta} - \tilde{f}_\eta \right\} dx d\eta = 0. \end{aligned}$$

Now, let us estimate each term of (3.9). Denote

$$\begin{cases} I_1 = \int_{\mathbb{T} \times \mathbb{R}^+} Z^m [(\tilde{u}\omega)_x + (\bar{v}\omega)_\eta] \langle \eta \rangle^{2l} Z^m \omega dx d\eta, \\ I_2 = - \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega Z^m \left( \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta + \frac{1}{\bar{\rho}} \omega_{\eta\eta} + \tilde{f}_\eta \right) dx d\eta, \\ I_3 = \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega Z^m \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta dx d\eta. \end{cases}$$

(2) *Estimate of  $I_1$ .* Obviously, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{T} \times \mathbb{R}^+} Z^m [(\tilde{u}\omega)_x + (\tilde{v}\omega)_\eta] \langle \eta \rangle^{2l} Z^m \omega dx d\eta \\ &= \int_{\mathbb{T} \times \mathbb{R}^+} Z^m \left[ \left( -\frac{\bar{\rho}_t + \bar{\rho}V}{\bar{\rho}} \right) \omega + \left( -\frac{\bar{\rho}_x \tilde{u}}{\bar{\rho}} \right) \omega + \tilde{u}\omega_x + \tilde{v}\omega_\eta \right] \langle \eta \rangle^{2l} Z^m \omega dx d\eta \\ &\triangleq I_1^1 + I_1^2 + I_1^3 + I_1^4, \end{aligned}$$

and

$$\begin{aligned} |I_1^1| &= \left| \sum_{m_1+m_2=m} C_m^{m_1} \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} \left( Z^{m_1} \left( -\frac{\bar{\rho}_t + \bar{\rho}V}{\bar{\rho}} \right) \right) (Z^{m_2}\omega)(Z^m\omega) dx d\tau \right| \\ &\lesssim \|\omega\|_{H_{co,l}^m}^2, \\ |I_1^2| &= \left| \sum_{m_1+m_2=m} C_m^{m_1} \int_{\mathbb{T} \times \mathbb{R}^+} Z^{m_1} \left[ \left( \frac{\bar{\rho}_x(\tilde{u} - U(x,t))}{\bar{\rho}} \right) \right. \right. \\ &\quad \left. \left. + \frac{\bar{\rho}_x U}{\bar{\rho}} \right] Z^{m_2}\omega \langle \eta \rangle^{2l} Z^m \omega dx d\eta \right| \\ &\lesssim \|\tilde{u} - U\|_{H_{co,l}^m} \|\omega\|_{L^\infty} \|\omega\|_{H_{co,l}^m} + (1 + \|\tilde{u}\|_{L^\infty}) \|\omega\|_{H_{co,l}^m}^2. \end{aligned}$$

On the other hand, one has

$$\begin{aligned} I_1^3 &= \sum_{m_1+m_2=m, m_2 < m} C_m^{m_1} \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} (Z^{m_1}\tilde{u})(Z^{m_2}\omega_x)(Z^m\omega) dx d\eta \\ &\quad + \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} \tilde{u} (Z^m\omega_x)(Z^m\omega) dx d\eta \triangleq I_1^{3a} + I_1^{3b} \end{aligned}$$

and

$$\begin{aligned} I_1^4 &= \sum_{m_1+m_2=m, m_2 < m} C_m^{m_1} \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} (Z^{m_1}\tilde{v})(Z^{m_2}\omega_\eta)(Z^m\omega) dx d\eta \\ &\quad + \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} \tilde{v} (Z^m\omega_\eta)(Z^m\omega) dx d\eta \triangleq I_1^{4a} + I_1^{4b}. \end{aligned}$$

Note that  $I_1^{3a}$  can be estimated similarly to  $I_1^2$  and

$$|I_1^{4a}| \lesssim \|Z^m\omega\|_{L_t^2} (\|Z^m\tilde{v}\|_{L_\eta^\infty(L_x^2)} \|\omega_\eta\|_{L_\eta^2(L_x^\infty)} + \|Z\tilde{v}\|_{L^\infty} \|Z^{m-1}\omega_\eta\|_{L_t^2}).$$

By using  $\partial_\eta(\bar{\rho}\tilde{v}) + \partial_x(\bar{\rho}\tilde{u}) = -\bar{\rho}_t - \bar{\rho}V$ , integration by parts, and the commutator estimate (2.1), we obtain

$$\begin{aligned} |I_1^{3b} + I_1^{4b}| &\lesssim (1 + \|\tilde{u}\|_{L^\infty} + \|\tilde{v}\|_{L^\infty}) \|Z^m\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}^2 \\ &\quad + \|\tilde{v}\|_{L^\infty} \|\partial_\eta\omega\|_{H_{co,l}^{m-1}} \|Z^m\omega\|_{L_t^2(\mathbb{T} \times \mathbb{R}^+)}. \end{aligned}$$

Moreover, the definition of the operator  $Z_2$  gives that

$$\begin{aligned} \int_{\mathbb{T} \times \mathbb{R}^+} Z^m (\eta V \omega)_\eta \langle \eta \rangle^{2l} Z^m \omega dx d\eta &= \int_{\mathbb{T} \times \mathbb{R}^+} Z^m (V\omega + V\eta\omega_\eta) \langle \eta \rangle^{2l} Z^m \omega dx d\eta \\ &\lesssim \|\omega\|_{H_{co,l}^m}^2. \end{aligned}$$

(3) *Estimate of  $I_2$ .* First, by using the boundary condition given in (3.2), we have

$$\begin{aligned} I_2 &= \int_{\mathbb{T} \times \mathbb{R}^+} (\langle \eta \rangle^{2l} Z^m \omega)_\eta Z^m \left( \frac{2}{\rho} \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} + \frac{1}{\rho} \omega_\eta + \tilde{f} \right) dx d\eta \\ &\quad - \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega [Z^m, \partial_\eta] \left( \frac{2}{\rho} \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} + \frac{1}{\rho} \omega_\eta + \tilde{f} \right) dx d\eta \\ &\triangleq I_2^1 + I_2^2 + I_2^3 + I_2^4 \end{aligned}$$

with  $I_2^4$  being the terms involving  $\tilde{f}$ ,

$$\begin{aligned} I_2^1 &= \int_{\mathbb{T} \times \mathbb{R}^+} (\langle \eta \rangle^{2l} Z^m \omega)_\eta Z^m \left( \frac{2}{\rho} \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right) dx d\eta \\ &\quad - \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega [Z^m, \partial_\eta] \left( \frac{2}{\rho} \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right) dx d\eta, \\ I_2^2 &= \int_{\mathbb{T} \times \mathbb{R}^+} (\langle \eta \rangle^{2l} Z^m \omega)_\eta Z^m \left( \frac{1}{\rho} \omega_\eta \right) dx d\eta, \end{aligned}$$

and

$$I_2^3 = - \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega [Z^m, \partial_\eta] \left( \frac{1}{\rho} \omega_\eta \right) dx d\eta.$$

It is straightforward to show that

$$\begin{aligned} |I_2^1| &\leq \left| \int_{\mathbb{T} \times \mathbb{R}^+} Z^m \left( \frac{2}{\rho} \chi \omega \right) \langle \eta \rangle^{2l} \left( Z^m \omega_\eta + \frac{2l}{\langle \eta \rangle} Z^m \omega + [Z^m, \partial_\eta] \omega \right) dx d\eta \right| \\ &\quad + \|Z^m \omega\|_{L_t^2} \left\| \frac{2}{\rho} (\chi \omega)_\eta \right\|_{H_{co,l}^{m-1}} \\ &\lesssim \left\| \frac{2}{\rho} \chi \omega \right\|_{H_{co,l}^m} (\|Z^m \omega_\eta\|_{L_t^2} + \|Z^m \omega\|_{L_t^2} + \|\partial_\eta \omega\|_{H_{co,l}^{m-1}}) \\ &\quad + \|Z^m \omega\|_{L_t^2} \left\| \frac{2}{\rho} (\chi \omega)_\eta \right\|_{H_{co,l}^{m-1}} \\ &\lesssim (\|\chi\|_{l^\infty} \|\omega\|_{H_{co,l}^m} + \|\chi\|_{D_{co,0}^m} \|\omega\|_{L_{\eta,t}^2(L_x^\infty)}) (\|\omega\|_{H_{co,l}^m} + \|\partial_\eta \omega\|_{H_{co,l}^{m-1}}) \\ &\quad + \|Z^m \omega\|_{L_t^2} (\|\partial_\eta \chi\|_{D_{co,0}^{m-1,0}} \|\omega\|_{L_{\eta,t}^2(L_x^\infty)} + \|\partial_\eta \chi\|_{L^\infty} \|\omega\|_{H_{co,l}^{m-1}}) \\ &\quad + \|\chi\|_{D_{co,0}^{m-1,0}} \|\omega_\eta\|_{L_{\eta,t}^2(L_x^\infty)} + \|\chi\|_{L^\infty} \|\omega_\eta\|_{H_{co,l}^{m-1}}, \\ I_2^2 &= \int_{\mathbb{T} \times \mathbb{R}^+} \left( Z^m \frac{\omega_\eta}{\rho} \right) (2l \langle \eta \rangle^{2l-1} Z^m \omega + \langle \eta \rangle^{2l} (Z^m \omega)_\eta) dx d\eta \\ &\gtrsim \frac{1}{2} \|Z^m \omega_\eta\|_{L_t^2}^2 - \|\omega\|_{H_{co,l}^m}^2 - \|\omega_\eta\|_{H_{co,l}^{m-1}}^2, \end{aligned}$$

and

$$|I_2^3| \leq \left\| \partial_\eta \left( \frac{\omega_\eta}{\rho} \right) \right\|_{H_{co,l}^{m-1}} \|Z^m \omega\|_{L_t^2}.$$

From (3.2), we have

$$\begin{aligned} \left\| \partial_\eta \left( \frac{\omega_\eta}{\bar{\rho}} \right) \right\|_{H_{co,l}^{m-1}} &= \left\| \omega_t + (\tilde{u}\omega)_x + (\eta V\omega)_\eta + (\bar{v}\omega)_\eta - \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta \right. \\ &\quad \left. + \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}} \omega - \tilde{f}_\eta \right\|_{H_{co,l}^{m-1}}. \end{aligned}$$

The terms on the right-hand side of the above equation can be estimated as

$$\begin{aligned} &\left\| \omega_t + (\eta V\omega)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}} \omega \right\|_{H_{co,l}^{m-1}} \\ &\lesssim \|\omega\|_{H_{co,l}^m}, \\ &\|(\tilde{u}\omega)_x + (\bar{v}\omega)_\eta - \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta\|_{H_{co,l}^{m-1}} \\ &= \left\| - \left( \frac{\bar{\rho}_t + \bar{\rho}V + \bar{\rho}_x \tilde{u}}{\bar{\rho}} + \frac{2}{\bar{\rho}} \left( \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta \right) \omega + \tilde{u}\omega_x + \left( \bar{v} - \frac{2}{\bar{\rho}} \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right) \omega_\eta \right\|_{H_{co,l}^{m-1}} \\ &\leq (1 + \|\tilde{u}\|_{L^\infty} + \|\chi_\eta\|_{L^\infty}) \|\omega\|_{H_{co,l}^m} + (\|\bar{v}\|_{L^\infty} + \|\chi\|_{L^\infty}) \|\omega_\eta\|_{H_{co,l}^{m-1}} \\ &\quad + \|\chi_\eta\|_{D_{co,l}^{m-1}} \|\omega\|_{L_{\eta,l}^\infty(L_x^2)} + \|\tilde{u} - U\|_{H_{co,l}^{m-1}} (\|\omega\|_{L^\infty} + \|\omega_x\|_{L^\infty}) \\ &\quad + (\|\bar{v}\|_{D_{co,l}^{m,0}} + \|\chi\|_{D_{co,l}^{m,0}}) \|\omega_\eta\|_{L_{\eta,l}^2(L_x^\infty)}, \end{aligned}$$

and

$$\begin{aligned} &\left\| \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta \right\|_{H_{co,l}^{m-1}} \\ &\leq (\|\xi_1\|_{H_{co,l}^{m-1}} + \|\tilde{u} - U\|_{H_{co,l}^{m-1}}) \|\omega\|_{L_t^\infty} + (\|\xi_1\|_{L^\infty} + \|\tilde{u}\|_{L^\infty}) \|\omega\|_{H_{co,l}^{m-1}} \\ &\quad + \|(\xi_1)_\eta, (\tilde{u} - U)_\eta\|_{H_{co,l}^{m-1}} \|\omega\|_{L_{\eta,l}^2(L_x^\infty)} \\ &\quad + \|(\xi_1)_\eta, (\tilde{u} - U)_\eta\|_{L_{\eta,l}^2(L_x^\infty)} \|\omega\|_{H_{co,l}^{m-1}}. \end{aligned}$$

(4) *Estimate of  $I_3$ .* Decompose  $I_3$  into

$$I_3 = I_3^1 + I_3^2$$

with

$$I_3^1 = \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega Z^m \left( \xi_1 \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta dx d\eta$$

and

$$I_3^2 = - \int_{\mathbb{T} \times \mathbb{R}^+} \langle \eta \rangle^{2l} Z^m \omega Z^m \left( \frac{\tilde{u} \bar{\rho}_x}{\bar{\rho}} \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta dx d\eta.$$

For  $l > 1/2$ , we have

$$\begin{aligned}
 |I_3^1| &\leq \left| \int_{\mathbb{T} \times \mathbb{R}^+} Z^m \left( \xi_1 \int_0^\eta \omega d\tilde{\eta} \right) (\langle \eta \rangle^{2l} Z^m \omega)_\eta dx d\eta \right| \\
 &\quad + \left| \int_{\mathbb{T} \times \mathbb{R}^+} [Z^m, \partial_\eta] \left( \xi_1 \int_0^\eta \omega d\tilde{\eta} \right) (\langle \eta \rangle^{2l} Z^m \omega) dx d\eta \right| \\
 &\lesssim (\|\xi_1\|_{L^2_{\eta,t}(L^\infty_x)} \|\omega\|_{H^m_{co,t}} + \|Z^m \xi_1\|_{L^2_t} \|\omega\|_{L^2_{\eta,t}(L^\infty_x)}) (\|Z^m \omega\|_{L^2_t} + \|\omega_\eta\|_{H^m_{co,t}}) \\
 &\quad + \|Z^m \omega\|_{L^2_t} \left\| \partial_\eta \left( \xi_1 \int_0^\eta \omega d\tilde{\eta} \right) \right\|_{H^{m-1}_{co,t}} \\
 &\lesssim (\|\xi_1\|_{L^2_{\eta,t}(L^\infty_x)} \|\omega\|_{H^m_{co,t}} + \|Z^m \xi_1\|_{L^2_t} \|\omega\|_{L^2_{\eta,t}(L^\infty_x)}) (\|Z^m \omega\|_{L^2_t} + \|\omega_\eta\|_{H^m_{co,t}}) \\
 &\quad + \|Z^m \omega\|_{L^2_t} (\|\xi_1\|_{H^{m-1}_{co,t}} \|\omega\|_{L^\infty_t} + \|\xi_1\|_{L^\infty_t} \|\omega\|_{H^{m-1}_{co,t}}) \\
 &\quad + \|(\xi_1)_\eta\|_{H^{m-1}_{co,t}} \|\omega\|_{L^2_{\eta,t}(L^\infty_x)} + \|(\xi_1)_\eta\|_{L^2_{\eta,t}(L^\infty_x)} \|\omega\|_{H^{m-1}_{co,t}}
 \end{aligned}$$

and

$$\begin{aligned}
 |I_3^2| &= \left| \int_{\mathbb{T} \times \mathbb{R}^+} Z^m \left( \frac{\bar{\rho}_x(\tilde{u} - U + U)}{\bar{\rho}} \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta \langle \eta \rangle^{2l} Z^m \omega dx d\eta \right| \\
 &\lesssim \left( \|Z^m(\tilde{u} - U)\|_{L^2_t} \|\omega\|_{L^2_{\eta,t}(L^\infty_x)} \right. \\
 &\quad + \|\tilde{u} - U\|_{L^2_{\eta,t}(L^\infty_x)} \|\omega\|_{H^m_{co,t}} (\|Z^m \omega\|_{L^2_t} + \|\omega_\eta\|_{H^m_{co,t}}) \\
 &\quad + \|Z^m \omega\|_{L^2_t} (\|\partial_\eta(\tilde{u} - U)\|_{L^2_{\eta,t}(L^\infty_x)} \|\omega\|_{H^{m-1}_{co,t}} \\
 &\quad + \|\partial_\eta(\tilde{u} - U)\|_{H^{m-1}_{co,t}} \|\omega\|_{L^2_{\eta,t}(L^\infty_x)} \\
 &\quad \left. + \|\tilde{u} - U\|_{H^{m-1}_{co,t}} \|\omega\|_{L^\infty_t} + \|\tilde{u} - U\|_{L^\infty_t} \|\omega\|_{H^{m-1}_{co,t}} + \|Z^m \omega\|_{L^2_t} \right).
 \end{aligned}$$

Summarizing the above estimates over  $m$  and choosing suitable constants, it follows

$$\begin{aligned}
 (3.10) \quad \frac{d}{dt} \|\omega\|_{H^m_{co,t}}^2 + \|\omega_\eta\|_{H^m_{co,t}}^2 &\lesssim \|\tilde{f}\|_{H^m_{co,t}}^2 + \|\tilde{f}_\eta\|_{H^{m-1}_{co,t}}^2 + (1 + \lambda_{3,1}^2) \|\omega\|_{H^m_{co,t}}^2 \\
 &\quad + (\lambda_{m,0}^2 + \lambda_{m-1,1}^2 + 1) \|\omega\|_{\mathcal{B}_l^{3,1}}^2,
 \end{aligned}$$

where we have used the inequalities

$$\|\omega, \omega_x\|_{L^\infty(\mathbb{T} \times \mathbb{R}^+)}, \quad \|\omega, \omega_\eta\|_{L^2_{\eta,t}(L^\infty_x)} \leq C \|\omega\|_{\mathcal{B}_l^{3,1}}.$$

And this completes the proof of the lemma.  $\square$

*Remark 3.1.* Similarly to the above proof, one can obtain

$$(3.11) \quad \|\omega\|_{\mathcal{B}_l^{m,1}}^2 \leq C (\|\tilde{f}\|_{H^m_{co,t}}^2 + \|\tilde{f}_\eta\|_{H^{m-1}_{co,t}}^2), \quad 0 \leq m \leq 3.$$

When  $m = 0$ , the term  $\|\tilde{f}_\eta\|_{H^{m-1}_{co,t}}^2$  is not in (3.11). By combining (3.10), (3.11), and using Gronwall's inequality, we get

$$(3.12) \quad \|\omega\|_{\mathcal{B}_l^{m,1}}^2 \leq C (\|\tilde{f}\|_{\mathcal{B}_l^{m,0}}^2 + \|\tilde{f}\|_{\mathcal{B}_l^{m-1,1}}^2) + (1 + \lambda_{m-1,1}^2 + \lambda_{m,0}^2) (\|\tilde{f}\|_{\mathcal{B}_l^{3,0}}^2 + \|\tilde{f}\|_{\mathcal{B}_l^{2,1}}^2).$$

We are now ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* It remains to estimate the higher order normal derivatives.

From the equation given in (3.2), we have

$$(3.13) \quad \begin{aligned} \frac{1}{\bar{\rho}} \omega_{\eta\eta} &= \omega_t + (\tilde{u}\omega)_x + (\eta V\omega)_\eta + (\bar{v}\omega)_\eta - \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta \\ &+ \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}} \omega - \tilde{f}_\eta. \end{aligned}$$

Applying the conormal operator  $Z^m$  to (3.13) gives

$$(3.14) \quad \begin{aligned} \|\omega\|_{\mathcal{B}_t^{m,2}} &\leq \|\bar{\rho}\omega_t\|_{\mathcal{B}_t^{m,0}} + \|\bar{\rho}((\tilde{u}\omega)_x + (\bar{v}\omega)_\eta)\|_{\mathcal{B}_t^{m,0}} + \|\bar{\rho}(\eta V\omega)_\eta\|_{\mathcal{B}_t^{m,0}} + \|2\chi\omega\|_{\mathcal{B}_t^{m,1}} \\ &+ \left\| \bar{\rho} \partial_\eta \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right) \right\|_{\mathcal{B}_t^{m,0}} + \|\bar{\rho}_t \omega\|_{\mathcal{B}_t^{m,0}} + \|\bar{\rho} \partial_\eta \tilde{f}\|_{\mathcal{B}_t^{m,0}}. \end{aligned}$$

We estimate each term on the right-hand side of the above inequality. By using Lemma 2.1, we get

$$\|\bar{\rho}\omega_t\|_{\mathcal{B}_t^{m,0}} + \|(\eta V\omega)_\eta\|_{\mathcal{B}_t^{m,0}} + \|\bar{\rho}_t \omega\|_{\mathcal{B}_t^{m,0}} \lesssim \|\omega\|_{\mathcal{B}_t^{m+1,0}}.$$

Obviously, it holds

$$\|\bar{\rho}((\tilde{u}\omega)_x + (\bar{v}\omega)_\eta)\|_{\mathcal{B}_t^{m,0}} = \|(\bar{\rho}_t + \bar{\rho}V + \bar{\rho}_x \tilde{u})\omega - \bar{\rho} \tilde{u} \omega_x - \bar{\rho} \bar{v} \omega_\eta\|_{\mathcal{B}_t^{m,0}},$$

where

$$\begin{aligned} \|(\bar{\rho}_t + \bar{\rho}V)\omega\|_{\mathcal{B}_t^{m,0}} &\lesssim \|\omega\|_{\mathcal{B}_t^{m,0}}, \\ \|\bar{\rho} \bar{v} \omega_\eta\|_{\mathcal{B}_t^{m,0}} &\lesssim \|\bar{v}\|_{L^\infty} \|\omega_\eta\|_{\mathcal{B}_t^{m,0}} + \|\bar{v}\|_{\mathcal{D}_{co,0}^{m,0}} \|\omega_\eta\|_{L_{\eta,t}^2(L_x^\infty)}, \end{aligned}$$

and

$$\begin{aligned} \|\bar{\rho} \tilde{u} \omega_x\|_{\mathcal{B}_t^{m,0}} &= \|\bar{\rho}(\tilde{u} - U)\omega_x + \bar{\rho}U\omega_x\|_{\mathcal{B}_t^{m,0}} \\ &\lesssim \|\tilde{u} - U\|_{\mathcal{B}_t^{m,0}} \|\omega_x\|_{L^\infty} + (\|\tilde{u} - U\|_{L^\infty} + 1) \|\omega\|_{\mathcal{B}_t^{m+1,0}}. \end{aligned}$$

The term  $\|\bar{\rho}_x \tilde{u} \omega\|_{\mathcal{B}_t^{m,0}}$  can be estimated similarly. Moreover, we have

$$\|2\chi\omega\|_{\mathcal{B}_t^{m,1}} \lesssim \|\chi\|_{L^\infty} \|\omega\|_{\mathcal{B}_t^{m,1}} + \|\chi\|_{\mathcal{D}_0^{m,1}} \|\omega\|_{L_{\eta,t}^2(L_x^\infty)}.$$

And

$$\left\| \bar{\rho} \partial_\eta \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right) \right\|_{\mathcal{B}_t^{m,0}} = \left\| \bar{\rho} \left( \xi_\eta \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} + \xi \omega \right) \right\|_{\mathcal{B}_t^{m,0}},$$

where

$$\begin{aligned} \left\| \bar{\rho} \left( \xi_\eta \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right) \right\|_{\mathcal{B}_t^{m,0}} &\lesssim \|\xi_{1\eta}\|_{\mathcal{B}_t^{m,0}} \|\omega\|_{L_{\eta,t}^2(L_x^\infty)} + \|\xi_{1\eta}\|_{L_{\eta,t}^2(L_{x,t}^\infty)} \|\omega\|_{\mathcal{B}_t^{m,0}} \\ &+ \|\tilde{u} - U\|_{\mathcal{B}_t^{m,1}} \|\omega\|_{L_{\eta,t}^2(L_x^\infty)} + \|\tilde{u}\|_{L_t^\infty} \|\omega\|_{\mathcal{B}_t^{m,1}} \end{aligned}$$

and

$$\begin{aligned} \|\xi\omega\|_{\mathcal{B}_i^{m,0}} &\lesssim \|\xi_1\|_{\mathcal{B}_i^{m,0}}\|\omega\|_{L_{\eta,t}^\infty} + \|\xi_1\|_{L_i^\infty}\|\omega\|_{\mathcal{B}_i^{m,0}} \\ &\quad + \|\tilde{u} - U\|_{\mathcal{B}_i^{m,0}}\|\omega\|_{L^\infty} + (\|\tilde{u}\|_{L^\infty} + 1)\|\omega\|_{\mathcal{B}_i^{m,1}}. \end{aligned}$$

Plugging the above estimates into (3.14) yields

$$\|\omega\|_{\mathcal{B}_i^{m,2}} \lesssim \lambda_{3,1}(\|\omega\|_{\mathcal{B}_i^{m,1}} + \|\omega\|_{\mathcal{B}_i^{m+1,0}}) + \lambda_{m,1}\|\omega\|_{\mathcal{B}_i^{3,1}} + \|\tilde{f}\|_{\mathcal{B}_i^{m,1}}.$$

Next, for any fixed  $n \geq 3$ , applying the differential operator  $Z^m \partial_\eta^{n-2}$  on (3.13) gives

$$\begin{aligned} \|\omega\|_{\mathcal{B}_i^{m,n}} &= \left\| \bar{\rho} \left( \omega_t + (\tilde{u}\omega)_x + (\eta V\omega)_\eta + (\bar{v}\omega)_\eta - \frac{2}{\bar{\rho}} \left( \omega \frac{\partial_\eta^2 \tilde{u}}{\partial_\eta \tilde{u}} \right)_\eta \right. \right. \\ &\quad \left. \left. + \left( \xi \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}} \omega - \tilde{f}_\eta \right) \right\|_{\mathcal{B}_i^{m,n-2}}, \end{aligned}$$

where

$$\begin{aligned} &\left\| \bar{\rho}(\omega_t + (\eta V\omega)_\eta - \frac{\bar{\rho}_t}{\bar{\rho}}\omega) \right\|_{\mathcal{B}_i^{m,n-2}} \\ &\lesssim \|\omega\|_{\mathcal{B}_i^{m+1,n-2}}, \\ &\|\bar{\rho}(\tilde{u}\omega)_x + \bar{\rho}(\bar{v}\omega)_\eta\|_{\mathcal{B}_i^{m,n-2}} \\ &\lesssim \|\omega\|_{\mathcal{B}_i^{m,n-2}} + \|\tilde{u} - U\|_{\mathcal{B}_i^{m,n-2}}\|\omega_x\|_{L^\infty} + (\|\tilde{u} - U\|_{L^\infty} + 1)\|\omega\|_{\mathcal{B}_i^{m+1,n-2}} \\ &\quad + \|\bar{v}\|_{L^\infty}\|\omega_\eta\|_{\mathcal{B}_i^{m,n-2}} + \|\bar{v}\|_{\mathcal{D}_0^{m,n-2}}\|\omega_\eta\|_{L_{\eta,t}^2(L_x^\infty)}, \\ &\|2(\omega\chi)_\eta\|_{\mathcal{B}_i^{m,n-2}} \\ &\lesssim \|\omega\|_{\mathcal{B}_i^{m,n-1}}\|\chi\|_{L^\infty} + \|\chi\|_{\mathcal{D}_0^{m,n-1}}\|\omega\|_{L_{\eta,t}^2(L_x^\infty)}, \end{aligned}$$

and

$$\begin{aligned} \left\| \left( \xi_1 \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta \right\|_{\mathcal{B}_i^{m,n-2}} &= \left\| \xi_{1\eta} \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} + \xi_1 \omega \right\|_{\mathcal{B}_i^{m,n-2}} \\ &\lesssim \|\xi_{1\eta}\|_{\mathcal{B}_i^{m,n-2}}\|\omega\|_{L_{\eta,t}^2(L_x^\infty)} + \|\xi_1\eta\|_{L_{\eta,t}^2(L_x^\infty)}\|\omega\|_{\mathcal{B}_i^{m,n-2}} \\ &\quad + \|\xi_1\|_{\mathcal{B}_i^{m,n-2}}\|\omega\|_{L_{\eta,t}^\infty} + \|\xi_1\|_{L_i^\infty}\|\omega\|_{\mathcal{B}_i^{m,n-2}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\left\| \left( \frac{\tilde{u}\bar{\rho}_x}{\bar{\rho}} \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta \right\|_{\mathcal{B}_i^{m,n-2}} \\ &= \left\| \left( \frac{(\tilde{u} - U)\bar{\rho}_x}{\bar{\rho}} \right)_\eta \int_0^\eta \omega(t, x, \tilde{\eta}) d\tilde{\eta} + \frac{(\tilde{u} - U)\bar{\rho}_x}{\bar{\rho}}\omega + \frac{U\bar{\rho}_x}{\bar{\rho}}\omega \right\|_{\mathcal{B}_i^{m,n-2}} \\ &\lesssim \|\tilde{u} - U\|_{\mathcal{B}_i^{m,n-1}}\|\omega\|_{L_{\eta,t}^2(L_x^\infty)} + \|(\tilde{u} - U)_\eta\|_{L_{\eta,t}^2(L_x^\infty)}\|\omega\|_{\mathcal{B}_i^{m,n-2}} \\ &\quad + \|\tilde{u} - U\|_{\mathcal{B}_i^{m,n-2}}\|\omega\|_{L_{\eta,t}^\infty} + (\|\tilde{u} - U\|_{L_i^\infty} + 1)\|\omega\|_{\mathcal{B}_i^{m,n-2}}. \end{aligned}$$



Thus, we obtain

$$\|\omega\|_{\mathcal{B}_t^{m,n}} \lesssim (\lambda_{3,1} + 1)(\|\omega\|_{\mathcal{B}_t^{m+1,n-2}} + \|\omega\|_{\mathcal{B}_t^{m,n-1}}) + \lambda_{m,n-1}\|\omega\|_{\mathcal{B}_t^{3,1}} + \|\tilde{f}\|_{\mathcal{B}_t^{m,n-1}}.$$

By induction on the index  $n$ , we conclude the estimate (3.5). And this completes the proof of Theorem 3.1.

**4. Iteration scheme and convergence.** Based on the energy estimate (3.5) on the solution to the linearized equations obtained in the previous section, we now study the well-posedness of the nonlinear problem (1.1) by using a suitable linear iteration scheme. From (3.5), there is a loss of regularity in the solutions to the linearized problem (3.1) with respect to both of the background states and initial data. Hence, as in [1], we apply the Nash–Moser–Hörmander iteration scheme. As we explained in section 1, we do not have the divergence free condition, and the far-field state is not uniform. Thus, the shear flow is no longer the special exact solution to the compressible Prandtl equations (1.1) in contrast to the incompressible problem studied in [1]. Thus, to start the Nash–Moser–Hörmander iteration, we need to construct a proper zeroth order approximate solution satisfying the nonlinear compressible Prandtl equations with enough decay in  $\eta$ . The construction will be given in subsection 4.1. Then, in subsection 4.2 we present the Nash–Moser–Hörmander iteration scheme for the problem (1.1). The estimates of the approximate solutions are obtained in subsection 4.3. In subsection 4.4, we conclude the convergence of iteration for the existence and uniqueness of the solution to the nonlinear problem (1.1)–(1.3).

**4.1. The zeroth order approximate solution.** In this subsection, we construct the initial approximate solution in the following three subsections.

**4.1.1. Compatibility conditions and initial data.** Set

$$u = U(t, x) + \bar{u}, \quad v = V(t, x)\eta + \bar{v}.$$

By using Bernoulli’s law (1.4), from (1.1) we know that  $(\bar{u}, \bar{v})$  satisfies

$$(4.1) \quad \begin{cases} \bar{u}_t + U_x \bar{u} + U \bar{u}_x + \bar{u} \bar{u}_x + (\bar{v} + V\eta) \bar{u}_\eta - \frac{1}{\bar{\rho}} \partial_\eta^2 \bar{u} = 0, \\ \partial_\eta(\bar{\rho} \bar{v}) + \partial_x(\bar{\rho} \bar{u}) = 0, \\ \bar{u}(t, x, \eta)|_{t=0} = u_0 - U(0, x), \quad \bar{v}|_{\eta=0} = 0. \end{cases}$$

Denote

$$\bar{u}^j(x, \eta) = \partial_t^j \bar{u}(t, x, \eta)|_{t=0}, \quad \bar{v}^j(x, \eta) = \partial_t^j \bar{v}(t, x, \eta)|_{t=0}.$$

From the compatibility condition of (4.1),  $\{\bar{u}^j, \bar{v}^j\}_{j \leq k_0}$  is in turn given explicitly by  $u_0(x, \eta), U(0, z)$ , and  $V(0, x)$ .

We define the first approximate solution  $(\bar{u}, \bar{v})$  of (4.1) as follows:

$$(4.2) \quad u^a(t, x, \eta) = \sum_{j=0}^{k_0} \frac{t^j}{j!} \bar{u}^j(x, \eta), \quad v^a(t, x, \eta) = -\frac{1}{\bar{\rho}} \int_0^\eta (\bar{\rho} u^a)_x(t, x, \tilde{\eta}) d\tilde{\eta}.$$

From (H3) and (H4) in the main assumptions (H), it follows that

$$(4.3) \quad \max_{0 \leq t \leq T} \|\langle \eta \rangle^{\gamma + \alpha_2} D^\alpha u^a(t, \cdot)\|_{L^2(\mathbb{T} \times \mathbb{R}^+)} \leq CC_0, \quad |\alpha| \leq 2k_0,$$

for a fixed  $T > 0$ , where  $C$  depends on  $\sigma_0$  and the Sobolev norms of  $\partial_t^k(\bar{\rho}, U, V, U_x)$ ,  $k \leq k_0$ . Setting

$$(4.4) \quad u^{a1}(t, x, \eta) = U(t, x) + u^a(t, x, \eta), \quad v^{a1}(t, x, \eta) = V(t, x)\eta + v^a(t, x, \eta),$$

then,  $(u^{a1}, v^{a1})$  is an approximate solution to the problem (1.1) satisfying compatibility conditions up to order  $k_0$  and initial data.

**4.1.2. Improving decay in  $\eta$ .** Note that the approximate solution  $(u^{a1}, v^{a1})$  satisfies

$$(4.5) \quad \lim_{\eta \rightarrow +\infty} u^{a1}(t, x, \eta) = U(t, x),$$

and the divergence constraint

$$(4.6) \quad \partial_x(\bar{\rho}u^{a1}) + \partial_\eta(\bar{\rho}v^{a1}) = -\bar{\rho}_t$$

for all  $t \geq 0$ . However, the error

$$f^{a1} = \left( \partial_t + u^{a1}\partial_x + v^{a1}\partial_\eta - \frac{1}{\bar{\rho}(t, x)}\partial_\eta^2 \right) u^{a1} + P_x$$

does not have enough decay compared with  $\partial_\eta u^{a1}$  as  $\eta \rightarrow +\infty$ . Since this property is essential for the convergence of the Nash–Moser–Hörmander iteration scheme of the nonlinear problem given in the next section, we need to modify the approximate solution  $(u^{a1}, v^{a1})$  as follows.

From (1.1),  $\partial_\eta u$  satisfies

$$\begin{cases} (\partial_t + u\partial_x + v\partial_\eta - \frac{1}{\bar{\rho}(t, x)}\partial_\eta^2)u_\eta + (u_x + v_\eta)u_\eta = 0, \\ \partial_\eta u_\eta|_{\eta=0} = \bar{\rho}P_x. \end{cases}$$

This motivates us to consider the following initial-boundary value problem for a linear degenerate parabolic equation:

$$(4.7) \quad \begin{cases} \phi_t + (u^{a1}\phi)_x + (v^{a1}\phi)_\eta - \frac{1}{\bar{\rho}}\partial_\eta^2\phi = 0, \\ \partial_\eta\phi|_{\eta=0} = (\bar{\rho}P_x)(t, x), \\ \phi|_{t=0} = (\partial_\eta u_0)(x, \eta). \end{cases}$$

Suppose that the solution  $\phi$  of (4.7) is obtained. Define an approximate solution  $(u^{a2}, v^{a2})$  as

$$(4.8) \quad u^{a2} = U(t, x) - \int_\eta^\infty \phi(t, x, \tilde{\eta})d\tilde{\eta}, \quad v^{a2} = V\eta + \frac{1}{\bar{\rho}} \int_0^\eta \left( \bar{\rho} \int_{\tilde{\eta}}^\infty \phi(t, x, s)ds \right)_x d\tilde{\eta}.$$

It is straightforward to verify that the compatibility conditions of (1.1), the far-field condition (4.5), and the divergence constraint (4.6) still hold for  $(u^{a2}, v^{a2})$ . Moreover, it satisfies the equation,

$$(4.9) \quad u_t^{a2} + u^{a2}u_x^{a2} + v^{a2}u_\eta^{a2} + P_x - \frac{1}{\bar{\rho}}\partial_\eta^2 u^{a2} = f^0,$$

where

$$f^0 = - \int_\eta^\infty \left( \left[ U - u^{a1} - \int_{\tilde{\eta}}^\infty \phi(t, x, s)ds \right] \phi \right)_x d\tilde{\eta} - (v^{a2} - v^{a1})\phi,$$

which will be shown to decay faster than  $\partial_\eta u^{a2}$ .

From the boundedness of  $u^a$  given in (4.3) and some elementary weighted energy estimates on the solution to (4.7), we have the following.

PROPOSITION 4.1. *Under the main assumptions (H) on the initial data, there exists a unique solution  $\phi(t, x, \eta)$  to (4.7). Moreover, there is  $T > 0$  such that  $\phi$  satisfies*

$$(4.10) \quad \begin{cases} \max_{0 \leq t \leq T} \|\phi(t)\|_{H_\gamma^{2k_0}} \leq C_1, & \phi(t, x, \eta) \geq \frac{C_2}{(1 + \eta)^{\gamma+2}} \quad \forall (t, x, \eta) \in [0, T] \times \mathbb{T} \times \mathbb{R}^+, \\ \|(1 + \eta)^{\gamma+2+\alpha_2} D^\alpha \phi\|_{L^\infty([0, T] \times \mathbb{T} \times \mathbb{R}^+)} \leq C_3, & |\alpha| \leq k_0, \end{cases}$$

where

$$\|\phi(t)\|_{H_\gamma^{2k_0}} = \sum_{|\alpha| \leq 2k_0} \|(1 + \eta)^{\gamma+\alpha_2} D^\alpha \phi(t)\|_{L^2(\mathbb{T} \times \mathbb{R}^+)}$$

with  $D^\alpha = \partial_{t,x}^{\alpha_1} \partial_\eta^{\alpha_2}$ ,  $\alpha = (\alpha_1, \alpha_2)$ .

*Proof.* The proof is divided into two steps.

(1) Applying the operator  $D^\alpha = \partial_{t,x}^{\alpha_1} \partial_\eta^{\alpha_2}$  to (4.7), multiplying the resulting equation by  $(1 + \eta)^{2\gamma+2\alpha_2} D^\alpha \phi$ , and integrating it over  $\mathbb{T} \times \mathbb{R}^+$ , we obtain

$$(4.11) \quad \frac{1}{2} \frac{d}{dt} \|(1 + \eta)^{\gamma+\alpha_2} D^\alpha \phi(t)\|_{L^2}^2 + \left\| \frac{1}{\sqrt{\rho}} (1 + \eta)^{\gamma+\alpha_2} D^\alpha \phi_\eta(t) \right\|_{L^2}^2 = \sum_{i=1}^7 I_i,$$

where

$$I_1 = \int_{\mathbb{T} \times \mathbb{R}^+} (u_x^{a1} + v_\eta^{a1})(1 + \eta)^{2\gamma+2\alpha_2} (D^\alpha \phi)^2 dx d\eta,$$

$$I_2 = \int_{\mathbb{T} \times \mathbb{R}^+} (u^{a1} D^\alpha \phi_x + v^{a1} D^\alpha \phi_\eta)(1 + \eta)^{2\gamma+2\alpha_2} D^\alpha \phi dx d\eta,$$

$$I_3 = 2(\gamma + \alpha_2) \int_{\mathbb{T} \times \mathbb{R}^+} \frac{1}{\rho} (1 + \eta)^{2\gamma+2\alpha_2-1} (D^\alpha \phi_\eta)(D^\alpha \phi) dx d\eta,$$

$$I_4 = \sum_{0 < \beta \leq \alpha} C_\alpha^\beta \int_{\mathbb{T} \times \mathbb{R}^+} (1 + \eta)^{2\gamma+2\alpha_2} D^\beta (u_x^{a1} + v_\eta^{a1})(D^{\alpha-\beta} \phi)(D^\alpha \phi) dx d\eta,$$

$$I_5 = \sum_{0 < \beta \leq \alpha} C_\alpha^\beta \int_{\mathbb{T} \times \mathbb{R}^+} (1 + \eta)^{2\gamma+2\alpha_2} (D^\beta u^{a1} D^{\alpha-\beta} \phi_x + D^\beta v^{a1} D^{\alpha-\beta} \phi_\eta) D^\alpha \phi dx d\eta,$$

$$I_6 = \sum_{0 < \beta \leq \alpha, \beta_2=0} C_\alpha^\beta \int_{\mathbb{T} \times \mathbb{R}^+} (1 + \eta)^{2\gamma+2\alpha_2} \left( D^\beta \frac{1}{\rho} \right) (\partial_\eta^2 D^{\alpha-\beta} \phi)(D^\alpha \phi) dx d\eta,$$

and

$$I_7 = \int_{\mathbb{T}} \frac{1}{\rho} (\partial_\eta D^\alpha \phi D^\alpha \phi)|_{\eta=0} dx.$$

It is straightforward to show

$$|I_1| \leq \|u_x^{a1}, v_\eta^{a1}\|_{L^\infty} \|(1 + \eta)^{\gamma+\alpha_2} D^\alpha \phi\|_{L^2}^2,$$

$$\begin{aligned} |I_2| &= \left| \int_{\mathbb{T} \times \mathbb{R}^+} (u^{a1} D^\alpha \phi_x + v^{a1} D^\alpha \phi_\eta)(1 + \eta)^{2\gamma+2\alpha_2} D^\alpha \phi dx d\eta \right| \\ &\lesssim (\|u_x^{a1}\|_{L^\infty} + \|v_\eta^{a1}\|_{L^\infty} + \|v^{a1}/(1 + \eta)\|_{L^\infty}) \|(1 + \eta)^{\gamma+\alpha_2} D^\alpha \phi\|_{L^2}^2 \end{aligned}$$

by integration by parts, and using  $v^{a_1}|_{\eta=0} = 0$ , and

$$\begin{aligned} |I_3| &= 2(\gamma + \alpha_2) \left| \int_{\mathbb{T} \times \mathbb{R}^+} \frac{1}{\bar{\rho}} D^\alpha \phi_\eta (1 + \eta)^{2\gamma + 2\alpha_2 - 1} D^\alpha \phi dx d\eta \right| \\ &\leq \frac{1}{8} \left\| \frac{1}{\sqrt{\bar{\rho}}} (1 + \eta)^{\gamma + \alpha_2} D^\alpha \phi_\eta \right\|_{L^2}^2 + C \left\| \frac{1}{\bar{\rho}} \right\|_{L^\infty} \|(1 + \eta)^{\gamma + \alpha_2} D^\alpha \phi\|_{L^2}^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |I_4| &= \left| \sum_{0 < \beta \leq \alpha} C_\alpha^\beta \int_{\mathbb{T} \times \mathbb{R}^+} (1 + \eta)^{\beta_2} D^\beta \left( \frac{-\bar{\rho}_t - \bar{\rho}_x u^{a_1}}{\bar{\rho}} \right) \right. \\ &\quad \left. (1 + \eta)^{\gamma + \alpha_2 - \beta_2} D^{\alpha - \beta} \phi (1 + \eta)^{\gamma + \alpha_2} D^\alpha \phi dx d\eta \right| \\ &\leq C \left\| \sum_\beta D^\beta \left( \frac{\bar{\rho}_t + \bar{\rho}_x U}{\bar{\rho}} \right) \right\|_{L^\infty} \|\phi\|_{H_\gamma^{2k_0}}^2 + |\tilde{I}_4| \end{aligned}$$

by noting that  $\beta_2 = 0$  for the operator  $D^\beta$  acting on  $\frac{\bar{\rho}_t + \bar{\rho}_x U}{\bar{\rho}}$ . Hence,

$$|\tilde{I}_4| \leq C \|(1 + \eta)^{\beta_2} \left( D^\beta \frac{\bar{\rho}_x (u^{a_1} - U)}{\bar{\rho}} \right)\|_{L^\infty} \|\phi\|_{H_\gamma^{2k_0}}^2 \quad \text{for } |\beta| \leq k_0,$$

and

$$\begin{aligned} |\tilde{I}_4| &\leq C \left\| (1 + \eta)^{\beta_2} \left( D^\beta \frac{\bar{\rho}_x (u^{a_1} - U)}{\bar{\rho}} \right) \right\|_{L^2} \|(1 + \eta)^{\gamma + \alpha_2 - \beta_2} D^{\alpha - \beta} \phi\|_{L^\infty} \\ &\quad \times \|(1 + \eta)^{\gamma + \alpha_2} D^\alpha \phi\|_{L^2} \\ &\leq C \left\| (1 + \eta)^{\beta_2} \left( D^\beta \frac{\bar{\rho}_x (u^{a_1} - U)}{\bar{\rho}} \right) \right\|_{L^2} \|\phi\|_{H_\gamma^{2k_0}}^2, \quad \text{for } |\beta| > k_0 \end{aligned}$$

by using the weighted Sobolev embedding. Similarly, one has

$$\begin{aligned} |I_5| &= \left| \sum_{0 < \beta \leq \alpha} \int_{\mathbb{T} \times \mathbb{R}^+} C_\alpha^\beta (D^\beta u^{a_1} D^{\alpha - \beta} \phi_x + D^\beta v^{a_1} D^{\alpha - \beta} \phi_\eta) \right. \\ &\quad \left. \times (1 + \eta)^{2\gamma + 2\alpha_2} D^\alpha \phi dx d\eta \right| \\ &\leq |I_5^1| + |I_5^2|, \end{aligned}$$

where

$$|I_5^1| \leq \begin{cases} C(\|D^\beta (u^{a_1} - U)(1 + \eta)^{\beta_2}\|_{L^\infty} + \|D^\beta U\|_{L^\infty}) \|\phi\|_{H_\gamma^{2k_0}}^2 & \text{for } 1 \leq |\beta| \leq k_0, \\ C(\|D^\beta (u^{a_1} - U)(1 + \eta)^{\beta_2}\|_{L^2} + \|D^\beta U\|_{L^\infty}) \|\phi\|_{H_\gamma^{2k_0}}^2 & \text{for } |\beta| > k_0, \end{cases}$$

and

$$\begin{aligned}
 |I_5^2| &\leq \begin{cases} C\|D^\beta v^{a1}(1+\eta)^{\beta_2-1}\|_{L^\infty}\|\phi\|_{H_\gamma^{2k_0}}^2 & \text{for } 1 \leq |\beta| \leq k_0, \\ C\|D^\beta v^{a1}(1+\eta)^{\beta_2-1}\|_{L^2}\|\phi\|_{H_\gamma^{2k_0}}^2 & \text{for } \beta_2 \neq 0, |\beta| > k_0, \end{cases} \\
 |I_5^2| &\leq C\|D^\beta v^{a1}(1+\eta)^{-1}\|_{L_\eta^\infty(L_x^2)}\|D^{\alpha-\beta}\phi_\eta(1+\eta)^{\gamma+\alpha_2+1}\|_{L_\eta^2(L_x^\infty)} \\
 &\quad \times \|D^\alpha\phi(1+\eta)^{\gamma+\alpha_2}\|_{L^2} \\
 &\leq C\|D^\beta v^{a1}(1+\eta)^{-1}\|_{L_\eta^\infty(L_x^2)}\|\phi\|_{H_\gamma^{2k_0}}^2 \quad \text{for } \beta_2 = 0, |\beta| > k_0.
 \end{aligned}$$

For the term  $I_6$ , by integration by parts, we have

$$\begin{aligned}
 |I_6| &= \sum_{0 < \beta \leq \alpha, \beta_2 = 0} \left( \left| \int_{\mathbb{T} \times \mathbb{R}^+} C_\alpha^\beta (2\gamma + 2\alpha_2)(1+\eta)^{2\gamma+2\alpha_2-1} \left( D^\beta \frac{1}{\bar{\rho}} \right) \right. \right. \\
 &\quad \left. \left( \partial_\eta D^{\alpha-\beta}\phi \right) (D^\alpha\phi) dx d\eta + \int_{\mathbb{T} \times \mathbb{R}^+} C_\alpha^\beta \right. \\
 &\quad \left. \times (1+\eta)^{2\gamma+2\alpha_2} \left( D^\beta \frac{1}{\bar{\rho}} \right) (\partial_\eta D^{\alpha-\beta}\phi) (D^\alpha\phi_\eta) dx d\eta \right. \\
 &\quad \left. - \int_{\mathbb{T}} C_\alpha^\beta \left( D^\beta \frac{1}{\bar{\rho}} \right) (\partial_\eta D^{\alpha-\beta}\phi) (D^\alpha\phi) \Big|_{\eta=0} dx \right) \\
 &\leq \sum_{0 < \beta \leq \alpha, \beta_2 = 0} \left( \left| \int_{\mathbb{T}} C_\alpha^\beta \left( D^\beta \frac{1}{\bar{\rho}} \right) (\partial_\eta D^{\alpha-\beta}\phi) (D^\alpha\phi) \Big|_{\eta=0} dx \right| \right. \\
 &\quad \left. + C\|(1+\eta)^{\gamma+\alpha_2} D^{\alpha-\beta}\phi_\eta\|_{L^2}^2 \right) \\
 &\quad + \frac{1}{8} \left\| \frac{1}{\sqrt{\bar{\rho}}} (1+\eta)^{\gamma+\alpha_2} D^\alpha\phi_\eta \right\|_{L^2}^2 + \|\phi\|_{H_\gamma^{2k_0}}^2.
 \end{aligned}$$

It remains to handle the boundary integration terms on the right-hand side of the above estimate and  $I_7$ . For illustration, we only estimate  $I_7$ .

First, noticing that

$$\partial_\eta\phi|_{\eta=0} = \bar{\rho}P_x,$$

and applying the operator  $\partial_\eta$  on (4.7), we obtain

$$\frac{1}{\bar{\rho}}\partial_\eta^3\phi = (\phi_\eta)_t + (u_x^{a1}\phi)_\eta + (u^{a1}\phi_x)_\eta + (v^{a1}\phi)_{\eta\eta}.$$

Taking this equation on the boundary  $\{\eta = 0\}$  and using the boundary condition, we get

$$\begin{aligned}
 \frac{1}{\bar{\rho}}\partial_\eta^3\phi|_{\eta=0} &= (\bar{\rho}P_x)_t + u_x^{a1}\bar{\rho}P_x + u_{x\eta}^{a1}\phi|_{\eta=0} + u^{a1}(\bar{\rho}P_x)_x \\
 &\quad + u_\eta^{a1}\phi_x|_{\eta=0} + 2v_\eta^{a1}\bar{\rho}P_x + v_{\eta\eta}^{a1}\phi|_{\eta=0}.
 \end{aligned}$$

By induction, for positive integer  $k$ , we have

$$\begin{aligned} \frac{1}{\rho} \partial_\eta^{2k+1} \phi|_{\eta=0} &= (\partial_\eta^{2k-1} \phi)_t + \partial_\eta^{2k-1} (u_x^{a1} \phi) + \partial_\eta^{2k-1} (u^{a1} \phi_x) + \partial_\eta^{2k} (v^{a1} \phi) \\ &= [F(D_{|\alpha| \leq 2k-2}^\alpha u^{a1}, D_{|\beta| \leq 2k-2}^\beta v^{a1}, D_{|\gamma| \leq 2k-3}^\gamma \phi, D^\pi(\bar{\rho}, P_x))]_t \\ &\quad + \sum_{i=1}^{2k-1} C_{2k-1}^i \partial_\eta^i u_x^{a1} \partial_\eta^{2k-1-i} \phi \\ &\quad + \sum_{j=1}^{2k-1} C_{2k-1}^j \partial_\eta^j u^{a1} \partial_\eta^{2k-1-j} \phi_x + \sum_{s=1}^{2k} C_{2k}^s \partial_\eta^s v^{a1} \partial_\eta^{2k-s} \phi \\ &= G(D_{|\alpha| \leq 2k}^\alpha u^{a1}, D_{|\beta| \leq 2k}^\beta v^{a1}, D_{|\gamma| \leq 2k-1}^\gamma \phi, D^\pi(\bar{\rho}, P_x)), \end{aligned}$$

where  $F, G$  are polynomial functions. Hence, the normal derivative of  $\phi$  can be reduced by two orders using the boundary condition and (4.7). Therefore, we can use the trace estimate to control the boundary integral.

Thus, by summarizing the above estimates, and taking summation over  $|\alpha| \leq 2k_0$  for (4.11), it follows

$$\frac{d}{dt} \|\phi(t)\|_{H_\gamma^{2k_0}}^2 + \sum_{|\alpha| \leq 2k_0} \left\| \frac{1}{\sqrt{\rho}} (1 + \eta)^{\gamma + \alpha_2} D^\alpha \phi_\eta(t) \right\|_{L^2}^2 \leq C \|\phi(t)\|_{H_\gamma^{2k_0}}^2$$

which implies the first boundedness estimate given in (4.10) by using the Gronwall inequality.

- (2) Next, we apply the maximal principle to prove the second estimate given in (4.10).

From (4.7),  $y(t, x, \eta) \triangleq (1 + \eta)^{\gamma+2} \phi$  satisfies the following degenerate parabolic equation,

$$\begin{aligned} y_t + \left( u_x^{a1} + v_\eta^{a1} - \frac{v^{a1}(2 + \gamma)}{1 + \eta} - \frac{(\gamma + 2)(\gamma + 3)}{\bar{\rho}(1 + \eta)^2} \right) y \\ + u^{a1} y_x + \left( v^{a1} + \frac{2(\gamma + 2)}{\bar{\rho}(1 + \eta)} \right) y_\eta - \frac{1}{\rho} \partial_\eta^2 y = 0. \end{aligned}$$

By the maximal principle (see also Lemma E.2 in [9]), we have

$$\min_{\mathbb{T} \times \mathbb{R}^+} y(t) \geq (1 - \lambda t e^{\lambda t}) k(t)$$

with

$$k(t) = \min \left\{ \min_{\mathbb{T} \times \mathbb{R}^+} y|_{t=0}, \min_{[0,t] \times \mathbb{T}} y|_{\eta=0} \right\}$$

for a fixed  $\lambda \geq \left\| \left( u_x^{a1} + v_\eta^{a1} - \frac{v^{a1}(2+\gamma)}{1+\eta} - \frac{(\gamma+2)(\gamma+3)}{\bar{\rho}(1+\eta)^2} \right) \right\|_{L^\infty}$ .

It follows from assumption (H2) on the initial data that

$$\min_{\mathbb{T} \times \mathbb{R}^+} y(0) \geq \sigma_0 > 0.$$

It suffices to derive the lower bound on  $\min_{[0,t] \times \mathbb{T}} y|_{\eta=0}$ . Notice that  $y|_{\eta=0} = \phi|_{\eta=0}$ , the first boundedness estimate of (4.10), and the Sobolev inequality give

$$\|\phi_t|_{\eta=0}\|_{L^\infty} \leq C.$$

Consequently,

$$\phi(t)|_{\eta=0} \geq \phi(0, x, \eta)|_{\eta=0} - Ct \geq \sigma_0 - Ct.$$

Thus, we have the lower bound given in the second estimate in (4.10) provided that  $t$  is suitably small. The third estimate in (4.10) can also be proved by the maximal principle similarly (also refer to Lemma E.1 in [9]). Then the proof of this proposition is completed.  $\square$

**4.1.3. Boundary condition.** It is noted that the approximate solution  $u^{a2}$  does not satisfy the original boundary condition, that is,  $u^{a2}|_{\eta=0} \neq 0$ . For this, set

$$\zeta(t, x) \triangleq u^{a2}|_{\eta=0} = U(t, x) - \int_0^\infty \phi(t, x, \eta) d\eta.$$

$\zeta(t, x)$  is uniformly continuous and bounded due to (4.10). By the compatibility condition of the initial data, we have  $\zeta(0, x) = 0$ . Consequently,  $|\zeta(t, x)| \leq \varepsilon_0$ ,  $t \in [0, t_0]$ , with  $\varepsilon_0 \rightarrow 0$  as  $t_0$  tends to zero.

In addition, there exists a smooth monotone decreasing function  $\psi(\eta) \subseteq [0, 1]$ ,  $\eta \geq 0$  such that  $\text{supp} \psi \subseteq [0, 1]$ ,  $\psi(0) = 1$ , and  $|\psi'(\eta)| < C$ . Note that there exists a positive constant  $a_0$  such that  $\phi(t, x, \eta) > a_0$ ,  $\eta \in [0, 1]$ .

Now, define

$$(4.12) \quad u^{a3} = u^{a2} - \zeta(t, x)\psi(\eta), \quad v^{a3} = -\frac{1}{\bar{\rho}} \int_0^\eta (\bar{\rho}(u^{a2}(t, x, \tilde{\eta}) - U(x, t) - \zeta(t, x)\psi(\tilde{\eta}))_x d\tilde{\eta}.$$

It is direct to check that  $u^{a3}(t, x, \eta)|_{\eta=0} = 0$ , and

$$(4.13) \quad u_\eta^{a3}(t, x, \eta) = \phi(t, x, \eta) - \zeta(t, x)\psi'(\eta) > \frac{\phi}{2} > 0,$$

provided that  $t \in [0, t_0]$  with  $t_0$  being suitably small. And the profile  $(u^{a3}, v^{a3})$  satisfies

$$(4.14) \quad \begin{cases} u_t^{a3} + u^{a3}u_x^{a3} + v^{a3}u_\eta^{a3} + P_x - \frac{1}{\bar{\rho}}\partial_\eta^2 u^{a3} = f^a, \\ \partial_\eta(\bar{\rho}v^{a3}) + \partial_x(\bar{\rho}u^{a3}) = -\bar{\rho}_t, \\ u^{a3}(0, x, \eta) = u_0(x, \eta) \end{cases}$$

with  $f^a = f^0 - \bar{f}^0$ , where

$$\begin{aligned} \bar{f}^0 = & \zeta_t \psi + \zeta \psi u_x^{a2} + u^{a2} \zeta_x \psi - \zeta \zeta_x \psi^2 + v^{a2} \zeta \psi' \\ & - \frac{u_\eta^{a2}(\bar{\rho}\zeta)_x}{\bar{\rho}} \int_0^\eta \psi(\tilde{\eta}) d\tilde{\eta} + \frac{\zeta \psi'(\bar{\rho}\zeta)_x}{\bar{\rho}} \int_0^\eta \psi(\tilde{\eta}) d\tilde{\eta} - \frac{\psi'' \zeta}{\bar{\rho}}. \end{aligned}$$

*Remark 4.1.* The approach of constructing the zeroth approximate solution to (1.1) introduced above can also be applied to the incompressible Prandtl equations.

**4.2. The Nash–Moser–Hömander iteration scheme.** We now construct the approximate solution sequence of (1.1) by using the Nash–Moser–Hömander iteration scheme. The procedure mainly follows the one given in [1]. Thus, we will only present the main steps.

Denote the linearized operator  $\mathcal{P}'$  around  $(\hat{\omega}, \hat{q})$  of (1.1) by

$$\mathcal{P}'_{(\hat{\omega}, \hat{q})}(\omega, q) = \partial_t \omega + \hat{\omega} \omega_x + \hat{q} \omega_\eta + \omega \hat{\omega}_x + q \hat{\omega}_\eta - \frac{1}{\bar{\rho}} \partial_\eta^2 \omega.$$

Suppose that the approximate solutions  $(u^k, v^k)$  of (1.1) have been constructed for all  $k \leq n$ , with  $u^0 = u^{a3}$  and  $v^0 = v^{a3}$  being defined in subsection 4.1.3; we construct the  $(n+1)$ th approximate solution  $(u^{n+1}, v^{n+1})$  as follows:

$$(4.15) \quad u^{n+1} = u^n + \delta u^n = u^{a3} + \tilde{u}^n + \delta u^n, \quad v^{n+1} = v^n + \delta v^n = v^{a3} + \tilde{v}^n + \delta v^n,$$

where the increment  $(\delta u^n, \delta v^n)$  is the solution to the following initial-boundary value problem

$$(4.16) \quad \begin{cases} \mathcal{P}'_{(u_{\theta_n}^n, v_{\theta_n}^n)}(\delta u^n, \delta v^n) = f^n, \\ \partial_\eta(\bar{\rho}(t, x) \delta v^n) + \partial_x(\bar{\rho}(t, x) \delta u^n) = 0, \\ \delta u^n|_{\eta=0} = \delta v^n|_{\eta=0} = 0, \quad \lim_{\eta \rightarrow +\infty} \delta u^n = 0, \\ \delta u^n|_{t=0} = 0. \end{cases}$$

Here,  $u_{\theta_n}^n = u^{a3} + S_{\theta_n} \tilde{u}^n$  and  $v_{\theta_n}^n = v^{a3} + S_{\theta_n} \tilde{v}^n$  with  $\theta_n = \sqrt{\theta_0^2 + n}$  for any  $n \geq 1$  and a large fixed constant  $\theta_0$ . The smoothing operator  $S_\theta$  is defined by

$$(S_\theta f)(t, x, \eta) = \iiint j_\theta(\tau) j_\theta(\xi) j_\theta(\mu) \tilde{f}(t - \tau + \theta^{-1}, x - \xi, \eta - \mu + \theta^{-1}) d\tau d\xi d\mu$$

for a function  $f$  defined on  $\Omega = [0, +\infty[ \times \mathbb{T}_x \times \mathbb{R}_\eta^+$  with  $\tilde{f}$  being the zero extension of  $f$  to  $\mathbb{R}^3$ , and the mollifier  $j_\theta(\tau) = \theta j(\theta\tau)$  with  $j \in C_0^\infty(\mathbb{R})$  being a nonnegative function satisfying  $\text{Supp } j \subseteq [-1, 1]$  and  $\|j\|_{L^1} = 1$ .

In order to show that the approximate solution  $(u^n, v^n)$  converges to the solution of the nonlinear problem (1.1), we need to define the source term  $f^n$  properly for the problem (4.16).

To do this, denoting the nonlinear operator on the left-hand side of (1.1) by  $\mathcal{P}(\omega, q)$ , obviously, the following identity holds:

$$(4.17) \quad \mathcal{P}(u^{n+1}, v^{n+1}) - \mathcal{P}(u^n, v^n) = \mathcal{P}'_{(u_{\theta_n}^n, v_{\theta_n}^n)}(\delta u^n, \delta v^n) + e_n,$$

where

$$e_n = e_n^1 + e_n^2,$$

with  $e_n^1$  being the error term from the Newton iteration,

$$(4.18) \quad \begin{aligned} e_n^1 &= \mathcal{P}(u^n + \delta u^n, v^n + \delta v^n) - \mathcal{P}(u^n, v^n) - \mathcal{P}'_{(u^n, v^n)}(\delta u^n, \delta v^n) \\ &= \delta u^n \partial_x(\delta u^n) + \delta v^n \partial_\eta(\delta u^n), \end{aligned}$$

and  $e_n^2$  being the error from mollifying the coefficients,

$$(4.19) \quad \begin{aligned} e_n^2 &= \mathcal{P}'_{(u^n, v^n)}(\delta u^n, \delta v^n) - \mathcal{P}'_{(u_{\theta_n}^n, v_{\theta_n}^n)}(\delta u^n, \delta v^n) \\ &= ((1 - S_{\theta_n})(u^n - u^{a3})) \partial_x(\delta u^n) + \delta u^n \partial_x((1 - S_{\theta_n})(u^n - u^{a3})) \\ &\quad + ((1 - S_{\theta_n})(v^n - v^{a3})) \partial_\eta \delta u^n + \delta v^n \partial_\eta((1 - S_{\theta_n})(u^n - u^{a3})). \end{aligned}$$



Taking the summation of (4.17) over all  $n \in \mathbb{N}$  leads to

$$(4.20) \quad \mathcal{P}(u^{n+1}, v^{n+1}) = \sum_{j=0}^n (\mathcal{P}'_{(u_{\theta_j}, v_{\theta_j})}(u^j, v^j) + e_j) + f^a$$

with  $f^a = \mathcal{P}(u^{a3}, v^{a3})$ .

It is obvious that if the approximate solution  $(u^n, v^n)$  converges to the solution to (1.1), then the right-hand side of (4.20) must converge to zero as  $n$  tends to  $+\infty$ . In this way, it is convenient to require that  $(\delta u^n, \delta v^n)$  ( $n \geq 0$ ) satisfies the equation

$$\mathcal{P}'_{(u_{\theta_n}, v_{\theta_n})}(\delta u^n, \delta v^n) = f^n,$$

where  $f^n$  is defined by

$$(4.21) \quad \sum_{j=0}^n f^j = -S_{\theta_n} \left( \sum_{j=0}^{n-1} e_j \right) - S_{\theta_n} f^a$$

inductively, that is,

$$(4.22) \quad \begin{cases} f^0 = -S_{\theta_0} f^a, & f^1 = (S_{\theta_0} - S_{\theta_1}) f^a + S_{\theta_0} f^a, \\ f^n = (S_{\theta_{n-1}} - S_{\theta_n}) \left( \sum_{j=0}^{n-2} e_j \right) - S_{\theta_n} e_{n-1} + (S_{\theta_{n-1}} - S_{\theta_n}) f^a & \forall n \geq 2 \end{cases}$$

with  $f^a$  given in (4.14).

We now give some properties of the smoothing operator in the following lemma, which also can be found in section 4.1 of [1].

LEMMA 4.1. *The smoothing operator  $\{S_\theta\}_{\theta>0} : \mathcal{A}_l^0(\Omega) \rightarrow \cap_{s \geq 0} \mathcal{A}_l^s(\Omega)$  satisfies the following estimates:*

$$(4.23) \quad \begin{cases} \|S_\theta v\|_{\mathcal{A}_l^s} \leq C_j \theta^{(s-\alpha)_+} \|v\|_{\mathcal{A}_l^\alpha} & \text{for all } s, \alpha \geq 0, \\ \|(1 - S_\theta)v\|_{\mathcal{A}_l^s} \leq C_j \theta^{(s-\alpha)} \|v\|_{\mathcal{A}_l^\alpha} & \text{for all } 0 \leq s \leq \alpha, \end{cases}$$

and

$$(4.24) \quad \|(S_{\theta_n} - S_{\theta_{n-1}})v\|_{\mathcal{A}_l^s} \leq C_j \Delta\theta_n^{s-\alpha} \|v\|_{\mathcal{A}_l^\alpha} \text{ for all } s, \alpha \geq 0,$$

where  $\Delta\theta_n = \theta_{n+1} - \theta_n$ , and the constant  $C_j$  depends only on the mollifier function  $j \in C_0^\infty(\mathbb{R})$ .

**4.3. Estimates of the approximate solutions.** To study the solutions  $(\delta u^n, \delta v^n)$  to the problem (4.16) with  $f^n$  given in (4.22), as in section 3, set

$$(4.25) \quad \omega^n = \partial_\eta \left( \frac{\bar{\rho} \delta u^n}{\partial_\eta u_{\theta_n}^n} \right).$$

Then  $\omega^n$  satisfies

$$(4.26) \quad \begin{cases} \partial_t \omega^n + \partial_x (u_{\theta_n}^n \omega^n) + \partial_\eta (v_{\theta_n}^n \omega^n) - \frac{2}{\bar{\rho}} (\omega^n \chi^n)_\eta + \left( \xi^n \int_0^\eta \omega^n(t, x, \tilde{\eta}) d\tilde{\eta} \right)_\eta \\ \quad - \frac{\bar{\rho}_t}{\bar{\rho}} \omega^n - \frac{1}{\bar{\rho}} \omega_{\eta\eta}^n = \tilde{f}_\eta^n, \\ \frac{1}{\bar{\rho}} (\omega_\eta^n + 2\omega^n \chi^n)|_{\eta=0} = -\tilde{f}^n|_{\eta=0}, \\ \omega^n|_{t=0} = 0, \end{cases}$$

where

$$\chi^n = \frac{\partial_\eta^2 u_{\theta_n}^n}{\partial_\eta u_{\theta_n}^n}, \quad \xi^n = \frac{(\partial_t + u_{\theta_n}^n \partial_x + v_{\theta_n}^n \partial_\eta - \frac{1}{\bar{\rho}} \partial_\eta^2) \partial_\eta u_{\theta_n}^n}{\partial_\eta u_{\theta_n}^n} - \frac{u_{\theta_n}^n \bar{\rho}_x}{\bar{\rho}} \triangleq \xi_1^n - \xi_2^n,$$

and

$$(4.27) \quad \tilde{f}^n = \frac{\bar{\rho} f^n}{\partial_\eta u_{\theta_n}^n}.$$

Similarly to (3.3)–(3.4), we define

$$\lambda_{k_1, k_2} = \|u_{\theta_n}^n - u^{a3}\|_{\mathcal{B}_t^{k_1, k_2}} + \|Z^{k_1} \partial_\eta^{k_2} v_{\theta_n}^n\|_{L_\eta^\infty(L_{t,x}^2)} + \|Z^{k_1} \partial_\eta^{k_2} \chi^n\|_{L_\eta^\infty(L_{t,x}^2)} + \|\xi_1^n\|_{\mathcal{B}_t^{k_1, k_2}}$$

and

$$\lambda_k^n = \sum_{k_1 + [(k_2 + 1)/2] \leq k} \lambda_{k_1, k_2}^n.$$

Applying Theorem 3.1 to the linearized problem (4.26), we have the following.

**THEOREM 4.1.** *Suppose the known functions  $(\bar{\rho}, U, V)(t, x)$  satisfy the same assumptions as in Theorem 1.1, and the main assumptions (H) are satisfied. Then for any fixed  $l > 1/2$ , the following estimate holds for the solution of the problem (4.26),*

$$(4.28) \quad \|\omega^n\|_{\mathcal{A}_t^k} \leq C_1(\lambda_4^n) \|\tilde{f}^n\|_{\mathcal{A}_t^k} + C_2(\lambda_4^n) \lambda_k^n \|\tilde{f}^n\|_{\mathcal{A}_t^3}.$$

Similarly to the Lemma 5.3 in [1], we also have

$$\left\| \frac{f^a}{\partial_\eta u^{a3}} \right\|_{\mathcal{A}_t^{k_0}([0, T] \times \mathbb{T} \times \mathbb{R}^+)} \leq C\varepsilon,$$

because of the construction of  $(u^{a3}, v^{a3})$  and the estimates in Proposition 4.1, where  $\varepsilon$  comes from the smallness of the integral interval of time. Then, as in [1], by studying estimates of  $\tilde{f}^n$  and using an induction argument, we have the following.

**THEOREM 4.2.** *Under the same assumptions as those in Theorem 4.1, there exists a positive constant  $C_0$  such that*

$$(4.29) \quad \|\omega^n\|_{\mathcal{A}_t^k} \leq C_0 \varepsilon \theta_n^{\max\{3 - \bar{k}, k - \bar{k}\}} \Delta \theta_n$$

holds for all  $n \geq 0, 0 \leq k \leq k_0$ , and  $\bar{k} \geq 6$ ; here  $\theta_n = \sqrt{\theta_0^2 + n}$  and  $\Delta \theta_n = \theta_{n+1} - \theta_n$ .

Using the transformation (4.25), we can obtain the following.

**COROLLARY 4.1.** *Under the same assumptions as those in Theorem 4.2, the following estimates hold:*

$$(4.30) \quad \|\delta u^n\|_{\mathcal{A}_t^k} \leq C\varepsilon \theta_n^{\max\{3 - \bar{k}, k - \bar{k}\}}, \quad 0 \leq k \leq k_0,$$

and

$$(4.31) \quad \|\delta v^n\|_{\mathcal{D}_0^k} \leq C_1 \varepsilon \theta_n^{\max\{3 - \bar{k}, k + 1 - \bar{k}\}}, \quad 0 \leq k \leq k_0 - 1.$$

**4.4. Existence of the solution to the nonlinear problem.** To show the existence of the solution to the nonlinear boundary layer equations (1.1), we need to show the convergence of the iteration scheme (4.15)–(4.16). From this iteration, we know that the approximate solutions  $(u^{n+1}, v^{n+1})$  solve the following problem:

$$(4.32) \quad \begin{cases} \mathcal{P}(u^{n+1}, v^{n+1}) = (1 - S_{\theta_n}) \sum_{j=0}^n e_j + S_{\theta_n} e_n + (1 - S_{\theta_n}) f^a, \\ \partial_x(\bar{\rho} u^{n+1}) + \partial_\eta(\bar{\rho} v^{n+1}) = -\bar{\rho}_t, \\ u^{n+1}|_{\eta=0} = v^{n+1}|_{\eta=0} = 0, \quad \lim_{\eta \rightarrow +\infty} u^{n+1} = U(t, x), \\ u^{n+1}|_{t=0} = u_0(x, \eta). \end{cases}$$

From the estimates given in Corollary 4.1, we know that there exist functions  $u \in \mathcal{A}_l^{\bar{k}-2}$  and  $v \in \mathcal{D}_0^{\bar{k}-3}$ , such that  $u^n$  converges to  $u$  in  $\mathcal{A}_l^{\bar{k}-2}$  and  $v^n$  converges to  $v$  in  $\mathcal{D}_0^{\bar{k}-3}$ . In order to show the function pair  $(u, v)$  is indeed a solution to the system (1.1), it suffices to show that the right-hand side in (4.32)<sub>1</sub> converges to zero as  $n$  tends to  $+\infty$ . First, by using Lemma 4.1,

$$\left\| (1 - S_{\theta_n}) \left( f^a + \sum_{j=0}^n e_j \right) \right\|_{\mathcal{A}_l^{\bar{k}}} \leq C \theta_n^{-1} \left( \|f^a\|_{\mathcal{A}_l^{k+1}} + \left\| \sum_{j=0}^n e_j \right\|_{\mathcal{A}_l^{k+1}} \right).$$

Then it suffices to show that  $\|\sum_{j=0}^n e_j\|_{\mathcal{A}_l^{k+1}}$  converges. From the definition of  $e_j = e_j^1 + e_j^2$  given in (4.18)–(4.19), we have

$$\begin{aligned} \|e_j^1\|_{\mathcal{A}_l^{k+1}} &\leq C(\|\delta u^j\|_{L^\infty} \|\delta u^j\|_{\mathcal{A}_l^{k+2}} + \|\delta v^j\|_{L^\infty} \|\delta u^j\|_{\mathcal{A}_l^{k+2}} + \|\delta v^j\|_{\mathcal{D}_0^{k+2}} \|\delta u^j\|_{L_{\eta,t}^2(L_{i,x}^\infty)}) \\ &\leq C\varepsilon^2 \theta_j^{3-\bar{k}+\max\{3-\bar{k}, k+2-\bar{k}\}} (\Delta\theta_j)^2 \leq C\varepsilon^2 \theta_j^{k+5-2\bar{k}} \Delta\theta_j, \end{aligned}$$

and

$$\begin{aligned} \|e_j^2\|_{\mathcal{A}_l^{k+1}} &\leq \|(1 - S_{\theta_j})(u^j - u^{a3})\partial_\eta(\delta v^j)\|_{\mathcal{A}_l^{k+1}} + 2 \left\| \frac{\bar{\rho}_x}{\bar{\rho}} (1 - S_{\theta_j})(u^j - u^{a3})(\delta u^j) \right\|_{\mathcal{A}_l^{k+1}} \\ &\quad + \|\partial_\eta((1 - S_{\theta_j})(v^j - v^{a3}))(\delta u^j)\|_{\mathcal{A}_l^{k+1}} \\ &\quad + \|((1 - S_{\theta_j})(v^j - v^{a3}))\partial_\eta(\delta u^j)\|_{\mathcal{A}_l^{k+1}} \\ &\quad + \|\partial_\eta((1 - S_{\theta_j})(u^j - u^{a3}))(\delta v^j)\|_{\mathcal{A}_l^{k+1}} \\ &\leq C \left( \|u^j - u^{a3}\|_{L_{\eta,t}^2(L_{i,x}^\infty)} \|\delta v^j\|_{\mathcal{D}_0^{k+2}} + \|u^j - u^{a3}\|_{\mathcal{A}_l^{k+1}} \|\partial_\eta(\delta v^j)\|_{L^\infty} \right. \\ &\quad + \|u^j - u^{a3}\|_{\mathcal{A}_l^{k+1}} \left\| \frac{\bar{\rho}_x}{\bar{\rho}} \delta u \right\|_{L^\infty} + \|u^j - u^{a3}\|_{L^\infty} \left\| \frac{\bar{\rho}_x}{\bar{\rho}} \delta u \right\|_{\mathcal{A}_l^{k+1}} \\ &\quad + \|\delta u\|_{\mathcal{A}_l^{k+1}} \|\partial_\eta(v^j - v^{a3})\|_{L^\infty} + \|\delta u\|_{L_{\eta,t}^2(L_{i,x}^\infty)} \|(v^j - v^{a3})\|_{\mathcal{D}_0^{k+2}} \\ &\quad + \|\delta u\|_{\mathcal{A}_l^{k+2}} \|(v^j - v^{a3})\|_{L^\infty} + \|\partial_\eta \delta u\|_{L_{\eta,t}^2(L_{i,x}^\infty)} \|(v^j - v^{a3})\|_{\mathcal{D}_0^{k+1}} \\ &\quad \left. + \|\delta v^j\|_{\mathcal{D}_0^{k+1}} \|\partial_\eta(u^j - u^{a3})\|_{L_{\eta,t}^2(L_{i,x}^\infty)} + \|\delta v^j\|_{L^\infty} \|u^j - u^{a3}\|_{\mathcal{A}_l^{k+2}} \right) \\ &\leq C\varepsilon^2 \theta_j^{k+3-\bar{k}} \Delta\theta_j. \end{aligned}$$

Thus, we get that

$$\sum_{j=0}^{+\infty} \|e_j\|_{\mathcal{A}_t^k} \leq C \sum_{j=0}^{+\infty} \theta_j^{k+3-\bar{k}} \Delta\theta_j \leq CC_0$$

for  $k \leq \bar{k} - 5$ .

Therefore, the right-hand side of (4.32)<sub>1</sub> tends to zero as  $n$  tends to  $+\infty$ . The uniqueness of classical solutions to (1.1) can be proved as in [1]. Then we complete the proof of Theorem 1.1.

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