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ANALYSIS OF AN SDG METHOD FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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Abstract. In this paper, we analyze a staggered discontinuous Galerkin (SDG) method for the incompressible Navier–Stokes equations. The method is based on a novel splitting of the nonlinear convection term and results in a skew-symmetric discretization of it. As a result, the SDG discretization has a better conservation property and numerical stability property. The aim of this paper is to develop a mathematical theory for this method. In particular, we will show that the SDG method is well-posed and has an optimal rate of convergence. A superconvergence result will also be shown.

Key words. SDG, Navier–Stokes equations

AMS subject classifications. 65N30, 65M60, 35L65

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1. Introduction.

We consider the incompressible Navier–Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ in two and three space dimensions:

$$\begin{aligned} -\mu \boldsymbol{\Delta} \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where μ is the kinematic viscosity, p is the pressure, $\boldsymbol{u}$ is the velocity, and $\boldsymbol{f}$ is a given source term. We assume the usual zero average condition $\int_\Omega p \, dx = 0$ for the pressure. In our numerical approximation, the above nonlinear problem is solved by considering a sequence of linear problems

$$\begin{aligned} -\mu \boldsymbol{\Delta} \boldsymbol{u}^{n+1} + ((\Pi \boldsymbol{u}^n) \cdot \nabla) \boldsymbol{u}^{n+1} + \nabla p^{n+1} &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where Π, to be specified, is a projection operator which maps weakly divergence-free vector fields into strongly divergence-free vector fields. The above linear problem will be solved numerically, and it will be shown that the corresponding sequence of numerical solutions $\{\boldsymbol{u}_h^n\}_{n \geq 0}$ converges to an approximate solution of (1.1) under some appropriate conditions. In view of (1.2), one needs a discretization of the following

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Oseen problem:

$$\begin{aligned} -\mu\boldsymbol{\Delta u} + \boldsymbol{V}\cdot\nabla\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $\boldsymbol{V}$ is a given strongly divergence-free velocity field. The goal of this paper is to analyze a staggered discontinuous Galerkin (SDG) method for the problem (1.1), which is solved by the sequence of problems (1.2). We remark that, similar to [14, 7], the resulting method has a skew-symmetric discretization of the convection term. In addition, it is reported in [7] that this property enhances the stability of the iteration defined in (1.2).

The discontinuous Galerkin methods and mixed methods have been proved to be very useful methods for fluid dynamics, and there are many successful schemes in the literature. See the classic mixed methods [21, 3, 20], the stabilized methods proposed in [26, 25, 28], and the DG methods [2, 27, 6, 16, 17, 22, 37, 36, 34] and mixed methods [4, 5, 24]. In [15, 18, 35], the hybridizable discontinuous Galerkin (HDG) methods are constructed and analyzed for the Stokes and the Navier–Stokes equations. Optimal convergence and superconvergence are also obtained for these HDG methods. A high order DG method in the vorticity stream-function formulation in two spatial dimension is developed in [33]. Recently, a class of discontinuous Galerkin methods based on staggered grids are proposed for the Stokes system [31, 32]. The method differs from other DG methods in the way that the basis functions for the velocity and the pressure are locally conforming, and this fact allows better conservation. It is shown in [10] that the HDG and the SDG methods for the Stokes system are related in the sense that the limit of a single-face HDG method is a hybridized SDG method. In this paper, we analyze an SDG method for the incompressible Navier–Stokes equation. This method is developed in [7] and is shown to give excellent performance. The SDG method considered in this paper has some similarities to the HDG method proposed in [35]. The difference is that our SDG method is based on a novel splitting of the convection term and gives a skew-symmetric discretization of the convection term. The purpose of this paper is to derive well-posedness, and a stability estimate, optimal convergence estimate, and superconvergence estimate for the SDG method in [7].

The paper is organized as follows. In section 2, we will give the derivation of the SDG method for the incompressible Navier–Stokes equations. In section 3, we will present the existence, uniqueness, and boundedness results for the numerical solution of SDG method for the incompressible Navier–Stokes equations. We will also show optimal convergence and superconvergence of the approximation to velocity in energy norm and L^2-norm, respectively. Section 4 is devoted to the proof of existence, uniqueness, and boundedness of a numerical solution of the SDG method. Sections 5 and 6 are devoted to the proof of optimal convergence and superconvergence of the approximation to velocity in energy norm and L^2-norm, respectively.

2. The SDG method. In this section, we will present the staggered meshes with finite element spaces and the SDG method originally proposed in [7]. The method is first constructed for the Oseen equation (1.3), and then the incompressible Navier–Stokes equations (1.1) by a type of fixed point iteration.

We will use the following convention. Scalar functions are denoted by lowercase letters, such as p and q. Vector fields are denoted by boldface lowercase letters, such as $\boldsymbol{u}$ and $\boldsymbol{v}$. We use capital letters, such as W and G, to denote matrix quantities.

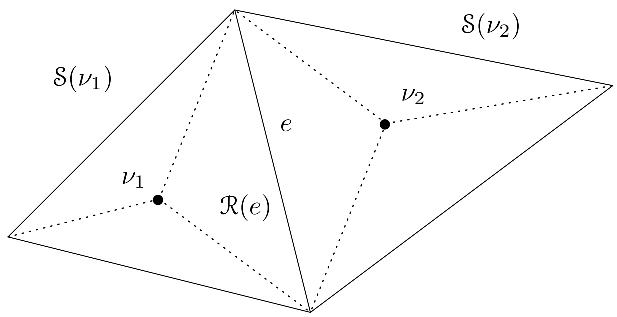

FIG. 1. *An illustration of the staggered mesh in two dimensions.*

2.1. Staggered meshes. We will first define the staggered mesh required in our method, following the general approach in [8, 11, 12, 13]. We will focus the use of unstructured mesh in this paper. For structured mesh, one can use the idea in [9]. Suppose the domain Ω is triangulated by a set of tetrahedra, denoted as $\mathcal{T}_u$, without hanging nodes. We use the notation $\mathcal{F}_u$ to denote the set of all faces in this triangulation and use the notation $\mathcal{F}_u^0$ to denote the subset of all interior faces in $\mathcal{F}_u$. For each tetrahedron from $\mathcal{T}_u$, we take an interior point ν, and divide this tetrahedron into four subtetrahedra by connecting the point ν to the four vertices of the tetrahedron. The union of these four subtetrahedra is called $\mathcal{S}(\nu)$. We introduce the notation $\mathcal{N}$ to denote the set of all such interior points ν. In addition, we use the notation $\mathcal{F}_p$ to denote all new faces generated by the subdivision of tetrahedra in $\mathcal{T}_u$ and use $\mathcal{T}$ to denote the new triangulation obtained by this subdivision process. Note that the interior points ν should be chosen so that $\mathcal{T}$ satisfies the standard shape regularity assumption. In addition, $\mathcal{F} = \mathcal{F}_u \cup \mathcal{F}_p$ denotes the set of all faces of $\mathcal{T}$ and $\mathcal{F}^0 = \mathcal{F}_u^0 \cup \mathcal{F}_p$ denotes the set of all interior faces of $\mathcal{T}$. For each face $e \in \mathcal{F}_u^0$, we let $\mathcal{R}(e)$ be the union of the two tetrahedra in $\mathcal{T}$ sharing the face e. When e is a boundary face, $\mathcal{R}(e)$ is the only tetrahedron in $\mathcal{T}$ having the face e. Figure 1 illustrates these definitions for the two-dimensional case. For each $e \in \mathcal{F}$, we let h_e be the diameter of the face e. The mesh size h is defined as $h = \max_{e\in\mathcal{F}} h_e$.

We remark that this kind of triangulation is also quite useful in other types of methods and applications. For example, in [1], it is used to prove some stability results for the quadratic velocity/linear pressure conforming finite elements. In [23], it is used to prove that weak symmetry implies strong symmetry in some discontinuous Galerkin formulation for the elasticity equations.

We will also define a unit normal vector $\boldsymbol{n}_e$ on each face e in $\mathcal{F}$ in the following way. If $e \in \mathcal{F}\backslash\mathcal{F}^0$ is a boundary face, then we define $\boldsymbol{n}_e$ as the unit normal vector of e pointing outside of Ω. If $e \in \mathcal{F}^0$ is an interior face, then we fix $\boldsymbol{n}_e$ as one of the two possible unit normal vectors on e. When it is clear that which face we are considering, we will use $\boldsymbol{n}$ instead of $\boldsymbol{n}_e$ to simplify the notation.

We will now discuss the discontinuous Galerkin spaces defined on the staggered mesh presented above. Let $k \geq 0$ be a nonnegative integer, representing the order of the polynomial approximation. Let $\tau \in \mathcal{T}$ and $e \in \mathcal{F}$. We define $P^k(\tau)$ and $P^k(e)$ as the spaces of polynomials of degree up to k on τ and e, respectively. We then define the following approximation spaces for our SDG method:

Locally $H^1(\Omega)$-conforming finite element space for velocity

$$\boldsymbol{U}^h = \{\boldsymbol{v} \, : \, \boldsymbol{v}|_\tau \in [P^k(\tau)]^d; \; \tau \in \mathcal{T}; \; \boldsymbol{v} \text{ is continuous over } e \in \mathcal{F}_u^0; \; \boldsymbol{v}|_{\partial\Omega} = 0\}. \tag{2.1}$$

Notice that if $\boldsymbol{v} \in \boldsymbol{U}^h$, then $\boldsymbol{v}|_{\mathcal{R}(e)} \in [H^1(\mathcal{R}(e))]^d$ for each face $e \in \mathcal{F}_u$. Furthermore, the condition $\boldsymbol{v}|_{\partial\Omega} = 0$ is equivalent to $\boldsymbol{v}|_e = 0$ for all $e \in \mathcal{F}_u \backslash \mathcal{F}_u^0$ since $\mathcal{F}_u$ contains all the boundary faces.

Next, we define the following:

Locally $H(div;\Omega)$-conforming finite element space

$$W^h = \{W \,:\, W|_\tau \in [P^k(\tau)]^{d\times d};\ \tau \in \mathcal{T};\ W\boldsymbol{n} \text{ is continuous over } e \in \mathcal{F}_p\}. \tag{2.2}$$

Notice that if $W \in W^h$, then $W|_{\mathcal{S}(\nu)} \in [H(\mathrm{div};\mathcal{S}(\nu))]^d$ for each $\nu \in \mathcal{N}$. Finally, we define the following space:

Locally $H^1(\Omega)$-conforming finite element space for pressure

$$P^h = \left\{q \,:\, q|_\tau \in P^k(\tau);\ \tau \in \mathcal{T};\ q \text{ is continuous over } e \in \mathcal{F}_p;\ \int_\Omega q\, dx = 0\right\}. \tag{2.3}$$

Note that this is a finite-dimensional subspace of $L_0^2(\Omega) := \{q \in L^2(\Omega)|\ \int_\Omega q\, dx = 0\}$.

Next, we need the following notation for jumps. The notation $[v]$ over a face e for a scalar-valued function v is defined as

$$[v]|_e := (v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-)\cdot\boldsymbol{n},$$

where v^+ and v^- denote the values of v taken from the triangles K^+ and K^- sharing a face e, respectively, the outward unit normal vectors $\boldsymbol{n}^\pm$ to the face e are similarly defined, and $\boldsymbol{n}$ is the unit normal vector chosen for the face e, which can be either of $\boldsymbol{n}^\pm$. The notation $[\boldsymbol{z}\cdot\boldsymbol{n}]$ for a vector-valued function $\boldsymbol{z}$ is similarly defined as

$$[\boldsymbol{z}\cdot\boldsymbol{n}]|_e := (\boldsymbol{n}\cdot\boldsymbol{n}^+)\boldsymbol{z}^+\cdot\boldsymbol{n}^+ + (\boldsymbol{n}\cdot\boldsymbol{n}^-)\boldsymbol{z}^-\cdot\boldsymbol{n}^-,$$

where $\boldsymbol{n}\cdot\boldsymbol{n}^\pm$ assumes minus one or plus one depending on the choice of $\boldsymbol{n}$. When e is located on the boundary of Ω, we only have K^+ having e as its face and choose $\boldsymbol{n} = \boldsymbol{n}^+$ and the other terms related to K^- do not appear in the definitions of $[v]$ and $[\boldsymbol{z}\cdot\boldsymbol{n}]$.

We will use the following norms will are useful in our analysis. First, the notation $\|\cdot\|_{0,\Omega}$ denotes the standard L^2 norm in Ω, and the notation $\|\cdot\|_{0,e}$ denotes the standard L^2 norm on the face e. In addition, $(\cdot,\cdot)_{0,\Omega}$ and $\langle\cdot,\cdot\rangle_{0,e}$ denote the corresponding inner products, respectively. Next, for any $(\boldsymbol{v}, G, \widetilde{G}) \in \boldsymbol{U}^h \times W^h \times W^h$, we define the following discrete energy norm:

$$\begin{aligned}&\left\|(\boldsymbol{v}, G, \widetilde{G})\right\|_h^2 \\ &\quad := \|\boldsymbol{v}\|_{0,\Omega}^2 + \|\nabla_h\boldsymbol{v}\|_{0,\Omega}^2 + \Sigma_{e\in\mathcal{F}_p} h_e^{-1}\|[\boldsymbol{v}]\|_e^2 + \mu^{-1}\left\|G + \frac{1}{2}\widetilde{G}\right\|_{0,\Omega}^2.\end{aligned} \tag{2.4}$$

For any vector field $\boldsymbol{V}$ whose components are piecewise polynomials on $\mathcal{T}$, we define the following discrete H^1 seminorm:

$$\|\boldsymbol{V}\|_U^2 := \|\nabla_h\boldsymbol{V}\|_{0,\Omega}^2 + \Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\boldsymbol{V}]\|_e^2. \tag{2.5}$$

Notice that, for any $\boldsymbol{v} \in \boldsymbol{U}^h$, $\|\boldsymbol{v}\|_U^2 = \|\nabla_h\boldsymbol{v}\|_{0,\Omega}^2 + \Sigma_{e\in\mathcal{F}_p} h_e^{-1}\|[\boldsymbol{v}]\|_e^2$.

2.2. The SDG method for the Oseen equations. Our SDG method will be first constructed for the Oseen problem (1.3). To do so, we introduce the following auxiliary variables:

$$W = \mu^{\frac{1}{2}} \nabla \boldsymbol{u} - \frac{1}{2}\mu^{-\frac{1}{2}} \boldsymbol{u} \otimes \boldsymbol{V}, \tag{2.6}$$

$$\widetilde{W} = \mu^{-\frac{1}{2}} \boldsymbol{u} \otimes \boldsymbol{V}. \tag{2.7}$$

Then the problem (1.3) can be reformulated as

$$\begin{aligned} -\mu^{\frac{1}{2}} \nabla \cdot W + \frac{1}{2}\mu^{-\frac{1}{2}} W\boldsymbol{V} + \frac{1}{4}\mu^{-\frac{1}{2}} \widetilde{W}\boldsymbol{V} + \nabla p = \boldsymbol{f} \quad &\text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad &\text{in } \Omega, \\ \boldsymbol{u} = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{2.8}$$

together with the constraint that $\int_\Omega p \, dx = 0$.

The SDG formulation for the Oseen problem (1.3) is based on the first order continuous formulation defined in (2.6), (2.7), and (2.8). The approximate velocity is denoted by $\boldsymbol{u}_h$ and will be obtained in the space $\boldsymbol{U}^h$, and the approximate pressure is denoted by p_h and will be obtained in the space P^h. The quantities W and $\widetilde{W}$ will be approximated in the same space W^h, and their approximations are denoted by W_h and $\widetilde{W}_h$, respectively. The SDG method for the Oseen problem (1.3) is to find $(\boldsymbol{u}_h, W_h, \widetilde{W}_h, p_h) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$ such that

$$B_h(W_h, \boldsymbol{v}) + \frac{1}{2} R_h \left(\boldsymbol{V}; W_h + \frac{1}{2}\widetilde{W}_h, \boldsymbol{v}\right) + b_h^*(p_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega}, \tag{2.9a}$$

$$B_h^*(\boldsymbol{u}_h, G) - \left(W_h + \frac{1}{2}\widetilde{W}_h, G\right)_{0,\Omega} = 0, \tag{2.9b}$$

$$R_h^* \left(\boldsymbol{V}; \boldsymbol{u}_h, \widetilde{G}\right) - \left(\widetilde{W}_h, \widetilde{G}\right)_{0,\Omega} = 0, \tag{2.9c}$$

$$b_h(\boldsymbol{u}_h, q) = 0 \tag{2.9d}$$

for any test function $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. In (2.9), the bilinear forms $B_h(W_h, \boldsymbol{v})$ and $B_h^*(\boldsymbol{u}_h, G)$ are defined as

$$\begin{aligned} B_h(W_h, \boldsymbol{v}) &= \mu^{\frac{1}{2}} \left(\int_\Omega W_h \cdot \nabla_h \boldsymbol{v} \, dx - \sum_{e \in \mathcal{F}_p} \int_e W_h \boldsymbol{n} \, [\boldsymbol{v}] \, d\sigma \right), \\ B_h^*(\boldsymbol{u}_h, G) &= \mu^{\frac{1}{2}} \left(- \int_\Omega \boldsymbol{u}_h \operatorname{div}_h G \, dx + \sum_{e \in \mathcal{F}_u^0} \int_e \boldsymbol{u}_h \, [G\boldsymbol{n}] \, d\sigma \right), \end{aligned} \tag{2.10}$$

and the bilinear forms $b_h^*(p_h, \boldsymbol{v})$ and $b_h(\boldsymbol{u}_h, q)$ are defined as

$$\begin{aligned} b_h^*(p_h, \boldsymbol{v}) &= - \int_\Omega p_h \operatorname{div}_h \boldsymbol{v} \, dx + \sum_{e \in \mathcal{F}_p} \int_e p_h [\boldsymbol{v} \cdot \boldsymbol{n}] \, d\sigma, \\ b_h(\boldsymbol{u}_h, q) &= \int_\Omega \boldsymbol{u}_h \cdot \nabla q \, dx - \sum_{e \in \mathcal{F}_u^0} \int_e \boldsymbol{u}_h \cdot \boldsymbol{n} [q] \, d\sigma. \end{aligned} \tag{2.11}$$

In addition, the bilinear forms $R_h(\boldsymbol{V}; W_h, \boldsymbol{v})$ and $R_h^*(\boldsymbol{V}; \boldsymbol{u}_h, \widetilde{G})$ are defined as

$$(2.12)\qquad \begin{aligned} R_h(\boldsymbol{V}; W_h, \boldsymbol{v}) &= \mu^{-\frac{1}{2}} \int_\Omega \boldsymbol{v}^\top W_h \boldsymbol{V}\, dx = \mu^{-\frac{1}{2}} (W_h, \boldsymbol{v} \otimes \boldsymbol{V})_{0,\Omega}, \\ R_h^*\left(\boldsymbol{V}; \boldsymbol{u}_h, \widetilde{G}\right) &= \mu^{-\frac{1}{2}} \int_\Omega \boldsymbol{u}_h^\top \widetilde{G} \boldsymbol{V}\, dx = \mu^{-\frac{1}{2}} \left(\widetilde{G}, \boldsymbol{u}_h \otimes \boldsymbol{V}\right)_{0,\Omega}. \end{aligned}$$

It is easy to verify that

$$(2.13a)\qquad B_h(W, \boldsymbol{v}) = B_h^*(\boldsymbol{v}, W), R_h(\boldsymbol{V}; W, \boldsymbol{v}) = R_h^*(\boldsymbol{V}; \boldsymbol{v}, W)\ \forall (W, \boldsymbol{v}) \in W^h \times \boldsymbol{U}^h,$$

$$(2.13b)\qquad b_h^*(\boldsymbol{v}, q) = b_h(q, \boldsymbol{v}) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{U}^h \times P^h.$$

The detailed derivation of SDG method (2.9) is shown in Appendix A.

If the given vector field $\boldsymbol{V}$ is divergence-free, the resulting discretization for the convection term $(\boldsymbol{V} \cdot \nabla)\boldsymbol{u}$ is skew-symmetric; see the proof of Lemma 4.1 and [7].

2.3. Postprocessing for the velocity. The SDG method (2.9) constructed in the last section will be applied to solve the incompressible Navier–Stokes equation (1.1) in an iterative way. In particular, the vector field $\boldsymbol{V}$ in (2.9) will be taken as the approximate velocity from the previous iteration. However, the approximate velocity field from the SDG method (2.9) is not strongly divergence-free. Therefore, in this section, we will define a postprocessing technique for the approximate velocity $\boldsymbol{u}_h$. The resulting postprocessed velocity is strongly divergence free.

We define $\boldsymbol{H}^h = \{\boldsymbol{u} \in H(\mathrm{div}, \Omega) : \boldsymbol{u}|_{\mathcal{S}(\nu)} \in RT^k(\mathcal{S}(\nu))\ \forall \nu \in \mathcal{N}\}$, where $RT^k(\mathcal{S}(\nu))$ is the standard kth order Raviart–Thomas space defined on the tetrahedron $\mathcal{S}(\nu)$. We remark that $\mathcal{S}(\nu)$ are tetrahedra of the partition $\mathcal{T}_u$. We define $\boldsymbol{\Pi}_{RT}^k$ the standard Raviart–Thomas projection onto $\boldsymbol{H}^h$ by

$$(2.14a)\qquad \int_e \left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h - \boldsymbol{u}_h\right) \cdot \boldsymbol{n}\, \mu\, d\sigma = 0 \quad \forall \mu \in P^k(e), e \in \partial \mathcal{S}(\nu),$$

$$(2.14b)\qquad \int_{\mathcal{S}(\nu)} \left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h - \boldsymbol{u}_h\right) \cdot \boldsymbol{v}\, dx = 0 \quad \forall \boldsymbol{v} \in [P^{k-1}(\mathcal{S}(\nu))]^d.$$

For the postprocessed velocity, we have Lemmas 2.1 and 2.2, whose proofs are given in Appendix B.

Lemma 2.1. *For any $\boldsymbol{u}_h \in \boldsymbol{U}^h$ which is weakly divergence-free in the sense of* (2.9d)*, we have*

$$\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h \in H(\mathrm{div}, \Omega) \qquad \textit{and} \qquad \nabla \cdot \boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h = 0.$$

Lemma 2.2. *There is a positive constant C_{RT} such that for any $\boldsymbol{u}_h \in \boldsymbol{U}^h$,*

$$(2.15a)\qquad \left\|\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h\right\|_{0,\Omega} \le C_{\mathrm{RT}} \|\boldsymbol{u}_h\|_{0,\Omega},$$

$$(2.15b)\qquad \left\|\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h\right\|_U \le C_{\mathrm{RT}} \|\boldsymbol{u}_h\|_U,$$

where the $\|\cdot\|_U$ norm is defined in (2.5)*.*

2.4. The SDG method for the incompressible Navier–Stokes equations and the Picard iteration. Finally, the SDG method for the incompressible Navier–Stokes equations is to find $(\boldsymbol{u}_h, W_h, \widetilde{W}_h, p_h) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$ by solving (2.9) with $\boldsymbol{V} = \boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h$, namely,

$$B_h(W_h, \boldsymbol{v}) + \frac{1}{2} R_h \left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; W_h + \frac{1}{2}\widetilde{W}_h, \boldsymbol{v} \right) + b_h^*(p_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega}, \tag{2.16a}$$

$$B_h^*(\boldsymbol{u}_h, G) - \left(W_h + \frac{1}{2}\widetilde{W}_h, G \right)_{0,\Omega} = 0, \tag{2.16b}$$

$$R_h^* \left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \boldsymbol{u}_h, \widetilde{G} \right) - \left(\widetilde{W}_h, \widetilde{G} \right)_{0,\Omega} = 0, \tag{2.16c}$$

$$b_h(\boldsymbol{u}_h, q) = 0 \tag{2.16d}$$

for any $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$.

In practice, (2.16) can be solved by the following Picard iteration. Let $\boldsymbol{u}_h^{(0)} \in \boldsymbol{U}^h$ be an initial guess which satisfies (2.9d). Then for each $n \geq 0$, we perform the following two steps. First, we define $\boldsymbol{V}^{(n)} = \boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h^{(n)}$ to be the postprocessed velocity. Second, we find $(\boldsymbol{u}_h^{(n+1)}, W_h^{(n+1)}, \widetilde{W}_h^{n+1}, p_h^{(n+1)}) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$ by solving (2.9) with $\boldsymbol{V} = \boldsymbol{V}^{(n)}$.

3. Main results.

In this section, we will present our main results, namely, the existence, uniqueness, and boundedness of the SDG approximation (Theorem 3.1), optimal convergence of velocity in energy norm and pressure in L^2-norm (Theorem 3.2), and superconvergence of velocity in L^2-norm (Theorem 3.3). We will also present a numerical convergence study (section 3.2) to show the optimal and superconvergence properties.

3.1. Wellposedness and optimal error estimates.

In this section, we will summarize the theoretical results. First we have the following existence, uniqueness, and boundedness of the SDG approximation.

THEOREM 3.1. *If $\mu^{-2}\|\boldsymbol{f}\|_{0,\Omega}$ is small enough, the SDG method (2.16) has a unique solution $(\boldsymbol{u}_h, W_h, \widetilde{W}_h, p_h) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. In addition,*

$$\left\| \left(\boldsymbol{u}_h, W_h, \widetilde{W}_h \right) \right\|_h \leq C_s \mu^{-1} \|\boldsymbol{f}\|_{0,\Omega},$$

where C_s is a positive constant.

Next, we will state the main results for the optimal estimates. We define projections $\Pi_W : [H^1(\mathcal{T})]^{d\times d} \to W^h$ and $\Pi_{\boldsymbol{U}} : [H^1(\mathcal{T})]^d \to \boldsymbol{U}^h$ by

$$(\Pi_W W - W, G)_\tau = 0, \quad \forall G \in \left[P^{k-1}(\tau)\right]^{d\times d}, \tau \in \mathcal{T}, \tag{3.1a}$$

$$\langle (\Pi_W W - W)\boldsymbol{n}, \boldsymbol{\mu} \rangle_e = 0, \quad \forall \boldsymbol{\mu} \in \left[P^k(e)\right]^d, e \in \mathcal{F}_p, \tag{3.1b}$$

$$(\Pi_{\boldsymbol{U}} \boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v})_\tau = 0, \quad \forall \boldsymbol{v} \in \left[P^{k-1}(\tau)\right]^d, \tau \in \mathcal{T}, \tag{3.1c}$$

$$\langle \Pi_{\boldsymbol{U}} \boldsymbol{u} - \boldsymbol{u}, \boldsymbol{\mu} \rangle_e = 0, \quad \forall \boldsymbol{\mu} \in \left[P^k(e)\right]^d, e \in \mathcal{F}_u. \tag{3.1d}$$

These projections are well-defined (see [12]). We also define $\Pi_{\widetilde{W}} = \Pi_W$ and Π_P to be standard L^2-orthogonal projection onto P^h. The projection operator $\Pi_{\boldsymbol{U}}$ has the following approximation property:

$$\|\Pi_{\boldsymbol{U}} \boldsymbol{u} - \boldsymbol{u}\|_{0,\Omega} \leq C h^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(\Omega)^d}. \tag{3.2}$$

A similar estimate holds for the other projection operators $\Pi_W, \Pi_{\widetilde{W}}$, and Π_p. Moreover, the stability bound

$$\|\Pi_{\boldsymbol{U}}\boldsymbol{u}\|_U \le C_u\|\nabla\boldsymbol{u}\|_{0,\Omega} \tag{3.3}$$

holds (see [12]). From (1.3), it is also easy to see that

$$\|\nabla\boldsymbol{u}\|_{0,\Omega} \le C_\Omega\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}, \tag{3.4}$$

where C_Ω is the Poincaré constant. For our analysis, we need the following error quantities:

$$\begin{aligned} \epsilon_{\boldsymbol{u}} &= \Pi_{\boldsymbol{U}}\boldsymbol{u} - \boldsymbol{u}_h, & \epsilon_W &= \Pi_W W - W_h,\\ \epsilon_{\widetilde{W}} &= \Pi_{\widetilde{W}}\widetilde{W} - \widetilde{W}_h, & \epsilon_p &= \Pi_P p - p_h. \end{aligned} \tag{3.5}$$

We have Theorem 3.2, which states optimal convergence of $(\epsilon_{\boldsymbol{u}}, \epsilon_W, \epsilon_{\widetilde{W}})$ in energy norm (2.4) and ϵ_p in L^2-norm.

THEOREM 3.2. *Assume that $\mu^{-2}\|\boldsymbol{f}\|_{0,\Omega}$ is small enough. Furthermore, we assume that $\boldsymbol{u} \in [L^\infty(\Omega)]^d \cap [W^{1,3}(\Omega)]^d$. Then, there is a constant C such that*

$$\|(\epsilon_{\boldsymbol{u}}, \epsilon_W, \epsilon_{\widetilde{W}})\|_h + \|\epsilon_p\|_{0,\Omega} \le C\mu^{-\frac{1}{2}}h^{k+1},$$

where the constant C depends on $\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}$, $\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}$, $\|\boldsymbol{u}\|_{W^{1,3}(\Omega)^d}$, $\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}$, and $\|p\|_{H^{k+1}(\Omega)^d}$.

Furthermore, in Theorem 3.3, we show a superconvergence result for the velocity error $\epsilon_{\boldsymbol{u}}$ in the L^2-norm. The key idea of the proof is a duality argument. We need to consider the following dual problem:

$$-\mu\boldsymbol{\Delta}\boldsymbol{\phi} - \nabla\cdot(\boldsymbol{\phi}\otimes\boldsymbol{u}) - \frac{1}{2}(\nabla\boldsymbol{\phi})^\top\boldsymbol{u} - \frac{1}{2}(\nabla\boldsymbol{u})^\top\boldsymbol{\phi} - \nabla\psi = \boldsymbol{\theta} \quad \text{in } \Omega, \tag{3.6a}$$

$$\nabla\cdot\boldsymbol{\phi} = 0 \quad \text{in } \Omega, \tag{3.6b}$$

$$\boldsymbol{\phi} = 0 \quad \text{on } \partial\Omega. \tag{3.6c}$$

If we assume $\|\boldsymbol{u}\|_{H^1(\Omega)^d}$ is small enough, then the dual problem (3.6) has the unique solution $(\boldsymbol{\phi}, \psi) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$. If we further assume that $\boldsymbol{u} \in [L^\infty(\Omega)]^d$ and the domain Ω is nice enough, then we have

$$\|\boldsymbol{\phi}\|_{H^2(\Omega)^d} + \|\psi\|_{H^1(\Omega)} \le C\|\boldsymbol{\theta}\|_{0,\Omega}. \tag{3.7}$$

We remark that the constant C in (3.7) depends on μ.

THEOREM 3.3. *If we assume that the source $\boldsymbol{f}$ is small enough, $k \ge 1$, $\boldsymbol{u} \in [W^{1,\infty}(\Omega)]^d$, and (3.7) holds, then we have*

$$\|\epsilon_{\boldsymbol{u}}\|_{0,\Omega} \le Ch^{k+2}(\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)^d} + \|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d} + \|p\|_{H^{k+1}(\Omega)} + 1)\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}.$$

Here the constant C depends on μ, $\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}$, $\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}$, $\|\boldsymbol{u}\|_{W^{1,3}(\Omega)^d}$, $\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}$ and $\|p\|_{H^{k+1}(\Omega)^d}$.

TABLE 1
Errors and orders of convergence for the Kovasznay flow with $\mu = 0.02$.

Degree k	Mesh h^{-1}	$\Vert\mathbf{u}-\mathbf{u}_h\Vert$ error	order	$\Vert L-L_h\Vert$ error	order	$\Vert p-p_h\Vert$ error	order	$\Vert\mathbf{u}-\mathbf{u}_h^\star\Vert$ error	order
1	1	8.4614e-01	-	7.8514e+00	-	6.5830e-01	-	5.7762e-01	-
	2	5.9473e-01	0.51	4.7425e+00	0.73	3.7482e-01	0.81	2.6868e-01	1.10
	4	1.3188e-01	2.17	5.2739e+00	-0.15	9.1298e-02	2.04	7.8287e-02	1.78
	8	7.2896e-02	0.86	1.5283e+00	1.79	1.7270e-02	2.40	1.6481e-02	2.25
	16	1.8930e-02	1.95	4.7643e-01	1.68	4.8762e-03	1.82	2.4540e-03	2.75
	32	4.8487e-03	1.96	1.2956e-01	1.88	1.3110e-03	1.90	3.3624e-04	2.87
2	1	4.0832e-01	-	5.8908e+00	-	3.3072e-01	-	1.5437e-01	-
	2	1.6002e-01	1.35	4.8678e+00	0.28	1.2206e-01	1.44	1.4223e-01	0.12
	4	5.5665e-02	1.52	1.2010e+00	2.02	1.7931e-02	2.77	2.0182e-02	2.82
	8	5.9827e-03	3.22	3.0338e-01	1.99	3.6898e-03	2.28	2.1581e-03	3.23
	16	7.0311e-04	3.09	3.3325e-02	3.19	3.8700e-04	3.25	1.3734e-04	3.97
	32	8.6765e-05	3.02	4.1195e-03	3.02	4.7171e-05	3.04	9.5710e-06	3.84

3.2. A numerical example. In this section, we will present a numerical example to show the rate of convergence and the rate of superconvergence of our proposed scheme. We consider the incompressible Navier–Stokes equation

$$
\begin{aligned}
-\mu\boldsymbol{\Delta}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega,\\
\operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial\Omega
\end{aligned}
\tag{3.8}
$$

with viscosity μ and inhomogeneous Dirichlet boundary condition. Our numerical scheme can be easily modified for this equation. For the simulations, we consider the Kovasznay flow and take the analytic solution as

$$
\begin{aligned}
u_1(x,y) &= 1 - e^{\lambda x}\cos(2\pi y),\\
u_2(x,y) &= \frac{\lambda}{2\pi}e^{\lambda x}\sin(2\pi y),\\
p(x,y) &= -\frac{1}{2}e^{2\lambda x} + \bar{p},
\end{aligned}
\tag{3.9}
$$

where $\lambda = \frac{-8\pi^2}{\mu^{-1}+\sqrt{\mu^{-2}+64\pi^2}}$ and $\bar{p}$ is a constant chosen such that $\int_\Omega p\,dx = 0$. We take the computational domain $\Omega = (-\frac{1}{2},\frac{3}{2})\times(0,2)$. By setting $\mu = 0.02$, we obtain the source function $\boldsymbol{f}$ and the boundary data $\boldsymbol{g}$. The initial guess of the Picard iterations is the zero field.

In Table 1, we present the convergence rates with polynomial degree $k = 1, 2$. In particular, the errors of the approximations for velocity, velocity gradient, and pressure all converge in the order $k+1$ and the postprocessed velocity converges in the order $k+2$. For the construction of the postprocessed velocity $\mathbf{u}_h^\star$, see [7].

4. Proof of existence, uniqueness, and boundedness of the SDG method. In this section, we will prove Theorem 3.1 in the following steps. We will first prove the stability of the SDG method for the Oseen problem (2.9), which is independent of $\boldsymbol{V}$. It implies the boundedness of numerical solution of SDG method (2.16) for the incompressible Navier–Stokes equations. Then, we will show that the iterative process resulting from the SDG method for the incompressible Navier–Stokes equations defined in (2.16) is a contraction mapping, which implies the SDG scheme has a unique solution.

First, by the inf-sup condition (see [12]) of SDG spaces $\boldsymbol{U}^h$ and W^h,

$$\mu^{\frac{1}{2}}\|\boldsymbol{v}\|_U \leq C_{is} \sup_{G\in W^h} \frac{B_h^*(\boldsymbol{u}_h, G)}{\|G\|_{0,\Omega}}, \tag{4.1}$$

where C_{is} is the inf-sup constant. Then, by (2.9b) and (4.1), we have

$$\|\boldsymbol{u}_h\|_U \leq C_{is}\mu^{-\frac{1}{2}} \left\| W_h + \frac{1}{2}\widetilde{W}_h \right\|_{0,\Omega}. \tag{4.2}$$

The following Lemma 4.1 gives stability bound in terms of the energy norm defined in (2.4). Due to (4.3), it is easy to see that if $(\boldsymbol{u}_h, W_h, \widetilde{W}_h, p_h) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$ is a solution of SDG method (2.16), then

$$\left\| \left(\boldsymbol{u}_h, W_h, \widetilde{W}_h\right) \right\|_h \leq C_s \mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}.$$

Lemma 4.1. *Let $(\boldsymbol{u}_h, W_h, \widetilde{W}_h, p_h)$ be the solution of the SDG method* (2.9) *with a given divergence-free vector field $\boldsymbol{V}$. Then we have the following stability bound:*

$$\left\| \left(\boldsymbol{u}_h, W_h, \widetilde{W}_h\right) \right\|_h \leq C_s \mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}, \tag{4.3}$$

where the positive constant C_s is independent of $\boldsymbol{V}$.

Proof. In (2.9), we take the test functions in the following way:

$$\boldsymbol{v} = \boldsymbol{u}_h, \quad G = -W_h, \quad \widetilde{G} = -\frac{1}{2}\left(W_h + \frac{1}{2}\widetilde{W}_h\right), \quad q = -p_h.$$

Then we have

$$B_h(W_h, \boldsymbol{u}_h) + \frac{1}{2}R_h\left(\boldsymbol{V}; W_h + \frac{1}{2}\widetilde{W}_h, \boldsymbol{u}_h\right) + b_h^*(p_h, \boldsymbol{u}_h) = (\boldsymbol{f}, \boldsymbol{u}_h)_{0,\Omega}, \tag{4.4a}$$

$$-B_h^*(\boldsymbol{u}_h, W_h) + \left(W_h + \frac{1}{2}\widetilde{W}_h, W_h\right)_{0,\Omega} = 0, \tag{4.4b}$$

$$-\frac{1}{2}R_h^*\left(\boldsymbol{V}; \boldsymbol{u}_h, W_h + \frac{1}{2}\widetilde{W}_h\right) + \frac{1}{2}\left(\widetilde{W}_h, W_h + \frac{1}{2}\widetilde{W}_h\right)_{0,\Omega} = 0, \tag{4.4c}$$

$$-b_h(\boldsymbol{u}_h, p_h) = 0. \tag{4.4d}$$

According to (2.13), the sum of all equations in (4.4) is

$$\left\| W_h + \frac{1}{2}\widetilde{W}_h \right\|_{0,\Omega}^2 = (\boldsymbol{f}, \boldsymbol{u}_h)_{0,\Omega}. \tag{4.5}$$

The result follows from (4.5), (4.2), and (2.4). ∎

Notice that (2.9) defines a mapping which maps a vector field $\boldsymbol{u}_h^{(n)} \in \boldsymbol{U}^h$ to another vector field $\boldsymbol{u}_h^{(n+1)} \in \boldsymbol{U}^h$. Next, we will prove that this mapping is a contraction (see inequality (4.14)). We will need a discrete Sobolev inequality from [19]:

$$\|\boldsymbol{v}\|_{L^6(\Omega)^d} \leq C_p\left(\|\nabla_h \boldsymbol{v}\|_{0,\Omega}^2 + \Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\boldsymbol{v}]\|_{0,e}^2\right)^{1/2} \tag{4.6}$$

for any $\boldsymbol{v}$ that is piecewise polynomial on $\mathcal{T}$.

LEMMA 4.2. *Let $\boldsymbol{V}^{(1)}$ and $\boldsymbol{V}^{(2)}$ be two divergence-free vector fields such that they are piecewise polynomials on $\mathcal{T}$. We define*

$$\delta = \frac{1}{2} C_s^2 C_p^2 \mu^{-2} \|\boldsymbol{f}\|_{0,\Omega}. \tag{4.7}$$

Then we have

$$\left\| \boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)} \right\|_U^2 \leq \delta^2 \left\| \boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)} \right\|_U^2 .$$

Here, $(\boldsymbol{u}_h^{(1)}, W_h^{(1)}, \widetilde{W}_h^{(1)}, p_h^{(1)})$ and $(\boldsymbol{u}_h^{(2)}, W_h^{(2)}, \widetilde{W}_h^{(2)}, p_h^{(2)})$ are the solutions of (2.9) *with $\boldsymbol{V} = \boldsymbol{V}^{(1)}$ and $\boldsymbol{V} = \boldsymbol{V}^{(2)}$, respectively.*

Proof. Since $(\boldsymbol{u}_h^{(1)}, W_h^{(1)}, \widetilde{W}_h^{(1)}, p_h^{(1)})$ is the solution of (2.9) with $\boldsymbol{V} = \boldsymbol{V}^{(1)}$, we have

$$\begin{aligned} B_h\left(W_h^{(1)}, \boldsymbol{v}\right) + \frac{1}{2} R_h\left(\boldsymbol{V}^{(1)}; W_h^{(1)} + \frac{1}{2}\widetilde{W}_h^{(1)}, \boldsymbol{v}\right) + b_h^*\left(p_h^{(1)}, \boldsymbol{v}\right) &= (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega}, \\ B_h^*\left(\boldsymbol{u}_h^{(1)}, G\right) - \left(W_h^{(1)} + \frac{1}{2}\widetilde{W}_h^{(1)}, G\right)_{0,\Omega} &= 0, \\ R_h^*\left(\boldsymbol{V}^{(1)}; \boldsymbol{u}_h^{(1)}, \widetilde{G}\right) - \left(\widetilde{W}_h^{(1)}, \widetilde{G}\right)_{0,\Omega} &= 0, \\ b_h\left(\boldsymbol{u}_h^{(1)}, q\right) &= 0, \end{aligned} \tag{4.8}$$

and since $(\boldsymbol{u}_h^{(2)}, W_h^{(2)}, \widetilde{W}_h^{(2)}, p_h^{(2)})$ is the solution of (2.9) with $\boldsymbol{V} = \boldsymbol{V}^{(2)}$, we have

$$\begin{aligned} B_h\left(W_h^{(2)}, \boldsymbol{v}\right) + \frac{1}{2} R_h\left(\boldsymbol{V}^{(2)}; W_h^{(2)} + \frac{1}{2}\widetilde{W}_h^{(2)}, \boldsymbol{v}\right) + b_h^*(p_h^{(2)}, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega}, \\ B_h^*\left(\boldsymbol{u}_h^{(2)}, G\right) - \left(W_h^{(2)} + \frac{1}{2}\widetilde{W}_h^{(2)}, G\right)_{0,\Omega} &= 0, \\ R_h^*\left(\boldsymbol{V}^{(2)}; \boldsymbol{u}_h^{(2)}, \widetilde{G}\right) - \left(\widetilde{W}_h^{(2)}, \widetilde{G}\right)_{0,\Omega} &= 0, \\ b_h\left(\boldsymbol{u}_h^{(2)}, q\right) &= 0 \end{aligned} \tag{4.9}$$

for any test function $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. Subtracting (4.8) by (4.9), we have

$$\begin{aligned} &B_h(W_h^{(1)} - W_h^{(2)}, \boldsymbol{v}) \\ &\quad + \frac{1}{2} R_h\left(\boldsymbol{V}^{(1)}; \left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right), \boldsymbol{v}\right) \\ &\quad + b_h^*\left(p_h^{(1)} - p_h^{(2)}, \boldsymbol{v}\right) = -\frac{1}{2} R_h\left(\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}; W_h^{(2)} + \frac{1}{2}\widetilde{W}_h^{(2)}, \boldsymbol{v}\right), \end{aligned} \tag{4.10a}$$

$$B_h^*\left(\boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}, G\right) - \left(\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right), G\right)_{0,\Omega} = 0, \tag{4.10b}$$

$$\begin{aligned} &R_h^*\left(\boldsymbol{V}^{(1)}; \boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}, \widetilde{G}\right) - \left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}, \widetilde{G}\right)_{0,\Omega} \\ &\quad = -R_h^*\left(\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}; \boldsymbol{u}_h^{(2)}, \widetilde{G}\right), \end{aligned} \tag{4.10c}$$

$$b_h\left(\boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}, q\right) = 0 \tag{4.10d}$$

for any $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. For (4.10), we take the test functions in the following way:

$$\boldsymbol{v} = \boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}, \quad G = -\left(W_h^{(1)} - W_h^{(2)}\right),$$
$$\widetilde{G} = -\frac{1}{2}\left(\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right), \quad q = -\left(p_h^{(1)} - p_h^{(2)}\right).$$

Using (2.13), the sum of all equations in (4.10) with the test functions chosen in the above way is

$$\begin{aligned} &\left\|\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega}^2 \\ &= -\frac{1}{2}R_h\left(\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}; W_h^{(2)} + \frac{1}{2}\widetilde{W}_h^{(2)}, \boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}\right) \\ &\quad + \frac{1}{2}R_h^*\left(\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}; \boldsymbol{u}_h^{(2)}, \left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right). \end{aligned} \tag{4.11}$$

By (4.10b) and (4.2), we have

$$\begin{aligned} &\left\|\left(\boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}, W_h^{(1)} - W_h^{(2)}, \widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_h \\ &\quad \leq C_s^{\frac{1}{2}}\mu^{-\frac{1}{2}}\left\|\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega}, \end{aligned} \tag{4.12}$$

where we remark that the above constant is the same as the one in Lemma 4.1. So, to prove the desired result, we need to estimate the right-hand side of (4.11).

By the Hölder's inequality, we can estimate (4.11) in the following way:

$$\begin{aligned} &\left\|\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega}^2 \\ &\leq \frac{1}{2}\mu^{-\frac{1}{2}}\left\|\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}\right\|_{L^4(\Omega)} \\ &\cdot\left\{\left\|W_h^{(2)} + \frac{1}{2}\widetilde{W}_h^{(2)}\right\|_{0,\Omega}\left\|\boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}\right\|_{L^4(\Omega)}\right. \\ &\quad \left. + \left\|\boldsymbol{u}_h^{(2)}\right\|_{L^4(\Omega)}\left\|\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega}\right\}. \end{aligned}$$

Using the Sobolev inequality (4.6) and the definition of the norm (2.5), the above becomes

$$\begin{aligned} &\left\|\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega}^2 \\ &\leq \frac{C_p^2}{2}\mu^{-\frac{1}{2}}\left\|\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}\right\|_U \\ &\cdot\left\{\left\|W_h^{(2)} + \frac{1}{2}\widetilde{W}_h^{(2)}\right\|_{0,\Omega}\left\|\boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}\right\|_U\right. \\ &\quad \left. + \left\|\boldsymbol{u}_h^{(2)}\right\|_U\left\|\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega}\right\}. \end{aligned}$$

Using (4.12), we have

$$\left\|(W_h^{(1)} - W_h^{(2)}) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega} \leq \frac{1}{2} C_s^{\frac{1}{2}} C_p^2 \mu^{-\frac{1}{2}} \left\|\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}\right\|_U \left\|\left(\boldsymbol{u}_h^{(2)}, W_h^{(2)}, \widetilde{W}_h^{(2)}\right)\right\|_h.$$

By Lemma 4.1, we obtain

$$\left\|\left(W_h^{(1)} - W_h^{(2)}\right) + \frac{1}{2}\left(\widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_{0,\Omega} \leq \frac{1}{2} C_s^{\frac{3}{2}} C_p^2 \mu^{-\frac{3}{2}} \|\boldsymbol{f}\|_{0,\Omega} \left\|\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}\right\|_U.$$

Finally, by using (4.12) again, we have

$$\left\|\left(\boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}, W_h^{(1)} - W_h^{(2)}, \widetilde{W}_h^{(1)} - \widetilde{W}_h^{(2)}\right)\right\|_h \leq \frac{1}{2} C_s^2 C_p^2 \mu^{-2} \|\boldsymbol{f}\|_{0,\Omega} \left\|\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)}\right\|_U.$$

The proof is complete by using the definition of the norm (2.4) and the definition of δ in (4.7). ∎

Now, we are ready the prove Theorem 3.1. To begin, we let $\widetilde{\boldsymbol{u}}_h^{(1)}$ and $\widetilde{\boldsymbol{u}}_h^{(2)}$ be vector fields in $\boldsymbol{U}^h$ and assume that they satisfy (2.9d). If $\|\boldsymbol{f}\|_{0,\Omega}$ is small enough such that

$$\gamma := \frac{C_s^2 C_p^2 C_{\mathrm{RT}}}{2\mu^2} \|\boldsymbol{f}\|_{0,\Omega} < 1, \tag{4.13}$$

then we have

$$\left\|\boldsymbol{u}_h^{(1)} - \boldsymbol{u}_h^{(2)}\right\|_U^2 \leq \gamma^2 \left\|\widetilde{\boldsymbol{u}}_h^{(1)} - \widetilde{\boldsymbol{u}}_h^{(2)}\right\|_U^2. \tag{4.14}$$

Here, $(\boldsymbol{u}_h^{(1)}, W_h^{(1)}, \widetilde{W}_h^{(1)}, p_h^{(1)})$ and $(\boldsymbol{u}_h^{(2)}, W_h^{(2)}, \widetilde{W}_h^{(2)}, p_h^{(2)})$ are the solutions of (2.9) with

$$\boldsymbol{V} = \boldsymbol{\Pi}_{RT}^k \widetilde{\boldsymbol{u}}_h^{(1)}, \quad \boldsymbol{V} = \boldsymbol{\Pi}_{RT}^k \widetilde{\boldsymbol{u}}_h^{(2)},$$

respectively. We remark that the above is an immediate consequence of Lemmas 2.2, 4.1, and 4.2. Notice that inequality (4.14) implies that the SDG method (2.16) has a unique solution. To see this, we first define the following compact set

$$Z = \left\{\boldsymbol{v} \in \boldsymbol{U}^h \,:\, \|\boldsymbol{v}\|_U \leq C_s \mu^{-1} \|\boldsymbol{f}\|_{0,\Omega}\right\}. \tag{4.15}$$

Notice that the Picard iteration introduced in section 2.4 defines a mapping such that if $\boldsymbol{u}_h^{(n)} \in Z$, then $\boldsymbol{u}_h^{(n+1)} \in Z$ by Lemma 4.1. From (4.14), we see that this mapping is also a contraction. Hence, the mapping has a unique fixed point. So, we can conclude that the proof of Theorem 3.1 is complete.

5. Proof of optimal error estimate in energy norm. In this section, we will prove Theorem 3.2. We assume that $\mu^{-2}\|\boldsymbol{f}\|_{0,\Omega}$ is small enough such that (4.13) and

$$\frac{1}{2} C_p^2 C_u C_{\mathrm{RT}} C_{is} \mu^{-1} \|\nabla \boldsymbol{u}\|_{0,\Omega} < 1 \tag{5.1}$$

hold. We remark that C_p is the Sobolev constant in (4.6), C_u is the stability constant in (3.3), C_{RT} is the stability constant in (2.15), and C_{is} is the inf-sup constant in (4.1).

5.1. Preliminaries. First, we will use the following error equations (5.2) for the SDG method, whose proof is given in Appendix C:

$$B_h(\epsilon_W,\boldsymbol{v})+\frac{1}{2}R_h\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}},\boldsymbol{v}\right)+b_h^*(\epsilon_p,\boldsymbol{v})=b_h^*(\Pi_P p-p,\boldsymbol{v})$$
$$-\frac{1}{2}\mu^{1/2}R_h(\boldsymbol{u};\nabla\boldsymbol{u},\boldsymbol{v})+\frac{1}{2}\mu^{1/2}R_h\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_W\nabla\boldsymbol{u},\boldsymbol{v}\right), \tag{5.2a}$$

$$B_h^*(\epsilon_{\boldsymbol{u}},G)-\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}},G\right)_{0,\Omega}=-\mu^{1/2}(\Pi_W\nabla\boldsymbol{u}-\nabla\boldsymbol{u},G), \tag{5.2b}$$

$$R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\epsilon_{\boldsymbol{u}},\widetilde{G}\right)-\left(\epsilon_{\widetilde{W}},\widetilde{G}\right)_{0,\Omega}=-\left(\Pi_W\widetilde{W}-\widetilde{W},\widetilde{G}\right)_{0,\Omega}$$
$$-R_h^*\left(\boldsymbol{u};\boldsymbol{u},\widetilde{G}\right)+R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_{\boldsymbol{U}}\boldsymbol{u},\widetilde{G}\right), \tag{5.2c}$$

$$b_h(\epsilon_{\boldsymbol{u}},q)=0 \tag{5.2d}$$

for any $(\boldsymbol{v},G,\widetilde{G},q)\in\boldsymbol{U}^h\times W^h\times W^h\times P^h$. Second, we need the following technical lemma for two error bounds about the operator R_h.

LEMMA 5.1. *Let $\boldsymbol{u}\in W^{1,3}(\Omega)^d\cap L^\infty(\Omega)^d$ and $\boldsymbol{u}_h\in\boldsymbol{U}^h$. Then for any $(\boldsymbol{v},G)\in\boldsymbol{U}^h\times W^h$, the following two inequalities hold:*

$$\begin{aligned}
&\left|R_h^*(\boldsymbol{u};\boldsymbol{u},G)-R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\boldsymbol{v},G\right)\right|\\
&\quad\le C\mu^{-\frac{1}{2}}\|G\|_{0,\Omega}\Big\{\|\boldsymbol{u}_h\|_{\boldsymbol{U}}\|\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{U}}\\
&\qquad+\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}\Big(\|\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}\|_{0,\Omega}\\
&\qquad\quad+\left\|\boldsymbol{u}-\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}\right\|_{0,\Omega}+h\|\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}\|_{\boldsymbol{U}}\Big)\Big\}\\
&\quad+C_p^2C_uC_{\mathrm{RT}}\mu^{-\frac{1}{2}}\|G\|_{0,\Omega}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{U}}
\end{aligned} \tag{5.3a}$$

and

$$\begin{aligned}
&\left|R_h(\boldsymbol{u};\nabla\boldsymbol{u},\boldsymbol{v})-R_h\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_W\nabla\boldsymbol{u},\boldsymbol{v}\right)\right|\\
&\quad\le C\mu^{-\frac{1}{2}}\|\boldsymbol{v}\|_{\boldsymbol{U}}\Big\{\|\boldsymbol{u}\|_{W^{1,3}(\Omega)^d}\left(\left\|\boldsymbol{u}-\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}\right\|_{0,\Omega}+h\|\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}\|_{\boldsymbol{U}}\right)\\
&\qquad+\|\boldsymbol{u}_h\|_{\boldsymbol{U}}\|\nabla\boldsymbol{u}-\Pi_W\nabla\boldsymbol{u}\|_{0,\Omega}\Big\}\\
&\quad+C_p^2C_{\mathrm{RT}}\mu^{-\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{U}}.
\end{aligned} \tag{5.3b}$$

Proof. We first consider the proof of (5.3a). By a direct computation, we have

$$\begin{aligned}
&R_h^*(\boldsymbol{u};\boldsymbol{u},G)-R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\boldsymbol{v},G\right)\\
&\quad=R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}-\boldsymbol{v},G\right)+R_h^*\left(\boldsymbol{u};\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u},G\right)\\
&\qquad+R_h^*\left(\boldsymbol{\Pi}_{RT}^k(\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u}-\boldsymbol{u}_h);\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u},G\right)\\
&\qquad+R_h^*\left(\boldsymbol{u}-\boldsymbol{\Pi}_{RT}^k\boldsymbol{u},\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u},G\right)+R_h^*\left(\boldsymbol{\Pi}_{RT}^k(\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u});\boldsymbol{\Pi}_{\boldsymbol{U}}\boldsymbol{u},G\right).
\end{aligned}$$

Then by (4.6), (2.15b), and the fact that

$$\|\boldsymbol{\Pi_U u}\|_{L^\infty(\Omega)^d} \leq C\|\boldsymbol{u}\|_{L^\infty(\Omega)^d},$$
$$\left\|\boldsymbol{\Pi}_{RT}^k(\boldsymbol{u}-\boldsymbol{\Pi_U u})\right\|_{0,\Omega} \leq Ch\|\nabla_h(\boldsymbol{u}-\boldsymbol{\Pi_U u})\|_{0,\Omega} \leq Ch\|\boldsymbol{u}-\boldsymbol{\Pi_U u}\|_{\boldsymbol{U}},$$

we obtain (5.3a). To show (5.3b), we notice, by a direct computation, that

$$\begin{aligned} &R_h(\boldsymbol{u};\nabla\boldsymbol{u},\boldsymbol{v}) - R_h\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_W\nabla\boldsymbol{u},\boldsymbol{v}\right)\\ &\quad= R_h\left(\boldsymbol{u}-\boldsymbol{\Pi}_{RT}^k\boldsymbol{u};\nabla\boldsymbol{u},\boldsymbol{v}\right) + R_h\left(\boldsymbol{\Pi}_{RT}^k(\boldsymbol{u}-\boldsymbol{\Pi_U u});\nabla\boldsymbol{u},\boldsymbol{v}\right)\\ &\qquad+ R_h\left(\boldsymbol{\Pi}_{RT}^k(\boldsymbol{\Pi_U u}-\boldsymbol{u}_h);\nabla\boldsymbol{u},\boldsymbol{v}\right) + R_h\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\nabla\boldsymbol{u}-\Pi_W\nabla\boldsymbol{u},\boldsymbol{v}\right).\end{aligned}$$

Then by similar arguments as above, it is easy to see that (5.3b) holds. ∎

Next, we prove the following error bound for $\epsilon_{\widetilde{W}}$.

Lemma 5.2. *Assume that* $\boldsymbol{u} \in [H^{k+2}(\Omega)]^d$. *Then, there is a constant* C *such that*

$$\|\epsilon_{\widetilde{W}}\|_{0,\Omega} \leq C\mu^{-\frac{1}{2}}\left((\|\nabla\boldsymbol{u}\|_{0,\Omega}+\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega})\|\epsilon_{\boldsymbol{u}}\|_{\boldsymbol{U}} + h^{k+1}\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}\right).$$

Proof of Lemma 5.2. Taking $\widetilde{G} = \epsilon_{\widetilde{W}}$ in (5.2c), we have

$$\left(\epsilon_{\widetilde{W}},\epsilon_{\widetilde{W}}\right)_{0,\Omega} = \left(\Pi_W\widetilde{W}-\widetilde{W},\epsilon_{\widetilde{W}}\right)_{0,\Omega} + R_h^*\left(\boldsymbol{u};\boldsymbol{u},\epsilon_{\widetilde{W}}\right) - R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\boldsymbol{u}_h,\epsilon_{\widetilde{W}}\right).$$

Obviously,

$$\begin{aligned}\left(\Pi_W\widetilde{W}-\widetilde{W},\epsilon_{\widetilde{W}}\right)_{0,\Omega} &\leq Ch^{k+1}\left\|\widetilde{W}\right\|_{H^{k+1}(\Omega)^d}\ \|\epsilon_{\widetilde{W}}\|_{0,\Omega}\\ &\leq C\mu^{-\frac{1}{2}}h^{k+1}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}\ \|\epsilon_{\widetilde{W}}\|_{0,\Omega}.\end{aligned}$$

By (5.3a), (3.2), approximation properties of $\boldsymbol{\Pi}_{RT}^k$, and (4.3), we have

$$\begin{aligned}&R_h^*(\boldsymbol{u};\boldsymbol{u},\epsilon_{\widetilde{W}}) - R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\boldsymbol{u}_h,\epsilon_{\widetilde{W}}\right)\\ &\quad\leq C\mu^{-\frac{1}{2}}\|\epsilon_{\widetilde{W}}\|_{0,\Omega}\Big(\left(\|\nabla\boldsymbol{u}\|_{0,\Omega}+\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}\right)\|\epsilon_{\boldsymbol{u}}\|_{\boldsymbol{U}} + h^{k+1}\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)^d}\Big).\end{aligned}$$

This completes the proof. ∎

5.2. Proof of Theorem 3.2. With the above preliminary results, we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. By the inf-sup condition (4.1), the error equation (5.2b), and the approximation property of Π_W, we have

$$\|\epsilon_{\boldsymbol{u}}\|_{\boldsymbol{U}} \leq C_{is}\mu^{-\frac{1}{2}}\left(\left\|\epsilon_W+\frac{1}{2}\epsilon_{\tilde{W}}\right\|_{0,\Omega} + \mu^{\frac{1}{2}}h^{k+1}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}\right). \tag{5.4}$$

In (5.2), we take the test functions as follows:

$$\boldsymbol{v}=\epsilon_{\boldsymbol{u}},\quad G=-\epsilon_W,\quad \widetilde{G}=-\frac{1}{2}\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right),\quad q=-\epsilon_p.$$

According to (2.13), the sum of all equations in (5.2) with the above choice of test functions is

(5.5)

$$\begin{aligned}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right\|_{0,\Omega}^2 &= b_h^*(\Pi_P p-p,\epsilon_{\boldsymbol{u}})-\frac{1}{2}\mu^{\frac{1}{2}}R_h(\boldsymbol{u};\nabla\boldsymbol{u},\epsilon_{\boldsymbol{u}})\\ &\quad+\frac{1}{2}\mu^{\frac{1}{2}}R_h\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_W\nabla\boldsymbol{u},\epsilon_{\boldsymbol{u}}\right)\\ &\quad+\mu^{\frac{1}{2}}(\Pi_W\nabla\boldsymbol{u}-\nabla\boldsymbol{u},\epsilon_W)+\left(\Pi_W\widetilde{W}-\widetilde{W},\frac{1}{2}\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right)\right)_{0,\Omega}\\ &\quad+R_h^*\left(\boldsymbol{u};\boldsymbol{u},\frac{1}{2}\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right)\right)\\ &\quad-R_h^*\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_{\boldsymbol{U}}\boldsymbol{u},\frac{1}{2}\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right)\right).\end{aligned}$$

Next, we will estimate the right-hand side of (5.5) term by term.

(i) (The first term on the right-hand side of (5.5)) By the standard estimate for the operator Π_P and the definition of the U-norm, the first term on the right-hand side of (5.5) can be estimated as

$$b_h^*(\Pi_P p-p,\epsilon_{\boldsymbol{u}})\le Ch^{k+1}\|p\|_{H^{k+1}(\Omega)}\|\epsilon_{\boldsymbol{u}}\|_U.$$

(ii) (The second and the third terms on the right-hand side of (5.5)) By (5.3b), (3.2), approximation properties of $\boldsymbol{\Pi}_{RT}^k$, and (4.3),

$$\begin{aligned}&\left|-\frac{1}{2}\mu^{\frac{1}{2}}R_h(\boldsymbol{u};\nabla\boldsymbol{u},\epsilon_{\boldsymbol{u}})+\frac{1}{2}\mu^{\frac{1}{2}}R_h\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_W\nabla\boldsymbol{u},\epsilon_{\boldsymbol{u}}\right)\right|\\ &\quad\le Ch^{k+1}\|\epsilon_{\boldsymbol{u}}\|_{\boldsymbol{U}}\left(\|\boldsymbol{u}\|_{W^{1,3}(\Omega)^d}+\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}\right)\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}.\end{aligned}$$

(iii) (The fourth term on the right-hand side of (5.5)) By Lemma 5.2,

$$\begin{aligned}&\mu^{\frac{1}{2}}(\Pi_W\nabla\boldsymbol{u}-\nabla\boldsymbol{u},\epsilon_W)\\ &\quad\le C\mu^{\frac{1}{2}}h^{k+1}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}\left(\left\|\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right\|_{0,\Omega}+\|\epsilon_{\widetilde{W}}\|_{0,\Omega}\right)\\ &\quad\le Ch^{k+1}\|\nabla\boldsymbol{u}\|_{0,\Omega}\left(\|\boldsymbol{u}\|_{H^2(\Omega)^d}+\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}\right)\\ &\qquad\left(\mu^{\frac{1}{2}}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right\|_{0,\Omega}+\|\epsilon_{\boldsymbol{u}}\|_{\boldsymbol{U}}\right)\\ &\qquad+Ch^{2k+2}\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}^2.\end{aligned}$$

(iv) (The fifth term on the right-hand side of (5.5)) By the Cauchy–Schwarz inequality, it is easy to see that

$$\begin{aligned}&\left(\Pi_W\widetilde{W}-\widetilde{W},\frac{1}{2}\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right)\right)_{0,\Omega}\\ &\quad\le Ch^{k+1}\left\|\widetilde{W}\right\|_{H^{k+1}(\Omega)^{d\times d}}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right\|_{0,\Omega}\\ &\quad\le C\mu^{-\frac{1}{2}}h^{k+1}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right\|_{0,\Omega}.\end{aligned}$$

(v) (The sixth and the seventh terms on the right-hand side of (5.5)) By (5.3a), (3.2), (5.4), and approximation properties of $\mathbf{\Pi}_{RT}^k$,

$$\begin{aligned}
&\left| R_h^*\left(\boldsymbol{u};\boldsymbol{u},\frac{1}{2}\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right)\right) - R_h^*\left(\mathbf{\Pi}_{RT}^k\boldsymbol{u}_h;\Pi_{\boldsymbol{U}}\boldsymbol{u},\frac{1}{2}\left(\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}}\right)\right)\right| \\
&\quad \le C\mu^{-\frac{1}{2}}h^{k+1}\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\tilde{W}}\right\|_{0,\Omega}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)^d} \\
&\qquad + \frac{1}{2}C_p^2C_uC_{\mathrm{RT}}\mu^{-\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{0,\Omega}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\tilde{W}}\right\|_{0,\Omega}\|\epsilon_{\boldsymbol{u}}\|_{\boldsymbol{U}} \\
&\quad \le C\mu^{-\frac{1}{2}}h^{k+1}\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\tilde{W}}\right\|_{0,\Omega}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)^d} \\
&\qquad + C\mu^{-\frac{1}{2}}h^{k+1}\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\tilde{W}}\right\|_{0,\Omega} \\
&\qquad + \frac{1}{2}C_p^2C_uC_{\mathrm{RT}}C_{is}\mu^{-1}\|\nabla\boldsymbol{u}\|_{0,\Omega}\left\|\epsilon_W+\frac{1}{2}\epsilon_{\tilde{W}}\right\|_{0,\Omega}^2.
\end{aligned}$$

Finally, from the estimate in (v) above, it is important to assume

$$\frac{1}{2}C_p^2C_uC_{\mathrm{RT}}C_{is}\mu^{-1}\|\nabla\boldsymbol{u}\|_{0,\Omega} < 1$$

in order to obtain the desired bound. Combing all estimates above and (5.4), we have $\|(\epsilon_{\boldsymbol{u}},\epsilon_W,\epsilon_{\tilde{W}})\|_h \le C\mu^{-\frac{1}{2}}h^{k+1}$, where the constant C depends on $\mu^{-1}\|\boldsymbol{f}\|_{0,\Omega}$, $\|\boldsymbol{u}\|_{L^\infty(\Omega)^d}$, $\|\boldsymbol{u}\|_{W^{1,3}(\Omega)^d}$, $\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d}$ and $\|p\|_{H^{k+1}(\Omega)^d}$.

For the pressure p, we obtain an error bound by using the following inf-sup condition (see [31]):

$$\|p\|_{0,\Omega} \le C\sup_{\boldsymbol{v}\in\boldsymbol{U}^h}\frac{b_h(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_U} \tag{5.6}$$

for all $p\in P^h$. By (5.2a), we have

$$b_h^*(\epsilon_p,\boldsymbol{v}) = -B_h(\epsilon_W,\boldsymbol{v}) - \frac{1}{2}R_h\left(\mathbf{\Pi}_{RT}^k\boldsymbol{u}_h;\epsilon_W+\frac{1}{2}\epsilon_{\widetilde{W}},\boldsymbol{v}\right)+Q,$$

where Q is the right-hand side of (5.2a). The term Q can be estimated using the same idea as the estimates of the first three terms on the right-hand side of (5.5). Combining Lemma 5.2, Theorem 3.1, and the inf-sup condition (5.6), we obtain the desired result. ∎

6. Proof of superconvergence for velocity. In this section, we prove Theorem 3.3. We will consider an equivalent form of the dual problem (3.6). We define two auxiliary variables by $H=\mu^{\frac{1}{2}}\nabla\boldsymbol{\phi}+\mu^{-\frac{1}{2}}\frac{1}{2}\boldsymbol{\phi}\otimes\boldsymbol{u}$ and $\widetilde{H}=-\mu^{-\frac{1}{2}}\boldsymbol{\phi}\otimes\boldsymbol{u}$. Then, the dual problem (3.6) can be rewritten as

$$\begin{aligned}
-\mu^{\frac{1}{2}}\nabla\cdot H - \frac{1}{2}\mu^{-\frac{1}{2}}H\boldsymbol{u} - \frac{1}{4}\mu^{-\frac{1}{2}}\widetilde{H}\boldsymbol{u} - \frac{1}{2}(\nabla\boldsymbol{\phi})^\top\boldsymbol{u} - \frac{1}{2}(\nabla\boldsymbol{u})^\top\boldsymbol{\phi} - \nabla\psi &= \boldsymbol{\theta}, \\
H - \left(\mu^{\frac{1}{2}}\nabla\boldsymbol{\phi}+\frac{1}{2}\mu^{-\frac{1}{2}}\boldsymbol{\phi}\otimes\boldsymbol{u}\right) &= 0, \\
\widetilde{H}+\mu^{-\frac{1}{2}}\boldsymbol{\phi}\otimes\boldsymbol{u} &= 0.
\end{aligned} \tag{6.1}$$

We recall that $\mathcal{T}_u$ is the initial mesh defined in section 2 whose edges/faces set is $\mathcal{F}_u$. We define $\boldsymbol{\phi}_h$ as the standard Lagrange interpolation of $\boldsymbol{\phi}$ in $[H_0^1(\Omega)]^d \cap [P^1(\mathcal{T}_u)]^d$. In addition, we define H_h and $\widetilde{H}_h$ as the standard L^2-orthogonal projections of H and $\widetilde{H}$ in $[P^0(\mathcal{T}_u)]^{d\times d}$, respectively. Next, we define ψ_h the standard L^2-orthogonal projection of ψ in $P^0(\mathcal{T}_u)$. The following lemma (Lemma 6.1) gives an important relation for our convergence analysis, and the proof is given in Appendix D.

LEMMA 6.1. *The following error relation holds:*

$$(\boldsymbol{\theta}, \epsilon_{\boldsymbol{u}})_{0,\Omega} = D_1 + D_2 + D_3, \tag{6.2}$$

where D_1, D_2*, and* D_3 *are given in* (D.3), (D.4)*, and* (D.2)*, respectively.*

Finally, by using the error relation in Lemma 6.1, we can prove Theorem 3.3.

Proof of Theorem 3.3. Throughout the proof, we assume that the source $\boldsymbol{f}$ satisfies (4.13) and (5.1). First, by Theorem 3.2 and the assumption (3.7) of the dual problem (3.6), we obtain

$$|D_3| \le Ch^{k+2}\|\boldsymbol{\theta}\|_{0,\Omega}.$$

Next, we will show that

$$|D_i| \le Ch^{k+2}\|\boldsymbol{\theta}\|_{0,\Omega}, \qquad i = 1, 2.$$

Since the proof for $i = 1$ and $i = 2$ are similar, we will only present the proof for the bound for D_1.

We notice that D_1 can be rewritten as

$$\begin{aligned} 2D_1 = {}& \mu^{\frac{1}{2}} R_h(\boldsymbol{u}; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h) + \mu^{\frac{1}{2}} R_h\left(\boldsymbol{\Pi}_{RT}^k \epsilon_{\boldsymbol{u}}; \Pi_W \nabla\boldsymbol{u} - \nabla\boldsymbol{u}, \boldsymbol{\phi}_h\right) \\ & + \mu^{\frac{1}{2}} R_h\left(\Pi_{\boldsymbol{U}}\boldsymbol{u} - \boldsymbol{\Pi}_{RT}^k \Pi_{\boldsymbol{U}}\boldsymbol{u}; \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h\right) \\ & + \mu^{\frac{1}{2}} R_h(\boldsymbol{u} - \Pi_{\boldsymbol{U}}\boldsymbol{u}; \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h) \\ & + \mu^{\frac{1}{2}} R_h\left(\boldsymbol{\Pi}_{RT}^k \epsilon_{\boldsymbol{u}} - \epsilon_{\boldsymbol{u}}; \nabla\boldsymbol{u}, \boldsymbol{\phi}_h\right) + \mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla\boldsymbol{u}, \boldsymbol{\phi}_h - \boldsymbol{\phi}). \end{aligned} \tag{6.3}$$

In the following, we will estimate the terms on the right-hand side of (6.3).

(i) (The first term of (6.3)) We denote by $\bar{\boldsymbol{u}}_h$ the standard L^2-orthogonal projection in $[P^0(\mathcal{T}_u)]^d$. We denote by $\bar{\boldsymbol{\phi}}_h$ the standard L^2-orthogonal projection in $[P^0(\mathcal{T}_u)]^d$. Then, we have

$$\begin{aligned} & R_h(\boldsymbol{u}; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h) \\ & \quad = R_h(\boldsymbol{u} - \bar{\boldsymbol{u}}_h; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h) + R_h(\bar{\boldsymbol{u}}_h; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h) \\ & \quad = R_h(\boldsymbol{u} - \bar{\boldsymbol{u}}_h; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h) + R_h\left(\bar{\boldsymbol{u}}_h; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h - \bar{\boldsymbol{\phi}}_h\right) \\ & \qquad + R_h(\bar{\boldsymbol{u}}_h; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \bar{\boldsymbol{\phi}}_h) \\ & \quad = R_h(\boldsymbol{u} - \bar{\boldsymbol{u}}_h; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h) + R_h\left(\bar{\boldsymbol{u}}_h; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h - \bar{\boldsymbol{\phi}}_h\right). \end{aligned}$$

The last equality above is due to the definition of Π_W and the assumption that $k \ge 1$. Since

$$\begin{aligned} \|\boldsymbol{u} - \bar{\boldsymbol{u}}_h\|_{L^\infty(\Omega)} &\le Ch\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)}, \\ \|\bar{\boldsymbol{u}}_h\|_{L^\infty(\Omega)} &\le C\|\boldsymbol{u}\|_{L^\infty(\Omega)}, \end{aligned}$$

we have that

$$|R_h(\boldsymbol{u}; \nabla\boldsymbol{u} - \Pi_W \nabla\boldsymbol{u}, \boldsymbol{\phi}_h)| \le Ch^{k+2}\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)} \cdot \|\boldsymbol{u}\|_{H^{k+2}(\Omega)} \cdot \|\boldsymbol{\theta}\|_{0,\Omega}.$$

(ii) (The second term of (6.3)) For the second term, it is easy to see that

$$\left|R_h\left(\mathbf{\Pi}_{RT}^k\epsilon_{\boldsymbol{u}};\Pi_W\nabla\boldsymbol{u}-\nabla\boldsymbol{u},\boldsymbol{\phi}_h\right)\right|\le Ch^{2k+1}\|\boldsymbol{\theta}\|_{0,\Omega}.$$

(iii) (The third and the fourth terms of (6.3)) We denote by $\bar{G}_h$ the standard L^2-orthogonal projection of $\nabla\boldsymbol{u}$ in $[P^0(\mathcal{T}_u)]^{d\times d}$. By the definition of $\mathbf{\Pi}_{RT}^k$ and the assumption $k\ge 1$, we have that

$$\begin{aligned}&R_h\left(\Pi_{\boldsymbol{U}}\boldsymbol{u}-\mathbf{\Pi}_{RT}^k\Pi_{\boldsymbol{U}}\boldsymbol{u};\Pi_W\nabla\boldsymbol{u},\boldsymbol{\phi}_h\right)\\&\quad=R_h\left(\Pi_{\boldsymbol{U}}\boldsymbol{u}-\mathbf{\Pi}_{RT}^k\Pi_{\boldsymbol{U}}\boldsymbol{u};\Pi_W\nabla\boldsymbol{u},\boldsymbol{\phi}_h-\bar{\boldsymbol{\phi}}_h\right)\\&\qquad+R_h\left(\Pi_{\boldsymbol{U}}\boldsymbol{u}-\mathbf{\Pi}_{RT}^k\Pi_{\boldsymbol{U}}\boldsymbol{u};\Pi_W\nabla\boldsymbol{u}-\bar{G}_h,\bar{\boldsymbol{\phi}}_h\right).\end{aligned}$$

We notice that

$$\begin{aligned}&\left\|\Pi_{\boldsymbol{U}}\boldsymbol{u}-\mathbf{\Pi}_{RT}^k\Pi_{\boldsymbol{U}}\boldsymbol{u}\right\|_{0,\Omega}\\&\quad\le\|\Pi_{\boldsymbol{U}}\boldsymbol{u}-\boldsymbol{u}\|_{0.\Omega}+\left\|\mathbf{\Pi}_{RT}^k\boldsymbol{u}-\boldsymbol{u}\right\|_{0,\Omega}+\left\|\mathbf{\Pi}_{RT}^k(\Pi_U\boldsymbol{u}-\boldsymbol{u})\right\|_{0,\Omega}\\&\quad=\|\Pi_{\boldsymbol{U}}\boldsymbol{u}-\boldsymbol{u}\|_{0.\Omega}+\left\|\mathbf{\Pi}_{RT}^k\boldsymbol{u}-\boldsymbol{u}\right\|_{0,\Omega},\end{aligned}$$

where the last equality is due to the definition of Π_U in (3.1c). So, we have

$$\left\|\Pi_{\boldsymbol{U}}\boldsymbol{u}-\mathbf{\Pi}_{RT}^k\Pi_{\boldsymbol{U}}\boldsymbol{u}\right\|_{0,\Omega}\le Ch^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}.$$

Using $\|\Pi_W\nabla\boldsymbol{u}\|_{L^\infty(\Omega)}\le C\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)}$ and $\|\bar{\phi}_h\|_{L^\infty(\Omega)}\le C\|\boldsymbol{\theta}\|_{0,\Omega}$, we have that

$$\begin{aligned}&|R_h(\Pi_{\boldsymbol{U}}\boldsymbol{u}-\mathbf{\Pi}_{RT}^k\Pi_{\boldsymbol{U}}\boldsymbol{u};\Pi_W\nabla\boldsymbol{u},\boldsymbol{\phi}_h)|\\&\quad\le Ch^{k+2}\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)}\cdot\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\cdot\|\boldsymbol{\theta}\|_{0,\Omega}.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}&|R_h\left(\boldsymbol{u}-\Pi_{\boldsymbol{U}}\boldsymbol{u};\Pi_W\nabla\boldsymbol{u},\boldsymbol{\phi}_h\right)|\\&\quad\le Ch^{k+2}\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)}\cdot\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\cdot\|\boldsymbol{\theta}\|_{0,\Omega}.\end{aligned}$$

(iv) (The fifth term of (6.3)) We notice that

$$\begin{aligned}\left\|\mathbf{\Pi}_{RT}^k\epsilon_{\boldsymbol{u}}-\epsilon_{\boldsymbol{u}}\right\|_{0,\Omega}&\le Ch\big(\|\nabla_h\epsilon_{\boldsymbol{u}}\|_{0,\Omega}^2+\Sigma_{e\in\mathcal{F}_p}h_e^{-1}\|[\epsilon_{\boldsymbol{u}}]\|_{0,e}^2\big)^{1/2}\\&\le Ch^{k+2}\|\boldsymbol{u}\|_{H^{k+2}(\Omega)}.\end{aligned}$$

So, the fifth term can be controlled as

$$\left|R_h\left(\mathbf{\Pi}_{RT}^k\epsilon_{\boldsymbol{u}}-\epsilon_{\boldsymbol{u}};\nabla\boldsymbol{u},\boldsymbol{\phi}_h\right)\right|\le Ch^{k+2}\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)}\cdot\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\cdot\|\boldsymbol{\theta}\|_{0,\Omega}.$$

(v) (The sixth term of (6.3)) We note that the sixth term can be estimated in the following way:

$$|R_h\left(\epsilon_{\boldsymbol{u}};\nabla\boldsymbol{u},\boldsymbol{\phi}_h-\boldsymbol{\phi}\right)|\le Ch^{k+2}\|\boldsymbol{u}\|_{W^{1,\infty}(\Omega)}\cdot\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\cdot\|\boldsymbol{\theta}\|_{0,\Omega}.$$

Finally, we see that Theorem 3.3 is proved by combining the estimates above. ∎

Appendix A. Derivation of the SDG method for the Oseen equations. We derive the SDG formulation for the Oseen problem (1.3) based on the first order continuous formulation defined in (2.6), (2.7), and (2.8). The idea is to multiply test functions to (2.6), (2.7), and (2.8) and then integrate the resulting equations on appropriate elements. First, we multiply (2.6) by a test function G which is smooth in $\mathcal{S}(\nu)$ and integrate the resulting equation over $\mathcal{S}(\nu)$ for $\nu \in \mathcal{N}$. Then we obtain

$$\int_{\mathcal{S}(\nu)} W : G \, dx = -\int_{\mathcal{S}(\nu)} \mu^{\frac{1}{2}} \boldsymbol{u} \cdot (\operatorname{div} G) \, dx + \int_{\partial\mathcal{S}(\nu)} \mu^{\frac{1}{2}} \boldsymbol{u} \cdot (G\boldsymbol{n}) \, d\sigma - \frac{1}{2} \int_{\mathcal{S}(\nu)} \widetilde{W} : G \, dx. \tag{A.1}$$

Summing over all $\mathcal{S}(\nu)$, we have

$$\int_{\Omega} \left(W + \frac{1}{2}\widetilde{W} \right) : G \, dx = -\int_{\Omega} \mu^{\frac{1}{2}} \boldsymbol{u} \cdot (\operatorname{div} G) \, dx + \sum_{\nu \in \mathcal{N}} \left(\int_{\partial\mathcal{S}(\nu)} \mu^{\frac{1}{2}} \boldsymbol{u} \cdot (G\boldsymbol{n}) \, d\sigma \right). \tag{A.2}$$

We notice that the matrix function G is discontinuous across $\partial\mathcal{S}(\nu)$ and that the faces in $\partial\mathcal{S}(\nu)$ lie in the set $\mathcal{F}_u$. Using these facts and the fact that $\boldsymbol{u} = 0$ on $\partial\Omega$, we have

$$\int_{\Omega} \left(W + \frac{1}{2}\widetilde{W} \right) : G \, dx = -\int_{\Omega} \mu^{\frac{1}{2}} \boldsymbol{u} \cdot (\operatorname{div} G) \, dx + \sum_{e \in \mathcal{F}_u^0} \left(\int_e \mu^{\frac{1}{2}} \boldsymbol{u} \cdot [G\boldsymbol{n}] \, d\sigma \right). \tag{A.3}$$

Next, we multiply (2.7) by a test function $\boldsymbol{\psi}$ which is smooth in $\mathcal{S}(\nu)$ and integrate the resulting equation over $\mathcal{S}(\nu)$ for $\nu \in \mathcal{N}$. Then we have

$$\int_{\mathcal{S}(\nu)} \widetilde{W} : \boldsymbol{\psi} \, dx = \mu^{-\frac{1}{2}} \int_{\mathcal{S}(\nu)} (\boldsymbol{u} \otimes \boldsymbol{V}) \boldsymbol{\psi} \, dx. \tag{A.4}$$

Summing the above equations over all $\nu \in \mathcal{N}$, we obtain

$$\int_{\Omega} \widetilde{W} : \boldsymbol{\psi} \, dx = \mu^{-\frac{1}{2}} \int_{\Omega} (\boldsymbol{u} \otimes \boldsymbol{V}) \boldsymbol{\psi} \, dx. \tag{A.5}$$

Now, we derive a weak formulation for (2.8). We multiply the first equation of (2.8) by a test function $\boldsymbol{\phi}$ which is smooth in $\mathcal{R}(e)$ and integrate the resulting equation over $\mathcal{R}(e)$ for $e \in \mathcal{F}_u$. Then we have

$$\begin{aligned} &\mu^{\frac{1}{2}} \int_{\mathcal{R}(e)} W : \nabla\boldsymbol{\phi} \, dx - \mu^{\frac{1}{2}} \int_{\partial\mathcal{R}(e)} (W\boldsymbol{n})\boldsymbol{\phi} \, d\sigma \\ &\quad + \frac{\mu^{-\frac{1}{2}}}{2} \int_{\mathcal{R}(e)} (W\boldsymbol{V}) \cdot \boldsymbol{\phi} \, dx + \frac{\mu^{-\frac{1}{2}}}{4} \int_{\mathcal{R}(e)} \left(\widetilde{W}\boldsymbol{V} \right) \cdot \boldsymbol{\phi} \, dx \\ &\quad - \int_{\mathcal{R}(e)} p(\operatorname{div} \boldsymbol{\phi}) + \int_{\partial\mathcal{R}(e)} p(\boldsymbol{\phi} \cdot \boldsymbol{n}) \, d\sigma = \int_{\mathcal{R}(e)} \boldsymbol{f} \cdot \boldsymbol{\phi} \, dx. \end{aligned} \tag{A.6}$$

We assume that the test function $\boldsymbol{\phi} = 0$ on $\partial\Omega$. Notice that the faces of $\partial\mathcal{R}(e)$ lie in $\mathcal{F}_p$ and that $\boldsymbol{\phi}$ is discontinuous across $\partial\mathcal{R}(e)$. So, summing (A.6) over all $e \in \mathcal{F}_u$, we have

(A.7)
$$\mu^{\frac{1}{2}}\int_\Omega W:\nabla\boldsymbol{\phi}\,dx-\mu^{\frac{1}{2}}\sum_{e\in\mathcal{F}_p}\int_e(W\boldsymbol{n})[\boldsymbol{\phi}]\,d\sigma+\frac{\mu^{-\frac{1}{2}}}{2}\int_\Omega\left(\left(W+\frac{1}{2}\widetilde{W}\right)\boldsymbol{V}\right)\cdot\boldsymbol{\phi}\,dx$$
$$-\int_\Omega p(\operatorname{div}\boldsymbol{\phi})+\sum_{e\in\mathcal{F}_p}\int_e p[\boldsymbol{\phi}\cdot\boldsymbol{n}]\,d\sigma=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\phi}\,dx.$$

Finally, we multiply the second equation in (2.8) by a test function q which is smooth in $\mathcal{S}(\nu)$ and integrate the resulting equation over $\mathcal{S}(\nu)$ for $\nu\in\mathcal{N}$. Then we obtain

$$-\int_{\mathcal{S}(\nu)}\boldsymbol{u}\cdot\nabla q\,dx+\int_{\partial\mathcal{S}(\nu)}(\boldsymbol{u}\cdot\boldsymbol{n})q\,d\sigma=0. \tag{A.8}$$

Notice again the fact that the faces in $\partial\mathcal{S}(\nu)$ lie in $\mathcal{F}_u$ and that $\boldsymbol{u}=0$ on $\partial\Omega$. Summing (A.8) over all $\nu\in\mathcal{N}$, we obtain

$$-\int_\Omega\boldsymbol{u}\cdot\nabla q\,dx+\sum_{e\in\mathcal{F}_u^0}\int_e(\boldsymbol{u}\cdot\boldsymbol{n})[q]\,d\sigma=0. \tag{A.9}$$

We recall that (A.3), (A.5), (A.7), and (A.9) give the weak formulations for the equations in (2.6), (2.7), and (2.8). Our SDG method is based on (A.3), (A.5), (A.7), and (A.9).

Appendix B. Properties of the postprocessing projection $\boldsymbol{\Pi}_{RT}^{k}$. Here, we prove Lemmas 2.1 and 2.2.

Proof of Lemma 2.1. Thanks to (2.14a) and $\boldsymbol{u}_h\in\boldsymbol{U}^h$, it is easy to see that $\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h\in H(\mathrm{div};\Omega)$. In order to show $\nabla\cdot\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h=0$, it is enough to show that for any $q\in P^k(\mathcal{S}(\nu))$,

$$\left(\nabla\cdot\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h,q\right)_{0,\mathcal{S}(\nu)}=0.$$

Using Green's identity, we need to show that

$$-\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h,\nabla q\right)_{0,\mathcal{S}(\nu)}+\left\langle\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h\cdot\boldsymbol{n},q\right\rangle_{0,\partial\mathcal{S}(\nu)}=0\quad\forall q\in P^k(\mathcal{S}(\nu)).$$

Due to (2.14), it is equivalent to show that

$$-(\boldsymbol{u}_h,\nabla q)_{0,\mathcal{S}(\nu)}+\langle\boldsymbol{u}_h\cdot\boldsymbol{n},q\rangle_{0,\partial\mathcal{S}(\nu)}=0\quad\forall q\in P^k(\mathcal{S}(\nu)).$$

Then by (2.9d) and (2.11), the above equation holds. Hence, we conclude that $\nabla\cdot\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h=0$. ∎

Proof of Lemma 2.2. We denote by $\widetilde{\boldsymbol{u}}_h$ the standard L^2-orthogonal projection of $\boldsymbol{u}_h$ onto $[P^k(\mathcal{S}(\nu))]^d$. By (2.14), we have

$$\int_e\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h\right)\cdot\boldsymbol{n}\,\mu\,d\sigma=\int_e(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h)\cdot\boldsymbol{n}\,\mu\,d\sigma\quad\forall\mu\in P^k(e),e\in\partial\mathcal{S}(\nu),$$
$$\int_{\mathcal{S}(\nu)}\left(\boldsymbol{\Pi}_{RT}^k\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}_h\right)\cdot\boldsymbol{v}\,dx=0\quad\forall\boldsymbol{v}\in\left[P^{k-1}(\mathcal{S}(\nu))\right]^d.$$

By a standard scaling argument, we have

$$\left\|\mathbf{\Pi}_{RT}^{k}\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h\right\|_{0,\mathcal{S}(\nu)} \le C h_{\mathcal{S}(\nu)}^{1/2} \|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h\|_{0,\partial\mathcal{S}(\nu)}, \tag{B.1}$$

$$\left\|\nabla_h \left(\mathbf{\Pi}_{RT}^{k}\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h\right)\right\|_{0,\mathcal{S}(\nu)} + h_{\mathcal{S}(\nu)}^{-1/2}\left\|\mathbf{\Pi}_{RT}^{k}\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h\right\|_{0,\partial\mathcal{S}(\nu)} \le C h_{\mathcal{S}(\nu)}^{-1/2} \|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h\|_{0,\partial\mathcal{S}(\nu)}, \tag{B.2}$$

where $h_{\mathcal{S}(\nu)}$ is the diameter of $\mathcal{S}(\nu)$. We denote by $\boldsymbol{v}_h$ the Oswald interpolation in [29, 30] of $\boldsymbol{u}_h$ on $[H_0^1(\Omega)]^d \cap [P^{k+1}(\mathcal{T}_h)]^d$, and $\widetilde{\boldsymbol{v}}_h$ the standard L^2-orthogonal projection of $\boldsymbol{v}_h$ onto $[P^k(\mathcal{S}(\nu))]^d$. Then we have the following estimates:

$$\begin{aligned}
\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{0,\Omega}^2 &\le C\Sigma_{e\in\mathcal{F}_p} h_e \|[\boldsymbol{u}_h]\|_{0,e}^2,\\
\Sigma_{\tau\in\mathcal{T}} h_\tau^{-2}\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{0,\tau}^2 &\le C\Sigma_{e\in\mathcal{F}_p} h_e^{-1} \|[\boldsymbol{u}_h]\|_{0,e}^2,\\
\|\nabla_h(\boldsymbol{u}_h - \boldsymbol{v}_h)\|_{0,\Omega}^2 &\le C\Sigma_{e\in\mathcal{F}_p} h_e^{-1} \|[\boldsymbol{u}_h]\|_{0,e}^2,
\end{aligned}$$

where h_τ is the diameter of $\tau \in \mathcal{T}$. Thus we obtain

$$\begin{aligned}
\|\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h\|_{0,\Omega} &\le \|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{u}}_h - \widetilde{\boldsymbol{v}}_h\|_{0,\Omega} + \|\widetilde{\boldsymbol{v}}_h - \boldsymbol{v}_h\|_{0,\Omega}\\
&\le C\Big(\|\boldsymbol{u}_h\|_{0,\Omega} + \big(\Sigma_{e\in\mathcal{F}_p} h_e \|[\boldsymbol{u}_h]\|_e^2\big)^{1/2}\Big)\\
&\le C\|\boldsymbol{u}_h\|_{0,\Omega}.
\end{aligned}$$

Combining this and (B.1), we obtain (2.15a). On the other hand, we show (2.15b), and we first notice that

$$\begin{aligned}
&\Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h]\|_{0,e}^2\\
&\quad\le C\Big\{\Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\boldsymbol{u}_h - \boldsymbol{v}_h]\|_{0,e}^2 + \Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\widetilde{\boldsymbol{u}}_h - \widetilde{\boldsymbol{v}}_h]\|_{0,e}^2 + \Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\widetilde{\boldsymbol{v}}_h - \boldsymbol{v}_h]\|_{0,e}^2\Big\}.
\end{aligned} \tag{B.3}$$

Since $\boldsymbol{v}_h \in [H_0^1(\Omega)]^d$ and $\boldsymbol{u}_h \in \boldsymbol{U}^h$, we have

$$\Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\boldsymbol{u}_h - \boldsymbol{v}_h]\|_{0,e}^2 = \Sigma_{e\in\mathcal{F}_p} h_e^{-1}\|[\boldsymbol{u}_h]\|_e^2.$$

Next, we have

$$\begin{aligned}
\Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\widetilde{\boldsymbol{u}}_h - \widetilde{\boldsymbol{v}}_h]\|_{0,e}^2 &\le C\Sigma_{\tau\in\mathcal{T}} h_\tau^{-2}\|\widetilde{\boldsymbol{u}}_h - \widetilde{\boldsymbol{v}}_h\|_{0,\tau}^2\\
&\le C\Sigma_{\tau\in\mathcal{T}} h_\tau^{-2}\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{0,\tau}^2\\
&\le C\Sigma_{e\in\mathcal{F}_p} h_e^{-1}\|[\boldsymbol{u}_h]\|_e^2.
\end{aligned}$$

For the last term in (B.3), we obtain

$$\Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\widetilde{\boldsymbol{v}}_h - \boldsymbol{v}_h]\|_{0,e}^2 \le C\Sigma_{\tau\in\mathcal{T}} h_\tau^{-2}\|\widetilde{\boldsymbol{v}}_h - \boldsymbol{v}_h\|_{0,\tau}^2 \le C\Sigma_{\tau\in\mathcal{T}}\|\nabla\boldsymbol{v}_h\|_{0,\tau}^2.$$

Combining the above results in (B.3), we finally obtain

$$\Sigma_{e\in\mathcal{F}} h_e^{-1}\|[\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h]\|_{0,e}^2 \le C\Big(\|\nabla_h\boldsymbol{u}_h\|_{0,\Omega}^2 + \Sigma_{e\in\mathcal{F}_p} h_e^{-1}\|[\boldsymbol{u}_h]\|_e^2\Big).$$

Using a similar argument, we can show that

$$\|\nabla_h(\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_h)\|_{0,\Omega}^2 \le C\Big(\|\nabla_h\boldsymbol{u}_h\|_{0,\Omega}^2 + \Sigma_{e\in\mathcal{F}_p} h_e^{-1}\|[\boldsymbol{u}_h]\|_e^2\Big).$$

Hence, we conclude that (2.15b) holds. ∎

Appendix C. Error equations. In this section, we derive the error equations (5.2). From the derivation of the SDG method (2.9) and (A.3), (A.5), (A.7), and (A.9), it is easy to see that the exact solution $(\boldsymbol{u}, W, \widetilde{W}, p)$ satisfies

$$B_h(W, \boldsymbol{v}) + \frac{1}{2} R_h \left(\boldsymbol{u}; W + \frac{1}{2}\widetilde{W}, \boldsymbol{v}\right) + b_h^*(p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega}, \tag{C.1a}$$

$$B_h^*(\boldsymbol{u}, G) - \left(W + \frac{1}{2}\widetilde{W}, G\right)_{0,\Omega} = 0, \tag{C.1b}$$

$$R_h^*\left(\boldsymbol{u}; \boldsymbol{u}, \widetilde{G}\right) - \left(\widetilde{W}, \widetilde{G}\right)_{0,\Omega} = 0, \tag{C.1c}$$

$$b_h(\boldsymbol{u}, q) = 0 \tag{C.1d}$$

for any test function $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. By (3.1), the system (C.1) can be written as

$$\begin{aligned}
B_h(\Pi_W W, \boldsymbol{v}) + \frac{1}{2} R_h \left(\boldsymbol{u}; W + \frac{1}{2}\widetilde{W}, \boldsymbol{v}\right) + b_h^*(\Pi_P p, \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} + b_h^*(\Pi_P p - p, \boldsymbol{v}),\\
B_h^*(\Pi_{\boldsymbol{U}} \boldsymbol{u}, G) - \left(\Pi_W W + \frac{1}{2}\Pi_{\widetilde{W}} \widetilde{W}, G\right)_{0,\Omega} &= -(\Pi_W \nabla \boldsymbol{u} - \nabla \boldsymbol{u}, G),\\
R_h^*\left(\boldsymbol{u}; \boldsymbol{u}, \widetilde{G}\right) - \left(\Pi_{\widetilde{W}} \widetilde{W}, \widetilde{G}\right)_{0,\Omega} &= -\left(\Pi_W \widetilde{W} - \widetilde{W}, \widetilde{G}\right)_{0,\Omega},\\
b_h(\Pi_{\boldsymbol{U}} \boldsymbol{u}, q) &= 0
\end{aligned}$$

for any test function $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. Next, we subtract the above equations by (2.16) to obtain

$$\begin{aligned}
&B_h(\epsilon_W, \boldsymbol{v}) + \frac{1}{2} R_h \left(\boldsymbol{u}; W + \frac{1}{2}\widetilde{W}, \boldsymbol{v}\right) - \frac{1}{2} R_h \left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; W_h + \frac{1}{2}\widetilde{W}_h, \boldsymbol{v}\right) + b_h^*(\epsilon_p, \boldsymbol{v})\\
&\qquad\qquad = b_h^*(\Pi_P p - p, \boldsymbol{v}),\\
&B_h^*(\epsilon_{\boldsymbol{u}}, G) - \left(\epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, G\right)_{0,\Omega} = -(\Pi_W \nabla \boldsymbol{u} - \nabla \boldsymbol{u}, G),\\
&R_h^*\left(\boldsymbol{u}; \boldsymbol{u}, \widetilde{G}\right) - R_h^*\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \boldsymbol{u}_h, \widetilde{G}\right) - \left(\epsilon_{\widetilde{W}}, \widetilde{G}\right)_{0,\Omega} = -\left(\Pi_W \widetilde{W} - \widetilde{W}, \widetilde{G}\right)_{0,\Omega},\\
&b_h(\epsilon_{\boldsymbol{u}}, q) = 0
\end{aligned}$$

for any $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$, and consequently, we have

$$\begin{aligned}
&B_h(\epsilon_W, \boldsymbol{v}) + \frac{1}{2} R_h \left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, \boldsymbol{v}\right) + b_h^*(\epsilon_p, \boldsymbol{v}) = b_h^*(\Pi_P p - p, \boldsymbol{v})\\
&\qquad - \frac{1}{2} R_h \left(\boldsymbol{u}; W + \frac{1}{2}\widetilde{W}, \boldsymbol{v}\right) + \frac{1}{2} R_h \left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_W W + \frac{1}{2}\Pi_{\widetilde{W}} \widetilde{W}, \boldsymbol{v}\right),\\
&B_h^*(\epsilon_{\boldsymbol{u}}, G) - \left(\epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, G\right)_{0,\Omega} = -(\Pi_W \nabla \boldsymbol{u} - \nabla \boldsymbol{u}, G),\\
&R_h^*\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \epsilon_{\boldsymbol{u}}, \widetilde{G}\right) - \left(\epsilon_{\widetilde{W}}, \widetilde{G}\right)_{0,\Omega} = -\left(\Pi_W \widetilde{W} - \widetilde{W}, \widetilde{G}\right)_{0,\Omega} - R_h^*\left(\boldsymbol{u}; \boldsymbol{u}, \widetilde{G}\right)\\
&\qquad + R_h^*\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_{\boldsymbol{U}} \boldsymbol{u}, \widetilde{G}\right),\\
&b_h(\epsilon_{\boldsymbol{u}}, q) = 0
\end{aligned}$$

for any $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. Finally, using (3.1), we obtain the required error equations (5.2) for the SDG method (2.16).

Appendix D. Proof of Lemma 6.1. In this section, we give the proof for Lemma 6.1. Using the definitions of the bilinear forms (2.10), (2.11), and (2.12), it is easy to see that the following four relations hold:

$$\begin{aligned}
B_h(H, \boldsymbol{v}) - \frac{1}{2} R_h\left(\boldsymbol{u}; H + \frac{1}{2}\widetilde{H}, \boldsymbol{v}\right) - b_h^*(\psi, \boldsymbol{v}) \\
= \frac{1}{2}\mu^{\frac{1}{2}} R_h(\boldsymbol{v}; \nabla\boldsymbol{\phi}, \boldsymbol{u}) + \frac{1}{2}\mu^{\frac{1}{2}} R_h(\boldsymbol{v}; \nabla\boldsymbol{u}, \boldsymbol{\phi}) + (\boldsymbol{\theta}, \boldsymbol{v})_{0,\Omega}&, \\
B_h^*(\boldsymbol{\phi}, G) - \left(H + \frac{1}{2}\widetilde{H}, G\right)_{0,\Omega} = 0&, \\
R_h^*\left(\boldsymbol{u}; \boldsymbol{\phi}, \widetilde{G}\right) + \left(\widetilde{H}, \widetilde{G}\right)_{0,\Omega} = 0&, \\
b_h(\boldsymbol{\phi}, q) = 0&
\end{aligned}$$

for any $(\boldsymbol{v}, G, \widetilde{G}, q) \in \boldsymbol{U}^h \times W^h \times W^h \times P^h$. By taking $\boldsymbol{v} = \epsilon_{\boldsymbol{u}}$, $G = \epsilon_W$, $\widetilde{G} = \frac{1}{2}(\epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}})$, $q = \epsilon_p$, and summing up the above equations, we obtain

$$\begin{aligned}
&B_h^*(\boldsymbol{\phi}, \epsilon_W) + \frac{1}{2} R_h^*\left(\boldsymbol{u}; \boldsymbol{\phi}, \epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}\right) + b_h(\boldsymbol{\phi}, \epsilon_p) \\
&\quad + B_h(H, \epsilon_{\boldsymbol{u}}) - \left(\epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, H\right)_{0,\Omega} \\
&\quad - \frac{1}{2} R_h\left(\boldsymbol{u}; H + \frac{1}{2}\widetilde{H}, \epsilon_{\boldsymbol{u}}\right) + \frac{1}{2}\left(\epsilon_{\widetilde{W}}, H + \frac{1}{2}\widetilde{H}\right)_{0,\Omega} \\
&\quad - b_h^*(\psi, \epsilon_{\boldsymbol{u}}) \\
&\quad - \frac{1}{2}\mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla\boldsymbol{\phi}, \boldsymbol{u}) - \frac{1}{2}\mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla\boldsymbol{u}, \boldsymbol{\phi}) = (\boldsymbol{\theta}, \epsilon_{\boldsymbol{u}}).
\end{aligned}$$

So, we have

$$\begin{aligned}
&B_h(\epsilon_W, \boldsymbol{\phi}) + \frac{1}{2} R_h\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, \boldsymbol{\phi}\right) + b_h^*(\epsilon_p, \boldsymbol{\phi}) \\
&\quad + B_h^*(\epsilon_{\boldsymbol{u}}, H) - \left(\epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, H\right)_{0,\Omega} \\
&\quad - \frac{1}{2} R_h^*\left(\boldsymbol{u}; \epsilon_{\boldsymbol{u}}, H + \frac{1}{2}\widetilde{H}\right) + \frac{1}{2}\left(\epsilon_{\widetilde{W}}, H + \frac{1}{2}\widetilde{H}\right)_{0,\Omega} \\
&\quad - b_h(\epsilon_{\boldsymbol{u}}, \psi) \\
&\quad - \frac{1}{2}\mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla\boldsymbol{\phi}, \boldsymbol{u}) - \frac{1}{2}\mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla\boldsymbol{u}, \boldsymbol{\phi}) \\
&= (\boldsymbol{\theta}, \epsilon_{\boldsymbol{u}})_{0,\Omega} + \frac{1}{2} R_h\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h - \boldsymbol{u}; \epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, \boldsymbol{\phi}\right).
\end{aligned} \tag{D.1}$$

Using the error equations (4.10) and the system (D.1), we have

$$(\boldsymbol{\theta}, \epsilon_{\boldsymbol{u}})_{0,\Omega} = E_1 + E_2 + D_3,$$

where

$$
\begin{aligned}
E_1 = &-b_h^*(\Pi_P p - p, \boldsymbol{\phi}_h) + \frac{1}{2}\mu^{\frac{1}{2}} R_h(\boldsymbol{u}; \nabla \boldsymbol{u}, \boldsymbol{\phi}_h) - \frac{1}{2}\mu^{\frac{1}{2}} R_h\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_W \nabla \boldsymbol{u}, \boldsymbol{\phi}_h\right) \\
&+ \mu^{\frac{1}{2}}(\Pi_W \nabla \boldsymbol{u} - \nabla \boldsymbol{u}, H_h) + \frac{1}{2}\left(\Pi_W \widetilde{W} - \widetilde{W}, H_h + \frac{1}{2}\widetilde{H}_h\right)_{0,\Omega} \\
&+ \frac{1}{2} R_h^*\left(\boldsymbol{u}; \boldsymbol{u}, H_h + \frac{1}{2}\widetilde{H}_h\right) - \frac{1}{2} R_h^*\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_{\boldsymbol{U}} \boldsymbol{u}, H_h + \frac{1}{2}\widetilde{H}_h\right), \\
E_2 = &-\frac{1}{2}\mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla \boldsymbol{\phi}, \boldsymbol{u}) - \frac{1}{2}\mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla \boldsymbol{u}, \boldsymbol{\phi}),
\end{aligned}
$$

and

(D.2)

$$
\begin{aligned}
D_3 = &B_h(\epsilon_W, \boldsymbol{\phi} - \boldsymbol{\phi}_h) + \frac{1}{2} R_h\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, \boldsymbol{\phi} - \boldsymbol{\phi}_h\right) + b_h^*(\epsilon_p, \boldsymbol{\phi} - \boldsymbol{\phi}_h) \\
&+ B_h^*(\epsilon_{\boldsymbol{u}}, H - H_h) - \left(\epsilon_W + \frac{1}{2}\epsilon_{\widetilde{W}}, H - H_h\right)_{0,\Omega} \\
&- \frac{1}{2} R_h^*\left(\boldsymbol{u}; \epsilon_{\boldsymbol{u}}, \left(H + \frac{1}{2}\widetilde{H}\right) - \left(H_h + \frac{1}{2}\widetilde{H}_h\right)\right) \\
&+ \frac{1}{2}\left(\epsilon_{\widetilde{W}}, \left(H + \frac{1}{2}\widetilde{H}\right) - \left(H_h + \frac{1}{2}\widetilde{H}_h\right)\right)_{0,\Omega} - b_h(\epsilon_{\boldsymbol{u}}, \psi - \psi_h).
\end{aligned}
$$

By using the definitions of Π_W and Π_P, we see that E_1 can be further simplified as

$$
\begin{aligned}
E_1 = &\frac{1}{2}\mu^{\frac{1}{2}} R_h(\boldsymbol{u}; \nabla \boldsymbol{u}, \boldsymbol{\phi}_h) - \frac{1}{2}\mu^{\frac{1}{2}} R_h\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_W \nabla \boldsymbol{u}, \boldsymbol{\phi}_h\right) \\
&+ \frac{1}{2} R_h^*\left(\boldsymbol{u}; \boldsymbol{u}, H_h + \frac{1}{2}\widetilde{H}_h\right) - \frac{1}{2} R_h^*\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_{\boldsymbol{U}} \boldsymbol{u}, H_h + \frac{1}{2}\widetilde{H}_h\right).
\end{aligned}
$$

Finally, by a direct computation, we see that $E_1 + E_2 = D_1 + D_2$, where

(D.3)

$$
D_1 = \frac{1}{2}\mu^{\frac{1}{2}}\left(R_h(\boldsymbol{u}; \nabla \boldsymbol{u}, \boldsymbol{\phi}_h) - R_h\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_W \nabla \boldsymbol{u}, \boldsymbol{\phi}_h\right) - R_h(\epsilon_{\boldsymbol{u}}; \nabla \boldsymbol{u}, \boldsymbol{\phi})\right),
$$

$$
\begin{aligned}
D_2 = &\frac{1}{2}\Bigg(R_h^*\left(\boldsymbol{u}; \boldsymbol{u}, H_h + \frac{1}{2}\widetilde{H}_h\right) - R_h^*\left(\boldsymbol{\Pi}_{RT}^k \boldsymbol{u}_h; \Pi_{\boldsymbol{U}} \boldsymbol{u}, H_h + \frac{1}{2}\widetilde{H}_h\right) \\
&- \mu^{\frac{1}{2}} R_h(\epsilon_{\boldsymbol{u}}; \nabla \boldsymbol{\phi}, \boldsymbol{u})\Bigg).
\end{aligned}
$$

(D.4)

This completes the proof.

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