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SPECTRUM STRUCTURE AND BEHAVIORS OF THE VLASOV–MAXWELL–BOLTZMANN SYSTEMS*

HAI-LIANG LI†, TONG YANG‡, AND MINGYING ZHONG§

Abstract. The spectrum structures and behaviors of the Vlasov–Maxwell–Boltzmann (VMB) systems for both two species and one species are studied in this paper. The analysis shows the effect of the Lorentz force induced by the electromagnetic field leads to some different structure of spectrum from the classical Boltzmann equation and the closely related VPB system. And the significant difference between the two-species VMB model and one-species VMB model are given. The structure in high frequency illustrates the hyperbolic structure of the Maxwell equation. Furthermore, the long time behaviors and the optimal convergence rates to the equilibrium of the VMB systems for both two species and one species are established based on the spectrum analysis, and in particular the phenomena of the electric field dominating and magnetic field dominating are observed for the one-species VMB system.

Key words. Vlasov–Maxwell–Boltzmann system, Lorentz force, spectrum structure, optimal convergence rates

AMS subject classifications. 76P05, 82C40, 82D05

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1. Introduction.

The Vlasov–Maxwell–Boltzmann system is a fundamental model in plasma physics for describing the time evolution of dilute charged particles, such as electrons and ions, under the influence of the self-induced Lorentz forces governed by Maxwell equations; cf. [2] for derivation and the physical background. In the literatures, there are two basic models: one is called the two-species Vlasov–Maxwell–Boltzmann system that describes both the time evolution of ions and electrons

$$
\begin{cases}
\partial_t F_+ + v\cdot\nabla_x F_+ + (E+v\times B)\cdot\nabla_v F_+ = \mathcal{Q}(F_+,F_+)+\mathcal{Q}(F_+,F_-),\\
\partial_t F_- + v\cdot\nabla_x F_- - (E+v\times B)\cdot\nabla_v F_- = \mathcal{Q}(F_-,F_+)+\mathcal{Q}(F_-,F_-),\\
\partial_t E = \nabla_x\times B - \displaystyle\int_{\mathbb{R}^3}(F_+-F_-)v\,dv,\\
\partial_t B = -\nabla_x\times E,\\
\nabla_x\cdot E = \displaystyle\int_{\mathbb{R}^3}(F_+-F_-)dv, \quad \nabla_x\cdot B = 0,
\end{cases}
\tag{1.1}
$$

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where $F_\pm = F_\pm(t,x,v)$ are the number density distribution functions of charged particles, and $E(t,x)$, $B(t,x)$ denote the electro and magnetic fields, respectively. Here, the operator $\mathcal{Q}(\cdot,\cdot)$ describing the binary elastic collisions is given by

$$\mathcal{Q}(F,G)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-v_*)\cdot\omega|(F(v_*')G(v')-F(v_*)G(v))dv_*d\omega, \tag{1.2}$$

where

$$v'=v-[(v-v_*)\cdot\omega]\omega,\quad v_*'=v_*+[(v-v_*)\cdot\omega]\omega,\quad \omega\in\mathbb{S}^2.$$

The other one, called the one-species Valsov–Maxwell–Boltzmann system, takes account of the fact that the ion is much heavier than the electron and the electron moves faster than the ion so that the time evolution of electron can be considered under a fixed background of ion distribution

$$\begin{cases}\partial_t F+v\cdot\nabla_x F+(E+v\times B)\cdot\nabla_v F=\mathcal{Q}(F,F),\\ \partial_t E=\nabla_x\times B-\displaystyle\int_{\mathbb{R}^3}Fvdv,\\ \partial_t B=-\nabla_x\times E,\\ \nabla_x\cdot E=\displaystyle\int_{\mathbb{R}^3}Fdv-n_b,\quad \nabla_x\cdot B=0,\end{cases} \tag{1.3}$$

where the time evolution of electrons is considered under the influence of a fixed ions background $n_b(x)$, and $F = F(t,x,v)$ is the number density function of electrons. And the operator $\mathcal{Q}(\cdot,\cdot)$ is defined by (1.2).

The Vlasov–Maxwell–Boltzmann system has been intensively studied and much important progress has been made in [7, 6, 10, 11, 19]. For instance, Guo [10] has first established the global existence of classical solutions in three-dimensional torus when the initial data is a small perturbation of a global Maxwellian, and Strain [19] proved the corresponding global existence result of classical solutions in $\mathbb{R}^3$. The diffusive limit for the two-species Vlasov–Maxwell–Boltzmann system was shown in [11]. The recent important investigation of long time behavior of global solution near the global Maxwellian studied in [7, 6] shed light on the complicacy of the Vlasov–Maxwell–Boltzmann (VMB) system. Therein, it was shown by the method of compensated functions in [6] that the total energy of the linearized one-species VMB system decays at the rate $(1+t)^{-\frac{3}{8}}$ (but the decay rate of the nonlinear system has not been obtained since the decay rate of the linear system obtained therein seems to be not enough to deal with the time evolution of the nonlinear terms) and in [7] that the total energy of the nonlinear two-species VMB system decays at the rate $(1+t)^{-\frac{3}{4}}$.

A natural and interesting question follows then: what is the main characters of the structures and time-asymptotical behaviors of the VMB system on the transport of charged particles under the influence of electromagnetic fields governed by the Maxwell equation and/or mutual interaction between charged particles? One of the methods to investigate this properties is the analysis of spectrum structures of the VMB system. However, in contrast to the works on Boltzmann equation [8, 15, 16, 20, 21, 22] and VPB system [13, 14], the spectrum of the linearized VMB system around a global equilibrium has not been given despite its importance. The main purpose of this paper is to fill in this gap.

The main purpose of this paper is to consider the spectrum structures of the linearized systems for the above VMB systems (1.1) and (1.3) around a global equilibrium so that some specific properties influenced by the electromagnetic fields and/or the mutual interaction between charged particles are revealed. As it has already been studied in [3, 13, 14] about the structures and behaviors of both one-species and two-species VPB systems, the influence of electric field gives rise to some complicated phenomena on the transport of charged particles. Indeed, it was shown in [3, 13] that the global distribution function to the one-species VPB system tends to the global Maxwellian at the optimal rate $(1+t)^{-\frac{1}{4}}$ in $\mathbb{R}^3$ which is slower than the Boltzmann equation and is caused mainly by the slower but optimal decay $(1+t)^{-\frac{1}{4}}$ of the electric fields. On the other hand, the global distribution function and electric field of the two-species VPB system was proved in [23, 24, 14] to converge to the equilibrium at the optimal rate $(1+t)^{-\frac{3}{4}}$ for the distribution functions and $e^{-\mathcal{O}(1)t}$ for the electric field, where the key issue lies in the fact that the mutual interaction of charged particles leads to spectral gap.

In the present paper, we shall establish the structure of the spectrum in detail for both two-species and one-species linearized VMB systems, analyze the corresponding semigroups to the linearized operators and show the optimal decay rates of global solutions to the linearized VMB systems, and finally obtain the (optimal) time-asymptotical behaviors of global solutions to the nonlinear VMB systems of both two-species and one-species types. To be more precise, we first establish the structure of the spectrum in detail for two linearized VMB systems and reveal the effect of electromagnetic fields on the distribution of spectrum of linearized operators. Indeed, the influence of electromagnetic fields on the transport of one-species charged particles causes the linearized VMB system to admit higher order eigenvalues (spectrum) $\lambda_6 = \lambda_7 = -\mathcal{O}(1)|\xi|^4$ at lower frequency $0 < |\xi| \ll 1$ besides those behaving like $\lambda_j = j\mathrm{i} - \mathcal{O}(1)|\xi|^2$ for $j = 0, \pm 1$ (refer to Theorem 2.7). However, the mutual interaction of particles with different type of charges cancels this particular influence of electromagnetic fields and only the spectrum like $\lambda_j = -\mathcal{O}(1)|\xi|^2$ is kept finally. In addition, the appearance of electromagnetic fields causes the additional spectrum (eigenvalues) around $\pm\mathrm{i}|\xi|$ and in particular $\mathrm{Re}\lambda = -\mathcal{O}(1)|\xi|^{-1}$ at high frequency $|\xi| \gg 1$ for both one-species and two-species VMB systems. This unfortunately leads to the loss of regularity of global solutions (refer to Theorems 2.2 and 2.7 for details).

Then, in terms of the analysis on spectrum structures and the semigroups of both two-species and one-species linearized VMB systems, we are further able to establish the optimal time convergence rates of the global solutions for both linearized systems and nonlinear systems. For two-species VMB system, we can observe some phenomena of wave propagation and magnetic field domination on long time behaviors of charge transport due to the effect of magnetic filed and mutual interaction between the particles of two species. Indeed, for the global solution (f_1, f_2, E, B) to the linearized two-species VMB system, we can show that the distribution function f_1, corresponding to the total summation of the distribution functions between the two species, is governed by the linearized Boltzmann equation and its optimal time decay rate $(1+t)^{-\frac{3}{4}}$ in L^2-norm has been already established for instance in [20, 26]. Meanwhile, the magnetic field B is also shown to tend to zero at the optimal time decay rates $(1+t)^{-\frac{3}{4}}$ in L^2-norm, but the distribution function f_2, corresponding to the difference of the distribution functions between the two species, and the electric field E decay at the faster optimal time rate $(1+t)^{-\frac{5}{4}}$ in L^2-norm. In particular, the macroscopic part of f_2 decays exponentially and the microscopic part of f_2 decays at the optimal rate $(1+t)^{-\frac{5}{4}}$ (refer to Theorem 2.4 for details). Here we recall that the macroscopic part and microscopic part related to f_1 decay at the different optimal rates $(1+t)^{-\frac{3}{4}}$ and $(1+t)^{-\frac{5}{4}}$, respectively, as shown in [26]. These optimal algebraic time decay rates also established for the distribution solution (f_1, f_2, E, B) to the nonlinear two-species VMB system, where the distribution function f_2 and the electric field E converge to zero state at the faster optimal time rate $(1+t)^{-\frac{5}{4}}$ than the optimal rate $(1+t)^{-\frac{3}{4}}$ for the function f_1 and the magnetic field B (refer to Theorems 2.5 and 2.6 for details).

For the one-species VMB system, some more subtle phenomena on long time behaviors of charge transport are observed. Indeed, we can show that there are different long time behaviors of global solutions to the one-species VMB system characterized and dominated by the effect of either the magnetic field or the electric field, which depends on whether the relation $\nabla_x \cdot E_0 = n_0$ holds or not with n_0 denoting the first moment of the initial distribution. We fist prove the phenomena of electric field dominating in the case that $\nabla_x \cdot E_0 \neq n_0$ for the linearized VMB system. Namely, we show that both the distribution function f and the electric field E tend to the equilibrium state at the optimal decay rate $(1+t)^{-\frac{1}{4}}$ in L^2-norm which is slower than the faster optimal convergence rate $(1+t)^{-\frac{3}{8}}$ of the magnetic field B to equilibrium state, and in particular the macroscopic density, momentum, and energy of the distribution function f, decay at the optimal rates $(1+t)^{-\frac{3}{4}}$, $(1+t)^{-\frac{1}{4}}$, and $(1+t)^{-\frac{3}{4}}$, respectively (refer to Theorem 2.9 for details). This phenomena of electric field dominating has not been observed before. However, one cannot extend this linear theory to the nonlinear VMB system although the global existence of strong solution can be established (refer to Theorem 2.11 for details), because these optimal time decay rates are too weak to be employed to control the expected long time rates of nonlinear terms (refer to Remark 2.13 for some verification).

In the case that $\nabla_x \cdot E_0 = n_0$, we prove the phenomena of magnetic field dominating for both linearized and nonlinear VMB systems. Indeed, we are able to show that the magnetic field B tends to zero at the optimal time rate $(1+t)^{-\frac{3}{8}}$ in L^2-norm, which is slower than the optimal time decay rate $(1+t)^{-\frac{5}{8}}$ of the distribution function f and the optimal time decay rate $(1+t)^{-\frac{3}{4}}$ of the electric field E. In particular, the macroscopic density, momentum, and energy related to f are shown to decay at the different optimal rates $(1+t)^{-\frac{5}{4}}$, $(1+t)^{-\frac{5}{8}}$, and $(1+t)^{-\frac{3}{4}}$, respectively (refer to Theorem 2.10 for details). Furthermore, we can also study rigorously the time-asymptotical behaviors of global solutions to the nonlinear VMB system and in particular obtain the optimal time decay rate $(1+t)^{-\frac{5}{8}}$ of the distribution function f, the optimal time decay rate $(1+t)^{-\frac{3}{8}}$ of the magnetic fields B, and the faster time decay rate $(1+t)^{-\frac{3}{4}}\ln(1+t)$ of electric field E (refer to Theorem 2.11 for details). This gives more information than those obtained by the energy method in [7, 6] where only the upper bound of the decay rate of total energy was obtained. Here, we should mention that the time-convergence rate $(1+t)^{-\frac{3}{8}}$ of the total energy of global solution for the linearized one-species VMB system in [7] indeed corresponds to the case of the magnetic field dominating phenomena but without the above analysis given in detail in the present paper. In particular, we obtain the optimal decay rates of the solution to the nonlinear one-species VMB system which was not solved in [7].

The rest of this paper will be organized as follows. In section 2, the main results on spectrum structures and time-asymptotic behaviors of global solutions for two-species VMB and one-species VMB systems are stated in sections 2.1 and 2.2, respectively. In sections 3 and 4, the spectrum structures of the two linearized systems for both two-species and one-species charge motion will be analyzed with detailed description in low and high frequency regions. Based on this analysis on the linearized operators, the decomposition of the corresponding semigroups generated by these operators will be given in section 5 together with the optimal convergence rates to the equilibrium in time. The optimal convergence rates of the global solution to the original nonlinear system will be studied in the last section.

2. Main results.

2.1. Two-species VMB system. In order to study the spectrum structure of the system (1.1), it is convenient to consider the following Cauchy problem for $F_1 = F_+ + F_-$ and $F_2 = F_+ - F_-$ that takes care of the cancellation in the original system:

(2.1)
$$\begin{cases} \partial_t F_1 + v \cdot \nabla_x F_1 + (E + v \times B) \cdot \nabla_v F_2 = \mathcal{Q}(F_1, F_1), \\ \partial_t F_2 + v \cdot \nabla_x F_2 + (E + v \times B) \cdot \nabla_v F_1 = \mathcal{Q}(F_2, F_1), \\ \partial_t E = \nabla_x \times B - \int_{\mathbb{R}^3} F_2 v dv, \\ \partial_t B = -\nabla_x \times E, \\ \nabla_x \cdot E = \int_{\mathbb{R}^3} F_2 dv, \quad \nabla_x \cdot B = 0, \\ F_1(0,x,v) = F_{1,0}, \quad F_2(0,x,v) = F_{2,0}, \quad E(0,x) = E_0(x), \quad B(0,x) = B_0(x). \end{cases}$$

In the following, we will consider the spectrum of the operator by linearizing the system (2.1) around an equilibrium state $(F_1^*, F_2^*, E^*, B^*) = (M(v), 0, 0, 0)$ with $M(v)$ being the normalized Maxwellian given by

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^3.$$

Set

$$F_1 = M + \sqrt{M} f_1, \quad F_2 = \sqrt{M} f_2.$$

Then the system (2.1) for (F_1, F_2, E, B) can be written as the following system for (f_1, f_2, E, B):

$$\partial_t f_1 + v \cdot \nabla_x f_1 - L f_1 = \frac{1}{2}(v \cdot E) f_2 - (E + v \times B) \cdot \nabla_v f_2 + \Gamma(f_1, f_1), \tag{2.2}$$

$$\partial_t f_2 + v \cdot \nabla_x f_2 - L_1 f_2 - v\sqrt{M} \cdot E = \frac{1}{2}(v \cdot E) f_1 - (E + v \times B) \cdot \nabla_v f_1 + \Gamma(f_2, f_1), \tag{2.3}$$

$$\partial_t E = \nabla_x \times B - \int_{\mathbb{R}^3} f_2 v \sqrt{M} dv, \tag{2.4}$$

$$\partial_t B = -\nabla_x \times E, \tag{2.5}$$

$$\nabla_x \cdot E = \int_{\mathbb{R}^3} f_2 \sqrt{M} dv, \quad \nabla_x \cdot B = 0, \tag{2.6}$$

$$f_1(0,x,v) = f_{1,0} =: \frac{F_{1,0} - M}{\sqrt{M}}, \quad f_2(0,x,v) = f_{2,0} =: \frac{F_{2,0}}{\sqrt{M}}, \quad E(0,x) = E_0, \quad B(0,x) = B_0, \tag{2.7}$$

where

$$Lf = \frac{1}{\sqrt{M}}[\mathcal{Q}(M, \sqrt{M} f) + \mathcal{Q}(\sqrt{M} f, M)],$$

$$L_1 f = \frac{1}{\sqrt{M}} \mathcal{Q}(\sqrt{M} f, M), \quad \Gamma(f,g) = \frac{1}{\sqrt{M}} \mathcal{Q}(\sqrt{M} f, \sqrt{M} g).$$

The linearized collision operators L and L_1 can be written as (cf. [1, 25])

$$\begin{aligned}
(Lf)(v) &= (Kf)(v) - \nu(v) f(v), \quad (L_1 f)(v) = (K_1 f)(v) - \nu(v) f(v), \\
\nu(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| M_* d\omega dv_*, \\
(Kf)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| (\sqrt{M_*'} f' + \sqrt{M'} f_*' - \sqrt{M} f_*) \sqrt{M_*} d\omega dv_* \\
&= \int_{\mathbb{R}^3} k(v, v_*) f(v_*) dv_*, \\
(K_1 f)(v) &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \sqrt{M_*'} \sqrt{M_*} f' d\omega dv_* = \int_{\mathbb{R}^3} k_1(v, v_*) f(v_*) dv_*,
\end{aligned}$$

where $\nu(v)$ is called the collision frequency, and K and K_1 are self-adjoint compact operators on $L^2(\mathbb{R}^3_v)$ with real symmetric integral kernels $k(v, v_*)$ and $k_1(v, v_*)$. The null space of the operator L, denoted by N_0, is a subspace spanned by the orthonoraml basis $\{\chi_j, \ j = 0, 1, \ldots, 4\}$ given by

$$\chi_0 = \sqrt{M}, \quad \chi_j = v_j \sqrt{M} \ (j = 1, 2, 3), \quad \chi_4 = \frac{(|v|^2 - 3)\sqrt{M}}{\sqrt{6}}; \tag{2.8}$$

and the null space of the operator L_1, denoted by N_1, is spanned only by $\sqrt{M}$.

For later use, denote by $L^2(\mathbb{R}^3)$ the Hilbert space of complex valued functions on $\mathbb{R}^3$ with the inner product and the norm given by

$$(f, g) = \int_{\mathbb{R}^3} f(v) \overline{g(v)} dv, \quad \|f\| = \left(\int_{\mathbb{R}^3} |f(v)|^2 dv \right)^{1/2}.$$

And let $\mathrm{P}_0, P_{\mathrm{d}}$ be the projection operators from $L^2(\mathbb{R}^3_v)$ to the subspace N_0, N_1 with

$$\mathrm{P}_0 f = \sum_{i=0}^{4} (f, \chi_i) \chi_i, \quad \mathrm{P}_1 = I - \mathrm{P}_0, \tag{2.9}$$

$$P_{\mathrm{d}} f = (f, \sqrt{M}) \sqrt{M}, \quad P_r = I - P_{\mathrm{d}}. \tag{2.10}$$

From Boltzmann's H-theorem, the linearized collision operators L and L_1 are nonpositive. Precisely, there is a constant $\mu > 0$ such that

$$(Lf, f) \le -\mu \|\mathrm{P}_1 f\|^2, \quad f \in D(L), \tag{2.11}$$

$$(L_1 f, f) \le -\mu \|P_r f\|^2, \quad f \in D(L_1), \tag{2.12}$$

where $D(L)$ and $D(L_1)$ are the domains of L and L_1 given by

$$D(L) = D(L_1) = \left\{ f \in L^2(\mathbb{R}^3) \,|\, \nu(v) f \in L^2(\mathbb{R}^3) \right\}.$$

In addition, for the hard sphere model, ν satisfies

$$\nu_0 (1 + |v|) \le \nu(v) \le \nu_1 (1 + |v|). \tag{2.13}$$

Without the loss of generality, we choose $\nu(0) \ge \nu_0 \ge \mu > 0$ throughout this paper.

From the system (2.2)–(2.6) for (f_1, f_2, E, B), we have the following decoupled linearized system for f_1 and (f_2, E, B):

$$\partial_t f_1 + v \cdot \nabla_x f_1 - L f_1 = 0, \tag{2.14}$$

$$\partial_t f_2 + v \cdot \nabla_x f_2 - L_1 f_2 - v\sqrt{M} \cdot E = 0, \tag{2.15}$$

$$\partial_t E = \nabla_x \times B - \int_{\mathbb{R}^3} f_2 v \sqrt{M} dv, \tag{2.16}$$

$$\partial_t B = -\nabla_x \times E, \tag{2.17}$$

$$\nabla_x \cdot E = \int_{\mathbb{R}^3} f_2 \sqrt{M} dv, \quad \nabla_x \cdot B = 0. \tag{2.18}$$

Equation (2.14) is simply the linearized Boltzmann equation around a global Maxwellian and thus its spectrum structure has been well established since the 1970s. Therefore, we need only study the spectrum structure of the linear system (2.15)–(2.18) on (f_2, E, B).

For convenience of notation, rewrite the linearized system for $f_1 \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ and $U = (f_2, E, B)^T \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) \times L^2(\mathbb{R}^3_x) \times L^2(\mathbb{R}^3_x)$ as

$$\partial_t f_1 = \mathbb{B}_0 f_1, \quad f_1(0, x, v) = f_{1,0}(x, v), \tag{2.19}$$

and

$$\begin{cases} \partial_t U = \mathbb{A}_0 U, \quad t > 0, \\ \nabla_x \cdot E = (f_2, \sqrt{M}), \quad \nabla_x \cdot B = 0, \\ U(0, x, v) = U_0(x, v) = (f_{2,0}, E_0, B_0), \end{cases} \tag{2.20}$$

where the operators $\mathbb{B}_0$ and $\mathbb{A}_0$ are operators on $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ and $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) \times L^2(\mathbb{R}^3_x) \times L^2(\mathbb{R}^3_x)$ defined by

$$\mathbb{B}_0 = L - (v \cdot \nabla_x), \tag{2.21}$$

$$\mathbb{A}_0 = \begin{pmatrix} L_1 - (v \cdot \nabla_x) & v\sqrt{M} & 0 \\ -P_m & 0 & \nabla_x \times \\ 0 & -\nabla_x \times & 0 \end{pmatrix}, \tag{2.22}$$

with

$$P_m f = (f, v\sqrt{M}). \tag{2.23}$$

Take the Fourier transform in (2.20) with respect to x to have

$$\begin{cases} \partial_t \hat{U} = \hat{\mathbb{A}}_0(\xi) \hat{U}, \quad t > 0, \\ \mathrm{i}(\xi \cdot \hat{E}) = (\hat{f}_2, \sqrt{M}), \quad \mathrm{i}(\xi \cdot \hat{B}) = 0, \\ \hat{U}(0, \xi, v) = \hat{U}_0(\xi, v) = (\hat{f}_{2,0}, \hat{E}_0, \hat{B}_0), \end{cases} \tag{2.24}$$

where $\hat{\mathbb{A}}_0(\xi)$ is the Fourier transform of $\mathbb{A}_0$.

Note that it is difficult to study the spectrum structure of the system (2.24) directly because of the constraints on $\hat{E}$ and $\hat{B}$ given in the second and third equations. One of the key observations in this paper is that by using the identity $F = (F \cdot y)y - y \times y \times F$ for any $F \in \mathbb{R}^3$ and $y \in \mathbb{S}^2$, we can first solve for $\hat{V} = (\hat{f}_2, \omega \times \hat{E}, \omega \times \hat{B})$ with

$\omega = \xi/|\xi|$ so that by combining these two constraints, we have the full information on $\hat{U}$. For this reason, we consider the following system for $\hat{V}$:

$$\begin{cases} \partial_t \hat{V} = \hat{\mathbb{A}}_1(\xi)\hat{V}, \quad t > 0, \\ \hat{V}(0,\xi,v) = \hat{V}_0(\xi,v) = (\hat{f}_{2,0}, \omega \times \hat{E}_0, \omega \times \hat{B}_0), \end{cases} \tag{2.25}$$

with

$$\hat{\mathbb{A}}_1(\xi) = \begin{pmatrix} \hat{B}_1(\xi) & -v\sqrt{M}\cdot\omega\times & 0 \\ -\omega \times P_m & 0 & \mathrm{i}\xi\times \\ 0 & -\mathrm{i}\xi\times & 0 \end{pmatrix}. \tag{2.26}$$

Here, for $\xi \neq 0$,

$$\hat{B}_1(\xi) = L_1 - \mathrm{i}(v\cdot\xi) - \tfrac{\mathrm{i}(v\cdot\xi)}{|\xi|^2} P_{\mathrm{d}}. \tag{2.27}$$

Before further discussion, we will give a remark on the eigenvalues and eigenfunctions of the original system and the reduced system (2.25).

Remark 2.1. Set $F_+ = \frac{1}{2}M + \sqrt{M}f_+$, $F_- = \frac{1}{2}M + \sqrt{M}f_-$ to have

$$\begin{cases} \partial_t f_\pm + v\cdot\nabla_x f_\pm - \dfrac{1}{2}(L \pm L_1)f_\pm - \dfrac{1}{2}(L \mp L_1)f_\pm \mp \dfrac{1}{2}v\sqrt{M}\cdot E = 0, \\ \partial_t E = \nabla_x \times B - \displaystyle\int_{\mathbb{R}^3}(f_+ - f_-)v\sqrt{M}dv, \quad \partial_t B = -\nabla_x \times E, \\ \nabla_x \cdot E = \displaystyle\int_{\mathbb{R}^3}(f_+ - f_-)\sqrt{M}dv, \quad \nabla_x\cdot B = 0. \end{cases} \tag{2.28}$$

The eigenvalues of the system (2.28) are the same as those of (2.19) and (2.25), and the eigenfunctions of the system (2.28) can be obtained as linear combinations of those for (2.19) and (2.25). In fact, let λ be an eigenvalue with the corresponding eigenfunction denoted by ϕ of (2.19), and let β be an eigenvalue with the corresponding eigenvector denoted by $\Psi = (\psi, E, B)$ of (2.25). Then $U = (\phi,\phi,0,0)$ is the corresponding eigenvector with the eigenvalue λ, and $V = (\psi, -\psi, -\frac{\mathrm{i}\xi}{|\xi|^2}(\psi,\chi_0) - \frac{\xi}{|\xi|}\times E, -\frac{\xi}{|\xi|}\times B)$ is the corresponding eigenvector with the eigenvalue β.

By the above argument, from now on, we will focus on the system (2.20). It is interesting to find out that the spectrum structure depends on the decomposition of the asymptotics in low frequency and high frequency like the classical Boltzmann equation and the VPB system, but depends on the low-intermediate-high frequencies. This is mainly due to the hyperbolic structure of the Maxwell equations, in particular the effect of the magnetic field on the velocity field. More precisely, the spectrum of linearized operators contains an eigenvalue in low frequency located in a small neighborhood of the origin, and two eigenvalues in high frequency located in two small neighborhoods centered at the points $\lambda = \pm\mathrm{i}|\xi|$, respectively. Except for these eigenvalues, there is a spectral gap for the intermediate frequency. Precisely, we have the following result on the spectrum structure.

THEOREM 2.2. *There exist two constants $r_0 > 0$ and $b_2 > 0$ so that the spectrum $\lambda \in \sigma(\hat{\mathbb{A}}_1(\xi)) \subset \mathbb{C}$ for $s = |\xi| \le r_0$ consists of two points $\{\lambda_j(s),\ j = 1,2\}$ in the domain $\mathrm{Re}\lambda > -b_2$. The spectrum $\lambda_j(s)$ are C^∞ functions of s for $|s| \le r_0$. In particular, the eigenvalues admit the following asymptotic expansion for $|s| \le r_0$:*

$$\lambda_1(s) = \lambda_2(s) = -a_1 s^2 + o(s^2),$$

where $a_1 > 0$ *is a constant defined in Theorem* 3.12.

There exists a constant $r_1 > 0$ *such that the spectrum* $\beta \in \sigma(\hat{\mathbb{A}}_1(\xi)) \subset \mathbb{C}$ *for* $s = |\xi| > r_1$ *consists of four eigenvalues* $\{\beta_j(s),\ j = 1, 2, 3, 4\}$ *in the domain* $\mathrm{Re}\lambda > -\mu/2$. *In particular, the eigenvalues satisfy*

$$\beta_1(s) = \beta_2(s) = -\mathrm{i}s + O(s^{-1/2}),$$
$$\beta_3(s) = \beta_4(s) = \mathrm{i}s + O(s^{-1/2}),$$
$$\frac{c_1}{s} \le -\mathrm{Re}\beta_j(s) \le \frac{c_2}{s}$$

for two positive constants c_1 *and* c_2.

For any $r_1 > r_0 > 0$, *there exists* $\alpha = \alpha(r_0, r_1) > 0$ *such that for* $r_0 \le |\xi| \le r_1$,

$$\sigma(\hat{\mathbb{A}}_1(\xi)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda(\xi) \le -\alpha\}.$$

Based on the spectrum structure given in the above theorem, the semigroup generated by the linearized operator of the VMB system can be decomposed in three parts according to the frequency in low, high, and intermediate regions so that the optimal time decay rates of the solution can be obtained for the linearized system. Note that a higher regularity on the initial data is needed because of the spectrum structure in the high frequency region. With the estimates on the semigroup, the optimal decay in time of the solution to the original nonlinear problem can also be obtained. Before stating results on nonlinear problem, let us first introduce the following notations.

Notation. The Fourier transform of $f = f(x, v)$ is denoted by $\hat{f}(\xi, v) = \mathcal{F}f(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x, v) e^{-\mathrm{i}x\cdot\xi} dx$.

Set a weight function $w(v)$ by

$$w(v) = (1 + |v|^2)^{1/2},$$

so that the Sobolev spaces H^N and H^N_w are given by

$$H^N = \{\, f \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) \,|\, \|f\|_{H^N} < \infty \,\}, \quad H^N_w = \{\, f \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) \,|\, \|f\|_{H^N_w} < \infty \,\},$$

equipped with the norms

$$\|f\|_{H^N} = \sum_{|\alpha|+|\beta|\le N} \|\partial^\alpha_x \partial^\beta_v f\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)}, \quad \|f\|_{H^N_w} = \sum_{|\alpha|+|\beta|\le N} \|w\partial^\alpha_x \partial^\beta_v f\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)}.$$

For $q \ge 1$, denote

$$L^{2,q} = L^2(\mathbb{R}^3_v, L^q(\mathbb{R}^3_x)), \quad \|f\|_{L^{2,q}} = \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f(x, v)|^q dx \right)^{2/q} dv \right)^{1/2}.$$

In the following, denote by $\|\cdot\|_{L^2_{x,v}}$ and $\|\cdot\|_{L^2_{\xi,v}}$ the norms of the function spaces $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ and $L^2(\mathbb{R}^3_\xi \times \mathbb{R}^3_v)$, respectively, and by $\|\cdot\|_{L^2_x}$, $\|\cdot\|_{L^2_\xi}$, and $\|\cdot\|_{L^2_v}$ the norms of the function spaces $L^2(\mathbb{R}^3_x)$, $L^2(\mathbb{R}^3_\xi)$, and $L^2(\mathbb{R}^3_v)$, respectively. For any integer $m \ge 1$, denote by $\|\cdot\|_{H^m_x}$ and $\|\cdot\|_{L^2_v(H^m_x)}$ the norms in the spaces $H^m(\mathbb{R}^3_x)$ and $L^2(\mathbb{R}^3_v, H^m(\mathbb{R}^3_x))$, respectively. Moreover, introduce a weighted Hilbert space $L^2_\xi(\mathbb{R}^3_v)$ for $\xi \ne 0$ defined by

$$L^2_\xi(\mathbb{R}^3) = \left\{ f \in L^2(\mathbb{R}^3_v) \,|\, \|f\|_\xi = \sqrt{(f, f)_\xi} < \infty \right\},$$

equipped with the inner product

$$(f,g)_\xi = (f,g) + \frac{1}{|\xi|^2}(P_{\rm d}f, P_{\rm d}g).$$

For any vectors $U=(f,E_1,B_1), V=(g,E_2,B_2)\in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3\times\mathbb{C}^3$, define a weighted inner product and the corresponding norm by

$$(U,V)_\xi = (f,g)_\xi + (E_1,E_2) + (B_1,B_2), \quad \|U\|_\xi = \sqrt{(U,U)_\xi},$$

and another L^2 inner product and norm by

$$(U,V) = (f,g) + (E_1,E_2) + (B_1,B_2), \quad \|U\| = \sqrt{(U,U)}.$$

For simplicity, denote

$$\|U\|^2_{Z^2} = \|f\|^2_{L^2_{x,v}} + \|E\|^2_{L^2_x} + \|B\|^2_{L^2_x}, \quad \|U\|^2_{Z^1} = \|f\|^2_{L^{2,1}} + \|E\|^2_{L^1_x} + \|B\|^2_{L^1_x}.$$

With the above preparation, we first state the estimates on the semigroup to linearized system.

THEOREM 2.3. *The semigroup $S(t,\xi)=e^{t\hat{\mathbb{A}}_1(\xi)}$ with $s=|\xi|\neq 0$ and $\omega=\xi/|\xi|$ can be decomposed into*

$$S(t,\xi)U = S_1(t,\xi)U + S_2(t,\xi)U + S_3(t,\xi)U, \quad U\in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi, \quad t>0,$$

which has the following properties:

$$S_1(t,\xi)U = \sum_{j=1}^{2} e^{t\lambda_j(s)}(U,\Psi^*_j(s,\omega))\Psi_j(s,\omega)1_{\{|\xi|\le r_0\}},$$
$$S_2(t,\xi)U = \sum_{j=1}^{4} e^{t\beta_j(s)}(U,\Phi^*_j(s,\omega))\Phi_j(s,\omega)1_{\{|\xi|\ge r_1\}},$$

where $\mathbb{C}^3_\xi=\{y\in\mathbb{C}^3 : y\cdot\xi=0\}$. Here, $(\lambda_j(s),\Psi_j(s,\omega))$ and $(\beta_j(s),\Phi_j(s,\omega))$ are the eigenvalues and eigenvectors of the operator $\hat{\mathbb{A}}_1(\xi)$ in low and high frequency regions with properties given in Theorems 3.12 *and* 3.15. *And $S_3(t,\xi)U =: S(t,\xi)U - S_1(t,\xi)U - S_2(t,\xi)U$ satisfies that there is a constant $\kappa_0>0$ independent of ξ such that*

$$\|S_3(t,\xi)U\|_\xi \le Ce^{-\kappa_0 t}\|U\|_\xi, \quad t\ge 0.$$

Then, we have the optimal time convergence rates of global solutions to linearized system (2.20).

THEOREM 2.4. *Let $(f_2(t),E(t),B(t))$ be a solution of the system* (2.20). *If the initial data $U_0=(f_0,E_0,B_0)\in L^2(\mathbb{R}^3_v;H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x))\times H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)\times H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)$ for $l\ge 0$, then it holds for any $\alpha,\alpha'\in\mathbb{N}^3$ with $\alpha'\le\alpha$ that*

$$\begin{aligned}\|\partial^\alpha_x f_2(t)\|_{L^2_{x,v}} \le{}& C(1+t)^{-\frac54-\frac k2}(\|\partial^\alpha_x U_0\|_{Z^2} + \|\partial^{\alpha'}_x U_0\|_{Z^1})\\ &+ C(1+t)^{-(m+\frac12)}\|\nabla^m_x\partial^\alpha_x U_0\|_{Z^2},\\ \|\partial^\alpha_x E(t)\|_{L^2_x} \le{}& C(1+t)^{-\frac54-\frac k2}(\|\partial^\alpha_x U_0\|_{Z^2} + \|\partial^{\alpha'}_x U_0\|_{Z^1}) + C(1+t)^{-m}\|\nabla^m_x\partial^\alpha_x U_0\|_{Z^2},\end{aligned}$$

$$\|\partial_x^\alpha B(t)\|_{L^2_x} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'} U_0\|_{Z^1}) + C(1+t)^{-m}\|\nabla_x^m \partial_x^\alpha U_0\|_{Z^2},$$

where $k = |\alpha - \alpha'|$ and $m \geq 0$. In particular, it holds for $f_2 = P_{\rm d} f_2 + P_r f_2$ that

$$\begin{aligned}
\|\partial_x^\alpha P_{\rm d} f_2(t)\|_{L^2_{x,v}} &\leq C e^{-\kappa_0 t}\|\partial_x^\alpha U_0\|_{Z^2},\\
\|\partial_x^\alpha P_r f_2(t)\|_{L^2_{x,v}} &\leq C(1+t)^{-\frac{5}{4}-\frac{k}{2}}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'} U_0\|_{Z^1})\\
&\quad + C(1+t)^{-(m+\frac{1}{2})}\|\nabla_x^m \partial_x^\alpha U_0\|_{Z^2}.
\end{aligned}$$

Furthermore, if we assume that $l \geq 2$ and the Fourier transform $\hat{B}_0(\xi)$ of initial magnetic field $B_0(x)$ satisfies that $\inf_{|\xi|\leq r_0} |\frac{\xi}{|\xi|} \times \hat{B}_0(\xi)| \geq d_0 > 0$, then

$$\begin{aligned}
C_1(1+t)^{-\frac{5}{4}} &\leq \|f_2(t)\|_{L^2_{x,v}} \leq C_2(1+t)^{-\frac{5}{4}},\\
C_1(1+t)^{-\frac{5}{4}} &\leq \|P_r f_2(t)\|_{L^2_{x,v}} \leq C_2(1+t)^{-\frac{5}{4}},\\
C_1(1+t)^{-\frac{5}{4}} &\leq \|E(t)\|_{L^2_x} \leq C_2(1+t)^{-\frac{5}{4}},\\
C_1(1+t)^{-\frac{3}{4}} &\leq \|B(t)\|_{L^2_x} \leq C_2(1+t)^{-\frac{3}{4}}
\end{aligned}$$

for $t > 0$ large enough with $C_2 \geq C_1 > 0$ two constants.

Finally, for the two-species system we have the optimal time convergence rates of global solutions to the original nonlinear system (2.2)–(2.7) as follows.

THEOREM 2.5. *When $(f_{1,0}, f_{2,0}) \in H^{N+5}_w \cap L^{2,1}$ and $(E_0, B_0) \in H^{N+5}(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)$ for $N \geq 4$ satisfying $\|(f_{1,0}, f_{2,0})\|_{H^{N+5}_w \cap L^{2,1}} + \|(E_0, B_0)\|_{H^{N+5}(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)} \leq \delta_0$ with $\delta_0 > 0$ being small, there exists a globally unique solution (f_1, f_2, E, B) to the system (2.2)–(2.7) satisfying*

$$\begin{aligned}
\|\partial_x^k f_1(t)\|_{L^2_{x,v}} + \|\partial_x^k B(t)\|_{L^2_x} &\leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}},\\
\|\partial_x^k f_2(t)\|_{L^2_{x,v}} + \|\partial_x^k E(t)\|_{L^2_x} &\leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}
\end{aligned}$$

for $k = 0, 1$. In particular, it holds for $f_1 = \mathrm{P}_0 f_1 + \mathrm{P}_1 f_1$ and $f_2 = P_{\rm d} f_2 + P_r f_2$ that

$$\begin{aligned}
\|\partial_x^k (f_1(t), \chi_j)\|_{L^2_x} &\leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 0, 1, 2, 3, 4,\\
\|\partial_x^k \mathrm{P}_1 f_1(t)\|_{L^2_{x,v}} &\leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}},\\
\|\partial_x^k P_{\rm d} f_2(t)\|_{L^2_{x,v}} &\leq C\delta_0(1+t)^{-2-\frac{k}{2}},\\
\|\partial_x^k P_r f_2(t)\|_{L^2_{x,v}} + \|\partial_x^k E(t)\|_{L^2_x} &\leq C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}},\\
\|\partial_x^k B(t)\|_{L^2_x} &\leq C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}},\\
\|(\mathrm{P}_1 f_1, P_r f_2)\|_{H^N_w} + \|\nabla_x(\mathrm{P}_0 f_1 P_{\rm d} f_2)\|_{L^2_v(H^{N-1}_x)} + \|\nabla_x(E, B)\|_{H^{N-1}_x} &\leq C\delta_0(1+t)^{-\frac{5}{4}}
\end{aligned}$$

for $k = 0, 1$.

THEOREM 2.6. *Under the conditions given in Theorem 2.5, if we further assume that there exist positive constants $d_0, d_1 > 0$ and a small constant $r_0 > 0$ so that the Fourier transform $(\hat{f}_{1,0}, \hat{f}_{2,0}, \hat{E}_0, \hat{B}_0)$ of the initial data $(f_{1,0}, f_{2,0}, E_0, B_0)$ satisfies that $\inf_{|\xi|\leq r_0} |(\hat{f}_{1,0}(\xi), \chi_0)| \geq d_0$, $\inf_{|\xi|\leq r_0} |(\hat{f}_{1,0}(\xi), \chi_4)| \geq d_1 \sup_{|\xi|\leq r_0} |(\hat{f}_{1,0}(\xi), \chi_0)|$, $\sup_{|\xi|\leq r_0} |(\hat{f}_{1,0}(\xi), v\sqrt{M})| = 0$, and $\inf_{|\xi|\leq r_0} |\frac{\xi}{|\xi|} \times \hat{B}_0(\xi)| \geq d_0$, then the global solution (f_1, f_2, E, B) to the system (2.2)–(2.7) satisfies*

$$C_1\delta_0(1+t)^{-\frac{3}{4}} \leq \|f_1(t)\|_{L^2_{x,v}} \leq C_2\delta_0(1+t)^{-\frac{3}{4}},$$

$$C_1\delta_0(1+t)^{-\frac{5}{4}} \le \|f_2(t)\|_{L^2_{x,v}} \le C_2\delta_0(1+t)^{-\frac{5}{4}},$$
$$C_1\delta_0(1+t)^{-\frac{5}{4}} \le \|E(t)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac{5}{4}},$$
$$C_1\delta_0(1+t)^{-\frac{3}{4}} \le \|B(t)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac{3}{4}}$$

for $t>0$ large with two constants $C_2>C_1>0$, and in particular

$$C_1\delta_0(1+t)^{-\frac{3}{4}} \le \|(f_1(t),\chi_j)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac{3}{4}},$$
$$C_1\delta_0(1+t)^{-\frac{5}{4}} \le \|\mathrm{P}_1 f_1(t)\|_{L^2_{x,v}} \le C_2\delta_0(1+t)^{-\frac{5}{4}},$$
$$C_1\delta_0(1+t)^{-\frac{5}{4}} \le \|P_r f_2(t)\|_{L^2_{x,v}} \le C_2\delta_0(1+t)^{-\frac{5}{4}}$$

for $j=0,1,2,3,4$.

2.2. One-species VMB system. We now turn to the one-species VMB system where the time evolution of electrons is considered under the influence of a fixed ion background $n_b(x)$. That is, consider

$$\begin{cases} \partial_t F + v\cdot\nabla_x F + (E+v\times B)\cdot\nabla_v F = \mathcal{Q}(F,F),\\ \partial_t E = \nabla_x\times B - \int_{\mathbb{R}^3} Fv dv,\\ \partial_t B = -\nabla_x\times E,\\ \nabla_x\cdot E = \int_{\mathbb{R}^3} F dv - n_b, \quad \nabla_x\cdot B = 0,\\ F(0,x,v) = F_0(x,v), \quad E_0(0,x) = E_0(x), \quad B_0(0,x) = B_0(x). \end{cases} \tag{2.29}$$

Here, $F=F(t,x,v)$ is the number density function of electrons, and $n_b(x)>0$ is assumed to be a constant representing the spatially uniform density of the ionic background. Take $n_b=1$ without loss of generality.

The one-species VMB system (1.3) has a stationary solution $(F^*,E^*,B^*) = (M(v),0,0)$ with $M(v)$ being the normalized global Maxwellian defined above. Set

$$F = M + \sqrt{M} f.$$

Then the one-species VMB system (2.29) for (F,E,B) is reformulated in terms of (f,E,B) into

$$\partial_t f + v\cdot\nabla_x f - Lf - v\sqrt{M}\cdot E = \frac{1}{2}(v\cdot E)f - (E+v\times B)\cdot\nabla_v f + \Gamma(f,f), \tag{2.30}$$

$$\partial_t E = \nabla_x\times B - \int_{\mathbb{R}^3} fv\sqrt{M} dv, \tag{2.31}$$

$$\partial_t B = -\nabla_x\times E, \tag{2.32}$$

$$\nabla_x\cdot E = \int_{\mathbb{R}^3} f\sqrt{M} dv, \quad \nabla_x\cdot B = 0, \tag{2.33}$$

$$f(0,x,v) = f_0 =: \frac{F_0 - M}{\sqrt{M}}, \quad E_0(0,x) = E_0(x), \quad B_0(0,x) = B_0(x). \tag{2.34}$$

Consider the linearized one-speices VMB system:

$$\partial_t f + v\cdot\nabla_x f - Lf - v\sqrt{M}\cdot E = 0, \tag{2.35}$$

$$\partial_t E = \nabla_x \times B - \int_{\mathbb{R}^3} f v \sqrt{M} dv, \tag{2.36}$$

$$\partial_t B = -\nabla_x \times E, \tag{2.37}$$

$$\nabla_x \cdot E = \int_{\mathbb{R}^3} f \sqrt{M} dv, \quad \nabla_x \cdot B = 0. \tag{2.38}$$

For convenience of notation, rewrite the linearized system for $U = (f, E, B)^T \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) \times L^2(\mathbb{R}^3_x) \times L^2(\mathbb{R}^3_x)$ as

$$\begin{cases} \partial_t U = \mathbb{A}_2 U, \quad t > 0, \\ \nabla_x \cdot E = (f, \sqrt{M}), \quad \nabla_x \cdot B = 0, \\ U(0, x, v) = U_0(x, v) = (f_0, E_0, B_0), \end{cases} \tag{2.39}$$

with the operator $\mathbb{A}_2$ defined by

$$\mathbb{A}_2 = \begin{pmatrix} L - v \cdot \nabla_x & v\sqrt{M} & 0 \\ -P_m & 0 & \nabla_x \times \\ 0 & -\nabla_x \times & 0 \end{pmatrix}. \tag{2.40}$$

Similarly, consider the following system for $\hat{V} = (\hat{f}, \omega \times \hat{E}, \omega \times \hat{B})$:

$$\begin{cases} \partial_t \hat{V} = \hat{\mathbb{A}}_3(\xi) \hat{V}, \quad t > 0, \\ \hat{V}(0, \xi, v) = \hat{V}_0(\xi, v) = (\hat{f}_0, \omega \times \hat{E}_0, \omega \times \hat{B}_0), \end{cases} \tag{2.41}$$

with

$$\hat{\mathbb{A}}_3(\xi) = \begin{pmatrix} \hat{B}_2(\xi) & -v\sqrt{M} \cdot \omega \times & 0 \\ -\omega \times P_m & 0 & \mathrm{i}\xi \times \\ 0 & -\mathrm{i}\xi \times & 0 \end{pmatrix}. \tag{2.42}$$

Here, for $\xi \neq 0$,

$$\hat{B}_2(\xi) = L - \mathrm{i}(v \cdot \xi) - \tfrac{\mathrm{i}(v \cdot \xi)}{|\xi|^2} P_{\mathrm{d}}. \tag{2.43}$$

Different from the two-species VMB, the spectrum of the linearized operator of one-species VMB consists of nine eigenvalues in low frequency located in small neighborhoods of three points $0, \pm\mathrm{i}$, and two eigenvalues in high frequency located in two small neighborhoods centered at the points $\lambda = \pm\mathrm{i}|\xi|$, respectively, in complex plane. Except for these eigenvalues, there is a spectral gap for the intermediate frequency. More precisely, we have the following theorem.

THEOREM 2.7. *There exists a constant $r_0 > 0$ so that the spectrum $\lambda \in \sigma(\hat{\mathbb{A}}_3(\xi)) \subset \mathbb{C}$ for $s = |\xi| \leq r_0$ consists of nine points $\{\lambda_j(s), \ -1 \leq j \leq 7\}$ in the domain $\mathrm{Re}\lambda > -\mu/2$. The spectrum $\lambda_j(s)$ are C^∞ functions of s for $|s| \leq r_0$. In particular, the eigenvalues $\lambda_j(s)$ have the following asymptotic expansions when $|s| \leq r_0$:*

$$\begin{cases} \lambda_{\pm 1}(s) = \pm\,\mathrm{i} + (-a_1 \pm \mathrm{i}b_1)s^2 + o(s^2), \quad \overline{\lambda_1} = \lambda_{-1}, \\ \lambda_0(s) = -\,a_0 s^2 + o(s^2), \\ \lambda_2(s) = \lambda_3(s) = -\mathrm{i} + (-a_2 - \mathrm{i}b_2)s^2 + o(s^2), \quad \overline{\lambda_2} = \lambda_4, \\ \lambda_4(s) = \lambda_5(s) = \mathrm{i} + (-a_2 + \mathrm{i}b_2)s^2 + o(s^2), \\ \lambda_6(s) = \lambda_7(s) = -a_3 s^4 + o(s^4), \end{cases}$$

where $a_j > 0$ $(0 \le j \le 3)$ *and* $b_j > 0$ $(1 \le j \le 2)$ *are constants defined in Theorem* 4.9.

There exists a constant $r_1 > 0$ *such that the spectrum* $\beta \in \sigma(\hat{\mathbb{A}}_3(\xi)) \subset \mathbb{C}$ *for* $s = |\xi| > r_1$ *consists of four eigenvalues* $\{\beta_j(s),\ j = 1,2,3,4\}$ *in the domain* $\mathrm{Re}\lambda > -\mu/2$. *In particular, the eigenvalues satisfy*

$$\beta_1(s) = \beta_2(s) = -\mathrm{i}s + O(s^{-1/2}),$$
$$\beta_3(s) = \beta_4(s) = \mathrm{i}s + O(s^{-1/2}),$$
$$\frac{c_1}{s} \le -\mathrm{Re}\beta_j(s) \le \frac{c_2}{s}$$

for two positive constants c_1 *and* c_2.

For any $r_1 > r_0 > 0$, *there exists* $\alpha = \alpha(r_0, r_1) > 0$ *such that for* $r_0 \le |\xi| \le r_1$,

$$\sigma(\hat{\mathbb{A}}_3(\xi)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda(\xi) \le -\alpha\}.$$

Based on the spectrum of the linearized operator of one-species VMB system, we are able to analyze the corresponding semigroup for (2.41) below.

THEOREM 2.8. *The semigroup* $S(t,\xi) = e^{t\hat{\mathbb{A}}_3(\xi)}$ *with* $s = |\xi| \neq 0$ *and* $\omega = \xi/|\xi|$ *satisfies*

$$S(t,\xi)U = S_1(t,\xi)U + S_2(t,\xi)U + S_3(t,\xi)U, \quad U \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi, \quad t > 0,$$

where

$$S_1(t,\xi)U = \sum_{j=-1}^{7} e^{t\lambda_j(s)}(U, \Psi^*_j(s,\omega))_\xi \Psi_j(s,\omega) 1_{\{|\xi| \le r_0\}},$$
$$S_2(t,\xi)U = \sum_{j=1}^{4} e^{t\beta_j(s)}(U, \Phi^*_j(s,\omega)) \Phi_j(s,\omega) 1_{\{|\xi| \ge r_1\}}.$$

Here, $(\lambda_j(s), \Psi_j(s,\omega))$ *and* $(\beta_j(s), \Phi_j(s,\omega))$ *are the eigenvalues and eigenvectors of the operator* $\hat{\mathbb{A}}_3(\xi)$ *given in Theorems* 4.9 *and* 4.10 *for* $|\xi| \le r_0$ *and* $|\xi| > r_1$, *respectively, and* $S_3(t,\xi)U =: S(t,\xi)U - S_1(t,\xi)U - S_2(t,\xi)U$ *satisfies that there exists a constant* $\kappa_0 > 0$ *independent of* ξ *so that*

$$\|S_3(t,\xi)U\|_\xi \le Ce^{-\kappa_0 t}\|U\|_\xi, \quad t \ge 0.$$

Then, we have the optimal time convergence rates of global solutions to linearized system.

THEOREM 2.9 (electric field dominating). *If the initial data* $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3_v; H^l(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)) \times H^l(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x) \times H^l(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)$ *for* $l \ge 0$ *with* $\nabla_x \cdot E_0 \neq (f_0, \chi_0)$ *is being held, then the unique solution* $(f(t), E(t), B(t))$ *to the system* (2.39) *exists globally in time and satisfies for any* $\alpha, \alpha' \in \mathbb{N}^3$ *with* $\alpha' \le \alpha$ *that*

(2.44)
$$\|\partial^\alpha_x f(t)\|_{L^2_{x,v}} \le C\delta(\alpha,\alpha')[(1+t)^{-\frac14-\frac{k}{2}} + (1+t)^{-\frac58-\frac{k}{4}}] + C(1+t)^{-m-\frac12}\|\nabla^m_x \partial^\alpha_x U_0\|_{Z^2},$$
(2.45)
$$\|\partial^\alpha_x E(t)\|_{L^2_x} \le C\delta(\alpha,\alpha')[(1+t)^{-\frac14-\frac{k}{2}} + (1+t)^{-\frac98-\frac{k}{4}}] + C(1+t)^{-m}\|\nabla^m_x \partial^\alpha_x U_0\|_{Z^2},$$

(2.46)
$$\|\partial_x^\alpha B(t)\|_{L^2_x} \le C\delta(\alpha,\alpha')(1+t)^{-\frac{3}{8}-\frac{k}{4}} + C(1+t)^{-m}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},$$

where $\delta(\alpha,\alpha') =: \|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1}$, $k = |\alpha-\alpha'|$, *and* $m \ge 0$. *In particular, it holds for* $f = \mathrm{P}_0 f + \mathrm{P}_1 f$ *that*

$$\|\partial_x^\alpha (f(t),\chi_0)\|_{L^2_x} \le C\delta(\alpha,\alpha')(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{2.47}$$

$$\begin{aligned}\|\partial_x^\alpha (f(t),v\sqrt{M})\|_{L^2_x} \le\,& C\delta(\alpha,\alpha')[(1+t)^{-\frac{1}{4}-\frac{k}{2}} + (1+t)^{-\frac{5}{8}-\frac{k}{4}}]\\ &+ C(1+t)^{-m-\frac{1}{2}}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},\end{aligned} \tag{2.48}$$

$$\|\partial_x^\alpha (f(t),\chi_4)\|_{L^2_x} \le C\delta(\alpha,\alpha')(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{2.49}$$

$$\begin{aligned}\|\partial_x^\alpha \mathrm{P}_1 f(t)\|_{L^2_{x,v}} \le\,& C\delta(\alpha,\alpha')[(1+t)^{-\frac{3}{4}-\frac{k}{2}} + (1+t)^{-\frac{7}{8}-\frac{k}{4}}]\\ &+ C(1+t)^{-m-\frac{1}{2}}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2}.\end{aligned} \tag{2.50}$$

Furthermore, if we assume further that $l \ge 1$ *and there exist two constants* $d_0, d_1 > 0$ *such that the Fourier transform* $\hat{U}_0 = (\hat{f}_0, \hat{E}_0, \hat{B}_0)$ *of the initial data* U_0 *satisfies that* $\inf_{|\xi|\le r_0} |(\hat{f}_0(\xi),\chi_0)| \ge d_0$, $\inf_{|\xi|\le r_0} |\frac{\xi}{|\xi|}\times \hat{B}_0(\xi)| \ge d_0$, $\inf_{|\xi|\le r_0} |(\hat{f}_0(\xi),\chi_4)| \ge d_1 \sup_{|\xi|\le r_0} |(\hat{f}_0(\xi),\chi_0)|$, *and* $\sup_{|\xi|\le r_0} |(\hat{f}_0(\xi), v\sqrt{M})| = 0$, *then the following optimal time decay rates hold:*

$$C_1(1+t)^{-\frac{1}{4}} \le \|f(t)\|_{L^2_{x,v}} \le C_2(1+t)^{-\frac{1}{4}}, \tag{2.51}$$

$$C_1(1+t)^{-\frac{1}{4}} \le \|E(t)\|_{L^2_x} \le C_2(1+t)^{-\frac{1}{4}}, \tag{2.52}$$

$$C_1(1+t)^{-\frac{3}{8}} \le \|B(t)\|_{L^2_x} \le C_2(1+t)^{-\frac{3}{8}} \tag{2.53}$$

for $t > 0$ *being large enough and* $C_2 \ge C_1 > 0$ *being two constants, and in particular*

$$C_1(1+t)^{-\frac{3}{4}} \le \|(f(t),\chi_0)\|_{L^2_x} \le C_2(1+t)^{-\frac{3}{4}}, \tag{2.54}$$

$$C_1(1+t)^{-\frac{1}{4}} \le \|(f(t),v\sqrt{M})\|_{L^2_x} \le C_2(1+t)^{-\frac{1}{4}}, \tag{2.55}$$

$$C_1(1+t)^{-\frac{3}{4}} \le \|(f(t),\chi_4)\|_{L^2_x} \le C_2(1+t)^{-\frac{3}{4}}, \tag{2.56}$$

$$C_1(1+t)^{-\frac{3}{4}} \le \|\mathrm{P}_1 f(t)\|_{L^2_{x,v}} \le C_2(1+t)^{-\frac{3}{4}}. \tag{2.57}$$

THEOREM 2.10 (magnetic field dominating). *If the initial data* $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3_v; H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)) \times H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x) \times H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)$ *for* $l \ge 0$ *with* $\nabla_x\cdot E_0 = (f_0,\chi_0)$ *is being held, then there exists globally in time a unique solution* $(f(t),E(t),B(t))$ *to the system* (2.39) *which satisfies for any* $\alpha,\alpha' \in \mathbb{N}^3$ *with* $\alpha' \le \alpha$ *that*

(2.58)
$$\|\partial_x^\alpha f(t)\|_{L^2_{x,v}} \le C\delta(\alpha,\alpha')(1+t)^{-\frac{5}{8}-\frac{k}{4}} + C(1+t)^{-m-\frac{1}{2}}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},$$

(2.59)
$$\|\partial_x^\alpha E(t)\|_{L^2_x} \le C\delta(\alpha,\alpha')[(1+t)^{-\frac{3}{4}-\frac{k}{2}} + (1+t)^{-\frac{9}{8}-\frac{k}{4}}] + C(1+t)^{-m}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},$$

(2.60)
$$\|\partial_x^\alpha B(t)\|_{L^2_x} \le C\delta(\alpha,\alpha')(1+t)^{-\frac{3}{8}-\frac{k}{4}} + C(1+t)^{-m}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},$$

where $\delta(\alpha,\alpha') =: \|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1}$, $k = |\alpha-\alpha'|$, *and* $m \ge 0$. *In particular, it holds for* $f = \mathrm{P}_0 f + \mathrm{P}_1 f$ *that*

$$\|\partial_x^\alpha (f(t),\chi_0)\|_{L^2_x} \le C\delta(\alpha,\alpha')(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \tag{2.61}$$

$$\|\partial_x^\alpha(f(t),v\sqrt{M})\|_{L^2_x} \le C\delta(\alpha,\alpha')(1+t)^{-\frac58-\frac k4}+C(1+t)^{-m-\frac12}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2}, \tag{2.62}$$

$$\|\partial_x^\alpha(f(t),\chi_4)\|_{L^2_x} \le C\delta(\alpha,\alpha')(1+t)^{-\frac34-\frac k2}, \tag{2.63}$$

$$\|\partial_x^\alpha \mathrm{P}_1 f(t)\|_{L^2_{x,v}} \le C\delta(\alpha,\alpha')(1+t)^{-\frac78-\frac k4}+C(1+t)^{-m-\frac12}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2}. \tag{2.64}$$

Furthermore, if we also assume that $l\ge 2$ and there exists a constant $d_0>0$ such that the Fourier transform $\hat U_0=(\hat f_0,\hat E_0,\hat B_0)$ of the initial data U_0 satisfies that $\inf_{|\xi|\le r_0}|\hat E_0(\xi)\cdot\frac{\xi}{|\xi|}|\ge d_0$, $\inf_{|\xi|\le r_0}|\frac{\xi}{|\xi|}\times\hat B_0(\xi)|\ge d_0$, $\inf_{|\xi|\le r_0}|(\hat f_0(\xi),\chi_4)|\ge d_0$, and $\sup_{|\xi|\le r_0}|(\hat f_0(\xi),v\sqrt{M})|=0$, then the global solution $(f(t),E(t),B(t))$ satisfies the following optimal time decay rates:

$$C_1(1+t)^{-\frac58} \le \|f(t)\|_{L^2_{x,v}} \le C_2(1+t)^{-\frac58}, \tag{2.65}$$

$$C_1(1+t)^{-\frac34} \le \|E(t)\|_{L^2_x} \le C_2(1+t)^{-\frac34}, \tag{2.66}$$

$$C_1(1+t)^{-\frac38} \le \|B(t)\|_{L^2_x} \le C_2(1+t)^{-\frac38} \tag{2.67}$$

for $t>0$ being large enough and $C_2\ge C_1>0$ being two constants, and in particular

$$C_1(1+t)^{-\frac54} \le \|(f(t),\chi_0)\|_{L^2_x} \le C_2(1+t)^{-\frac54}, \tag{2.68}$$

$$C_1(1+t)^{-\frac58} \le \|(f(t),v\sqrt{M})\|_{L^2_x} \le C_2(1+t)^{-\frac58}, \tag{2.69}$$

$$C_1(1+t)^{-\frac34} \le \|(f(t),\chi_4)\|_{L^2_x} \le C_2(1+t)^{-\frac34}, \tag{2.70}$$

$$C_1(1+t)^{-\frac78} \le \|\mathrm{P}_1 f(t)\|_{L^2_{x,v}} \le C_2(1+t)^{-\frac78}. \tag{2.71}$$

Finally, we have the time convergence rates of global solutions to the nonlinear system (2.30)–(2.34) as follows.

Theorem 2.11. *Assume that the initial data $f_0\in H^{N+3}_w\cap L^{2,1}$ and $(E_0,B_0)\in H^{N+3}(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)$ for $N\ge 4$ satisfy $\|f_0\|_{H^{N+3}_w\cap L^{2,1}}+\|(E_0,B_0)\|_{H^{N+3}_x\cap L^1_x}\le\delta_0$ with $\delta_0>0$ being small enough. Then, the unique solution (f,E,B) to the VMB system (2.30)–(2.34) exists globally in time and belongs to $H^{N+3}_w\times H^{N+3}_x\times H^{N+3}_x$.*

Moreover, if it also holds that $\nabla_x\cdot E_0=(f_0,\chi_0)$, then the global solution satisfies for $k=0,1$ that

$$\|\partial_x^k f(t)\|_{L^2_{x,v}} \le C\delta_0(1+t)^{-\frac58-\frac k4}, \tag{2.72}$$

$$\|\partial_x^k E(t)\|_{L^2_x} \le C\delta_0(1+t)^{-\frac34-\frac k4}\ln(1+t), \tag{2.73}$$

$$\|\partial_x^k B(t)\|_{L^2_x} \le C\delta_0(1+t)^{-\frac38-\frac k4}, \tag{2.74}$$

and in particular

$$\|\partial_x^k(f(t),\chi_0)\|_{L^2_x} \le C\delta_0(1+t)^{-1-\frac k4}, \tag{2.75}$$

$$\|\partial_x^k(f(t),v\sqrt{M})\|_{L^2_x} \le C\delta_0(1+t)^{-\frac58-\frac k4}, \tag{2.76}$$

$$\|\partial_x^k(f(t),\chi_4)\|_{L^2_x} \le C\delta_0(1+t)^{-\frac34-\frac k2}, \tag{2.77}$$

$$\|\partial_x^k\mathrm{P}_1 f(t)\|_{L^2_{x,v}} \le C\delta_0(1+t)^{-\frac78-\frac k4}, \tag{2.78}$$

$$\|\mathrm{P}_1 f\|_{H^N_w}+\|\nabla_x\mathrm{P}_0 f\|_{L^2_v(H^{N-1}_x)}+\|\nabla_x(E,B)\|_{H^{N-1}_x} \le C\delta_0(1+t)^{-\frac58}. \tag{2.79}$$

Furthermore, if there also exist a constant $d_0>0$ and a small constant $r_0>0$ so that the Fourier transform $\hat U_0=(\hat f_0,\hat E_0,\hat B_0)$ of the initial data $U_0=(f_0,E_0,B_0)$ satisfies that $\inf_{|\xi|\le r_0}|\hat E_0(\xi)\cdot\frac{\xi}{|\xi|}|\ge d_0$, $\inf_{|\xi|\le r_0}|\frac{\xi}{|\xi|}\times\hat B_0(\xi)|\ge d_0$, $\inf_{|\xi|\le r_0}|(\hat f_0(\xi),\chi_4)|\ge$

d_0, and $\sup_{|\xi|\le r_0}|(\hat f_0(\xi), v\sqrt{M})| = 0$, *then the global solution* (f,E,B) *satisfies*

$$C_1\delta_0(1+t)^{-\frac58} \le \|f(t)\|_{L^2_{x,v}} \le C_2\delta_0(1+t)^{-\frac58}, \tag{2.80}$$

$$C_1\delta_0(1+t)^{-\frac38} \le \|B(t)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac38} \tag{2.81}$$

for $t>0$ *large enough with two constants* $C_2>C_1$, *and in particular*

$$C_1\delta_0(1+t)^{-\frac58} \le \|(f(t), v\sqrt{M})\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac58}, \tag{2.82}$$

$$C_1\delta_0(1+t)^{-\frac34} \le \|(f(t), \chi_4)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac34}, \tag{2.83}$$

$$C_1\delta_0(1+t)^{-\frac78} \le \|\mathrm{P}_1 f(t)\|_{L^2_{x,v}} \le C_2\delta_0(1+t)^{-\frac78}. \tag{2.84}$$

Remark 2.12. Let us give an example of the initial function (f_0,E_0,B_0) which satisfies the assumptions of Theorem 2.11. For a positive constant d_0, we define (f_0,E_0,B_0) as

$$\begin{aligned} f_0(x,v) &= \frac{1}{(2\pi)^{3/2}} d_0 e^{r_0^2/2}\int_{\mathbb{R}^3}|\xi|e^{-|\xi|^2/2}e^{\mathrm{i}x\cdot\xi}d\xi\chi_0(v) + d_0e^{r_0^2/2}e^{-|x|^2/2}\chi_4(v),\\ E_0(x) &= \frac{1}{(2\pi)^{3/2}} d_0 e^{r_0^2/2}\int_{\mathbb{R}^3}\left(\frac{\xi}{|\xi|} + \frac{(-\xi_2,\xi_1,0)}{(\xi_1^2+\xi_2^2)^{1/2}}\right)e^{-|\xi|^2/2}e^{\mathrm{i}x\cdot\xi}d\xi,\\ B_0(x) &= \frac{1}{(2\pi)^{3/2}} d_0 e^{r_0^2/2}\int_{\mathbb{R}^3}\frac{(-\xi_2,\xi_1,0)}{(\xi_1^2+\xi_2^2)^{1/2}}e^{-|\xi|^2/2}e^{\mathrm{i}x\cdot\xi}d\xi. \end{aligned}$$

Remark 2.13. In the case of $\nabla_x\cdot E_0 \ne (f_0,\sqrt{M})$, the solutions of the nonlinear VMB system have no decay rates. The main reason is that the magnetic field decays too slowly. For instance, we assume that f,E,B decay at the same rates $(1+t)^{-\frac14}$, $(1+t)^{-\frac14}$, and $(1+t)^{-\frac38}$, respectively, as the linear solution; then we can obtain that the key quadratic nonlinear term decays at most at the rate $(1+t)^{-\frac12}$. Thus, making use of the Duhamel's principle to represent the solution via the semigroup, the corresponding nonlinear term to magnetic field B generated from the nonhomogeneous source decays as

$$\int_0^t (1+t-s)^{-\frac58}(1+s)^{-\frac12}ds \le C(1+t)^{-\frac18}.$$

Thus, the bootstrap argument breaks down and one cannot expect the magnetic field to decay as $(1+t)^{-\frac38}$ in the nonlinear case.

3. Spectral analysis for two-species VMB system.

For the study on the spectrum structure of the two-species VMB, in the following subsection, we will first investigate some properties of the operator $\hat{\mathbb{A}}_1(\xi)$ that lead to the description of its spectra and resolvent. The asymptotics of its eigenvalues and eigenfunctions in low and high frequency regions will be given in the next two subsections.

3.1. Spectrum structure.

First of all, note that P_{d} is a self-adjoint operator satisfying $(P_{\mathrm{d}}f, P_{\mathrm{d}}g) = (P_{\mathrm{d}}f, g) = (f, P_{\mathrm{d}}g)$. Hence,

$$(f,g)_\xi = \left(f, g + \frac{1}{|\xi|^2}P_{\mathrm{d}}g\right) = \left(f + \frac{1}{|\xi|^2}P_{\mathrm{d}}f, g\right). \tag{3.1}$$

By (3.1), we have for any $f, g \in L^2_\xi(\mathbb{R}^3_v) \cap D(\hat{B}_1(\xi))$,

$$(\hat{B}_1(\xi)f, g)_\xi = \left(\hat{B}_1(\xi)f, g + \frac{1}{|\xi|^2}P_{\rm d}g\right) = \left(f, (L_1 + {\rm i}(v\cdot\xi) + \frac{{\rm i}(v\cdot\xi)}{|\xi|^2}P_{\rm d})g\right)_\xi$$
$$= (f, \hat{B}_1(-\xi)g)_\xi. \tag{3.2}$$

Also, note that $\hat{B}_1(\xi)$ is a linear operator from the space $L^2_\xi(\mathbb{R}^3)$ to itself, and for any $y \in \mathbb{C}^3_\xi$,

$$\frac{\xi}{|\xi|} \times \frac{\xi}{|\xi|} \times y = -y. \tag{3.3}$$

Since $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ is an invariant subspace of the operator $\hat{\mathbb{A}}_1(\xi)$, $\hat{\mathbb{A}}_1(\xi)$ can be regarded as a linear operator on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. First, we have the following lemma.

LEMMA 3.1. *The operator $\hat{\mathbb{A}}_1(\xi)$ generates a strongly continuous contraction semigroup on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ satisfying*

$$\|e^{t\hat{\mathbb{A}}_1(\xi)}U\|_\xi \le \|U\|_\xi \quad \text{for } t > 0,\ U \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi. \tag{3.4}$$

Proof. We first show that both $\hat{\mathbb{A}}_1(\xi)$ and $\hat{\mathbb{A}}_1(\xi)^*$ are dissipative operators on $L^2_\xi(\mathbb{R}^3_v)$. By (3.2), we obtain for any $U, V \in L^2_\xi(\mathbb{R}^3_v) \cap D(\hat{B}_1(\xi)) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ that $(\hat{\mathbb{A}}_1(\xi)U, V)_\xi = (U, \hat{\mathbb{A}}_1(\xi)^*V)_\xi$ with $\hat{\mathbb{A}}_1(\xi)^* = \hat{\mathbb{A}}_1(-\xi)$.

Direct computation shows the dissipation of both $\hat{\mathbb{A}}_1(\xi)$ and $\hat{\mathbb{A}}_1(\xi)^*$, namely

$${\rm Re}(\hat{\mathbb{A}}_1(\xi)U, U)_\xi = {\rm Re}(\hat{\mathbb{A}}_1(\xi)^*U, U)_\xi = (L_1 f, f) \le 0.$$

Since $\hat{\mathbb{A}}_1(\xi)$ is a densely defined closed operator, it follows from Corollary 4.4 on p. 15 of [18] that the operator $\hat{\mathbb{A}}_1(\xi)$ generates a C_0-contraction semigroup on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. ∎

Define a 6×6 matrix by

$$B_3(\xi) = \begin{pmatrix} 0 & {\rm i}\xi\times \\ -{\rm i}\xi\times & 0 \end{pmatrix}_{6\times 6}. \tag{3.5}$$

Since $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ is an invariant subspace of the operator $B_3(\xi)$, we can regard $B_3(\xi)$ as an operator on $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. Then we have the following lemma.

LEMMA 3.2. *For any $\lambda \ne \pm{\rm i}|\xi|$, the operator $\lambda - B_3(\xi)$ is invertible on $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and satisfies*

$$\|(\lambda - B_3(\xi))^{-1}\| = \max_{j=\pm 1} |\lambda - j{\rm i}|\xi||^{-1}. \tag{3.6}$$

Proof. First, we compute the eigenvalues of the operator $B_3(\xi)$. For this, consider

$$(\lambda - B_3(\xi))X = 0, \quad X = (X_1, X_2) \in \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi.$$

It follows that

$$\lambda X_1 - {\rm i}\xi \times X_2 = 0, \tag{3.7}$$

$$\lambda X_2 + \mathrm{i}\xi \times X_1 = 0. \tag{3.8}$$

Multiplying (3.7) by λ and using (3.8) and (3.3) gives

$$\lambda^2 X_1 + |\xi|^2 X_1 = 0, \tag{3.9}$$

which implies that $\lambda_j = j\mathrm{i}|\xi|$ for $j = \pm 1$ are the eigenvalues of $B_3(\xi)$. Thus $\lambda - B_3(\xi)$ is invertible on $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ for any $\lambda \neq \pm\mathrm{i}|\xi|$. Since

$$(\mathrm{i}B_3(\xi)X, Y) = (X, \mathrm{i}B_3(\xi)Y) \quad \forall\ X, Y \in \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi, \tag{3.10}$$

it follows that $\mathrm{i}B_3(\xi)$ is a self-adjoint operator on $\mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and satisfies (3.6). ∎

Denote by $\rho(\hat{\mathbb{A}}_1(\xi))$ the resolvent set and by $\sigma(\hat{\mathbb{A}}_1(\xi))$ the spectrum set of $\hat{\mathbb{A}}_1(\xi)$. We have the following lemma.

LEMMA 3.3. *For each $\xi \neq 0$, the spectrum set $\sigma(\hat{\mathbb{A}}_1(\xi))$ of the operator $\hat{\mathbb{A}}_1(\xi)$ in the domain* $\mathrm{Re}\lambda \geq -\nu_0 + \delta$ *for any constant $\delta > 0$ consists of isolated eigenvalues $\Sigma =: \{\lambda_j(\xi)\}$ with* $\mathrm{Re}\lambda_j(\xi) < 0$.

Proof. Define

$$G_1(\xi) = \begin{pmatrix} c(\xi) & 0 & 0 \\ 0 & 0 & \mathrm{i}\xi\times \\ 0 & -\mathrm{i}\xi\times & 0 \end{pmatrix}, \quad G_2(\xi) = \begin{pmatrix} K_1 - \frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}} & -v\sqrt{M}\cdot\omega\times & 0 \\ -\omega\times P_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.11}$$

$$c(\xi) = -\nu(v) - \mathrm{i}(v\cdot\xi). \tag{3.12}$$

It is obvious that $\lambda - G_1(\xi)$ is invertible for $\mathrm{Re}\lambda > -\nu_0$ and $\lambda \neq \pm\mathrm{i}|\xi|$. Since $G_2(\xi)$ is a compact operator on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ for any fixed $\xi \neq 0$, $\hat{\mathbb{A}}_1(\xi)$ is a compact perturbation of $G_1(\xi)$. Hence, by Theorem 5.35 on p. 244 of [12], $\hat{\mathbb{A}}_1(\xi)$ and $G_1(\xi)$ have the same essential spectrum where $\sigma_{\mathrm{e}}(G_1(\xi)) = \mathrm{Ran}(c(\xi))$ and $\sigma_{\mathrm{d}}(G_1(\xi)) = \pm\mathrm{i}|\xi|$. Thus the spectrum of $\hat{\mathbb{A}}_1(\xi)$ in the domain $\mathrm{Re}\lambda > -\nu_0$ consists of discrete eigenvalues $\lambda_j(\xi)$ with possible accumulation points only on the line $\mathrm{Re}\lambda = -\nu_0$.

We claim that for any discrete eigenvalue $\lambda(\xi)$ of $\hat{\mathbb{A}}_1(\xi)$ in the region $\mathrm{Re}\lambda \geq -\nu_0+\delta$ for any constant $\delta > 0$, it holds that $\mathrm{Re}\lambda(\xi) < 0$ for $\xi \neq 0$. Indeed, set $\xi = s\omega$ and let $U = (f, E, B) \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ be the eigenvector corresponding to the eigenvalue λ so that

$$\begin{cases} \lambda f = L_1 f - \mathrm{i}s(v\cdot\omega)(f + \frac{1}{s^2}P_{\mathrm{d}}f) - v\sqrt{M}\cdot(\omega\times E), \\ \lambda E = -\omega\times(f, v\sqrt{M}) + \mathrm{i}\xi\times B, \\ \lambda B = -\mathrm{i}\xi\times E. \end{cases} \tag{3.13}$$

Taking the inner product $(\cdot,\cdot)_\xi$ of (3.13) with U, we have

$$(L_1 f, f) = \mathrm{Re}\lambda\Big(\|f\|^2 + \frac{1}{s^2}\|P_{\mathrm{d}}f\|^2 + |E|^2 + |B|^2\Big),$$

which together with (2.12) implies $\mathrm{Re}\lambda \leq 0$.

Furthermore, if there exists an eigenvalue λ with $\mathrm{Re}\lambda = 0$, then it follows from the above that $(L_1 f, f) = 0$, namely $f = C_0\sqrt{M} \in N_1$. By substituting this into (3.13), we obtain

$$\lambda C_0\sqrt{M} = -\mathrm{i}(v\cdot\omega)\Big(s + \frac{1}{s}\Big)C_0\sqrt{M} - v\sqrt{M}\cdot(\omega\times E), \tag{3.14}$$

which implies that $C_0 = 0$ and $\omega \times E = 0$. Therefore, $f \equiv 0$ and $E \equiv 0$. By substituting this into (3.13), we obtain $B \equiv 0$. This is a contradiction and thus it holds $\mathrm{Re}\lambda < 0$ for any discrete eigenvalue $\lambda \in \sigma(\hat{\mathbb{A}}_1(\xi))$. ∎

Now denote by T a linear operator on $L^2(\mathbb{R}^3_v)$ or $L^2_\xi(\mathbb{R}^3_v)$, and we define the corresponding norms of T by

$$\|T\| = \sup_{\|f\|=1} \|Tf\|, \quad \|T\|_\xi = \sup_{\|f\|_\xi=1} \|Tf\|_\xi.$$

Obviously,

$$(1+|\xi|^{-2})^{-1}\|T\| \le \|T\|_\xi \le (1+|\xi|^{-2})\|T\|. \tag{3.15}$$

Also, if T is a linear operator on $L^2(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ or $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$, then

$$\|T\| = \sup_{\|U\|=1} \|TU\|, \quad \|T\|_\xi = \sup_{\|U\|_\xi=1} \|TU\|_\xi.$$

We will make use of the following decomposition associated with the operator $\hat{\mathbb{A}}_1(\xi)$ for $|\xi| > 0$:

$$\lambda - \hat{\mathbb{A}}_1(\xi) = \lambda - G_1(\xi) - G_2(\xi) = (I - G_2(\xi)(\lambda - G_1(\xi))^{-1})(\lambda - G_1(\xi)), \tag{3.16}$$

where $G_1(\xi), G_2(\xi)$ are defined by (3.11). For $\mathrm{Re}\lambda > -\nu_0$ and $\lambda \ne \pm\mathrm{i}|\xi|$, we have

$$(\lambda - G_1(\xi))^{-1} = \begin{pmatrix} (\lambda - c(\xi))^{-1} & 0 \\ 0 & (\lambda - B_3(\xi))^{-1} \end{pmatrix}_{7\times 7}, \tag{3.17}$$

$$G_2(\xi)(\lambda - G_1(\xi))^{-1} = \begin{pmatrix} X_1(\lambda,\xi) & X_2(\lambda,\xi) \\ X_3(\lambda,\xi) & 0 \end{pmatrix}_{7\times 7}, \tag{3.18}$$

where $B_3(\xi)$ is defined in (3.5), and

$$X_1(\lambda,\xi) = \Big(K_1 - \frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}}\Big)(\lambda - c(\xi))^{-1}, \tag{3.19}$$

$$X_2(\lambda,\xi) = (v\sqrt{M}\cdot\omega\times, 0_{1\times 3})_{1\times 6}(\lambda - B_3(\xi))^{-1}, \tag{3.20}$$

$$X_3(\lambda,\xi) = \begin{pmatrix} -\omega\times P_m(\lambda - c(\xi))^{-1} \\ 0_{3\times 1} \end{pmatrix}_{6\times 1}. \tag{3.21}$$

Let K_1, K_4 be the operators on the space X and Y, and let K_2, K_3 be the operators $Y \to X$ and $X \to Y$, respectively. Let K be a matrix operator on $X \times Y$ defined by

$$K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}.$$

Then, we have the following lemma.

LEMMA 3.4. *If the norms of K_1, K_2, K_3, and K_4 satisfy*

$$\|K_1\| < 1, \quad \|K_4\| < 1, \quad \|K_2\|\|K_3\| < (1-\|K_1\|)(1-\|K_4\|),$$

then the operator $I + K$ is invertible on $X \times Y$.

Proof. Decompose $I+K$ into

$$I+K=\begin{pmatrix} I+K_1 & K_2 \\ 0 & I+K_4 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ K_3 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} I+K_1 & K_2 \\ 0 & I+K_4 \end{pmatrix}^{-1}=\begin{pmatrix} (I+K_1)^{-1} & -(I+K_1)^{-1}K_2(I+K_4)^{-1} \\ 0 & (I+K_4)^{-1} \end{pmatrix},$$

it follows that

$$I+K=\begin{pmatrix} I+K_1 & K_2 \\ 0 & I+K_4 \end{pmatrix}\begin{pmatrix} I-(I+K_1)^{-1}K_2(I+K_4)^{-1}K_3 & 0 \\ (I+K_4)^{-1}K_3 & I \end{pmatrix}$$

is invertible on $X\times Y$ because $\|(I+K_1)^{-1}K_2(I+K_4)^{-1}K_3\|<1$. ◻

LEMMA 3.5. *There exists a constant $C>0$ such that:*

1. *If $x>-\nu_0$, we have*

$$\|K_1(x+\mathrm{i}y-c(\xi))^{-1}\|\le C(x+\nu_0)^{-11/13}(1+|\xi|)^{-2/13}. \tag{3.22}$$

2. *If $x>-\nu_0$ and $|y|\ge(2|\xi|)^{5/3}(x+\nu_0)^{-2/3}$, we have*

$$\|K_1(x+\mathrm{i}y-c(\xi))^{-1}\|\le C(x+\nu_0)^{-3/5}(1+|y|)^{-2/5}. \tag{3.23}$$

3. *If $x>-\nu_0$, we have*

$$\|P_m(x+\mathrm{i}y-c(\xi))^{-1}\|_{L^2(\mathbb{R}^3)\to\mathbb{C}^3}\le C(x+\nu_0)^{-1/2}(1+|\xi|)^{-1/2}, \tag{3.24}$$

$$\|P_m(x+\mathrm{i}y-c(\xi))^{-1}\|_{L^2(\mathbb{R}^3)\to\mathbb{C}^3}\le C(1+(x+\nu_0)^{-1})(|\xi|+1)|y|^{-1}. \tag{3.25}$$

4. *If $x>-\nu_0$, we have*

$$\|(v\cdot\xi)|\xi|^{-2}P_{\mathrm{d}}(x+\mathrm{i}y-c(\xi))^{-1}\|\le C(x+\nu_0)^{-1}|\xi|^{-1}, \tag{3.26}$$

$$\|(v\cdot\xi)|\xi|^{-2}P_{\mathrm{d}}(x+\mathrm{i}y-c(\xi))^{-1}\|\le C(1+(x+\nu_0)^{-1})(|\xi|^{-1}+1)|y|^{-1}. \tag{3.27}$$

Proof. The proof of (3.22), (3.23), (3.26), and (3.27) can be found in Lemma 2.3 of [13]. We need only prove (3.24) and (3.25). Since

$$\begin{aligned}
&\|(x-\mathrm{i}y+\nu(v)-\mathrm{i}(v\cdot\xi))^{-1}v\sqrt{M}\|^2\\
&\le C\int_{\mathbb{R}^3}\frac{1}{(x+\nu_0)^2+(y+(v\cdot\xi))^2}e^{-\frac{|v|^2}{4}}dv=C\int_{\mathbb{R}^3}\frac{1}{(x+\nu_0)^2+(y+v_1|\xi|)^2}e^{-\frac{|v|^2}{4}}dv\\
&=C\frac{1}{|\xi|}\int_{\mathbb{R}^3}\frac{1}{(x+\nu_0)^2+v_1^2}e^{-\frac{(v_1-y)^2}{4|\xi|^2}}e^{-\frac{v_2^2}{4}}e^{-\frac{v_3^2}{4}}dv\le C(x+\nu_0)^{-1}|\xi|^{-1},
\end{aligned}$$

we obtain

$$\begin{aligned}
|P_m(x+\mathrm{i}y-c(\xi))^{-1}f|&\le\|(x-\mathrm{i}y+\nu(v)-\mathrm{i}(v\cdot\xi))^{-1}v\sqrt{M}\|\|f\|\\
&\le C(x+\nu_0)^{-1/2}|\xi|^{-1/2}\|f\|.
\end{aligned}$$

This proves (3.24). Since $P_m(\lambda-c(\xi))^{-1}=\frac{1}{\lambda}P_m+\frac{1}{\lambda}P_m(\lambda-c(\xi))^{-1}c(\xi)$, it follows that $\|P_m(\lambda-c(\xi))^{-1}\|\le|\lambda|^{-1}+C(x+\nu_0)^{-1}(1+|\xi|)|\lambda|^{-1}$, which proves (3.25). ◻

With Lemma 3.5, we can investigate the spectrum set of the operator $\hat{\mathbb{A}}_1(\xi)$ in the intermediate and high frequency regions.

LEMMA 3.6. *For the high and intermediate frequencies, the following statements hold.*

1. *For any $\delta_1,\delta_2>0$, there exists $R_1=R_1(\delta_1,\delta_2)>0$ such that for $|\xi|>R_1$,*

$$\sigma(\hat{\mathbb{A}}_1(\xi))\cap\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda\ge-\nu_0+\delta_1\}\subset\bigcup_{j=\pm1}\{\lambda\in\mathbb{C}\,|\,|\lambda-j\mathrm{i}|\xi||\le\delta_2\}. \tag{3.28}$$

2. *For any $r_1>r_0>0$, there exists $\alpha=\alpha(r_0,r_1)>0$ such that for $r_0\le|\xi|\le r_1$,*

$$\sigma(\hat{\mathbb{A}}_1(\xi))\subset\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda(\xi)\le-\alpha\}. \tag{3.29}$$

Proof. We prove (3.28) first. By Lemma 3.5, (3.6), and (3.15), there is $R_1=R_1(\delta_1,\delta_2)>0$ such that for $\mathrm{Re}\lambda\ge-\nu_0+\delta_1$, $\min_{j=\pm1}|\lambda-j\mathrm{i}|\xi||>\delta_2$, and $|\xi|>R_1$,

$$\|(K_1-\tfrac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}})(\lambda-c(\xi))^{-1}\|_\xi\le1/2,\quad\|P_m(\lambda-c(\xi))^{-1}\|_{L^2_\xi(\mathbb{R}^3)\to\mathbb{C}^3}\le\delta_2/4,$$
$$\|(\lambda-B_3(\xi))^{-1}\|\le\delta_2^{-1},$$

which leads to

$$\|X_1(\lambda,\xi)\|_\xi\le1/2,\quad\|X_2(\lambda,\xi)\|_{\mathbb{C}^6\to L^2_\xi(\mathbb{R}^3)}\|X_3(\lambda,\xi)\|_{L^2_\xi(\mathbb{R}^3)\to\mathbb{C}^6}\le1/4.$$

This and Lemma 3.4 imply that the operator $I-G_2(\xi)(\lambda-G_1(\xi))^{-1}$ is invertible on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$, and thus $\lambda-\hat{\mathbb{A}}_1(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$. Therefore, we have $\rho(\hat{\mathbb{A}}_1(\xi))\supset\{\lambda\in\mathbb{C}\,|\,\min_{j=\pm1}|\lambda-j\mathrm{i}|\xi||>\delta_2,\mathrm{Re}\lambda\ge-\mu+\delta_1\}$ for $|\xi|>R_1$ which gives (3.28).

Next, we turn to prove (3.29). By Lemma 3.5, (3.6), and (3.15), there exists $y_1=y_1(r_0,r_1,\delta_1)>0$ large enough such that for $\mathrm{Re}\lambda\ge-\nu_0+\delta_1$, $|\mathrm{Im}\lambda|>y_1$, and $r_0\le|\xi|\le r_1$,

$$\|(K_1-\tfrac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}})(\lambda-c(\xi))^{-1}\|_\xi\le1/6,\quad\|P_m(\lambda-c(\xi))^{-1}\|_{L^2_\xi(\mathbb{R}^3)\to\mathbb{C}^3}\le1/6,$$
$$\|(\lambda-B_3(\xi))^{-1}\|\le1/6,$$

which leads to

$$\|X_1(\lambda,\xi)\|_\xi+\|X_2(\lambda,\xi)\|_{\mathbb{C}^6\to L^2_\xi(\mathbb{R}^3)}+\|X_3(\lambda,\xi)\|_{L^2_\xi(\mathbb{R}^3)\to\mathbb{C}^6}\le1/2.$$

This implies that the operator $I-G_2(\xi)(\lambda-G_1(\xi))^{-1}$ is invertible on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$, which together with (3.16) yields that $\lambda-\hat{\mathbb{A}}_1(\xi)$ is also invertible on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ when $\mathrm{Re}\lambda\ge-\nu_0+\delta_1$, $|\mathrm{Im}\lambda|>y_1$, and $r_0\le|\xi|\le r_1$. Note that it satisfies

$$(\lambda-\hat{\mathbb{A}}_1(\xi))^{-1}=(\lambda-G_1(\xi))^{-1}(I-G_2(\xi)(\lambda-G_1(\xi))^{-1})^{-1}. \tag{3.30}$$

Based on the above argument, it is sufficient to prove (3.29) holds for $|\mathrm{Im}\lambda|\le y_1$. If it does not hold, then there exists a sequence of $\{(\xi_n,\lambda_n,f_n,E_n,B_n)\}$ satisfying $|\xi_n|\in[r_0,r_1]$, $(f_n,E_n,B_n)\in L^2(\mathbb{R}^3)\times\mathbb{C}^3_{\xi_n}\times\mathbb{C}^3_{\xi_n}$ with $\|f_n\|+|E_n|+|B_n|=1$, and $\lambda_n\in\sigma(\hat{\mathbb{A}}_1(\xi_n))$ with $|\mathrm{Im}\lambda_n|\le y_1$ and $\mathrm{Re}\lambda_n\to0$ as $n\to\infty$. That is,

$$\begin{cases}\lambda_nf_n=(L_1-\mathrm{i}(v\cdot\xi_n)-\frac{\mathrm{i}(v\cdot\xi_n)}{|\xi_n|^2}P_{\mathrm{d}})f_n-v\sqrt{M}\cdot(\omega_n\times E_n),\\ \lambda_nE_n=-\omega_n\times(f_n,v\sqrt{M})+\mathrm{i}\xi_n\times B_n,\quad\lambda_nB_n=-\mathrm{i}\xi_n\times E_n.\end{cases}$$

Taking the inner product $(\cdot,\cdot)_{\xi_n}$ between the above system and (f_n,E_n,B_n) and choosing the real part, we have

$$\mathrm{Re}\lambda_n(\|f_n\|^2_{\xi_n}+|E_n|^2+|B_n|^2)=(Lf_n,f_n)\le-\mu\|\mathrm{P}_1f_n\|^2,$$

This implies that $\lim_{n\to\infty}\|\mathrm{P}_1 f_n\|^2 = 0$ and hence $\lim_{n\to\infty}\|\mathrm{P}_0 f_n\|^2 = 1$. Since P_0 is a compact operator, there exists a subsequence n_j of n and $f_0 \neq 0 \in N_0$ such that $\mathrm{P}_0 f_{n_j} \to f_0$ as $j\to\infty$. Thus we have $f_{n_j} \to f_0$ in L^2 as $j \to \infty$. Since $|E_n|+|B_n| \le 1$, $|\xi_n| \in [r_0, r_1]$, $|\mathrm{Im}\lambda_n| \le y_1$, and $\mathrm{Re}\lambda_n \to 0$, we can extract a subsequence of (still denoted by) $(\lambda_{n_j}, \xi_{n_j}, E_{n_j}, B_{n_j})$ such that $(\lambda_{n_j}, \xi_{n_j}, E_{n_j}, B_{n_j}) \to (\lambda_0, \xi_0, E_0, B_0)$ with $\mathrm{Re}\lambda_0 = 0$ and $|\xi_0| \in [r_0, r_1]$. It follows that $\hat{\mathbb{A}}_1(\xi_0)U_0 = \lambda_0 U_0$ with $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3)\times\mathbb{C}^3_{\xi_0}\times\mathbb{C}^3_{\xi_0}$ and λ_0 is an eigenvalue of $\hat{\mathbb{A}}_1(\xi_0)$ with $\mathrm{Re}\lambda_0 = 0$. This contradicts the fact that $\mathrm{Re}\lambda(\xi) < 0$ for $\xi \neq 0$ obtained by Lemma 3.3. Thus, the proof of the lemma is completed. ∎

We now investigate the spectrum and resolvent sets of $\hat{\mathbb{A}}_1(\xi)$ in low frequency. For this, we decompose $\lambda - \hat{\mathbb{A}}_1(\xi)$ into

$$\lambda - \hat{\mathbb{A}}_1(\xi) = \lambda - G_3(\xi) - G_4(\xi), \tag{3.31}$$

where

$$G_3(\xi) = \begin{pmatrix} Q(\xi) & 0 & 0 \\ 0 & 0 & \mathrm{i}\xi\times \\ 0 & -\mathrm{i}\xi\times & 0 \end{pmatrix}, \quad G_4(\xi) = \begin{pmatrix} Q_1(\xi) & -v\sqrt{M}\cdot\omega\times & 0 \\ -\omega\times P_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.32}$$

$$Q(\xi) = L_1 - \mathrm{i}P_r(v\cdot\xi)P_r, \quad Q_1(\xi) = \mathrm{i}P_{\mathrm{d}}(v\cdot\xi)P_r + \mathrm{i}P_r(v\cdot\xi)\left(1+\frac{1}{|\xi|^2}\right)P_{\mathrm{d}}. \tag{3.33}$$

LEMMA 3.7. *Let $\xi \neq 0$ and $Q(\xi)$ defined by (3.33). We have*

1. *If $\lambda \neq 0$, then*

$$\|\lambda^{-1}P_r(v\cdot\xi)\left(1+\frac{1}{|\xi|^2}\right)P_{\mathrm{d}}\|_\xi \le C(|\xi|+1)|\lambda|^{-1}. \tag{3.34}$$

2. *If $\mathrm{Re}\lambda > -\mu$, then the operator $\lambda P_r - Q(\xi)$ is invertible on $N_1^\perp$ and satisfies*

$$\|(\lambda P_r - Q(\xi))^{-1}\| \le (\mathrm{Re}\lambda+\mu)^{-1}, \tag{3.35}$$

$$\|P_{\mathrm{d}}(v\cdot\xi)P_r(\lambda P_r - Q(\xi))^{-1}P_r\|_\xi \le C(1+|\lambda|)^{-1}[(\mathrm{Re}\lambda+\mu)^{-1}+1](1+|\xi|)^2, \tag{3.36}$$

$$\|P_m(\lambda P_r - Q(\xi))^{-1}P_r\|_{L^2_\xi(\mathbb{R}^3)\to\mathbb{C}^3} \le C(1+|\lambda|)^{-1}[(\mathrm{Re}\lambda+\mu)^{-1}+1](1+|\xi|). \tag{3.37}$$

Proof. The proof of (3.34)–(3.36) can be found in [14]. Repeating the same argument as that of estimate (3.25), we can obtain (3.37). Hence, we omit the details for brevity. ∎

By Lemmas 3.3–3.7, we analyze the spectral and resolvent sets of the operator $\hat{\mathbb{A}}_1(\xi)$.

LEMMA 3.8. *For any $\delta_1, \delta_2 > 0$, there are two constants $r_1 = r_1(\delta_1,\delta_2), y_1 = y_1(\delta_1,\delta_2) > 0$ such that for all $|\xi| \neq 0$ the resolvent set of $\hat{\mathbb{A}}_1(\xi)$ satisfies*

$$\rho(\hat{\mathbb{A}}_1(\xi)) \supset \begin{cases} \{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda \ge -\mu+\delta_1,\ |\mathrm{Im}\lambda| \ge y_1\}\cup\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda>0\}, & |\xi|\le r_1, \\ \{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda \ge -\mu+\delta_1,\ \min\limits_{j=\pm1}|\lambda - j\mathrm{i}|\xi|| \ge \delta_2\}\cup\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda>0\}, & |\xi|\ge r_1. \end{cases} \tag{3.38}$$

Proof. By Lemma 3.6, there exists $r_1 = r_1(\delta_1, \delta_2) > 0$ so that the second part of (3.38) holds. Thus we need only prove the first part of (3.38). By Lemma 3.7, we have for $\mathrm{Re}\lambda > -\mu$ and $\lambda \neq 0$ that the operator $\lambda - Q(\xi) = \lambda P_{\mathrm{d}} + \lambda P_r - Q(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)$ and it satisfies

$$(\lambda P_{\mathrm{d}} + \lambda P_r - Q(\xi))^{-1} = \lambda^{-1} P_{\mathrm{d}} + (\lambda P_r - Q(\xi))^{-1} P_r,$$

because the operator λP_{d} is orthogonal to $\lambda P_r - Q(\xi)$. Thus, for $\mathrm{Re}\lambda > -\mu$ and $\lambda \neq 0, \pm\mathrm{i}|\xi|$, the operator $\lambda - G_3(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and satisfies

$$(3.39)\qquad (\lambda - G_3(\xi))^{-1} = \begin{pmatrix} \lambda^{-1} P_{\mathrm{d}} + (\lambda P_r - Q(\xi))^{-1} P_r & 0 \\ 0 & (\lambda - B_3(\xi))^{-1} \end{pmatrix}_{7\times 7}.$$

Therefore, we can rewrite (3.31) as

$$(3.40)\qquad \lambda - \hat{\mathbb{A}}_1(\xi) = (I - G_4(\xi)(\lambda - G_3(\xi))^{-1})(\lambda - G_3(\xi)),$$

where

$$(3.41)\qquad G_4(\xi)(\lambda - G_4(\xi))^{-1} = \begin{pmatrix} X_4(\lambda,\xi) & X_2(\lambda,\xi) \\ X_5(\lambda,\xi) & 0 \end{pmatrix}_{7\times 7},$$

$$(3.42)\qquad X_4(\lambda,\xi) = \mathrm{i} P_{\mathrm{d}}(v\cdot\xi) P_r (\lambda P_r - Q(\xi))^{-1} P_r + \mathrm{i}\lambda^{-1} P_r (v\cdot\xi)\left(1 + \frac{1}{|\xi|^2}\right) P_{\mathrm{d}},$$

$$(3.43)\qquad X_5(\lambda,\xi) = \begin{pmatrix} -\omega \times P_m (\lambda P_r - Q(\xi))^{-1} P_r \\ 0_{3\times 1} \end{pmatrix}_{6\times 1}.$$

For $|\xi| \leq r_1$, by (3.6) and (3.34)–(3.36) we can choose $y_1 = y_1(\delta_1, r_1) > 0$ such that it holds for $\mathrm{Re}\lambda \geq -\mu + \delta_1$ and $|\mathrm{Im}\lambda| \geq y_1$ that

$$(3.44)\qquad \|X_4(\lambda,\xi)\|_\xi + \|X_2(\lambda,\xi)\|_{\mathbb{C}^6 \to L^2_\xi(\mathbb{R}^3)} + \|X_5(\lambda,\xi)\|_{L^2_\xi(\mathbb{R}^3)\to\mathbb{C}^6} \leq 1/2.$$

This implies that the operator $I - G_4(\xi)(\lambda - G_3(\xi))^{-1}$ is invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and thus $\lambda - \hat{\mathbb{A}}_1(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ and satisfies

$$(3.45)\qquad (\lambda - \hat{\mathbb{A}}_1(\xi))^{-1} = (\lambda - G_3(\xi))^{-1}(I - G_4(\xi)(\lambda - G_3(\xi))^{-1})^{-1}.$$

Therefore, $\rho(\hat{\mathbb{A}}_1(\xi)) \supset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\mu + \delta_1, |\mathrm{Im}\lambda| \geq y_1\}$ for $|\xi| \leq r_1$. This completes the proof of the lemma. ∎

3.2. Asymptotics in low frequency. In this subsection, we study in the low frequency region, the asymptotics of the eigenvalues and eigenvectors of the operator $\hat{\mathbb{A}}_1(\xi)$. In terms of (2.27), the eigenvalue problem $\hat{\mathbb{A}}_1(\xi)U = \lambda U$ for $U = (f, X, Y) \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ can be written as

$$(3.46)\qquad \lambda f = (L_1 - \mathrm{i}(v\cdot\xi) - \tfrac{\mathrm{i}(v\cdot\xi)}{|\xi|^2} P_{\mathrm{d}}) f - v\sqrt{M}\cdot(\omega \times X),$$

$$(3.47)\qquad \begin{aligned} \lambda X &= -\omega \times (f, v\sqrt{M}) + \mathrm{i}\xi \times Y, \\ \lambda Y &= -\mathrm{i}\xi \times X, \quad |\xi| \neq 0. \end{aligned}$$

Let f be the eigenfunction of (3.46). We rewrite f in the form $f = f_0 + f_1$, where $f_0 = P_{\mathrm{d}} f = C_0\sqrt{M}$ and $f_1 = (I - P_{\mathrm{d}}) f = P_r f$. Then (3.46) gives

$$(3.48)\qquad \lambda f_0 = -P_{\mathrm{d}}[\mathrm{i}(v\cdot\xi)(f_0 + f_1)],$$

$$\lambda f_1 = L_1 f_1 - P_r[\mathrm{i}(v\cdot\xi)(f_0+f_1)] - \frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2} f_0 - v\sqrt{M}\cdot(\omega\times X). \tag{3.49}$$

By Lemma 3.7, (3.33), and (3.49), the microscopic part f_1 can be represented by

$$f_1 = -(\lambda P_r - Q(\xi))^{-1} P_r\left[\mathrm{i}(v\cdot\xi)\left(1+\frac{1}{|\xi|^2}\right) f_0 + v\sqrt{M}\cdot(\omega\times X)\right], \quad \mathrm{Re}\lambda > -\mu. \tag{3.50}$$

Substituting (3.50) into (3.48) and (3.47), we obtain the eigenvalue problem for (λ, C_0, X, Y) as

$$\begin{aligned}\lambda C_0 =& \left(1+\frac{1}{|\xi|^2}\right)(R(\lambda,\xi)(v\cdot\xi)\sqrt{M}, (v\cdot\xi)\sqrt{M})C_0\\ &+ (R(\lambda,\xi)(v\sqrt{M}\cdot(\omega\times X)), (v\cdot\xi)\sqrt{M}),\end{aligned} \tag{3.51}$$

$$\begin{aligned}\lambda X =& -\omega\times \mathrm{i}\left(1+\frac{1}{|\xi|^2}\right)(R(\lambda,\xi)(v\cdot\xi)\sqrt{M}, v\sqrt{M})C_0\\ &- \omega\times(R(\lambda,\xi)(v\sqrt{M}\cdot(\omega\times X)), v\sqrt{M}) + \mathrm{i}\xi\times Y,\end{aligned} \tag{3.52}$$

$$\lambda Y = -\,\mathrm{i}\xi\times X, \tag{3.53}$$

where $R(\lambda,\xi) = -(\lambda P_r - Q(\xi))^{-1} = [L_1 - \lambda P_r - \mathrm{i}P_r(v\cdot\xi)P_r]^{-1}$.

By changing variable $(v\cdot\xi)\to|\xi|v_1$ and using the rotational invariance of the operator L_1, we have the following transformation.

LEMMA 3.9. *Let $e_1=(1,0,0)$, $\xi = s\omega$ with $s\in\mathbb{R}$, $\omega\in\mathbb{S}^2$. Then*

$$(R(\lambda,\xi)\chi_i,\chi_j) = \omega_i\omega_j(R(\lambda,se_1)\chi_1,\chi_1) + (\delta_{ij}-\omega_i\omega_j)(R(\lambda,se_1)\chi_2,\chi_2). \tag{3.54}$$

With (3.54), the equations (3.51)–(3.53) can be simplified as

$$\lambda C_0 = (1+s^2)(R(\lambda,se_1)\chi_1,\chi_1)C_0, \tag{3.55}$$

$$\lambda X = (R(\lambda,se_1)\chi_2,\chi_2)X + \mathrm{i}\xi\times Y, \tag{3.56}$$

$$\lambda Y = -\,\mathrm{i}\xi\times X. \tag{3.57}$$

Multiplying (3.56) by λ and using (3.57) and (3.3), we obtain

$$(\lambda^2 - (R(\lambda,se_1)\chi_2,\chi_2)\lambda + s^2)X = 0. \tag{3.58}$$

Denote

$$D_0(\lambda,s) =: \lambda - (1+s^2)(R(\lambda,se_1)\chi_1,\chi_1), \tag{3.59}$$

$$D_1(\lambda,s) =: \lambda^2 - (R(\lambda,se_1)\chi_2,\chi_2)\lambda + s^2. \tag{3.60}$$

The following result on $D_0(\lambda,s)=0$ was proved in [14] in the study on the bipolar VPB system.

LEMMA 3.10 (see [14]). *There are constants $b_0>0$ and $r_0>0$ with r_0 being small such that the equation $D_0(\lambda,s)=0$ has no solution for $\mathrm{Re}\lambda\geq -b_0$ and $|s|\leq r_0$.*

LEMMA 3.11. *There are constants $b_1, r_0, r_1 > 0$ such that the equation $D_1(\lambda,s)=0$ with $\mathrm{Re}\lambda\geq -b_1$ has only one solution $\lambda(s)$ for $(s,\lambda)\in[-r_0,r_0]\times B_{r_1}(0)$ and it satisfies*

$$\lambda(0)=0, \quad \lambda'(0)=0, \quad \lambda''(0) = \frac{2}{(L_1^{-1}\chi_2,\chi_2)}.$$

Proof. Since

$$D_1(0,0)=0,\quad \partial_s D_1(0,0)=0,\quad \partial_\lambda D_1(0,0)=-(L_1^{-1}\chi_2,\chi_2), \tag{3.61}$$

the application of the implicit function theorem implies that there exist constants $r_0, r_1>0$ and a unique C^∞ function $\lambda_0(s)$ such that $D_1(\lambda_0(s),s)=0$ for $(s,\lambda)\in[-r_0,r_0]\times B_{r_1}(0)$. In particular,

$$\lambda_0(0)=0 \quad\text{and}\quad \lambda_0'(0)=-\frac{\partial_s D(0,0)}{\partial_\lambda D(0,0)}=0. \tag{3.62}$$

A direct computation gives $\partial_s^2 D(0,0)=2$, which together with (3.61) yield

$$\lambda_0''(0)=-\frac{\partial_s^2 D(0,0)}{\partial_\lambda D(0,0)}=\frac{2}{(L_1^{-1}\chi_2,\chi_2)}. \tag{3.63}$$

Let

$$D_2(\lambda,s)=\frac{D_1(\lambda,s)}{\lambda-\lambda_0(s)}.$$

Similarly to Lemma 3.10, we can prove that there is $b_1>0$ so that $D_2(\lambda,0)=\lambda-(R(\lambda,0)\chi_2,\chi_2)\neq 0$ for $\mathrm{Re}\lambda\geq -b_1$. This and $\lim_{|\lambda|\to\infty}|D_2(\lambda,0)|=\infty$ imply that there is a constant $\delta>0$ such that $|D_2(\lambda,0)|\geq\delta$ for $\mathrm{Re}\lambda\geq -b_1$. Since

$$\begin{aligned} D_2(\lambda,s)&=\frac{D_1(\lambda,s)-D_1(\lambda_0(s),s)}{\lambda-\lambda_0(s)}\\ &=\lambda-(R(\lambda,se_1)\chi_2,\chi_2)+\lambda_0(s)+\lambda_0(s)\frac{([R(\lambda,se_1)-R(\lambda_0(s),se_1)]\chi_2,\chi_2)}{\lambda-\lambda_0(s)}, \end{aligned}$$

it follows that $|D_2(\lambda,s)-D_2(\lambda,0)|\leq C(|s|+|\lambda_0(s)|)\to 0,\ s\to 0$. Thus for $r_0>0$ small enough,

$$|D_2(\lambda,s)|\geq |D_2(\lambda,0)|-|D_2(\lambda,s)-D_2(\lambda,0)|>0,\quad \mathrm{Re}\lambda>-b_1,\quad |s|\leq r_0.$$

We then conclude that the equation $D(\lambda,s)=0$ with $\mathrm{Re}\lambda>-b_1$ has only one solution $\lambda_0(s)$ for $s\in[-r_0,r_0]$. ∎

With Lemmas 3.10–3.11, we are able to construct the eigenvector $\Psi_j(s,\omega)$ corresponding to the eigenvalue λ_j in the low frequency. Indeed, we have the following theorem.

THEOREM 3.12. *There exist two constants* $r_0>0$ *and* $b_2>0$ *so that the spectrum* $\lambda\in\sigma(\hat{\mathbb{A}}_1(\xi))\subset\mathbb{C}$ *for* $\xi=s\omega$ *with* $|s|\leq r_0$ *and* $\omega\in\mathbb{S}^2$ *consists of two points* $\{\lambda_j(s),\ j=1,2\}$ *in the domain* $\mathrm{Re}\lambda>-b_2$. *The spectrum* $\lambda_j(s)$ *and the corresponding eigenvector* $\Psi_j(s,\omega)=(\psi_j,X_j,Y_j)(s,\omega)$ *are* C^∞ *functions of* s *for* $|s|\leq r_0$. *In particular, the eigenvalues admit the following asymptotic expansion for* $|s|\leq r_0$*:*

$$\lambda_1(s)=\lambda_2(s)=-a_1s^2+o(s^2), \tag{3.64}$$

where

$$a_1=-\frac{1}{(L_1^{-1}\chi_2,\chi_2)}>0. \tag{3.65}$$

The eigenvectors $\Psi_j = (\psi_j, X_j, Y_j)$ *are orthogonal to each other and satisfy*

$$\text{(3.66)}\quad \begin{cases} (\Psi_i(s,\omega), \Psi_j^*(s,\omega)) = (\psi_i, \overline{\psi_j}) - (X_i, \overline{X_j}) - (Y_i, \overline{Y_j}) = \delta_{ij}, \\ (\psi_j, X_j, Y_j)(s,\omega) = (\psi_{j,0}, X_{j,0}, Y_{j,0})(\omega) + (\psi_{j,1}, X_{j,1}, Y_{j,1})(\omega)s + O(s^2), \end{cases}$$

where $\Psi_j^* = (\overline{\psi_j}, -\overline{X_j}, -\overline{Y_j})$*, and the coefficients* $(\psi_{j,n}, X_{j,n}, Y_{j,n})$ *are given by*

$$\text{(3.67)}\quad \begin{cases} \psi_{j,0} = 0, \quad P_{\rm d}\psi_{j,n} = 0 \ \ (n \ge 0), \quad X_{j,0} = 0, \quad Y_{j,0} = \mathrm{i}W^j, \\ \psi_{j,1} = -a_1 L_1^{-1} P_r (v \cdot W^j)\sqrt{M}, \quad X_{j,1} = a_1 \omega \times W^j, \quad Y_{j,1} = 0. \end{cases}$$

Here, W^j $(j = 1, 2)$ *are two orthonormal vectors satisfying* $W^j \cdot \omega = 0$.

Proof. The eigenvalues $\lambda_j(s)$ and the eigenvectors $\Psi_j(s,\omega) = (\psi_j, X_j, Y_j)(s,\omega)$, $j = 1, 2$, can be constructed as follows. Let $b_2 = \min\{b_0, b_1\}$ and take $\lambda_j = \lambda(s)$ to be the solution of the equation $D_1(\lambda, s) = 0$ defined in Lemma 3.11, and choose $C_0 = 0$, and $X_j = \omega \times W^j$ with W^j, $j = 1, 2$, being two linearly independent vectors satisfying $W^j \cdot \omega = 0$ and $W^1 \cdot W^2 = 0$. The corresponding eigenvectors $\Psi_j(s,\omega) = (\psi_j, X_j, Y_j)(s,\omega)$ are defined by

$$\text{(3.68)}\quad \begin{cases} \psi_j(s,\omega) = -[L_1 - \lambda_j P_r - \mathrm{i}sP_r(v\cdot\omega)P_r]^{-1} P_r(v \cdot W^j)\sqrt{M}, \\ X_j(s,\omega) = \omega \times W^j, \quad Y_j(s,\omega) = \dfrac{\mathrm{i}s}{\lambda_j} W^j, \quad j = 1, 2, \end{cases}$$

which satisfy $(\Psi_1(s,\omega), \Psi_2^*(s,\omega)) = 0$. We can normalize them by taking

$$(\Psi_j(s,\omega), \Psi_j^*(s,\omega)) = 1, \quad j = 1, 2.$$

The coefficients $W^j = b_1(s)T^j(\omega)$ for $j = 1, 2$ with $b_1 \in \mathbb{R}$ and $T^j = (T_1^j, T_2^j, T_3^j) \in \mathbb{S}^2$ are determined by the normalization condition as

$$\text{(3.69)}\quad \begin{cases} b_1(s)^2\left(D_1(s) - 1 + \dfrac{s^2}{\lambda_1(s)^2}\right) = 1, \\ T^1 \cdot \omega = T^2 \cdot \omega = T^1 \cdot T^2 = 0, \end{cases}$$

where $D_1(s) = (R(\lambda_1(s), se_1)\chi_1, R(\overline{\lambda_1(s)}, -se_1)\chi_1)$.

To study the asymptotic expression of eigenvectors in the low frequency, we can take Taylor expansion of both eigenvalues and eigenvectors as

$$\lambda_j(s) = \sum_{n=0}^{2} \lambda_{j,n} s^n + O(s^3), \quad (\psi_j, X_j, Y_j)(s,\omega) = \sum_{n=0}^{1} (\psi_{j,n}, X_{j,n}, Y_{j,n})(\omega)s^n + O(s^2).$$

Substituting the above expansions into (3.69), we obtain $b_1(0) = 0$, $b_1'(0) = a_1$, and $b_1(-s) = -b_1(s)$. This and (3.68) give the expansion of $\Psi_j(s,\omega)$ for $j = 1, 2$, stated in (3.67). This completes the proof of the theorem. ∎

3.3. Asymptotics in high frequency. We now turn to study the asymptotic expansions of the eigenvalues and eigenvectors in the high frequency region. First, recalling the eigenvalue problem

$$\text{(3.70)}\quad \lambda f = B_1(\xi)f - v\sqrt{M} \cdot (\omega \times X),$$

$$\lambda X = -\omega \times (f, v\sqrt{M}) + \mathrm{i}\xi \times Y, \qquad (3.71)$$
$$\lambda Y = -\mathrm{i}\xi \times X, \quad |\xi| \neq 0.$$

By Lemma 3.5, there is $R_0 > 0$ large enough such that the operator $\lambda - B_1(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3)$ for $\mathrm{Re}\lambda \geq -\nu_0/2$ and $|\xi| > R_0$. Then it follows from (3.70) that

$$f = (B_1(\xi) - \lambda)^{-1} v\sqrt{M} \cdot (\omega \times X), \quad |\xi| > R_0. \qquad (3.72)$$

Substituting (3.72) into (3.71) and using the transformation

$$((B_1(\xi) - \lambda)^{-1}\chi_i, \chi_j) = \omega_i\omega_j((B_1(|\xi|e_1) - \lambda)^{-1}\chi_1, \chi_1) + (\delta_{ij} - \omega_i\omega_j)((B(|\xi|e_1) - \lambda)^{-1}\chi_2, \chi_2),$$

we obtain

$$\lambda X = ((B_1(|\xi|e_1) - \lambda)^{-1}\chi_2, \chi_2)X + \mathrm{i}\xi \times Y, \qquad (3.73)$$
$$\lambda Y = -\mathrm{i}\xi \times X, \quad |\xi| > R_0. \qquad (3.74)$$

Multiplying (3.73) by λ and using (3.74) and (3.3), we obtain

$$(\lambda^2 - ((B_1(|\xi|e_1) - \lambda)^{-1}\chi_2, \chi_2)\lambda + |\xi|^2)X = 0, \quad |\xi| > R_0. \qquad (3.75)$$

Denote

$$D(\lambda, s) = \lambda^2 - ((B_1(se_1) - \lambda)^{-1}\chi_2, \chi_2)\lambda + s^2, \quad s > R_0. \qquad (3.76)$$

Similar to the proof of Lemma 3.5, we can obtain the following.

LEMMA 3.13. *If $x > -\nu_0$, we have*

$$\|(x + \mathrm{i}y - c(\xi))^{-1}K_1\| \leq C(x + \nu_0)^{-11/13}(1 + |\xi|)^{-2/13}, \qquad (3.77)$$
$$\|(x + \mathrm{i}y - c(\xi))^{-1}(v \cdot \xi)|\xi|^{-2}P_{\mathrm{d}}\| \leq C(x + \nu_0)^{-1}|\xi|^{-1}, \qquad (3.78)$$
$$\|(x + \mathrm{i}y - c(\xi))^{-1}v\sqrt{M}\| \leq C(x + \nu_0)^{-1/2}(1 + |\xi|)^{-1/2}. \qquad (3.79)$$

As consequence, it holds that for $|\xi| > R_0$,

$$\|(x + \mathrm{i}y - B_1(\xi))^{-1}v\sqrt{M}\| \leq C(x + \nu_0)^{-1/2}(1 + |\xi|)^{-1/2}. \qquad (3.80)$$

We now study (3.76) as follows.

LEMMA 3.14. *There is a constant $R_1 > 0$ such that the equation $D(\lambda, s) = 0$ has two solutions $\lambda_j(s)$, $j = \pm1$, for $s > R_1$ satisfying*

$$|\lambda_j(s) - j\mathrm{i}s| \leq Cs^{-1/2} \to 0 \quad as \quad s \to \infty.$$

In particular, there are two constants $c_1, c_2 > 0$ such that

$$c_1\frac{1}{s} \leq -\mathrm{Re}\lambda_j(s) \leq c_2\frac{1}{s}, \quad j = \pm1. \qquad (3.81)$$

Proof. For any fixed $s > R_0$, we define a function of λ as

$$G^1_j(\lambda) = \frac{1}{2}\Big(R_{22}(\lambda, s) + j\sqrt{R_{22}(\lambda, s)^2 - 4s^2}\Big), \quad j = \pm1, \;\; s > R_0, \qquad (3.82)$$

where $R_{22}(\lambda,s)=((B_1(se_1)-\lambda)^{-1}\chi_2,\chi_2)$. It is straightforward to verify that a solution of $D(\lambda,s)=0$ for any fixed $s>R_0$ is a fixed point of $G^1_j(\lambda)$.

Consider an equivalent equation of (3.82) as

(3.83)
$$G^2_j(\beta)=G^1_j(\lambda)-j\mathrm{i}s=\frac{1}{2}\Big(B_j(\beta,s)+\frac{jB_j(\beta,s)^2}{\sqrt{B_j(\beta,s)^2-4s^2+2\mathrm{i}s}}\Big),\quad j=\pm1,\ \ s>R_0,$$

where $B_j(\beta,s)=((B_1(se_1)-j\mathrm{i}s-\beta)^{-1}\chi_2,\chi_2)$. By (3.80), when $R_1>0$ is large enough and $\delta>0$ is small enough, it holds for $s>R_1$ and $|\beta|\le\delta$ that

$$|G^2_j(\beta)|\le\delta,\quad |G^2_j(\beta_1)-G^2_j(\beta_2)|\le\frac{1}{2}|\beta_1-\beta_2|.$$

Thus $G^2_j(\beta)$ is a contraction mapping on $B_\delta(0)$ and there is a unique fixed point $\beta_j(s)$ of $G^2_j(\beta)$. Thus $\lambda_j(s)=j\mathrm{i}s+\beta_j(s)$ is the solution of $D(\lambda,s)=0$ and $|\beta_j(s)|\le Cs^{-1/2}$ because $|B_j(\beta,s)|\le Cs^{-1/2}$ due to (3.80).

We now turn to prove (3.81). For this, we decompose $\beta_j(s)$, $j=\pm1$, into

$$\beta_j(s)=\frac{1}{2}B_j(\beta_j,s)+\frac{1}{2}\frac{jB_j(\beta_j,s)^2}{\sqrt{B_j(\beta_j,s)^2-4s^2+2\mathrm{i}s}}=I_1+I_2. \tag{3.84}$$

First, we estimate I_1. Since

$$-\mathrm{Re}I_1=-\frac{1}{2}((L_1-\mathrm{Re}\beta_j)g_j,g_j)=-\frac{1}{2}(L_1g_j,g_j)+O\bigg(\frac{1}{\sqrt{s}}\bigg)(g_j,g_j),$$

with $g_j=(L_1-\mathrm{i}(v_1+j)s-\mathrm{i}\frac{v_1}{s}P_{\mathrm{d}}-\beta_j)^{-1}\chi_2\in N_1^\perp$ for $j=\pm1$, we obtain

$$C_0(g_j,g_j)\le-\mathrm{Re}I_1\le C_1(\nu(v)g_j,g_j). \tag{3.85}$$

Note that

$$\bigg(L_1-\mathrm{i}(v_1\pm1)s-\mathrm{i}\frac{v_1}{s}P_{\mathrm{d}}-\beta_{\pm1}\bigg)^{-1}=(-\nu-\mathrm{i}(v_1\pm1)s)^{-1}+Z(\beta_{\pm1},s),$$

with

$$\begin{aligned}Z(\beta_{\pm1},s)&=Y(\beta_{\pm1},s)(I+Y(\beta_{\pm1},s))^{-1}(-\nu-\mathrm{i}(v_1\pm1)s)^{-1},\\ Y(\beta_{\pm1},s)&=(-\nu-\mathrm{i}(v_1\pm1)s)^{-1}\bigg(K_1-\mathrm{i}\frac{v_1}{s}P_{\mathrm{d}}-\beta_{\pm1}\bigg),\end{aligned}$$

we have

$$\begin{aligned}(\nu(v)g_{\pm1},g_{\pm1})&\le2(\nu(\nu+\mathrm{i}(v_1\pm1)s)^{-1}\chi_2,(\nu+\mathrm{i}(v_1\pm1)s)^{-1}\chi_2)\\&\quad+2(\nu Z(\beta_{\pm1},s)\chi_2,Z(\beta_{\pm1},s)\chi_2),\\ (g_{\pm1},g_{\pm1})&\ge\frac{1}{2}((\nu+\mathrm{i}(v_1\pm1)s)^{-1}\chi_2,(\nu+\mathrm{i}(v_1\pm1)s)^{-1}\chi_2)\\&\quad-(Z(\beta_{\pm1},s)\chi_2,Z(\beta_{\pm1},s)\chi_2).\end{aligned}$$

In the following, we will prove

$$I_3=:((\nu+\mathrm{i}(v_1\pm1)s)^{-1}\chi_2,(\nu+\mathrm{i}(v_1\pm1)s)^{-1}\chi_2)\ge\frac{C_3}{s}, \tag{3.86}$$

for some constant $C_3 > 0$. Indeed, by changing variables $(u_1, u_2, u_3) = ((v_1 \pm 1)s, v_2, v_3)$, we obtain for $s > 1$ that

$$\begin{aligned} I_3 &\geq \int_{\mathbb{R}^3} \frac{1}{\nu_1^2(1+|v|^2) + (v_1 \pm 1)^2 s^2} v_2^2 M dv \\ &\geq \frac{1}{s} \int_{\mathbb{R}^3} \frac{1}{\nu_1^2(1 + (\frac{u_1}{s} \mp 1)^2 + u_2^2 + u_3^2) + u_1^2} u_2^2 e^{-\frac{1}{2}(\frac{u_1}{s} \mp 1)^2} e^{-\frac{u_2^2}{2}} e^{-\frac{u_3^2}{2}} du \\ &\geq \frac{1}{s} \int_{\mathbb{R}^3} \frac{1}{\nu_1^2(3 + 2u_1^2 + u_2^2 + u_3^2) + u_1^2} u_2^2 e^{-u_1^2 - 1} e^{-\frac{u_2^2}{2}} e^{-\frac{u_3^2}{2}} du \geq \frac{C_3}{s}. \end{aligned}$$

As for

$$I_4 =: (\nu(\nu + \mathrm{i}(v_1 \pm 1)s)^{-1} \chi_2, (\nu + \mathrm{i}(v_1 \pm 1)s)^{-1} \chi_2), \tag{3.87}$$

by the changing variables $(u_1, u_2, u_3) = ((v_1 \pm 1)s, v_2, v_3)$, we obtain for $s > 1$ that

$$\begin{aligned} I_4 &\leq \int_{\mathbb{R}^3} \frac{\nu_0^2(1+|v|^2)}{\nu_1^2 + (v_1 \pm 1)^2 s^2} v_2^2 M dv \leq C \int_{\mathbb{R}^3} \frac{1}{\nu_1^2 + (v_1 \pm 1)^2 s^2} v_2^2 e^{-\frac{|v|^2}{4}} dv \\ &\leq C \frac{1}{s} \int_{\mathbb{R}^3} \frac{1}{\nu_1^2 + u_1^2} u_2^2 e^{-\frac{1}{4}(\frac{u_1}{s} \mp 1)^2} e^{-\frac{u_2^2}{4}} e^{-\frac{u_3^2}{4}} du \\ &\leq C \frac{1}{s} \int_{\mathbb{R}^3} \frac{1}{\nu_1^2 + u_1^2} u_2^2 e^{-\frac{u_2^2}{4}} e^{-\frac{u_3^2}{4}} du \leq \frac{C_4}{s}, \end{aligned}$$

where $C_4 > 0$ is a constant.

By Lemma 3.13, for any $0 < \epsilon \ll 1$ there exists $s > R_1$ such that

$$\begin{aligned} \|Y(\beta_{\pm 1}, s)\| &\leq \|(\nu + \mathrm{i}(v_1 \pm 1)s)^{-1} \left(K_1 - \frac{\mathrm{i}v_1}{s} P_{\mathrm{d}} \right) \| \\ &\qquad + O\left(\frac{1}{\sqrt{s}} \right) \|(\nu + \mathrm{i}(v_1 \pm 1)s)^{-1}\| \leq \epsilon, \\ \|\nu Y(\beta_{\pm 1}, s)\| &\leq \|\nu(\nu + \mathrm{i}(v_1 \pm 1)s)^{-1}\| \left\| K_1 - \mathrm{i}\frac{v_1}{s} P_{\mathrm{d}} - \beta_{\pm 1} \right\| \leq C. \end{aligned}$$

This and (3.87) lead to

$$\begin{aligned} &(\nu Z(\beta_{\pm 1}, s)\chi_2, Z(\beta_{\pm 1}, s)\chi_2) \\ &\leq \|\nu Y(\beta_{\pm 1}, s)\| \|Y(\beta_{\pm 1}, s)\| \|(I + Y(\beta_{\pm 1}, s))^{-1}\|^2 \|(\nu + \mathrm{i}(v_1 \pm 1)s)^{-1}\chi_2\|^2 \\ &\leq C\epsilon I_4 \leq \frac{C\epsilon}{s}. \end{aligned}$$

Thus, by combining with (3.85), (3.86), and (3.87), there exist constants $C_5, C_6 > 0$ such that

$$\frac{C_5}{s} \leq -\mathrm{Re} I_1 \leq \frac{C_6}{s}. \tag{3.88}$$

For I_2, we have

$$|I_2| \leq \frac{C}{s} |B_{\pm 1}(\beta_{\pm 1}, s)|^2 \leq \frac{C}{s^2}. \tag{3.89}$$

Combining (3.84), (3.88), and (3.89), we obtain (3.81). The proof of the lemma is then complete. □

The following theorem gives the asymptotic expansions of eigenvalues and eigenvectors in the high frequency region.

THEOREM 3.15. *There exists a constant* $r_1 > 0$ *such that the spectrum* $\sigma(\hat{\mathbb{A}}_1(\xi)) \subset \mathbb{C}$ *for* $\xi = s\omega$ *with* $s = |\xi| > r_1$ *and* $\omega \in \mathbb{S}^2$ *consists of four eigenvalues* $\{\beta_j(s),\ j = 1,2,3,4\}$ *in the domain* $\mathrm{Re}\lambda > -\mu/2$. *In particular, the eigenvalues satisfy*

$$\beta_1(s) = \beta_2(s) = -\mathrm{i}s + O(s^{-1/2}), \tag{3.90}$$

$$\beta_3(s) = \beta_4(s) = \mathrm{i}s + O(s^{-1/2}), \tag{3.91}$$

$$\frac{c_1}{s} \le -\mathrm{Re}\beta_j(s) \le \frac{c_2}{s} \tag{3.92}$$

for two positive constants c_1 *and* c_2. *The eigenvectors* $\Phi_j(s,\omega) = (\phi_j, X_j, Y_j)(s,\omega)$ *are orthogonal to each other and satisfy*

$$(\Phi_i(s,\omega), \Phi_j^*(s,\omega)) = (\phi_i, \overline{\phi_j}) - (X_i, \overline{X_j}) - (Y_i, \overline{Y_j}) = \delta_{ij}, \quad 1 \le i, j \le 4, \tag{3.93}$$

where $\Phi_j^* = (\overline{\phi_j}, -\overline{X_j}, -\overline{Y_j})$. *Moreover,*

$$\|\phi_j(s,\omega)\| = O\left(\frac{1}{\sqrt{s}}\right),\ P_{\mathrm{d}}\phi_j(s,\omega) = 0,\ X_j(s,\omega) = O(1)\mathrm{i}(\omega\times W^j),\ Y_j(s,\omega) = O(1)\mathrm{i}W^j. \tag{3.94}$$

Here, W^j $(j = 1,2,3,4)$ *are vectors satisfying* $W^j \cdot \omega = 0$, $W^1 \cdot W^2 = 0$, $W^1 = W^3, W^2 = W^4$ *and the normalization condition* (3.96).

Proof. The eigenvalue $\beta_j(s)$ and the eigenvector $\Phi_j(s,\omega) = (\phi_j, X_j, Y_j)(s,\omega)$ can be constructed as follows. For $j = 1,2,3,4$, we take $\beta_1 = \beta_2 = \lambda_{-1}(s)$ and $\beta_3 = \beta_4 = \lambda_1(s)$ to be the solution of the equation $D(\lambda, s) = 0$ defined in Lemma 3.14. Choose $X_j = \omega \times W^j$ with $W^1 = W^3$ and $W^2 = W^4$ as linearly independent vectors so that $W^j \cdot \omega = 0$ and $W^1 \cdot W^2 = 0$. The corresponding eigenvectors $\Phi_j(s,\omega) = (\phi_j, X_j, Y_j)(s,\omega)$, $1 \le j \le 4$, are defined by

$$\begin{cases} \phi_j(s,\omega) = -\left[L_1 - \beta_j - \mathrm{i}s(v\cdot\omega) - \dfrac{\mathrm{i}(v\cdot\omega)}{s}P_{\mathrm{d}}\right]^{-1}(v\cdot W^j)\sqrt{M}, \\ X_j(s,\omega) = \omega \times W^j, \quad Y_j(s,\omega) = \dfrac{\mathrm{i}s}{\beta_j}W^j, \end{cases} \tag{3.95}$$

which satisfy $(\Phi_1(s,\omega), \Phi_2^*(s,\omega)) = (\Phi_3(s,\omega), \Phi_4^*(s,\omega)) = 0$.

Rewrite the eigenvalue problem as

$$\hat{\mathbb{A}}_1(\xi)\Phi_j(s,\omega) = \beta_j(s)\Phi_j(s,\omega), \quad 1 \le j \le 4,\ |s| \le r_0.$$

By taking the inner product $(\cdot,\cdot)_\xi$ of it with $\Phi_j^*(s,\omega)$, and using the fact that

$$(\hat{\mathbb{A}}_1(\xi)U, V)_\xi = (U, \hat{\mathbb{A}}_1(\xi)^*V)_\xi, \quad U, V \in D(\hat{B}_1(\xi)) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi,$$
$$\hat{\mathbb{A}}_1(\xi)^*\Phi_j^*(s,\omega) = \overline{\beta_j(s)}\Phi_j^*(s,\omega),$$

we have

$$(\beta_j(s) - \beta_k(s))(\Phi_j(s,\omega), \Phi_j^*(s,\omega))_\xi = 0, \quad 1 \le j,k \le 4.$$

Since $\beta_j(s) \ne \beta_k(s)$ for $j = 1,2$, $k = 3,4$ and $P_{\mathrm{d}}\phi_j(s,\omega) = 0$, we have the orthogonal relation

$$(\Phi_j(s,\omega), \Phi_k^*(s,\omega))_\xi = (\Phi_j(s,\omega), \Phi_k^*(s,\omega)) = 0, \quad 1 \le j \ne k \le 4.$$

This can be normalized so that

$$(\Phi_j(s,\omega),\Phi_j^*(s,\omega))=1,\quad j=1,2,3,4.$$

Precisely, denote $W^j=b_j(s)T^j(\omega)$ for $j=1,2,3,4$, with $b_j\in\mathbb{R}$ and $T^j=(T_1^j,T_2^j,T_3^j)\in\mathbb{S}^2$, then the coefficients b_j and T_j are determined by the normalization condition

$$\begin{cases} b_j(s)^2\left(D_j(s)-1+\dfrac{s^2}{\beta_j(s)^2}\right)=1, \quad j=1,2,3,4,\\ T^j\cdot\omega=T^1\cdot T^2=0,\quad T^1=T^3,\ T^2=T^4, \end{cases} \tag{3.96}$$

where $D_j(s)=((B_1(se_1)-\beta_j(s))^{-1}\chi_1,(B_1(-se_1)-\overline{\beta_j(s)})^{-1}\chi_1)$. By substituting (3.80), (3.90), and (3.91) into (3.96) and (3.95), we obtain (3.94) so that the proof of the theorem is complete. □

4. Spectral analysis for the one-species VMB system. In this section, we will study the spectrum structure of the linearized system (2.41) of one-species VMB. It is interesting to find out that its structure is very different in the low frequency region from the case of two species. And this essential difference comes from the lack of the cancellation in the one-species system.

4.1. Spectrum structure. Since $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ is the invariant subspace of the operator $\hat{\mathbb{A}}_3(\xi)$, we can regard $\hat{\mathbb{A}}_3(\xi)$ as a linear operator on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$. We have for any $U,V\in L^2_\xi(\mathbb{R}^3_v)\cap D(\hat{B}_2(\xi))\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$,

$$(\hat{\mathbb{A}}_3(\xi)U,V)_\xi=(U,\hat{\mathbb{A}}_3(-\xi)V)_\xi.$$

Denote by $\rho(\hat{\mathbb{A}}_3(\xi))$ the resolvent set and by $\sigma(\hat{\mathbb{A}}_3(\xi))$ the spectrum set of $\hat{\mathbb{A}}_3(\xi)$. Similar to Lemmas 3.1 and 3.3, we have the following lemmas.

LEMMA 4.1. *The operator $\hat{\mathbb{A}}_3(\xi)$ generates a strongly continuous contraction semigroup on $L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ satisfying*

$$\|e^{t\hat{\mathbb{A}}_3(\xi)}U\|_\xi\le\|U\|_\xi\quad \textit{for } t>0,\ U\in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi. \tag{4.1}$$

LEMMA 4.2. *For each $\xi\neq0$, the spectrum set $\sigma(\hat{\mathbb{A}}_3(\xi))$ of the operator $\hat{\mathbb{A}}_3(\xi)$ in the domain $\mathrm{Re}\lambda\ge-\nu_0+\delta$ for any constant $\delta>0$ consists of isolated eigenvalues $\Sigma=:\{\lambda_j(\xi)\}$ with $\mathrm{Re}\lambda_j(\xi)<0$.*

We will make use of the following decomposition associated with the operator $\hat{\mathbb{A}}_3(\xi)$ for $|\xi|>0$:

$$\lambda-\hat{\mathbb{A}}_3(\xi)=\lambda-G_1(\xi)-G_5(\xi)=(I-G_5(\xi)(\lambda-G_1(\xi))^{-1})(\lambda-G_1(\xi)), \tag{4.2}$$

where $G_1(\xi)$ is defined by (3.11), and

$$G_5(\xi)=\begin{pmatrix} K-\frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\rm d} & -v\sqrt{M}\cdot\omega\times & 0\\ -\omega\times P_m & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{4.3}$$

Here,

$$G_5(\xi)(\lambda-G_1(\xi))^{-1}=\begin{pmatrix} X_1^1(\lambda,\xi) & X_2(\lambda,\xi)\\ X_3(\lambda,\xi) & 0\end{pmatrix}_{7\times7}, \tag{4.4}$$

where $X_2(\lambda,\xi)$ and $X_3(\lambda,\xi)$ are defined by (3.20) and (3.21), and

$$X_1^1(\lambda,\xi)=\left(K-\frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}}\right)(\lambda-c(\xi))^{-1}. \tag{4.5}$$

By a similar argument as the one for Lemma 3.6, we can obtain the spectrum of the operator $\hat{\mathbb{A}}_3(\xi)$ in the intermediate and high frequency regions.

Lemma 4.3. *In the high and intermediate regions of the frequency, we have the following:*

1. *For any $\delta_1,\delta_2>0$, there exists $r_1=r_1(\delta_1,\delta_2)>0$ so that for $|\xi|>r_1$,*

$$\sigma(\hat{\mathbb{A}}_3(\xi))\cap\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda\ge-\nu_0+\delta_1\}\subset\bigcup_{j=\pm1}\{\lambda\in\mathbb{C}\,|\,|\lambda-j\mathrm{i}|\xi||\le\delta_2\}. \tag{4.6}$$

2. *For any $r_1>r_0>0$, there exists $\alpha=\alpha(r_0,r_1)>0$ so that for all $r_0\le|\xi|\le r_1$,*

$$\sigma(\hat{\mathbb{A}}_3(\xi))\subset\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda(\xi)\le-\alpha\}. \tag{4.7}$$

Then, we need only study the spectrum and resolvent sets of $\hat{\mathbb{A}}_3(\xi)$ in the low frequency region. For this, we decompose $\lambda-\hat{\mathbb{A}}_3(\xi)$ as

$$\lambda-\hat{\mathbb{A}}_3(\xi)=\lambda P_A-G_6(\xi)+\lambda P_B-G_7(\xi)+G_8(\xi)+G_9(\xi), \tag{4.8}$$

where

$$G_6(\xi)=\begin{pmatrix} B_4(\xi) & -v\sqrt{M}\cdot\omega\times & 0\\ -\omega\times P_m & 0 & \mathrm{i}\xi\times\\ 0 & -\mathrm{i}\xi\times & 0\end{pmatrix},\quad G_7(\xi)=\begin{pmatrix} B_5(\xi) & 0 & 0\\ 0&0&0\\0&0&0\end{pmatrix}, \tag{4.9}$$

$$G_8(\xi)=\begin{pmatrix} \mathrm{i}\mathrm{P}_1(v\cdot\xi)\mathrm{P}_0 & 0 & 0\\ 0&0&0\\0&0&0\end{pmatrix},\quad G_9(\xi)=\begin{pmatrix} \mathrm{i}\mathrm{P}_0(v\cdot\xi)\mathrm{P}_1 & 0 & 0\\ 0&0&0\\0&0&0\end{pmatrix}, \tag{4.10}$$

$$B_4(\xi)=\mathrm{i}\mathrm{P}_0(v\cdot\xi)\mathrm{P}_0-\frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}},\quad B_5(\xi)=L-\mathrm{i}\mathrm{P}_1(v\cdot\xi)\mathrm{P}_1, \tag{4.11}$$

and P_A, P_B are the orthogonal projection operators defined by

$$P_A=\begin{pmatrix} \mathrm{P}_0 & 0 & 0\\ 0 & I_{3\times3} & 0\\ 0&0&I_{3\times3}\end{pmatrix},\quad P_B=\begin{pmatrix} \mathrm{P}_1 & 0 & 0\\ 0&0&0\\0&0&0\end{pmatrix}. \tag{4.12}$$

It is straightforward to verify that $G_6(\xi)$ is a linear operator from the space $N_0\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ to itself, which admits nine eigenvalues $\alpha_j(\xi)$ satisfying

$$\begin{cases}\alpha_j(\xi)=0,\quad j=0,2,3,\quad \alpha_{\pm1}(\xi)=\pm\mathrm{i}\sqrt{1+\frac{5}{3}|\xi|^2},\\ \alpha_4(\xi)=\alpha_5(\xi)=-\mathrm{i}\sqrt{1+|\xi|^2},\quad \alpha_6(\xi)=\alpha_7(\xi)=\mathrm{i}\sqrt{1+|\xi|^2}.\end{cases} \tag{4.13}$$

Lemma 4.4. *Let $\xi\ne0$. We have the following properties for the linear operators $G_6(\xi)$ and $G_7(\xi)$ defined by* (4.9)*:*

1. *If* $\lambda \neq \alpha_j(\xi)$, *then the operator* $\lambda P_A - G_6(\xi)$ *is invertible on* $N_0 \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ *and satisfies*

$$\|(\lambda P_A - G_6(\xi))^{-1}\|_\xi = \max_{-1\le j\le 7} \left(|\lambda - \alpha_j(\xi)|^{-1}\right), \tag{4.14}$$

$$\|G_8(\xi)(\lambda P_A - G_6(\xi))^{-1}P_A\|_\xi \le C|\xi| \max_{-1\le j\le 7} \left(|\lambda - \alpha_j(\xi)|^{-1}\right), \tag{4.15}$$

where $\alpha_j(\xi)$, $-1 \le j \le 7$, *are the eigenvalues of* $G_6(\xi)$ *defined by* (4.13).

2. *If* $\mathrm{Re}\lambda > -\mu$, *then the operator* $\lambda P_B - G_7(\xi)$ *is invertible on* $N_0^\perp \times \{0\} \times \{0\}$ *and satisfies*

$$\|(\lambda P_B - G_7(\xi))^{-1}\| \le (\mathrm{Re}\lambda + \mu)^{-1}, \tag{4.16}$$

$$\|G_9(\xi)(\lambda P_B - G_7(\xi))^{-1}P_B\|_\xi \le C(1+|\lambda|)^{-1}[(\mathrm{Re}\lambda+\mu)^{-1}+1](|\xi|+|\xi|^2). \tag{4.17}$$

Proof. Since the operator $\mathrm{i}G_6(\xi)$ is self-adjoint on $N_0 \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$, namely,

$$(\mathrm{i}G_6(\xi)U, V)_\xi = (U, \mathrm{i}G_6(\xi)V)_\xi, \quad U, V \in N_0 \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi, \tag{4.18}$$

we can prove (4.14)–(4.15). And by the dissipative of the operator $G_7(\xi)$ on $N_0^\perp \times \{0\} \times \{0\}$, we can prove (4.16)–(4.17). ∎

With the help of Lemmas 4.2–4.4, we obtain the spectral and resolvent sets of the operator $\hat{\mathbb{A}}_3(\xi)$ by applying a similar argument to that of Lemma 3.8.

LEMMA 4.5. *For any* $\delta_1, \delta_2 > 0$, *there are* $r_1 = r_1(\delta_1,\delta_2), r_2 = r_2(\delta_1,\delta_2), y_1 = y_1(\delta_1,\delta_2) > 0$ *such that we have the following:*

1. *it holds for all* $|\xi| \neq 0$ *that the resolvent set of* $\hat{\mathbb{A}}_3(\xi)$ *contains the regions*

$$\rho(\hat{\mathbb{A}}_3(\xi)) \supset \begin{cases} \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \ge -\mu+\delta_1,\ |\mathrm{Im}\lambda| \ge y_1\} \cup \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > 0\}, \\ |\xi| \le r_1, \\ \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \ge -\mu+\delta_1,\ \min\limits_{j=\pm1} |\lambda - j\mathrm{i}|\xi|| \ge \delta_2\} \cup \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda > 0\}, \\ |\xi| \ge r_1; \end{cases} \tag{4.19}$$

2. *it holds for* $0 < |\xi| \le r_2$ *that the spectrum set of* $\hat{\mathbb{A}}_3(\xi)$ *is located in the domain*

$$\sigma(\hat{\mathbb{A}}_3(\xi)) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \ge -\mu+\delta_1\} \subset \bigcup_{j=-1}^{7} \{\lambda \in \mathbb{C} \,|\, |\lambda - \alpha_j(\xi)| \le \delta_2\}, \tag{4.20}$$

where $\alpha_j(\xi)$, $-1 \le j \le 7$, *are the eigenvalues of* $G_6(\xi)$ *defined in* (4.13).

4.2. Asymptotics in low frequency. We study the low frequency asymptotics of the eigenvalues and eigenvectors of the operator $\hat{\mathbb{A}}_3(\xi)$ in this subsection. In terms of (2.43), the eigenvalue problem $\hat{\mathbb{A}}_3(\xi)U = \lambda U$ for $U = (f, X, Y) \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$ can be written as

$$\begin{cases} \lambda f = (L - \mathrm{i}(v\cdot\xi) - \frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}})f - v\sqrt{M}\cdot(\omega \times X), \\ \lambda X = -\omega \times (f, v\sqrt{M}) + \mathrm{i}\xi \times Y, \\ \lambda Y = -\mathrm{i}\xi \times X, \quad |\xi| \neq 0. \end{cases} \tag{4.21}$$

By macro-micro decomposition, the eigenfunction f of (4.21) can be decomposed into $f = f_0 + f_1 =: \mathrm{P}_0 f + \mathrm{P}_1 f$. Then the first equation of (4.21) gives rise to

$$\lambda f_0 = -\mathrm{P}_0[\mathrm{i}(v\cdot\xi)(f_0+f_1)] - \frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}} f_0 - v\sqrt{M}\cdot(\omega\times X), \tag{4.22}$$

$$\lambda f_1 = Lf_1 - \mathrm{P}_1[\mathrm{i}(v\cdot\xi)(f_0+f_1)] \implies f_1 = -\mathrm{i}(\lambda\mathrm{P}_1 - B_5(\xi))^{-1}\mathrm{P}_1(v\cdot\xi)f_0. \tag{4.23}$$

Substituting (4.23) into (4.22), we obtain the eigenvalue problem for (f_0, X, Y) as

$$\begin{cases} \lambda f_0 = -\mathrm{i}\mathrm{P}_0(v\cdot\xi)f_0 - \frac{\mathrm{i}(v\cdot\xi)}{|\xi|^2}P_{\mathrm{d}} f_0 + \mathrm{P}_0[(v\cdot\xi)R^1(\lambda,\xi)\mathrm{P}_1(v\cdot\xi)f_0] - v\sqrt{M}\cdot(\omega\times X),\\ \lambda X = -\omega\times(f_0, v\sqrt{M}) + \mathrm{i}\xi\times Y,\\ \lambda Y = -\mathrm{i}\xi\times X, \end{cases} \tag{4.24}$$

where $R^1(\lambda,\xi) = -(\lambda\mathrm{P}_1 - B_5(\xi))^{-1} = [L - \lambda\mathrm{P}_1 - \mathrm{i}\mathrm{P}_1(v\cdot\xi)\mathrm{P}_1]^{-1}$.

To solve the eigenvalue problem (4.24), we write $f_0 \in N_0$ as $f_0 = \sum_{j=0}^4 W_j\chi_j$. Taking the inner product of the first equation of (4.24) and χ_j for $j = 0,1,2,3,4$, respectively, we have the equations about λ and (W_0, W, W_4, X, Y) with $W =: (W_1, W_2, W_3)$ and $X = (X_1, X_2, X_3), Y = (Y_1, Y_2, Y_3) \in \mathbb{C}^3_\xi$ for $\mathrm{Re}\lambda > -\mu$:

$$\lambda W_0 = -\mathrm{i}(W\cdot\xi), \tag{4.25}$$

$$\lambda W_i = -\mathrm{i}W_0\left(\xi_i + \frac{\xi_i}{|\xi|^2}\right) - \mathrm{i}\sqrt{\frac{2}{3}}W_4\xi_i - (\omega\times X)_i + \sum_{j=1}^4 W_j(R^1(\lambda,\xi)\mathrm{P}_1(v\cdot\xi)\chi_j, (v\cdot\xi)\chi_i), \tag{4.26}$$

$$\lambda W_4 = -\mathrm{i}\sqrt{\frac{2}{3}}(W\cdot\xi) + \sum_{j=1}^4 W_j(R^1(\lambda,\xi)\mathrm{P}_1(v\cdot\xi)\chi_j, (v\cdot\xi)\chi_4), \tag{4.27}$$

$$\lambda X = -\omega\times W + \mathrm{i}\xi\times Y, \tag{4.28}$$

$$\lambda Y = -\mathrm{i}\xi\times X. \tag{4.29}$$

We apply the following transform to simplify the system (4.25)–(4.27).

Lemma 4.6 (see [13]). *Let $e_1 = (1,0,0)$, $\xi = s\omega$ with $s\in\mathbb{R}$, $\omega = (\omega_1,\omega_2,\omega_3)\in\mathbb{S}^2$. Then, it holds for $1\le i,j\le 3$ and $Re\lambda > -\mu$ that*

$$(R^1(\lambda,\xi)\mathrm{P}_1(v\cdot\xi)\chi_j, (v\cdot\xi)\chi_i) = s^2(\delta_{ij} - \omega_i\omega_j)(R^1(\lambda, se_1)\mathrm{P}_1(v_1\chi_2), v_1\chi_2) + s^2\omega_i\omega_j(R^1(\lambda, se_1)\mathrm{P}_1(v_1\chi_1), v_1\chi_1), \tag{4.30}$$

$$(R^1(\lambda,\xi)\mathrm{P}_1(v\cdot\xi)\chi_4, (v\cdot\xi)\chi_i) = s^2\omega_i(R^1(\lambda, se_1)\mathrm{P}_1(v_1\chi_4), v_1\chi_1), \tag{4.31}$$

$$(R^1(\lambda,\xi)\mathrm{P}_1(v\cdot\xi)\chi_i, (v\cdot\xi)\chi_4) = s^2\omega_i(R^1(\lambda, se_1)\mathrm{P}_1(v_1\chi_1), v_1\chi_4), \tag{4.32}$$

$$(R^1(\lambda,\xi)\mathrm{P}_1(v\cdot\xi)\chi_4, (v\cdot\xi)\chi_4) = s^2(R^1(\lambda, se_1)\mathrm{P}_1(v_1\chi_4), v_1\chi_4). \tag{4.33}$$

With the help of (4.30)–(4.33), the system (4.25)–(4.27) can be simplified as

$$\lambda W_0 = -\mathrm{i}s(W\cdot\omega), \tag{4.34}$$

$$\lambda W_i = -\mathrm{i}W_0\left(s + \frac{1}{s}\right)\omega_i - \mathrm{i}s\sqrt{\frac{2}{3}}W_4\omega_i + s^2(W\cdot\omega)\omega_i R_{11}$$

$$+ s^2(W_i - (W\cdot\omega)\omega_i)R_{22} + s^2 W_4\omega_i R_{41} - (\omega\times X)_i, \quad i=1,2,3, \tag{4.35}$$

$$\lambda W_4 = -\mathrm{i}s\sqrt{\frac{2}{3}}(W\cdot\omega) + s^2(W\cdot\omega)R_{14} + s^2 W_4 R_{44}, \tag{4.36}$$

where

$$R_{ij} = R_{ij}(\lambda,s) =: (R^1(\lambda, se_1)\mathrm{P}_1(v_1\chi_i), v_1\chi_j). \tag{4.37}$$

Multiplying (4.35) by ω_i and making the summation of resulted equations with respect to $i=1,2,3$, we have

$$\lambda(W\cdot\omega) = -\mathrm{i}W_0\left(s+\frac{1}{s}\right) - \mathrm{i}s\sqrt{\frac{2}{3}}W_4 + s^2(W\cdot\omega)R_{11} + s^2 W_4 R_{41}. \tag{4.38}$$

Denote by $U=(W_0, W\cdot\omega, W_4)$ a vector in $\mathbb{R}^3$. The system (4.34), (4.36), and (4.38) can be written as $\mathbb{M}U=0$ with the matrix $\mathbb{M}$ defined by

$$\mathbb{M} = \begin{pmatrix} \lambda & \mathrm{i}s & 0 \\ \mathrm{i}(s+\frac{1}{s}) & \lambda - s^2R_{11} & \mathrm{i}s\sqrt{\frac{2}{3}} - s^2R_{41} \\ 0 & \mathrm{i}s\sqrt{\frac{2}{3}} - s^2R_{14} & \lambda - s^2R_{44} \end{pmatrix}. \tag{4.39}$$

The equation $\mathbb{M}U=0$ admits a nontrivial solution $U\neq 0$ for $\mathrm{Re}\lambda > -\mu$ if and only if it holds that $\det(\mathbb{M})=0$ for $\mathrm{Re}\lambda>-\mu$.

Furthermore, by taking $\omega\times$ to (4.35) and using (3.3), we have

$$(\lambda - s^2R_{22})(\omega\times W) = X. \tag{4.40}$$

Then we multiply (4.28) by $\lambda(\lambda - s^2R_{22})$ and using (4.40), (4.29), and (3.3) we have

$$(\lambda^3 - s^2R_{22}\lambda^2 + (1+s^2)\lambda - s^4R_{22})X = 0. \tag{4.41}$$

Denote

$$D(\lambda,s) = \det(\mathbb{M}), \quad D_0(\lambda,s) =: \lambda^3 - s^2R_{22}\lambda^2 + (1+s^2)\lambda - s^4R_{22}. \tag{4.42}$$

The following two lemmas are about the solutions to the equations $D(\lambda,s)=0$ and $D_0(\lambda,s)=0$.

Lemma 4.7 (see [13]). *There exist two small constants $r_0>0$ and $r_1>0$ so that the equation $D(\lambda,s)=0$ admits C^∞ solution $\lambda_j(s)$ $(j=-1,0,1)$ for $(s,\lambda_j)\in[-r_0,r_0]\times B_{r_1}(j\mathrm{i})$ that satisfy*

$$\lambda_j(0) = j\mathrm{i}, \quad \lambda_j'(0)=0, \tag{4.43}$$

$$\begin{aligned}\lambda_{\pm1}''(0) =& 2(L(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_1), (L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_1)) \\ &\pm 2\mathrm{i}\left(\|(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_1)\|^2 + \frac{5}{3}\right),\end{aligned} \tag{4.44}$$

$$\lambda_0''(0) = 2(L^{-1}\mathrm{P}_1(v_1\chi_4), v_1\chi_4). \tag{4.45}$$

Moreover, $\lambda_j(s)$ is an even function and satisfies

$$\overline{\lambda_j(s)} = \lambda_{-j}(-s) = \lambda_{-j}(s) \quad \textit{for} \quad j=0,\pm1. \tag{4.46}$$

In particular, $\lambda_0(s)$ is a real function.

LEMMA 4.8. *There exist two small constants $r_0 > 0$ and $r_1 > 0$ so that the equation $D_0(\lambda, s) = 0$ admits C^∞ solution $\lambda_j(s)$ $(j = -1, 0, 1)$ for $(s, \lambda_j) \in [-r_0, r_0] \times B_{r_1}(j\mathrm{i})$ that satisfy*

$$\lambda_j(0) = j\mathrm{i}, \quad \lambda_j'(0) = 0, \quad \lambda_0''(0) = \lambda_0'''(0) = 0, \tag{4.47}$$

$$\begin{aligned}\lambda_{\pm1}''(0) =& 2(L(L + \mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2), (L + \mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2))\\ &\pm 2\mathrm{i}(\|(L + \mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2)\|^2 + 1),\end{aligned} \tag{4.48}$$

$$\lambda_0^{(4)}(0) = 24(L^{-1}\mathrm{P}_1(v_1\chi_2), v_1\chi_2). \tag{4.49}$$

Moreover, $\lambda_j(s)$ is an even function and satisfies

$$\overline{\lambda_j(s)} = \lambda_{-j}(-s) = \lambda_{-j}(s) \quad \textit{for} \quad j = 0, \pm1. \tag{4.50}$$

In particular, $\lambda_0(s)$ is a real function.

Proof. Since

$$D_0(j\mathrm{i}, 0) = 0, \quad \partial_s D_0(j\mathrm{i}, 0) = 0, \quad \partial_\lambda D_0(j\mathrm{i}, 0) = 1 - 3j^2 \neq 0, \quad j = -1, 0, 1, \tag{4.51}$$

the implicit function theorem implies that there exist small constants $r_0, r_1 > 0$ and a unique C^∞ function $\lambda_j(s)$: $[-r_0, r_0] \to B_{r_1}(j\mathrm{i})$ such that $D_0(\lambda_j(s), s) = 0$ for $s \in [-r_0, r_0]$, and in particular

$$\lambda_j(0) = j\mathrm{i} \quad \text{and} \quad \lambda_j'(0) = -\frac{\partial_s D_0(j\mathrm{i},0)}{\partial_\lambda D_0(j\mathrm{i},0)} = 0, \quad j = 0, \pm1.$$

Direct computation gives

$$\begin{aligned}&\partial_s^2 D_0(\pm\mathrm{i}, 0) = 2((L \mp \mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2), v_1\chi_2) \pm 2\mathrm{i},\\ &\partial_s^2 D_0(0, 0) = \partial_s^3 D_0(0, 0) = 0, \quad \partial_s^4 D_0(0, 0) = 24(L^{-1}\mathrm{P}_1(v_1\chi_2), v_1\chi_2),\end{aligned}$$

which together with (4.51) yields

$$\begin{cases} \lambda_0''(0) = \lambda_0'''(0) = 0, \quad \lambda_0^{(4)}(0) = -\frac{\partial_s^4 D_0(0,0)}{\partial_\lambda D_0(0,0)} = 24(L^{-1}\mathrm{P}_1(v_1\chi_2), v_1\chi_2), \\ \lambda_{\pm1}''(0) = -\frac{\partial_s^2 D_0(\pm\mathrm{i},0)}{\partial_\lambda D_0(\pm\mathrm{i},0)} = 2((L \mp \mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2), v_1\chi_2) \pm 2\mathrm{i}. \end{cases} \tag{4.52}$$

Finally, since $D_0(\lambda, s) = D_0(\lambda, -s)$, $\overline{D_0(\lambda, s)} = D_0(\overline{\lambda}, -s)$, we can obtain (4.50) by using the fact that $\lambda_{\pm1}(s) = \pm\mathrm{i} + O(s^2)$ and $\lambda_0(s) = O(s^4)$ as $s \to 0$. ∎

With the help of Lemmas 4.7–4.8, we are able to construct the eigenvector $\Psi_j(s, \omega)$ corresponding to the eigenvalue λ_j at the low frequency. Indeed, we have the following theorem.

THEOREM 4.9. *There exists a constant $r_0 > 0$ so that the spectrum $\lambda \in \sigma(\hat{\mathbb{A}}_3(\xi)) \subset \mathbb{C}$ for $\xi = s\omega$ with $|s| \le r_0$ and $\omega \in \mathbb{S}^2$ consists of nine points $\{\lambda_j(s),\ -1 \le j \le 7\}$ in the domain $\mathrm{Re}\lambda > -\mu/2$. The spectrum $\lambda_j(s)$ and the corresponding eigenvector $\Psi_j(s, \omega) = (\psi_j, X_j, Y_j)(s, \omega)$ are C^∞ functions of s for $|s| \le r_0$. In particular, the eigenvalues admit the following asymptotic expansion for $|s| \le r_0$:*

$$\begin{cases} \lambda_{\pm1}(s) = \pm\mathrm{i} + (-a_1 \pm \mathrm{i}b_1)s^2 + o(s^2), \quad \overline{\lambda_1} = \lambda_{-1}, \\ \lambda_0(s) = -a_0 s^2 + o(s^2), \\ \lambda_2(s) = \lambda_3(s) = -\mathrm{i} + (-a_2 - \mathrm{i}b_2)s^2 + o(s^2), \quad \overline{\lambda_2} = \lambda_4, \\ \lambda_4(s) = \lambda_5(s) = \mathrm{i} + (-a_2 + \mathrm{i}b_2)s^2 + o(s^2), \\ \lambda_6(s) = \lambda_7(s) = -a_3 s^4 + o(s^4), \end{cases} \tag{4.53}$$

where $a_j>0$ $(0\le j\le 3)$ *and* $b_j>0$ $(1\le j\le 2)$ *are given by*

$$\begin{cases} a_0=-(L^{-1}\mathrm{P}_1(v_1\chi_4),v_1\chi_4), \quad a_3=-(L^{-1}\mathrm{P}_1(v_1\chi_2),v_1\chi_2),\\ a_1=-(L(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_1),(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_1)),\\ a_2=-(L(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2),(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2)),\\ b_1=\|(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_1)\|^2+\frac53, \quad b_2=\|(L+\mathrm{iP}_1)^{-1}\mathrm{P}_1(v_1\chi_2)\|^2+1. \end{cases} \tag{4.54}$$

The eigenvectors $\Psi_j=(\psi_j,X_j,Y_j)$ *are orthogonal to each other and satisfy*

$$\begin{cases} (\Psi_i(s,\omega),\Psi_j^*(s,\omega))_\xi=(\psi_i,\overline{\psi_j})_\xi-(X_i,\overline{X_j})-(Y_i,\overline{Y_j})=\delta_{ij},\\ (\psi_j,X_j,Y_j)(s,\omega)=\sum_{n=0}^2(\psi_{j,n},X_{j,n},Y_{j,n})(\omega)s^n+o(s^2), \end{cases} \tag{4.55}$$

where $\Psi_j^*=(\overline{\psi_j},-\overline{X_j},-\overline{Y_j})$, *and the coefficients* $(\psi_{j,n},X_{j,n},Y_{j,n})$ *are given by*

(4.56)

$$\begin{cases} \psi_{0,0}=\chi_4, \quad \psi_{0,1}=\mathrm{i}L^{-1}\mathrm{P}_1(v\cdot\omega)\chi_4, \quad (\psi_{0,2},\sqrt M)=-\sqrt{\frac23}, \quad X_0=Y_0\equiv 0;\\ \psi_{\pm1,0}=\frac{\sqrt2}{2}(v\cdot\omega)\sqrt M, \quad (\psi_{\pm1,2},\sqrt M)=0, \quad X_{\pm1}=Y_{\pm1}\equiv 0,\\ \psi_{\pm1,1}=\mp\frac{\sqrt2}{2}\sqrt M\mp\frac{\sqrt3}{3}\chi_4+\frac{\sqrt2}{2}\mathrm{i}(L\mp\mathrm{iP}_1)^{-1}\mathrm{P}_1(v\cdot\omega)^2\sqrt M;\\ \psi_{j,0}=\frac{\sqrt2}{2}(v\cdot W^j)\sqrt M, \quad (\psi_j,\sqrt M)=(\psi_j,\chi_4)\equiv 0,\\ \psi_{j,1}=\mathrm{i}\frac{\sqrt2}{2}L^{-1}\mathrm{P}_1[(v\cdot\omega)(v\cdot W^j)\sqrt M], \quad j=2,3,4,5,\\ X_{j,0}=-\mathrm{i}\frac{\sqrt2}{2}\omega\times W^j,\ Y_{j,0}=0,\ j=2,3, \quad X_{j,0}=\mathrm{i}\frac{\sqrt2}{2}\omega\times W^j,\ Y_{j,0}=0,\ j=4,5;\\ \psi_{j,0}=0, \quad (\psi_j,\sqrt M)=(\psi_j,\chi_4)\equiv 0,\\ \psi_{j,1}=(v\cdot W^j)\sqrt M, \quad X_{j,0}=X_{j,1}=X_{j,2}=0, \quad Y_{j,0}=\mathrm{i}W^j, \quad j=6,7. \end{cases}$$

Here, W^j $(j=2,3,4,5,6,7)$ *are normal vectors satisfying* $W^j\cdot\omega=0$, $W^1\cdot W^2=0$, $W^2=W^4=W^6, W^3=W^5=W^7$.

Proof. The eigenvalues $\lambda_j(s)$ and the eigenvectors $\Psi_j(s,\omega)=(\psi_j,X_j,Y_j)(s,\omega)$, $-1\le j\le 7$, can be constructed as follows. For $2\le j\le 7$, we take $\lambda_2=\lambda_3=\lambda_{-1}(s)$, $\lambda_4=\lambda_5=\lambda_1(s)$, $\lambda_6=\lambda_7=\lambda_0(s)$ to be the solution of the equation $D_0(\lambda,s)=0$ defined in Lemma 4.8, and choose $W_0=W_4=0$, and $W=W^j$ to be the linearly independent vector so that $W^j\cdot\omega=0$, $W^2\cdot W^3=W^4\cdot W^5=W^6\cdot W^7=0$, and $W^2=W^4=W^6, W^3=W^5=W^7$. And the corresponding eigenvectors $\Psi_j(s,\omega)=(\psi_j,X_j,Y_j)(s,\omega)$ are defined by

$$\begin{cases} \psi_j(s,\omega)=(v\cdot W^j)\sqrt M+\mathrm{i}s[L-\lambda_j\mathrm{P}_1-\mathrm{i}s\mathrm{P}_1(v\cdot\omega)\mathrm{P}_1]^{-1}\mathrm{P}_1[(v\cdot\omega)(v\cdot W^j)\sqrt M],\\ X_j(s,\omega)=(\lambda_j-s^2R_{22}(\lambda_j,s))(\omega\times W^j), \quad Y_j(s,\omega)=\dfrac{\mathrm{i}s}{\lambda_j}(\lambda_j-s^2R_{22}(\lambda_j,s))W^j, \end{cases}$$

which satisfy the orthonormal relation $(\Psi_2,\Psi_3^*)_\xi=(\Psi_4,\Psi_5^*)_\xi=(\Psi_6,\Psi_7^*)_\xi=0$.

For $j=-1,0,1$, we choose $\lambda_j=\lambda_j(s)$ to be a solution of $D(\lambda,s)=0$ given by Lemma 4.7, and choose $X=Y=0$, and denote by $\{a_j,b_j,d_j\}=:\{W_0^j,(W\cdot\omega)^j,W_4^j\}$ a solution of system (4.34), (4.36), and (4.38) for $\lambda=\lambda_j(s)$. Then we define $\Psi_j(s,\omega)=(\psi_j(s,\omega),0,0)$ $(j=-1,0,1)$, where

$$\begin{cases} \psi_j(s,\omega)=\mathrm{P}_0\psi_j(s,\omega)+\mathrm{P}_1\psi_j(s,\omega),\\ \mathrm{P}_0\psi_j(s,\omega)=a_j(s)\chi_0+b_j(s)(v\cdot\omega)\sqrt M+d_j(s)\chi_4,\\ \mathrm{P}_1\psi_j(s,\omega)=\mathrm{i}s[L-\lambda_j\mathrm{P}_1-\mathrm{i}s\mathrm{P}_1(v\cdot\omega)\mathrm{P}_1]^{-1}\mathrm{P}_1[(v\cdot\omega)\mathrm{P}_0\psi_j(s,\omega)]. \end{cases}$$

Then following the similar argument as Theorem 3.12, we can obtain the expansion of $\Psi_j(s,\omega)$ in (4.55) and (4.56). Hence, we omit the detail for brevity. ∎

4.3. Asymptotics in high frequency. The structure of the spectrum for the one-species VMB in the high frequency region is similar to the two-species VMB so that we only sketch the key points here. Recalling the eigenvalue problem

$$\begin{aligned}\lambda f &= B_2(\xi)f - v\sqrt{M}\cdot(\omega\times X),\\ \lambda X &= -\omega\times(f, v\sqrt{M}) + \mathrm{i}\xi\times Y,\\ \lambda Y &= -\mathrm{i}\xi\times X, \quad |\xi|\neq 0.\end{aligned}$$

Similar to the two-species case, we obtain

$$(\lambda^2 - ((B_2(|\xi|e_1)-\lambda)^{-1}\chi_2,\chi_2)\lambda + |\xi|^2)X = 0, \quad |\xi| > R_0. \tag{4.57}$$

Denote

$$D(\lambda,s) = \lambda^2 - ((B_2(se_1)-\lambda)^{-1}\chi_2,\chi_2)\lambda + s^2, \quad s > R_0. \tag{4.58}$$

As section 2.3, we can obtain the following theorem.

THEOREM 4.10. *There exists a constant $r_1>0$ so that the spectrum $\sigma(A_2(\xi))\subset\mathbb{C}$ for $\xi = s\omega$ with $s=|\xi|>r_1$ and $\omega\in\mathbb{S}^2$ consists of four points $\{\beta_j(s),\ j=1,2,3,4\}$ in the domain $\mathrm{Re}\beta > -\mu/2$. In particular, the eigenvalues satisfy*

$$\beta_1(s) = \beta_2(s) = -\mathrm{i}s + O(s^{-1/2}), \tag{4.59}$$

$$\beta_3(s) = \beta_4(s) = \mathrm{i}s + O(s^{-1/2}), \tag{4.60}$$

$$c_1\frac{1}{s} \le -\mathrm{Re}\beta_j(s) \le c_2\frac{1}{s}. \tag{4.61}$$

The eigenvectors $\Phi_j = (\phi_j, X_j, Y_j)$ are orthogonal to each other and satisfy

$$(\Phi_i(s,\omega),\Phi_j^*(s,\omega)) = (\psi_i,\overline{\psi_j}) - (X_i,\overline{X_j}) - (Y_i,\overline{Y_j}) = \delta_{ij}, \quad 1\le i,j\le 4, \tag{4.62}$$

where $\Phi_j^* = (\overline{\phi_j}, -\overline{X_j}, -\overline{Y_j})$, and

$$\begin{cases}\|\phi_j(s,\omega)\| = O(\frac{1}{\sqrt{s}}), & (\phi_j(s,\omega),\chi_0) = (\phi_j(s,\omega),\chi_4) = 0,\\ X_j(s,\omega) = O(1)\mathrm{i}(\omega\times W^j), & Y_j(s,\omega) = O(1)\mathrm{i}W^j.\end{cases} \tag{4.63}$$

Here, W^j $(j=1,2,3,4)$ are normal vectors satisfying $W^j\cdot\omega = 0,\ W^1\cdot W^2 = 0,$ $W^1 = W^3, W^2 = W^4$.

5. The linearized system. In this section, we consider the Cauchy problems (2.20) and (2.39) for the linearized VMB equations in two- and one-species and give the optimal time decay rates of the solution based on spectrum structures obtained in the previous sections.

5.1. Semigroup for two-species VMB system. Before giving the theorem on the semigroup, we first prepare some lemmas on its properties.

LEMMA 5.1 (see [14]). *The operator $Q(\xi) = L_1 - \mathrm{i}P_r(v\cdot\xi)P_r$ generates a strongly continuous contraction semigroup on $N_1^\perp$, which satisfies for any $t>0$ and $f\in N_1^\perp\cap L^2(\mathbb{R}^3_v)$ that*

$$\|e^{tQ(\xi)}f\| \le e^{-\mu t}\|f\|. \tag{5.1}$$

In addition, for any $x > -\mu$ and $f \in N_1^\perp \cap L^2(\mathbb{R}^3_v)$ it holds that

$$\int_{-\infty}^{+\infty} \|[(x+\mathrm{i}y)P_r - Q(\xi)]^{-1} f\|^2 dy \le \pi(x+\mu)^{-1}\|f\|^2. \tag{5.2}$$

LEMMA 5.2. *The operator $B_3(\xi)$ generates a strongly continuous unitary semigroup on $\mathbb{C}^6$, which satisfies for any $t > 0$ and $U \in \mathbb{C}^6$ that*

$$|e^{tB_3(\xi)}U| = |U|. \tag{5.3}$$

In addition, for any $x \neq 0$ and $U \in \mathbb{C}^6$ it holds that

$$\int_{-\infty}^{+\infty} |[(x+\mathrm{i}y) - B_3(\xi)]^{-1} U|^2 dy \le \pi|x|^{-1}|U|^2. \tag{5.4}$$

Proof. Since $\mathrm{i}B_3(\xi)$ is a self-adjoint operator on $\mathbb{C}^6$ satisfying (3.10), we can prove (5.3) and (5.4) by applying a similar argument as the one for Lemma 3.2 in [13]. ∎

LEMMA 5.3. *Let $r_0 > 0$ and $b_2 > 0$ be given in Lemma 3.12. Let $\alpha = \alpha(r_0, r_1) > 0$ with $r_1 > r_0$ and $\alpha(r_0, r_1)$ defined in Lemma 3.6. Then*

$$\sup_{0<|\xi|<r_0, y\in\mathbb{R}} \left\| \left(I - G_4(\xi)\left(-\frac{b_2}{2} + \mathrm{i}y - G_3(\xi)\right)^{-1}\right)^{-1} \right\|_\xi \le C, \tag{5.5}$$

$$\sup_{r_0<|\xi|<r_1, y\in\mathbb{R}} \left\| \left(I - G_2(\xi)\left(-\frac{\alpha}{2} + \mathrm{i}y - G_1(\xi)\right)^{-1}\right)^{-1} \right\| \le C. \tag{5.6}$$

Proof. Let $\lambda = x + \mathrm{i}y$. We prove (5.5) first. By (3.34) and (3.36), there exists $R > 0$ large enough such that if $\mathrm{Re}\lambda \ge -\mu/2$, $|\mathrm{Im}\lambda| \ge R$, and $|\xi| \le r_0$, then (3.44) holds. This yields

$$\|(I - G_4(\xi)(\lambda - G_3(\xi))^{-1})^{-1}\|_\xi \le 2.$$

Thus, it remains to prove (5.5) for $|y| \le R$. We will prove it by contradiction. Indeed, if (5.5) does not hold for $|y| \le R$, namely, there are subsequences $\{\xi_n\}$, $\{\lambda_n = b_2/2 + \mathrm{i}y_n\}$ with $|\xi_n| \le r_0$, $|y_n| \le R$, and $U_n = (f_n, E_n^1, B_n^1)$, $V_n = (g_n, E_n^2, B_n^2)$ with $\|U_n\|_{\xi_n} \to 0$ $(n \to \infty)$, $\|V_n\|_{\xi_n} = 1$ such that

$$(I - G_4(\xi)(\lambda - G_3(\xi))^{-1})^{-1} U_n = V_n.$$

Let

$$\begin{aligned}(a_n, b_n)^T = (\lambda_n - B_1(\xi_n))^{-1}(E_n^2, B_n^2)^T \Longleftrightarrow (E_n^2, B_n^2)^T \\ = \lambda_n(a_n, b_n)^T - (\mathrm{i}\xi_n \times b_n, -\mathrm{i}\xi_n \times a_n)^T.\end{aligned}$$

Then

$$P_{\mathrm{d}} f_n = P_{\mathrm{d}} g_n + \mathrm{i}P_{\mathrm{d}}(v\cdot\xi_n)P_r(\lambda_n P_r - Q(\xi_n))^{-1}P_r g_n, \tag{5.7}$$

$$P_r f_n = P_r g_n + \mathrm{i}\lambda_n^{-1} P_r(v\cdot\xi_n)\left(1 + \frac{1}{|\xi_n|^2}\right) P_{\mathrm{d}} g_n - v\sqrt{M}\cdot(\omega_n \times a_n), \tag{5.8}$$

$$E_n^1 = \lambda_n a_n - \mathrm{i}\xi_n \times b_n - \omega_n \times ((\lambda_n P_r - Q(\xi_n))^{-1} P_r g_n, v\sqrt{M}), \tag{5.9}$$

$$B_n^1 = \lambda_n b_n + \mathrm{i}\xi_n \times a_n. \tag{5.10}$$

Substituting (5.8) into (5.7) and (5.9), we obtain

$$
\begin{aligned}
P_{\rm d} f_n =& P_{\rm d} g_n + \mathrm{i} P_{\rm d}(v\cdot\xi_n)P_r(\lambda_n P_r - Q(\xi_n))^{-1}P_r f_n \\
&+ \lambda_n^{-1}P_{\rm d}(v\cdot\xi_n)P_r(\lambda_n P_r - Q(\xi_n))^{-1}P_r(v\cdot\xi_n)\left(1+\frac{1}{|\xi_n|^2}\right)P_{\rm d} g_n, \qquad (5.11)
\end{aligned}
$$

$$
\begin{aligned}
E_n^1 =& \lambda_n a_n - \mathrm{i}\xi_n\times b_n - \omega_n\times((\lambda_n P_r - Q(\xi_n))^{-1}P_r f_n, v\sqrt{M}) \\
&- \omega_n\times((\lambda_n P_r - Q(\xi_n))^{-1}v\sqrt{M}\cdot(\omega_n\times a_n), v\sqrt{M}). \qquad (5.12)
\end{aligned}
$$

Since $\|f_n\|_{\xi_n} + |E_n^1| + |B_n^1| \to 0$ as $n\to\infty$, it follows from (5.11), (5.12), and (5.10) that

$$
\begin{aligned}
&\lim_{n\to\infty}\sqrt{1+\frac{1}{|\xi_n|^2}}\frac{|C_n|}{|\lambda_n|}|\lambda_n + (|\xi_n|^2+1)((\lambda_n P_r - Q(|\xi_n|e_1))^{-1}\chi_1,\chi_1)| \to 0, \\
&\lim_{n\to\infty}|\lambda_n a_n - \mathrm{i}\xi_n\times b_n - ((\lambda_n P_r - Q(|\xi_n|e_1))^{-1}\chi_2,\chi_2)a_n| = 0, \\
&\lim_{n\to\infty}|\lambda_n b_n + \mathrm{i}\xi_n\times a_n| = 0,
\end{aligned}
$$

where $C_n = (g_n,\sqrt{M})$. Since $\sqrt{1+\frac{1}{|\xi_n|^2}}|C_n| \le 1$, $|\xi_n|\le r_0$, $|y_n|\le R$, and $|(a_n.b_n)| \le 2b_2^{-1}|(E_n^2,B_n^2)| \le 2b_2^{-1}$, there is a subsequence $\{(\xi_{n_j},\lambda_{n_j},C_{n_j})\}$ such that $\sqrt{1+\frac{1}{|\xi_{n_j}|^2}}C_{n_j} \to A_0, a_{n_j}\to a_0, b_{n_j}\to b_0, \xi_{n_j}\to\xi_0, \lambda_{n_j}\to\lambda_0 = b_2/2+\mathrm{i}y \ne 0$. Thus

$$
\frac{|A_0|}{|\lambda_0|}|\lambda_0 + (|\xi_0|^2+1)((\lambda_0 P_r - Q(|\xi_0|e_1))^{-1}\chi_1,\chi_1)| = 0, \qquad (5.13)
$$

$$
\lambda_0 a_0 - \mathrm{i}\xi_0\times b_0 + ((\lambda_0 P_r - Q(|\xi_0|e_1))^{-1}\chi_2,\chi_2)a_0 = 0, \qquad (5.14)
$$

$$
\lambda_0 b_0 + \mathrm{i}\xi_0\times a_0 = 0. \qquad (5.15)
$$

It is straightforward to verify that $(A_0,a_0,b_0)\ne 0$. Indeed, otherwise, we have $\lim_{j\to\infty}(\sqrt{1+\frac{1}{|\xi_{n_j}|^2}}C_{n_j},a_{n_j},b_{n_j}) = 0$. This and (5.8) lead to $\lim_{j\to\infty}\|P_r g_{n_j}\| = 0$ and $\lim_{j\to\infty}(E_{n_j}^2,B_{n_j}^2) = 0$. Thus $\lim_{j\to\infty}\|V_{n_j}\|_{\xi_{n_j}} = 0$, which contradicts $\|V_n\|_{\xi_n} = 1$. Therefore, (5.13), (5.14), and (5.15) imply that λ_0 is an eigenvalue of $\hat{B}(\xi_0)$ with $\mathrm{Re}\lambda_0 = -b_2/2$ and $|\xi_0|\le r_0$, which contradicts Theorem 3.12 since we can assume $\mathrm{Re}\lambda_j(s)\ne -b_2/2$ by taking a smaller r_0 if necessary.

By an argument similar to the one for Lemma 3.6, we can prove (5.6). And this completes the proof of the lemma. ∎

With the help of Lemmas 3.5–3.8 and Lemmas 5.1–5.3, we have a decomposition of the semigroup $S(t,\xi) = e^{t\hat{\mathbb{A}}_1(\xi)}$.

THEOREM 5.4. *The semigroup $S(t,\xi) = e^{t\hat{\mathbb{A}}_1(\xi)}$ with $s=|\xi|\ne 0$ and $\omega = \xi/|\xi|$ has the following decomposition:*

$$
S(t,\xi)U = S_1(t,\xi)U + S_2(t,\xi)U + S_3(t,\xi)U, \quad U\in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi, \quad t>0, \qquad (5.16)
$$

where

$$
S_1(t,\xi)U = \sum_{j=1}^{2} e^{t\lambda_j(s)}(U,\Psi_j^*(s,\omega))\Psi_j(s,\omega)1_{\{|\xi|\le r_0\}}, \qquad (5.17)
$$

$$
S_2(t,\xi)U = \sum_{j=1}^{4} e^{t\beta_j(s)}(U,\Phi_j^*(s,\omega))\Phi_j(s,\omega)1_{\{|\xi|\ge r_1\}}, \qquad (5.18)
$$

with $(\lambda_j(s), \Psi_j(s,\omega))$ *and* $(\beta_j(s), \Phi_j(s,\omega))$ *being the eigenvalue and eigenvector of the operator* $\hat{\mathbb{A}}_1(\xi)$ *given by Theorems* 3.12 *and* 3.15 *for* $|\xi| \le r_0$ *and* $|\xi| > r_1$*, respectively, and* $S_3(t,\xi) =: S(t,\xi) - S_1(t,\xi) - S_2(t,\xi)$ *satisfies that there exists a constant* $\kappa_0 > 0$ *independent of* ξ *such that*

$$\|S_3(t,\xi)U\|_\xi \le Ce^{-\kappa_0 t}\|U\|_\xi, \quad t \ge 0. \tag{5.19}$$

Proof. Since $D(\hat{B}_1(\xi)^2)$ is dense in $L^2_\xi(\mathbb{R}^3_v)$, by Theorem 2.7 in [18], it is sufficient to prove the above decomposition for $U = (f, E, B) \in D(\hat{B}_1(\xi)^2) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi$. By Corollary 7.5 in [18], the semigroup $e^{t\hat{\mathbb{A}}_1(\xi)}$ can be represented by

$$e^{t\hat{\mathbb{A}}_1(\xi)}U = \frac{1}{2\pi \mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} e^{\lambda t}(\lambda - \hat{\mathbb{A}}_1(\xi))^{-1}Ud\lambda, \quad U \in D(\hat{B}_1(\xi)^2) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi,\ \kappa > 0. \tag{5.20}$$

It remains to analyze the resolvent $(\lambda - \hat{\mathbb{A}}_1(\xi))^{-1}$ for $\xi \in \mathbb{R}^3$ in order to obtain the decomposition (5.16) for the semigroup $e^{t\hat{\mathbb{A}}_1(\xi)}$.

By (3.45), we rewrite $(\lambda - \hat{\mathbb{A}}_1(\xi))^{-1}$ for $|\xi| \le r_0$ as

$$(\lambda - \hat{\mathbb{A}}_1(\xi))^{-1} = (\lambda - G_3(\xi))^{-1} + Z_1(\lambda,\xi), \tag{5.21}$$

with

$$Z_1(\lambda,\xi) = (\lambda - G_3(\xi))^{-1}[I - Y_1(\lambda,\xi)]^{-1}Y_1(\lambda,\xi),$$
$$Y_1(\lambda,\xi) = G_4(\xi)(\lambda - G_3(\xi))^{-1}.$$

Substituting (5.21) into (5.20), we have the following decomposition of the semigroup $e^{t\hat{\mathbb{A}}_1(\xi)}$:

$$e^{t\hat{\mathbb{A}}_1(\xi)}U = (e^{tQ(\xi)}P_r f, 0, 0) + \frac{1}{2\pi \mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} e^{\lambda t}Z_2(\lambda,\xi)Ud\lambda, \quad |\xi| \le r_0, \tag{5.22}$$

where

$$Z_2(\lambda,\xi) = Z_1(\lambda,\xi) + Y_2(\lambda,\xi) \quad \text{with} \quad Y_2(\lambda,\xi) = \begin{pmatrix} \lambda^{-1}P_{\mathrm{d}} & 0 \\ 0 & (\lambda - B_3(\xi))^{-1} \end{pmatrix}.$$

To estimate the last term on the right-hand side of (5.22), let us denote

$$X_{\kappa,N} = \frac{1}{2\pi \mathrm{i}}\int_{-N}^{N} e^{(\kappa+\mathrm{i}y)t}Z_2(\kappa + iy,\xi)U1_{|\xi|\le r_0}dy, \tag{5.23}$$

where the constant $N > 0$ is chosen large enough so that $N > y_1$ with y_1 defined in Lemma 3.8. Since $Z_2(\lambda,\xi)$ is analytic in the domain $\mathrm{Re}\lambda > -b_2/2$ with only finite singularities at $\lambda = \lambda_j(s) \in \sigma(\hat{\mathbb{A}}_1(\xi))$ for $j = 1, 2$, we can shift the integration (5.23) from the line $\mathrm{Re}\lambda = \kappa > 0$ to $\mathrm{Re}\ \lambda = -b_2/2$. Then

$$X_{\kappa,N} = X_{-\frac{b_2}{2},N} + H_N + 2\pi \mathrm{i}\sum_{j=1}^{2}\mathrm{Res}\left\{e^{\lambda t}Z_2(\lambda,\xi)U; \lambda_j(s)\right\}1_{|\xi|\le r_0}, \tag{5.24}$$

where $\mathrm{Res}\{f(\lambda); \lambda_j\}$ is the residue of f at $\lambda = \lambda_j$ and

$$H_N = \frac{1}{2\pi \mathrm{i}}\left(\int_{-\frac{b_2}{2}+\mathrm{i}N}^{\kappa+\mathrm{i}N} - \int_{-\frac{b_2}{2}-\mathrm{i}N}^{\kappa-\mathrm{i}N}\right)e^{\lambda t}Z_2(\lambda,\xi)U1_{|\xi|\le r_0}d\lambda.$$

The right-hand side of (5.24) is estimated as follows. By Lemma 3.7, we have

$$\|H_N\|_\xi \to 0 \quad \text{as} \quad N \to \infty. \tag{5.25}$$

By the Cauchy theorem, we obtain

$$\lim_{N\to\infty}\left|\int_{-\frac{b_2}{2}-\mathrm{i}N}^{-\frac{b_2}{2}+\mathrm{i}N} e^{\lambda t}\lambda^{-1}d\lambda\right| = \lim_{N\to\infty}\left\|\int_{-\frac{b_2}{2}-\mathrm{i}N}^{-\frac{b_2}{2}+\mathrm{i}N} e^{\lambda t}(\lambda - B_3(\xi))^{-1}d\lambda\right\| = 0, \tag{5.26}$$

which leads to

$$\lim_{N\to\infty} X_{-\frac{b_2}{2},N}(t) =: X_{-\frac{b_2}{2},\infty}(t) = \int_{-\frac{b_2}{2}-\mathrm{i}\infty}^{-\frac{b_2}{2}+\mathrm{i}\infty} e^{\lambda t} Z_1(\lambda,\xi)U d\lambda. \tag{5.27}$$

By (5.5), (5.2), and (5.4), we have for any $U, V \in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$,

$$\begin{aligned}|(X_{-\frac{b_2}{2},\infty}(t)U,V)_\xi| &\le Ce^{-\frac{b_2 t}{2}}\int_{-\infty}^{+\infty}\|[\lambda - G_3(\xi)]^{-1}U\|_\xi\|[\overline{\lambda} - G_3(-\xi)]^{-1}V\|_\xi dy\\ &\le Ce^{-\frac{b_2 t}{2}}(\|f\|^2_\xi + |(E,B)|^2), \quad \lambda = -\frac{b_2}{2} + \mathrm{i}y,\end{aligned}$$

which yields $|(X_{-\frac{b_2}{2},\infty}(t)U,V)_\xi| \le Ce^{-\frac{b_2 t}{2}}\|U\|_\xi\|V\|_\xi$, and

$$\|X_{-\frac{b_2}{2},\infty}(t)\|_\xi \le Ce^{-\frac{b_2 t}{2}}. \tag{5.28}$$

Since $\lambda_j(s)\in\rho(Q(\xi))$, by a similar argument as Theorem 3.4 in [13], we can prove

$$\begin{aligned}\mathrm{Res}\{e^{\lambda t}Z_2(\lambda,\xi)U;\lambda_j(s)\} &= \mathrm{Res}\{e^{\lambda t}(\lambda - \hat{\mathbb{A}}_1(\xi))^{-1}U;\lambda_j(s)\}\\ &= e^{\lambda_j(s)t}(U,\Psi^*_j(s,\omega))\Psi_j(s,\omega).\end{aligned} \tag{5.29}$$

Therefore, we conclude from (5.22)–(5.29) that

$$\begin{aligned}e^{t\hat{\mathbb{A}}_1(\xi)}U &= (e^{tQ(\xi)}P_r f,0,0) + X_{-\frac{b_2}{2},\infty}(t)\\ &\quad + \sum_{j=1}^{2} e^{t\lambda_j(s)}(U,\Psi^*_j(s,\omega))\Psi_j(s,\omega)1_{\{|\xi|\le r_0\}}, \ \ |\xi|\le r_0.\end{aligned} \tag{5.30}$$

By (3.30), we have for $|\xi| > r_0$ that

$$(\lambda - \hat{\mathbb{A}}_1(\xi))^{-1} = (\lambda - G_1(\xi))^{-1} + Z_3(\lambda,\xi), \tag{5.31}$$

with the operator $Z_3(\lambda,\xi)$ defined by

$$\begin{aligned}&Z_3(\lambda,\xi) = (\lambda - G_1(\xi))^{-1}[I - Y_3(\lambda,\xi)]^{-1}Y_3(\lambda,\xi),\\ &Y_3(\lambda,\xi) =: G_2(\xi)(\lambda - G_1(\xi))^{-1}.\end{aligned}$$

Substituting (5.31) into (5.20) yields

$$e^{t\hat{\mathbb{A}}_1(\xi)}U = (e^{tc(\xi)}f,0,0) + \frac{1}{2\pi\mathrm{i}}\int_{\kappa-\mathrm{i}\infty}^{\kappa+\mathrm{i}\infty} e^{\lambda t}Z_4(\lambda,\xi)U d\lambda, \quad |\xi| > r_1. \tag{5.32}$$

Here,

$$Z_4(\lambda,\xi)=Z_3(\lambda,\xi)-Y_4(\lambda,\xi) \quad \text{with} \quad Y_4(\lambda,\xi)=\begin{pmatrix} 0 & 0 \\ 0 & (\lambda-B_3(\xi))^{-1} \end{pmatrix}.$$

Similarly, in order to estimate the last term on the right-hand side of (5.32), let us denote

$$Y_{\kappa,N}=\frac{1}{2\pi \mathrm{i}}\int_{-N}^{N} e^{(\kappa+\mathrm{i}y)t} Z_4(\kappa+\mathrm{i}y,\xi)U1_{\{|\xi|>r_0\}}dy, \tag{5.33}$$

where the constant $N>y_1$ for $|\xi|\le r_1$ and $N>2|\xi|$ for $|\xi|\ge r_1$ with y_1,r_1 defined in Lemma 3.8. Since the operator $Z_4(\lambda,\xi)$ is analytic in the domain $\mathrm{Re}\lambda\ge -\kappa_0=: -\alpha(r_0,r_1)/2>0$ and $r_0<|\xi|<r_1$ with the constant $\alpha(r_0,r_1)>0$ defined in Lemma 3.6, and is analytic except only finite singularities at $\lambda=\lambda_j(s)\in\sigma(\hat{\mathbb{A}}_1(\xi))$ for $j=1,2,3,4$, in the domain $\mathrm{Re}\lambda\ge-\mu/2$ and $|\xi|\ge r_1$, we can shift the integration of (5.33) for $r_0<|\xi|<r_1$ from the line $\mathrm{Re}\lambda=\kappa>0$ to $\mathrm{Re}\lambda=-\kappa_0$, and shift the integration of (5.33) for $|\xi|\ge r_1$ from the line $\mathrm{Re}\lambda=\kappa>0$ to $\mathrm{Re}\lambda=-\mu/2$ to obtain

$$Y_{\kappa,N}1_{\{r_0<|\xi|<r_1\}}=Y_{-\kappa_0,N}1_{\{r_0<|\xi|<r_1\}}+I_N1_{\{r_0<|\xi|<r_1\}}, \tag{5.34}$$

$$\begin{aligned} Y_{\kappa,N}1_{\{|\xi|\ge r_1\}}&=Y_{-\frac{\mu}{2},N}1_{\{|\xi|\ge r_1\}}+I_N1_{\{|\xi|\ge r_1\}}\\ &\quad+2\pi\mathrm{i}\sum_{j=1}^{4}\mathrm{Res}\left\{e^{\lambda t}Z_4(\lambda,\xi)U;\beta_j(s)\right\}1_{\{|\xi|\ge r_1\}}, \end{aligned} \tag{5.35}$$

with

$$I_N=\frac{1}{2\pi\mathrm{i}}\left(\int_{-\kappa_0+\mathrm{i}N}^{\kappa+\mathrm{i}N}-\int_{-\kappa_0-\mathrm{i}N}^{\kappa-\mathrm{i}N}\right)e^{\lambda t}Z_4(\lambda,\xi)U1_{\{|\xi|\ge r_0\}}d\lambda.$$

By Lemmas 3.5 and 5.3, it is straightforward to verify

$$\|I_N\|\to 0 \quad \text{as} \quad N\to\infty, \tag{5.36}$$

$$\sup_{r_0<|\xi|<r_1,y\in\mathbb{R}}\|[I-Y_3(-\kappa_0+\mathrm{i}y,\xi)]^{-1}\|\le C, \quad \sup_{|\xi|>r_1,y\in\mathbb{R}}\left\|\left[I-Y_3\Big(-\frac{\mu}{2}+\mathrm{i}y,\xi\Big)\right]^{-1}\right\|\le 2. \tag{5.37}$$

Then by (5.33) and (5.26), we have for any $U,V\in L^2(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$,

$$\begin{aligned} |(Y_{-\kappa_0,\infty}(t)U1_{\{r_0<|\xi|<r_1\}},V)|&\le Ce^{-\kappa_0 t}\int_{-\infty}^{+\infty}\|(\lambda-G_1(\xi))^{-1}U\|\|(\overline{\lambda}-G_1(-\xi))^{-1}V\|dy\\ &\le C(\nu_0-\kappa_0)^{-1}e^{-\frac{\mu}{2}t}\|U\|\|V\|, \quad \lambda=-\kappa_0+\mathrm{i}y, \end{aligned} \tag{5.38}$$

$$\begin{aligned} |(Y_{-\frac{\mu}{2},\infty}(t)U1_{\{|\xi|\ge r_1\}},V)|&\le Ce^{-\frac{\mu}{2}t}\int_{-\infty}^{+\infty}\|(\lambda-G_1(\xi))^{-1}U\|\|(\overline{\lambda}-G_1(-\xi))^{-1}V\|dy\\ &\le C\Big(\nu_0-\frac{\mu}{2}\Big)^{-1}e^{-\frac{\mu}{2}t}\|U\|\|V\|, \quad \lambda=-\frac{\mu}{2}+\mathrm{i}y, \end{aligned} \tag{5.39}$$

where we have used (5.4) and the fact (cf. Lemma 2.2.13 of [22]) that

$$\int_{-\infty}^{+\infty}\|(x+\mathrm{i}y-c(\xi))^{-1}f\|^2dy\le\pi(x+\nu_0)^{-1}\|f\|^2, \quad x>-\nu_0.$$

From (5.38), (5.39), and the fact that $\|f\|^2 \le \|f\|_\xi^2 \le (1+r_0^{-2})\|f\|^2$ for $|\xi|>r_0$, we have

$$\|Y_{-\kappa_0,\infty}(t)1_{\{r_0<|\xi|<r_1\}}\|_\xi \le Ce^{-\kappa_0 t}, \quad \|Y_{-\frac{\mu}{2},\infty}(t)1_{\{|\xi|\ge r_1\}}\|_\xi \le Ce^{-\frac{\mu}{2}t}. \tag{5.40}$$

By $\lambda_j(s)\in\rho(c(\xi))$, we can prove

$$\begin{aligned}\operatorname{Res}\{e^{\lambda t}Z_4(\lambda,\xi)U;\beta_j(s)\} &= \operatorname{Res}\{e^{\lambda t}(\lambda-\hat{\mathbb{A}}_1(\xi))^{-1}U;\beta_j(s)\}\\ &= e^{\beta_j(s)t}(U,\Phi_j^*(s,\omega))\Phi_j(s,\omega).\end{aligned} \tag{5.41}$$

Therefore, we conclude from (5.32)–(5.41) that

$$\begin{aligned} e^{t\hat{\mathbb{A}}_1(\xi)}U &= (e^{tc(\xi)}f,0,0) + Y_{-\kappa_0,\infty}(t)1_{\{r_0<|\xi|<r_1\}} + Y_{-\frac{\mu}{2},\infty}(t)1_{\{|\xi|\ge r_1\}}\\ &\quad + \sum_{j=1}^4 e^{t\beta_j(s)}(U,\Phi_j^*(s,\omega))\Phi_j(s,\omega)1_{\{|\xi|>r_1\}}, \quad |\xi|>r_0. \end{aligned} \tag{5.42}$$

Combining (5.30) and (5.42) gives (5.16), where $S_1(t,\xi)f$, $S_2(t,\xi)f$ are given by (5.17) and (5.18), and $S_3(t,\xi)f$ is the remainder term defined by

$$\begin{aligned} S_3(t,\xi)U &= (e^{tQ(\xi)}P_r f,0,0)1_{\{|\xi|\le r_0\}} + X_{-\frac{b_2}{2},\infty}(t)1_{\{|\xi|\le r_0\}}\\ &\quad + (e^{tc(\xi)}f,0,0)1_{\{|\xi|\ge r_1\}} + Y_{-\kappa_0,\infty}(t)1_{\{r_0<|\xi|<r_1\}} + Y_{-\frac{\mu}{2},\infty}(t)1_{\{|\xi|\ge r_1\}}. \end{aligned}$$

In particular, $S_3(t,\xi)U$ satisfies (5.19) because of (5.1), (5.28), (5.40), and the estimate $\|e^{tc(\xi)}1_{\{|\xi|>r_0\}}\|_\xi \le Ce^{-\nu_0 t}$ coming from (3.12) and (2.13). This completes the proof of the theorem. ∎

5.2. Optimal convergence rates for two-species VMB system. Based on the decomposition of the semigroup given in the previous subsection, we now study the optimal convergence rates of the solution of the linearized system to the equilibrium.

Set $U=(f,E,B)$ with $f=f(x,v)$, $E=E(x)$, and $B=B(x)$ and denote $D^l=\{U\in L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)\times L^2(\mathbb{R}^3_x)\times L^2(\mathbb{R}^3_x)\,|\,\|U\|_{D^l}<\infty\}$ $(Z^2=D^0)$ with the norm $\|\cdot\|_{D^l}$ defined by

$$\begin{aligned}\|U\|_{D^l} &= \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^l\|\hat{U}\|^2 d\xi\right)^{1/2}\\ &= \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^l\left(\int_{\mathbb{R}^3}|\hat{f}|^2dv+|\hat{E}|^2+|\hat{B}|^2\right)d\xi\right)^{1/2},\end{aligned}$$

where $\hat{f}=\hat{f}(\xi,v)$, $\hat{E}=\hat{E}(\xi)$, and $\hat{B}=\hat{B}(\xi)$.

For any $U_0=(f_0,E_0,B_0)\in L^2(\mathbb{R}^3_v;H^l(\mathbb{R}^3_x))\times H^l(\mathbb{R}^3_x)\times H^l(\mathbb{R}^3_x)$, set

$$e^{t\hat{\mathbb{A}}_0(\xi)}\hat{U}_0 = \left((e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0)_1,\ -\frac{\mathrm{i}\xi}{|\xi|^2}((e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0)_1,\sqrt{M}) - \frac{\xi}{|\xi|}\times(e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0)_2,\ -\frac{\xi}{|\xi|}\times(e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0)_3\right) \tag{5.43}$$

with

$$\begin{aligned} e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0 &= ((e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0)_1,(e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0)_2,(e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0)_3)\in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi,\\ \hat{V}_0 &= \left(\hat{f}_0,\frac{\xi}{|\xi|}\times\hat{E}_0,\frac{\xi}{|\xi|}\times\hat{B}_0\right). \end{aligned}$$

Then $e^{t\mathbb{A}_0}U_0$ is the solution of the system (2.20). By Lemma 3.1, it holds that

$$\|e^{t\mathbb{A}_0}U_0\|_{D^l} = \int_{\mathbb{R}^3}(1+|\xi|^2)^l\|e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0\|_\xi^2 d\xi \le \int_{\mathbb{R}^3}(1+|\xi|^2)^l\|\hat{V}_0\|_\xi^2 d\xi = \|U_0\|_{D^l}.$$

This means that the linear operator $\mathbb{A}_0$ generates a strongly continuous contraction semigroup $e^{t\mathbb{A}_0}$ on D^l, and therefore, $U(t) = e^{t\mathbb{A}_0}U_0$ is a global solution to (2.20) for the linearized VMB equations with initial data $U_0 \in D^l$.

First of all, we have the upper bounds of the time decay rates given in the next theorem.

THEOREM 5.5. *Let* $(f_2(t), E(t), B(t))$ *be a solution of the system* (2.20). *If the initial data* $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3_v; H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)) \times H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)\times H^l(\mathbb{R}^3_x)\cap L^1(\mathbb{R}^3_x)$, *then it holds for any* $\alpha, \alpha' \in \mathbb{N}^3$ *with* $\alpha' \le \alpha$ *that*

$$\|\partial_x^\alpha P_{\rm d} f(t)\|_{L^2_{x,v}} \le Ce^{-\kappa_0 t}\|\partial_x^\alpha U_0\|_{Z^2}, \tag{5.44}$$

$$\begin{aligned}\|\partial_x^\alpha P_r f(t)\|_{L^2_{x,v}} &\le C(1+t)^{-(\frac54+\frac k2)}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1})\\ &\quad + C(1+t)^{-(m+\frac12)}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},\end{aligned} \tag{5.45}$$

$$\begin{aligned}\|\partial_x^\alpha E(t)\|_{L^2_x} &\le C(1+t)^{-(\frac54+\frac k2)}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1})\\ &\quad + C(1+t)^{-m}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},\end{aligned} \tag{5.46}$$

$$\begin{aligned}\|\partial_x^\alpha B(t)\|_{L^2_x} &\le C(1+t)^{-(\frac34+\frac k2)}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1})\\ &\quad + C(1+t)^{-m}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},\end{aligned} \tag{5.47}$$

where $k = |\alpha-\alpha'|$ *and* $m \ge 0$.

Proof. By (5.43) and Theorem 5.4, we have for $\omega = \xi/|\xi|$ that

$$\begin{aligned}(\hat{f}_2(t), \omega\times\hat{E}(t), \omega\times\hat{B}(t)) &= e^{t\hat{\mathbb{A}}_1(\xi)}\hat{V}_0 = S_1(t,\xi)\hat{V}_0 + S_2(t,\xi)\hat{V}_0 + S_3(t,\xi)\hat{V}_0\\ &= \sum_{k=1}^3 (h_k(t), H_k(t), J_k(t)),\end{aligned} \tag{5.48}$$

where $S_k(t,\xi)\hat{V}_0 = (h_k(t), H_k(t), J_k(t)) \in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi,\ k=1,2,3,$ and $\hat{V}_0 = (\hat{f}_0, \omega\times\hat{E}_0, \omega\times\hat{B}_0)$. Then

$$\begin{aligned}\|\partial_x^\alpha P_r f_2(t)\|_{L^2_{x,v}} &= \|\xi^\alpha P_r\hat{f}_2(t)\|_{L^2_{\xi,v}}\\ &\le \|\xi^\alpha P_r h_1(t)\|_{L^2_{\xi,v}} + \|\xi^\alpha P_r h_2(t)\|_{L^2_{\xi,v}} + \|\xi^\alpha P_r h_3(t)\|_{L^2_{\xi,v}},\end{aligned} \tag{5.49}$$

$$\begin{aligned}\|\partial_x^\alpha E(t)\|_{L^2_x} &= \|\xi^\alpha\hat{E}(t)\|_{L^2_\xi} = (\|\xi^\alpha|\xi|^{-1}(\hat{f}(t),\chi_0)\|^2_{L^2_\xi} + \|\xi^\alpha(\omega\times\hat{E})(t)\|^2_{L^2_\xi})^{1/2}\\ &\le \|\frac{\xi^\alpha}{|\xi|}(h_1(t),\chi_0)\|_{L^2_\xi} + \left\|\frac{\xi^\alpha}{|\xi|}(h_2(t),\chi_0)\right\|_{L^2_\xi} + \left\|\frac{\xi^\alpha}{|\xi|}(h_3(t),\chi_0)\right\|_{L^2_\xi}\\ &\quad + \|\xi^\alpha H_1(t)\|_{L^2_\xi} + \|\xi^\alpha H_2(t)\|_{L^2_\xi} + \|\xi^\alpha H_3(t)\|_{L^2_\xi},\end{aligned} \tag{5.50}$$

$$\begin{aligned}\|\partial_x^\alpha B(t)\|_{L^2_x} &= \|\xi^\alpha\hat{B}(t)\|_{L^2_\xi} = \|\xi^\alpha(\omega\times\hat{B})(t)\|_{L^2_\xi}\\ &\le \|\xi^\alpha J_1(t)\|_{L^2_\xi} + \|\xi^\alpha J_2(t)\|_{L^2_\xi} + \|\xi^\alpha J_3(t)\|_{L^2_\xi}.\end{aligned} \tag{5.51}$$

By (5.19), we can estimate the terms on the right-hand side of (5.49)–(5.51) as follows:

$$\int_{\mathbb{R}^3}(\xi^\alpha)^2(\|h_3(t)\|^2_{L^2_v} + \frac{1}{|\xi|^2}|(h_3(t),\sqrt{M})|^2 + |H_3(t)|^2 + |J_3(t)|^2)d\xi$$

$$\leq C\int_{\mathbb{R}^3} e^{-2\kappa_0 t}(\xi^\alpha)^2\left(\|\hat{f}_0\|^2_{L^2_v}+\frac{1}{|\xi|^2}|(\hat{f}_0,\sqrt{M})|^2+|\omega\times\hat{E}_0|^2+|\omega\times\hat{B}_0|^2\right)d\xi$$

$$\leq Ce^{-2\kappa_0 t}(\|\partial^\alpha_x f_0\|^2_{L^2_{x,v}}+\|(f_0,\sqrt{M})\|^2_{L^1_x}+\|\partial^\alpha_x E_0\|^2_{L^2_x}+\|\partial^\alpha_x B_0\|^2_{L^2_x}). \tag{5.52}$$

In the low frequency region, by (5.17), we have

$$S_1(t,\xi)\hat{V}_0=\sum_{j=1}^2 e^{t\lambda_j(|\xi|)}\big\{[(\hat{f}_0,\overline{\psi_{j,0}})-(\omega\times\hat{E}_0,\overline{X_{j,0}})-(\omega\times\hat{B}_0,\overline{Y_{j,0}})](\psi_{j,0},X_{j,0},Y_{j,0})$$
$$+|\xi|((T_j(\xi)\hat{V}_0)_1,(T_j(\xi)\hat{V}_0)_2,(T_j(\xi)\hat{V}_0)_3)\big\}1_{\{|\xi|\leq r_0\}},$$

where $T_j(\xi),\ j=1,2$, are the linear operators with the norm $\|T_j(\xi)\|$ being uniformly bounded for $|\xi|\leq r_0$.

By (3.66) and (3.67), we have

$$(h_1(t),\sqrt{M})=0,\ \ P_r h_1(t)=|\xi|\sum_{j=1}^2 e^{\lambda_j(|\xi|)t}(T_j(\xi)\hat{V}_0)_1, \tag{5.53}$$

$$H_1(t)=|\xi|\sum_{j=1}^2 e^{\lambda_j(|\xi|)t}(T_j(\xi)\hat{V}_0)_2, \tag{5.54}$$

$$J_1(t)=\sum_{j=1}^2 e^{\lambda_j(|\xi|)t}(\omega\times\hat{B}_0,W^j)W^j+|\xi|\sum_{j=1}^2 e^{\lambda_j(|\xi|)t}(T_j(\xi)\hat{V}_0)_3, \tag{5.55}$$

where $W^j,\ j=1,2$, is given by (3.67). Since

$$\mathrm{Re}\lambda_j(|\xi|)=a_j|\xi|^2(1+O(|\xi|))\leq-\eta_1|\xi|^2,\quad |\xi|\leq r_0, \tag{5.56}$$

where $\eta_1>0$ denotes a generic constant that will also be used later, we obtain by (5.53)–(5.55) that

$$\|\xi^\alpha P_r h_1(t)\|^2_{L^2_{\xi,v}}\leq C(1+t)^{-(5/2+k)}(\|\partial^{\alpha'}_x f_0\|^2_{L^{2,1}}+\|\partial^{\alpha'}_x E_0\|^2_{L^1_x}+\|\partial^{\alpha'}_x B_0\|^2_{L^1_x}), \tag{5.57}$$

$$\|\xi^\alpha H_1(t)\|^2_{L^2_\xi}\leq C(1+t)^{-(5/2+k)}(\|\partial^{\alpha'}_x f_0\|^2_{L^{2,1}}+\|\partial^{\alpha'}_x E_0\|^2_{L^1_x}+\|\partial^{\alpha'}_x B_0\|^2_{L^1_x}), \tag{5.58}$$

$$\|\xi^\alpha J_1(t)\|^2_{L^2_\xi}\leq C(1+t)^{-(3/2+k)}(\|\partial^{\alpha'}_x f_0\|^2_{L^{2,1}}+\|\partial^{\alpha'}_x E_0\|^2_{L^1_x}+\|\partial^{\alpha'}_x B_0\|^2_{L^1_x}), \tag{5.59}$$

with $k=|\alpha-\alpha'|$.

In the high frequency region, by (5.18), we have

$$S_2(t,\xi)\hat{V}_0=\sum_{j=1}^4 e^{t\beta_j(|\xi|)}[(\hat{f}_0,\overline{\phi_j})$$
$$-(\omega\times\hat{E}_0,\overline{X_j})-(\omega\times\hat{B}_0,\overline{Y_j})](\phi_j,X_j,Y_j)(s,\omega)1_{\{|\xi|\geq r_1\}}, \tag{5.60}$$

and in particular $(h_2(t),\sqrt{M})=0$. Since

$$\mathrm{Re}\beta_j(|\xi|)\leq-c_1|\xi|^{-1},\quad |\xi|\geq r_1, \tag{5.61}$$

we obtain by (5.60) and (3.94) that

$$\|\xi^\alpha P_r h_2(t)\|^2_{L^2_{\xi,v}}\leq C\sup_{|\xi|\geq r_1}\frac{1}{|\xi|^{2m+1}}e^{-\frac{2c_1 t}{|\xi|}}\int_{|\xi|\geq r_1}(\xi^\alpha)^2|\xi|^{2m}\|\hat{U}_0\|^2 d\xi, \tag{5.62}$$

$$\|\xi^\alpha H_2(t)\|_{L^2_\xi}^2 \le C \sup_{|\xi|\ge r_1} \frac{1}{|\xi|^{2m}} e^{-\frac{2c_1 t}{|\xi|}} \int_{|\xi|\ge r_1} (\xi^\alpha)^2 |\xi|^{2m} \|\hat{U}_0\|^2 d\xi, \tag{5.63}$$

$$\|\xi^\alpha J_2(t)\|_{L^2_\xi}^2 \le C \sup_{|\xi|\ge r_1} \frac{1}{|\xi|^{2m}} e^{-\frac{2c_1 t}{|\xi|}} \int_{|\xi|\ge r_1} (\xi^\alpha)^2 |\xi|^{2m} \|\hat{U}_0\|^2 d\xi. \tag{5.64}$$

Since

$$\sup_{|\xi|\ge r_1} \frac{1}{|\xi|^{2m}} e^{-\frac{2c_1 t}{|\xi|}} \le C(1+t)^{-2m},$$

it follows from (5.62)–(5.64) that

$$\|\xi^\alpha P_r h_2(t)\|_{L^2_{\xi,v}}^2 \le C(1+t)^{-(2m+1)} \|\nabla_x^m \partial_x^\alpha U_0\|_{Z^2}^2, \tag{5.65}$$

$$\|\xi^\alpha H_2(t)\|_{L^2_\xi}^2 \le C(1+t)^{-2m} \|\nabla_x^m \partial_x^\alpha U_0\|_{Z^2}^2, \tag{5.66}$$

$$\|\xi^\alpha J_2(t)\|_{L^2_\xi}^2 \le C(1+t)^{-2m} \|\nabla_x^m \partial_x^\alpha U_0\|_{Z^2}^2. \tag{5.67}$$

The combination of (5.49)–(5.52), (5.57)–(5.59), and (5.65)–(5.67) leads to (5.44)–(5.47). And this completes the proof of the theorem. ∎

In fact, the above time decay rates are optimal as shown in the next theorem.

THEOREM 5.6. *Let $(f_2(t), E(t), B(t))$ be a solution of the system* (2.20). *Assume that the initial data $U_0 = (f_0, E_0, B_0) \in L^2(\mathbb{R}^3_v; H^l(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)) \times H^l(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x) \times H^l(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)$ for $l \ge 2$ and $\hat{U}_0 = (\hat{f}_0, \hat{E}_0, \hat{B}_0)$ satisfies that $\inf_{|\xi|\le r_0} |\frac{\xi}{|\xi|} \times \hat{B}_0| \ge d_0 > 0$; then*

$$C_1(1+t)^{-\frac{5}{4}} \le \|P_r f_2(t)\|_{L^2_{x,v}} \le C_2(1+t)^{-\frac{5}{4}}, \tag{5.68}$$

$$C_1(1+t)^{-\frac{5}{4}} \le \|E(t)\|_{L^2_x} \le C_2(1+t)^{-\frac{5}{4}}, \tag{5.69}$$

$$C_1(1+t)^{-\frac{3}{4}} \le \|B(t)\|_{L^2_x} \le C_2(1+t)^{-\frac{3}{4}} \tag{5.70}$$

for $t > 0$ large enough with $C_2 \ge C_1 > 0$ as two generic constants.

Proof. By Theorem 5.5, we need only show the lower bounds of the time decay rates for the solution $(f(t), E(t), B(t))$ under the assumptions of Theorem 5.6. Indeed, in terms of Theorem 5.5, we have

$$\begin{aligned} \|P_r f_2(t)\|_{L^2_{x,v}} &\ge \|P_r h_1(t)\|_{L^2_{\xi,v}} - \|P_r h_2(t)\|_{L^2_{\xi,v}} - \|P_r h_3(t)\|_{L^2_{\xi,v}} \\ &\ge \|P_r h_1(t)\|_{L^2_\xi} - C(1+t)^{-2} - Ce^{-Ct}, \end{aligned} \tag{5.71}$$

$$\begin{aligned} \|E(t)\|_{L^2_x} &\ge \frac{\sqrt{2}}{2}\left(\left\|\frac{1}{|\xi|}(h_1(t),\chi_0)\right\|_{L^2_\xi} - \left\|\frac{1}{|\xi|}(h_2(t),\chi_0)\right\|_{L^2_\xi} - \left\|\frac{1}{|\xi|}(h_3(t),\chi_0)\right\|_{L^2_\xi}\right) \\ &\quad + \frac{\sqrt{2}}{2}(\|H_1(t)\|_{L^2_\xi} - \|H_2(t)\|_{L^2_\xi} - \|H_3(t)\|_{L^2_\xi}) \\ &\ge \frac{\sqrt{2}}{2}\left(\left\|\frac{1}{|\xi|}(h_1(t),\chi_0)\right\|_{L^2_\xi} + \|H_1(t)\|_{L^2_\xi}\right) - C(1+t)^{-2} - Ce^{-Ct}, \end{aligned} \tag{5.72}$$

$$\begin{aligned} \|B(t)\|_{L^2_x} &\ge \|J_1(t)\|_{L^2_\xi} - \|J_2(t)\|_{L^2_\xi} - \|J_3(t)\|_{L^2_\xi} \\ &\ge \|J_1(t)\|_{L^2_\xi} - C(1+t)^{-2} - Ce^{-Ct}, \end{aligned} \tag{5.73}$$

where we have used (5.52) and (5.65)–(5.67) for $|\alpha| = 0$.

By (5.53) and $\lambda_1(|\xi|)=\lambda_2(|\xi|)$, we have

$$P_r h_1(t) = \mathrm{i}a_1|\xi|e^{\lambda_1(|\xi|)t}\sum_{j=1,2}(\omega\times\hat{B}_0,W^j)L_1^{-1}(v\cdot W^j)\chi_0+|\xi|^2T_3(t,\xi)\hat{U}_0, \tag{5.74}$$

where $T_3(t,\xi)\hat{U}_0$ is a linear operator satisfying $\|T_3(t,\xi)\hat{U}_0\|^2\le Ce^{-2\eta_1|\xi|^2t}\|\hat{U}_0\|^2$. Since the terms $L_1^{-1}(v\cdot W^1)\sqrt{M}$ and $L_1^{-1}(v\cdot W^2)\sqrt{M}$ are orthogonal, it follows from (5.74) that

$$\|P_rh_1(t)\|^2_{L^2_v}\ge\frac{1}{2}|\xi|^2a_1^2\|L_1^{-1}\chi_1\|^2_{L^2_v}e^{2\mathrm{Re}\lambda_1(|\xi|)t}|\omega\times\hat{B}_0|^2-C|\xi|^4e^{-2\eta_1|\xi|^2t}\|\hat{U}_0\|^2. \tag{5.75}$$

Since

$$\mathrm{Re}\lambda_j(|\xi|)=a_j|\xi|^2(1+O(|\xi|))\ge-\eta_2|\xi|^2,\quad |\xi|\le r_0,$$

for some constant $\eta_2>0$, we obtain by (5.75) that

$$\begin{aligned}\|P_rh_1(t)\|^2_{L^2_{\xi,v}}\ge\ &\frac{1}{2}d_0^2a_1^2\|L_1^{-1}\chi_1\|^2_{L^2_v}\int_{|\xi|\le r_0}|\xi|^2e^{-2\eta_2|\xi|^2t}d\xi\\&-C\int_{|\xi|\le r_0}e^{-2\eta_1|\xi|^2t}|\xi|^4\|\hat{U}_0\|^2d\xi\\&=:\ \frac{1}{2}d_0^2a_1^2\|L_1^{-1}\chi_1\|^2_{L^2_v}I_1-C\|U_0\|^2_{Z^1}(1+t)^{-7/2}.\end{aligned} \tag{5.76}$$

Moreover, for time $t\ge t_0=\frac{1}{r_0^2}$, we have

$$I_1=\int_{|\xi|\le r_0}|\xi|^2e^{-2\eta_2|\xi|^2t}d\xi=4\pi t^{-5/2}\int_0^{r_0\sqrt{t}}e^{-2\eta_2r^2}r^4dr\ge C_0(1+t)^{-5/2}, \tag{5.77}$$

where $C_0>0$ denotes a generic positive constant. We can substitute (5.76) and (5.77) into (5.71) to obtain (5.68).

By (5.54), we have

$$H_1(t)=\mathrm{i}a_1|\xi|e^{\lambda_1(|\xi|)t}\sum_{j=1,2}(\omega\times\hat{B}_0,W^j)(\omega\times W^j)+|\xi|^2T_4(t,\xi)\hat{U}_0,\quad (h_1(t),\sqrt{M})=0,$$

where $T_4(t,\xi)$ is a linear operator satisfying $\|T_3(t,\xi)\hat{U}_0\|^2\le Ce^{-2\eta_1|\xi|^2t}\|\hat{U}_0\|^2$. Then

$$|H_1(t)|^2\ge\frac{1}{2}|\xi|^2a_1^2e^{2\mathrm{Re}\lambda_1(|\xi|)t}|\omega\times\hat{B}_0|^2-C|\xi|^4e^{-2\eta_1|\xi|^2t}|\hat{U}_0|^2.$$

Similar to (5.77), we get

$$\begin{aligned}\|H_1(t)\|^2_{L^2_\xi}&\ge\frac{1}{2}d_0^2a_1^2\int_{|\xi|\le r_0}|\xi|^2e^{-2\eta_2|\xi|^2t}d\xi-C\int_{|\xi|\le r_0}|\xi|^4e^{-2\eta_1|\xi|^2t}\|\hat{U}_0\|^2d\xi\\&\ge C_3(1+t)^{-5/2}-C(1+t)^{-7/2},\end{aligned}$$

which together with (5.72) leads to (5.69) for $t>0$ being large enough.

By (5.55), we have

$$|J_1(t)|^2\ge\frac{1}{2}e^{2\mathrm{Re}\lambda_1(|\xi|)t}|\omega\times\hat{B}_0|^2-C|\xi|^2e^{-2\eta_1|\xi|^2t}|\hat{U}_0|^2,$$

which leads to

$$\begin{aligned}\|J_1(t)\|_{L^2_\xi}^2 &\ge \frac{1}{2}d_0^2\int_{|\xi|\le r_0} e^{-2\eta_2|\xi|^2t}d\xi - C\int_{|\xi|\le r_0}|\xi|^2e^{-2\eta_1|\xi|^2t}\|\hat{U}_0\|^2d\xi\\ &\ge C_3(1+t)^{-3/2} - C(1+t)^{-5/2}.\end{aligned}$$

This and (5.73) give (5.70) for $t>0$ large enough. The proof is then complete. ∎

For comparison, we include the known time decay rates of the global solution to the linearized Boltzmann equation (2.14) as follows; see [20, 26] and references therein.

THEOREM 5.7. *Assume that* $f_0 \in L^2_v(H^N_x)\cap L^{2,q}$ *for* $N\ge 1$ *and* $q\in[1,2]$. *Then there is a globally unique solution* $f(x,v,t) = e^{t\mathbb{B}_0}f_0(x,v)$ *to the linearized Boltzmann equation* (2.14), *which satisfies for any* $\alpha,\alpha'\in\mathbb{N}^3$ *with* $|\alpha|\le N$, $\alpha'\le\alpha$ *and* $k=|\alpha-\alpha'|$ *that*

(5.78)
$$\|(\partial_x^\alpha e^{t\mathbb{B}_0}f_0,\chi_j)\|_{L^2_x} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{2}}(\|\partial_x^\alpha f_0\|_{L^2_{x,v}} + \|\partial_x^{\alpha'}f_0\|_{L^{2,q}}),\ j=0,1,2,3,4,$$

(5.79)
$$\|\mathrm{P}_1(\partial_x^\alpha e^{t\mathbb{B}_0}f_0)\|_{L^2_{x,v}} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k+1}{2}}(\|\partial_x^\alpha f_0\|_{L^2_{x,v}} + \|\partial_x^{\alpha'}f_0\|_{L^{2,q}}).$$

In addition, assume that $f_0\in L^2_v(H^N_x)\cap L^{2,1}$ *for* $N\ge1$ *and there exist positive constants* $d_0,d_1>0$ *and a small constant* $r_0>0$ *so that the Fourier transform* $\hat{f}_0(\xi,v)$ *of the initial data* f_0 *satisfies that* $\inf_{|\xi|\le r_0}|(\hat{f}_0,\sqrt{M})|\ge d_0$, $\inf_{|\xi|\le r_0}|(\hat{f}_0,\chi_4)|\ge d_1\sup_{|\xi|\le r_0}|(\hat{f}_0,\sqrt{M})|$, *and* $\sup_{|\xi|\le r_0}|(\hat{f}_0,v\sqrt{M})|=0$. *Then there exist two positive constants* $C_2\ge C_1$ *such that the global solution* $f(x,v,t)=e^{t\mathbb{B}_0}f_0(x,v)$ *satisfies*

$$C_1(1+t)^{-\frac{3}{4}} \le \|(e^{t\mathbb{B}_0}f_0,\chi_j)\|_{L^2_x} \le C_2(1+t)^{-\frac{3}{4}},\quad j=0,1,2,3,4, \tag{5.80}$$

$$C_1(1+t)^{-\frac{5}{4}} \le \|\mathrm{P}_1(e^{t\mathbb{B}_0}f_0)\|_{L^2_{x,v}} \le C_2(1+t)^{-\frac{5}{4}} \tag{5.81}$$

for $t>0$ *sufficiently large.*

5.3. Corresponding results for one-species VMB system. Corresponding to the linearized two-species VMB, the decomposition and estimation on the semigroup together with the optimal convergence rates can be obtained for the one-species VMB based on its spectrum structure. However, in the following, we can see that the estimates are very different from the case of two-species mainly due to the lack of cancellation in the one-species model.

LEMMA 5.8 (see [13]). *The operator* $B_5(\xi) = L - \mathrm{i}\mathrm{P}_1(v\cdot\xi)\mathrm{P}_1$ *generates a strongly continuous contraction semigroup on* $N_0^\perp$ *for any fixed* $|\xi|\ne0$, *which satisfies for any* $t>0$ *and* $f\in N_0^\perp\cap L^2(\mathbb{R}^3_v)$ *that*

$$\|e^{tB_5(\xi)}f\| \le e^{-\mu t}\|f\|. \tag{5.82}$$

In addition, for any $x>-\mu$ *and* $f\in N_0^\perp\cap L^2(\mathbb{R}^3_v)$, *it holds that*

$$\int_{-\infty}^{+\infty}\|[(x+\mathrm{i}y)\mathrm{P}_1 - B_5(\xi)]^{-1}f\|^2dy \le \pi(x+\mu)^{-1}\|f\|^2. \tag{5.83}$$

LEMMA 5.9. *The operator $G_6(\xi)$ generates a strongly continuous unitary group on $N_0\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ for any fixed $|\xi|\neq 0$, which satisfies for $t>0$ and $U\in N_0\cap L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ that*

$$\|e^{tG_6(\xi)}U\|_\xi=\|U\|_\xi. \tag{5.84}$$

In addition, for any $x\neq 0$ and $f\in N_0\cap L^2_\xi(\mathbb{R}^3_v)$, it holds that

$$\int_{-\infty}^{+\infty}\|[(x+\mathrm{i}y)P_A-G_6(\xi)]^{-1}U\|^2_\xi dy=\pi|x|^{-1}\|U\|^2_\xi. \tag{5.85}$$

Proof. Since the operator $\mathrm{i}G_6(\xi)$ is self-adjoint on $N_0\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi$ with respect to the inner product $(\cdot,\cdot)_\xi$ defined by (4.18), we can prove (5.84) and (5.85) by applying an argument similar to that of Lemma 3.2 in [13]. ∎

By a similar argument as for Lemma 5.3, we have the next lemma.

LEMMA 5.10. *Given any constant $r_0>0$. Let $\alpha=\alpha(r_0,r_1)>0$ with $r_1>r_0$ and $\alpha(r_0,r_1)$ defined in Lemma 4.3. Then*

$$\sup_{r_0<|\xi|<r_1,y\in\mathbb{R}}\left\|\left[I-G_5(\xi)\left(-\frac{\alpha}{2}+\mathrm{i}y-G_1(\xi)\right)^{-1}\right]^{-1}\right\|\leq C. \tag{5.86}$$

With the help of Lemmas 4.3–4.5 and Lemmas 5.8–5.10, we have the decomposition of the semigroup $S(t,\xi)=e^{t\hat{\mathbb{A}}_3(\xi)}$ as follows; the details of the proof are omitted for brevity.

THEOREM 5.11. *The semigroup $S(t,\xi)=e^{t\hat{\mathbb{A}}_3(\xi)}$ with $s=|\xi|\neq 0$ and $\omega=\xi/|\xi|$ has the following decomposition:*

$$S(t,\xi)U=S_1(t,\xi)U+S_2(t,\xi)U+S_3(t,\xi)U,\quad U\in L^2_\xi(\mathbb{R}^3_v)\times\mathbb{C}^3_\xi\times\mathbb{C}^3_\xi,\quad t>0, \tag{5.87}$$

where

$$S_1(t,\xi)U=\sum_{j=-1}^{7}e^{t\lambda_j(s)}(U,\Psi^*_j(s,\omega))_\xi\Psi_j(s,\omega)1_{\{|\xi|\leq r_0\}}, \tag{5.88}$$

$$S_2(t,\xi)U=\sum_{j=1}^{4}e^{t\beta_j(s)}(U,\Phi^*_j(s,\omega))\Phi_j(s,\omega)1_{\{|\xi|\geq r_1\}}, \tag{5.89}$$

with $(\lambda_j(s),\Psi_j(s,\omega))$ and $(\beta_j(s),\Phi_j(s,\omega))$ being the eigenvalue and eigenvector of the operator $\hat{\mathbb{A}}_3(\xi)$ given by Theorems 4.9 and 4.10 for $|\xi|\leq r_0$ and $|\xi|>r_1$, respectively, and $S_3(t,\xi)=:S(t,\xi)-S_1(t,\xi)-S_2(t,\xi)$ satisfies that there exists a constant $\kappa_0>0$ independent of ξ such that

$$\|S_3(t,\xi)U\|_\xi\leq Ce^{-\kappa_0 t}\|U\|_\xi,\quad t\geq 0. \tag{5.90}$$

Based on the decomposition of the semigroup given in Theorem 5.11, we now study the optimal convergence rates of the solutions to the linearized system around an equilibrium.

For any $U_0=(f_0,E_0,B_0)\in L^2(\mathbb{R}^3_v;H^l(\mathbb{R}^3_x))\times H^l(\mathbb{R}^3_x)\times H^l(\mathbb{R}^3_x)$, define

$$e^{t\hat{\mathbb{A}}_2(\xi)}\hat{U}_0=\Big((e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0)_1,\ -\frac{\mathrm{i}\xi}{|\xi|^2}((e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0)_1,\sqrt{M})-\frac{\xi}{|\xi|}\times(e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0)_2,-\frac{\xi}{|\xi|}\times(e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0)_3\Big),$$

with

$$e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0 = ((e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0)_1, (e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0)_2, (e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0)_3) \in L^2_\xi(\mathbb{R}^3_v) \times \mathbb{C}^3_\xi \times \mathbb{C}^3_\xi,$$
$$\hat{V}_0 = \left(\hat{f}_0, \frac{\xi}{|\xi|} \times \hat{E}_0, \frac{\xi}{|\xi|} \times \hat{B}_0\right).$$

Then $e^{t\mathbb{A}_2}U_0$ is the solution of the system (2.39). By Lemma 4.1, it holds that

$$\|e^{t\mathbb{A}_2}U_0\|_{D^l} = \int_{\mathbb{R}^3}(1+|\xi|^2)^l\|e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0\|^2_\xi d\xi \le \int_{\mathbb{R}^3}(1+|\xi|^2)^l\|\hat{V}_0\|^2_\xi d\xi = \|U_0\|_{D^l}.$$

This implies that the linear operator $\mathbb{A}_2$ generates a strongly continuous contraction semigroup $e^{t\mathbb{A}_2}$ in D^l, and therefore, $U(t) = e^{t\mathbb{A}_2}U_0$ is a global solution to the IVP (2.39) for the linearized one-species VMB equation for $U_0 \in D^l$.

Proof of Theorem 2.9. By Theorem 5.11, we have for $\omega = \xi/|\xi|$ that

$$(\hat{f}(t), \omega \times \hat{E}(t), \omega \times \hat{B}(t)) = e^{t\hat{\mathbb{A}}_3(\xi)}\hat{V}_0 = S_1(t,\xi)\hat{V}_0 + S_2(t,\xi)\hat{V}_0 + S_3(t,\xi)\hat{V}_0$$
$$= \sum_{k=1}^3 (h_k(t), H_k(t), J_k(t)),$$

with $\hat{V}_0 = (\hat{f}_0, \omega \times \hat{E}_0, \omega \times \hat{B}_0)$. Note that

(5.91)
$$\|\partial^\alpha_x(f(t),\chi_j)\|_{L^2_x} \le \|\xi^\alpha(h_1(t),\chi_j)\|_{L^2_\xi} + \|\xi^\alpha(h_2(t),\chi_j)\|_{L^2_\xi} + \|\xi^\alpha(h_3(t),\chi_j)\|_{L^2_\xi},$$

$$\|\partial^\alpha_x E(t)\|_{L^2_x} \le \left\|\frac{\xi^\alpha}{|\xi|}(h_1(t),\chi_0)\right\|_{L^2_\xi} + \left\|\frac{\xi^\alpha}{|\xi|}(h_2(t),\chi_0)\right\|_{L^2_\xi} + \left\|\frac{\xi^\alpha}{|\xi|}(h_3(t),\chi_0)\right\|_{L^2_\xi}$$
$$+ \|\xi^\alpha H_1(t)\|_{L^2_\xi} + \|\xi^\alpha H_2(t)\|_{L^2_\xi} + \|\xi^\alpha H_3(t)\|_{L^2_\xi}, \tag{5.92}$$

$$\|\partial^\alpha_x B(t)\|_{L^2_x} \le \|\xi^\alpha J_1(t)\|_{L^2_\xi} + \|\xi^\alpha J_2(t)\|_{L^2_\xi} + \|\xi^\alpha J_3(t)\|_{L^2_\xi}. \tag{5.93}$$

By (5.90), we can estimate the last term on the right-hand side of (5.91)–(5.93) as follows:

$$\int_{\mathbb{R}^3}(\xi^\alpha)^2\left(\|h_3(t)\|^2_{L^2_v} + \frac{1}{|\xi|^2}|(h_3(t),\sqrt{M})|^2 + |H_3(t)|^2 + |J_3(t)|^2\right)d\xi$$
$$\le Ce^{-2\kappa_0 t}(\|\partial^\alpha_x f_0\|^2_{L^2_{x,v}} + \|(f_0,\sqrt{M})\|^2_{L^1_x} + \|\partial^\alpha_x E_0\|^2_{L^2_x} + \|\partial^\alpha_x B_0\|^2_{L^2_x}). \tag{5.94}$$

In the low frequency region, by (5.88), we have

$$S_1(t,\xi)\hat{V}_0 = \sum_{j=-1}^7 e^{t\lambda_j(|\xi|)}[(\hat{f}_0,\overline{\psi_j})_\xi - (\omega\times\hat{E}_0,\overline{X_j}) - (\omega\times\hat{B}_0,\overline{Y_j})](\psi_j,X_j,Y_j)1_{\{|\xi|\le r_0\}}.$$

From (4.55) and (4.56), the macroscopic part and microscopic part of $h_1(t)$ and $H_1(t), J_1(t)$ satisfy

(5.95)
$$(h_1(t),\sqrt{M}) = \frac{1}{2}\sum_{j=\pm1} e^{\lambda_j(|\xi|)t}\hat{n}_0 + |\xi|(T_1(t,\xi)\hat{V}_0,\sqrt{M}),$$

$$(h_1(t), v\sqrt{M}) = \frac{1}{2}\sum_{j=\pm1} e^{\lambda_j(|\xi|)t}\Big(\hat{m}_0\cdot\omega - \frac{j}{|\xi|}\hat{n}_0\Big)\omega + \frac{1}{2}(e^{\lambda_2(|\xi|)t} + e^{\lambda_4(|\xi|)t})(\hat{m}_0 - (\hat{m}_0\cdot\omega)\omega) + \frac{1}{2}(e^{\lambda_2(|\xi|)t} - e^{\lambda_4(|\xi|)t})(\hat{E}_0 - (\hat{E}_0\cdot\omega)\omega) + \mathrm{i}|\xi|e^{\lambda_6(|\xi|)t}(\omega\times\hat{B}_0) + |\xi|(T_1(t,\xi)\hat{V}_0, v\sqrt{M}) + |\xi|^2(R_1(t,\xi)\hat{V}_0, v\sqrt{M}), \tag{5.96}$$

$$(h_1(t), \chi_4) = \sqrt{\frac{1}{6}}\sum_{j=\pm1} e^{\lambda_j(|\xi|)t}\hat{n}_0 + e^{\lambda_0(|\xi|)t}\Big(\hat{q}_0 - \sqrt{\frac{2}{3}}\hat{n}_0\Big) + |\xi|(T_1(t,\xi)\hat{V}_0, \chi_4), \tag{5.97}$$

$$\mathrm{P}_1 h_1(t) = -\frac{1}{2}\sum_{j=\pm1} e^{t\lambda_j(|\xi|)}\hat{n}_0 j\mathrm{i}(L - j\mathrm{i}\mathrm{P}_1)^{-1}\mathrm{P}_1(v\cdot\omega)^2\sqrt{M} + |\xi|\mathrm{P}_1(T_1(t,\xi)\hat{V}_0) + |\xi|^2\mathrm{P}_1(R_1(t,\xi)\hat{V}_0), \tag{5.98}$$

$$H_1(t) = -\frac{1}{2}\mathrm{i}(e^{\lambda_2(|\xi|)t} - e^{\lambda_4(|\xi|)t})(\omega\times\hat{m}_0) - \frac{1}{2}(e^{\lambda_2(|\xi|)t} + e^{\lambda_4(|\xi|)t})(\omega\times\hat{E}_0) + |\xi|T_2(t,\xi)\hat{V}_0 + |\xi|^3R_2(t,\xi)\hat{V}_0, \tag{5.99}$$

$$J_1(t) = e^{\lambda_6(|\xi|)t}(\omega\times\hat{B}_0) + \mathrm{i}|\xi|e^{\lambda_6(|\xi|)t}(\hat{m}_0 - (\hat{m}_0\cdot\omega)\omega) + |\xi|T_3(t,\xi)\hat{V}_0 + |\xi|^2R_3(t,\xi)\hat{V}_0, \tag{5.100}$$

where $(\hat{n}_0, \hat{m}_0, \hat{q}_0)$ is the Fourier transform of the macroscopic density, momentum, and energy (n_0, m_0, q_0) of the initial data f_0 defined by $(n_0, m_0, q_0) =: ((f_0, \chi_0), (f_0, v\sqrt{M}), (f_0, \chi_4))$, and $T_j(t,\xi), R_j(t,\xi),\ j = 1,2,3$, are linear operators satisfying $\|T_j(t,\xi)\| \le Ce^{-\eta_1|\xi|^2 t}$ and $\|R_j(t,\xi)\| \le Ce^{-\eta_1|\xi|^4 t}$ for $|\xi| \le r_0$.

Since

$$\mathrm{Re}\lambda_j(|\xi|) = a_j|\xi|^2(1 + O(|\xi|)) \le -\eta_1|\xi|^2, \quad |\xi| \le r_0,\ j = -1,0,1,2,3,4,5,$$
$$\mathrm{Re}\lambda_k(|\xi|) = a_k|\xi|^2(1 + O(|\xi|)) \le -\eta_1|\xi|^4, \quad |\xi| \le r_0,\ k = 6,7,$$

where $\eta_1 > 0$ denotes a generic constant, we obtain by (5.93)–(5.100) that

$$\|\xi^\alpha(h_1(t), \sqrt{M})\|^2_{L^2_\xi} \le C(1+t)^{-(3/2+k)}\|\partial_x^{\alpha'}U_0\|^2_{Z^1}, \tag{5.101}$$

$$\|\xi^\alpha(h_1(t), v\sqrt{M})\|^2_{L^2_\xi} \le C[(1+t)^{-(1/2+k)} + (1+t)^{-(5/4+k/2)}]\|\partial_x^{\alpha'}U_0\|^2_{Z^1}, \tag{5.102}$$

$$\|\xi^\alpha(h_1(t), \chi_4)\|^2_{L^2_\xi} \le C(1+t)^{-(3/2+k)}\|\partial_x^{\alpha'}U_0\|^2_{Z^1}, \tag{5.103}$$

$$\|\xi^\alpha\mathrm{P}_1h_1(t)\|^2_{L^2_{\xi,v}} \le C[(1+t)^{-(3/2+k)} + (1+t)^{-(7/4+k/2)}]\|\partial_x^{\alpha'}U_0\|^2_{Z^1}, \tag{5.104}$$

$$\|\xi^\alpha H_1(t)\|^2_{L^2_\xi} \le C[(1+t)^{-(3/2+k)} + (1+t)^{-(9/4+k/2)}]\|\partial_x^{\alpha'}U_0\|^2_{Z^1}, \tag{5.105}$$

$$\|\xi^\alpha J_1(t)\|^2_{L^2_\xi} \le C[(1+t)^{-(3/4+k/2)} + (1+t)^{-(5/2+k)}]\|\partial_x^{\alpha'}U_0\|^2_{Z^1}, \tag{5.106}$$

with $k = |\alpha - \alpha'|$.

In the high frequency region, by (5.89), we have

$$S_2(t,\xi)\hat{V}_0 = \sum_{j=1}^{4} e^{t\beta_j(|\xi|)}[(\hat{f}_0, \overline{\phi_j}) - (\omega\times\hat{E}_0, \overline{X_j}) - (\omega\times\hat{B}_0, \overline{Y_j})](\phi_j, X_j, Y_j)1_{\{|\xi|\ge r_1\}}, \tag{5.107}$$

and in particular $(h_2(t),\chi_0)=(h_2(t),\chi_4)=0$. Since

$$\mathrm{Re}\beta_j(|\xi|)\le -c_1|\xi|^{-1},\quad |\xi|\ge r_1,$$

we obtain by (5.107) that

$$\|\xi^\alpha(h_2(t),v\sqrt{M})\|^2_{L^2_\xi}+\|\xi^\alpha \mathrm{P}_1h_2(t)\|^2_{L^2_{\xi,v}}\le C(1+t)^{-(2m+1)}\|\nabla^m_x\partial^\alpha_x U_0\|^2_{Z^2}, \tag{5.108}$$

$$\|\xi^\alpha H_2(t)\|^2_{L^2_\xi}+\|\xi^\alpha J_2(t)\|^2_{L^2_\xi}\le C(1+t)^{-2m}\|\nabla^m_x\partial^\alpha_x U_0\|^2_{Z^2}. \tag{5.109}$$

Combining (5.91)–(5.94), (5.101)–(5.106), and (5.108)–(5.109) leads to (2.44)–(2.50).

Now we turn to show the lower bound of time decay rates for the global solution under the assumptions of Theorem 2.9. Note that

(5.110)

$$\|(f(t),\chi_j)\|_{L^2_x}\ge \|(h_1(t),\chi_j)\|_{L^2_\xi}-C(1+t)^{-1}-Ce^{-Ct},$$

$$\|E(t)\|_{L^2_x}\ge \frac{\sqrt2}{2}\left(\left\|\frac{1}{|\xi|}(h_1(t),\sqrt{M})\right\|_{L^2_\xi}+\|H_1(t)\|_{L^2_\xi}\right)-C(1+t)^{-1}-Ce^{-Ct}, \tag{5.111}$$

$$\|B(t)\|_{L^2_x}\ge \|J_1(t)\|_{L^2_\xi}-C(1+t)^{-1}-Ce^{-Ct}, \tag{5.112}$$

where we have used (5.94) and (5.108)–(5.109) for $|\alpha|=0$.

By (5.95) and that fact that $\lambda_{-1}(|\xi|)=\overline{\lambda_1(|\xi|)}$, we have

$$|(h_1(t),\sqrt{M})|^2\ge \frac12 e^{2\mathrm{Re}\lambda_1(|\xi|)t}\cos^2(\mathrm{Im}\lambda_1(|\xi|)t)|\hat n_0|^2-C|\xi|^2e^{-2\eta_1|\xi|^2t}\|\hat V_0\|^2. \tag{5.113}$$

Since

$$\cos^2(\mathrm{Im}\lambda_1(|\xi|)t)\ge\frac12\cos^2[(1+b_1|\xi|^2)t]-O([|\xi|^3t]^2),$$

and

$$\mathrm{Re}\lambda_j(|\xi|)=a_j|\xi|^2(1+O(|\xi|))\ge-\eta_2|\xi|^2,\quad |\xi|\le r_0,\ j=-1,0,1,2,3,4,5,$$

for some constant $\eta_2>0$, we obtain by (5.113) that

$$\begin{aligned}\|(h_1(t),\sqrt{M})\|^2_{L^2_\xi}&\ge\frac{d_0^2}{4}\int_{|\xi|\le r_0}e^{-2\eta_2|\xi|^2t}\cos^2(t+b_1|\xi|^2t)d\xi-C(1+t)^{-5/2}\\ &=: I_1-C(1+t)^{-5/2}.\end{aligned} \tag{5.114}$$

It holds for $t\ge t_0=:\frac{L^2}{r_0^2}$ with the constant $L\ge\sqrt{\frac{4\pi}{b_1}}$ that

$$\begin{aligned}I_1&=\frac{d_0^2}{4}t^{-3/2}\int_{|\zeta|\le r_0\sqrt t}e^{-2\eta|\zeta|^2}\cos^2(t+b_1|\zeta|^2)d\zeta\\ &\ge t^{-3/2}\frac{\pi d_0^2L}{2}e^{-2\eta L^2}\int_{L/2}^L r\cos^2(t+b_1r^2)dr\\ &\ge(1+t)^{-3/2}\frac{\pi d_0^2L}{4b_1}e^{-2\eta L^2}\int_0^\pi\cos^2 y dy\ge C_3(1+t)^{-3/2},\end{aligned} \tag{5.115}$$

where $C_3 > 0$ denotes a generic positive constant. We can substitute (5.114) and (5.115) into (5.110) with $j = 0$ to obtain (2.54).

By (5.96), we have

$$|(h_1(t), v\sqrt{M})|^2 \geq \frac{1}{2|\xi|^2} e^{2\mathrm{Re}\lambda_1(|\xi|)t} \sin^2(\mathrm{Im}\lambda_1(|\xi|)t)|\hat{n}_0|^2 \\ - Ce^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2 - C|\xi|^2 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2.$$

Then

$$\|(h_1(t), v\sqrt{M})\|^2_{L^2_\xi} \geq \frac{d_0^2}{4}\int_{|\xi|\leq r_0} \frac{1}{|\xi|^2} e^{-2\eta_2|\xi|^2 t}\sin^2(t + b_1|\xi|^2 t)d\xi \\ - C\int_{|\xi|\leq r_0} e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2 d\xi - C\int_{|\xi|\leq r_0} |\xi|^2 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2 d\xi \\ \geq C_3(1+t)^{-1/2} - C(1+t)^{-3/2} - C(1+t)^{-5/4},$$

which together with (5.110) for $j = 1, 2, 3$ leads to (2.55) for $t > 0$ being large enough.

By (5.97) and the fact that $\lambda_0(|\xi|)$ is real, we have

$$|(h_1(t), \chi_4)|^2 \geq \frac{1}{2} e^{2\lambda_0(|\xi|)t}|\hat{q}_0|^2 - Ce^{-2\eta_1|\xi|^2 t}(|\hat{n}_0|^2 + |\xi|^2\|\hat{V}_0\|^2),$$

which leads to

$$\|(h_1(t), \chi_4)\|^2_{L^2_\xi} \geq \frac{1}{2}\int_{|\xi|\leq r_0} e^{-2\eta_2|\xi|^2 t}|\hat{q}_0|^2 d\xi - C\int_{|\xi|\leq r_0} e^{-2\eta_1|\xi|^2 t}(|\hat{n}_0|^2 + |\xi|^2\|\hat{V}_0\|^2)d\xi \\ \geq C_3\left(\inf_{|\xi|\leq r_0}|\hat{q}_0(\xi)|^2 - d_1\sup_{|\xi|\leq r_0}|\hat{n}_0(\xi)|\right)(1+t)^{-3/2} - C(1+t)^{-5/2}.$$

This and (5.110) with $j = 4$ lead to (2.56) for $t > 0$ being large enough.

By (5.98), we have

$$\|\mathrm{P}_1 h_1(t)\|^2_{L^2_v} \geq \frac{1}{2}|\hat{n}_0|^2 e^{2\mathrm{Re}\lambda_1(|\xi|)t}\|\sin(\mathrm{Im}\lambda_1(|\xi|)t)L\Psi + \cos(\mathrm{Im}\lambda_1(|\xi|)t)\Psi\|^2_{L^2_v} \\ - C|\xi|^2 e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2 - C|\xi|^4 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2,$$

where $\Psi = (L - \mathrm{i}\mathrm{P}_1)^{-1}(L + \mathrm{i}\mathrm{P}_1)^{-1}\mathrm{P}_1(v\cdot\omega)^2\sqrt{M} \in N_0^\perp$ is a nonzero real function. Then for time $t \geq t_0 =: \frac{L^2}{r_0^2}$ with $L \geq \sqrt{\frac{4\pi}{b_1}}$,

$$\|\mathrm{P}_1 h_1(t)\|^2_{L^2_{\xi,v}} \geq \frac{d_0^2}{4}\int_{|\xi|\leq r_0} e^{-2\eta_2|\xi|^2 t}\|\sin(t + b_1|\xi|^2 t)L\Psi + \cos(t + b_1|\xi|^2 t)\Psi\|^2_{L^2_v} d\xi \\ - C(1+t)^{-5/2} - C(1+t)^{-7/4} \\ \geq C_3(1+t)^{-3/2} - C(1+t)^{-5/2} - C(1+t)^{-7/4}. \tag{5.116}$$

We can substitute (5.116) into (5.110) imply (2.57) for sufficiently large $t > 0$.

By (5.99), we obtain

$$\frac{1}{|\xi|^2}|(h_1(t), \sqrt{M})|^2 + |H_1(t)|^2$$

$$\geq \frac{1}{2|\xi|^2} e^{2\mathrm{Re}\lambda_1(|\xi|)t} \cos^2(\mathrm{Im}\lambda_1(|\xi|)t)|\hat{n}_0|^2 - Ce^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2 - C|\xi|^6 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2,$$

which leads to

$$\left\| \frac{1}{|\xi|}(h_1(t), \sqrt{M}) \right\|_{L^2_\xi}^2 + \|H_1(t)\|_{L^2_\xi}^2 \geq C_3(1+t)^{-1/2} - C(1+t)^{-3/2} - C(1+t)^{-9/4}.$$

This together with (5.111) lead to (2.52) for sufficiently large $t > 0$.

Finally, by (5.100) and the fact that $\lambda_6(|\xi|) = \lambda_7(|\xi|)$ are real, we obtain

$$|J_1(t)|^2 \geq \frac{1}{2} e^{2\lambda_6(|\xi|)t} |\omega \times \hat{B}_0|^2 - C|\xi|^4 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2 - C|\xi|^2 e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2.$$

Since

$$\mathrm{Re}\lambda_j(|\xi|) = a_j|\xi|^4(1 + O(|\xi|)) \geq -\eta_2|\xi|^4, \quad |\xi| \leq r_0, \ j = 6, 7,$$

for some constant $\eta_2 > 0$, we have

$$\|J_1(t)\|_{L^2_\xi}^2 \geq C_3(1+t)^{-3/4} - C(1+t)^{-7/4} - C(1+t)^{-5/2}.$$

This together with (5.112) leads to (2.53). The proof is then complete. ∎

Proof of Theorem 2.10. In the case of $\nabla_x \cdot E_0 = (f_0, \sqrt{M})$, the high frequency term is $S_2(t,\xi)\hat{V}_0$ are the same as those in the case of $\nabla_x \cdot E_0 \neq (f_0, \sqrt{M})$. The different part lies in the expansions of the low frequency term $S_1(t,\xi)\hat{V}_0$. That is, in the low frequency region, by (5.88), we have

$$\begin{aligned} S_1(t,\xi)\hat{V}_0 = &\sum_{j=-1}^{7} e^{t\lambda_j(|\xi|)} \\ &\times \left[(\hat{f}_0, \overline{\psi_j}) + \frac{1}{|\xi|^2} \mathrm{i}(\hat{E}_0 \cdot \xi)(\psi_j, \sqrt{M}) - (\omega \times \hat{E}_0, \overline{X_j}) - (\omega \times \hat{B}_0, \overline{Y_j}) \right] \\ &\times (\psi_j, X_j, Y_j) 1_{\{|\xi| \leq r_0\}}, \end{aligned}$$

where we have used $(\hat{f}_0, \sqrt{M}) = \mathrm{i}(\hat{E}_0 \cdot \xi)$ to replace $(\hat{f}_0, \sqrt{M})$. From (4.55) and (4.56), the macroscopic part and microscopic part of $h_1(t)$ and $H_1(t), J_1(t)$ satisfy

(5.117)
$$(h_1(t), \sqrt{M}) = -\frac{1}{2}|\xi| \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} [j(\hat{m}_0 \cdot \omega) - \mathrm{i}(\hat{E}_0 \cdot \omega)] + |\xi|^2 T_4(t,\xi)\hat{V}_0,$$

$$\begin{aligned} (h_1(t), v\sqrt{M}) = &\frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)t} [(\hat{m}_0 \cdot \omega) - j\mathrm{i}(\hat{E}_0 \cdot \omega)]\omega \\ &+ \frac{1}{2}(e^{\lambda_2(|\xi|)t} + e^{\lambda_4(|\xi|)t})(\hat{m}_0 - (\hat{m}_0 \cdot \omega)\omega) \\ &+ \frac{1}{2}(e^{\lambda_2(|\xi|)t} - e^{\lambda_4(|\xi|)t})(\hat{E}_0 - (\hat{E}_0 \cdot \omega)\omega) + \mathrm{i}|\xi| e^{\lambda_6(|\xi|)t}(\omega \times \hat{B}_0) \\ &+ |\xi| T_5(t,\xi)\hat{V}_0 + |\xi|^2 R_4(t,\xi)\hat{V}_0, \end{aligned} \tag{5.118}$$

$$(h_1(t), \chi_4) = e^{\lambda_0(|\xi|)t}\hat{q}_0 + |\xi| T_6(t,\xi)\hat{V}_0, \tag{5.119}$$

$$\mathrm{P}_1 h_1(t) = |\xi|^2 \sum_{j=6,7} e^{\lambda_j(|\xi|)t} (\omega \times \hat{B}_0, W^j) L^{-1} \mathrm{P}_1 (v \cdot \omega)(v \cdot W^j)\sqrt{M}$$

$$+ |\xi| T_7(t,\xi)\hat{V}_0 + |\xi|^3 R_5(t,\xi)\hat{V}_0, \tag{5.120}$$

$H_1(t), J_1(t)$ are the same as (5.99) and (5.100), and $T_i(t,\xi)$, $i = 4,5,6,7$, and $R_k(t,\xi)$, $k = 4,5$, are the linear operators satisfying $\|T_i(t,\xi)\| \le Ce^{-\eta_1|\xi|^2 t}$ for $i = 4,5,6,7$ and $\|R_k(t,\xi)\| \le Ce^{-\eta_1|\xi|^4 t}$ for $k = 4,5$. Thus we obtain by (5.117)–(5.120) that

$$\|\xi^\alpha (h_1(t), \sqrt{M})\|^2_{L^2_\xi} \le C(1+t)^{-(5/2+k)} \|\partial_x^{\alpha'} U_0\|^2_{Z^1}, \tag{5.121}$$

$$\|\xi^\alpha (h_1(t), v\sqrt{M})\|^2_{L^2_\xi} \le C[(1+t)^{-(3/2+k)} + (1+t)^{-(5/4+k/2)}] \|\partial_x^{\alpha'} U_0\|^2_{Z^1}, \tag{5.122}$$

$$\|\xi^\alpha (h_1(t), \chi_4)\|^2_{L^2_\xi} \le C(1+t)^{-(3/2+k)} \|\partial_x^{\alpha'} U_0\|^2_{Z^1}, \tag{5.123}$$

$$\|\xi^\alpha \mathrm{P}_1 h_1(t)\|^2_{L^2_{\xi,v}} \le C[(1+t)^{-(5/2+k)} + (1+t)^{-(7/4+k/2)}] \|\partial_x^{\alpha'} U_0\|^2_{Z^1}, \tag{5.124}$$

with $k = |\alpha - \alpha'|$. Thus by (5.121)–(5.124), (5.105)–(5.106), and (5.108)–(5.109), we can prove (2.58)–(2.64).

Now we turn to show the lower bound of time decay rates for the global solution under the assumptions of Theorem 2.9. By (5.117)–(5.119), we have

$$\begin{aligned}
|(h_1(t), \sqrt{M})|^2 &\ge \frac{1}{2}|\xi|^2 e^{2\mathrm{Re}\lambda_1(|\xi|)t} \sin^2(\mathrm{Im}\lambda_1(|\xi|)t)|\hat{E}_0\cdot\omega|^2 - C|\xi|^4 e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2,\\
|(h_1(t), v\sqrt{M})|^2 &\ge \frac{1}{2}|\xi|^2 e^{2\lambda_6(|\xi|)t}|\omega\times\hat{B}_0|^2 - Ce^{-2\eta_1|\xi|^2 t}(|\hat{E}_0|^2 + |\xi|^2\|\hat{V}_0\|^2)\\
&\quad - C|\xi|^4 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2,\\
|(h_1(t), \chi_4)|^2 &\ge \frac{1}{2} e^{2\lambda_0(|\xi|)t}|\hat{q}_0|^2 - C|\xi|^2 e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2,
\end{aligned}$$

which lead to

$$\|(h_1(t), \sqrt{M})\|^2_{L^2_\xi} \ge C_3(1+t)^{-5/2} - C(1+t)^{-7/2}, \tag{5.125}$$

$$\|(h_1(t), v\sqrt{M})\|^2_{L^2_\xi} \ge C_3(1+t)^{-5/4} - C(1+t)^{-7/4} - C(1+t)^{-3/2}, \tag{5.126}$$

$$\|(h_1(t), \chi_4)\|^2_{L^2_\xi} \ge C_3(1+t)^{-3/2} - C(1+t)^{-5/2}. \tag{5.127}$$

We can substitute (5.125)–(5.127) into (5.110) to obtain (2.68)–(2.70) for $t > 0$ being large enough.

By (5.120), we have

$$\begin{aligned}
\|\mathrm{P}_1 h_1(t)\|^2_{L^2_v} \ge \frac{1}{2}\|L^{-1}\mathrm{P}_1(v_1\chi_2)\|^2_{L^2_v}|\xi|^4 e^{2\lambda_6(|\xi|)t}|\omega\times\hat{B}_0|^2 - C|\xi|^6 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2\\
- C|\xi|^2 e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2.
\end{aligned}$$

This leads to

$$\|\mathrm{P}_1 h_1(t)\|^2_{L^2_{\xi,v}} \ge C_3(1+t)^{-7/4} - C(1+t)^{-9/4} - C(1+t)^{-5/2},$$

which together with (5.110) implies (2.71) for sufficiently large $t > 0$.

Finally, by (5.99) and (5.100) we obtain

$$\frac{1}{|\xi|^2}|(h_1(t), \sqrt{M})|^2 + |H_1(t)|^2 \ge \frac{1}{2} e^{2\mathrm{Re}\lambda_1(|\xi|)t}\sin^2(\mathrm{Im}\lambda_1(|\xi|)t)|\hat{E}_0\cdot\omega|^2$$

$$
\begin{aligned}
&\quad - C|\xi|^2 e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2 - C|\xi|^6 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2,\\
|J_1(t)|^2 &\ge \frac{1}{2} e^{2\lambda_6(|\xi|)t}|\omega\times \hat{B}_0|^2\\
&\quad - C|\xi|^4 e^{-2\eta_1|\xi|^4 t}\|\hat{V}_0\|^2 - C|\xi|^2 e^{-2\eta_1|\xi|^2 t}\|\hat{V}_0\|^2,
\end{aligned}
$$

which lead to

$$
\begin{aligned}
\left\|\frac{1}{|\xi|}(h_1(t),\sqrt{M})\right\|^2_{L^2_\xi} + \|H_1(t)\|^2_{L^2_\xi} &\ge C(1+t)^{-3/2} - C(1+t)^{-5/2} - C(1+t)^{-9/4},\\
\|J_1(t)\|^2_{L^2_\xi} &\ge C(1+t)^{-3/4} - C(1+t)^{-7/4} - C(1+t)^{-5/2}.
\end{aligned}
$$

These together with (5.111) and (5.112) lead to (2.66) and (2.67). The proof is then complete. ∎

6. The nonlinear system. In this section, we prove the large time decay rates of the solution to the Cauchy problem for VMB systems with the estimates on the linearized problem obtained in section 5.

6.1. Energy estimates for two-species VMB system. We first obtain some energy estimates. Let N be a positive integer and $U = (f_1, f_2, E, B)$, and

(6.1)
$$
E_{N,k}(U) = \sum_{|\alpha|+|\beta|\le N} \|w^k \partial_x^\alpha \partial_v^\beta (f_1, f_2)\|^2_{L^2_{x,v}} + \sum_{|\alpha|\le N} \|\partial_x^\alpha (E,B)\|^2_{L^2_x},
$$

$$
\begin{aligned}
H_{N,k}(U) &= \sum_{|\alpha|+|\beta|\le N} \|w^k \partial_x^\alpha \partial_v^\beta (\mathrm{P}_1 f_1, P_r f_2)\|^2_{L^2_{x,v}} + \sum_{1\le|\alpha|\le N} \|\partial_x^\alpha (E,B)\|^2_{L^2_x} + \|E\|^2_{L^2_x}\\
&\quad + \sum_{|\alpha|\le N-1} \|\partial_x^\alpha \nabla_x (\mathrm{P}_0 f_1, P_{\mathrm{d}} f_2)\|^2_{L^2_{x,v}} + \|P_{\mathrm{d}} f_2\|^2_{L^2_{x,v}},
\end{aligned} \tag{6.2}
$$

$$
\begin{aligned}
D_{N,k}(U) &= \sum_{|\alpha|+|\beta|\le N} \|w^{\frac12+k} \partial_x^\alpha \partial_v^\beta (\mathrm{P}_1 f_1, P_r f_2)\|^2_{L^2_{x,v}} + \sum_{1\le|\alpha|\le N-1} \|\partial_x^\alpha B\|^2_{L^2_x}\\
&\quad + \sum_{|\alpha|\le N-1} (\|\partial_x^\alpha \nabla_x (\mathrm{P}_0 f_1, P_{\mathrm{d}} f_2)\|^2_{L^2_{x,v}} + \|\partial_x^\alpha E\|^2_{L^2_x}) + \|P_{\mathrm{d}} f_2\|^2_{L^2_{x,v}}
\end{aligned} \tag{6.3}
$$

for $k \ge 0$. For brevity, we write $E_N(U) = E_{N,0}(U)$, $H_N(U) = H_{N,0}(U)$, and $D_N(U) = D_{N,0}(U)$ for $k = 0$.

First, by taking the inner product between χ_j ($j = 0,1,2,3,4$) and (2.2), we obtain a compressible Euler–Maxwell type system

$$
\partial_t n_1 + \mathrm{div}_x m_1 = 0, \tag{6.4}
$$

$$
\partial_t m_1 + \nabla_x n_1 + \sqrt{\frac{2}{3}}\nabla_x q_1 = n_2 E + m_2 \times B - \int_{\mathbb{R}^3} v\cdot\nabla_x(\mathrm{P}_1 f_1) v\sqrt{M}dv, \tag{6.5}
$$

$$
\partial_t q_1 + \sqrt{\frac{2}{3}}\mathrm{div}_x m_1 = \sqrt{\frac{2}{3}} E\cdot m_2 - \int_{\mathbb{R}^3} v\cdot\nabla_x(\mathrm{P}_1 f_1)\chi_4 dv, \tag{6.6}
$$

where

$$
(n_1, m_1, q_1) = ((f_1,\sqrt{M}), (f_1, v\sqrt{M}), (f_1,\chi_4)), \quad (n_2, m_2) = ((f_2,\sqrt{M}), (f_2, v\sqrt{M})).
$$

Taking the microscopic projection P_1 on (2.2), we have

$$\partial_t(\mathrm{P}_1 f_1) + \mathrm{P}_1(v\cdot\nabla_x \mathrm{P}_1 f_1) - L(\mathrm{P}_1 f_1) = -\mathrm{P}_1(v\cdot\nabla_x \mathrm{P}_0 f_1) + \mathrm{P}_1 G_1, \tag{6.7}$$

where the nonlinear term G_1 is denoted by

$$G_1 = \frac{1}{2}(v\cdot E) f_2 - (E + v\times B)\cdot\nabla_v f_2 + \Gamma(f_1, f_1). \tag{6.8}$$

By (6.7), we can express the microscopic part $\mathrm{P}_1 f_1$ as

$$\mathrm{P}_1 f_1 = L^{-1}[\partial_t(\mathrm{P}_1 f_1) + \mathrm{P}_1(v\cdot\nabla_x \mathrm{P}_1 f_1) - \mathrm{P}_1 G_1] + L^{-1}\mathrm{P}_1(v\cdot\nabla_x \mathrm{P}_0 f_1). \tag{6.9}$$

Substituting (6.9) into (6.4)–(6.6), we obtain a compressible Navier–Stokes–Maxwell type system

$$\partial_t n_1 + \mathrm{div}_x m_1 = 0, \tag{6.10}$$

$$\partial_t m_1 + \partial_t R_1 + \nabla_x n_1 + \sqrt{\frac{2}{3}}\nabla_x q_1 = \kappa_1\Big(\Delta_x m_1 + \frac{1}{3}\nabla_x \mathrm{div}_x m_1\Big) + n_2 E + m_2\times B + R_2, \tag{6.11}$$

$$\partial_t q_1 + \partial_t R_3 + \sqrt{\frac{2}{3}}\mathrm{div}_x m_1 = \kappa_2\Delta_x q_1 + \sqrt{\frac{2}{3}} E\cdot m_2 + R_4, \tag{6.12}$$

where the viscosity and heat conductivity coefficients $\kappa_1, \kappa_2 > 0$ and the remainder terms R_1, R_2, R_3, R_4 are given by

$$\begin{aligned} &\kappa_1 = -(L^{-1}\mathrm{P}_1(v_1\chi_2), v_1\chi_2), \quad \kappa_2 = -(L^{-1}\mathrm{P}_1(v_1\chi_4), v_1\chi_4),\\ &R_1 = (v\cdot\nabla_x L^{-1}\mathrm{P}_1 f_1, v\sqrt{M}), \quad R_2 = -(v\cdot\nabla_x L^{-1}(\mathrm{P}_1(v\cdot\nabla_x\mathrm{P}_1 f_1) - \mathrm{P}_1 G_1), v\sqrt{M}),\\ &R_3 = (v\cdot\nabla_x L^{-1}\mathrm{P}_1 f_1, \chi_4), \quad R_4 = -(v\cdot\nabla_x L^{-1}(\mathrm{P}_1(v\cdot\nabla_x\mathrm{P}_1 f_1) - \mathrm{P}_1 G_1), \chi_4). \end{aligned}$$

By taking the inner product between $\sqrt{M}$ and (2.3), we obtain

$$\partial_t n_2 + \mathrm{div}_x m_2 = 0. \tag{6.13}$$

Taking the microscopic projection P_r on (2.3), we have

$$\partial_t(P_r f_2) + P_r(v\cdot\nabla_x P_r f_2) - v\sqrt{M}\cdot E - L_1(P_r f_2) = -P_r(v\cdot\nabla_x P_{\mathrm{d}} f_2) + P_r G_2, \tag{6.14}$$

where the nonlinear term G_2 is denoted by

$$G_2 = \frac{1}{2}(v\cdot E) f_1 - (E + v\times B)\cdot\nabla_v f_1 + \Gamma(f_2, f_1). \tag{6.15}$$

By (6.14), we can express the microscopic part $P_r f_2$ as

$$P_r f_2 = L_1^{-1}[\partial_t(P_r f_2) + P_r(v\cdot\nabla_x P_r f_2) - P_r G_2] + L_1^{-1} P_r(v\cdot\nabla_x P_{\mathrm{d}} f_2) - L_1^{-1}(v\sqrt{M}\cdot E). \tag{6.16}$$

Substituting (6.16) into (6.13) and (2.4), we obtain

$$\partial_t n_2 + \partial_t \mathrm{div}_x R_5 = -\kappa_3 n_2 + \kappa_3\Delta_x n_2 - \mathrm{div}_x R_6, \tag{6.17}$$

$$\partial_t E + \partial_t R_5 = \nabla_x \times B + \kappa_3 \nabla_x n_2 - \kappa_3 E + R_6, \tag{6.18}$$

$$\partial_t B = -\nabla_x \times E, \tag{6.19}$$

where the viscosity coefficient $\kappa_3 > 0$ and the remainder terms R_5, R_6 are defined by

$$\kappa_3 = -(L_1^{-1}\chi_1, \chi_1), \quad R_5 = (L_1^{-1}P_r f_2, v\sqrt{M}),$$
$$R_6 = (L_1^{-1}(P_r(v \cdot \nabla_x P_r f_2) - P_r G_2), v\sqrt{M}).$$

The following lemma is from [4, 9].

LEMMA 6.1 (see [4, 9]). *It holds that*

$$\|\nu^k \partial_v^\beta \Gamma(f,g)\|_{L_v^2} \le C \sum_{\beta_1+\beta_2 \le \beta} (\|\partial_v^{\beta_1} f\|_{L_v^2} \|\nu^{k+1}\partial_v^{\beta_2} g\|_{L_v^2} + \|\nu^{k+1}\partial_v^{\beta_1} f\|_{L_v^2} \|\partial_v^{\beta_2} g\|_{L_v^2})$$

for $k \ge -1$, *and*

$$\|\Gamma(f,g)\|_{L^{2,1}} \le C(\|f\|_{L_{x,v}^2}\|\nu g\|_{L_{x,v}^2} + \|\nu f\|_{L_{x,v}^2}\|g\|_{L_{x,v}^2}).$$

LEMMA 6.2 (macroscopic dissipation). *Let* (n_1, m_1, q_1) *and* (n_2, E, B) *be the strong solutions to* (6.10)–(6.12) *and* (6.17)–(6.19), *respectively. Then, there are two constants* $s_0, s_1 > 0$ *such that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\le|\alpha|\le N-1} s_0\bigg(\|\partial_x^\alpha(n_1,m_1,q_1)\|_{L_x^2}^2 + 2\int_{\mathbb{R}^3}\partial_x^\alpha R_1 \partial_x^\alpha m_1 dx + 2\int_{\mathbb{R}^3}\partial_x^\alpha R_3 \partial_x^\alpha q_1 dx\bigg)\\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\le|\alpha|\le N-1} 4\int_{\mathbb{R}^3}\partial_x^\alpha m_1 \partial_x^\alpha \nabla_x n_1 dx + \sum_{k\le|\alpha|\le N-1}\|\partial_x^\alpha \nabla_x(n_1,m_1,q_1)\|_{L_x^2}^2\\
&\le C\sqrt{E_N(U)}D_N(U) + C\sum_{k\le|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x \mathrm{P}_1 f_1\|_{L_{x,v}^2}^2,
\end{aligned} \tag{6.20}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\le|\alpha|\le N-1} s_1\bigg(\|\partial_x^\alpha(n_2,E,B)\|_{L_x^2}^2 + 2\int_{\mathbb{R}^3}\partial_x^\alpha \mathrm{div}_x R_5 \partial_x^\alpha n_2 dx + 2\int_{\mathbb{R}^3}\partial_x^\alpha R_5 \partial_x^\alpha E dx\bigg)\\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\le|\alpha|\le N-2} 4\int_{\mathbb{R}^3}\partial_x^\alpha E \partial_x^\alpha(\nabla_x \times B) dx\\
&\quad + \sum_{k\le|\alpha|\le N-1}(\|\partial_x^\alpha n_2\|_{L_x^2}^2 + \|\partial_x^\alpha\nabla_x n_2\|_{L_x^2}^2 + \|\partial_x^\alpha E\|_{L_x^2}^2) + \sum_{k+1\le|\alpha|\le N-1}\|\partial_x^\alpha B\|_{L_x^2}^2\\
&\le C E_N(U)D_N(U) + C\sum_{k\le|\alpha|\le N}\|\partial_x^\alpha P_r f_2\|_{L_{x,v}^2}^2,
\end{aligned} \tag{6.21}$$

with $0 \le k \le N-2$.

Proof. First of all, we prove (6.20). Taking the inner product between $\partial_x^\alpha m_1$ and ∂_x^α(6.11) with $|\alpha| \le N-1$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha m_1\|_{L_x^2}^2 + \int_{\mathbb{R}^3}\partial_x^\alpha \partial_t R_1 \partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3}\partial_x^\alpha\nabla_x n_1 \partial_x^\alpha m_1 dx\\
&\quad + \sqrt{\frac{2}{3}}\int_{\mathbb{R}^3}\partial_x^\alpha\nabla_x q_1 \partial_x^\alpha m_1 dx + \kappa_1\bigg(\|\partial_x^\alpha\nabla_x m_1\|_{L_x^2}^2 + \frac{1}{3}\|\partial_x^\alpha \mathrm{div}_x m_1\|_{L_x^2}^2\bigg)
\end{aligned} \tag{6.22}$$

$$= \int_{\mathbb{R}^3} \partial_x^\alpha (n_2 E)\partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3} \partial_x^\alpha (m_2 \times B)\partial_x^\alpha m_1 dx + \int_{\mathbb{R}^3} \partial_x^\alpha R_2 \partial_x^\alpha m_1 dx.$$

For the second and third terms on the left-hand side of (6.22), we use (6.4) and (6.5) to get

$$\begin{aligned}\int_{\mathbb{R}^3} \partial_x^\alpha \partial_t R_1 \partial_x^\alpha m_1 dx &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx \\ &\quad - \int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha \Big[-\nabla_x n_1 - \sqrt{\frac{2}{3}}\nabla_x q_1 + n_2 E \\ &\quad + m_2 \times B - (v\cdot\nabla_x \mathrm{P}_1 f_1, v\sqrt{M})\Big] dx \\ &\geq \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx - \epsilon(\|\partial_x^\alpha \nabla_x n_1\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2) \\ &\quad - C\sqrt{E_N(U)}D_N(U) - \frac{C}{\epsilon}\|\partial_x^\alpha \nabla_x \mathrm{P}_1 f_1\|_{L_x^2}^2, \end{aligned} \tag{6.23}$$

and

$$\int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x n_1 \partial_x^\alpha m_1 dx = -\int_{\mathbb{R}^3} \partial_x^\alpha n \partial_x^\alpha \mathrm{div}_x m dx = \int_{\mathbb{R}^3} \partial_x^\alpha n_1 \partial_x^\alpha \partial_t n_1 dx = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha n_1\|_{L_x^2}^2. \tag{6.24}$$

The first and second terms on the right-hand side of (6.22) are bounded by $C\sqrt{E_N(U)}D_N(U)$. By Lemma 6.1, the last term can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^3} \partial_x^\alpha R_2 \partial_x^\alpha m_1 dx &\leq C\|\partial_x^\alpha \nabla_x \mathrm{P}_1 f_1\|_{L_{x,v}^2}\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2} + C(\|\partial_x^\alpha (E f_2)\|_{L_{x,v}^2} \\ &\quad + \|\partial_x^\alpha (B f_2)\|_{L_{x,v}^2} + \|w^{-\frac{1}{2}}\partial_x^\alpha \Gamma(f_1,f_1)\|_{L_{x,v}^2})\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2} \\ &\leq \frac{\kappa_1}{2}\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2}^2 + C\|\partial_x^\alpha \nabla_x \mathrm{P}_1 f_1\|_{L_{x,v}^2}^2 + C\sqrt{E_N(U)}D_N(U). \end{aligned} \tag{6.25}$$

Therefore, it follows from (6.22)–(6.25) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_x^\alpha m_1\|_{L_x^2}^2 + \|\partial_x^\alpha n_1\|_{L_x^2}^2) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \partial_x^\alpha R_1 \partial_x^\alpha m_1 dx + \sqrt{\frac{2}{3}}\int_{\mathbb{R}^3} \partial_x^\alpha \nabla_x q_1 \partial_x^\alpha m_1 dx \\ &\quad + \frac{\kappa_1}{2}\left(\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2}^2 + \frac{1}{3}\|\partial_x^\alpha \mathrm{div}_x m_1\|_{L_x^2}^2\right) \\ &\leq C\sqrt{E_N(U)}D_N(U) + \frac{C}{\epsilon}\|\partial_x^\alpha \nabla_x \mathrm{P}_1 f_1\|_{L_{x,v}^2}^2 + \epsilon(\|\partial_x^\alpha \nabla_x n_1\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2). \end{aligned} \tag{6.26}$$

Similarly, taking the inner product between $\partial_x^\alpha q_1$ and $\partial_x^\alpha(6.12)$ with $|\alpha| \leq N-1$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha q_1\|_{L_x^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \partial_x^\alpha R_3 \partial_x^\alpha q_1 dx + \sqrt{\frac{2}{3}}\int_{\mathbb{R}^3} \partial_x^\alpha \mathrm{div}_x m_1 \partial_x^\alpha q_1 dx + \frac{\kappa_2}{2}\|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2 \\ &\quad \leq C\sqrt{E_N(U)}D_N(U) + \frac{C}{\epsilon}\|\partial_x^\alpha \nabla_x \mathrm{P}_1 f_1\|_{L_{x,v}^2}^2 + \epsilon\|\partial_x^\alpha \nabla_x m_1\|_{L_x^2}^2. \end{aligned} \tag{6.27}$$

Again, taking the inner product between $\partial_x^\alpha \nabla_x n_1$ and $\partial_x^\alpha(6.5)$ with $|\alpha| \le N-1$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \partial_x^\alpha m_1 \partial_x^\alpha \nabla_x n_1 dx + \frac{1}{2}\|\partial_x^\alpha \nabla_x n_1\|_{L_x^2}^2$$
$$\le C\sqrt{E_N(U)}D_N(U) + \|\partial_x^\alpha \mathrm{div}_x m_1\|_{L_x^2}^2 + \|\partial_x^\alpha \nabla_x q_1\|_{L_x^2}^2 + C\|\partial_x^\alpha \nabla_x \mathrm{P}_1 f_1\|_{L_{x,v}^2}^2. \tag{6.28}$$

Taking the summation of $2s_0\sum_{k\le|\alpha|\le N-1}[(6.26)+(6.27)] + 4\sum_{k\le|\alpha|\le N-1}(6.28)$ with $s_0>0$ large enough, $\epsilon>0$ small enough, and $0\le k\le N-1$, we obtain (6.20).

Next, we turn to show (6.21). Taking the inner product between $\partial_x^\alpha n_2$ and $\partial_x^\alpha(6.17)$ with $|\alpha| \le N-1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha n_2\|_{L_x^2}^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\partial_x^\alpha \mathrm{div}_x R_5 \partial_x^\alpha n_2 dx + \kappa_3\|\partial_x^\alpha n_2\|_{L_x^2}^2 + \kappa_3\|\partial_x^\alpha \nabla_x n_2\|_{L_x^2}^2$$
$$\le C\|\partial_x^\alpha \nabla_x P_r f_2\|_{L_{x,v}^2}^2 + CE_N(U)D_N(U). \tag{6.29}$$

Similarly, taking the inner product between $\partial_x^\alpha E$ and $\partial_x^\alpha(6.18)$ with $|\alpha| \le N-1$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha (E,B)\|_{L_x^2}^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\partial_x^\alpha R_5 \partial_x^\alpha E dx + \kappa_3\|\partial_x^\alpha E\|_{L_x^2}^2 + \kappa_3\|\partial_x^\alpha n_2\|_{L_x^2}^2$$
$$\le \frac{C}{\epsilon}(\|\partial_x^\alpha P_r f_2\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha \nabla_x P_r f_2\|_{L_{x,v}^2}^2) + \epsilon\|\partial_x^\alpha(\nabla_x\times B)\|_{L_x^2}^2$$
$$+ CE_N(U)D_N(U). \tag{6.30}$$

And taking the inner product between $\partial_x^\alpha \nabla_x\times B$ and $\partial_x^\alpha(6.18)$ with $|\alpha| \le N-1$ gives

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\partial_x^\alpha E \partial_x^\alpha(\nabla_x\times B)dx - \|\partial_x^\alpha(\nabla_x\times E)\|_{L_x^2}^2 + \|\partial_x^\alpha(\nabla_x\times B)\|_{L_x^2}^2$$
$$= \int_{\mathbb{R}^3}\partial_x^\alpha m_2 \partial_x^\alpha(\nabla_x\times B)dx.$$

This and the fact that $\|\nabla_x\times B\|_{L_x^2}^2 = \|\nabla_x B\|_{L_x^2}^2$ imply that

$$-2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\partial_x^\alpha E \partial_x^\alpha(\nabla_x\times B)dx + \|\partial_x^\alpha \nabla_x B\|_{L_x^2}^2 \le C\|\partial_x^\alpha P_r f_2\|_{L_{x,v}^2}^2 + 2\|\partial_x^\alpha(\nabla_x\times E)\|_{L_x^2}^2. \tag{6.31}$$

Taking the summation of $s_1\sum_{k\le|\alpha|\le N-1}[(6.29)+(6.30)] + 2\sum_{k\le|\alpha|\le N-2}(6.31)$ with $s_1>0$ large enough, $\epsilon>0$ small enough, and $0\le k\le N-1$, we obtain (6.21). And this completes the proof of the lemma. □

In the following, we shall estimate the microscopic terms to close the energy estimate.

LEMMA 6.3 (microscopic dissipation). *Let $N\ge 4$ and let (f_1,f_2,E,B) be a strong solution to VMB system* (2.2)–(2.7). *Then, there are constants $p_k>0$, $1\le k\le N$, such that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|(f_1,f_2)\|_{L_{x,v}^2}^2 + \|(E,B)\|_{L_x^2}^2) + \mu\|w^{\frac{1}{2}}(\mathrm{P}_1 f_1, P_r f_2)\|_{L_{x,v}^2}^2$$
$$\le C\sqrt{E_N(U)}D_N(U), \tag{6.32}$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le N}(\|\partial_x^\alpha(f_1,f_2)\|_{L_{x,v}^2}^2 + \|\partial_x^\alpha(E,B)\|_{L_x^2}^2)$$

$$+\mu\sum_{1\le|\alpha|\le N}\|w^{\frac12}\partial_x^\alpha(\mathrm{P}_1f_1,P_rf_2)\|^2_{L^2_{x,v}}\le C\sqrt{E_N(U)}D_N(U), \tag{6.33}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le k\le N}p_k\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_v^\beta(\mathrm{P}_1f_1,P_rf_2)\|^2_{L^2_{x,v}}\\&+\mu\sum_{\substack{|\beta|\ge1\\|\alpha|+|\beta|\le N}}\|w^{\frac12}\partial_x^\alpha\partial_v^\beta(\mathrm{P}_1f_1,P_rf_2)\|^2_{L^2_{x,v}}\\&\le C\sum_{|\alpha|\le N-1}(\|\partial_x^\alpha\nabla_x(f_1,f_2)\|^2_{L^2_{x,v}}+\|\partial_x^\alpha E\|^2_{L^2_x})+C\sqrt{E_N(U)}D_N(U).\end{aligned} \tag{6.34}$$

Proof. Taking the inner product between $\partial_x^\alpha f_1$ and $\partial_x^\alpha(2.2)$ with $|\alpha|\le N$ $(N\ge4)$, we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha f_1\|^2_{L^2_{x,v}}-\int_{\mathbb{R}^3}(L\partial_x^\alpha f_1)\partial_x^\alpha f_1dxdv\\&=\frac12\int_{\mathbb{R}^6}\partial_x^\alpha(v\cdot Ef_2)\partial_x^\alpha f_1dxdv-\int_{\mathbb{R}^6}\partial_x^\alpha((E+v\times B)\cdot\nabla_vf_2)\partial_x^\alpha f_1dxdv\\&\quad+\int_{\mathbb{R}^6}\partial_x^\alpha\Gamma(f_1,f_1)\partial_x^\alpha f_1dxdv\\&=:I_1+I_2+I_3.\end{aligned} \tag{6.35}$$

The terms I_1 and I_3 are bounded by $C\sqrt{E_N(U)}D_N(U)$. For I_2, it holds that

$$I_2\le C\sqrt{E_N(U)}D_N(U)-\int_{\mathbb{R}^6}(E+v\times B)\partial_x^\alpha\nabla_vf_2\partial_x^\alpha f_1dxdv. \tag{6.36}$$

Therefore, it follows from (6.35)–(6.36) that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha f_1\|^2_{L^2_{x,v}}+\mu\|w^{\frac12}\partial_x^\alpha\mathrm{P}_1f_1\|^2_{L^2_{x,v}}&\le C\sqrt{E_N(U)}D_N(U)\\&\quad-\int_{\mathbb{R}^6}(E+v\times B)\partial_x^\alpha\nabla_vf_2\partial_x^\alpha f_1dxdv.\end{aligned} \tag{6.37}$$

Similarly, taking the inner product between $\partial_x^\alpha f_2$ and $\partial_x^\alpha(2.3)$ with $|\alpha|\le N$ $(N\ge4)$, we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_x^\alpha f_2\|^2_{L^2_{x,v}}+\|\partial_x^\alpha(E,B)\|^2_{L^2_x})+\mu\|w^{\frac12}\partial_x^\alpha P_rf_2\|^2_{L^2_{x,v}}\\&\le C\sqrt{E_N(U)}D_N(U)-\int_{\mathbb{R}^6}(E+v\times B)\partial_x^\alpha\nabla_vf_1\partial_x^\alpha f_2dxdv.\end{aligned} \tag{6.38}$$

Taking the summation of (6.37)+(6.38) for $|\alpha|=0$ and $1\le|\alpha|\le N$, we obtain (6.32) and (6.33), respectively.

In order to close the energy estimate, we need to estimate the terms $\partial_x^\alpha\nabla_vf$ with $|\alpha|\le N-1$. For this, we rewrite (2.2) and (2.3) as

$$\begin{aligned}&\partial_t(\mathrm{P}_1f_1)+v\cdot\nabla_x\mathrm{P}_1f_1+(E+v\times B)\cdot\nabla_vP_rf_2-L(\mathrm{P}_1f_1)\\&=\Gamma(f_1,f_1)+\frac12v\cdot EP_rf_2-\mathrm{P}_1(v\cdot\nabla_x\mathrm{P}_0f_1)\\&\quad+\mathrm{P}_0\bigg(v\cdot\nabla_x\mathrm{P}_1f_1-\frac12v\cdot EP_rf_2+(E+v\times B)\cdot\nabla_vP_rf_2\bigg),\end{aligned} \tag{6.39}$$

and

$$\begin{aligned}&\partial_t(P_rf_2)+v\cdot\nabla_xP_rf_2-v\sqrt{M}\cdot E+(E+v\times B)\cdot\nabla_v\mathrm{P}_1f_1+L_1(P_rf_2)\\&=\Gamma(f_2,f_1)+\frac{1}{2}v\cdot E\mathrm{P}_1f_1+P_{\mathrm{d}}(v\cdot\nabla_xP_rf_2)\\&\quad-\Big(v\cdot\nabla_xP_{\mathrm{d}}f_2-\frac{1}{2}v\cdot E\mathrm{P}_0f_1+(E+v\times B)\cdot\nabla_v\mathrm{P}_0f_1\Big).\end{aligned}\tag{6.40}$$

Let $1\le k\le N$, and choose α,β with $|\beta|=k$ and $|\alpha|+|\beta|\le N$. Taking the inner product between $\partial_x^\alpha\partial_v^\beta\mathrm{P}_1f_1$ and (6.39), between $\partial_x^\alpha\partial_v^\beta P_rf_2$ and (6.40), respectively, and then taking summation of the resulted equations, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_v^\beta(\mathrm{P}_1f_1,P_rf_2)\|^2_{L^2_{x,v}}+\mu\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|w^{\frac12}\partial_x^\alpha\partial_v^\beta(\mathrm{P}_1f_1,P_rf_2)\|^2_{L^2_{x,v}}\\&\le C\sum_{|\alpha|\le N-k}(\|\partial_x^\alpha\nabla_x(f_1,f_2)\|^2_{L^2_{x,v}}+\|\partial_x^\alpha E\|^2_{L^2_x})\\&+C_k\sum_{\substack{|\beta|\le k-1\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_v^\beta(\mathrm{P}_1f_1,P_rf_2)\|^2_{L^2_{x,v}}+C\sqrt{E_N(U)}D_N(U).\end{aligned}\tag{6.41}$$

Then taking summation $\sum_{1\le k\le N}p_k(6.41)$ with constants p_k chosen by

$$\mu p_k\ge 2\sum_{1\le j\le N-k}p_{k+j}C_{k+j},\quad 1\le k\le N-1,\quad p_N=1,$$

we obtain (6.34). The proof of the lemma is then complete. ∎

With the help of Lemmas 6.2–6.3, we have the next lemma.

LEMMA 6.4. *For $N\ge 4$, there are two equivalent energy functionals $E_N^f(\cdot)\sim E_N(\cdot)$, $H_N^f(\cdot)\sim H_N(\cdot)$ such that the following holds. If $E_N(U_0)$ is sufficiently small, then the Cauchy problem (2.2)–(2.7) of the two-species VMB system admits a unique global solution $U=(f_1,f_2,E,B)$ satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_N^f(U)(t)+\mu D_N(U)(t)\le 0,\tag{6.42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}H_N^f(U)(t)+\mu D_N(U)(t)\le C\|\nabla_x(n_1,m_1,q_1)\|^2_{L^2_x}+C\|\nabla_xB\|^2_{L^2_x}.\tag{6.43}$$

Proof. Assume that

$$E_N(U)(t)\le\delta$$

for $\delta>0$ being small.

Taking the summation of $A_1[(6.20)+(6.21)]+A_2[(6.32)+(6.33)]+(6.34)$ with $A_2>C_0A_1>0$ large enough and taking $k=0$ in (6.20) and (6.21), we obtain (6.42).

Taking the inner product between E and (6.18), between f_2 and (2.3), respectively, we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|E\|^2_{L^2_x}+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}R_5E dx+\kappa_3\|E\|^2_{L^2_x}+\kappa_3\|n_2\|^2_{L^2_x}\\&\le\epsilon\|E\|^2_{L^2_x}+\frac{C}{\epsilon}\|\nabla_xB\|^2_{L^2_x}+C(\|P_rf_2\|^2_{L^2_{x,v}}+\|\nabla_xP_rf_2\|^2_{L^2_{x,v}})\end{aligned}$$

$$+ C\sqrt{E_N(U)}D_N(U), \tag{6.44}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|f_2\|^2_{L^2_{x,v}} + \|E\|^2_{L^2_x}) - \int_{\mathbb{R}^3}(L_1 f_2)f_2 dxdv \le \epsilon\|E\|^2_{L^2_x} + \frac{C}{\epsilon}\|\nabla_x B\|^2_{L^2_x} + C\sqrt{E_N(U)}D_N(U). \tag{6.45}$$

Taking the summation of $s_2(6.44) + (6.45)$ with $s_2 > 0$ large enough and $\epsilon > 0$ small enough, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}s_2(\|f_2\|^2_{L^2_{x,v}} + \|E\|^2_{L^2_x}) + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} R_5 E dx + \kappa_3\|E\|^2_{L^2_x}\\ &\quad + \kappa_3\|n_2\|^2_{L^2_x} + \mu\|w^{\frac{1}{2}}P_r f_2\|^2_{L^2_{x,v}}\\ &\le C\|\nabla_x B\|^2_{L^2_x} + C\|\nabla_x P_r f_2\|^2_{L^2_{x,v}} + C\sqrt{E_N(U)}D_N(U).\end{aligned} \tag{6.46}$$

Taking the inner product between (6.39) and $\mathrm{P}_1 f_1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{P}_1 f_1\|^2_{L^2_{x,v}} + \|w^{\frac{1}{2}}\mathrm{P}_1 f_1\|^2_{L^2_{x,v}} \le C\|\nabla_x \mathrm{P}_0 f_1\|^2_{L^2_{x,v}} + E_N(U)D_N(U). \tag{6.47}$$

Taking the summation of $A_3[(6.20) + (6.21)] + A_4[(6.46) + (6.47)] + A_5(6.33) + (6.34)$ with $A_5 > C_0 A_4, A_4 > C_1 A_3$ large enough and taking $k = 1$ in (6.20) and (6.21), we obtain (6.43). ∎

Repeating the proofs of Lemmas 6.2–6.3, we can show the next lemma.

LEMMA 6.5. *For $N \ge 4$, there are the equivalent energy functionals $E^f_{N,1}(\cdot) \sim E_{N,1}(\cdot)$, $H^f_{N,1}(\cdot) \sim H_{N,1}(\cdot)$ such that if $E_{N,1}(U_0)$ is sufficiently small, then the solution $U = (f_1, f_2, E, B)(t, x, v)$ to the two-species VMB system* (2.2)–(2.7) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}E^f_{N,1}(U)(t) + \mu D_{N,1}(U)(t) \le 0, \tag{6.48}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}H^f_{N,1}(U)(t) + \mu D_{N,1}(U)(t) \le C\|\nabla_x(n_1, m_1, q_1)\|^2_{L^2_x} + C\|\nabla_x B\|^2_{L^2_x}. \tag{6.49}$$

6.2. Convergence rates for two-species VMB system. Based on the above energy estimates and the convergence rates of the solution to the linearized system, the convergence rates of the solution to the two-species VMB can be summarized in the following theorem.

THEOREM 6.6. *Under the assumptions of Theorem* 2.5, *there exists a globally unique solution (f_1, f_2, E, B) to the system* (2.2)–(2.7) *satisfying*

$$\|\partial^k_x(f_1(t), \chi_j)\|_{L^2_x} \le C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad j = 0, 1, 2, 3, 4, \tag{6.50}$$

$$\|\partial^k_x \mathrm{P}_1 f_1(t)\|_{L^2_{x,v}} \le C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \tag{6.51}$$

$$\|\partial^k_x(f_2(t), \sqrt{M})\|_{L^2_x} \le C\delta_0(1+t)^{-2-\frac{k}{2}}, \tag{6.52}$$

$$\|\partial^k_x P_r f_2(t)\|_{L^2_{x,v}} + \|\partial^k_x E(t)\|_{L^2_x} \le C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \tag{6.53}$$

$$\|\partial^k_x B(t)\|_{L^2_{x,v}} \le C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{6.54}$$

$$\|(\mathrm{P}_1 f_1, P_r f_2)(t)\|_{H^N_w} + \|\nabla_x(\mathrm{P}_0 f_1, P_{\mathrm{d}} f_2)(t)\|_{L^2_v(H^{N-1}_x)} \tag{6.55}$$

$$+\|\nabla_x(E,B)(t)\|_{H^{N-1}_x} \le C\delta_0(1+t)^{-\frac{5}{4}}$$

for $k=0,1$.

Proof. Let (f_1,f_2,E,B) be a solution to the Cauchy problem (2.2)–(2.7) for $t>0$. We can represent the solution in terms of the semigroups $e^{t\mathbb{B}_0}$ and $e^{t\mathbb{A}_0}$ as

$$f_1(t) = e^{t\mathbb{B}_0} f_{1,0} + \int_0^t e^{(t-s)\mathbb{B}_0} G_1(s)ds, \tag{6.56}$$

$$(f_2,E,B)(t) = e^{t\mathbb{A}_0}(f_{2,0},E_0,B_0) + \int_0^t e^{(t-s)\mathbb{A}_0}(G_2(s),0,0)ds, \tag{6.57}$$

where the nonlinear terms G_1 and G_2 are given by (6.8) and (6.15), respectively. Define a functional $Q(t)$ for any $t>0$ by

$$\begin{aligned} Q(t) = \sup_{0\le s\le t} \sum_{k=0}^{1} \Bigg\{ & \sum_{j=0}^{4} \|\partial_x^k (f_1(s),\chi_j)\|_{L^2_x}(1+s)^{\frac{3}{4}+\frac{k}{2}} + \|\partial_x^k \mathrm{P}_1 f_1(s)\|_{L^2_{x,v}}(1+s)^{\frac{5}{4}+\frac{k}{2}} \\ & + \|\partial_x^k P_{\mathrm{d}} f_2(s)\|_{L^2_{x,v}}(1+s)^{2+\frac{k}{2}} + (\|\partial_x^k P_r f_2(s)\|_{L^2_{x,v}} + \|\partial_x^k E(s)\|_{L^2_x})(1+s)^{\frac{5}{4}+\frac{k}{2}} \\ & + \|\partial_x^k B(s)\|_{L^2_x}(1+s)^{\frac{3}{4}+\frac{k}{2}} + H_{N,1}(U)(s)(1+s)^{\frac{5}{2}} \Bigg\}. \end{aligned}$$

We claim that it holds under the assumptions of Theorem 6.6 that

$$Q(t) \le C\delta_0. \tag{6.58}$$

It is straightforward to verify that the estimates (6.50)–(6.55) follow from (6.58). Hence, it remains to prove (6.58).

By Lemma 6.1, we can estimate the nonlinear term $G_1(s), G_2(s)$ for $0\le s\le t$ in terms of $Q(t)$ as

$$\begin{aligned} \|G_1(s)\|_{L^2_{x,v}} &\le C\{\|wf_1\|_{L^{2,3}}\|f_1\|_{L^{2,6}} + \|E\|_{L^3_x}(\|wf_2\|_{L^{2,6}} + \|\nabla_v f_2\|_{L^{2,6}}) \\ &\quad + \|B\|_{L^3_x}\|w\nabla_v f_2\|_{L^{2,6}}\} \\ &\le C(1+s)^{-2}Q(t)^2, \end{aligned} \tag{6.59}$$

$$\begin{aligned} \|G_1(s)\|_{L^{2,1}} &\le C\{\|f_1\|_{L^2_{x,v}}\|wf_1\|_{L^2_{x,v}} + \|E\|_{L^3_x}(\|wf_2\|_{L^2_{x,v}} + \|\nabla_v f_1\|_{L^2_{x,v}}) \\ &\quad + \|B\|_{L^3_x}\|w\nabla_v f_2\|_{L^2_{x,v}}\} \\ &\le C(1+s)^{-\frac{3}{2}}Q(t)^2, \end{aligned} \tag{6.60}$$

and similarly

$$\|G_2(s)\|_{L^2_{x,v}} \le C(1+s)^{-2}Q(t)^2, \tag{6.61}$$

$$\|G_2(s)\|_{L^{2,1}} \le C(1+s)^{-\frac{3}{2}}Q(t)^2, \tag{6.62}$$

$$\|G_2\|_{L^2_v(H^k_x)} \le C(1+s)^{-\frac{5}{2}}Q(t)^2 \tag{6.63}$$

for $1\le k\le N-1$. First, we consider the time decay rate of the macroscopic density, momentum, and energy of f_1. It follows from (5.78), (6.59), and (6.60) that

$$\|(\nabla_x^k f_1(t),\chi_j)\|_{L^2_x} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^k f_{1,0}\|_{L^2_{x,v}} + \|f_{1,0}\|_{L^{2,1}})$$

$$
\begin{aligned}
&+ C\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^k G_1(s)\|_{L^2_{x,v}} + \|G_1(s)\|_{L^{2,1}})ds \\
&\le C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} + C\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{k}{2}}(1+s)^{-\frac{3}{2}}Q(t)^2 ds \\
&\le C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} + C(1+t)^{-\frac{3}{4}-\frac{k}{2}}Q(t)^2, \quad k=0,1. \qquad (6.64)
\end{aligned}
$$

Second, we estimate the microscopic part $\mathrm{P}_1 f_1(t)$ as follows. By (5.79), (6.59), and (6.60), we have

$$
\begin{aligned}
\|\mathrm{P}_1 f_1(t)\|_{L^2_{x,v}} &\le C(1+t)^{-\frac{5}{4}}(\|f_{1,0}\|_{L^2_{x,v}} + \|f_{1,0}\|_{L^{2,1}}) \\
&\quad + C\int_0^t (1+t-s)^{-\frac{5}{4}}(\|G_1(s)\|_{L^2_{x,v}} + \|G_1(s)\|_{L^{2,1}})ds \\
&\le C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}Q(t)^2, \qquad (6.65)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla_x \mathrm{P}_1 f_1(t)\|_{L^2_{x,v}} &\le C(1+t)^{-\frac{7}{4}}(\|\nabla_x f_{1,0}\|_{L^2_{x,v}} + \|f_{1,0}\|_{L^{2,1}}) \\
&\quad + C\int_0^{t/2} (1+t-s)^{-\frac{7}{4}}(\|\nabla_x G_1(s)\|_{L^2_{x,v}} + \|G_1(s)\|_{L^{2,1}})ds \\
&\quad + C\int_{t/2}^t (1+t-s)^{-\frac{5}{4}}(\|\nabla_x G_1(s)\|_{L^2_{x,v}} + \|\nabla_x G_1(s)\|_{L^{2,1}})ds \\
&\le C\delta_0(1+t)^{-\frac{7}{4}} + C(1+t)^{-\frac{7}{4}}Q(t)^2, \qquad (6.66)
\end{aligned}
$$

where we have used

$$
\|\nabla_x G_1(s)\|_{L^2_{x,v}} + \|\nabla_x G_1(s)\|_{L^{2,1}} \le C(1+s)^{-2}Q(t)^2.
$$

Next, we consider the time decay rate of the f_2, E, B. Denote $U_{2,0} = (f_{2,0}, E_0, B_0)$. By (5.45), (5.46), (6.61), (6.62), and (6.63), we have

$$
\begin{aligned}
\|P_r f_2(t)\|_{L^2_{x,v}} + \|E(t)\|_{L^2_x} &\le C(1+t)^{-\frac{5}{4}}(\|U_{2,0}\|_{Z^2} + \|U_{2,0}\|_{Z^1} + \|\nabla_x^2 U_{2,0}\|_{Z^2}) \\
&\quad + C\int_0^t (1+t-s)^{-\frac{5}{4}}(\|G_2\|_{L^2_{x,v}} + \|G_2\|_{L^{2,1}})ds \\
&\quad + C\int_0^t (1+t-s)^{-2}\|\nabla_x^2 G_2\|_{L^2_{x,v}} ds \\
&\le C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}Q(t)^2, \qquad (6.67)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\nabla_x P_r f_2(t)\|_{L^2_{x,v}} + \|\nabla_x E(t)\|_{L^2_x} &\le C(1+t)^{-\frac{7}{4}}(\|\nabla_x U_{2,0}\|_{Z^2} + \|U_{2,0}\|_{Z^1} + \|\nabla_x^3 U_{2,0}\|_{Z^2}) \\
&\quad + C\int_0^{t/2} (1+t-s)^{-\frac{7}{4}}(\|\nabla_x G_2\|_{L^2_{x,v}} + \|G_2\|_{L^{2,1}})ds \\
&\quad + C\int_{t/2}^t (1+t-s)^{-\frac{5}{4}}(\|\nabla_x G_2\|_{L^2_{x,v}} + \|\nabla_x G_2\|_{L^{2,1}})ds \\
&\quad + C\int_0^t (1+t-s)^{-2}\|\nabla_x^3 G_2\|_{L^2_{x,v}} ds
\end{aligned}
$$

$$\leq C\delta_0(1+t)^{-\frac{7}{4}} + C(1+t)^{-\frac{7}{4}}Q(t)^2. \tag{6.68}$$

By (5.47), (6.61), (6.62), and (6.63), we have

$$\begin{aligned}\|\nabla_x^k B(t)\|_{L^2_x} \leq & C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^k U_{2,0}\|_{Z^2} + \|U_{2,0}\|_{Z^1} + \|\nabla_x^2 U_{2,0}\|_{Z^2}) \\ & + C\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^k G_2\|_{L^2_{x,v}} + \|G_2\|_{L^{2,1}})ds \\ & + C\int_0^t (1+t-s)^{-2}\|\nabla_x^{k+2} G_2\|_{L^2_{x,v}} ds \\ \leq & C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} + C(1+t)^{-\frac{3}{4}-\frac{k}{2}}Q(t)^2, \quad k=0,1. \end{aligned} \tag{6.69}$$

Finally, we estimate the higher order terms. Since

$$d_1 H^f_{N,1}(U) \leq D_{N,1}(U) + C\sum_{|\alpha|=N} \|\partial_x^\alpha (E,B)\|^2_{L^2_x} \tag{6.70}$$

for $d_1 > 0$, we still need to estimate the decay rate of $\|\partial_x^\alpha(E,B)\|^2_{L^2_x}$ for $|\alpha| = N$. By Theorem 1.3 in [7], one has

$$E_{k,1}(U)(t) \leq C(1+t)^{-3/2}(E_{k+2,1}(U_0) + (\delta_0 + Q(t)^2)^2) \tag{6.71}$$

for any integer $k \geq 4$, where $E_{k,1}$ is defined by (6.1). Then

$$\begin{aligned}\|\nabla_x^N (E,B)(t)\|_{L^2_x} \leq & C(1+t)^{-\frac{5}{4}}(\|\nabla_x^N U_{2,0}\|_{Z^2} + \|U_{2,0}\|_{Z^1} + \|\nabla_x^{N+2} U_{2,0}\|_{Z^2}) \\ & + C\int_0^t (1+t-s)^{-\frac{5}{4}}(\|\nabla_x^N G_2(s)\|_{L^2_{x,v}} + \|G_2(s)\|_{L^{2,1}})ds \\ & + C\int_0^t (1+t-s)^{-2}\|\nabla_x^{N+2} G_2(s)\|_{L^2_{x,v}} ds \\ \leq & C\delta_0(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}(\delta_0 + Q(t)^2)^2, \end{aligned} \tag{6.72}$$

where we have used (6.71) to obtain

$$\|G_2(s)\|_{L^2_v(H^{N+2}_x)} \leq CE_{N+3,1}(U)(s) \leq C(E_{N+5,1}(U_0) + (\delta_0 + Q(t)^2)^2)(1+s)^{-\frac{3}{2}}.$$

Then, by (6.49) and (6.70), we have

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} H^f_{N,1}(U)(t) + d_1\mu H^f_{N,1}(U)(t) \\ & \leq C\|\nabla_x(n_1,m_1,q_1)(t)\|^2_{L^2_x} + C\|\nabla_x B(t)\|^2_{L^2_x} + C\sum_{|\alpha|=N}\|\partial_x^\alpha(E,B)(t)\|^2_{L^2_x}. \end{aligned} \tag{6.73}$$

This and (6.72) give

$$\begin{aligned} H^f_{N,1}(U)(t) \leq & e^{-d_1\mu t}H^f_{N,1}(U_0) + C\int_0^t e^{-d_1\mu(t-s)}\|\nabla_x(n_1,m_1,q_1,B)(s)\|^2_{L^2_x} ds \\ & + C\int_0^t e^{-d_1\mu(t-s)} \sum_{|\alpha|=N}\|\partial_x^\alpha(E,B)(s)\|^2_{L^2_x} ds \\ \leq & \ C(1+t)^{-\frac{5}{2}}(\delta_0 + Q(t)^2 + (\delta_0+Q(t)^2)^2. \end{aligned} \tag{6.74}$$

By summing (6.64)–(6.69) and (6.74), we have

$$Q(t) \le C(\delta_0 + Q(t)^2) + C(\delta_0 + Q(t)^2)^2,$$

which yields (6.58) when $\delta_0 > 0$ is chosen small enough. This completes the proof of the theorem. ∎

Indeed, some of the above convergence rates can be shown to be optimal even for the nonlinear system.

THEOREM 6.7. *Under the assumption of Theorem* 2.6*, the global solution* (f_1, f_2, E, B) *to the two-species VMB system* (2.2)–(2.7) *satisfies*

$$C_1\delta_0(1+t)^{-\frac{3}{4}} \le \|(f_1(t), \chi_j)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac{3}{4}}, \quad j = 0,1,2,3,4, \tag{6.75}$$

$$C_1\delta_0(1+t)^{-\frac{5}{4}} \le \|\mathrm{P}_1 f_1(t)\|_{L^2_{x,v}} \le C_2\delta_0(1+t)^{-\frac{5}{4}}, \tag{6.76}$$

$$C_1\delta_0(1+t)^{-\frac{5}{4}} \le \|P_r f_2(t)\|_{L^2_{x,v}} \le C_2\delta_0(1+t)^{-\frac{5}{4}}, \tag{6.77}$$

$$C_1\delta_0(1+t)^{-\frac{5}{4}} \le \|E(t)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac{5}{4}}, \tag{6.78}$$

$$C_1\delta_0(1+t)^{-\frac{3}{4}} \le \|B(t)\|_{L^2_x} \le C_2\delta_0(1+t)^{-\frac{3}{4}} \tag{6.79}$$

for $t > 0$ *large with two constants* $C_2 > C_1$.

Proof. By (6.56), (6.57), and Theorems 5.7 and 6.6, we can establish the lower bounds of the time decay rates of macroscopic density, momentum, and energy of the global solution (f_1, f_2, E, B) to the system (2.2)–(2.7) and its microscopic part for $t > 0$ large enough. For example, it holds that

$$\begin{aligned} \|(f_1(t), \chi_j)\|_{L^2_x} &\ge \|(e^{t\mathbb{B}_0} f_{1,0}, \chi_j)\|_{L^2_x} - \int_0^t \|(e^{(t-s)\mathbb{B}_0} G_1(s), \chi_j)\|_{L^2_x} ds \\ &\ge C_1\delta_0(1+t)^{-\frac{3}{4}} - C_2\delta_0^2(1+t)^{-\frac{3}{4}}, \\ \|E(t)\|_{L^2_x} &\ge \|(e^{t\mathbb{A}_0}(f_{2,0}, E_0, B_0))_2\|_{L^2_x} - \int_0^t \|(e^{(t-s)\mathbb{A}_0}(G_2(s), 0, 0))_2\|_{L^2_x} ds \\ &\ge C_1\delta_0(1+t)^{-\frac{5}{4}} - C_2\delta_0^2(1+t)^{-\frac{5}{4}}. \end{aligned}$$

Therefore, for $\delta_0 > 0$ small and $t > 0$ large, we can prove (6.75)–(6.79). ∎

6.3. Corresponding results on one-species VMB system.

Finally, we give the corresponding results on the one-species VMB. Let N be a positive integer, and $U = (f, E, B)$, and

$$\begin{aligned} E^1_{N,k}(U) &= \sum_{|\alpha|+|\beta|\le N} \|w^k \partial^\alpha_x \partial^\beta_v f\|^2_{L^2_{x,v}} + \sum_{|\alpha|\le N} \|\partial^\alpha_x (E,B)\|^2_{L^2_x}, \\ H^1_{N,k}(U) &= \sum_{|\alpha|+|\beta|\le N} \|w^k \partial^\alpha_x \partial^\beta_v \mathrm{P}_1 f\|^2_{L^2_{x,v}} + \sum_{1\le|\alpha|\le N} \|\partial^\alpha_x (E,B)\|^2_{L^2_x} \\ &\quad + \sum_{|\alpha|\le N-1} \|\partial^\alpha_x \nabla_x \mathrm{P}_0 f\|^2_{L^2_{x,v}} + \|P_{\mathrm{d}} f\|^2_{L^2_{x,v}}, \\ D^1_{N,k}(U) &= \sum_{|\alpha|+|\beta|\le N} \|w^{\frac{1}{2}+k} \partial^\alpha_x \partial^\beta_v \mathrm{P}_1 f\|^2_{L^2_{x,v}} + \sum_{|\alpha|\le N-1} \|\partial^\alpha_x \nabla_x \mathrm{P}_0 f\|^2_{L^2_{x,v}} + \|P_{\mathrm{d}} f\|^2_{L^2_{x,v}} \\ &\quad + \sum_{1\le|\alpha|\le N-1} \|\partial^\alpha_x E\|^2_{L^2_x} + \sum_{2\le|\alpha|\le N-1} \|\partial^\alpha_x B\|^2_{L^2_x} \end{aligned}$$

for $k \geq 0$. For brevity, we write $E^1_N(U) = E^1_{N,0}(U)$, $H^1_N(U) = H^1_{N,0}(U)$ and $D^1_N(U) = D^1_{N,0}(U)$ for $k = 0$.

Applying an argument similar to that used to derive (6.10)–(6.12) and making use of the system (2.30)–(2.33), we can obtain a compressible Navier–Stokes–Maxwell type equations with inhomogeneous terms for the macroscopic density, momentum, and energy $(n, m, q) =: ((f, \chi_0), (f, v\chi_0), (f, \chi_4))$ and E, B as follows:

$$\partial_t n + \mathrm{div}_x m = 0, \tag{6.80}$$

(6.81)
$$\partial_t m + \partial_t R_1 + \nabla_x n + \sqrt{\frac{2}{3}}\nabla_x q - E = \kappa_1\left(\Delta_x m + \frac{1}{3}\nabla_x \mathrm{div}_x m\right) + nE + m \times B + R_2,$$

$$\partial_t q + \partial_t R_3 + \sqrt{\frac{2}{3}}\mathrm{div}_x m = \kappa_2 \Delta_x q + \sqrt{\frac{2}{3}} E \cdot m + R_4, \tag{6.82}$$

$$\partial_t E = \nabla_x \times B - m, \tag{6.83}$$

$$\partial_t B = -\nabla_x \times E, \tag{6.84}$$

where the viscosity and heat conductivity coefficients $\kappa_1, \kappa_2 > 0$ and the remainder terms R_1, R_2, R_3, R_4 are defined by

$$\begin{aligned}
&\kappa_1 = -(L^{-1}\mathrm{P}_1(v_1\chi_2), v_1\chi_2), \quad \kappa_2 = -(L^{-1}\mathrm{P}_1(v_1\chi_4), v_1\chi_4),\\
&R_1 = (v \cdot \nabla_x L^{-1}\mathrm{P}_1 f, v\sqrt{M}), \quad R_2 = -(v \cdot \nabla_x L^{-1}(\mathrm{P}_1(v \cdot \nabla_x \mathrm{P}_1 f) - \mathrm{P}_1 G), v\sqrt{M}),\\
&R_3 = (v \cdot \nabla_x L^{-1}\mathrm{P}_1 f, \chi_4), \quad R_4 = -(v \cdot \nabla_x L^{-1}(\mathrm{P}_1(v \cdot \nabla_x \mathrm{P}_1 f) - \mathrm{P}_1 G), \chi_4).
\end{aligned}$$

Here,

$$G = \frac{1}{2}(v \cdot E)f - (E + v \times B) \cdot \nabla_v f + \Gamma(f, f). \tag{6.85}$$

Similar to section 5.1, we have the energy estimates of the one-species VMB system (2.30)–(2.34) as follows.

Lemma 6.8 (macroscopic dissipation). *Let (n, m, q, E, B) be the strong solutions to* (6.80)–(6.84). *Then, there are two constants $s_0, s_1 > 0$ such that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\le|\alpha|\le N-1} s_0\left(\|\partial^\alpha_x(n,m,q,E,B)\|^2_{L^2_x} + 2\int_{\mathbb{R}^3}\partial^\alpha_x R_1 \partial^\alpha_x m dx + 2\int_{\mathbb{R}^3}\partial^\alpha_x R_3 \partial^\alpha_x q dx\right)\\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\le|\alpha|\le N-1} s_1\int_{\mathbb{R}^3}\partial^\alpha_x m \partial^\alpha_x \nabla_x n dx - \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+1\le|\alpha|\le N-1} 8\int_{\mathbb{R}^3}\partial^\alpha_x m \partial^\alpha_x E dx\\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+1\le|\alpha|\le N-2} 2\int_{\mathbb{R}^3}\partial^\alpha_x E \partial^\alpha_x(\nabla_x \times B)dx - \frac{\mathrm{d}}{\mathrm{d}t}s_0\sqrt{\frac{2}{3}}\delta_{k0}\int_{\mathbb{R}^3} m^2 q dx\\
&\quad + \sum_{k\le|\alpha|\le N-1}(\|\partial^\alpha_x \nabla_x(n,m,q)\|^2_{L^2_x} + \|\partial^\alpha_x n\|^2_{L^2_x}) + \sum_{k+1\le|\alpha|\le N-1}\|\partial^\alpha_x E\|^2_{L^2_x}\\
&\quad + \sum_{k+2\le|\alpha|\le N-1}\|\partial^\alpha_x B\|^2_{L^2_x}\\
&\qquad \le C\left(\sqrt{E^1_N(U)} + E^1_N(U)\right)D^1_N(U) + C\sum_{k\le|\alpha|\le N}\|\partial^\alpha_x \mathrm{P}_1 f\|^2_{L^2_{x,v}},
\end{aligned}$$

with $0 \le k \le N-3$ *and* $\delta_{k0} = 1$ *if* $k = 0$ *and* $\delta_{k0} = 0$ *if* $k \ne 0$.

LEMMA 6.9 (microscopic dissipation). *Let* $N \ge 4$ *and let* (f, E, B) *be a strong solution to VMB system* (2.30)–(2.34). *Then, there are constants* $p_k > 0$, $1 \le k \le N$, *such that*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|f\|^2_{L^2_{x,v}} + \|(E,B)\|^2_{L^2_x} - \sqrt{\frac{2}{3}}\int_{\mathbb{R}^3} m^2 q dx\right) + \mu\|w^{\frac12}\mathrm{P}_1 f\|^2_{L^2_{x,v}} \\
&\qquad \le C\left(\sqrt{E^1_N(U)} + E^1_N(U)\right) D^1_N(U), \\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le N}(\|\partial^\alpha_x f\|^2_{L^2_{x,v}} + \|\partial^\alpha_x (E,B)\|^2_{L^2_x}) + \mu\sum_{1\le|\alpha|\le N}\|w^{\frac12}\partial^\alpha_x \mathrm{P}_1 f\|^2_{L^2_{x,v}} \\
&\qquad \le C\sqrt{E^1_N(U)}D^1_N(U), \\
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le k\le N} p_k \sum_{\substack{|\beta|=k\\ |\alpha|+|\beta|\le N}}\|\partial^\alpha_x\partial^\beta_v \mathrm{P}_1 f\|^2_{L^2_{x,v}} + \mu\sum_{1\le k\le N} p_k \sum_{\substack{|\beta|=k\\ |\alpha|+|\beta|\le N}}\|w^{\frac12}\partial^\alpha_x\partial^\beta_v \mathrm{P}_1 f\|^2_{L^2_{x,v}} \\
&\le C\sum_{|\alpha|\le N-1}\|\partial^\alpha_x \nabla_x f\|^2_{L^2_{x,v}} + C\sqrt{E^1_N(U)}D^1_N(U).
\end{aligned}$$

LEMMA 6.10. *Let* $N \ge 4$. *Then, there are two equivalent energy functionals* $E^{1,f}_N(\cdot) \sim E^1_N(\cdot)$, $H^{1,f}_N(\cdot) \sim H^1_N(\cdot)$ *such that the following holds. If* $E^1_N(U_0)$ *is sufficiently small, then the Cauchy problem* (2.30)–(2.34) *of the one-species VMB system admits a unique global solution* $U = (f, E, B)$ *satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{1,f}_N(U)(t) + \mu D^1_N(U)(t) \le 0, \tag{6.86}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}H^{1,f}_N(U)(t) + \mu D^1_N(U)(t) \le C\|\nabla_x(n,m,q)\|^2_{L^2_x} + \|\nabla_x E\|^2_{L^2_x} + \|\nabla^2_x B\|^2_{L^2_x}. \tag{6.87}$$

LEMMA 6.11. *Let* $N \ge 4$. *There are the equivalent energy functionals* $E^{1,f}_{N,1}(\cdot) \sim E^1_{N,1}(\cdot)$, $H^{1,f}_{N,1}(\cdot) \sim H^1_{N,1}(\cdot)$ *such that if* $E^1_{N,1}(U_0)$ *is sufficiently small, then the solution* $U = (f, E, B)$ *to the one-species VMB system* (2.30)–(2.34) *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}E^{1,f}_{N,1}(U)(t) + \mu D^1_{N,1}(U)(t) \le 0, \tag{6.88}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}H^{1,f}_{N,1}(U)(t) + \mu D^1_{N,1}(U)(t) \le C\|\nabla_x(n,m,q)\|^2_{L^2_x} + \|\nabla_x E\|^2_{L^2_x} + \|\nabla^2_x B\|^2_{L^2_x}. \tag{6.89}$$

With these energy estimates, we can prove Theorem 2.11 for the nonlinear one-species VMB system (2.30)–(2.34).

Proof of Theorem 2.11. Let f, E, B be a solution to the problem (2.30)–(2.34) for $t > 0$. We can represent its solution in terms of the semigroup $e^{t\mathbb{A}_2}$ by

$$(f, E, B)(t) = e^{t\mathbb{A}_2}(f_0, E_0, B_0) + \int_0^t e^{(t-s)\mathbb{A}_2}(G, 0, 0)(s)ds, \tag{6.90}$$

where the nonlinear term G is given by (6.85). Define a functional $Q(t)$ for any $t > 0$ as

$$Q(t) = \sup_{0\le s\le t}\sum_{k=0}^{1}\Big\{(1+s)^{1+\frac{k}{4}}\|\partial^k_x(f(s), \sqrt{M})\|_{L^2_x} + (1+s)^{\frac58+\frac{k}{4}}\|\partial^k_x(f(s), v\sqrt{M})\|_{L^2_x}$$

$$+ (1+s)^{\frac{3}{4}+\frac{k}{2}}\|\partial_x^k(f(s),\chi_4)\|_{L^2_x} + (1+s)^{\frac{7}{8}+\frac{k}{4}}\|\partial_x^k \mathrm{P}_1 f(s)\|_{L^2_{x,v}}$$
$$+ (1+s)^{\frac{3}{4}+\frac{k}{4}}\ln(1+t)\|\partial_x^k E(s)\|_{L^2_x} + (1+s)^{\frac{3}{8}+\frac{k}{4}}\|\partial_x^k B(s)\|_{L^2_x}$$
$$+ (1+s)^{\frac{5}{8}}(\|\mathrm{P}_1 f(s)\|_{H^N_w} + \|\nabla_x \mathrm{P}_0 f(s)\|_{L^2(\mathbb{R}^3_v,H^{N-1}_x)} + \|\nabla_x(E,B)(s)\|_{H^{N-1}_x})\Big\}.$$

In the case of $\nabla_x\cdot E_0 = (f_0,\sqrt{M})$ and $B_0 = 0$, we can obtain by (5.117)–(5.120) and (5.99)–(5.100) that

$$\|\partial_x^\alpha(f(t),\chi_j)\|_{L^2_x} \le C[(1+t)^{-\frac{3}{4}-\frac{k}{2}} + (1+t)^{-\frac{7}{8}-\frac{k}{4}}](\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1})$$
$$+ C(1+t)^{-m-\frac{1}{2}}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},\quad j=1,2,3, \tag{6.91}$$

$$\|\partial_x^\alpha(f(t),\chi_4)\|_{L^2_x} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}(f_0,\chi_4)\|_{L^1_x} + \|\partial_x^{\alpha'}\nabla_x U_0\|_{Z^1}), \tag{6.92}$$

$$\|\partial_x^\alpha \mathrm{P}_1 f(t)\|_{L^2_{x,v}} \le C(1+t)^{-\frac{9}{8}-\frac{k}{4}}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1})$$
$$+ C(1+t)^{-m-\frac{1}{2}}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2}, \tag{6.93}$$

$$\|\partial_x^\alpha B(t)\|_{L^2_x} \le C(1+t)^{-\frac{5}{8}-\frac{k}{4}}(\|\partial_x^\alpha U_0\|_{Z^2} + \|\partial_x^{\alpha'}U_0\|_{Z^1}) \tag{6.94}$$
$$+ +C(1+t)^{-m}\|\nabla_x^m\partial_x^\alpha U_0\|_{Z^2},$$

where $(f,E,B) = e^{t\mathbb{A}_2}U_0$ with $U_0 = (f_0,E_0,B_0)$, $\alpha'\le\alpha$, and $k = |\alpha-\alpha'|$.

By Lemma 6.1, we can estimate the nonlinear term $G(s)$ for $0\le s\le t$ in terms of $Q(t)$ as

$$\|G(s)\|_{L^2_{x,v}} \le C\{\|wf\|_{L^{2,3}}\|f\|_{L^{2,6}} + \|E\|_{L^3_x}(\|wf\|_{L^{2,6}} + \|\nabla_v f\|_{L^{2,6}}) + \|B\|_{L^6_x}\|w\nabla_v f\|_{L^{2,3}}\} \le C(1+s)^{-5/4}Q(t)^2, \tag{6.95}$$

$$\|G(s)\|_{L^{2,1}} \le C\{\|f\|_{L^2_{x,v}}\|wf\|_{L^2_{x,v}} + \|E\|_{L^3_x}(\|wf\|_{L^2_{x,v}} + \|\nabla_v f\|_{L^2_{x,v}}) + \|B\|_{L^2_x}\|w\nabla_v f\|_{L^2_{x,v}}\} \le C(1+s)^{-1}Q(t)^2, \tag{6.96}$$

and similarly

$$\|G(s)\|_{L^2_v(H^k_x)} \le C(1+s)^{-5/4}Q(t)^2 \tag{6.97}$$

for $1\le k\le N-1$. Then, it follows from (2.61), (6.95), and (6.96) that

$$\|\nabla_x^k(f(t),\sqrt{M})\|_{L^2_x} \le C(1+t)^{-\frac{5}{4}-\frac{k}{2}}(\|\nabla_x^k U_0\|_{Z^2} + \|U_0\|_{Z^1})$$
$$+ C\int_0^t(1+t-s)^{-\frac{5}{4}}(\|\nabla_x^k G(s)\|_{L^2_{x,v}} + \|\nabla_x^k G(s)\|_{L^{2,1}})ds$$
$$\le C\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}} + C(1+t)^{-1-\frac{k}{4}}Q(t)^2,\quad k=0,1. \tag{6.98}$$

Similarly, in terms of (2.62) and (6.91) we have

$$\|\nabla_x^k(f(t),v\sqrt{M})\|_{L^2_x} \le C(1+t)^{-\frac{5}{8}-\frac{k}{4}}(\|\nabla_x^k U_0\|_{Z^2} + \|U_0\|_{Z^1} + \|\nabla_x^2 U_0\|_{Z^2})$$
$$+ C\int_0^t(1+t-s)^{-\frac{3}{4}-\frac{3k}{8}}(\|\nabla_x^k G(s)\|_{L^2_{x,v}} + \|\nabla_x^k G(s)\|_{L^{2,1}})ds$$

$$
\begin{aligned}
&+ C\int_0^t (1+t-s)^{-2}\|\nabla_x^{2+k}G\|_{L^2_{x,v}}ds\\
&\le C\delta_0(1+t)^{-\frac{5}{8}-\frac{k}{4}} + C(1+t)^{-\frac{5}{8}-\frac{k}{4}}Q(t)^2, \quad k=0,1.
\end{aligned} \tag{6.99}
$$

In terms of (2.63) and (6.92), we can estimate the macroscopic energy $(f(t),\chi_4)$ as

$$
\begin{aligned}
\|\nabla_x^k(f(t),\chi_4)\|_{L^2_x} &\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^kU_0\|_{Z^2}+\|U_0\|_{Z^1})\\
&\quad + C\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^kG(s)\|_{L^2_{x,v}}\\
&\quad + \|(G(s),\chi_4)\|_{L^1_x} + \|\nabla_xG(s)\|_{L^{2,1}})ds\\
&\le C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} + C(1+t)^{-\frac{3}{4}-\frac{k}{2}}Q(t)^2, \quad k=0,1.
\end{aligned} \tag{6.100}
$$

In addition, the microscopic part $\mathrm{P}_1f(t)$ can be estimated by (2.64) and (6.93) as follows:

$$
\begin{aligned}
\|\nabla_x^k\mathrm{P}_1f(t)\|_{L^2_{x,v}} &\le C(1+t)^{-\frac{7}{8}-\frac{k}{4}}(\|\nabla_x^kU_0\|_{Z^2}+\|U_0\|_{Z^1}+\|\nabla_x^2U_0\|_{Z^2})\\
&\quad + C\int_0^t (1+t-s)^{-\frac{9}{8}}(\|\nabla_x^kG(s)\|_{L^2_{x,v}} + \|\nabla_x^kG(s)\|_{L^{2,1}})ds\\
&\quad + C\int_0^t (1+t-s)^{-2}\|\nabla_x^{2+k}G\|_{L^2_{x,v}}ds\\
&\le C\delta_0(1+t)^{-\frac{7}{8}-\frac{k}{4}} + C(1+t)^{-\frac{7}{8}-\frac{k}{4}}Q(t)^2, \quad k=0,1.
\end{aligned} \tag{6.101}
$$

Moreover, the electricity potential E is bounded by

$$
\begin{aligned}
\|\nabla_x^kE(t)\|_{L^2_x} &\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^kU_0\|_{Z^2}+\|U_0\|_{Z^1}+\|\nabla_x^2U_0\|_{Z^2})\\
&\quad + C\int_0^t (1+t-s)^{-\frac{3}{4}-\frac{k}{2}}(\|\nabla_x^kG(s)\|_{L^2_{x,v}} + \|G(s)\|_{L^{2,1}})ds\\
&\quad + C\int_0^t (1+t-s)^{-2}\|\nabla_x^{2+k}G\|_{L^2_{x,v}}ds\\
&\le C\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} + C(1+t)^{-\frac{3}{4}-\frac{k}{4}}\ln(1+t)Q(t)^2
\end{aligned} \tag{6.102}
$$

for $k=0,1$. And the magnetic potential B is bounded by

$$
\begin{aligned}
\|\nabla_x^kB(t)\|_{L^2_x} &\le C(1+t)^{-\frac{3}{8}-\frac{k}{4}}(\|\nabla_x^kU_0\|_{Z^2}+\|U_0\|_{Z^1}+\|\nabla_x^{k+1}U_0\|_{Z^2})\\
&\quad + C\int_0^t (1+t-s)^{-\frac{5}{8}-\frac{k}{4}}(\|G(s)\|_{L^2_{x,v}} + \|G(s)\|_{L^{2,1}})ds\\
&\quad + C\int_0^t (1+t-s)^{-1}\|\nabla_x^{k+1}G\|_{L^2_{x,v}}ds\\
&\le C\delta_0(1+t)^{-\frac{3}{8}-\frac{k}{4}} + C(1+t)^{-\frac{3}{8}-\frac{k}{4}}Q(t)^2
\end{aligned} \tag{6.103}
$$

for $k=0,1,2$.

Next, we estimate the higher order terms as below. By a similar argument as in Theorem 1.3 in [7], we have

$$
E^1_{k,1}(U)(t) \le C(1+t)^{-3/4}(E^1_{k+1,1}(U_0) + (\delta_0+Q(t)^2)^2) \tag{6.104}
$$

for any integer $k \geq 4$. Then

$$\begin{aligned}\|\nabla_x^N(E,B)(t)\|_{L^2_x} \leq & C(1+t)^{-\frac{5}{8}}(\|\nabla_x^N U_0\|_{Z^2}+\|U_0\|_{Z^1}+\|\nabla_x^{N+1}U_0\|_{Z^2}) \\ & + C\int_0^t (1+t-s)^{-\frac{7}{8}}(\|\nabla_x^N G(s)\|_{L^2_{x,v}}+\|G(s)\|_{L^{2,1}})ds \\ & + C\int_0^t (1+t-s)^{-1}\|\nabla_x^{N+1}G(s)\|_{L^2_{x,v}}ds \\ \leq & C\delta_0(1+t)^{-\frac{5}{8}}+C(1+t)^{-\frac{5}{8}}(\delta_0+Q(t)^2)^2, \end{aligned} \tag{6.105}$$

where we have used

$$\|G(s)\|_{L^2_v(H^{N+1}_x)} \leq E^1_{N+2,1}(U)(s) \leq C(E^1_{N+3,1}(U_0)+(\delta_0+Q(t)^2)^2)(1+s)^{-\frac{3}{4}}.$$

By (6.89), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}H^{1,f}_{N,1}(U)(t)+d_1\mu H^{1,f}_{N,1}(U)(t) \\ &\leq C(\|\nabla_x(n,m,q,E,B)(t)\|^2_{L^2_x}+\|\nabla_x^2 B(t)\|^2_{L^2_x})+C\sum_{|\alpha|=N}\|\partial_x^\alpha(E,B)(t)\|^2_{L^2_x},\end{aligned} \tag{6.106}$$

and (6.105) leads to

$$\begin{aligned}H^{1,f}_{N,1}(U)(t) \leq & e^{-d_1\mu t}H^{1,f}_{N,1}(U_0)+C\int_0^t e^{-d_1\mu(t-s)}\sum_{|\alpha|=N}\|\partial_x^\alpha(E,B)(s)\|^2_{L^2_x}ds \\ & + C\int_0^t e^{-d_1\mu(t-s)}(\|\nabla_x(n,m,q,E,B)(s)\|^2_{L^2_x}+\|\nabla_x^2B(s)\|^2_{L^2_x})ds \\ \leq & C(1+t)^{-\frac{5}{4}}(\delta_0+Q(t)^2+(\delta_0+Q(t)^2)^2. \end{aligned} \tag{6.107}$$

By summing (6.98)–(6.103) and (6.107), we have

$$Q(t) \leq C(\delta_0+Q(t)^2)+C(\delta_0+Q(t)^2)^2,$$

from which (2.72)–(2.79) can be verified provided that $\delta_0 > 0$ is small enough. Similarly, as Theorem 2.6, we can prove (2.80)–(2.84). ∎

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