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Published in:
SIAM Journal on Applied Mathematics

Published: 01/01/2015

Document Version:
Final Published version, also known as Publisher's PDF, Publisher's Final version or Version of Record

Publication record in CityU Scholars:
Go to record

Published version (DOI):
10.1137/130920265

Publication details:
Huang, S.-J., Dai, H.-H., & Kong, D.-X. (2015). Global structure stability for the wave catching-up phenomenon in a prestressed two-material bar. *SIAM Journal on Applied Mathematics*, *75*(2), 585-604. https://doi.org/10.1137/130920265

SIAM J. APPL. MATH.
Vol. 75, No. 2, pp. 585–604

GLOBAL STRUCTURE STABILITY FOR THE WAVE CATCHING-UP PHENOMENON IN A PRESTRESSED TWO-MATERIAL BAR*

SHOU-JUN HUANG†, HUI-HUI DAI‡, AND DE-XING KONG§

Abstract. Shock waves in a structure can result in the detachment of an interface and induce microcracks. In a recent study [Huang et al., *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 468 (2012), pp. 3882–3901], it was shown that for certain nonlinearly elastic materials it is possible to generate a phenomenon in which a tensile wave can catch the first transmitted compressive wave (so the former can be undermined) in an initially stress-free two-material bar. In this study, we consider the wave catching-up phenomenon in a nonlinearly elastic prestressed two-material bar. We use the same method as that used by Huang et al. in the previously mentioned paper to construct solutions. Our main focus is on proving the global structure stability of the solutions in a prestressed (or initially stress-free) two-material bar. We first reduce the corresponding initial boundary value problem into several typical free boundary problems based on the formulation of Riemann invariants. Then, using a constructive method and carefully treating the complexity arising from multiple reflections of waves at the interface in the two-material bar, we successfully prove the global structure stability of the wave catching-up phenomenon.

Key words. prestress, global structure stability, wave catching-up phenomenon, two-material bar, impact

AMS subject classifications. 35L50, 35L65, 74J30, 74M20

DOI. 10.1137/130920265

1. Introduction.

When a structure is subjected to an impact, shock waves may induce microcracking and the subsequent dynamic pressure can cause the collapse of the entire structure. Composite materials, due to their effective energy-absorbing capabilities, are increasingly used for impact-resistance purposes. There has been extensive research on such waves in composite materials. Based on linear elasticity, Achenbach [1] studied the relationship between incident, reflected, and transmitted waves in a multilayer composite structure. Andrianov et al. [2] used a homogenization method to deduce the average dynamic properties of layered nonlinear Murnaghan materials. Parnell [20] considered wave propagation in a nonlinearly elastic composite bar with the aim of determining the effective incremental response. In addition, many other authors used numerical methods to simulate waves in composite materials. For instance, Clements, Johnson, and Hixson [5] adopted the method of cells to compute stress waves in laminated materials and the numerical results successfully explained some of the experimental features. A. Berezovski, M. Berezovski, and Engelbrecht [4] used a finite volume method to study waves in layered heterogeneous materials

*Received by the editors May 8, 2013; accepted for publication (in revised form) January 2, 2015; published electronically March 26, 2015. This work was supported by a GRF grant from Hong Kong Research Grants Council (Project CityU 100911), the National Natural Science Foundation of China (Grants 11271323, 11101001, 11301006), and Anhui Provincial Natural Science Foundation (Grant 1408085MA01).
http://www.siam.org/journals/siap/75-2/92026.html
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under an impact. However, those results were for given materials and geometries, with little discussion of how to adjust material properties and geometries to achieve better impact-resistance performance.

Recently, Huang et al. [10] considered wave propagation in a two-material bar subjected to an impact at one end. Their focus was on whether and how material nonlinearity can be exploited to generate a phenomenon in which a destructive tensile wave can catch a transmitted compressive wave. They succeeded in constructing the physical solution for the wave catching-up phenomenon in an initially stress-free two-material bar. To shed light on how the parameters influence the wave catching-up phenomenon, the corresponding asymptotic and numerical solutions were also provided. The study in [10] may provide a new way of designing certain structures for impact protection purposes. Moreover, a comprehensive numerical study based on a discontinuous Galerkin scheme was conducted by Jian, Xu and Dai [11], who showed that both tensive and compressive strains can be reduced by up to $\frac{4}{5}$.

A critical requirement for the abovementioned phenomenon, as shown in Huang et al. [10], is the convexity of the stress-strain curve around zero strain. However, for some well-used hyperelastic models, such as the Arruda–Boyce model (Arruda and Boyce [3]) and the Gent model (Gent [9]), the stress-strain curve only becomes convex after the strain reaches a certain value. Based on this consideration, we consider wave propagation in a prestressed two-material bar and show that due to prestress or prestrain, the wave catching-up phenomenon can still happen.

Wave propagation in prestressed materials has many applications, such as the nondestructive testing of materials that are often prestressed. The influences of prestresses have been examined in many different contexts. Fu and Devenish [8] studied the effects of prestresses on the propagation of surface waves in an incompressible half-space. Destrade and Scott [7] focused on surface waves in a deformed isotropic hyperelastic material subjected to an isotropic internal constraint. Recently, Parnell [21] illustrated the realization of object cloaking from antiplane elastic waves by employing nonlinear elastic prestress in a neo-Hookean material. Later, Norris and Parnell [18] considered cloaking in hyperelastic materials with prestresses by transformation elasticity theory, which is often developed to study linear anisotropic elasticity. It was noted that applying transformation elasticity theory made it possible to convert physical material domains into new configurations without altering wave properties, enabling cloaking and other related effects (see also details in Norris and Shuvalov [19]). Parnell, Norris, and Shearer [22] proved that the elastic prestress of neo-Hookean hyperelastic materials can be used to generate finite cloaks and make objects almost invisible to incoming antiplane elastic waves. For other contributions to this area, we refer to Nemat-Nasser and Amirkhizi [17], Parnell [20], and the references therein.

Here, the prestress is used to generate the convexity of the stress-strain curve, inducing the wave catch-up phenomenon in a nonlinearly elastic two-material bar subjected to an end impact. We first concentrate on the construction of solutions, which are realized by suitable transformations reducing the present formulation to that in Huang et al. [10]. Then, we focus on proving the global structure stability of solutions for the wave catching-up phenomenon. The constructed wave patterns induced by the impact in a nonlinearly elastic prestressed two-material bar are globally structurally stable in a certain reasonable sense. Some studies on the global structure stability of nonlinear elastic waves governed by a hyperbolic system have been conducted. For example, Dai and Kong [6] proved the global structure stability of impact-generated tensile waves in a bar composed of a rubber-like material, for which the solution was constructed by Knowles [14]. In a more general mathematical framework, Kong [12]–[13] showed that the Lax's Riemann solution (see [15]) of general quasilinear hyperbolic systems of conservation laws is globally structurally stable if and only if it contains only nondegenerate shocks and contact discontinuities, but no rarefaction waves or other weak discontinuities. For more results on structure stability for hyperbolic conservation laws, we refer to [12] and the references therein.

The remainder of this paper is organized as follows. In section 2, we derive the analytical solutions for the wave catching-up phenomenon in a nonlinearly elastic prestressed two-material bar by the method introduced in [10]. In section 3, we introduce the concept of *global structure stability*, and present the main results; that is, if a compressive impact is suitably perturbed, then the solution to the corresponding initial boundary value problem is structurally stable. Section 4 is devoted to proving the structure stability results. The method of proof in this section is constructive and the key point is how to handle the complexity that occurs at the interface of the two-material bar, which is due to multiple reflections of waves, and how to solve the typical free boundary problems. The paper is concluded in section 5.

2. Wave catching-up phenomenon in a prestressed two-material bar. We consider a two-material bar, which in the stress-free reference configuration B_0 has a layer composed of a linearly elastic material of finite length h_0 (called material 1), and a second part composed of a nonlinearly elastic material of infinite length (called material 2). Denote the material coordinate for a particle in B_0 by X_0. When a given deformation is imposed on the body, a particle located at X_0 in B_0 moves to a new position $\bar{X}$ in the configuration $\bar{B}$ (see Figure 1). Thus, we have $\bar{X} = \bar{X}(X_0)$.

In this study, our first objective is to investigate wave propagation in a prestressed two-material bar $\bar{B}$ due to a compressive impact. Here, the two parts are assumed to remain in perfect contact throughout the time interval of interest. During the motion caused by an impact, the entire assembly is assumed to undergo uniaxial deformation. It is well known that using B_0 as the reference configuration, in a Lagrangian description, the governing system of equations of motion reads

$$\rho v_t = \sigma_{X_0}, \quad \gamma_t = v_{X_0}, \tag{2.1}$$

where γ, v, and σ are smooth functions of (t, X_0) and denote the strain, particle velocity, and nominal stress, respectively. The constant density ρ and stress-strain relation are defined as

$$\rho = \begin{cases} \rho_1^0, & 0 \le X_0 < h_0, \\ \rho_2^0, & h_0 < X_0 < \infty, \end{cases} \qquad \sigma = \begin{cases} E_1^0 \gamma, & 0 \le X_0 < h_0, \\ E_2^0 f(\gamma), & h_0 < X_0 < \infty, \end{cases} \tag{2.2}$$

where E_1^0 is the Young's modulus of material 1 and E_2^0 can be regarded as the Young's modulus of material 2 for small strains if we impose that $f(0) = 0$ and $f'(0) = 1$.

The given prestrains due to the deformation in configuration $\bar{B}$ are assumed to be piecewise constant with γ_1 in material 1 and γ_2 in material 2. It should be noted that the stretch must be positive; otherwise, a material line will have a negative or zero

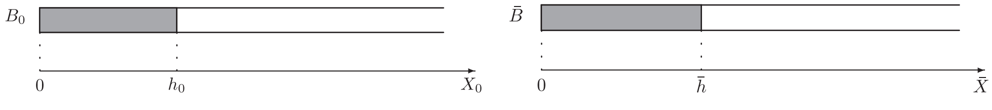

FIG. 1. *The unstressed configuration B_0 and stressed configuration $\bar{B}$ of the composite bar.*

length in the deformed configuration. We impose that $\gamma_1 > -1$ and $\gamma_2 > -1$. The equilibrium equation is trivially satisfied by this deformation provided that $E_1^0\gamma_1 = E_2^0 f(\gamma_2)$. For the configuration $\bar{B}$, we set the initial time to $t = 0$ and an impulsive load described by a rectangular loading function is imposed at the end, such that $\bar{X} = 0$ with duration T. The prescribed stress is supposed to be a negative constant A. Then, the initial and boundary conditions are given by

$$v(0, X_0) = 0 \ \text{ for } X_0 > 0, \ \gamma(0, X_0) = \begin{cases} \gamma_1, & 0 \le X_0 < h_0, \\ \gamma_2, & h_0 < X_0, \end{cases} \tag{2.3}$$

and

$$\sigma(t, 0) = \begin{cases} E_1^0\gamma_1 + A, & 0 < t \le T, \\ E_1^0\gamma_1, & T < t. \end{cases} \tag{2.4}$$

Next, we rewrite the above initial boundary value problem using the spatial variable $\bar{X}$ in $\bar{B}$. For convenience of notation, from now on the strain, velocity, and nominal stress superimposed on $\bar{B}$ are still denoted by γ, v, and σ, respectively, without superscripts, the material coordinate $\bar{X}$ in reference configuration $\bar{B}$ is represented by x, and $\bar{h}$ is denoted by h. Then, by some analysis (see details in Appendix A), (2.1) can be reduced into the following forms,

$$\rho v_t = \sigma_x, \quad \gamma_t = v_x, \tag{2.5}$$

where γ, v, and σ are smooth functions of (t, x), and

$$\rho = \begin{cases} \rho_1, & 0 \le x < h, \\ \rho_2, & h < x < \infty, \end{cases} \qquad \sigma = \begin{cases} E_1\gamma, & 0 \le x < h, \\ E_2 g(\gamma), & h < x < \infty, \end{cases} \tag{2.6}$$

where

$$\rho_1 = \frac{\rho_1^0}{1+\gamma_1}, \ \rho_2 = \frac{\rho_2^0}{1+\gamma_2}, \ E_1 = (1+\gamma_1)E_1^0, \ E_2 = (1+\gamma_2)f'(\gamma_2)E_2^0, \tag{2.7}$$

and

$$h = (1+\gamma_1)h_0, \ \ g(\gamma) = \frac{f(\gamma_2 + (1+\gamma_2)\gamma) - f(\gamma_2)}{(1+\gamma_2)f'(\gamma_2)}. \tag{2.8}$$

Moreover, the initial and boundary conditions (2.3)–(2.4) can be rewritten as

$$v(0, x) = 0, \ \gamma(0, x) = 0 \text{ for } x > 0, \text{ and } \sigma(t, 0) = \begin{cases} A, & 0 < t \le T, \\ 0, & t > T. \end{cases} \tag{2.9}$$

At a moving strain discontinuity $x = s(t)$, the jump conditions are

$$\dot{s}[\gamma] + [v] = 0, \quad \dot{s}[\rho\, v] + [\sigma] = 0, \tag{2.10}$$

where $[F] = F(t, s(t)+0) - F(t, s(t)-0)$, $\dot{s} = ds(t)/dt$ denotes the Lagrangian velocity of discontinuity.

Therefore, the propagation of waves due to an impact in a prestressed two-material bar $\bar{B}$ can be described by the initial boundary value problem (2.5) and (2.9). It is easy to note that $g(0) = 0$ and $g'(0) = 1$. For a positive stretch, we have $\gamma > -1$. As in [10], we impose the following conditions on $g(\gamma)$, which can be easily converted into conditions on $f(\gamma)$ using $(2.8)_2$:

(H_1) $g(\gamma)$ is smooth enough in $(-1, \infty)$;

(H_2) letting α_1, β_1 be assigned constants consistent with $-1 \le \alpha_1 < 0 < \beta_1 \le \infty$, it holds that $g'(\gamma) > 0, g''(\gamma) > 0$ for $\gamma \in (\alpha_1, \beta_1) \subset (-1, \infty)$.

Thus, we consider the case that $g(\gamma)$ is a monotonic increasing and convex function.

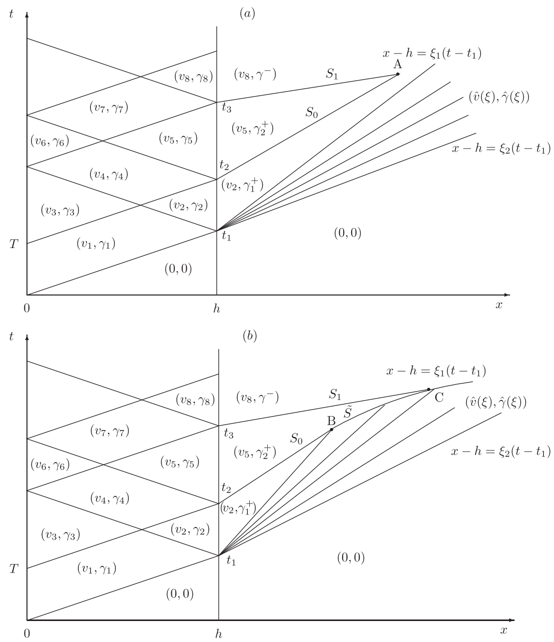

FIG. 2. *The wave catching-up phenomenon in the prestressed composite bar.* (a) *Case* I: $T \geq T_*$*, the shock wave* S_1 *first captures the shock wave* S_0*;* (b) *Case* II: $T < T_*$*, the shock wave* S_0 *first penetrates the rarefaction wave and then is caught by the shock wave* S_1*, where* $T_* \doteq 2t_1 s_1(s_0 - c(\gamma_1^+))/[s_0(s_1 - c(\gamma_1^+))], c(\gamma) = (\sigma'(\gamma)/\rho_2)^{1/2}, \xi_1 = c(\gamma_1^+), \xi_2 = c_2, t_1 = h/c_1, t_2 = T + t_1 = (1+\theta)t_1, \theta = T/t_1, t_3 = 3t_1$.

We point out that, except for some modifications on material parameters (cf. (2.5), (2.6) and (2.1), (2.2)), the above initial boundary value problem is the same as that for wave propagation in an initially stress-free two-material bar, which has been studied in [10]. In that paper, it was shown that the third transmitted tensile wave can catch the first transmitted compressive wave in material 2 (such that the magnitude of the former can be decreased) under some conditions on the stress-strain relation.

Hence, all of the results for the wave catching-up phenomenon in an initially stress-free two-material bar in Huang et al. [10] can be immediately applied in the present case of a prestressed two-material bar. Thus, we mainly list the results without derivations.

As shown in [10], if material 2 is linearly elastic, then there is no wave catching-up phenomenon, as the transmitted waves in material 2 are all linear with the same velocity. For the third transmitted wave in material 2, the corresponding strain is $\gamma^- = \frac{2(\beta-1)}{(\beta+1)^2}\frac{A}{E_2}$, where $\beta = \frac{\rho_1 c_1}{\rho_2 c_2}$ denotes the ratio of the impedances of prestressed materials 1 and 2, and $c_i = (\frac{E_i}{\rho_i})^{\frac{1}{2}} (i = 1, 2)$ are the sound wave speeds in the prestressed materials 1 and 2, respectively. Obviously, the tensile wave (i.e., $\gamma^- > 0$) can arise when $\beta < 1$.

As in [10], if material 2 is nonlinearly elastic, then there are two cases in which the wave catching-up phenomenon can happen. Case I: the third transmitted shock wave, denoted by S_1, first captures the second shock wave denoted by S_0. Case II: the second transmitted shock wave S_0 first penetrates the rarefaction wave (see Figure 2). The transmitted waves are composed of a rarefaction wave followed by two subsequent shock waves. The main unknowns are the strains γ_1^+, γ_2^+, and γ^- and the speeds of shock waves S_0 and S_1 and varying-speed shock wave $\tilde{S}$, which can be obtained readily via the following equations:

$$g(\gamma_1^+) + \beta \int_0^{\gamma_1^+} \sqrt{g'(\gamma)} d\gamma - \frac{2A}{E_2} = 0, \tag{2.11}$$

$$g(\gamma_2^+) - \beta \int_{\gamma_1^+}^{0} \sqrt{g'(\gamma)} d\gamma - \beta \operatorname{sgn}\left(\gamma_1^+ - \gamma_2^+\right) \sqrt{\left[g(\gamma_1^+) - g(\gamma_2^+)\right]\left(\gamma_1^+ - \gamma_2^+\right)} = 0, \tag{2.12}$$

$$(\gamma^- - \gamma_2^+)\sqrt{\frac{g(\gamma^-) - g(\gamma_2^+)}{\gamma^- - \gamma_2^+}} + \frac{1}{\beta}\left[g(\gamma^-) + g(\gamma_1^+)\right] + (\gamma_2^+ - \gamma_1^+)\sqrt{\frac{g(\gamma_1^+) - g(\gamma_2^+)}{\gamma_1^+ - \gamma_2^+}} = 0, \tag{2.13}$$

$$\frac{s_0}{c_2} = \sqrt{\frac{g(\gamma_1^+) - g(\gamma_2^+)}{\gamma_1^+ - \gamma_2^+}}, \quad \frac{s_1}{c_2} = \sqrt{\frac{g(\gamma^-) - g(\gamma_2^+)}{\gamma^- - \gamma_2^+}}, \tag{2.14}$$

and

$$\left.\begin{aligned} &\tfrac{dx(t)}{dt} = s(\gamma) = c_2\sqrt{\tfrac{g(\gamma)-g(\gamma_2^+)}{\gamma-\gamma_2^+}}, \\ &\tfrac{x-h}{t-t_1} = c(\gamma) = c_2\sqrt{g'(\gamma)}, \\ &c(\gamma_1^+) \le \tfrac{x-h}{t-t_1} \le c_2 \ (\text{equivalently, } \gamma_1^+ \le \gamma \le 0), \\ &x(t_{_B}) = x_{_B}, \end{aligned}\right\} \tag{2.15}$$

where s_0 and s_1 are the speeds of shock waves S_0 and S_1, respectively.

The solutions to (2.11)–(2.13) can be uniquely solved and satisfy (cf. [10])

$$-1 < \gamma_1^+ < \gamma_2^+ < 0 < \gamma^-, \tag{2.16}$$

provided that some suitable assumptions are imposed on A and $\beta < 1$. The speeds of shock waves s_0 and s_1 can, in turn, be obtained from (2.14). Moreover, the Lax's entropy condition (cf. [15]) also holds: $c(\gamma_2^+) > s_0 > c(\gamma_1^+)$, $\quad c(\gamma^-) > s_1 > c(\gamma_2^+)$. The rarefaction waves in cases I and II are determined by

$$c(\hat{\gamma}(\xi)) = \xi, \quad \hat{v}'(\xi) = -\xi\hat{\gamma}'(\xi), \tag{2.17}$$

where $\xi = (x-h)/(t-t_1)$, $c(\gamma) = \sqrt{\sigma'(\gamma)/\rho_2} = c_2\sqrt{g'(\gamma)}$, $\xi_1 = c(\gamma_1^+)$, $\xi_2 = c_2$.

When the amplitude of impact is relatively small, that is, $\delta = -2A/E_2 > 0$ is small, asymptotic solutions can be constructed (see [10]). For the present case of a prestressed two-material bar, we have

$$\gamma_1^+ = -\frac{1}{\beta+1}\delta - \frac{\beta+2}{4(\beta+1)^3}\frac{(1+\gamma_2)f''(\gamma_2)}{f'(\gamma_2)}\delta^2 + O(\delta^3), \tag{2.18}$$

$$\gamma_2^+ = -\frac{\beta(1+\gamma_2)^2[f''(\gamma_2)]^2}{96(1+\beta)^4[f'(\gamma_2)]^2}\delta^3 + O(\delta^4), \tag{2.19}$$

$$\gamma^- = \frac{1-\beta}{(1+\beta)^2}\delta - \frac{(1+\gamma_2)f''(\gamma_2)}{4(1+\beta)^5 f'(\gamma_2)}(\beta^3 + 2\beta^2 - \beta + 2)\delta^2 + O(\delta^3), \tag{2.20}$$

$$\begin{aligned} \frac{s_0}{c_2} &= 1 - \frac{(1+\gamma_2)f''(\gamma_2)}{4(1+\beta)f'(\gamma_2)}\delta + O(\delta^2), \\ \frac{s_1}{c_2} &= 1 + \frac{(1-\beta)(1+\gamma_2)f''(\gamma_2)}{4(1+\beta)^2 f'(\gamma_2)}\delta + O(\delta^2). \end{aligned} \tag{2.21}$$

From expression (2.21), we can see that $(1+\gamma_2)f''(\gamma_2)/f'(\gamma_2) = g''(0) > 0$ slows down the second compressive shock and speeds up the third tensile shock, allowing the catching-up phenomenon to take place.

In terms of solutions, the results for a prestressed two-material bar and an initially stress-free two-material bar are parallel, so we do not discuss those further and the readers are referred to [10] for details. Nevertheless, the two parameters $\beta(=\frac{\rho_1 c_1}{\rho_2 c_2})$ and $\alpha(=\frac{c_1}{c_2})$, which characterize the nondimensionalized equations (see Jiang, Xu, and Dai [11]), are different. More specifically, $\beta = [f'(\gamma_2)]^{-0.5}\beta^0$ and $\alpha = \frac{1+\gamma_1}{1+\gamma_2}\frac{1}{[f'(\gamma_2)]^{0.5}}\alpha^0$, where β^0 and α^0 are the quantities in the initial stress-free case. It can be seen that if $f'(\gamma_2) > 1$ the impedance β decreases, which results in the slowdown of the second compressive shock and the speedup of the third tensile shock wave. For $f'(\gamma_2) < 1$, the effect is opposite. The parameter α measures the catching point (see (4.10) and (4.11) of [10]). Roughly, the position of the catching point is inversely proportional to α. Thus, if $\frac{1+\gamma_1}{1+\gamma_2}\frac{1}{[f'(\gamma_2)]^{0.5}} > 1$, the catching takes place earlier and if it is less than 1, the effect is opposite.

Below, we turn our attention to the global structure stability of wave patterns for the wave catching-up phenomenon, which is the main focus of this study.

3. The global structure stability. The solutions presented in the previous section are for a rectangular impact. In reality, the form of impact is never so ideal. Now, we consider the case that the rectangular impact is perturbed by an arbitrary time-dependent disturbance and study the corresponding stability of the two wave patterns shown in Figure 2.

Given the similarities between the initial boundary value problem (2.5), (2.6), and (2.9), we can generally assume that the two wave patterns in cases I and II have the form $(v(\xi), \gamma(\xi))$ with $\xi = (x-h)/(t-t_1)$.

Similarly to [6], to state the concept of the *global structure stability* of similarity solution $(v(\xi), \gamma(\xi))$, we need to study an initial boundary value problem for (2.5) with the following conditions:

$$v(0,x) = 0,\ \gamma(0,x) = 0 \text{ for } x > 0, \text{ and } \sigma(t,0) = \begin{cases} A + \epsilon\sigma_0(t), & 0 \leqslant t \leqslant T, \\ 0, & t > T, \end{cases} \tag{3.1}$$

where $\epsilon > 0$ is a small parameter, and $\sigma_0(t)$ is a disturbance function.

Now, we introduce the concept of global structure stability (cf. [6] or [12]).

Definition 3.1. *The similarity solution* $(v(\xi), \gamma(\xi))$ *is called globally structurally stable with respect to* $\sigma_0(t)$ *if for every* $\kappa > 0$ *there exists* $\epsilon_0 = \epsilon_0(\kappa)$ *such that for any given* $\epsilon \in [0, \epsilon_0]$*, the initial boundary value problem* (2.5), (2.6), *and* (3.1) *admits a unique global piecewise* C^1 *solution* $(v(t,x), \gamma(t,x))$ *composed of a centered rarefaction wave followed by two subsequent shock waves; moreover, this solution has a global structure similar to that of* $(v(\xi), \gamma(\xi))$ *and it holds that*

$$|v(t,x) - v_2|, \quad |\gamma(t,x) - \gamma_1^+| \leqslant \kappa \quad \forall\, (t,x) \in \tilde{\Omega}_1, \tag{3.2}$$

$$|v(t,x) - v_5|, \quad |\gamma(t,x) - \gamma_2^+| \leqslant \kappa \quad \forall\, (t,x) \in \tilde{\Omega}_2, \tag{3.3}$$

$$|v(t,x) - v_8|, \quad |\gamma(t,x) - \gamma^-| \leqslant \kappa \quad \forall\, (t,x) \in \tilde{\Omega}_3, \tag{3.4}$$

where the $\tilde{\Omega}_i\ (i = 1, 2, 3)$ *denote the domains where* $(v(t,x), \gamma(t,x))$ *satisfies* (2.5) *in the classical sense (see Figure* 3*).*

In this study, we only consider the global structure stability for case II (see Figure 2(b)). The discussion for case I (see Figure 2(a)) is very similar, so the details

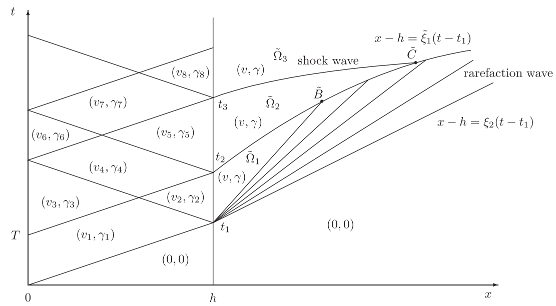

FIG. 3. *The perturbational wave catching-up phenomenon in the composite bar for case* II, *where* $\xi_2 = c_2, t_1 = h/c_1, t_2 = T + t_1 = (1+\theta)t_1, \theta = T/t_1, t_3 = 3t_1$.

are omitted. The term "global" in the above definition means that the constructed solution is valid only up to the catching point (see Figures 2 and 3).

The main result for structure stability is the following Theorem 3.2.

THEOREM 3.2. *Suppose that* $g(\gamma)$ *satisfies the assumptions* (H_1)–(H_2) *and* $\sigma_0(t)$ *is a* C^1 *smooth function defined on* $[0,T]$ *with bounded* C^1 *norm, i.e.,*

$$||\sigma_0(t)||_{C^1} \leqslant K \quad \text{for} \quad t \in [0,T], \tag{3.5}$$

where $||\cdot||_{C^1}$ *denotes the* C^1 *norm and* $K > 0$ *is a constant. Then, the unique similarity solution as shown in Figure* 2 *is globally structurally stable with respect to* $\sigma_0(t)$.

Remark 3.3. We can observe that the inequality (3.5) implies that

$$|\sigma_0'(t)| \leqslant \frac{KT}{t} \triangleq \frac{K_0}{t} \quad \text{for} \quad t \in (0,T], \tag{3.6}$$

which will be much more convenient for future argument.

Remark 3.4. If the two-material bar is stress free, i.e., $\gamma_1 = \gamma_2 = 0$, then the above structure stability results are also applicable.

4. Proofs of the structure stability results. Introduce Riemann invariants

$$p = v + \int_0^\gamma \sqrt{\sigma'(\tilde\gamma)/\rho}\, d\tilde\gamma, \quad q = v - \int_0^\gamma \sqrt{\sigma'(\tilde\gamma)/\rho}\, d\tilde\gamma. \tag{4.1}$$

Sometimes we denote the Riemann invariants (4.1) in material 2 by p^* and q^*.

By the above Riemann invariants, the system (2.5) can be written as

$$p_t + \lambda(p-q)p_x = 0, \quad q_t + \mu(p-q)q_x = 0, \tag{4.2}$$

where $\lambda(p-q) = -\sqrt{\sigma'(H(p-q))/\rho} = -\mu(p-q)$, and $\gamma = H(p-q)$ is the inverse function of

$$p - q = 2\int_0^\gamma \sqrt{\sigma'(\tilde\gamma)/\rho}\, d\tilde\gamma. \tag{4.3}$$

By the assumptions on $g(\gamma)$, we have $\sigma''(\gamma) > 0$ for the strains under consideration. Thus, it holds that $\lambda_p = \mu_q < 0$.

In the Riemann invariants (p,q), the initial boundary conditions (3.1) become

$$p(0,x) = 0, q(0,x) = 0 \ \text{for} \ x > 0, \quad (p-q)(t,0) = \frac{2c_1}{E_1}[A + \epsilon\sigma_0(t)] \ \text{for} \ t > 0. \tag{4.4}$$

Proof of Theorem 3.2. The method of proof is constructive and the key point is to employ the theory on typical free boundary problems to readily solve the wave catching-up phenomenon. For clarity, we divide the proof into the following four steps.

Step 1: set up the centered rarefaction wave.

Consider the initial boundary value problem (2.5) and (2.6) with conditions

$$v(0,x)=0,\ \gamma(0,x)=0 \text{ for } x>0, \text{ and } \sigma(t,0)=\begin{cases} A+\epsilon\sigma_0(0), & 0\leqslant t\leqslant T,\\ 0, & t>T,\end{cases} \tag{4.5}$$

where ϵ is a small parameter and the stress $\sigma_0(0)$ is the constant disturbance at time $t=0$. By the same method as that used in [10], we can solve the initial boundary value problem (2.5), (2.6), and (4.5). We still refer to Figure 2 for convenience of discussion.

On the domain Ω_1, the constant states denoted by $(v_{1,\epsilon}^+,\gamma_{1,\epsilon}^+)$ are determined by

$$v_{1,\epsilon}^+=\int_{\gamma_{1,\epsilon}^+}^0 c(\gamma)d\gamma, \tag{4.6}$$

$$g(\gamma_{1,\epsilon}^+)+\beta\int_0^{\gamma_{1,\epsilon}^+}\sqrt{g'(\gamma)}d\gamma-\frac{2}{E_2}\left[A+\epsilon\sigma_0(0)\right]=0. \tag{4.7}$$

Moreover, from (2.11), (4.7), and (4.6) we see that $|v_{1,\epsilon}^+-v_2|,\ |\gamma_{1,\epsilon}^+-\gamma_1^+|\leqslant K_1\epsilon$, where K_1 is a positive constant independent of (t,x) and ϵ. The centered rarefaction wave $(\hat v_\epsilon(\xi),\hat\gamma_\epsilon(\xi))$ is given by $c(\hat\gamma_\epsilon(\xi))=\xi$, $\hat v_\epsilon'(\xi)=-\xi\hat\gamma_\epsilon'(\xi)$, where $\xi=(x-h)/(t-t_1)$ and

$$c(\gamma_{1,\epsilon}^+)=\xi_\epsilon\leqslant\xi\leqslant\xi_2=c_2. \tag{4.8}$$

On the domains Ω_2 and Ω_3, the constant states are denoted by $(v_{2,\epsilon}^+,\gamma_{2,\epsilon}^+)$ and $(v_\epsilon^-,\gamma_\epsilon^-)$, respectively. The speeds of transmitted shock waves $\hat S_0$ and $\hat S_1$ and curved shock wave $\tilde S_\epsilon$ are represented by $s_{0,\epsilon}, s_{1,\epsilon}$, and $s_\epsilon'(t)$, respectively. By a similar argument, we have

$$|v_{2,\epsilon}^+-v_5|,\quad |v_\epsilon^--v_8|,\quad |\gamma_{2,\epsilon}^+-\gamma_2^+|,\quad |\gamma_\epsilon^--\gamma^-|\leqslant K_2\epsilon, \tag{4.9}$$

and

$$|s_{0,\epsilon}-s_0|,\quad |s_{1,\epsilon}-s_1|,\quad |s_\epsilon'(t)-\tilde s'(t)|\leqslant K_2\epsilon, \tag{4.10}$$

where $\tilde s'(t)$ is the speed of the curved shock wave (cf. [10]; see Figure 2(b)) and K_2 is a positive constant independent of (t,x) and ϵ.

Step 2: construct the unique continuous and piecewise C^1 solution in the domain $\tilde\Omega_1$.

We now consider the solution for system (2.5) on the domain $\tilde\Omega_1$ with the following boundary conditions:

$$v(t,h+\xi_\epsilon(t-t_1))=v_{1,\epsilon}^+,\quad \gamma(t,h+\xi_\epsilon(t-t_1))=\gamma_{1,\epsilon}^+. \tag{4.11}$$

However, the boundary conditions on the interface $x=h$ for system (2.5) are temporarily unclear and need further analysis. First, by the characteristic method, we have

$$q(t,x)=-\frac{2c_1}{E_1}\left[A+\epsilon\sigma_0(-x/c_1+t)\right] \tag{4.12}$$

on the domain of (v_2, γ_2) (see Figure 3). Moreover, by the first equation in (4.2), (4.6), and (4.11), we deduce that the Riemann invariant p^* in material 2 satisfies $p^*(t,x) \equiv 0$ on $\tilde{\Omega}_1$, i.e.,

$$v = -\int_0^{\gamma} c(\tilde{\gamma})d\tilde{\gamma} \text{ on } \tilde{\Omega}_1. \tag{4.13}$$

However, because the stress should remain continuous across the interface $x = h$, it holds that

$$v_2|_{x=h} = v|_{x=h}, \quad E_1\gamma_2|_{x=h} = E_2 g(\gamma)|_{x=h}. \tag{4.14}$$

Noting (4.12) and (4.13), we obtain from the first equation in (4.14) that

$$\left\{c_1\gamma_2 - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t - x/c_1)\right]\right\}\Big|_{x=h} = -\int_0^{\gamma} c(\tilde{\gamma})d\tilde{\gamma}\Big|_{x=h}. \tag{4.15}$$

Then, combining the second equation in (4.14) and (4.15) leads to

$$g(a) + \beta\int_0^a \sqrt{g'(\tilde{\gamma})}d\tilde{\gamma} - \frac{2}{E_2}\left[A + \epsilon\sigma_0(t - t_1)\right] = 0, \tag{4.16}$$

where $a = a(t) = \gamma(t,x)|_{x=h}$. Similar to (2.11), there is a unique solution $a < 0$ to (4.16) if the parameter ϵ is taken to be small enough. Moreover, by comparing (2.11) and (4.16) we observe that

$$|a - \gamma_1^+| \leqslant K_3\epsilon, \tag{4.17}$$

where K_3 is a constant independent of t, x, ϵ. Meanwhile, it follows from (4.16) that

$$\left(g'(a) + \beta\sqrt{g'(a)}\right)a'(t) = \frac{2\epsilon}{E_2}\sigma'(t - t_1), \tag{4.18}$$

which is useful in the following decay estimate for $a'(t)$.

Bearing in mind that $p^* = 0$, we can see that the other Riemann invariant $q^* \equiv -2\int_0^{\gamma} c(\tilde{\gamma})d\tilde{\gamma}$ and satisfies

$$q_t^* + \mu(-q^*)q_x^* = 0 \tag{4.19}$$

and

$$q^*|_{x=h} = -2\int_0^a c(\tilde{\gamma})d\tilde{\gamma}. \tag{4.20}$$

From (4.17) and (4.19)–(4.20), we easily find that

$$\left|q^*(t,x) + 2\int_0^{\gamma_1^+} c(\tilde{\gamma})d\tilde{\gamma}\right| \leqslant K_4\epsilon, \tag{4.21}$$

and hence

$$\left|\gamma(t,x) - \gamma_1^+\right| \leqslant K_4\epsilon, \tag{4.22}$$

where K_4 is a constant independent of (t,x) and ϵ. Now, (4.19)–(4.20) can be written as

$$\begin{cases} q_x^* + \frac{1}{\mu(-q^*)} q_t^* = 0 \ \text{ on } \ \tilde{\Omega}_1, \\ x = h : q^* = -2\int_0^a c(\tilde{\gamma}) d\tilde{\gamma}, \end{cases} \tag{4.23}$$

or equivalently,

$$\begin{cases} \gamma_x + \frac{1}{\mu(\gamma)} \gamma_t = 0 \ \text{ on } \ \tilde{\Omega}_1, \\ x = h : \ \gamma = a(t). \end{cases} \tag{4.24}$$

Making use of (4.18) and the boundedness of the C^1 norm for $\sigma(t)$, we can deduce that the lifespan of the solution for (4.24) is $t_1 + O(\epsilon^{-1})$, by the standard theory of hyperbolic equations (cf. [16]). This means that if ϵ is small enough, there is a unique C^1 solution to (4.24) (equivalently, (4.23)) on the domain $\tilde{\Omega}_1$.

At this stage, we have solved the desired solution (p^*, q^*) for the problem (4.2), (4.11), and (4.20) on the domain $\tilde{\Omega}_1$. Then, by (4.1), we obtain the unique global continuous and piecewise C^1 smooth solution of the problem (2.5), (2.6), and (3.1), which contains a centered rarefaction wave with the back state $(v(t,x), \gamma(t,x))$. Moreover, by (4.22) and (4.13) it is easy to see that

$$\left|\gamma(t,x) - \gamma_1^+\right|, \quad |v(t,x) - v_2| \leqslant K_5 \epsilon \tag{4.25}$$

on the domain $\tilde{\Omega}_1$ and

$$|\xi_\epsilon - \xi_1| \leqslant K_5 \epsilon, \tag{4.26}$$

where v_2, ξ_ϵ are constants, as shown in Figure 2 and (4.8), and K_5 is a constant independent of (t,x) and ϵ.

Step 3: construct the unique continuous and piecewise C^1 solution in the domain $\tilde{\Omega}_2$.

First, from (4.12) and the first equation in (4.2) we have

$$v_1(t,x) = -c_1 \gamma_1(t,x), \quad \gamma_1(t,x) = \frac{1}{E_1}\left[A + \epsilon \sigma_0(t - x/c_1)\right]. \tag{4.27}$$

In addition, by (4.14) it follows from the first equation in (4.2) that

$$p(t,x) = \frac{2c_1}{\alpha\beta} g(a(x/c_1 + t - t_1)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t - 2t_1 + x/c_1)\right] \tag{4.28}$$

on the domains of the states $(v_i, \gamma_i)(i = 2,4,6)$ (see Figure 3). Combining (4.12) and (4.28) yields

$$\begin{aligned} v_2(t,x) = {}& \frac{c_1}{\alpha\beta} g\left(a\left(\frac{x}{c_1} + t - t_1\right)\right) - \frac{2c_1 A}{E_1} \\ & - \frac{c_1}{E_1}\epsilon\left[\sigma_0\left(\frac{x}{c_1} + t - 2t_1\right) + \sigma_0\left(t - \frac{x}{c_1}\right)\right], \end{aligned} \tag{4.29}$$

$$\begin{aligned} \gamma_2(t,x) = {}& \frac{1}{\alpha\beta} g\left(a\left(\frac{x}{c_1} + t - t_1\right)\right) \\ & - \frac{\epsilon}{E_1}\left[\sigma_0\left(\frac{x}{c_1} + t - 2t_1\right) - \sigma_0\left(t - \frac{x}{c_1}\right)\right]. \end{aligned} \tag{4.30}$$

Obviously, it is easy to get

$$(v_3(t,x),\gamma_3(t,x))=(0,0). \tag{4.31}$$

Then, on the domains of the states (v_4,γ_4) and (v_5,γ_5), it can be observed that

$$q(t,x)\equiv 0. \tag{4.32}$$

We easily obtain from (4.28) and (4.32) that

$$\begin{aligned} v_4(t,x)&=c_1\gamma_4(t,x),\ \gamma_4(t,x)\\ &=\frac{1}{\alpha\beta}g\left(a\left(\frac{x}{c_1}+t-t_1\right)\right)-\frac{1}{E_1}\left[A+\epsilon\sigma_0\left(t-2t_1+\frac{x}{c_1}\right)\right]. \end{aligned} \tag{4.33}$$

For the state (v_5,γ_5), it follows from (4.32) that

$$v_5(t,x)=c_1\gamma_5(t,x). \tag{4.34}$$

Next, we derive the solutions on both sides of the interface $x=h$. Analogously to (4.14), on the boundary $x=h$ we have

$$v_5|_{x=h}=v|_{x=h},\quad E_1\gamma_5|_{x=h}=E_2 g(\gamma)|_{x=h}. \tag{4.35}$$

Making use of (4.34)–(4.35), we have a boundary condition for the system (2.5):

$$v|_{x=h}=\frac{c_1}{\alpha\beta}g(\gamma)|_{x=h}. \tag{4.36}$$

However, on the unknown free boundary $x=s(t)$ $(s(t_2)=h)$, we have

$$\begin{cases} \rho_2[v(t,s(t)-0)-v(t,s(t)+0)]^2\\ \quad =[\gamma(t,s(t)-0)-\gamma(t,s(t)+0)][\sigma(\gamma(t,s(t)-0))-\sigma(\gamma(t,s(t)+0))],\\ \frac{ds}{dt}=\sqrt{\frac{\sigma(\gamma(t,s(t)-0))-\sigma(\gamma(t,s(t)+0))}{\rho_2(\gamma(t,s(t)-0)-\gamma(t,s(t)+0))}}, \end{cases} \tag{4.37}$$

which is nothing but the jump conditions (2.10). It is worth pointing out that $\gamma(t,s(t)+0)$ in (4.37) has been solved in Steps 1 and 2. The above boundary conditions (4.36)–(4.37) and the system (2.5) formulate a typical free boundary problem. By taking the square root of $(4.37)_1$ (the negative sign is taken, because in the special case of $\sigma_0=0$ the corresponding sign is negative) and noting (4.13), we can rewrite (4.37) as

$$\begin{cases} v=-\int_0^{\gamma(t,s(t)+0)}c(\tilde{\gamma})d\tilde{\gamma}-\sqrt{[\gamma-\gamma(t,s(t)+0)][\sigma(\gamma)-\sigma(\gamma(t,s(t)+0))]/\rho_2},\\ \frac{ds}{dt}=\sqrt{\frac{\sigma(\gamma)-\sigma(\gamma(t,s(t)+0))}{\rho_2[\gamma-\gamma(t,s(t)+0)]}}, \end{cases} \tag{4.38}$$

where $v=v(t,s(t)-0),\gamma=\gamma(t,s(t)-0)$.

In the Riemann invariants (p,q), the typical free boundary problem (2.5), (4.36), and (4.38) on the domain $\tilde{\Omega}_2$ can be equivalently rewritten as a typical free boundary problem for (4.2) with the following boundary conditions:
on the fixed boundary $x=h$,

$$(p+q)(t,h)=\frac{2c_1}{\alpha\beta}g(H(p-q))\quad \forall t>t_2; \tag{4.39}$$

on the free boundary $x = s(t)\,(s(t_2) = h)$,

$$(4.40)\quad \begin{cases} p + q = -2\int_0^{\gamma(t,s(t)+0)} c(\tilde{\gamma})d\tilde{\gamma} \\ \qquad - 2\sqrt{[H(p-q) - \gamma(t,s(t)+0)][\sigma(H(p-q)) - \sigma(\gamma(t,s(t)+0))]/\rho_2}, \\ \frac{ds}{dt} = \sqrt{\frac{\sigma(H(p-q)) - \sigma(\gamma(t,s(t)+0))}{\rho_2[H(p-q) - \gamma(t,s(t)+0)]}}. \end{cases}$$

LEMMA 4.1. *There exists a unique function $x = s(t) \in C^1$ with $s(t_2) = h$ such that the typical free boundary problem* (4.2), (4.39), *and* (4.40) *has a unique global C^1 solution (p, q) on the domain $\tilde{\Omega}_2$. Moreover, it holds that*

$$(4.41)\quad |p(t,x) - \bar{p}|, \ \ |q(t,x) - \bar{q}| \leqslant K_6\epsilon,$$

$$(4.42)\quad \left|\frac{\partial p}{\partial t}(t,x)\right|, \ \left|\frac{\partial p}{\partial x}(t,x)\right|, \ \left|\frac{\partial q}{\partial t}(t,x)\right|, \ \left|\frac{\partial q}{\partial x}(t,x)\right| \leqslant \frac{K_6\epsilon}{t - t_1},$$

$$(4.43)\quad |s'(t) - s_{0,\epsilon}|, \ \ |s'(t) - s'_\epsilon(t)| \leqslant K_6\epsilon,$$

where K_6 is a positive constant independent of (t, x), and ϵ, $\bar{p}$, and $\bar{q}$ are given by

$$(4.44)\quad \bar{p} = v^+_{2,\epsilon} + \int_0^{\gamma^+_{2,\epsilon}} c(\tilde{\gamma})d\tilde{\gamma}, \quad \bar{q} = v^+_{2,\epsilon} - \int_0^{\gamma^+_{2,\epsilon}} c(\tilde{\gamma})d\tilde{\gamma}.$$

Proof. From (4.39) and the first equation in (4.40), we may get $q = \zeta(p)$ and $p = \eta(q, \gamma(t, s(t) + 0))$, respectively. Differentiating (4.39) with respect to p yields

$$(4.45)\quad \frac{\partial \zeta}{\partial p} = \frac{\frac{2c_1}{\alpha\beta} g' H' - 1}{\frac{2c_1}{\alpha\beta} g' H' + 1}.$$

Because $g' > 0$ and $H' > 0$, it follows from (4.45) that

$$(4.46)\quad \left|\frac{\partial \zeta}{\partial p}\right| < 1.$$

Now, we calculate $\frac{\partial \eta}{\partial q}$.

First, we consider the typical free boundary problem before the second transmitted shock wave begins to penetrate the rarefaction wave (see Figure 3). By the estimate (4.25) and the fact (2.16), it holds that

$$(4.47)\quad 0 > \gamma = H(p - q) > \gamma(t, s(t) + 0).$$

Differentiating the first equation in (4.40) with respect to q gives

$$(4.48)\quad \frac{\partial \eta}{\partial q} = \frac{b - 1}{b + 1},$$

where

$$(4.49)\quad b = H'(p - q)(A + B)\Delta^{-1},$$

in which

$$(4.50)\quad \begin{cases} A = \sigma(H(p-q)) - \sigma(\gamma(t,s(t)+0)), \\ B = [H(p-q) - \gamma(t,s(t)+0)]\,\sigma'(H(p-q)), \\ \Delta = \sqrt{[H(p-q) - \gamma(t,s(t)+0)]\,[\sigma(H(p-q)) - \sigma(\gamma(t,s(t)+0))]\,/\rho_2}. \end{cases}$$

Now, we estimate b. Because $\sigma'(H(p-q)) > 0,\ H'(p-q) > 0$, it follows from (4.47) that $A > 0,\ B > 0$. Then, we have

$$b > 0. \tag{4.51}$$

Thus, using (4.51), we obtain from (4.48) that

$$\left|\frac{\partial \eta}{\partial q}\right| < 1. \tag{4.52}$$

Next, we turn to study the situation in which the second transmitted shock wave begins to penetrate the rarefaction wave (see the curve $\tilde{B}\tilde{C}$ in Figure 3). This situation can happen because the velocity of the second transmitted shock wave is always greater than $c(\gamma_{1,\epsilon}^+)$ (i.e., ξ_ϵ), which can be seen from (2.16) and (4.25), provided that ϵ is sufficiently small.

The shock wave starts to penetrate the rarefaction wave at the intersection $\tilde{B}$ (see Figure 3). The propagation of the second shock wave after penetration is determined by

$$\begin{cases} \frac{ds}{dt} = c_2\sqrt{\frac{g(\gamma)-g(\gamma^+)}{\gamma-\gamma^+}}, \\ \frac{x-h}{t-t_1} = c(\gamma^+) = c_2\sqrt{g'(\gamma^+)}, \\ c(\gamma_{1,\epsilon}^+) \leqslant \frac{x-h}{t-t_1} \leqslant c_2 \text{ (equivalently, } \gamma_{1,\epsilon}^+ \leqslant \gamma^+ \leqslant 0), \\ x(t_{\tilde{B}}) = x_{\tilde{B}}, \end{cases} \tag{4.53}$$

where $\gamma = \gamma(t, s(t)-0), \gamma^+ = \gamma(t, s(t)+0)$. By a similar discussion in [10], we can show that the strain value ahead of this shock wave increases from $\gamma_{1,\epsilon}^+$ until the time that the strain value ahead of the shock equals the strain value behind the shock. Thus, when the strain values on both sides of the shock wave become the same, the speed of the shock equals the speed of the rarefaction wave, and then penetration stops. This means that $\gamma = \gamma^+$ after some time. Therefore, we can see from the first equation in (4.40) that

$$\frac{\partial \eta}{\partial q} = -1. \tag{4.54}$$

Combining (4.46), (4.52), and (4.54) leads to

$$\left|\frac{\partial \zeta}{\partial p} \cdot \frac{\partial \eta}{\partial q}\right| < 1, \tag{4.55}$$

which is the minimal characterizing condition (cf. [16]).

To employ Theorem 5.1 in Chapter 6 of Li [16] to complete the proof, we need to explore the decay properties of the first derivatives of $\gamma(t,x)$ on the free boundary $x = s(t)$ (see the boundary conditions in (4.40)). By the standard theory on hyperbolic equations (see, e.g., pp. 10–11 in [16]), we obtain from (4.24) that

$$\left|\frac{\partial \gamma}{\partial t}\right| \leqslant |a'(\theta)| \leqslant \frac{K_0 K_7}{\theta - t_1} \quad \text{for} \quad \theta > t_1, \tag{4.56}$$

where (θ, h) is a point on the boundary $x = h$ and K_7 is a suitably small positive constant independent of (t,x). In (4.56), we have made use of (3.6) and (4.18). The characteristic line of the problem (4.24) is given by $t - \theta = \frac{x-h}{c(a(\theta))}$, or equivalently,

$$\theta - t_1 = \left[1 - \frac{\xi}{c(a(\theta))}\right](t - t_1). \tag{4.57}$$

It is easy to see that

$$0 \leqslant \xi \leqslant \xi_\epsilon \tag{4.58}$$

on the free boundary $x = s(t)$. By (4.58) and the boundedness of $a(\theta)$ in (4.17), we can see that

$$K_8 < 1 - \frac{\xi}{c(a(\theta))} < K_9, \tag{4.59}$$

where K_8 and K_9 are positive constants independent of (t, x) and ϵ. Thus, by (4.57) and (4.59), it follows from (4.56) that

$$\left|\frac{\partial \gamma}{\partial t}\right| \leqslant \frac{K_0 K_{10}}{t - t_1} \tag{4.60}$$

on the free boundary $x = s(t)$. Using the boundedness of $1/\mu(\gamma)$, we obtain from the first equation in (4.24) that on the free boundary $x = s(t)$, it also holds that

$$\left|\frac{\partial \gamma}{\partial x}\right| \leqslant \frac{K_0 K_{10}}{t - t_1}, \tag{4.61}$$

where K_{10} is a suitably small constant independent of (t, x). Thus, the first derivatives of $\gamma(t, x)$ should decay on the boundary $x = s(t)$ like (4.60) and (4.61).

Finally, by (4.55) and (4.60)–(4.61), we can use Theorem 5.1 in Chapter 6 of Li [16] to obtain the results of Lemma 4.1. The proof of Lemma 4.1 is completed. ∎

In (v, γ)-coordinates, Lemma 4.1 can be stated as follows.

LEMMA 4.2. *Under the assumptions in Lemma* 4.1, *there exists a unique function* $x = s(t) \in C^1$ *with* $s(t_2) = h$, *such that the typical free boundary problem* (2.5), (4.36), *and* (4.38) *has a unique global* C^1 *solution* (v, γ) *on the domain* $\tilde{\Omega}_2$. *Moreover, it holds that*

$$|v - v_5|, \ \ |\gamma - \gamma_2^+| \leqslant K_{11}\epsilon, \tag{4.62}$$

$$\left|\frac{\partial v}{\partial t}(t,x)\right|, \ \left|\frac{\partial v}{\partial x}(t,x)\right|, \ \left|\frac{\partial \gamma}{\partial t}(t,x)\right|, \ \left|\frac{\partial \gamma}{\partial x}(t,x)\right| \leqslant \frac{K_{11}\epsilon}{t - t_1}, \tag{4.63}$$

$$|s'(t) - s_0|, \ \ |s'(t) - \tilde{s}'(t)| \leqslant K_{11}\epsilon, \tag{4.64}$$

where K_{11} *is a positive constant independent of* (t, x) *and* ϵ, v_5, γ_2^+, s_0, *and* $\tilde{s}'(t)$ *are all given constants and functions (see Figure* 2*).*

Step 4: construct the unique continuous and piecewise C^1 solution in the domain $\tilde{\Omega}_3$.

In the previous steps, we determine the strains and particle velocities in the domain $\tilde{\Omega}_2$. For convenience of discussion, we denote the strain values at the boundary $x = h$ by $b(t)(t_2 < t < t_3)$ and $a(t)(t_1 < t < t_2)$. Then, it is easy to obtain the expressions for the strains, particle velocities, and Riemann invariants in material 1, and we list the results below (cf. Figure 3).

On the domains of (v_5, γ_5), we have

$$p(t,x) = \frac{2c_1}{\alpha\beta} g(b(x/c_1 + t - t_1)), \quad v_5(t,x) = c_1\gamma_5(t,x), \quad \gamma_5(t,x) = \frac{1}{\alpha\beta} g(b(x/c_1 + t - t_1)). \tag{4.65}$$

On the domain of (v_6, γ_6),

$$p(t,x) = \frac{2c_1}{\alpha\beta} g(a(x/c_1 + t - t_1)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t - 2t_1 + x/c_1)\right], \tag{4.66}$$

and

$$\gamma_6 = 0, \quad v_6 = \frac{2c_1}{\alpha\beta} g(a(x/c_1 + t - t_1)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t - 2t_1 + x/c_1)\right], \tag{4.67}$$

and

$$q(t,x) = \frac{2c_1}{\alpha\beta} g(a(x/c_1 + t - t_1)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t - 2t_1 + x/c_1)\right]. \tag{4.68}$$

On the domain of (v_7, γ_7),

$$v_7 = \frac{c_1}{\alpha\beta} g(b(x/c_1 + t - t_1)) + \frac{c_1}{\alpha\beta} g(a(x/c_1 + t - t_1)) - \frac{c_1}{E_1}\left[A + \epsilon\sigma_0(x/c_1 + t - 2t_1)\right], \tag{4.69}$$

and

$$\gamma_7 = \frac{1}{\alpha\beta} g(b(x/c_1 + t - t_1)) - \frac{1}{\alpha\beta} g(a(x/c_1 + t - t_1)) + \frac{1}{E_1}\left[A + \epsilon\sigma_0(x/c_1 + t - 2t_1)\right]. \tag{4.70}$$

On the domain (v_8, γ_8), we have

$$q(t,x) = \frac{2c_1}{\alpha\beta} g(a(x/c_1 + t - t_1)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t - 2t_1 + x/c_1)\right] \tag{4.71}$$

and the formulas for (v_8, γ_8) are given later.

Because the stress should keep continuous at the interface of $x = h$, we have

$$v_8(t,x)|_{x=h} = v(t,x)|_{x=h}, \quad E_1\gamma_8(t,x)|_{x=h} = E_2 g(\gamma)|_{x=h}. \tag{4.72}$$

Then, it follows from (4.71)–(4.72) that on the fixed boundary $x = h$,

$$v - \frac{c_1}{\alpha\beta} g(\gamma) = \frac{2c_1}{\alpha\beta} g(a(t)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t - t_1)\right]. \tag{4.73}$$

On the unknown free boundary $x = s(t)(s(t_3) = h)$, we have

$$\begin{cases} \rho_2[v(t, s(t)-0) - v(t, s(t)+0)]^2 \\ \qquad = [\gamma(t, s(t)-0) - \gamma(t, s(t)+0)][\sigma(\gamma(t, s(t)-0)) - \sigma(\gamma(t, s(t)+0))], \\ \frac{ds}{dt} = \sqrt{\frac{\sigma(\gamma(t,s(t)-0)) - \sigma(\gamma(t,s(t)+0))}{\rho_2(\gamma(t,s(t)-0) - \gamma(t,s(t)+0))}}, \end{cases} \tag{4.74}$$

which is nothing but the jump conditions (2.10), $v(t, s(t)+0)$ and $\gamma(t, s(t)+0)$ have been solved in Step 3.

Taking the square root of $(4.74)_1$ (the negative sign is taken, because in the special case of $\sigma_0 = 0$ the corresponding sign is negative), we can rewrite (4.74) as

$$\begin{cases} v = v(t, s(t)+0) - \sqrt{[\gamma - \gamma(t, s(t)+0)][\sigma(\gamma) - \sigma(\gamma(t, s(t)+0))]/\rho_2}, \\ \frac{ds}{dt} = \sqrt{\frac{\sigma(\gamma) - \sigma(\gamma(t,s(t)+0))}{\rho_2[\gamma - \gamma(t,s(t)+0)]}}, \end{cases} \tag{4.75}$$

where $v = v(t, s(t)-0)$, $\gamma = \gamma(t, s(t)-0)$.

Thus, in the Riemann invariants (p, q), the typical free boundary problem (2.5), (4.73), and (4.75) on the domain $\tilde{\Omega}_3$ can be equivalently rewritten as a typical free boundary problem for (4.2) with the following boundary conditions:

on the fixed boundary $x = h$,

$$(p+q)(t,h) = \frac{c_1}{\alpha\beta} g(H(p-q)) + \frac{2c_1}{\alpha\beta} g(a(t)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0(t-t_1)\right]; \tag{4.76}$$

on the free boundary $x = s(t)(s(t_3) = h)$,

$$\begin{cases} p+q = 2v(t,s(t)+0) \\ \qquad - 2\sqrt{[H(p-q)-\gamma(t,s(t)+0)][\sigma(H(p-q))-\sigma(\gamma(t,s(t)+0))]/\rho_2}, \\ \frac{ds}{dt} = \sqrt{\frac{\sigma(H(p-q))-\sigma(\gamma(t,s(t)+0))}{\rho_2[H(p-q)-\gamma(t,s(t)+0)]}}. \end{cases} \tag{4.77}$$

We have the following lemma.

LEMMA 4.3. *Under the assumptions in Lemma* 4.1, *there exists a unique function* $x = s(t) \in C^1$ *with* $s(t_3) = h$ *such that the typical free boundary problem* (4.2), (4.76), *and* (4.77) *has a unique global* C^1 *solution* (p,q) *on the domain* $\tilde{\Omega}_3$. *Moreover, it holds that*

$$|p(t,x) - \hat{p}|, \quad |q(t,x) - \hat{q}| \leqslant K_{12}\epsilon, \tag{4.78}$$

$$\left|\frac{\partial p}{\partial t}(t,x)\right|, \quad \left|\frac{\partial p}{\partial x}(t,x)\right|, \quad \left|\frac{\partial q}{\partial t}(t,x)\right|, \quad \left|\frac{\partial q}{\partial x}(t,x)\right| \leqslant \frac{K_{12}\epsilon}{t-t_1}, \tag{4.79}$$

$$|s'(t) - s_{1,\epsilon}| \leqslant K_{12}\epsilon, \tag{4.80}$$

where K_{12} *is a positive constant independent of* (t,x) *and* ϵ, $\hat{p}$, *and* $\hat{q}$ *are given by*

$$\hat{p} = v_\epsilon^- + \int_0^{\gamma_\epsilon^-} c(\tilde{\gamma})d\tilde{\gamma}, \quad \hat{q} = v_\epsilon^- - \int_0^{\gamma_\epsilon^-} c(\tilde{\gamma})d\tilde{\gamma}. \tag{4.81}$$

The proof is very similar to that of Lemma 4.2, and thus, the details are omitted.

In (v,γ)-coordinates, Lemma 4.3 can be stated as follows.

LEMMA 4.4. *Under the assumptions in Lemma* 4.1, *there exists a unique function* $x = s(t) \in C^1$ *with* $s(t_3) = h$ *such that the typical free boundary problem* (2.5), (4.73), *and* (4.75) *has a unique global* C^1 *solution* (v,γ) *on the domain* $\tilde{\Omega}_3$. *Moreover, it holds that*

$$|v - v_8|, \quad |\gamma - \gamma^-| \leqslant K_{13}\epsilon, \tag{4.82}$$

$$\left|\frac{\partial v}{\partial t}(t,x)\right|, \quad \left|\frac{\partial v}{\partial x}(t,x)\right|, \quad \left|\frac{\partial \gamma}{\partial t}(t,x)\right|, \quad \left|\frac{\partial \gamma}{\partial x}(t,x)\right| \leqslant \frac{K_{13}\epsilon}{t-t_1}, \tag{4.83}$$

$$|s'(t) - s_1| \leqslant K_{13}\epsilon, \tag{4.84}$$

where K_{13} *is a positive constant independent of* (t,x) *and* ϵ, v_8, γ^-, *and* s_1 *are all given constants (see Figure* 2*).*

Now, we denote the strain and particle velocity at the boundary of domain $\tilde{\Omega}_3$ by $m(t), n(t)$, i.e., $\gamma(t,x)|_{x=h} = m(t), v(t,x)|_{x=h} = n(t)$. Then, by (4.72) we have

$$p(t,x)|_{x=h} = (v_8 + c_1\gamma_8)|_{x=h} = n(t) + \frac{c_1}{\alpha\beta} g(m(t)) \tag{4.85}$$

on the domain of (v_8, γ_8). So making use of the first equation in (4.2), we obtain

$$p(t,x) = n(x/c_1 + t - t_1) + \frac{c_1}{\alpha\beta} g(m(x/c_1 + t - t_1)). \tag{4.86}$$

Moreover, by (4.71) and (4.85), it is easy to derive that

$$n(t) = \frac{2c_1}{\alpha\beta} g(a(t)) + \frac{c_1}{\alpha\beta} g(m(t)) - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0 (t - t_1)\right]. \tag{4.87}$$

Thus, a combination of (4.71) and (4.86)–(4.87) leads to

$$\begin{aligned} v_8(t,x) = {} & \frac{c_1}{\alpha\beta} g\left(m\left(\frac{x}{c_1} + t - t_1\right)\right) + \frac{2c_1}{\alpha\beta} g\left(a\left(\frac{x}{c_1} + t - t_1\right)\right) \\ & - \frac{2c_1}{E_1}\left[A + \epsilon\sigma_0\left(\frac{x}{c_1} + t - 2t_1\right)\right] \end{aligned} \tag{4.88}$$

and

$$\gamma_8(t,x) = \frac{1}{\alpha\beta} g(m(x/c_1 + t - t_1)). \tag{4.89}$$

It is worth noting that the solution constructed for the initial boundary value problem (2.5), (2.6), and (3.1) satisfies the Lax's entropy condition

$$c(\gamma^-(t,x)) > \tilde{s}_1 > c(\gamma_2^+(t,x)), \quad c(\gamma_2^+(t,x)) > \tilde{s}_0 > c(\gamma_1^+(t,x)), \tag{4.90}$$

where $\gamma_1^+(t,x), \gamma_2^+(t,x)$, and $\gamma^-(t,x)$ denote the strains in the domains $\tilde{\Omega}_1, \tilde{\Omega}_2$, and $\tilde{\Omega}_3$, respectively and $\tilde{s}_0$ and $\tilde{s}_1$ denote the speeds of the second and third transmitted curved shock waves (see Figure 3). This fact is easy to verify because (4.90) can be deduced from (4.25), (4.62), (4.64), (4.82), and (4.84). Thus, by (4.90) we may say that the derived solution is physically admissible.

We can also show that the third transmitted curved shock wave captures the second curved shock wave in a finite time. The key point of proving this statement is to use the Lax's entropy condition (4.90) and the convexity of the stress-strain relation for material 2. Because the proof is very similar to that used in establishing the wave catching-up phenomenon shown in Figure 2 (or see [10]), the details are omitted here.

Finally, by the above argument and Lemmas 4.2 and 4.4, to complete the proof of Theorem 3.2, it suffices to show that in material 1, the solution of the initial boundary value problem (2.5), (2.6), and (3.1) (see Figure 3) is very *close* (see Definition 3.1) to the solution constructed in Figure 2 (cf. [10]). In fact, for these quantities $(v_i(t,x), \gamma_i(t,x))(i = 1, 2, \ldots, 8)$, by comparing (4.27), (4.29)–(4.31), (4.33), (4.65), (4.67), (4.69)–(4.70), and (4.88)–(4.89) with (3.4)–(3.8) in [10], we can easily verify the above claim. Therefore, the proof of Theorem 3.2 is finished. ∎

5. Conclusions. Wave propagation due to impact in a prestressed two-material bar is studied. We employ the same method as that used in [10] to construct the solution. It is shown that if the stress-strain curve of the second material is convex around a given prestrain, the phenomenon in which the third transmitted tensile wave catches the first transmitted compressive wave is possible. In practical applications, one may explore such a phenomenon for designing a structure with good impact-resistance capabilities.

From a mathematical perspective, it is important to know the stability of the wave pattern for such a wave catching-up phenomenon. If it is unstable, this phenomenon cannot be observed in reality. Here, we provide a mathematical proof for the global structure stability of constructed solutions for the wave catching-up phenomenon. A constructive method is used. We consider the relevant domains in the

x-t plane separately. In each of them, by the theory of typical free boundary problems and careful treatment of the complexity caused by wave reflections at interface, the stability is successfully proved. We point out that the global structure stability considered here is for a small disturbance superimposed on the end boundary condition. As a result, we do not impose any artificial constraints, compared with the results in [6]. The theoretical results on the solution and stability of the wave catching-up phenomenon in a two-material bar may provide guidance on a new way to design superior impact-protection structures.

Appendix A. In this appendix, we show how (2.1) can be reduced to (2.5). First, from the basic hypothesis in section 2 we have

$$\bar{X} = \begin{cases} (1+\gamma_1)X_0, & 0 \le X_0 \le h_0, \\ (1+\gamma_2)X_0 + (\gamma_1 - \gamma_2)h_0, & h_0 \le X_0. \end{cases} \tag{A.1}$$

The displacement from B_0 to $\bar{B}$ is $u_0(X_0) = \bar{X} - X_0$. Denote the position of a material point in the current configuration $\hat{B}$ as y. Then, the displacements from B_0 to $\hat{B}$ and from $\bar{B}$ to $\hat{B}$ are, respectively, $u(t, X_0) = y - X_0$ and $\bar{u}(t, \bar{X}) = y - \bar{X}$. It is easy to see that

$$u(t, X_0) = \bar{u}(t, \bar{X}) + u_0(X_0). \tag{A.2}$$

Then, a combination of (A.1) and (A.2) leads to the following relationships for strains:

$$\gamma(t, X_0) = \begin{cases} (1+\gamma_1)\bar{\gamma}(t, \bar{X}) + \gamma_1, & 0 \le X_0 \le h_0, \\ (1+\gamma_2)\bar{\gamma}(t, \bar{X}) + \gamma_2, & h_0 \le X_0, \end{cases} \tag{A.3}$$

where $\bar{\gamma} = \bar{u}_{\bar{X}}$. We denote the velocity and stress on $\bar{B}$ as $\bar{v} = \bar{u}_t$ and $\bar{\sigma}$, respectively. By the transformations (A.1)–(A.3), for material 1 we have from (2.1) that

$$\rho_1^0 v_t = \rho_1^0 \bar{v}_t = E_1^0 \gamma_{X_0} = E_1^0 (1+\gamma_1)\bar{\gamma}_{\bar{X}} \bar{X}_{X_0} = E_1^0 (1+\gamma_1)^2 \bar{\gamma}_{\bar{X}},$$

or equivalently, $\rho_1 \bar{v}_t = E_1 \bar{\gamma}_{\bar{X}}$ for $0 \le \bar{X} < \bar{h}$, where $\rho_1 = \rho_1^0/(1+\gamma_1)$, $E_1 = E_1^0(1+\gamma_1)$. Moreover, by $\gamma_t = v_{X_0}$ and $\gamma_t = (1+\gamma_1)\bar{\gamma}_t$, $v_{X_0} = \bar{v}_{\bar{X}} \bar{X}_{X_0} = (1+\gamma_1)\bar{v}_{\bar{X}}$, we have

$$\bar{\gamma}_t = \bar{v}_{\bar{X}} \quad \text{for} \quad 0 \le \bar{X} < \bar{h}. \tag{A.4}$$

However, for material 2 we note that

$$\sigma(\gamma) = E_2^0 f(\gamma(t, X_0)) = E_2^0 f(\gamma_2 + (1+\gamma_2)\bar{\gamma}(t, \bar{X})). \tag{A.5}$$

Then, it follows from the first equation in (2.1) and (A.5) that

$$\rho_2^0 v_t = \rho_2^0 \bar{v}_t = E_2^0 f(\gamma(t, X_0))_{X_0} = E_2^0 [f(\gamma_2 + (1+\gamma_2)\bar{\gamma}(t, \bar{X})) - f(\gamma_2)]_{X_0}. \tag{A.6}$$

Let $\tilde{f}(\bar{\gamma}) = f(\gamma_2 + (1+\gamma_2)\bar{\gamma}(t, \bar{X})) - f(\gamma_2)$. Then, we obtain from (A.6) that

$$\rho_2 \bar{v}_t = [E_2^0 \tilde{f}(\bar{\gamma})]_{\bar{X}}, \tag{A.7}$$

where $\rho_2 = \rho_2^0/(1+\gamma_2)$. By noting that $\tilde{f}(\bar{\gamma})_{\bar{\gamma}}|_{\bar{\gamma}=0} = (1+\gamma_2) f'(\gamma_2)$, we may introduce

$$g(\bar{\gamma}) = \tilde{f}(\bar{\gamma})/((1+\gamma_2) f'(\gamma_2)), \quad E_2 = E_2^0 (1+\gamma_2) f'(\gamma_2). \tag{A.8}$$

Now, (A.7) is reduced to $\rho_2 \bar{v}_t = \bar{\sigma}_{\bar{X}}$ for $\bar{h} < \bar{X} < \infty$, where $\bar{\sigma} = E_2 g(\bar{\gamma})$. Similarly, the second equation in (2.1) can be written as $\bar{\gamma}_t = \bar{v}_{\bar{X}}$ for $\bar{h} < \bar{X} < \infty$.

Thus, based on the above analysis, (2.1) can be converted to

$$\bar{\rho} \bar{v}_t = \bar{\sigma}_{\bar{X}}, \quad \bar{\gamma}_t = \bar{v}_{\bar{X}}, \tag{A.9}$$

where $\bar{\gamma}, \bar{v}, \bar{\sigma}$ are smooth functions of $(t, \bar{X})$ and $\bar{\rho}$ and $\bar{\sigma}$ are given in (2.6)–(2.8), respectively.

Acknowledgment. The authors would like to sincerely thank the anonymous referees for pertinent comments and valuable suggestions.

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