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PROBABILITY MEASURES WITH FINITE MOMENTS AND THE HOMOGENEOUS BOLTZMANN EQUATION*

YONG-KUM CHO†, YOSHINORI MORIMOTO‡, SHUAIKUN WANG§, AND TONG YANG§

Abstract. We characterize the class of probability measures possessing finite moments of an arbitrary positive order in terms of the symmetric difference operators of their Fourier transforms. As an application, we prove the continuity of probability densities associated with measure-valued solutions to the Cauchy problem for the homogeneous Boltzmann equation with Maxwellian molecules.

Key words. Boltzmann equation, characteristic function, Fourier transform, moment, probability measure, symmetric difference operator

AMS subject classifications. Primary, 35Q20, 76P05; Secondary, 35H20, 82B40, 82C40

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1. Introduction.

The spatially homogeneous Boltzmann equation states

$$\partial_t f(t,v) = Q(f,f)(t,v) \quad \text{for} \quad t>0,\, v\in\mathbb{R}^3, \tag{1.1}$$

which arises as a physical model for describing the behavior of a dilute gas by its density f under the simplified assumption that it depend only on the velocity v and time t. Here Q stands for the collision operator defined by

$$Q(f,g)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left[f(v')g(v'_*) - f(v)g(v_*)\right] d\sigma dv_* \tag{1.2}$$

for scalar-valued functions f, g on $\mathbb{R}^3$, where

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\,\sigma, \\ v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\,\sigma, \\ \mathbf{k} \;= \dfrac{v-v_*}{|v-v_*|}, \end{cases} \tag{1.3}$$

the collision kernel $B = B(|v-v_*|, \mathbf{k}\cdot\sigma)$ is a nonnegative measurable function, and $d\sigma$ denotes the area measure on the unit sphere $\mathbb{S}^2$.

Each pair (v', v'_*) represents the postcollision velocities of two gas molecules colliding with velocities (v, v_*) under binary and elastic collision dynamics. The collision

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†Department of Mathematics, Chung-Ang University, Seoul 156-756, Korea (ykcho@cau.ac.kr). The research of this author was supported by National Research Foundation of Korea grants funded by the Korean Government (20150238, 20150863).
‡Corresponding author. Graduate School of Human and Environmental Studies, Kyoto University, Kyoto, 606-8501, Japan (morimoto@math.h.kyoto-u.ac.jp). The research of this author was supported by Grant-in-Aid for Scientific Research 25400160, Japan Society for the Promotion of Science.
§Department of Mathematics, City University of Hong Kong, Hong Kong, People's Republic of China (shuaiwang4-c@my.cityu.edu.hk, matyang@cityu.edu.hk). The research of the fourth author was supported by the General Research Fund of Hong Kong, City U 11303614.

kernel B represents a model of collision dynamics which specifies the types of interaction potentials and impact parameters in terms of the relative velocity $|v - v_*|$ and the deviation angle $\theta \in [0, \pi]$ defined by $\cos\theta = \mathbf{k} \cdot \sigma$.

Of main interest are the inverse-power interacting potential models which specify the collision kernels in the form

$$B = |v - v_*|^\gamma \, b(\cos\theta) \tag{1.4}$$

with $\gamma > -3$. It is classified as soft potential if $-3 < \gamma < 0$, Maxwellian if $\gamma = 0$, and hard potential if $\gamma > 0$. The angular part b, defined implicitly, is known to be continuous or at least bounded away from $\theta = 0$ but develops a nonintegrable singularity near $\theta = 0$ in the sense that

$$b(\cos\theta) \, \sin\theta \, \sim \, \theta^{-1-\nu} \quad \text{for some } 0 < \nu < 2 \text{ as } \theta \to 0. \tag{1.5}$$

In this paper we shall only consider the Boltzmann equation (1.1) with the Maxwellian kernel. As is customary, replacing $b(\cos\theta)$ by

$$[b(\cos\theta) + b(\cos(\pi - \theta))] \, \mathbf{1}_{[0,\pi/2]}(\theta),$$

we shall assume $b(\cos\theta)$ is supported on $[0, \pi/2]$ (see [1, 16]). In addition, it will be assumed to satisfy the weak integrability condition

$$\int_0^{\pi/2} b(\cos\theta) \sin^{\alpha_0}\left(\frac{\theta}{2}\right) \sin\theta \, d\theta < \infty \quad \text{for some} \quad 0 < \alpha_0 < 2\,, \tag{1.6}$$

which is verified by the singular kernel b described as in (1.5) when $0 < \nu < \alpha_0$.

Our primary concern here is to study measure-valued solutions to the Cauchy problem for the Boltzmann equation (1.1) with the Maxwellian kernel and the initial data possessing finite (absolute) moments.

Let $P_0(\mathbb{R}^d)$ denote the set of all probability measures defined on $\mathbb{R}^d$. As is standard, we classify $P_0(\mathbb{R}^d)$ by the order of finite moments: For $k \in \mathbb{N}$ and $\alpha \in [0, 2)$ with $k + \alpha > 1$, we denote by $P_{2k-2+\alpha}(\mathbb{R}^d)$ the set of all probability measures $F \in P_0(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} |v|^{2k-2+\alpha} dF(v) < \infty \tag{1.7}$$

and the first-order moment vanishing condition

$$\int_{\mathbb{R}^d} v_j \, dF(v) = 0, \;\; j = 1, \ldots, d, \tag{1.8}$$

whenever $2k - 2 + \alpha > 1$ (either $k = 1, \alpha \in (1, 2)$ or $k \geq 2, \alpha \in [0, 2)$).

We recall that the Fourier transform of $F \in P_0(\mathbb{R}^d)$ is defined by

$$\widehat{F}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot v} dF(v) \qquad (\xi \in \mathbb{R}^d),$$

referred to as the characteristic function of F. Due to Bobylev's formula for the Fourier transform of the collision term (see [3, 4]), the Cauchy problem for the Boltzmann equation (1.1) with the Maxwellian kernel and the initial datum

$$f(0, v) = F_0 \in P_{2k-2+\alpha}(\mathbb{R}^3) \tag{1.9}$$

can be reformulated in terms of the characteristic functions of the unknown family of probability measures $(F_t)_{t>0}$ as follows:

$$\begin{cases} \partial_t\varphi(t,\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(t,\xi^+)\varphi(t,\xi^-) - \varphi(t,\xi)\right] d\sigma, \\ \varphi(0,\xi) = \varphi_0(\xi), \end{cases} \tag{1.10}$$

where $\varphi(t,\xi) = \widehat{F_t}(\xi)$, $\varphi_0(\xi) = \widehat{F_0}(\xi)$ for each $t > 0$, $\xi \in \mathbb{R}^3$, and

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}.$$

In studying the Cauchy problem (1.10) with an initial datum given in (1.9), it is essential to gain insight into the image spaces $\mathcal{F}\left(P_{2k-2+\alpha}(\mathbb{R}^d)\right)$, where $\mathcal{F}$ stands for the Fourier transform operator, and construct appropriate complete metrics on these spaces. In what follows, we shall denote by $\mathcal{K}(\mathbb{R}^d)$ the space of characteristic functions on $\mathbb{R}^d$, that is, $\mathcal{K}(\mathbb{R}^d) = \mathcal{F}\left(P_0(\mathbb{R}^d)\right)$.

Inspired by a series of earlier work by Toscani and coauthors [6, 8, 15], Cannone and Karch [5] introduced the subclasses of $\mathcal{K}(\mathbb{R}^d)$ defined as

$$\mathcal{K}^\alpha(\mathbb{R}^d) = \left\{\varphi \in \mathcal{K}(\mathbb{R}^d) \,:\, \sup_{\xi\in\mathbb{R}^d} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha} < \infty\right\} \tag{1.11}$$

which are shown to be monotone in $\alpha > 0$ in the sense

$$\{1\} \subset \mathcal{K}^\alpha(\mathbb{R}^d) \subset \mathcal{K}^\beta(\mathbb{R}^d) \subset \mathcal{K}(\mathbb{R}^d) \quad \text{for} \quad 0 < \beta \le \alpha \le 2$$

and $\mathcal{K}^\alpha(\mathbb{R}^d) = \{1\}$ for all $\alpha > 2$. By making use of the fact that each $\mathcal{K}^\alpha(\mathbb{R}^d)$ is complete with respect to the induced metric

$$\|\varphi - \tilde{\varphi}\|_\alpha = \sup_{\xi\in\mathbb{R}^d} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha} \tag{1.12}$$

(see [5, Proposition 3.10]), Cannone and Karch established the global existence and uniqueness of measure-valued solutions to the Cauchy problem (1.10) in the space $\mathcal{K}^\alpha(\mathbb{R}^3)$ under the kernel assumption (1.6) (see also [10]).

The space $\mathcal{K}^\alpha(\mathbb{R}^d)$ arises in connection with the Fourier image of $P_\alpha(\mathbb{R}^d)$. Indeed, Cannone and Karch proved $\mathcal{F}\left(P_\alpha(\mathbb{R}^d)\right) \subset \mathcal{K}^\alpha(\mathbb{R}^d)$ and observed that the inclusion is proper except for $\alpha = 2$. As an illustration, the characteristic function of α-stable symmetric Lévy process $\varphi_\alpha(\xi) = e^{-|\xi|^\alpha}$ with $\alpha \in (0,2)$ belongs to $\mathcal{K}^\alpha(\mathbb{R}^d)$, but $f_\alpha(v) = \mathcal{F}^{-1}(\varphi_\alpha)(v)$ does not belong to $P_\alpha(\mathbb{R}^d)$ (see [5, Remark 3.16]).

A precise characterization of $\mathcal{F}\left(P_\alpha(\mathbb{R}^d)\right)$ was discovered recently by Morimoto, Wang, and Yang [12, 13], which may be summarized as follows. Put

$$\widetilde{\mathcal{M}}^\alpha(\mathbb{R}^d) = \left\{\varphi \in \mathcal{K}(\mathbb{R}^d) : \|\operatorname{Re}\varphi - 1\|_{\mathcal{M}^\alpha} + \|\varphi - 1\|_\alpha < \infty\right\}, \tag{1.13}$$

where $\operatorname{Re}\varphi$ stands for the real part of φ and

$$\|\varphi - 1\|_\alpha = \sup_{\xi\in\mathbb{R}^d} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha},$$

$$\|\varphi - 1\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^d} \frac{|\varphi(\xi) - 1|}{|\xi|^{d+\alpha}}\, d\xi. \tag{1.14}$$

Then $\widetilde{\mathcal{M}}^\alpha(\mathbb{R}^d) = \mathcal{F}\left(P_\alpha(\mathbb{R}^d)\right)$ in the range $\alpha \in (0,2)$, and it becomes a complete metric space under each of the following metrics:

$$dis_{\alpha,\beta,\epsilon}(\varphi,\tilde{\varphi}) = \|\mathrm{Re}\ \varphi - \mathrm{Re}\ \tilde{\varphi}\|_{\mathcal{M}^\alpha} + \|\varphi - \tilde{\varphi}\|_\beta + \|\varphi - \tilde{\varphi}\|_\beta^\epsilon$$

with $0 < \beta < \alpha < 2,\ 0 < \epsilon < 1$, where

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^d} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^{d+\alpha}}\, d\xi, \tag{1.15}$$

and $\|\varphi - \tilde{\varphi}\|_\beta$ stands for the same metric defined as in (1.12).

As an application, the aforementioned authors established the global well-posedness of the Cauchy problem (1.10) in the space $\widetilde{\mathcal{M}}^\alpha(\mathbb{R}^3)$. We also remark that the family of metrics (1.15) in the one-dimensional case was introduced earlier by Baringhaus and Grübel [2] in their probabilistic work on random convex combinations.

In this paper we aim at characterizing the Fourier images of $P_{2k-2+\alpha}(\mathbb{R}^d)$ with an arbitrary positive integer k. In [7], Cho characterized the Fourier images of probability measures having finite moments of an arbitrary order, without the vanishing moment condition described as in (1.8), in terms of the forward difference operator and its iteration. As a modification of these results, we introduce a new set of characterizations as follows.

Let Δ denote the symmetric central difference operator which acts on each characteristic function φ on $\mathbb{R}^d$, $\varphi = \widehat{F}$ with $F \in P_0(\mathbb{R}^d)$, by the rule

$$\begin{aligned}\Delta\varphi(\xi) &= \frac{2\varphi(0) - \varphi(\xi) - \varphi(-\xi)}{4}\\ &= \frac{1 - \mathrm{Re}\ \varphi(\xi)}{2}\\ &= \int_{\mathbb{R}^d} \sin^2\left(\frac{v\cdot\xi}{2}\right) dF(v).\end{aligned} \tag{1.16}$$

We iterate Δ once with the increment ξ to define

$$\begin{aligned}\Delta^2\varphi(\xi) &= \frac{6\varphi(0) - 4\varphi(\xi) - 4\varphi(-\xi) + \varphi(2\xi) + \varphi(-2\xi)}{16}\\ &= \frac{3 - 4\,\mathrm{Re}\ \varphi(\xi) + \mathrm{Re}\ \varphi(2\xi)}{8}\\ &= \int_{\mathbb{R}^d} \sin^4\left(\frac{v\cdot\xi}{2}\right) dF(v).\end{aligned}$$

Repeating this procedure, we write Δ^k for the kth iterate of Δ given by

$$\begin{aligned}\Delta^k\varphi(\xi) &= \frac{1}{2}\sum_{j=0}^{k} c_{k,j}\ [\varphi(j\xi) + \varphi(-j\xi)]\\ &= \sum_{j=0}^{k} c_{k,j}\ \mathrm{Re}\ \varphi(j\xi)\\ &= \int_{\mathbb{R}^d} \sin^{2k}\left(\frac{v\cdot\xi}{2}\right) dF(v).\end{aligned} \tag{1.17}$$

It follows from the definition of the Fourier transform that the $c_{k,j}$ coincide with the coefficients of the trigonometric identity

$$\sin^{2k}\left(\frac{x}{2}\right) = \sum_{j=0}^{k} c_{k,j}\, \cos(jx) \qquad (x \in \mathbb{R}), \tag{1.18}$$

and an inductive calculation gives

$$c_{k,j} = \begin{cases} 2^{-2k}\binom{2k}{k} & \text{for} \quad j = 0, \\ (-1)^j 2^{-2k+1}\binom{2k}{k+j} & \text{for} \quad j = 1, \ldots, k. \end{cases} \tag{1.19}$$

We are now in a position to introduce the spaces of characteristic functions in question. For each positive integer k and $\alpha \in [0, 2)$ with $k + \alpha > 1$, we put $\alpha^* = \alpha$ if $k = 1$, $\alpha^* = 2$ if $k \geq 2$, and define

$$\mathcal{M}_k^\alpha(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{K}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}}\, d\xi + \|\varphi - 1\|_{\alpha^*} < \infty \right\}. \tag{1.20}$$

For each $0 < \beta < \alpha^*$, we also introduce the metric

$$\begin{aligned} dis_{k,\alpha,\beta}(\varphi, \tilde{\varphi}) &= \|\varphi - \tilde{\varphi}\|_{\mathcal{M}_k^\alpha} + \|\varphi - \tilde{\varphi}\|_\beta, \quad \text{where} \\ \|\varphi - \tilde{\varphi}\|_{\mathcal{M}_k^\alpha} &= \int_{\mathbb{R}^d} \frac{|\Delta^k \varphi(\xi) - \Delta^k \tilde{\varphi}(\xi)|}{|\xi|^{d+2k-2+\alpha}}\, d\xi. \end{aligned} \tag{1.21}$$

Concerning the image spaces of the $P_{2k-2+\alpha}(\mathbb{R}^d)$ under the Fourier transform and complete metric structures, we have the following result. As is customary, we shall write $\langle v \rangle = \sqrt{1 + |v|^2}$ for $v \in \mathbb{R}^d$ and use the notation $A \lesssim B$ to indicate the inequality $A \leq cB$ for a generic constant c.

THEOREM 1.1. *For each positive integer k and $\alpha \in [0, 2)$ with $k + \alpha > 1$,*

$$\mathcal{M}_k^\alpha(\mathbb{R}^d) = \mathcal{F}\left(P_{2k-2+\alpha}(\mathbb{R}^d)\right). \tag{1.22}$$

Moreover, the space $\mathcal{M}_k^\alpha(\mathbb{R}^d)$ is complete with respect to each metric $dis_{k,\alpha,\beta}$, and the condition $\lim_{n\to\infty} dis_{k,\alpha,\beta}(\varphi_n, \varphi) = 0$ with $\varphi_n, \varphi \in \mathcal{M}_k^\alpha(\mathbb{R}^d)$ implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \psi(v)\, dF_n(v) = \int_{\mathbb{R}^d} \psi(v)\, dF(v) \tag{1.23}$$

for every continuous function ψ on $\mathbb{R}^d$ satisfying $|\psi(v)| \lesssim \langle v \rangle^{2k-2+\alpha}$, where $F_n, F \in P_{2k-2+\alpha}(\mathbb{R}^d)$ is defined by $\widehat{F_n} = \varphi_n$, $\widehat{F} = \varphi$.

Remark 1.2.

(1) In the case $k = 1$, $0 < \alpha < 2$, the space $\mathcal{M}_k^\alpha(\mathbb{R}^d)$ coincides with $\widetilde{\mathcal{M}}^\alpha(\mathbb{R}^d)$, which is investigated in detail in the aforementioned work [12, 13]. Henceforth we shall focus on the case $k \geq 2$, $0 \leq \alpha < 2$.

(2) For a probability measure $F \in P_{2k-2+\alpha}(\mathbb{R})$, it is proven in [9] that

$$\int_0^\infty \left\{ 1 - \mathrm{Re}\ \varphi(t) + \sum_{j=1}^{k-1} \frac{\varphi^{(2j)}(0)}{(2k)!}\, t^{2j} \right\} \frac{dt}{t^{1+\alpha}} = C_{k,\alpha} \int_{-\infty}^\infty |x|^{2k-2+\alpha}\, dF(x)$$

for some constant $C_{k,\alpha} > 0$, where $\varphi = \widehat{F}$. However, this characterization is different from the one given in (1.20).

Concerning the Cauchy problem for the homogeneous Boltzmann equation with the Maxwellian kernel, we improve the previous results on the continuity of solutions established in [13]. As usual, $L^1_\alpha(\mathbb{R}^d)$ will denote the Lebesgue space of integrable functions on $\mathbb{R}^d$ with the weight $\langle v\rangle^\alpha$ for each $\alpha > 0$, and $H^\infty(\mathbb{R}^d)$ will denote the L^2-based Sobolev space of infinite order.

THEOREM 1.3. *Assume that b satisfies* (1.6). *Let k be a positive integer with $k \geq 2$, $\alpha \in [0,2)$, and $F_0 \in P_{2k-2+\alpha}(\mathbb{R}^3)$. Then there exists a unique measure valued solution $F_t \in C([0,\infty), P_{2k-2+\alpha}(\mathbb{R}^3))$ to the Cauchy problem* (1.1)–(1.9) *which preserves the energy and momentum for all time $t > 0$, that is,*

$$\int |v|^2 dF_t = \int |v|^2 dF_0, \quad \int v_j dF_t = 0 \quad \text{for } j = 1,2,3. \tag{1.24}$$

Furthermore, if b satisfies (1.5) *and if F_0 is not a single Dirac mass, then F_t admits the probability density $dF_t(v) = f(t,v)dv$ satisfying*

$$f \in C((0,\infty); L^1_{2k-2+\alpha}(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)).$$

Thanks to our new characterization of $P_{2k-2+\alpha}(\mathbb{R}^d)$ as stated in Theorem 1.1, Theorem 1.3 will be a simple corollary of the following Fourier version.

THEOREM 1.4. *Assume that b satisfies* (1.6) *and $0 < \alpha_0 < \beta < 2$. Let k be a positive integer with $k \geq 2$, $\alpha \in [0,2)$, and $\varphi_0 \in \mathcal{M}^\alpha_k(\mathbb{R}^3)$. Then there exists a unique classical solution $\varphi(t,\xi) \in C([0,\infty), \mathcal{M}^\alpha_k(\mathbb{R}^3))$ to the Cauchy problem* (1.10) *with the following time-continuity property: Given an arbitrary $T > 0$,*

$$||\varphi(t,\cdot) - \varphi(s,\cdot)||_\beta \leq |t-s| \cdot e^{\lambda_\beta T} ||\varphi_0 - 1||_\beta, \tag{1.25}$$

$$||\varphi(t,\cdot) - \varphi(s,\cdot)||_{\mathcal{M}^\alpha_k} \lesssim |t-s| \cdot \sup_{\tau\in[0,T]} \int \langle v\rangle^{2k-2+\alpha} dF_\tau \tag{1.26}$$

for all $s,t \in [0,T]$, where $\widehat{F}_t(\xi) = \varphi(t,\xi)$ and

$$\lambda_\beta = \int_0^{\pi/2} b(\cos\theta)\left[\sin^\beta\left(\frac{\theta}{2}\right) + \cos^\beta\left(\frac{\theta}{2}\right) - 1\right]\sin\theta d\theta. \tag{1.27}$$

Remark 1.5. The estimates (1.25) and (1.26) imply

$$dis_{k,\alpha,\beta}\left(\varphi(t,\cdot),\, \varphi(s,\cdot)\right) \leq C_{T,\varphi_0}|t-s|, \tag{1.28}$$

where the constant C_{T,φ_0} depends only on T and the initial datum.

2. Characterization of $P_{2k-2+\alpha}(\mathbb{R}^d)$.

Our proof of Theorem 1.1 is based on the following, which will also be applied frequently in other parts of our work.

LEMMA 2.1. *Let $k \in \mathbb{N}$, $\alpha \in [0,2)$ with $k+\alpha > 1$. Put*

$$c_{\alpha,d,M,k} = \int_{\{|\zeta|\leq M\}} \frac{\sin^{2k}(\mathbf{e}_1\cdot\zeta/2)}{|\zeta|^{d+2k-2+\alpha}} d\zeta \tag{2.1}$$

for each $M \in [1,\infty]$, where $\mathbf{e}_1 = (1,0,\ldots,0) \in \mathbb{R}^d$ and the integral on the right is taken over $\mathbb{R}^d$ in the limiting case $M = \infty$. If $\varphi \in \mathcal{M}^\alpha_k(\mathbb{R}^d)$ with $\varphi = \widehat{F}$ for a unique *probability measure F, then $F \in P_{2k-2+\alpha}(\mathbb{R}^d)$ and*

$$\int_{\{|v|\geq R\}} |v|^{2k-2+\alpha} dF(v) \leq \frac{1}{c_{\alpha,d,1,k}} \int_{\{|\xi|\leq 1/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi \,, \tag{2.2}$$

$$\int_{\mathbb{R}^d} |v|^{2k-2+\alpha} dF(v) \leq \frac{1}{c_{\alpha,d,\infty,k}} \int_{\mathbb{R}^d} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi \,, \tag{2.3}$$

where the first estimate holds for any $R > 0$.

Proof. It is evident that $0 < c_{\alpha,d,M,k} \leq c_{\alpha,d,\infty,k} < \infty$ for each $M \in [1,\infty]$ in the stated range of k, α. For $0 < \delta < 1 \leq M < \infty$ and $0 < R < \rho < \infty$, we use the defining relation (1.17) and interchange the order of integrations to find

$$\begin{aligned}\int_{\{|\xi|\leq M/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi &= \int_{\mathbb{R}^d} \left[\int_{\{|\xi|\leq M/R\}} \frac{\sin^{2k}(v\cdot\xi/2)}{|\xi|^{d+2k-2+\alpha}} d\xi \right] dF(v) \\ &\geq \int_{|v|\leq\rho} \left[\int_{\left\{\frac{\delta}{\rho}\leq|\xi|\leq\frac{M}{R}\right\}} \frac{\sin^{2k}(v\cdot\xi/2)}{|\xi|^{d+2k-2+\alpha}} d\xi \right] dF(v) \,.\end{aligned}$$

Changing variables $|v|\xi \to \zeta$ and using the rotation invariance, we note that the inner integral on the right is equal to

$$\begin{aligned}&|v|^{2k-2+\alpha} \int_{\left\{\frac{\delta|v|}{\rho}\leq|\zeta|\leq\frac{M|v|}{R}\right\}} \frac{\sin^{2k}(\mathbf{e}_1\cdot\zeta/2)}{|\zeta|^{d+2k-2+\alpha}} d\zeta \\ &\quad\geq |v|^{2k-2+\alpha} \mathbf{1}_{\{R\leq|v|\leq\rho\}} \int_{\{\delta\leq|\zeta|\leq M\}} \frac{\sin^{2k}(\mathbf{e}_1\cdot\zeta/2)}{|\zeta|^{d+2k-2+\alpha}} d\zeta \,,\end{aligned}$$

which implies

$$\begin{aligned}&\int_{\{|\xi|\leq M/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi \\ &\quad\geq \int_{\{R\leq|v|\leq\rho\}} |v|^{2k-2+\alpha} dF(v) \int_{\{\delta\leq|\zeta|\leq M\}} \frac{\sin^{2k}(\mathbf{e}_1\cdot\zeta/2)}{|\zeta|^{d+2k-2+\alpha}} d\zeta \,.\end{aligned}$$

Letting $\delta \downarrow 0$ and $\rho \to \infty$, we obtain

$$\int_{\{|\xi|\leq M/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi \geq c_{\alpha,d,M,k} \int_{\{|v|\geq R\}} |v|^{2k-2+\alpha} dF(v), \tag{2.4}$$

whence the first estimate (2.2) follows with the choice of $M = 1$. Letting $M \to \infty$ and $R \to 0$ in the estimate (2.4), we obtain (2.3).

It remains to verify that F satisfies the vanishing momentum condition (1.8) when $2k - 2 + \alpha > 1$. In this case, it follows from the definition that $\mathcal{M}_k^\alpha(\mathbb{R}^d) \subset \mathcal{K}^{\alpha^*}(\mathbb{R}^d)$ with $\alpha^* > 1$. By the same reasoning as presented in the proof of [13, Theorem 1.1], we see that $\varphi \in \mathcal{K}^{\alpha^*}(\mathbb{R}^d)$ with $\alpha^* > 1$ implies (1.8) for $F = \mathcal{F}^{-1}(\varphi)$. Indeed, let us assume $a = \int v dF \neq 0$ to the contrary. Since $\int \langle v\rangle^{\alpha^*} dF(v+a) < \infty$, we find $e^{i\xi\cdot a}\varphi(\xi) = \varphi_a(\xi) = \mathcal{F}(F(\cdot + a)) \in \mathcal{K}^{\alpha^*}(\mathbb{R}^d)$ by [5, Lemma 3.15]. Therefore we have

$$\begin{aligned}\sup_\xi \frac{|1-e^{-i\xi\cdot a}|}{|\xi|^{\alpha^*}} &\leq \sup_\xi \frac{|1-\varphi|}{|\xi|^{\alpha^*}} + \sup_\xi \frac{|\varphi - e^{-i\xi\cdot a}|}{|\xi|^{\alpha^*}} \\ &= \sup_\xi \frac{|1-\varphi|}{|\xi|^{\alpha^*}} + \sup_\xi \frac{|\varphi_a - 1|}{|\xi|^{\alpha^*}} < \infty \,,\end{aligned}$$

which gives a contradiction. ∎

Proof of Theorem 1.1. Concerning the Fourier image identification (1.22), Lemma 2.1 shows $\mathcal{M}_k^\alpha(\mathbb{R}^d) \subset \mathcal{F}\left(P_{2k-2+\alpha}(\mathbb{R}^d)\right)$. For the reverse inclusion, if $F \in P_{2k-2+\alpha}(\mathbb{R}^d)$ and $\varphi = \widehat{F}$, then it is straightforward to deduce

$$\int_{\mathbb{R}^d} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi = c_{\alpha,d,\infty,k} \int_{\mathbb{R}^d} |v|^{2k-2+\alpha} dF(v) \tag{2.5}$$

so that $\varphi \in \mathcal{M}_k^\alpha(\mathbb{R}^d)$. Thus $\mathcal{F}\left(P_{2k-2+\alpha}(\mathbb{R}^d)\right) \subset \mathcal{M}_k^\alpha(\mathbb{R}^d)$ and (1.22) follows.

As for the completeness of $\mathcal{M}_k^\alpha(\mathbb{R}^d)$, suppose that $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{M}_k^\alpha(\mathbb{R}^d)$ is a Cauchy sequence with respect to the metric $dis_{k,\alpha,\beta}$, that is,

$$dis_{k,\alpha,\beta}(\varphi_n, \varphi_m) \to 0 \qquad (n, m \to \infty).$$

Since $\mathcal{K}^\beta(\mathbb{R}^d)$ is a complete metric space under each metric defined in (1.12) (see [5, Proposition 3.10]), there exists a unique $\varphi \in \mathcal{K}^\beta(\mathbb{R}^d)$ such that $\varphi_n \to \varphi$ pointwise and $\|\varphi_n - \varphi\|_\beta \to 0$. The Cauchy condition implies

$$\sup_n \int_{\mathbb{R}^d} \frac{\Delta^k \varphi_n(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi \le C$$

for some constant $C > 0$. Since $\Delta^k \varphi_n \to \Delta^k \varphi$ pointwise, it is easy to observe $\varphi \in \mathcal{M}_k^\alpha(\mathbb{R}^d)$. To see the convergence $\|\varphi_n - \varphi\|_{\mathcal{M}_k^\alpha} \to 0$, let $\varepsilon > 0$ and choose an integer N so that $\|\varphi_n - \varphi_m\|_{\mathcal{M}_k^\alpha} < \varepsilon/2$ for all $n, m \ge N$. For each fixed $n \ge N$, we apply Fatou's lemma to find

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\left|\Delta^k \varphi_n(\xi) - \Delta^k \varphi(\xi)\right|}{|\xi|^{d+2k-2+\alpha}} d\xi &\le \liminf_{m\to\infty} \int_{\mathbb{R}^d} \frac{\left|\Delta^k \varphi_n(\xi) - \Delta^k \varphi_m(\xi)\right|}{|\xi|^{d+2k-2+\alpha}} d\xi \\ &\le \varepsilon/2\,, \end{aligned} \tag{2.6}$$

whence $\|\varphi_n - \varphi\|_{\mathcal{M}_k^\alpha} < \varepsilon$ for all $n \ge N$ and the desired convergence follows. Therefore the space $\mathcal{M}_k^\alpha(\mathbb{R}^d)$ is complete with respect to the metric $dis_{k,\alpha,\beta}$.

It remains to verify the convergence property (1.23) under the assumption stated as in Theorem 1.1. Let $\varepsilon > 0$ and fix a continuous function ψ on $\mathbb{R}^d$ such that $|\psi(v)| \le C|v|^{2k-2+\alpha}$ with a constant $C > 0$. For any $R \ge 1$, we have

$$\int_{\{|\xi|\le 1/R\}} \frac{\Delta^k \varphi_n(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi \le \int_{\{|\xi|\le 1/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\alpha}} d\xi + \|\varphi_n - \varphi\|_{\mathcal{M}_k^\alpha}\,.$$

It follows from (2.2) that there exist $R > 1$ and an integer N such that

$$\int_{\{|v|\ge R\}} |v|^{2k-2+\alpha} dF_n(v) + \int_{\{|v|\ge R\}} |v|^{2k-2+\alpha} dF(v) < \frac{\varepsilon}{2C}$$

for all $n \ge N$, which implies

$$\left| \int_{\{|v|\ge R\}} \psi(v) dF_n(v) - \int_{\{|v|\ge R\}} \psi(v) dF(v) \right| < \frac{\varepsilon}{2}\,.$$

On the other hand, the convergence $\|\varphi_n - \varphi\|_\beta \to 0$ implies the convergence $\varphi_n \to \varphi$ pointwise and uniformly on every compact subset of $\mathbb{R}^d$, and so $F_n \to F$

weakly, that is, (1.23) holds if ψ is bounded and continuous. Consequently, we may take N sufficiently large so that

$$\left|\int_{\{|v|\le R\}} \psi(v)dF_n(v) - \int_{\{|v|\le R\}} \psi(v)dF(v)\right| < \frac{\varepsilon}{2}$$

for all $n \ge N$. By adding the two estimates, we have

$$\left|\int \psi(v)dF_n(v) - \int \psi(v)dF(v)\right| < \varepsilon$$

if $n \ge N$ and (1.23) follows.

Our proof of Theorem 1.1 is now complete. ∎

3. Proof of Theorem 1.4. Let $k \ge 2$, $\alpha \in [0,2)$, and $\varphi_0 \in \mathcal{M}_k^\alpha(\mathbb{R}^3)$. In view of the inclusions

$$\mathcal{M}_k^\alpha(\mathbb{R}^3) \subset \mathcal{K}^2(\mathbb{R}^3) \subset \widetilde{\mathcal{M}}^{\beta_1}(\mathbb{R}^3) \quad \text{for some} \quad \beta_1 \in (\beta, 2),$$

it follows from [13, Theorem 1.8] that there exists a unique classical solution $\varphi(t,\xi) \in C([0,\infty), \widetilde{\mathcal{M}}^{\beta_1}(\mathbb{R}^3))$ to the Cauchy problem (1.10).

Since $F_0 \in P_{2k-2+\alpha}(\mathbb{R}^3)$, it is easy to deduce from [13, Corollary 1.7] and (2.5) that $\varphi(t,\xi) \in \mathcal{M}_k^\alpha(\mathbb{R}^3)$ for each $t > 0$. More precisely, there exists a constant $C_T > 0$ such that

$$\sup_{s\le\tau\le t} \int_{\mathbb{R}^3} \frac{\Delta^k\varphi(\tau,\xi)}{|\xi|^{3+2k-2+\alpha}}\, d\xi \le C_T \quad \text{for all } s,t \in [0,T]. \tag{3.1}$$

The continuity estimate (1.25) is a direct consequence of the formula

$$\varphi(t,\xi) - \varphi(s,\xi) = \int_s^t \int_{\mathbb{S}^2} b\Big(\frac{\xi\cdot\sigma}{|\xi|}\Big)\Big(\varphi(\tau,\xi^+)\varphi(\tau,\xi^-) - \varphi(\tau,\xi)\Big) d\sigma d\tau$$

and [10, Lemma 2.2]. To prove the second continuity estimate (1.26), we shall modify the ideas of [10, Lemma 2.2], [12, Lemma 3.4] and proceed as follows (see also [13, (1.23)]).

We use the identity (1.17) to write

$$\begin{aligned}
&\int \frac{|\Delta^k\varphi(t,\xi) - \Delta^k\varphi(s,\xi)|}{|\xi|^{3+2k-2+\alpha}} d\xi \\
&= \int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}} \left|\sum_{j=1}^k c_{k,j}\, [\mathrm{Re}\ \varphi(t,j\xi) - \mathrm{Re}\ \varphi(s,j\xi)]\right| \\
&= \int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}} \left|\sum_{j=1}^k c_{k,j}\, \mathrm{Re} \int_s^t \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\right. \\
&\qquad \left. \times \left[\varphi(\tau,j\xi^+)\varphi(\tau,j\xi^-) - \varphi(\tau,j\xi)\right] d\sigma d\tau \right|.
\end{aligned} \tag{3.2}$$

Setting $\zeta = \big(\xi^+ \cdot \frac{\xi}{|\xi|}\big)\frac{\xi}{|\xi|}$, $\tilde{\xi}^+ = \zeta - (\xi^+ - \zeta)$ (see Figure 1), we split the spherical

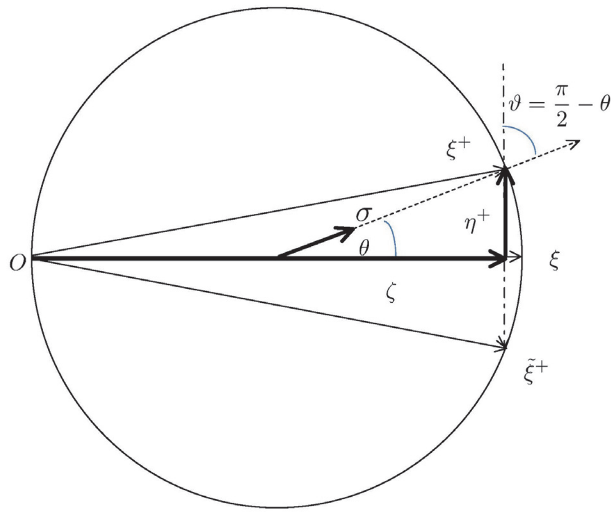

FIG. 1. $\theta = \langle \frac{\xi}{|\xi|}, \sigma \rangle$, $\vartheta = \langle \frac{\eta^+}{|\eta^+|}, \sigma \rangle$, $\eta^+ = \xi^+ - \zeta$.

integral on the right side of (3.2) into three parts:

$$I_{j,1} = \frac{1}{2} \int_{\mathbb{S}^2} b\Big(\frac{\xi \cdot \sigma}{|\xi|}\Big) \big[\varphi(\tau, j\xi^+) + \varphi(\tau, j\tilde{\xi}^+) - 2\varphi(\tau, j\zeta)\big] d\sigma,$$
$$I_{j,2} = \int_{\mathbb{S}^2} b\Big(\frac{\xi \cdot \sigma}{|\xi|}\Big) \big[\varphi(\tau, j\zeta) - \varphi(\tau, j\xi)\big] d\sigma,$$
$$I_{j,3} = \int_{\mathbb{S}^2} b\Big(\frac{\xi \cdot \sigma}{|\xi|}\Big) \varphi(\tau, j\xi^+) \big[\varphi(\tau, j\xi^-) - 1\big] d\sigma.$$

Representing $\varphi(\tau, \xi) = \widehat{F_\tau}(\xi)$ and summing over j, we find

$$\sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,1} = \frac{1}{2} \iint b\Big(\frac{\xi \cdot \sigma}{|\xi|}\Big) \Big(\sin^{2k} \frac{\xi^+ \cdot v}{2} + \sin^{2k} \frac{\tilde{\xi}^+ \cdot v}{2} - 2\sin^{2k} \frac{\zeta \cdot v}{2}\Big) dF_\tau d\sigma,$$
$$\sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,2} = \iint b\Big(\frac{\xi \cdot \sigma}{|\xi|}\Big) \Big(\sin^{2k} \frac{\zeta \cdot v}{2} - \sin^{2k} \frac{\xi \cdot v}{2}\Big) dF_\tau d\sigma.$$

In dealing with the first terms, we change variables and use the rotation invariance as before to observe

$$\int \frac{1}{|\xi|^{3+2k-2+\alpha}} \Big| \int_s^t \sum_{j=1}^k c_{k,j} \operatorname{Re} I_{j,1} d\tau \Big| d\xi$$
$$\leq \frac{1}{2} \int_s^t \iint b\Big(\frac{\xi \cdot \sigma}{|\xi|}\Big) \Big| \sin^{2k} \frac{\xi^+ \cdot \mathbf{e}_1}{2} + \sin^{2k} \frac{\tilde{\xi}^+ \cdot \mathbf{e}_1}{2} - 2\sin^{2k} \frac{\zeta \cdot \mathbf{e}_1}{2} \Big| d\sigma d\xi$$
$$\times \int |v|^{2k-2+\alpha} dF_\tau d\tau.$$

If we put $A=\zeta\cdot\mathbf{e}_1/2$ and $B=\eta^+\cdot\mathbf{e}_1/2$ with $\eta^+=\xi^+-\zeta$, then

$$\begin{aligned}&\sin^{2k}\frac{\xi^+\cdot\mathbf{e}_1}{2}+\sin^{2k}\frac{\tilde{\xi}^+\cdot\mathbf{e}_1}{2}-2\sin^{2k}\frac{\zeta\cdot\mathbf{e}_1}{2}\\&=(\sin A\cos B+\sin B\cos A)^{2k}+(\sin A\cos B-\sin B\cos A)^{2k}-2\sin^{2k}A\\&=2\sin^{2k}A(\cos^{2k}B-1)+2\sum_{j=1}^{k}\binom{2k}{2j}\sin^{2j}B\sin^{2k-2j}A\cos^{2k-2j}B\cos^{2j}A\,.\end{aligned}$$

In terms of the deviation angle $\xi\cdot\sigma=|\xi|\cos\theta$, it is plain to see

$$\sin^2\frac{\eta^+\cdot\mathbf{e}_1}{2}\lesssim|\xi|^2\sin^2\frac{\theta}{2}\,,$$

which yields

$$\begin{aligned}&\left|\sin^{2k}\frac{\xi^+\cdot\mathbf{e}_1}{2}+\sin^{2k}\frac{\tilde{\xi}^+\cdot\mathbf{e}_1}{2}-2\sin^{2k}\frac{\zeta\cdot\mathbf{e}_1}{2}\right|\\&\qquad\lesssim\min\left\{|\xi|^2\sin^2\frac{\theta}{2},\,1\right\}\cdot\mathbf{1}_{\{|\xi|\geq1\}}+|\xi|^{2k}\sin^2\frac{\theta}{2}\cdot\mathbf{1}_{\{|\xi|<1\}}\,.\end{aligned}$$

Making use of the assumption (1.6) on the kernel b, we estimate

$$\begin{aligned}&\int_{|\xi|>1}\frac{1}{|\xi|^{3+2k-2+\alpha}}\Big|\int_s^t\sum_{j=1}^{k}c_{k,j}\,\mathrm{Re}\ I_{j,1}d\tau\Big|d\xi\\&\lesssim\int_{|\xi|>1}\frac{1}{|\xi|^{3+2k-2+\alpha}}\int_0^{\pi/2}b(\cos\theta)\min\left\{|\xi|^2\sin^2\frac{\theta}{2},\,1\right\}\sin\theta\,d\theta d\xi\\&\lesssim\left[\int_0^{\pi/2}b(\cos\theta)\sin^{\alpha_0}\frac{\theta}{2}\sin\theta\,d\theta\right]\cdot\int_{|\xi|>1}\frac{|\xi|^{\alpha_0}}{|\xi|^{3+2k-2+\alpha}}d\xi<\infty.\end{aligned}$$

As the integral over $\{|\xi|<1\}$ can be estimated in the same way, we have

$$\begin{aligned}&\int\frac{1}{|\xi|^{3+2k-2+\alpha}}\Big|\int_s^t\sum_{j=1}^{k}c_{k,j}\,\mathrm{Re}\ I_{j,1}d\tau\Big|d\xi\\&\qquad\lesssim|t-s|\cdot\sup_{\tau\in[0,T]}\int|v|^{2k-2+\alpha}dF_\tau.\end{aligned}\tag{3.3}$$

Modifying the above arguments slightly, it is not hard to obtain (3.3) for the second terms involving $I_{j,2}$. As to the last terms involving $I_{j,3}$, we use the fact the solution conserves the momentum, i.e.,

$$\int v\,dF_t(v)=0\qquad(t\geq0).\tag{3.4}$$

Noting $\int dF_\tau(w)=1$, we rewrite $\mathrm{Re}\ I_{j,3}$ as

$$\begin{aligned}&\int b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\mathrm{Re}\ \Big(\varphi(\tau,j\xi^+)(\varphi(\tau,j\xi^-)-1)\Big)d\sigma\\&=\int b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)d\sigma\iint\mathrm{Re}\ \Big(e^{-i(j\xi^+\cdot v)}(e^{-i(j\xi^-)\cdot w}-1)\Big)dF_\tau(v)dF_\tau(w)\\&=\int b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)d\sigma\iint\big[\cos(j(\xi^+\cdot v+\xi^-\cdot w))-\cos(j\xi^+\cdot v)\big]dF_\tau(v)dF_\tau(w)\,.\end{aligned}$$

It follows from (1.18) that

$$\sum_{j=1}^{k} c_{k,j}\,\mathrm{Re}\, I_{j,3}$$
$$= \int b\Big(\frac{\xi\cdot\sigma}{|\xi|}\Big)d\sigma \iint \Big(\sin^{2k}\frac{\xi^+\cdot v+\xi^-\cdot w}{2} - \sin^{2k}\frac{\xi^+\cdot v}{2}\Big)dF_\tau(v)dF_\tau(w)\,.$$

Setting

$$J = \sin^{2k}\frac{\xi^+\cdot v+\xi^-\cdot w}{2} - \sin^{2k}\frac{\xi^+\cdot v}{2} \quad \text{and} \quad x = \frac{\xi^+\cdot v}{2}\,, \quad y = \frac{\xi^-\cdot w}{2}\,,$$

we decompose J in the form

$$\begin{aligned} J &= \Big(\sin x\cos y + \sin y \cos x\Big)^{2k} - \sin^{2k} x \\ &= 2k\sin y\sin^{2k-1} x\cos x + 2k \sin y(\cos^{2k-1} y - 1)\sin^{2k-1} x \cos x \\ &\quad + \sum_{\ell=2}^{2k}\binom{2k}{\ell}\sin^\ell y\cos^{2k-\ell} y \sin^{2k-\ell} x\cos^\ell x + \sin^{2k} x(\cos^{2k} y - 1) \\ &= J_{3,1}+J_{3,2}+J_{3,3}+J_{3,4}\,. \end{aligned}$$

As stated before, we use (3.4) to see

$$\Big|\iint J_{3,1}dF_\tau(v)dF_\tau(w)\Big|$$
$$= 2k\Big|\int \sin^{2k-1}\frac{\xi^+\cdot v}{2}\cos\frac{\xi^+\cdot v}{2}dF_\tau(v)\int\Big(\sin\frac{\xi^-\cdot w}{2} - \frac{\xi^-\cdot w}{2}\Big)dF_\tau(w)\Big|\,.$$

Using $|z-\sin z| \lesssim |z|\min\{|z|,1\}$ and changing variables $|v|\xi \to \zeta$, we estimate

$$\begin{aligned} &\int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}}\int b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Big|\iint J_{3,1}dF_\tau(v)dF_\tau(w)\Big|d\sigma \\ &\quad\lesssim \int\frac{\min\{|\zeta|^{2k-1},1\}d\zeta}{|\zeta|^{3+2k-2+\alpha}}\int b\left(\cos\theta\right)\theta^{\max\{\alpha_0,1\}}\sin\theta d\theta \\ &\quad\times\int |v|^{2k-2+\alpha}\Big[\int\Big(\frac{|\zeta||w|}{|v|}\Big)^{\max\{\alpha_0,1\}}dF_\tau(w)\Big]dF_\tau(v) \\ &\quad\lesssim \int\langle v\rangle^{2k-2+\alpha}dF_\tau(v)\int\langle w\rangle^2 dF_\tau(w)\,. \end{aligned}$$

Proceeding in a similar manner, it is not hard to estimate the integrals corresponding to $J_{3,2}$ and $J_{3,4}$ and end up with the same upper bounds.

For the integrals involving $J_{3,3}$, we change variables $|w|\xi\to\zeta$ for the terms with $\ell \geq k$ to find that

$$\int \frac{d\xi}{|\xi|^{3+2k-2+\alpha}}\int b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Big|\iint J_{3,3}dF_\tau(v)dF_\tau(w)\Big|d\sigma$$

is bounded above, modulo constants, by the following:

$$\begin{aligned}
&\int \frac{d\zeta}{|\zeta|^{3+2k-2+\alpha}} \int b\left(\cos\theta\right)\theta^{\alpha_0}\sin\theta d\theta \\
&\quad\times \iint |v|^{2k-2+\alpha} \sum_{\ell=2}^{k-1} \Big(|\zeta|^{2k-\ell}\mathbf{1}_{\{|\zeta|\le 1\}} + \mathbf{1}_{\{|\zeta|>1\}}\Big) \\
&\quad\times \Big(\Big(\frac{|w||\zeta|}{|v|}\Big)^{\ell}\mathbf{1}_{\{|\zeta|\le 1\}} + \Big(\frac{|w||\zeta|}{|v|}\Big)^{\alpha_0}\mathbf{1}_{\{|\zeta|>1\}}\Big) dF_\tau(v)dF_\tau(w) \\
&\ \ + \int \frac{d\zeta}{|\zeta|^{3+2k-2+\alpha}} \int b\left(\cos\theta\right)\theta^{\alpha_0}\sin\theta d\theta \\
&\quad\times \iint |w|^{2k-2+\alpha} \sum_{\ell=k}^{2k} \Big(\Big(\frac{|v||\zeta|}{|w|}\Big)^{2k-\ell}\mathbf{1}_{\{|\zeta|\le 1\}} + \mathbf{1}_{\{|\zeta|>1\}}\Big) \\
&\quad\times \Big(|\zeta|^{\ell}\mathbf{1}_{\{|\zeta|\le 1\}} + |\zeta|^{\alpha_0}\mathbf{1}_{\{|\zeta|>1\}}\Big) dF_\tau(v)dF_\tau(w) \\
&\lesssim \iint \Big(\langle v\rangle^{2k-2+\alpha}\langle w\rangle^2 + \langle w\rangle^{2k-2+\alpha}\langle v\rangle^2\Big) dF_\tau(v)dF_\tau(w).
\end{aligned}$$

Collecting these estimates, we are led to a (3.3)-type estimate, that is,

$$\begin{aligned}
&\int \frac{1}{|\xi|^{3+2k-2+\alpha}} \Big| \int_s^t \sum_{j=1}^{k} c_{k,j} \operatorname{Re} I_{j,3} d\tau \Big| d\xi \\
&\qquad \lesssim |t-s| \cdot \sup_{\tau\in[0,T]} \int \langle v\rangle^{2k-2+\alpha} dF_\tau.
\end{aligned}$$

As the second continuity estimate (1.26) is now verified, our proof of Theorem 1.4 is complete. ∎

4. Proof of Theorem 1.3. The global existence and uniqueness of the solution to the Cauchy problem (1.1)–(1.9) follow from Theorem 1.4. As the smoothing effect has already been proved in [13, Corollary 1.10] (see also [11, 14]), it remains to verify

$$f \in C((0,\infty); L^1_{2k-2+\alpha}(\mathbb{R}^3))$$

under the assumption (1.5) on the Maxwellian kernel b.

By the smoothing effect, for any $0 < t_0 < T < \infty$ and $N > 0$, there exists a constant $C_{N,t_0,T} > 0$ such that

$$\sup_{t_0\le\tau\le T} \int \langle\xi\rangle^N |\varphi(\tau,\xi)|^2 d\xi \le C_{N,t_0,T}\,. \tag{4.1}$$

Let $\varepsilon > 0$ be arbitrary and fix $t_1 \in (t_0, T)$. Since $\varphi(t_1,\cdot) \in \mathcal{M}^\alpha_k(\mathbb{R}^3)$, there exists $R > 1$ such that

$$\int_{\{|\xi|<1/R\}} \frac{\Delta^k \varphi(t_1,\xi)}{|\xi|^{3+2k-2+\alpha}} d\xi < c_{\alpha,3,1,k} \cdot \frac{\varepsilon}{2}\,.$$

By using this and (1.26), we can choose $\delta > 0$ so that $|t - t_1| < \delta$ implies

$$\int_{\{|\xi|<1/R\}} \frac{\Delta^k \varphi(t,\xi)}{|\xi|^{3+2k-2+\alpha}} d\xi < c_{\alpha,3,1,k} \cdot \varepsilon.$$

Due to the estimate (2.2), if $|t-t_1|<\delta$, then $\int_{\{|v|\geq R\}}|v|^{2k+\alpha-2}f(t,v)dv<\varepsilon$, and hence

$$\begin{aligned}&\int \langle v\rangle^{2k-2+\alpha}|f(t,v)-f(t_1,v)|\,dv\\&\qquad\leq \langle R\rangle^{2k-2+\alpha}\int_{\{|v|<R\}}|f(t,v)-f(t_1,v)|\,dv+4\varepsilon\\&\qquad\leq \frac{4\pi}{3}\langle R\rangle^{2k+1+\alpha}\,\|f(t,\cdot)-f(t_1,\cdot)\|_\infty+4\varepsilon.\end{aligned}\tag{4.2}$$

For $M>1$, we apply the Fourier inversion theorem and Hölder's inequality to find that $\|f(t,\cdot)-f(t_1,\cdot)\|_\infty$ is bounded above, modulo constants, by

$$\begin{aligned}\int|\varphi(t,\xi)-\varphi(t_1,\xi)|d\xi \leq \Big(\int_{|\xi|\geq M}\langle\xi\rangle^{-4}d\xi\Big)^{1/2}\Big(\int\langle\xi\rangle^4|\varphi(t,\xi)-\varphi(t_1,\xi)|^2d\xi\Big)^{1/2}\\+\ \frac{4\pi M^3}{3}\sup_{|\xi|\leq M}|\varphi(t,\xi)-\varphi(t_1,\xi)|\,.\end{aligned}$$

By using (4.1) and the established fact $\varphi\in C([0,\infty);\mathcal{K}^2(\mathbb{R}^3))$, we may choose M and $0<\delta'<\delta$ appropriately in such a way that $|t-t_1|<\delta'$ implies

$$\|f(t,\cdot)-f(t_1,\cdot)\|_\infty<\frac{3\varepsilon}{4\pi\langle R\rangle^{2k+1+\alpha}}.\tag{4.3}$$

Combining (4.2) and (4.3), if $|t-t_1|<\delta'$, then

$$\int\langle v\rangle^{2k-2+\alpha}|f(t,v)-f(t_1,v)|\,dv<5\varepsilon\,,$$

which proves the desired continuity of f at t_1.

Our proof of Theorem 1.3 is now complete. ∎

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