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Published in:
SIAM Journal on Mathematical Analysis

Published: 01/01/2016

Document Version:
Final Published version, also known as Publisher's PDF, Publisher's Final version or Version of Record

Publication record in CityU Scholars:
Go to record

Published version (DOI):
10.1137/16M1056237

Publication details:
BENSOUSSAN, A., MERTZ, L., & YAM, S. C. P. (2016). Nonlocal boundary value problems of a stochastic variational inequality modeling an elasto-plastic oscillator excited by a filtered noise. *SIAM Journal on Mathematical Analysis*, *48*(4), 2783-2805. https://doi.org/10.1137/16M1056237

SIAM J. MATH. ANAL.
Vol. 48, No. 4, pp. 2783–2805

NONLOCAL BOUNDARY VALUE PROBLEMS OF A STOCHASTIC VARIATIONAL INEQUALITY MODELING AN ELASTO-PLASTIC OSCILLATOR EXCITED BY A FILTERED NOISE*

A. BENSOUSSAN†, L. MERTZ‡, AND S. C. P. YAM§

Abstract. In the literature, failure risk analysis on most elasto-perfectly-plastic oscillators is essentially focused on those excited by *white noise*, which is rather restrictive from the modeling perspective. Our present article is motivated by the study of the probability distribution of the solution of a stochastic variational inequality modeling an elasto-plastic oscillator excited by a *filtered noise*. We introduce a class of partial differential equations (PDEs) with nonlocal Dirichlet conditions and we establish the unique existence of solutions of these PDEs by extending the method developed in [A. Bensoussan and J. Turi, *Applied and Numerical Partial Differential Equations*, Comput. Methods Appl. Sci. 15, Springer, New York, 2009, pp. 9–23]. A major mathematical challenge here is to carry out the analysis of boundary value problems for elliptic equations in dimension two rather than that in dimension one.

Key words. boundary value problem, variational inequality, elasto-plastic oscillator

AMS subject classifications. 74H50, 35R60, 60H10, 60H30, 74C05

DOI. 10.1137/16M1056237

1. Background and motivations.

For applications to earthquake engineering, elasto-plastic oscillators are crucial in understanding and predicting the risk of failure of mechanical systems subject to random excitations such as piping systems (see [9, 10, 11, 12, 13, 14, 15, 16, 18] and references therein). It is important to investigate their mathematical structures and properties which lead us to having a better understanding and being able to forecast the level of permanent (plastic) deformations that could result in high failure risk of the mechanical systems. In some of our earlier works, for elasto-perfectly-plastic (EPP) oscillators excited by *white noise* $\frac{dw}{dt}$, i.e., when $w(t)$ is a Wiener process, we have considered (a) theoretical aspects (long time behavior and cycle behavior) in [2, 3, 6, 7, 8], (b) numerical aspects for the ergodic measure in [5, 17], and (c) engineering applications for the statistics of plastic deformations in [16]. Yet, from the modeling viewpoint, considering the excitation as a

*Received by the editors January 11, 2016; accepted for publication (in revised form) April 14, 2016; published electronically August 30, 2016. The research of the second and third authors was supported by Hong Kong RGC GRF 502408, Chinese University of Hong Kong direct grant 2010/2011 project 2060422, and Chinese University of Hong Kong direct grant 2010/2011 project 2060444.
http://www.siam.org/journals/sima/48-4/M105623.html

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white noise may be viable only for a restricted class of physical seismic forcing. A natural generalization, which can also cater to the need for more tailor-made modeling design, is to consider the same set of mathematical problems for elasto-plastic oscillators excited by a *filtered noise* $\frac{d\xi_\alpha}{dt}$ with the dynamics of $\xi_\alpha(t)$ to be given in (1). Basically, it is well-known that the white noise is a signal whose correlation function $R(\tau)$ is the Dirac delta function $\delta(\tau)$. Therefore the Fourier transform of R satisfies

$$\hat{R}(\xi) = \int_{-\infty}^{\infty} \exp(-2\pi i\tau\xi)R(\tau)\mathrm{d}\tau = 1 \quad \forall \xi \in \mathbb{R}.$$

Since $\hat{R}(\xi)$ represents the contribution of the frequency ξ to the signal, all frequencies are equaly present. Here the filtered noise $\frac{d\xi_\alpha}{dt}$ is a signal whose correlation function $R_\alpha(\tau)$ is $\alpha \exp(-\alpha|\tau|) + \delta(\tau)$. Therefore the Fourier transform of R_α is

$$\hat{R}_\alpha(\xi) = \int_{-\infty}^{\infty} \exp(-2\pi i\tau\xi)R_\alpha(\tau)\mathrm{d}\tau = \frac{\alpha^2}{2\pi(\xi^2+\alpha^2)} + 1 \quad \forall \xi \in \mathbb{R}.$$

Thus, in contrast with the white noise case, we interpret $\frac{d\xi_\alpha}{dt}$ as a signal whose high frequencies are less present than low ones. In that sense, it is more realistic for applications. Also, it is one of the simplest ways to generalize the models of white noise driven elasto-plastic oscillators. In this context, an extension of [7] together with a Khasminskii approach has been provided in [4] on the existence and uniqueness of an invariant probability measure. In the present paper, our aim is to investigate some mathematical properties of the partial differential equations (PDEs) defining by duality the probability distribution of an EPP oscillator excited by a *filtered noise*.

1.1. The elasto-plastic model excited by a filtered noise. Here, we consider a parameter $\alpha > 0$ and the process $\xi_\alpha(t)$ as follows:

$$\xi_\alpha(t) = \int_0^t e^{-\alpha(t-s)}\mathrm{d}w(s). \tag{1}$$

It is well known that $\xi_\alpha(t)$ is an Ornstein–Uhlenbeck process which solves the following stochastic differential equation: $d\xi_\alpha(t) = -\alpha\xi_\alpha(t)\,dt + dw(t)$. Moreover, note that the case $\alpha = 0$ recovers the study which has been done for the EPP-oscillator driven by white noise. Let us give a brief presentation of the elasto-plastic model which has been proposed by Karnopp and Scharton [18]. Denote $x(t)$ the elasto-plastic displacement of the oscillator; then $x(t)$ satisfies the following equation:

$$\ddot{x} + c_0\dot{x} + \mathbf{F}\left(x(s), s \in (0,t)\right) = \dot{\xi}_\alpha \tag{2}$$

with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. Here c_0 is a damping coefficient, and $\mathbf{F}\left(x(s), s \in (0,t)\right)$ is a nonlinear restoring force of EPP type and satisfies

$$|\mathbf{F}\left(x(s), s \in (0,t)\right)| \leq kY$$

and

$$\mathbf{F}\left(x(s), s \in (0,t)\right) = \begin{cases} kY & \text{if} \quad x(t) - \Delta(t) = Y, \\ k(x(t) - \Delta(t)) & \text{if} \quad -Y < x(t) - \Delta(t) < Y, \\ -kY & \text{if} \quad x(t) - \Delta(t) = -Y, \end{cases}$$

where k is a stiffness coefficient, $\Delta(t)$ is the permanent (or plastic) deformation in $x(t)$, and Y is an elasto-plastic bound. Basically, there are two types of phases at any given time: either (1) the norm of the restoring force is strictly bounded above by kY (i.e., $|\mathbf{F}(x(s), s \in (0,t))| < kY$) when the system is at the elastic phase, or (2) the restoring force is equal to kY (i.e., $|\mathbf{F}(x(s), s \in (0,t))| = kY$) when it is at the plastic phase. Karnopp and Scharton [18] formulated a separation between the two states. They introduced an extra-state variable $z(t) := x(t) - \Delta(t)$ and remarked on the simple fact that, between two consecutive plastic phases, $z(t)$ behaves like a linear oscillator. Now, $x(t)$ can be decomposed in $z(t)+\Delta(t)$, where $z(t)$ is called the elastic component of $x(t)$ and $\Delta(t)$ is the plastic component. During the elastic phase, $\Delta(t)$ remains constant, $\dot{x} = \dot{z}$, and $z(t)$ satisfies

$$\ddot{z} + c_0\dot{z} + kz = \dot{\xi}_\alpha, \tag{3}$$

where the initial conditions are given by the previous plastic phase. But, during a plastic phase $z(t)$ remains constant, $\dot{x} = \dot{\Delta}$, and $\Delta(t)$ satisfies

$$\ddot{\Delta} + c_0\dot{\Delta} \pm kY = \dot{\xi}_\alpha, \tag{4}$$

where the initial conditions are given by the previous elastic phase. Finally, since the instants of phase transition cannot be anticipated, one has to follow in an alternative manner between (3) and (4) in order to solve (2).

1.2. Stochastic variational inequality for an elasto-plastic oscillator driven by a filtered noise. In a previous work [7], the authors introduced a stochastic variational inequality (SVI) in order to express the relationship between the restoring and the velocity in a new mathematical manner. In our context, by setting $y(t) := \dot{x}(t)$, we know that the couple $(y(t), z(t)) \in \mathbb{R} \times [-Y, Y]$ is the unique solution of the following SVI:

$$(\mathcal{SVI}, \alpha) \qquad \begin{cases} \dot{y}(t) = -(c_0 y(t) + kz(t)) + \dot{\xi}_\alpha(t), \\ (\dot{z}(t) - y(t))(\phi - z(t)) \geq 0 \quad \forall\, |\phi| \leq Y, \quad |z(t)| \leq Y, \end{cases}$$

with $y(0) = 0$, $z(0) = 0$. We refer to [1] for a purely mathematical presentation of SVIs. Next, Proposition 1 shows in which sense $(\mathcal{SVI}, \alpha)$ generalizes the models of white noise driven elasto-plastic oscillators. For the reader's convenience, we use the notation $(.)_\alpha$ to specify the dependency with respect to α. We show that the solution $(y_\alpha(t), z_\alpha(t))$ converges to $(y_0(t), z_0(t))$ as $\alpha \downarrow 0$ in the following sense.

PROPOSITION 1. *Fixing $T > 0$, we have the following estimate:*

$$\frac{1}{\alpha^2 T^2}\mathbb{E}\left\{\sup_{0\leq\tau\leq T}\left(|y_\alpha(\tau) - y_0(\tau)|^2 + k|z_\alpha(\tau) - z_0(\tau)|^2\right)\right\} \leq \frac{1}{4c_0} + o(\alpha T) \qquad \text{as } \alpha \downarrow 0.$$

Also, for any continuous and bounded function f on $\mathbb{R} \times (-Y, Y)$ and for any finite time $t > 0$, the convergence property holds:

$$\lim_{\alpha\to 0} \mathbb{E} f(y_\alpha(t), z_\alpha(t)) = \mathbb{E} f(y_0(t), z_0(t)), \qquad \lim_{\alpha\to 0} \mathbb{E}|f(y_\alpha(t), z_\alpha(t)) - f(y_0(t), z_0(t))| = 0$$

and

$$\lim_{\alpha\to 0} \mathbb{E}\int_0^\infty \exp(-\lambda t) f(y_\alpha(t), z_\alpha(t))\, dt = \mathbb{E}\int_0^\infty \exp(-\lambda t) f(y_0(t), z_0(t))\, dt.$$

Proof. Denote $\eta_\alpha(t) := y_\alpha(t) - y_0(t), \quad \zeta_\alpha(t) := z_\alpha(t) - z_0(t)$. As $(y_0(t), z_0(t))$ and $(y_\alpha(t), z_\alpha(t))$ have the same initial conditions, we have $\eta_\alpha(0) = \zeta_\alpha(0) = 0$, and from the equations on $y_\alpha(t)$ and $y_0(t)$ in $(\mathcal{SVI}, \alpha)$, we deduce

$$\dot{\eta}_\alpha(t) + c_0\eta_\alpha(t) = -\alpha\xi_\alpha(t) - k\zeta_\alpha(t). \tag{5}$$

In addition, from the inequalities on $z_\alpha(t)$ and $z_0(t)$ in $(\mathcal{SVI}, \alpha)$, we deduce $(\dot{z}_\alpha(t) - y_\alpha(t))(z_0(t) - z_\alpha(t)) \geq 0$ and $(\dot{z}_0(t) - y_0(t))(z_\alpha(t) - z_0(t)) \geq 0$, which implies

$$\eta^\alpha(t)\zeta^\alpha(t) \geq \zeta^\alpha(t)\dot{\zeta}^\alpha(t). \tag{6}$$

Therefore, multiplying (5) by $\eta^\alpha(t)$, we obtain $\eta_\alpha(t)\dot{\eta}_\alpha(t) + c_0\eta_\alpha^2(t) = -\alpha\eta_\alpha(t)\xi_\alpha(t) - k\eta_\alpha(t)\zeta_\alpha(t)$, and from (6), we obtain $\eta_\alpha(t)\dot{\eta}_\alpha(t) + c_0\eta_\alpha^2(t) + k\zeta_\alpha(t)\dot{\zeta}_\alpha(t) \leq -\alpha\eta_\alpha(t)\xi_\alpha(t)$. Then, for a $\tau > 0$ we integrate over $(0, \tau)$ and we get

$$\frac{1}{2}\eta_\alpha^2(\tau) + c_0\int_0^\tau \eta_\alpha^2(t)\mathrm{d}t + \frac{k}{2}\zeta_\alpha^2(\tau) \leq -\alpha\int_0^\tau \eta_\alpha(t)\xi_\alpha(t)\mathrm{d}t.$$

Now, observe that from the elementary inequality

$$\left|\int_0^\tau \eta_\alpha(t)\xi_\alpha(t)\mathrm{d}t\right| \leq \frac{c_0}{2\alpha}\int_0^\tau \eta_\alpha^2(t)\mathrm{d}t + \frac{\alpha}{2c_0}\int_0^\tau \xi_\alpha^2(t)\mathrm{d}t,$$

we deduce that

$$\eta_\alpha^2(\tau) + c_0\int_0^\tau \eta_\alpha^2(t)\mathrm{d}t + k\zeta_\alpha^2(t) \leq \frac{\alpha^2}{c_0}\int_0^\tau \xi_\alpha^2(t)\mathrm{d}t.$$

Then, considering $T > 0$, taking first the supremum over $(0, T)$ and the expectation, we deduce that

$$\mathbb{E}\left\{\sup_{0\leq\tau\leq T}(\eta_\alpha^2(\tau) + k\zeta_\alpha^2(\tau)) + c_0\int_0^T \eta_\alpha^2(t)\mathrm{d}t\right\} \leq \frac{\alpha^2}{c_0}\int_0^T \mathbb{E}\xi_\alpha^2(t)\mathrm{d}t.$$

But $\mathbb{E}\xi_\alpha^2(t) = \frac{1-\exp(-2\alpha t)}{2\alpha}$, and consequently,

$$\frac{\alpha^2}{c_0}\int_0^T \mathbb{E}\xi_\alpha^2(t)\mathrm{d}t = \frac{\alpha}{2c_0}T - \frac{1}{4c_0}(1 - \exp(-2\alpha T))$$

which leads to the following expansion:

$$\underbrace{\frac{\alpha^2}{c_0}\int_0^T \mathbb{E}\xi_\alpha^2(t)\mathrm{d}t}_{=:\gamma(\alpha T)} = \frac{1}{4c_0}(\alpha T)^2 - \frac{1}{3c_0}(\alpha T)^3 + \cdots + (-1)^k(\alpha T)^k\frac{2^{k-2}}{c_0 k!} + o((\alpha T)^k).$$

Finally, for the second part of the proposition, as the sequence $\alpha \to f(y_\alpha(t), z_\alpha(t))$ is bounded by $\sup|f|$, the result is a direct combination of the result above and an application of Lebesgue's dominated convergence theorem. ∎

1.3. Characterization of the solutions of $(\mathcal{SVI},\alpha)$ through PDEs with nonlocal boundary conditions. Let us first introduce some notation.

Notation 1. Define the domains

$$D := \mathbb{R}\times(-Y,Y),\quad D^+ := (0,+\infty)\times\{Y\},\quad D^- := (-\infty,0)\times\{-Y\}$$

and the operators

$$A\phi := -\frac{1}{2}\frac{\partial^2\phi}{\partial y^2} + (c_0 y + kz)\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial z},$$
$$B_+\phi := -\frac{1}{2}\frac{\partial^2\phi}{\partial y^2} + (c_0 y + kY)\frac{\partial\phi}{\partial y},$$
$$B_-\phi := -\frac{1}{2}\frac{\partial^2\phi}{\partial y^2} + (c_0 y - kY)\frac{\partial\phi}{\partial y}.$$

We further define the domains

$$\Delta := (-\infty,\infty)\times D,\quad \Delta^+ := (-\infty,\infty)\times D^+,\quad \Delta^- := (-\infty,\infty)\times D^-$$

and the operators

$$\mathbf{A}\phi := -\frac{1}{2}\frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\phi}{\partial\xi\partial y} + \alpha\xi\frac{\partial\phi}{\partial\xi} + \alpha\xi\frac{\partial\phi}{\partial y} + A\phi,$$
$$\mathbf{B}_+\phi := -\frac{1}{2}\frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\phi}{\partial\xi\partial y} + \alpha\xi\frac{\partial\phi}{\partial\xi} + \alpha\xi\frac{\partial\phi}{\partial y} + B_+\phi,$$
$$\mathbf{B}_-\phi := -\frac{1}{2}\frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\phi}{\partial\xi\partial y} + \alpha\xi\frac{\partial\phi}{\partial\xi} + \alpha\xi\frac{\partial\phi}{\partial y} + B_-\phi.$$

Moreover, define the space

$$L^\infty(\Delta) := \left\{u:\Delta\to\mathbb{R},\ \operatorname*{ess\,sup}_{(\xi,y,z)\in\Delta}|u|<\infty\right\}.$$

Our work is motivated by the study of the probability distribution of $(\xi_\alpha(t), y(t), z(t))$ and is in a straightforward line with [8], where nonlocal problems related to an elasto-plastic oscillator excited by white noise ($\alpha = 0$) have been studied. Letting $\lambda > 0$ and $g \in L^\infty(\Delta)$, we are interested in the problem of finding a function $u \in L^\infty(\Delta)$ which is continuous up to the boundary of Δ and which satisfies the following equations:

$$\lambda u + \mathbf{A}u = g \text{ in } \Delta,\quad \lambda u + \mathbf{B}_+u = g \text{ in } \Delta^+,\quad \lambda u + \mathbf{B}_-u = g \text{ in } \Delta^-.$$

Nonetheless, it seems challenging to address directly this problem since the crossed derivative between y and ξ makes it too degenerate. So, we introduce a slight modification by imposing a correlation structure between the noise involved in $y(t)$ and the noise in $\xi_\alpha(t)$ and considering the couple $(y^\rho(t), z^\rho(t))$ which satisfies

$$\begin{cases} \dot{y}^\rho(t) = -(\alpha\xi_\alpha(t) + c_0 y^\rho(t) + kz^\rho(t)) + \mathrm{d}w^\rho(t),\\ (\dot{z}^\rho(t) - y^\rho(t))(\phi - z^\rho(t)) \geq 0 \quad \forall\ |\phi|\leq Y,\quad |z^\rho(t)|\leq Y. \end{cases}$$

Here $w^\rho(t)$ is another Wiener process correlated to $w(t)$ in a sense that $\mathbb{E}w(t)w^\rho(t) = \rho t$, where ρ is arbitrarily close to 1 but strictly smaller. Therefore, in this framework, we are interested in the corresponding problem for $(\xi_\alpha(t), y^\rho(t), z^\rho(t))$. The latter can be found in Theorem 2, where a class of PDEs with nonlocal boundary conditions is studied in an appropriate Sobolev weighted space.

2. Main result. In this section, we present our main result. We now introduce a class of PDEs with nonlocal Dirichlet boundary condition. First, let us introduce more of the set-up of the problem.

Notation 2. Consider the operators with a correlation structure

$$\mathbf{A}^\rho \phi := \mathbf{A}\phi + (1-\rho)\frac{\partial^2 \phi}{\partial \xi \partial y}, \quad \mathbf{B}^\rho_\pm \phi := \mathbf{B}_\pm \phi + (1-\rho)\frac{\partial^2 \phi}{\partial \xi \partial y}.$$

Define the weight function $\theta_m(\xi, y) = \frac{1}{(1+\xi^2+y^2)^m}$ and the spaces

$$L^2_m(\Delta) := \left\{ u : \Delta \to \mathbb{R}, \int_\Delta |u(\xi,y,z)|^2 \theta_m(\xi,y) \mathrm{d}\xi \mathrm{d}y \mathrm{d}z < \infty \right\}, \quad m \geq 2.$$

The main result of this paper is the following.

Theorem 2. *Let $g \in L^\infty(\Delta)$ and $\lambda > 0$. There exists a unique function $u \in L^\infty(\Delta)$ bounded by $\frac{\|g\|_{L^\infty}}{\lambda}$ which satisfies the following condition, for $m \geq 3$, $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial y} \in L^2_m(\Delta)$,*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y| u^2(\xi, y, \pm Y) \theta_m(\xi, y) \, dy \, d\xi < \infty,$$

for $m \geq 4$, $y^2 \frac{\partial^2 u}{\partial z^2} \in L^2_m(\Delta)$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y|^3 \left\{ \left(\frac{\partial u}{\partial \xi}(\xi, y, \pm Y) \right)^2 + \left(\frac{\partial u}{\partial y}(\xi, y, \pm Y) \right)^2 \right\} \theta_m(\xi, y) \, dy \, d\xi < \infty,$$

and the following equations in the sense of the distributions:

$$(P_\lambda) \qquad \lambda u + \mathbf{A}^\rho u = g \text{ in } \Delta, \quad \lambda u + \mathbf{B}^\rho_+ u = g \text{ in } \Delta^+, \quad \lambda u + \mathbf{B}^\rho_- u = g \text{ in } \Delta^-.$$

If we further assume $\frac{\partial g}{\partial z} \in L^2_2(\Delta)$, then u has more regularity in the sense, for $m \geq 3$, $\frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial \xi \partial z}, \frac{\partial^2 u}{\partial y \partial z} \in L^2_m(\Delta)$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y| \left(\frac{\partial u}{\partial z}(\xi, y, \pm Y) \right)^2 \theta_m(\xi, y) \, dy \, d\xi < \infty,$$

for $m \geq 4$, $y^2 \frac{\partial^2 u}{\partial z^2} \in L^2_m(\Delta)$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y|^3 \left\{ \left(\frac{\partial u}{\partial z}(\xi, y, \pm Y) \right)^2 + \left(\frac{\partial^2 u}{\partial \xi \partial z}(\xi, y, \pm Y) \right)^2 + \left(\frac{\partial^2 u}{\partial y \partial z}(\xi, y, \pm Y) \right)^2 \right\} \theta_m(\xi, y) \, dy \, d\xi < \infty,$$

and equation (P_λ) is satisfied in a weak sense with a nonlocal boundary condition:

$$(\xi, y) \to u(\xi, y, Y) \quad \text{and} \quad (\xi, y) \to u(\xi, y, -Y) \qquad \text{are continuous.}$$

In addition, if g is a continuous function on $\bar{\Delta}$, then the probabilistic interpretation of u is given by

$$u(\xi, y, z) = \mathbb{E} \int_0^\infty e^{-\lambda t} g(\xi_\alpha(t), y^\rho(t), z^\rho(t)) \, dt. \tag{7}$$

Remark 3. Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|y|\left(\frac{\partial u}{\partial z}(\xi,y,\pm Y)\right)^2\theta_m(\xi,y)\mathrm{d}y\mathrm{d}\xi < \infty$, the function $u(\xi,y,\pm Y)$ satisfies

$$\lambda u-\frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}-\frac{1}{2}\frac{\partial^2 u}{\partial y^2}+(\alpha\xi+c_0y\pm kY)\frac{\partial u}{\partial y}+\alpha\xi\frac{\partial u}{\partial \xi}=y\frac{\partial u}{\partial z}(\xi,y,\pm Y)+g(\xi,y,\pm Y),$$
$$(\xi,y)\in\mathbb{R}^2,$$

where the right-hand side can be seen as a function of $L_m^2(\mathbb{R})$, and therefore we can deduce from an elliptic regularity argument that $(\xi,y)\to u(\xi,y,\pm Y)$ is a continuous function. For the reader's convenience, let us explain the nonlocal boundary condition. If the values of the function u were known at $y=0,z=Y$ and $y=0,z=-Y$, then problem (P_λ) would be an elliptic degenerate PDE with a local nonhomogeneous Dirichlet condition. Then, this PDE could be solved by standard arguments related to a certain weighted Sobolev space. But here the originality resides in the fact that these values are not input data of the problem but a part of it. In this sense, it is a nonlocal problem and that makes it difficult to solve.

Remark 4. In the context of the probabilistic interpretation above (7), we deduce that for any $(\xi,y,z)\in\bar{\Delta}$, $u(\xi,y,z)\to u^1(\xi,y,z)$ as $\rho\to 1$ where

$$u^1(\xi,y,z)=\mathbb{E}\int_0^\infty e^{-\lambda t}g(\xi_\alpha(t),y^{\rho=1}(t),z^{\rho=1}(t))\mathrm{d}t$$

and then u^1 is continuous on Δ up to the boundary. Moreover, u^1 satisfies the original problem (P_λ) with $\rho=1$ in the sense of the distributions.

3. Proof of Theorem 2. Without loss of generality of the case $0\le\rho<1$, we can proceed with the proof of Theorem 2 taking $\rho=0$. For the reader's convenience, we will replace the notation $(.)^\rho$ by $(.)$. The organization of the proof is as follows. In the first subsection, we undertake a preliminary study of the boundary conditions. In particular, we provide some estimates and a new formulation of the function u on the boundary. Next, in the second subsection, we obtain a priori estimates on u. In the third subsection, we justify existence of a solution for (P_λ) using the framework of a partial differential variational inequality. Finally, in the fourth subsection, we justify uniqueness, and in the fifth subsection a probabilistic interpretation of u is provided.

3.1. A preliminary study of the boundary conditions (P_λ). Here, we study the equation (P_λ) on the boundary $(\xi,y)\in\Delta^\pm$. Consider the spaces for $m\ge 2$

$$L_m^2(\Delta^\pm):=\left\{\psi:\Delta^\pm\to\mathbb{R},\int_{\Delta^\pm}|\psi(\xi,y)|^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y<\infty\right\},$$

for $m\ge 3$

$$\tilde{H}_m(\Delta^\pm):=\left\{\psi\in L_{m-1}^2(\Delta^\pm),\int_{\Delta^\pm}\left(\left|\frac{\partial\psi}{\partial\xi}(\xi,y)\right|^2+\left|\frac{\partial\psi}{\partial y}(\xi,y)\right|^2\right)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y<\infty\right\}$$

and

$$\tilde{H}_m^0(\Delta^\pm):=\{\psi\in\tilde{H}_m(\Delta^\pm),\quad \psi(\xi,0)=0,\quad \forall\xi\in(-\infty,\infty)\}.$$

Moreover, consider the subset of $L^\infty(\Delta)$

$$L_\gamma^\infty(\Delta^\pm):=\left\{\psi:\Delta^\pm\to\mathbb{R},\|u\|_\infty\le\gamma\right\},\quad \gamma:=\frac{\|g\|_\infty}{\lambda}.$$

PROPOSITION 5. *There exist unique functions $\beta^+(\xi,y)\in\tilde H^0_m(\Delta^+)$ and $\beta^-(\xi,y)\in\tilde H^0_m(\Delta^-)$ such that*

$$(p^+_\lambda)\qquad \lambda\beta^+ + \mathbf{B}_+\beta^+ = g_+ \quad \text{in} \quad \Delta^+,\quad \beta^+(\xi,0)=0,\quad \forall\xi\in(-\infty,\infty),$$

and

$$(p^-_\lambda)\qquad \lambda\beta^- + \mathbf{B}_-\beta^- = g_- \quad \text{in} \quad \Delta^-,\quad \beta^-(\xi,0)=0,\quad \forall\xi\in(-\infty,\infty).$$

Proof. We only proceed with the first equation (p^+_λ). We first provide the existence of a solution. We will give a variational formulation of the equation (p^+_λ). The first step of the proof consists in defining a mapping from $L^\infty(\Delta^+)$ to $H^0_m(\Delta^+)$. For $\eta,\zeta\in\tilde H^0_m(\Delta^+)$, consider the bilinear form

$$\begin{aligned} a_+(\eta,\zeta) :=& \lambda\int_{\Delta^+}\eta\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y + \frac12\int_{\Delta^+}\left(\frac{\partial\eta}{\partial\xi}\frac{\partial\zeta}{\partial\xi}+\frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial y}\right)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\\ &- m\int_{\Delta^+}\left(\xi\frac{\partial\eta}{\partial\xi}+y\frac{\partial\eta}{\partial y}\right)\frac{\zeta}{1+\xi^2+y^2}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\\ &+\int_{\Delta^+}(\alpha\xi+c_0y+kY)\frac{\partial\eta}{\partial y}\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y+\int_{\Delta^+}\alpha\xi\frac{\partial\eta}{\partial\xi}\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y. \end{aligned}\tag{8}$$

For λ and μ sufficiently large the modified bilinear form

$$a_+(\eta,\zeta)+\mu\int_{\Delta^+}\eta\zeta\theta_m(\xi,y)(1+\xi^2+y^2)\mathrm{d}\xi\mathrm{d}y$$

is continuous and coercive on $\tilde H^0_m(\Delta^+)$. Therefore, if $f\in L^\infty(\Delta^+)$, the linear form

$$\int_{\Delta^+}f\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$

is also obviously continuous on $\tilde H^0_m(\Delta^+)$ and then there exists one and only one solution $\eta\in\tilde H^0_m(\Delta^+)$ to the variational formulation

$$a_+(\eta,\zeta)+\mu\int_{\Delta^+}\eta\zeta\theta_m(\xi,y)(1+\xi^2+y^2)\mathrm{d}\xi\mathrm{d}y=\int_{\Delta^+}f\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\quad\forall\zeta\in\tilde H^0_m(\Delta^+).$$

Hence, we can define a map $\eta=T_\mu\chi$. If $\chi\in L^\infty(\Delta^+)$, η is the unique solution of the variational formulation, $\forall\zeta\in\tilde H^0_m(\Delta^+)$,

$$a_+(\eta,\zeta)+\mu\int_{\Delta^+}\eta\zeta\theta_m(\xi,y)(1+\xi^2+y^2)\mathrm{d}\xi\mathrm{d}y=\int_{\Delta^+}(g+\mu(1+\xi^2+y^2)\chi)\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y.\tag{9}$$

The second step of the proof consists in showing that T_μ maps a compact subset of $L^2_m(\Delta^+)$ into itself. Next, we can show that if $\|\chi\|_\infty\le\gamma$, then $\|\eta\|_\infty\le\gamma$. Indeed, the justification is standard and for completeness we recall it below. Take $\zeta=(\eta-\gamma)^+$ which belongs to $\tilde H^0_m(\Delta^+)$ and plug it into (9). Then we obtain

$$\begin{aligned} a_+(\eta,(\eta-\gamma)^+)&+\mu\int_{\Delta^+}\eta(\eta-\gamma)^+\theta_m(\xi,y)(1+\xi^2+y^2)\mathrm{d}\xi\mathrm{d}y\\ &=\int_{\Delta^+}(g+\mu(1+\xi^2+y^2)\chi)(\eta-\gamma)^+\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y. \end{aligned}$$

Hence,

$$
\begin{aligned}
&a_+((\eta-\gamma)^+,(\eta-\gamma)^+)+\mu\int_{\Delta^+}\left((\eta-\gamma)^+\right)^2\theta_m(\xi,y)(1+\xi^2+y^2)\mathrm{d}\xi\mathrm{d}y\\
&=\int_{\Delta^+}(g-\lambda\gamma)(\eta-\gamma)^+\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\\
&\quad+\mu\int_{\Delta^+}(\eta-\gamma)^+\theta_m(\xi,y)(1+\xi^2+y^2)(\chi-\gamma)\mathrm{d}\xi\mathrm{d}y.
\end{aligned}
$$

And then since $\|g\|_\infty\leq\gamma$ and $\|\chi\|_\infty\leq\gamma$ we have $\int_{\Delta^+}\left((\eta-\gamma)^+\right)^2\theta_m(\xi,y)(1+\xi^2+y^2)\mathrm{d}\xi\mathrm{d}y=0$, which implies $(\eta-\gamma)^+=0$ and $\eta\leq\gamma$ as well. Similarly, we can pick up $\zeta=(-\eta-\gamma)^+$ and we also have $-\gamma\leq\eta$. Moreover, from the coercivity of

$$
a_+(\eta,\eta)+\mu\|\eta\|^2_{L^2_{m-1}(\Delta^+)}\geq\nu\|\eta\|^2_{\tilde{H}^0_m(\Delta^+)},\quad \nu>0,
$$

and from the continuity

$$
a_+(\eta,\eta)+\mu\|\eta\|^2_{L^2_{m-1}(\Delta^+)}\leq\left(\|g\|_{L^2_m(\Delta^+)}+\mu\|\chi\|_{L^2_{m-1}(\Delta^+)}\right)\|\eta\|_{L^2_{m-1}(\Delta^+)},
$$

we deduce that there exists $M>0$ depending on the $L^2_m(\Delta^+)$ norm of g, γ, and μ such that

$$
\|\eta\|_{\tilde{H}^0_m(\Delta^+)}\leq M.
$$

Therefore we define the subset of $\tilde{H}^0_m(\Delta^+)$

$$
K=\left\{\zeta\in\tilde{H}^0_m(\Delta^+),\quad \|\zeta\|_\infty\leq\gamma,\quad \|\zeta\|_{\tilde{H}^0_m(\Delta^+)}\leq M\right\},
$$

which is closed, convex in $\tilde{H}^0_m(\Delta^+)$. In addition, this set is not empty since it contains 0. And we observe that T_μ maps K into itself. The third and last step of the proof consists in showing that T_μ has a fixed point which must satisfy (p^+_λ) in a weak sense. Indeed, K is a compact subset of $L^2_m(\Delta^+)$ and T_μ is continuous. Therefore, T_μ has a fixed point β^+ satisfying

$$
a_+(\beta^+,\zeta)=\int_{\Delta^+}g\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\quad\forall\zeta\in\tilde{H}^0_m(\Delta^+). \tag{10}
$$

To recover the solution of (p^+_λ), taking an arbitrary function $\psi\in\mathcal{D}(\Delta^+)$ and plugging it into (10), we have

$$
\lambda\beta^++\mathbf{B}_+\beta^+=g\quad\text{in}\quad\Delta^+
$$

in the sense of distributions. By the definition of $\tilde{H}^{+,0}_m(\Delta^+)$, $\beta^+(\xi,0)=0$ $\forall\xi\in\mathbb{R}$. Next, we provide uniqueness of a solution using a probabilistic argument. Assume there is a function $w(\xi,y)$ bounded by a constant C which is solution of

$$
\lambda w+\mathbf{B}_+w=0\quad\text{in}\quad\Delta^+,\quad w(\xi,0)=0.
$$

Due to the structure of the operator $\mathbf{B}_+$, it is reasonable to assume that w is C^2 and then writing w in terms of the following stochastic differential equation:

$$
\begin{aligned}
&\mathrm{d}y(t)=-(c_0y(t)+kY+\alpha\xi_\alpha(t))\mathrm{d}t+dw(t),\quad y(0)=y,\quad\text{and}\\
&\mathrm{d}\xi_\alpha(t)=-\alpha\xi_\alpha(t)\mathrm{d}t+dw_\rho(t),\quad \xi_\alpha(0)=\xi.
\end{aligned}
$$

Let $M>0$ and define $\tau_M:=\inf\{t>0,\quad y(t)\notin(0,M)\text{ or }|\xi_\alpha(t)|>M\}$. For every $T>0$, an application of Ito's formula with $T\wedge\tau_M$ gives

$$
w(\xi,y)=\mathbb{E}\exp(-\lambda T\wedge\tau_M)w(\xi_\alpha(T\wedge\tau_M),y(T\wedge\tau_M)).
$$

From the decomposition

$$\begin{aligned} w(\xi,y) &= \mathbb{E}\exp(-\lambda T)w(\xi_\alpha(T),y(T))\mathbf{1}_{\{\tau_M>T\}} \\ &\quad+\mathbb{E}\exp(-\lambda\tau_M)w(\xi_\alpha(\tau_M),y(\tau_M))\mathbf{1}_{\{\tau_M<T\}}, \\ &= \mathbb{E}\exp(-\lambda T)w(\xi_\alpha(T),y(T))\mathbf{1}_{\{\tau_M>T\}} \\ &\quad+\mathbb{E}\exp(-\lambda\tau_M)w(\xi_\alpha(\tau_M),M)\mathbf{1}_{\{\tau_M<T,y(\tau_M)=M\}} \\ &\quad+\underbrace{\mathbb{E}\exp(-\lambda\tau_M)w(\xi_\alpha(\tau_M),0)\mathbf{1}_{\{\tau_M<T,y(\tau_M)=0\}}}_{=0 \quad \text{boundary condition of } w} \\ &\quad+\mathbb{E}\exp(-\lambda\tau_M)w(\pm M,y(\tau_M))\mathbf{1}_{\{\tau_M<T,\xi_\alpha(\tau_M)=\pm M\}} \\ &\le C\exp(-\lambda T)\mathbb{P}(\tau_M>T)+C\mathbb{P}(\tau_M<T,y(\tau_M)=M) \\ &\quad+C\mathbb{P}(\tau_M<T,\xi_\alpha(\tau_M)=\pm M), \end{aligned}$$

we obtain

$$|w(\xi,y)|\le C\exp(-\lambda T)+\frac{C}{M^2}\mathbb{E}\sup_{0\le t\le T}|y(t)|^2+\frac{C}{M^2}\mathbb{E}\sup_{0\le t\le T}|\xi_\alpha(t)|^2.$$

The right-hand side in the inequality goes to 0 as first M goes to ∞ and then T goes to ∞. Therefore $w=0$. The proof is done. ∎

3.1.1. A new formulation of the boundary condition. Next, we can give a new formulation of the function u on the boundary. The boundary conditions on Δ^+ and Δ^- can be replaced by

$$(\mathcal{BC}) \qquad u(\xi,y,Y)-\beta^+(\xi,y)\in K_\lambda^+, \quad u(\xi,y,-Y)-\beta^-(\xi,y)\in K_\lambda^-,$$

where

$$K_\lambda^+ := \left\{ w\in\tilde{H}_m(\Delta^+), \quad a_+(w,\psi)=0 \quad \forall\psi\in\tilde{H}_m^0(\Delta^+), \quad \|w\|_\infty\le\frac{\|g\|_\infty}{\lambda}\right\},$$

and by symmetry of the problem,

$$K_\lambda^- := \left\{ w:\Delta^-\to\mathbb{R}, \quad \sigma(w)\in K_\lambda^+\right\},$$

where σ is a symmetric operator which is defined by $\sigma(w)(\xi,y)=w(-\xi,-y)$. These sets are not empty since they contain 0. The condition $(\mathcal{BC})$ is useful to propose a weak formulation of problem (P_λ) as presented below. Next, in order to establish an appropriate functional space for finding the solution, we derive a priori estimates in the next subsection.

3.2. A priori estimates. Consider a solution u of (P_λ) that we assume bounded. The L^∞ bound on u does not follow from energy estimates below. We multiply the first equation of (P_λ) on Δ by $u(\xi,y,z)\theta_m(\xi,y)$ and we integrate over Δ. Then we obtain

$$\begin{aligned} &\int_\Delta\left\{\lambda u^2+\frac{1}{2}\left(\frac{\partial u}{\partial\xi}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right\}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &-m\int_\Delta\left(\xi\frac{\partial u}{\partial\xi}+y\frac{\partial u}{\partial y}\right)\frac{u}{1+\xi^2+y^2}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0}y(u(\xi,y,Y))^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y \end{aligned}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y(u(\xi,y,-Y))^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$
$$+\int_{\Delta}(\alpha\xi+c_0y+kz)\frac{\partial u}{\partial y}u\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\alpha\int_{\Delta}\xi\frac{\partial u}{\partial \xi}u\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z$$
$$=\int_{\Delta} gu\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y(u(\xi,y,Y))^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$
$$(\mathcal{E}_1)\qquad -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} y(u(\xi,y,-Y))^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y.$$

Next, we test the same equation with $y^3\frac{\partial u}{\partial z}\theta_m(\xi,y)$ and we integrate over Δ. Then we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y^3\left(\lambda u(\xi,y,Y)^2+\frac{1}{2}\left(\frac{\partial u}{\partial \xi}(\xi,y,Y)\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial u}{\partial y}(\xi,y,Y)\right)^2\right)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$
$$-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} y^3\left(\lambda u(\xi,y,-Y)^2+\frac{1}{2}\left(\frac{\partial u}{\partial \xi}(\xi,y,-Y)\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial u}{\partial y}(\xi,y,-Y)\right)^2\right)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$
$$+\int_{\Delta}\left(\frac{3}{2}-\frac{my^2}{1+\xi^2+y^2}\right)\frac{\partial u}{\partial y}y^2\frac{\partial u}{\partial z}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z$$
$$-m\int_{\Delta}\frac{\partial u}{\partial \xi}y^2\frac{\partial u}{\partial z}\theta_m(\xi,y)\left(\frac{y\xi}{1+\xi^2+y^2}\right)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z$$
$$+\int_{\Delta}(\alpha\xi+c_0y+kz)\frac{\partial u}{\partial y}y^3\frac{\partial u}{\partial z}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\alpha\int_{\Delta}\xi\frac{\partial u}{\partial \xi}y^3\frac{\partial u}{\partial z}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z$$
$$=-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} y^3\left(\lambda u(\xi,y,Y)^2+\frac{1}{2}\left(\frac{\partial u}{\partial \xi}(\xi,y,Y)\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial u}{\partial y}(\xi,y,Y)\right)^2\right)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$
$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y^3\left(\lambda u(\xi,y,-Y)^2+\frac{1}{2}\left(\frac{\partial u}{\partial \xi}(\xi,y,-Y)\right)^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{\partial u}{\partial y}(\xi,y,-Y)\right)^2\right)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$
$$(\mathcal{E}_2)\qquad +\int_{\Delta} gy^3\frac{\partial u}{\partial z}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\int_{\Delta} y^4\left(\frac{\partial u}{\partial z}\right)^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z.$$

Then, from $(\mathcal{E}_1)$, since u is bounded, if $m\geq 3$ we easily deduce that $u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial y}\in L^2_m(\Delta)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|y|u^2(\xi,y,\pm Y)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y<\infty.$$

Also, from $(\mathcal{E}_2)$ we deduce that if $m \geq 4$, then $y^2 \frac{\partial u}{\partial z} \in L^2_m(\Delta)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y|^3 u^2(\xi, y, \pm Y)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y < \infty,$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y|^3 \left(\frac{\partial u}{\partial \xi}(\xi, y, \pm Y)\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y < \infty,$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y|^3 \left(\frac{\partial u}{\partial y}(\xi, y, \pm Y)\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y < \infty.$$

3.2.1. Further regularity. Next, for more estimates on $\frac{\partial u}{\partial z}$, we consider the PDE

$$\lambda v + \mathbf{A} v = \frac{\partial g}{\partial z} - k\frac{\partial u}{\partial y} \quad \text{in} \quad \Delta, \quad v = 0 \quad \text{in} \quad \Delta^{\pm}. \tag{11}$$

For a solution v to this PDE, we can obtain the analogues of $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$. Here, there are no integral terms on Δ^- and Δ^+. Formally, $v = \frac{\partial u}{\partial z}$. Then we obtain

$$\begin{aligned}
&\int_\Delta \left\{\lambda v^2 + \frac{1}{2}\left(\frac{\partial v}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2\right\}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&- m\int_\Delta \left(\xi\frac{\partial v}{\partial \xi} + y\frac{\partial v}{\partial y}\right)\frac{v}{1+\xi^2+y^2}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&- \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} y(v(\xi, y, Y))^2\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y + \frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y(v(\xi, y, -Y))^2\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\\
&+ \int_\Delta (\alpha\xi + c_0 y + kz)\frac{\partial v}{\partial y}v\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&+ \alpha\int_\Delta \xi\frac{\partial v}{\partial \xi}v\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z = \int_\Delta \left(\frac{\partial g}{\partial z} - k\frac{\partial u}{\partial y}\right)v\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z
\end{aligned} \tag{$\mathcal{E}_3$}$$

and

$$\begin{aligned}
&\int_\Delta \left(\frac{3}{2} - \frac{my^2}{1+\xi^2+y^2}\right)\frac{\partial v}{\partial y}y^2\frac{\partial v}{\partial z}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&- m\int_\Delta \frac{\partial v}{\partial \xi}y^2\frac{\partial v}{\partial z}\theta_m(\xi, y)\left(\frac{y\xi}{1+\xi^2+y^2}\right)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&+ \int_\Delta (\alpha\xi + c_0 y + kz)\frac{\partial v}{\partial y}y^3\frac{\partial v}{\partial z}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z + \alpha\int_\Delta \xi\frac{\partial v}{\partial \xi}y^3\frac{\partial v}{\partial z}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&= -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} y^3\left(\lambda v(\xi, y, Y)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial \xi}(\xi, y, Y)\right)^2\right.\\
&\qquad\qquad \left. + \frac{1}{2}\left(\frac{\partial v}{\partial y}(\xi, y, Y)\right)^2\right)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y
\end{aligned}$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y^3\left(\lambda v(\xi,y,-Y)^2+\frac{1}{2}\left(\frac{\partial v}{\partial \xi}(\xi,y,-Y)\right)^2 \right.$$
$$\left. +\frac{1}{2}\left(\frac{\partial v}{\partial y}(\xi,y,-Y)\right)^2\right)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y$$
$$+\int_\Delta\left(\frac{\partial g}{\partial z}-\frac{\partial u}{\partial y}\right)y^3\frac{\partial v}{\partial z}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\int_\Delta y^4\left(\frac{\partial v}{\partial z}\right)^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z. \quad (\mathcal{E}_4)$$

Therefore we deduce that if $m\geq 3$, then $\frac{\partial u}{\partial z},\frac{\partial^2 u}{\partial\xi\partial z},\frac{\partial^2 u}{\partial y\partial z}\in L^2_m(\Delta)$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|y|\left(\frac{\partial u}{\partial z}(\xi,y,\pm Y)\right)^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y<\infty,$$

and also that if $m\geq 4$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|y|^3\left\{\left(\frac{\partial u}{\partial z}(\xi,y,\pm Y)\right)^2+\left(\frac{\partial^2 u}{\partial \xi\partial z}(\xi,y,\pm Y)\right)^2\right.$$
$$\left.+\left(\frac{\partial^2 u}{\partial y\partial z}(\xi,y,\pm Y)\right)^2\right\}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y<\infty.$$

3.3. Existence of a solution for (P_λ). First, we proceed with a regularization approach by adding $\epsilon\frac{\partial^2}{\partial z^2}$ to the operator A. In that context, we show the existence of a solution to the regularized problem using a variational inequality. Then we provide uniform bounds on the solution with respect to the parameter ϵ. Finally, we extract a converging, subsequence whose limit is a solution to (P_λ).

Define

$$\tilde{H}_m(\Delta):=\left\{u\in L^2_{m-1}(\Delta),\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\in L^2_m(\Delta)\right\},\quad m\geq 3,$$

and

$$\tilde{K}_m:=\Big\{u\in\tilde{H}_m(\Delta),\quad u(\xi,y,Y)-\beta^+(\xi,y)\in K^+_\lambda \text{ for } y>0,$$
$$u(\xi,y,-Y)-\beta^-(\xi,y)\in K^-_\lambda \text{ for } y<0\Big\}.$$

It is clear that $\tilde{K}_m$ is a closed and convex set of $\tilde{H}_m(\Delta)$. To show it is nonempty, we take

$$\Psi(\xi,y,z):=\frac{Y+z}{2}\beta^+(\xi,y)\mathbf{1}_{\{y>0\}}+\frac{Y-z}{2}\beta^-(\xi,y)\mathbf{1}_{\{y<0\}},$$

then $\Psi\in\tilde{K}_m$. We also consider a bilinear form for $\eta,\zeta\in\tilde{H}_m(\Delta)$,

$$a(\eta,\zeta):=\lambda\int_\Delta\eta\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\frac{1}{2}\int_\Delta\left\{\frac{\partial\eta}{\partial\xi}\frac{\partial\zeta}{\partial\xi}+\frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial y}\right\}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z$$
$$-m\int_\Delta\left(\xi\frac{\partial\eta}{\partial\xi}+y\frac{\partial\eta}{\partial y}\right)\frac{\zeta}{1+\xi^2+y^2}\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z$$
$$+\int_\Delta(\alpha\xi+c_0y+kz)\frac{\partial\eta}{\partial y}\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z$$
$$+\int_\Delta\alpha\xi\frac{\partial\eta}{\partial\xi}\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z-\int_\Delta y\frac{\partial\eta}{\partial z}\zeta\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z, \quad (12)$$

and a linear form

$$l(\zeta) := \int_{\Delta} g(\xi, y, z)\zeta(\xi, y, z)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z.$$

3.3.1. Approximation. We look for a bounded function $u^\epsilon \in \tilde{K}_m$ satisfying

$$(P_\lambda^\epsilon) \qquad \begin{cases} \lambda u^\epsilon + \frac{\epsilon}{2}\frac{\partial^2 u^\epsilon}{\partial z^2} + \mathbf{A}u^\epsilon = g \text{ in } \Delta, \\ \frac{\partial u^\epsilon}{\partial z} = 0 \quad \text{for} \quad y < 0, z = Y, \\ \frac{\partial u^\epsilon}{\partial z} = 0 \quad \text{for} \quad y > 0, z = -Y. \end{cases}$$

We will give a variational formulation of (P_λ^ϵ). First, we define the bilinear form for $\eta, \zeta \in \tilde{H}_m(\Delta)$,

$$a_\epsilon(\eta, \zeta) := \frac{\epsilon}{2}\int_{\Delta} \frac{\partial \eta}{\partial z}\frac{\partial \zeta}{\partial z}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z + a(\eta, \zeta).$$

It is straightforward to check that $a_\epsilon(\eta, \zeta)$ is continuous on $\tilde{H}_m(\Delta)$, and if we consider the modified bilinear form

$$a_\epsilon(\eta, \zeta) + \mu\int_{\Delta}(1 + \xi^2 + y^2)\eta\zeta\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z,$$

then for λ and μ sufficiently large this modified bilinear form is coercive. Therefore, if $f \in L^\infty(\Delta)$, then there exists one and only one solution $\eta \in \tilde{K}_m$ to the variational inequality

$$\begin{aligned} &a_\epsilon(\eta, \zeta - \eta) + \mu\int_{\Delta}(1 + \xi^2 + y^2)\eta(\zeta - \eta)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &\geq \int_{\Delta} f(\zeta - \eta)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \ \forall \zeta \in \tilde{K}_m. \end{aligned}$$

Next, in a similar manner to what was done before, we proceed by defining a map S_μ. If $\chi \in L^\infty(\Delta)$, then $\eta \in \tilde{K}_m$ is the solution of the following variational inequality:

$$\begin{aligned} a_\epsilon(\eta, \zeta - \eta) + &\mu\int_{\Delta}(1 + \xi^2 + y^2)\eta(\zeta - \eta)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &\geq \int_{\Delta}(g + \mu(1 + \xi^2 + y^2)\chi)(\zeta - \eta)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \ \forall \zeta \in \tilde{K}_m. \end{aligned}$$

We can justify that if $\|\chi\|_{L^\infty} \leq \frac{\|g\|_\infty}{\lambda}$, then $\|\eta\|_{L^\infty} \leq \frac{\|g\|_\infty}{\lambda}$. Indeed, if we pick up $\zeta = \eta - (\eta - \gamma)^+$ which belongs to $\tilde{K}_m$ by definition of η and $\beta^\pm$ on the boundary, we obtain

$$\begin{aligned} &- a_\epsilon(\eta, (\eta - \gamma)^+) - \mu\int_{\Delta}(1 + \xi^2 + y^2)\eta(\eta - \gamma)^+\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &\geq - \int_{\Delta}\left(g + \mu(1 + \xi^2 + y^2)\chi\right)(\eta - \gamma)^+\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z. \end{aligned}$$

Thus

$$\begin{aligned} &a_\epsilon(\eta, (\eta - \gamma)^+) + \mu\int_{\Delta}(1 + \xi^2 + y^2)\eta(\eta - \gamma)^+\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &\leq \int_{\Delta}\left(g + \mu(1 + \xi^2 + y^2)\chi\right)(\eta - \gamma)^+\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \end{aligned}$$

and since

$$
\begin{aligned}
&a_\epsilon(\eta,(\eta-\gamma)^+) = a_\epsilon((\eta-\gamma)^+,(\eta-\gamma)^+) + \lambda\int_\Delta \gamma(\eta-\gamma)^+\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z,\\
&\int_\Delta (1+\xi^2+y^2)\eta(\eta-\gamma)^+\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&= \int_\Delta (1+\xi^2+y^2)\left((\eta-\gamma)^+\right)^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z + \lambda\int_\Delta \gamma(\eta-\gamma)^+\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z,
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
&a_\epsilon((\eta-\gamma)^+,(\eta-\gamma)^+) + \mu\int_\Delta (1+\xi^2+y^2)\left((\eta-\gamma)^+\right)^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&\le \int_\Delta (g-\lambda\gamma)(\eta-\gamma)^+\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z + \mu\int_\Delta (1+\xi^2+y^2)(\chi-\gamma)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

That implies $\|\eta\|_{L^\infty} \le \frac{\|g\|_\infty}{\lambda}$. In addition, from the coercivity of the modified bilinear form

$$
a_\epsilon(\eta,\eta) + \mu\int_\Delta (1+\xi^2+y^2)\eta^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \ge \nu_\epsilon\|\eta\|^2_{\tilde{H}_m(\Delta)}, \quad \nu_\epsilon > 0,
$$

and from the inequality

$$
\begin{aligned}
&a_\epsilon(\eta,\psi_0) + \mu\int_\Delta (1+\xi^2+y^2)\eta\psi_0\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&\quad - \int_\Delta (g+\mu(1+\xi^2+y^2)\chi)(\psi_0-\eta)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&\quad \ge a_\epsilon(\eta,\eta) + \mu\int_\Delta (1+\xi^2+y^2)\eta^2\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z,
\end{aligned}
$$

where $\psi_0 \in \tilde{K}_m$ is chosen arbitrarily, we can prove that there exists a constant M_ϵ depending on the L^2-norm of g, γ, and μ such that

$$
\|\eta\|_{\tilde{H}_m(\Delta)} \le M_\epsilon.
$$

Then, we define

$$
L_m^{M_\epsilon} := \left\{\zeta \in \bar{K}_m, \quad \|\zeta\|_\infty \le \gamma \text{ and } \|\zeta\|_{\tilde{H}_m(\Delta)} \le M_\epsilon\right\},
$$

then we must have $S_\mu(L_m^{M_\epsilon}) \subset L_m^{M_\epsilon}$. But $L_m^{M_\epsilon}$ is a compact subset of $L^2_m(\Delta)$ and S_μ is continuous. It implies that there exists a unique fixed point $u^\epsilon \in L_m^{M_\epsilon}$ satisfying

$$
(\mathcal{V}^\epsilon) \qquad a_\epsilon(u^\epsilon,\zeta-u^\epsilon) \ge \int_\Delta g(\zeta-u^\epsilon)\theta_m(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \ \forall\zeta \in \tilde{K}_m.
$$

Therefore u^ϵ is a weak solution of (P_λ^ϵ).

3.3.2. Uniform bounds with respect to ϵ. Next, we derive some uniform bounds with respect to ϵ for u^ϵ. Let us write out $(\mathcal{V}^\epsilon)$; we get that $\forall \zeta \in \tilde{K}_m$

$$\begin{aligned}
&\lambda \int_\Delta u^\epsilon(\zeta - u^\epsilon)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z + \frac{\epsilon}{2}\int_\Delta \frac{\partial u^\epsilon}{\partial z}\left(\frac{\partial \zeta}{\partial z} - \frac{\partial u^\epsilon}{\partial z}\right)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&+ \frac{1}{2}\int_\Delta \frac{\partial u^\epsilon}{\partial \xi}\left(\frac{\partial \zeta}{\partial \xi} - \frac{\partial u^\epsilon}{\partial \xi}\right)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z + \frac{1}{2}\int_\Delta \frac{\partial u^\epsilon}{\partial y}\left(\frac{\partial \zeta}{\partial y} - \frac{\partial u^\epsilon}{\partial y}\right)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&- m\int_\Delta \left(\xi\frac{\partial u^\epsilon}{\partial \xi} + y\frac{\partial u^\epsilon}{\partial y}\right)\frac{(\zeta - u^\epsilon)}{1 + \xi^2 + y^2}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&+ \int_\Delta (\alpha\xi + c_0 y + kz)\frac{\partial u^\epsilon}{\partial y}(\zeta - u^\epsilon)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&+ \int_\Delta \alpha\xi\frac{\partial u^\epsilon}{\partial \xi}(\zeta - u^\epsilon)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z - \int_\Delta y\frac{\partial u^\epsilon}{\partial z}(\zeta - u^\epsilon)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&\geq \int_\Delta g(\zeta - u^\epsilon)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z.
\end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}
&\int_\Delta \left\{\lambda u^2 + \frac{\epsilon}{2}\left(\frac{\partial u^\epsilon}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u^\epsilon}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial u^\epsilon}{\partial y}\right)^2\right\}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&- m\int_\Delta \left(\xi\frac{\partial u^\epsilon}{\partial \xi} + y\frac{\partial u^\epsilon}{\partial y}\right)\frac{u^\epsilon}{1 + \xi^2 + y^2}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&+ \int_\Delta (\alpha\xi + c_0 y + kz)\frac{\partial u^\epsilon}{\partial y}u^\epsilon\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&+ \int_\Delta \alpha\xi\frac{\partial u^\epsilon}{\partial \xi}u^\epsilon\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} y(u^\epsilon(\xi, y, Y))^2\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
&+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y(u^\epsilon(\xi, y, -Y))^2\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
&\leq \int_\Delta \left\{\lambda u\zeta + \frac{\epsilon}{2}\frac{\partial u^\epsilon}{\partial z}\frac{\partial \zeta}{\partial z} + \frac{1}{2}\frac{\partial u^\epsilon}{\partial \xi}\frac{\partial \zeta}{\partial \xi} + \frac{1}{2}\frac{\partial u^\epsilon}{\partial y}\frac{\partial \zeta}{\partial y}\right\}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&- m\int_\Delta \left(\xi\frac{\partial u^\epsilon}{\partial \xi} + y\frac{\partial u^\epsilon}{\partial y}\right)\frac{\zeta}{1 + \xi^2 + y^2}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&+ \int_\Delta (\alpha\xi + c_0 y + kz)\frac{\partial u^\epsilon}{\partial y}\zeta\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&+ \int_\Delta \alpha\xi\frac{\partial u^\epsilon}{\partial \xi}\zeta\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} y(u^\epsilon\zeta)(\xi, y, Y)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
&+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} y(u^\epsilon\zeta)(\xi, y, -Y)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y + \int_\Delta g u^\epsilon\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
&- \int_\Delta g\zeta\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \quad \forall \zeta \in \bar{K}_m.
\end{aligned}$$

Since u^ϵ is uniformly bounded, we deduce there exists a constant $C > 0$ which does not depend on ϵ such that

$$
\begin{aligned}
(14) \qquad & \epsilon \int_\Delta \left(\frac{\partial u^\epsilon}{\partial z}\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z < C, \int_\Delta \left(\frac{\partial u^\epsilon}{\partial \xi}\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z < C, \\
& \int_\Delta \left(\frac{\partial u^\epsilon}{\partial y}\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z < C, \\
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y|(u^\epsilon(\xi, y, \pm Y))^2\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y < C, \quad m \geq 3.
\end{aligned}
$$

Next, since $u^\epsilon \in \tilde{H}_m(\Delta)\cap L^\infty(\Delta)$, we can multiply the equation $\lambda u^\epsilon + \frac{\epsilon}{2}\frac{\partial^2 u^\epsilon}{\partial z^2} + \mathbf{A}u^\epsilon = g$ by $\frac{\partial u^\epsilon}{\partial z} y^3\theta_m(\xi, y)$ on both sides and integrate over Δ in order to get

$$
\begin{aligned}
& \int_\Delta \left(y^2\frac{\partial u^\epsilon}{\partial z}\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\
& - \frac{\lambda}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} (u^\epsilon(\xi, y, Y))^2 y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& + \frac{\lambda}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} (u^\epsilon(\xi, y, -Y))^2 y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& + \frac{\epsilon}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\frac{\partial u^\epsilon}{\partial z}\right)^2 (\xi, y, Y) y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& - \frac{\epsilon}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \left(\frac{\partial u^\epsilon}{\partial z}\right)^2 (\xi, y, -Y))^2 y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& - \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \left(\frac{\partial u^\epsilon}{\partial y}\right)^2 (\xi, y, Y) y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& + \frac{1}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\frac{\partial u^\epsilon}{\partial y}(\xi, y, -Y)\right)^2 y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& - \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \left(\frac{\partial u^\epsilon}{\partial \xi}\right)^2 (\xi, y, Y) y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& + \frac{1}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\frac{\partial u^\epsilon}{\partial \xi}(\xi, y, -Y)\right)^2 y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& = \frac{\lambda}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} (u^\epsilon(\xi, y, Y))^2 y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& - \frac{\lambda}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} (u^\epsilon(\xi, y, -Y))^2 y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& - \frac{\epsilon}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \left(\frac{\partial u^\epsilon}{\partial z}\right)^2 (\xi, y, Y) y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\
& + \frac{\epsilon}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\left(\frac{\partial u^\epsilon}{\partial z}\right)^2 (\xi, y, -Y)\right) y^3\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{\partial u^{\epsilon}}{\partial y}\right)^{2}(\xi,y,Y)y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\\
&-\frac{1}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{\partial u^{\epsilon}}{\partial y}(\xi,y,-Y)\right)^{2}y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\\
&+\frac{1}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{\partial u^{\epsilon}}{\partial \xi}\right)^{2}(\xi,y,Y)y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\\
&-\frac{1}{4}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{\partial u^{\epsilon}}{\partial \xi}(\xi,y,-Y)\right)^{2}y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\\
&-\int_{\Delta}g\frac{\partial u^{\epsilon}}{\partial z}y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\int_{\Delta}\alpha\xi\frac{\partial u^{\epsilon}}{\partial \xi}\frac{\partial u^{\epsilon}}{\partial z}y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&+\int_{\Delta}(\alpha\xi+c_{0}y+kz)\frac{\partial u^{\epsilon}}{\partial y}\frac{\partial u^{\epsilon}}{\partial z}y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z+\frac{3}{2}\int_{\Delta}\frac{\partial u^{\epsilon}}{\partial y}\frac{\partial u^{\epsilon}}{\partial z}y^{2}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z\\
&-m\int_{\Delta}\left(\xi\frac{\partial u^{\epsilon}}{\partial \xi}+y\frac{\partial u^{\epsilon}}{\partial y}\right)\frac{\partial u^{\epsilon}}{\partial z}\frac{y^{2}}{1+\xi^{2}+y^{2}}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

Now, observe that we must have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{0}\left(\frac{\partial u^{\epsilon}}{\partial z}\right)^{2}(\xi,y,Y)y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y=0\quad\text{and}\\
&\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{\partial u^{\epsilon}}{\partial z}\right)^{2}(\xi,y,-Y)y^{3}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y=0
\end{aligned}
$$

since

$$\frac{\partial u^{\epsilon}}{\partial z}=0\qquad\text{for }\operatorname{sign}(y)z=-Y.$$

Therefore, if $m\geq 4$, then there exists a constant C which does not depend on ϵ such that

$$
\begin{aligned}
&\int_{\Delta}\left(y^{2}\frac{\partial u^{\epsilon}}{\partial z}\right)^{2}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z<C,\\
&\epsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|y|^{3}\left(\frac{\partial u^{\epsilon}}{\partial z}(\xi,y,\pm Y)\right)^{2}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y<C,\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|y|^{3}\left(\frac{\partial u^{\epsilon}}{\partial \xi}(\xi,y,\pm Y)\right)^{2}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y<C,\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|y|^{3}\left(\frac{\partial u^{\epsilon}}{\partial y}(\xi,y,\pm Y)\right)^{2}\theta_{m}(\xi,y)\mathrm{d}\xi\mathrm{d}y<C.
\end{aligned}
$$

Further regularity. Next, we consider the PDE

$$\lambda v^{\epsilon}+\mathbf{A}v^{\epsilon}=\frac{\partial g}{\partial z}-k\frac{\partial u^{\epsilon}}{\partial y}\ \text{ in }\ \Delta,\quad \frac{\epsilon}{2}\frac{\partial v^{\epsilon}}{\partial z}+yv^{\epsilon}=0\ \text{ in }\ \Delta^{\pm},\ v^{\epsilon}=0\ \text{ in }\operatorname{sign}(y)z=-Y.$$

Formally, $v^\epsilon = \frac{\partial u^\epsilon}{\partial z}$. And then, we test $v^\epsilon \theta_m(\xi, y)$ with this PDE, we integrate over Δ, and we get

$$\begin{aligned}&\int_\Delta \left\{ \lambda(v^\epsilon)^2 + \frac{\epsilon}{2}\left(\frac{\partial v^\epsilon}{\partial z}\right)^2 + \frac{1}{2}\int_\Delta \left(\frac{\partial v^\epsilon}{\partial \xi}\right)^2 + \frac{1}{2}\int_\Delta \left(\frac{\partial v^\epsilon}{\partial y}\right)^2 \right\} \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &+ \frac{1}{2}\int_{-\infty}^{\infty}\int_0^{\infty} (v^\epsilon(\xi, y, Y))^2 |y| \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\ &+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{0} (v^\epsilon(\xi, y, -Y))^2 |y| \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y \\ &- m\int_\Delta \left(\xi \frac{\partial v^\epsilon}{\partial \xi} + y\frac{\partial v^\epsilon}{\partial y}\right) \frac{v^\epsilon}{1+\xi^2+y^2}\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &+ \int_\Delta (\alpha\xi + c_0 y + kz)\frac{\partial v^\epsilon}{\partial y} v^\epsilon \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \\ &+ \int_\Delta \alpha\xi \frac{\partial v^\epsilon}{\partial \xi} v^\epsilon \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z = \int_\Delta \left(\frac{\partial g}{\partial z} - k\frac{\partial u^\epsilon}{\partial z}\right)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z. \end{aligned} \tag{15}$$

Therefore, we deduce that if $m \geq 4$, then

$$\begin{aligned}&\int_\Delta \left(\frac{\partial u^\epsilon}{\partial z}\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z < C, \int_\Delta \left(\frac{\partial^2 u^\epsilon}{\partial \xi \partial z}\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z < C, \\ &\int_\Delta \left(\frac{\partial^2 u^\epsilon}{\partial y \partial z}\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z < C, \\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y| \left(\frac{\partial u^\epsilon}{\partial z}(\xi, y, \pm Y)\right)^2 \theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y < C. \end{aligned} \tag{16}$$

3.3.3. Passage to the limit as ϵ goes to 0. Finally, from the bounds above we know that up to a subsequence there exists a function $u \in L^2_m(\Delta)$ such that $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \in L^2_m(\Delta)$, where $\frac{\partial u^\epsilon}{\partial \xi} \to \frac{\partial u}{\partial \xi}, \frac{\partial u^\epsilon}{\partial y} \to \frac{\partial u}{\partial y}, \frac{\partial u^\epsilon}{\partial z} \to \frac{\partial u}{\partial z}$ weakly and $u^\epsilon \to u$ strongly in $L^2_m(\Delta)$. Moreover letting ϵ go to 0 in $(\mathcal{V}^\epsilon)$ we obtain that u satisfies

$$a(u, \zeta - u) \geq \int_\Delta g(\zeta - u)\theta_m(\xi, y)\mathrm{d}\xi\mathrm{d}y\mathrm{d}z \ \forall \zeta \in \tilde{K}_m. \tag{17}$$

That shows existence of a solution for (P_λ).

3.3.4. Existence of a solution to (P_λ) when $\lambda > 0$. We complete this section by showing that existence of a solution to (P_λ) holds for $\lambda > 0$. We proceed in a similar manner to what was done p. 14 in [8]. We consider a map T_α defined on $L^\infty(\Delta)$ by $u = T_\alpha w$, where $w \in L^\infty(\Delta)$, $\alpha > 0$ is sufficiently large, and u satisfies

$$\begin{aligned}&(\lambda + \alpha)u + \mathbf{A}u = g + \alpha w \text{ in } \Delta, \quad (\lambda + \alpha)u + \mathbf{B}_+ u = g + \alpha w \text{ in } \Delta^+, \\ &(\lambda + \alpha)u + \mathbf{B}_- u = g + \alpha w \text{ in } \Delta^-\end{aligned}$$

Then a fixed point of T_α is a solution of (P_λ). It is clear that T_α is a contraction in $L^\infty(\Delta)$ since

$$\|T_\alpha(w_1) - T_\alpha(w_2)\|_\infty \leq \frac{\alpha}{\alpha + \lambda}\|w_1 - w_2\|_\infty.$$

3.4. A probabilistic interpretation of u. Because $u \in L^{\infty}(\Delta), \frac{\partial u}{\partial y}, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial z} \in L^2_m(\Delta)$ and the operator's structure of $\mathbf{A}^{\rho}$, local H^2 estimates can be obtained on any subdomain U of Δ avoiding a neighborhood of $z = \pm Y, y = 0$. That provides the continuity of u up to the boundary of U. Moreover, as mentioned in Remark 3, $(\xi, y) \to u(\xi, y \pm Y)$ are also continuous. Therefore u is continuous everywhere on Δ up to the boundary. Hence we can apply a generalized version of Ito's lemma to get (7); see [1, 19]. We can write u in terms of $(\mathcal{SVI}, \alpha)$. Let $M > 0$ and $\theta_M := \inf\{t > 0,\ |y(t)| > M \text{ or } |\xi_\alpha(t)| > M\}$; then for every $T > 0$, an application of Ito's formula with $T \wedge \theta_M$ gives

$$\begin{aligned} u(\xi, y, z) &= \mathbb{E} \exp(-\lambda T \wedge \theta_M) u(\xi_\alpha(T \wedge \theta_M), y(T \wedge \theta_M), z(T \wedge \theta_M)) \\ &\quad + \mathbb{E} \int_0^{T \wedge \theta_M} \exp(-\lambda s) g(\xi_\alpha(s), y(s), z(s)) \mathrm{d}s. \end{aligned}$$

From the decomposition

$$\begin{aligned} & u(\xi, y, z) - \mathbb{E} \int_0^{T \wedge \theta_M} \exp(-\lambda s) g(\xi_\alpha(s), y(s), z(s)) \mathrm{d}s \\ &= \mathbb{E} \exp(-\lambda T) u(\xi_\alpha(T), y(T), z(t)) \mathbf{1}_{\{\theta_M > T\}} \\ &\quad + \mathbb{E} \exp(-\lambda \theta_M) u(\xi_\alpha(\theta_M), y(\theta_M), z(\theta_M)) \mathbf{1}_{\{\theta_M < T\}}, \end{aligned}$$

we obtain

$$\begin{aligned} & |u(\xi, y, z) - \mathbb{E} \int_0^{T \wedge \theta_M} \exp(-\lambda s) g(\xi_\alpha(s), y(s), z(s)) \mathrm{d}s| \leq C \exp(-\lambda T) \\ & + \frac{C}{M^2} \mathbb{E} \sup_{0 \leq t \leq T} |y(t)|^2 + \frac{C}{M^2} \mathbb{E} \sup_{0 \leq t \leq T} |\xi_\alpha(t)|^2. \end{aligned}$$

On the one hand, the left-hand side in the inequality goes to $|u(\xi, y, z) - \mathbb{E} \int_0^{\infty} \exp(-\lambda s) g(\xi_\alpha(s), y(s), z(s)) \mathrm{d}s|$, and on the other hand, the right-hand side in the inequality goes to 0, we take first M going to ∞, and then T going to ∞.

3.5. Uniqueness. We must prove that if u satisfies (P_λ) with $g = 0$, then $u = 0$. This is a consequence of the bound $\|u\|_\infty \leq \frac{\|g\|_\infty}{\lambda}$ and the probabilistic interpretation. Then the solution of (P_λ) is unique.

4. General case when $g \in L^{\infty}(\Delta)$ but $\frac{\partial g}{\partial z} \notin L^2_2(\Delta)$. When $g \in L^{\infty}(\Delta)$ but $\frac{\partial g}{\partial z} \notin L^2_2(\Delta)$, we can find a sequence of functions $g^n \to g$ in $L^2_2(\Delta)$ with $\frac{\partial g^n}{\partial z} \in L^2_2(\Delta)$ and $\|g^n\|_\infty \leq \|g\|_\infty$. Then denoting u^n the corresponding solution of (P_λ) with g^n at the right-hand side, we observe that u^n has a weak limit u in $L^2_m(\Delta)$. In general, estimates $(\mathcal{E}_3)$ and $(\mathcal{E}_4)$ are not satisfied by u, and thus we cannot write the variational inequality for u. However, the weak limit of u^n in the sense of the estimates $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ will satisfy the first equation of (P_λ) in the sense of distributions. Also, the boundary conditions are obtained by considering the weak limit in L^{∞}-star of $\chi^n_+ := u^n - \beta^+ \in K^+_\lambda, \chi^n_- := u^n - \beta^- \in K^-_\lambda$.

5. Appendix

5.1. Existence and uniqueness of a solution for $(\mathcal{SVI}, \alpha)$. One way to construct a solution to $(\mathcal{SVI}, \alpha)$ is to proceed by penalization. This is a natural approach in the context of processes with reflection [21]. Also, see [20] for the specific case of an SVI modeling an EPP oscillator excited by a white noise. The idea is simple. The variational inequality is replaced by an equation involving a nonlinear Lipschitz function depending on a parameter n penalizing the solution whenever it goes beyond a prespecified area $|z| > Y$, as shown below:

$$(\mathcal{SVI}, \alpha, n) \qquad \begin{cases} \mathrm{d}\xi_\alpha(t) = -\alpha\xi_\alpha(t)\mathrm{d}t + \mathrm{d}w(t), \\ \mathrm{d}y_{\alpha,n}(t) = -(c_0 y_{\alpha,n}(t) + kz_{\alpha,n}(t))\mathrm{d}t + \mathrm{d}\xi_\alpha(t), \\ \mathrm{d}z_{\alpha,n}(t) = y_{\alpha,n}(t)\mathrm{d}t - n(z_{\alpha,n}(t) - \pi(z_{\alpha,n}(t)))\mathrm{d}t, \end{cases}$$

where $\pi(\zeta) = \zeta$ if $|\zeta| \leq Y$ and $\pi(\zeta) = \mathrm{sign}(\zeta)$ otherwise. Thus, it can be shown that the penalized process converges as n goes to ∞ and then the limiting process is identified as a solution of $(\mathcal{SVI}, \alpha)$. Uniqueness can be obtained by the variational inequality structure. The proof can be done in a very similar to what was done in [20]. For the reader's convenience, we recall here the main ideas in two propositions.

The first step consists in showing that $(y_{\alpha,n}, z_{\alpha,n})$ is a Cauchy sequence.

Proposition 6. *Fix $T > 0$; the sequence $(y_{\alpha,n}, z_{\alpha,n})$ satisfies the following Cauchy property:* $\forall \epsilon > 0, \exists N_\epsilon, \forall n, m > N_\epsilon$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\{|y_{\alpha,n}(t) - y_{\alpha,m}(t)|^2 + |z_{\alpha,n}(t) - z_{\alpha,m}(t)|^2\right\}\right) < \epsilon.$$

We then identify a candidate, namely, $(\tilde{y}_\alpha, \tilde{z}_\alpha) \triangleq \lim_{n\to\infty}(y_{\alpha,n}, z_{\alpha,n})$. The second step consists in showing that $(\tilde{y}_\alpha, \tilde{z}_\alpha)$ together with

$$\tilde{\Delta}_\alpha(t) \triangleq \lim_{n\to\infty} n \int_0^t (z_{\alpha,n}(s) - \pi(z_{\alpha,n}(s)))\,\mathrm{d}s$$

solve a reflected problem, as shown in the proposition below.

Proposition 7. *The process $(\tilde{y}_\alpha, \tilde{z}_\alpha, \tilde{\Delta}_\alpha)$ satisfies the following properties:*

1. $\tilde{y}_\alpha(0) = y_0, \tilde{z}_\alpha(0) = z_0$,
2. $\tilde{y}_\alpha, \tilde{z}_\alpha$ *and* $\tilde{\Delta}_\alpha$ *are adapted and continuous,*
3. $|\tilde{z}_\alpha(t)| \leq Y \quad \forall t$ *a.s.,*
4. *it has bounded variations,*
5. $\int_{t_1}^{t_2} \mathbf{1}_{\{|\tilde{z}_\alpha(t)|<Y\}} d\tilde{\Delta}_\alpha(t) = 0 \ \forall t_1 < t_2$ *a.s.,*
6. $d\tilde{z}_\alpha(t) = \tilde{y}_\alpha(t)dt - \mathbf{1}_{\{|\tilde{z}_\alpha(t)|=Y\}} d\tilde{\Delta}_\alpha(t)$.

Finally, as shown in [1] (see p. 49), solving the reflected problem above is equivalent to solving $(\mathcal{SVI}, \alpha)$. Existence and uniqueness of a solution of the variant of $(\mathcal{SVI}, \alpha)$ with parameter ρ can be obtained in a very similar way.

5.2. $(y_\alpha(t), z_\alpha(t))$ depends continuously on ξ_α. The fact that $(y_\alpha(t), z_\alpha(t))$ is adapted can be seen from the penalization procedure. To be complete, we show that $(y_\alpha(t), z_\alpha(t))$ depends continuously on ξ_α. Fix $T > 0$ and $t \in [0, T]$ using the notation $b(y, z) = -c_0 y - kz$, the mapping $g(.) \to (y(.), z(.))$ of

$$\begin{cases} y(t) = y_0 + \int_0^t b(y(s), z(s))\mathrm{d}s + g(t), \\ \int_0^t (\mathrm{d}z(s) - y(s)\mathrm{d}s)(\varphi(s) - z(s)) \geq 0 \ \forall \varphi : [0, T] \to [-1, 1], \quad z(0) = z_0, \end{cases}$$

is a continuous map of $C[0,T] \to C[0,T]$. Indeed, the difference of two solutions $(y(.),z(.))$ and $(\eta(.),\zeta(.))$ corresponding to $g(.)$ and $h(.)$, respectively, satisfies

$$y(t)-\eta(t)=\int_0^t b(y(s),z(s))-b(\eta(s),\zeta(s))\mathrm{d}s+g(t)-h(t) \tag{18}$$

and

$$\begin{cases} \int_0^t(\mathrm{d}\zeta(s)-\eta(s)\mathrm{d}s)(z(s)-\zeta(s))\geq 0, \\ \int_0^t(\mathrm{d}z(s)-y(s)\mathrm{d}s)(\zeta(s)-z(s))\geq 0. \end{cases}$$

This implies

$$\int_0^t(z(s)-\zeta(s))\mathrm{d}(z(s)-\zeta(s))\leq\int_0^t(z(s)-\zeta(s))(y(s)-\eta(s))\mathrm{d}s,$$

which gives

$$|z(t)-\zeta(t)|^2\leq 2T\sup_{0\leq s\leq T}|z(s)-\zeta(s)|\sup_{0\leq s\leq T}|y(s)-\eta(s)|$$

and then, most importantly,

$$\sup_{0\leq s\leq T}|z(s)-\zeta(s)|\leq\sqrt{2T}\sup_{0\leq s\leq T}|y(s)-\eta(s)|. \tag{19}$$

Thus we deduce that

$$|y(t)-\eta(t)|\leq A_T\int_0^t\sup_{0\leq u\leq s}|y(u)-\eta(u)|\mathrm{d}s+\sup_{0\leq t\leq T}|g(t)-h(t)|.$$

Here A_T is a certain constant. Then, applying Gronwall's inequality and using (19), we deduce that

$$\sup_{0\leq t\leq T}(|y(t)-\eta(t)|+|z(t)-\zeta(t)|)\leq C_T\sup_{0\leq t\leq T}|g(t)-h(t)|.$$

Here C_T is a certain constant. That implies that $(y_\alpha(t),z_\alpha(t))$ depends continuously on ξ_α.

Acknowledgments. Phillip Yam expresses his sincere gratitude for the hospitality of both the Hausdorff Center for Mathematics of the University of Bonn and Mathematisches Forschungsinstitut Oberwolfach (MFO) in the German Black Forest during the preparation of the present work.

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