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INCOMPRESSIBLE RÉTHY FLOWS IN TWO DIMENSIONS*

JIANFENG CHENG†, LILI DU‡, and WEI XIANG§

Abstract. This paper deals with the mathematical theory of a two-dimensional incompressible flow with two free boundaries effusing from a semi-infinite nozzle around a given obstacle, which is named the Réthy flows. The interesting and old problem was suggested and tried by M. Réthy in 1895 for a flow around a symmetric wedge. Here, we are concerned with the well-posedness theory of the symmetric Réthy flow with more general geometric conditions to the nozzle and obstacle. Given a mass flux at the inlet of the nozzle, we established the existence of the incompressible symmetric Réthy flows, containing two free boundaries behind the obstacle. Furthermore, the location estimate, the deflection angle estimate of the Réthy flow in the far field, and the asymptotic behavior of the Réthy flow in the upstream and downstream are also obtained. Finally, some results on the uniqueness of the Réthy flow are established.

Key words. incompressible Réthy flows, jet and cavity, existence and uniqueness, free boundary, asymptotic behavior

AMS subject classifications. 76B10, 76B03, 35Q31, 35J25

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1. Introduction and main results.

1.1. Introduction. As mentioned in [25], "An interesting problem in hydrodynamics on the flow of a jet around a symmetric wedge" was first suggested by M. Réthy in [31] in 1895. His work was characterized by means of the Hodograph method to reduce the free surface problem into a solution of a first-order integral equation, and he gave a solution to the problem of the flow past a plate or a wedge symmetrically from a cylinder (see the intuitive Figure 1). As mentioned by G. Birkhoff and E. H. Zarantonello in their book [11] (in section 7, Chapter II) "Such a flow may be called a Réthy flow." The main purpose of this paper is to investigate the so-called Réthy flow in two dimensions and to establish the systematic well-posedness theory of a hydrodynamic flow with two free boundaries involving a jet and a cavity at the same time, effusing from a semi-infinite nozzle with variable cross-sections past a general obstacle. The problem which involves both jets and cavities has also been introduced by A. Friedman in the classical book [26] (see P267 in Chapter 3).

1.2. The statement of Réthy flow problem. First, we introduce an open semi-infinitely symmetric nozzle and a symmetric obstacle as follows. Denote N_1 : $y = f_1(x)$ as the nozzle wall, which satisfies

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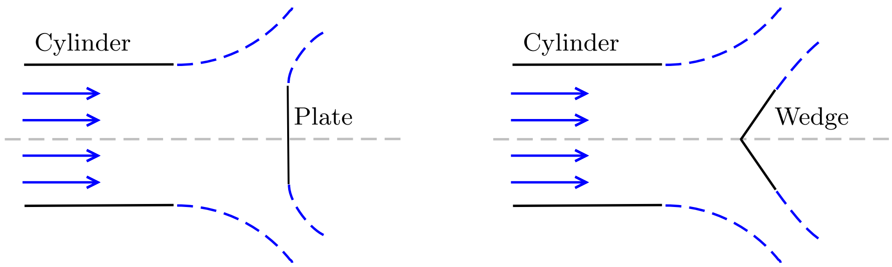

Fig. 1. *Réthy flow.*

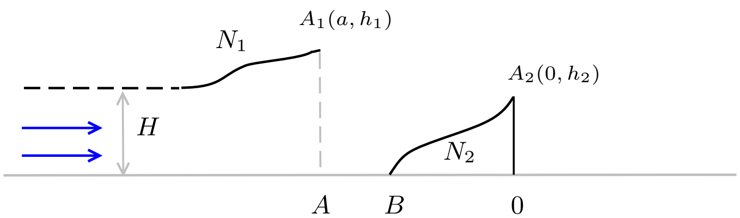

Fig. 2. *Symmetric open channel and obstacle.*

$$f_1(x) > 0 \ \text{ for } -\infty < x \le a, \quad f_1(a) = h_1, \quad \lim_{x\to-\infty} f_1(x) = H, \tag{1.1}$$

and $N_2 : y = f_2(x)$ as the obstacle, which satisfies

$$f_2(x) > 0 \text{ for } b \le x \le 0, \ f_2(b) = 0, \text{ and } f_2(0) = h_2, \tag{1.2}$$

where $f_1(x)$ and $f_2(x)$ are increasing and $C^{2,\alpha}$-smooth functions (see Figure 2). $B = (b, 0)$ is the vertex of the obstacle. Here, without loss of generality, we assume $a \le b$, which means the obstacle is behind of the mouth of the open channel. We would like to mention that the assumption is not essential, and the results in this paper are still valid even when the obstacle lies in the channel ($a \ge 0$).

Consider a two-dimensional symmetric irrotational inviscid fluid efflux from the open channel, and the governing system of equations is the following Euler system for incompressible fluids flow:

$$\begin{cases} u_x + v_y = 0, \\ uu_x + vu_y + p_x = 0, \\ uv_x + vv_y + p_y = 0, \end{cases} \tag{1.3}$$

where (u, v) and p denote the velocity field and the pressure of the incompressible fluid. The irrotational condition gives that

$$v_x - u_y = 0. \tag{1.4}$$

The nozzle wall N_1 and the obstacle N_2 are assumed to be solid, and thus

$$(u, v) \cdot \vec{n} = 0 \qquad \text{on} \ \ N_1 \cup N_2, \tag{1.5}$$

where $\vec{n}$ is the unit outer normal of the boundaries.

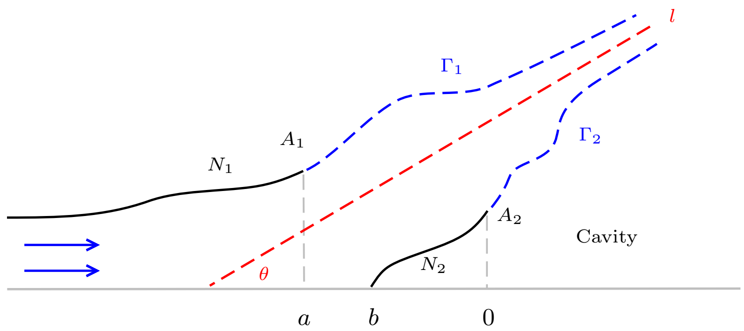

FIG. 3. *Réthy flow.*

The free boundaries Γ_1 and Γ_2 are material surfaces; then the velocity field still satisfies the perfect slip boundary condition (1.5) on Γ_1 and Γ_2. Furthermore, for the dynamic condition on the free boundaries, the classical assumption (neglecting the effects of gravity and surface tension) is that the pressure p is constant, say p_0 on the free boundaries. Hence, due to the Bernoulli's law for incompressible inviscid flows, the speed remains a positive constant λ called the cavity number on the free boundaries, namely,

$$\sqrt{u^2+v^2} = \lambda \quad \text{on} \quad \Gamma_1 \text{ and } \Gamma_2. \tag{1.6}$$

Moreover, we assume that the free boundary of the jet and the free boundary of the cavity are parallel to some direction, denoted as $(\cos\theta, \sin\theta)$ at the far field. And the direction $(\cos\theta, \sin\theta)$ is the so-called asymptotic direction of the Réthy flow in downstream, and θ is the so-called deflection angle of the Réthy flow (see Figure 3). It should be noted that the cavity number λ and the deflection angle θ of the free boundaries are not imposed a priori, which will be determined later by the Réthy flow itself.

It follows from the continuity equation and the boundary condition (1.5) that the mass flux crossing any section S transversal to the x-direction with $x = x_0 < a$ remains a positive constant m_0 called mass flux, that is,

$$m_0 = \int_S (u,v)\cdot \vec{l} dS = \int_0^{f_1(x_0)} u(x_0,y)dy, \tag{1.7}$$

where $\vec{l}$ is the unit normal of S in the positive x-direction.

The Réthy flow problem is stated as follows.

Réthy flow problem. Given a semi-infinitely long nozzle N_1, an obstacle N_2 as above, and an incoming mass flux $m_0 > 0$ of the incompressible flows in the inlet, does there exist a planar symmetric Réthy flow, such that the two free boundaries detach smoothly from the end point A_1 and the edge point A_2, respectively, and the pressure remains constant on the free boundaries Γ_1 and Γ_2 (see Figure 3)?

Furthermore, we give the definition of the solution to the planar symmetric Réthy flow problem in the following.

DEFINITION 1.1 (a solution to the Réthy flow problem). *A vector $(u, v, p, \Gamma_1, \Gamma_2)$ is called a solution to the Réthy flow problem, provided that the following hold.*

(1) *Two smooth curves Γ_1 and Γ_2 are given by some smooth functions $y = k_{1,\lambda,\theta}(x) \in C^1([a,+\infty))$ and $y = k_{2,\lambda,\theta}(x) \in C^1([0,+\infty))$, and there exists a*

pair (λ, θ) *such that*

$$f_1(a) = k_{1,\lambda,\theta}(a) \text{ and } f_2(0) = k_{2,\lambda,\theta}(0), \tag{1.8}$$

and

$$f_1'(a-0) = k_{1,\lambda,\theta}'(a+0) \text{ and } f_2'(0-0) = k_{2,\lambda,\theta}'(0+0). \tag{1.9}$$

Furthermore, $k_{i,\lambda,\theta}'(x) \to \tan\theta$ *as* $x \to +\infty$ *for* $i = 1, 2$.

(2) $(u, v, p) \in (C^{1,\alpha}(\Omega_0) \cap C(\overline{\Omega}_0))^3$ *solves the steady incompressible Euler system* (1.3)*, the boundary condition* (1.5)*, and the mass flux condition* (1.7)*, where* Ω_0 *is bounded by* $N_1, N_2, \Gamma_1, \Gamma_2$*, and* $I_b = \{(x, 0) \mid x \le b\}$.

(3) $\sqrt{u^2 + v^2} = \lambda$ *on* Γ_1 *and* Γ_2.

Remark 1.2. The condition (1.8) is the so-called continuous fit condition to the free boundaries of the Réthy flow, and the condition (1.9) is the so-called smooth fit condition to Γ_1 and Γ_2, which means that the free boundaries detach smoothly from the end point A_1 and the edge point A_2, respectively. From the mathematical point of view, the choice of the parameters λ and θ guarantees the two continuous fit conditions. In this situation, the Réthy flow problem seems to be well-posed, since two parameters λ and θ meet the two continuous fit conditions. As we shall see below, there exists a pair of the parameters (λ, θ), such that the continuous fit conditions are fulfilled to the Réthy flows, which is one of the key points in this paper.

1.3. Main results. The first result in this paper is the existence of the Réthy flow as follows.

THEOREM 1.3. *Given a semi-infinitely long nozzle* N_1*, an obstacle* N_2*, and a mass flux* $m_0 > 0$*, there exist a pair of parameters* (λ, θ) *with* $\lambda > 0$ *and a solution* $(u, v, p, \Gamma_1, \Gamma_2)$ *to the Réthy flow problem satisfying the conditions in Definition* 1.1*. Moreover, the deflection angle of the Réthy flow at the far field satisfies the estimate*

$$0 < \theta < \frac{\pi}{2}. \tag{1.10}$$

The horizontal velocity u *and the vertical velocity* v *satisfy that*

$$u > 0 \quad \text{in } \overline{\Omega_0} \setminus B \qquad \text{and} \qquad v > 0 \quad \text{in } \Omega_0, \tag{1.11}$$

respectively.

Remark 1.4. It should be mentioned that there does not exist a stagnation point in the Réthy flow established in Theorem 1.3 except for the vertex point B of the obstacle. Moreover, our results can be extended to the Réthy flow around a cusped obstacle ($f_2'(b) = 0$) and the only difference is that the cusped point B is not a stagnation point (see Figure 4).

Remark 1.5. The Réthy flows established in Theorem 1.3 possess the nonnegative vertical velocity, which implies that the cavity behind the obstacle is infinite. This result coincides with the assertion in [32, 33] by Villat that a symmetric finite cusped cavity behind a wedge is impossible and the conjecture that cusped cavities are mathematically impossible in general.

Theorem 1.3 implies that there exists a pair of parameters (λ, θ), such that there exists a Réthy flow, which satisfies the conditions (1)–(3) in Definition 1.1 with (1.10)

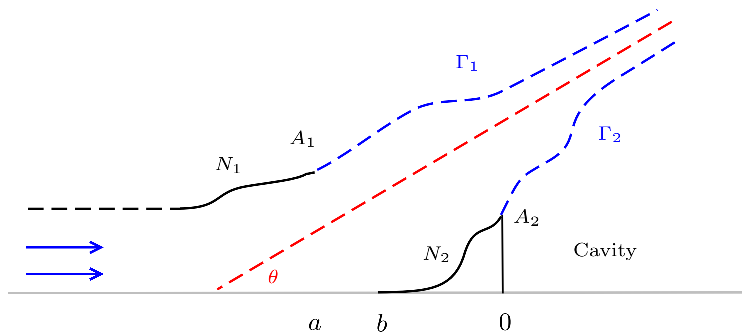

FIG. 4. *Réthy flow around a cusped obstacle.*

and (1.11). However, the uniqueness of the parameters (λ, θ) and the Réthy flow (u, v, p) with two free boundaries Γ_1 and Γ_2 are a totally open and challenging problem. We will first establish the uniqueness of the Réthy flow with two free boundaries, for some fixed parameters (λ, θ), and determine the position of the Réthy flow. Next, we will consider the uniqueness of the parameters (λ, θ) under the continuous fit condition (1.8). The result reads that the deflection angle θ is unique for some fixed cavity number λ.

THEOREM 1.6. *For any fixed pair of parameters (λ, θ) such that the free boundaries satisfy the conditions* (1.8) *and* (1.9), *the solution $(u, v, p, \Gamma_1, \Gamma_2)$ established in Theorem* 1.3 *is unique. Furthermore, for any parameter pairs (λ, θ_1) and (λ, θ_2) such that the free boundaries of the Réthy flow satisfy the conditions* (1.8) *and* (1.9), *$\theta_1 = \theta_2$.*

By (1.10), the deflection angle θ of the Réthy flow established in this paper lies in the interval $(0, \frac{\pi}{2})$. Moreover, as a byproduct in this paper, we will give a more precise estimate of the deflection angle θ for some special case $h_1 < h_2$.

THEOREM 1.7. *Assume $h_1 < h_2$ and for the pair of parameters (λ, θ) such that the free boundaries satisfy the conditions* (1.8) *and* (1.9), *if the obstacle N_2 lies below $\overline{A_1A_2}$, then the deflection angle θ obtained in Theorem* 1.3 *satisfies*

$$\theta_0 = \arctan \frac{h_2 - h_1}{a} < \theta < \frac{\pi}{2}. \tag{1.12}$$

Remark 1.8. The additional assumption $h_1 < h_2$ means that the height of A_1 is less than that of A_2; thus, we observed that if we restrict that deflection angle θ to be less than θ_0, then in fact the free boundary Γ_2 will vanish. Hence, we can show that the deflection angle in fact lies in the interval $(\theta_0, \frac{\pi}{2})$ (see Figure 5).

The uniqueness result in Theorem 1.6 implies that the location of the Réthy flow can be determined by the parameters (λ, θ). Finally, we will estimate the position of the Réthy flow and establish the asymptotic behavior of the Réthy flow in upstream and downstream.

THEOREM 1.9. *For any fixed pair of parameters (λ, θ) such that the free boundaries of the Réthy flow satisfy the conditions* (1.8) *and* (1.9), *then there exists a unique $\beta \in [h_2 - \frac{m_0}{2\lambda \cos\theta}, h_1 - a\tan\theta + \frac{m_0}{2\lambda\cos\theta}]$, such that*

$$k_{1,\lambda,\theta}(x) \to x\tan\theta + \beta + \frac{m_0}{2\lambda\cos\theta} \quad \textit{and} \quad k_{2,\lambda,\theta}(x) \to x\tan\theta + \beta - \frac{m_0}{2\lambda\cos\theta} \quad \textit{as } x \to +\infty.$$

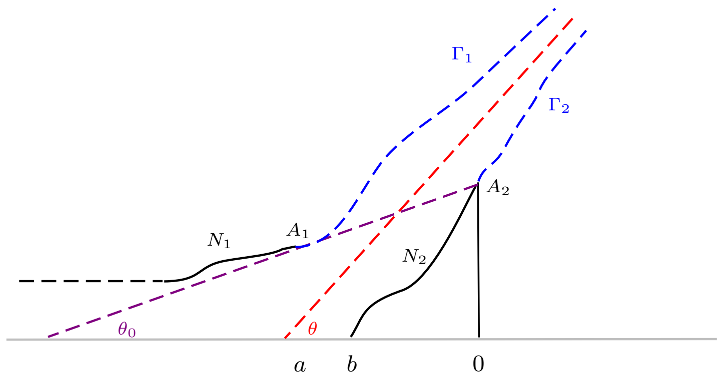

FIG. 5. *Deflection angle estimate.*

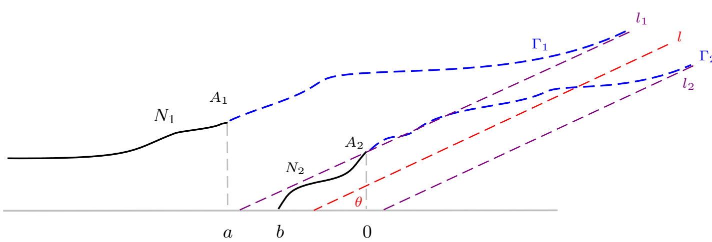

FIG. 6. *Lower critical case.*

Furthermore, the Réthy flow satisfies the following asymptotic behavior in the upstream:

$$(u(x,y), v(x,y), p(x,y)) \to (u_0, 0, p_1) \ \ and \ \ \nabla(u,v,p) \to 0$$

uniformly in any compact subset of $(0,H)$ *as* $x \to -\infty$, *where* $u_0 = -\frac{m_0}{H}$ *and* $p_1 = p_0 + \frac{\lambda^2}{2} - \frac{m_0^2}{2H^2}$.

Similarly, the Réthy flow satisfies the following asymptotic behavior in the downstream:

$$(u(x,y), v(x,y), p(x,y)) \to (\lambda\cos\theta, \lambda\sin\theta, p_0) \ \ and \ \ \nabla(u,v,p) \to 0 \ \ as \ \ x \to +\infty.$$

Remark 1.10. We would like to point out that the interval $[h_2 - \frac{m_0}{2\lambda\cos\theta}, h_1 - a\tan\theta + \frac{m_0}{2\lambda\cos\theta}]$ is well defined. In fact, it is easy to see it for the case $h_2 \le h_1$. On the other hand, for $h_1 < h_2$, this fact can be also verified due to the estimate (1.12).

Remark 1.11. In Figure 3, $l : y = x\tan\theta + \beta$ is the midline of the Réthy flow at the far field and $\frac{m_0}{\lambda}$ is the asymptotic width of the Réthy flow. Denote $l_1 : y = x\tan\theta + \beta + \frac{m_0}{2\lambda\cos\theta}$ and $l_2 : y = x\tan\theta + \beta - \frac{m_0}{2\lambda\cos\theta}$ to be the asymptotic lines of the free boundaries Γ_1 and Γ_2 at the far fields, respectively. And the lower critical case $\beta = h_2 - \frac{m_0}{2\lambda\cos\theta}$ means that l_1 passes through the point A_2, and the upper critical case $\beta = h_1 - a\tan\theta + \frac{m_0}{2\lambda\cos\theta}$ means that l_2 passes through the point A_1; see Figures 6 and 7, respectively.

Before we give the proof of the main results in this work, we would like to present the main ingredients of the proof and review the recent progress on the problem

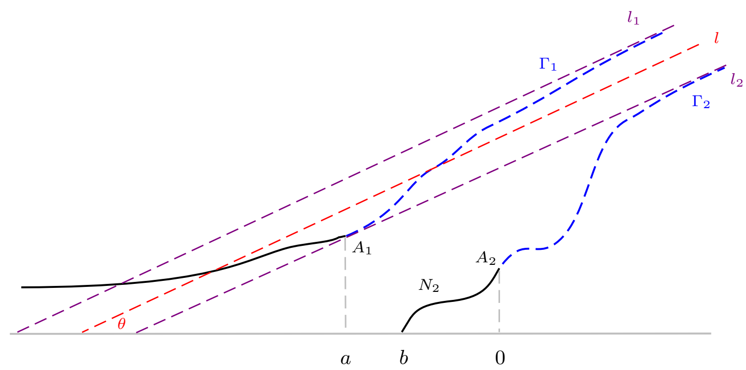

Fig. 7. *Upper critical case.*

with free streamlines as follows. Adapting the standard stream function formulation for two-dimensional incompressible ideal flows, we reduce the Réthy flow problem into a boundary value problem to Laplace equation with two parameters. However, the main obstacle to solving the boundary value problem is how to determine the free streamlines. To deal with the boundary value problem for the stream function in an unbounded domain, we approximate the possible fluid field Ω by a sequence of bounded domains Ω_L and introduce the variational method in the truncated domains. The variational method is effective in solving the elliptic problem with free boundaries and has been suggested by Garabedian to deal with the axially symmetric cavitational flows in [28]. Furthermore, it has been developed by Alt, Caffarelli, and Friedman in their elegant work in [1, 5] and applied to deal with jet flows in [2, 4], cavitational flows in [12, 27], jet flows with gravity in [3], and two phase fluids in [6, 7]. Combining it with the ideas and technique in [2, 4], we give the existence of the minimizers to the truncated variational problem. One of the key issues here is to establish the monotonicity of the stream function with respect to x and y directions, which implies the nonnegativity of the vertical and horizontal velocity. Another key point is how to choose the pair of parameters (λ, θ) to fulfill the continuous fit conditions (1.8). The continuous fit condition (1.8) can be obtained; it is based on the important properties of the free boundaries, such as the continuous dependence respect to the parameters λ and θ, some monotonicity of the stream function with respect to the deflection angle θ, and so on. As we mentioned before, the uniqueness and the asymptotic behaviors of the solutions to the Réthy flow problem and the parameters are not clear, even for the two-dimensional asymmetric jet flow in [2]. First, we establish the uniqueness of the Réthy flow for fixed parameters λ and θ. Second, we can determine the location of the Réthy flow in the downstream. The asymptotic behavior of the Réthy flow follows from the standard blow-up argument, which has been adapted to deal with the asymptotic behavior of compressible subsonic flows in infinitely long nozzles in [16, 20, 21, 22, 23, 36, 37].

There is another interesting problem of a flow passing through an infinitely long nozzle; the mathematical theory was suggested by Bers [10] in 1958. Recently, great progress has been made on the analysis of this problem. For the potential flow, the existence and uniqueness of multidimensional subsonic and subsonic-sonic flows through an infinitely long nozzle of arbitrary cross-sections have been established in [23, 35]; also see Chen et al. [15]. Xie and Xin [36, 37] first established the existence of global subsonic isentropic flows and obtained the critical upper bound of mass flux under the assumption that the derivative of the Bernoulli function equals zero on the two boundaries. Then the existence of subsonic-sonic full Euler flows in arbitrary infinitely long nozzles is obtained via the method of compensated compactness in [17] and [30]. Du, Xie, and Xin [22] then established the existence of global subsonic isentropic flows for large vorticity with the assumptions on the sign of the second derivatives of the horizontal velocity at the inlet, which has been extended to nonisentropic flows in [16, 24, 14]. For the existence and stability of vortex sheets, Bae [8] established the stability of a subsonic flat contact discontinuity in nozzles by the perturbation argument. Some further related results can be found in Bae and Feldman [9], Canic, Keyfitz, and Lieberman [13], Chen [18], Chen and Yuan [19], Yuan [38], and the references cited therein.

The remainder of the present paper is organized as follows. We first formulate the Réthy flow problem mathematically via the stream function approach into a boundary value problem with two undetermined parameters λ and θ in section 2. Due to the standard variational method, we establish the existence of the minimizer to the truncated variational problem. It is easy to check that the minimizer in fact solves the truncated boundary value problem in a weak sense. Moreover, some properties of the free boundaries to the minimizer will be obtained and we will choose a suitable pair of parameters (λ, θ) to satisfy the continuous fit conditions. The existence of the Réthy flow will be obtained in section 3. In section 4, we show the some results on uniqueness of the Réthy flow and the parameters. Then the estimate of the deflection angle θ is established in section 5. Finally, the location and the asymptotic behavior of the Réthy flow will be shown in section 6.

2. Mathematical setting of the Réthy flow problem.

2.1. Stream function approach. The first equation in (1.3) gives that there exists a stream function ψ such that

$$u = \psi_y \quad \text{and} \quad v = -\psi_x; \tag{2.1}$$

It is easy to check that

$$(u, v) \cdot \nabla \left(|\nabla\psi|^2 + p\right) = 0; \tag{2.2}$$

this implies that $|\nabla\psi|^2 + p$ remains a constant along each streamline. The equality (2.2) is the so-called Bernoulli law in steady incompressible flows. The irrotational condition (1.4) implies that the stream function satisfies the Laplace equation

$$\Delta\psi = 0. \tag{2.3}$$

Let I be the symmetric axis, $A_1 = (a, h_1)$ be the endpoint of the nozzle N_1, and $A_2 = (0, h_2)$ be the edge point of the obstacle N_2. Set $A = (a, 0)$, $B = (b, 0)$, and $O = (0, 0)$; Let $\overline{AA_1}$ be the half mouth of the nozzle and h_1 be the half width of the mouth. Denote the segments $I_0 = \{(x, 0) \mid x \geq 0\}$, $I_a = \{(a, y) \mid y \geq h_1\}$, $I_b = \{(x, 0) \mid x \leq b\}$, and $I_{h_2} = \{(x, h_2) \mid x \geq 0\}$. In this paper, we seek a planar symmetric Réthy flow with positive horizontal velocity and vertical velocity in the whole flow field; then the possible flow field is defined by Ω and bounded by N_1, N_2 I_a, I_b and I_{h_2} (see Figure 8).

Without loss of generality, we impose the Dirichlet boundary value conditions as follows:

$$\psi = m_0 \text{ on } N_1 \cup \Gamma_1 \text{ and } \psi = 0 \text{ on } I_b \cup N_2 \cup \Gamma_2.$$

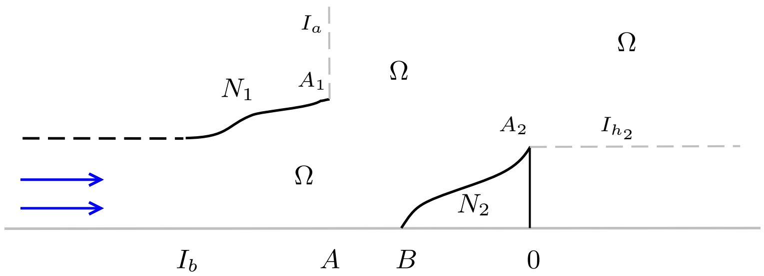

FIG. 8. *The possible fluid field* Ω.

Thus, the free boundaries of the plane symmetric Réthy flow can be defined by

$$\Gamma_1 = \Omega \cap \partial\{\psi < m_0\} \quad \text{and} \quad \Gamma_2 = \Omega \cap \partial\{\psi > 0\}. \tag{2.4}$$

On the free streamline Γ_1 and Γ_2, the pressure is assumed to be a constant p_0; then it follows from the Bernoulli law that the speed remains a constant on Γ_1 and Γ_2, namely,

$$|\nabla\psi| = \left|\frac{\partial\psi}{\partial\nu}\right| = \lambda \quad \text{on} \quad \Gamma_1 \cup \Gamma_2, \tag{2.5}$$

where ν is the unit outer normal of Γ_1 and Γ_2.

Hence, we formulate the following free boundary value problem for the stream function:

$$\begin{cases} \Delta\psi = 0 & \text{in} \ \ \Omega_0, \\ \dfrac{\partial\psi}{\partial\nu} = \lambda & \text{on} \ \ \Gamma_1, \ \dfrac{\partial\psi}{\partial\nu} = -\lambda \text{ on } \ \Gamma_2, \\ \psi = 0 & \text{on} \ \ I_b \cup N_2 \cup \Gamma_2, \quad \psi = m_0 \quad \text{on} \ \ N_1 \cup \Gamma_1. \end{cases} \tag{2.6}$$

Once the stream function is solved, the velocity field (u, v) can be obtained via (2.1) and the free boundaries Γ_1 and Γ_2 can be obtained by the definition (2.4). Furthermore, the cavity number λ and the deflection angle θ are determined by (2.5) and Γ_1 and Γ_2, respectively.

2.2. The truncated variational problem ($P_{\lambda,\theta,L}$). To solve the free boundary value problem (2.6), we introduce the following variational problem with two parameters $\lambda > 0$ and θ.

Define two domains D_1 and D_2 (see Figure 9) as

$$D_1 = \left\{ (x, y) \in \mathbb{R}^2 \ \middle|\ 0 < y < f_1(x), x < a; \tilde{f}_2(x) < y < h_1, a \leq x \leq +\infty \right\}$$

and

$$D_2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \ \middle|\ 0 < y < f_1(x), x \leq a \text{ and, } \ y > \tilde{f}_2(x), \ a \leq x < 0 \right\},$$

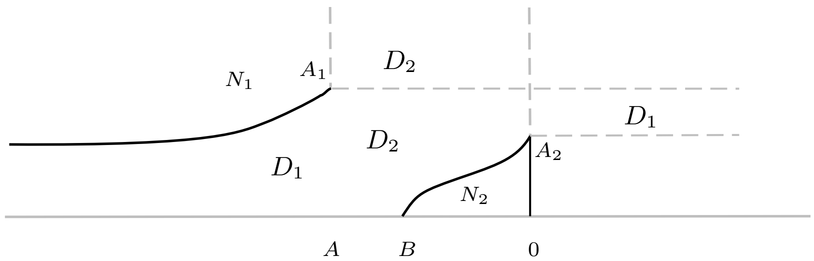

FIG. 9. *The domains D_1 and D_2.*

where $\tilde{f}_2(x)$ is defined as follows:

$$\tilde{f}_2(x) = \begin{cases} 0 & \text{if} \quad x \leq b, \\ f_2(x) & \text{if} \quad b \leq x < 0, \\ h_2 & \text{if} \ \ x \geq 0. \end{cases} \tag{2.7}$$

First, we introduce the admissible set as

$$K = \left\{ \psi \in H^1_{loc}(\Omega) | \ \phi_2 \leq \psi \leq \phi_1 \right\},$$

where the bounded functions ϕ_1 and ϕ_2 are defined as

$$\Delta\phi_1 = 0 \text{ in } \ D_1 \ \text{ and } \ 0 < \phi_1 < m_0 \text{ in } \ D_1,$$

$$\phi_1(x,y) = \begin{cases} 0 & \text{if} \quad (x,y) \text{ lies below } I_b \cup N_2 \cup I_{h_2}, \\ m_0 & \text{if} \quad (x,y) \text{ lies above } N_1, \\ m_0 & \text{if} \ \ (x,y) \in \Omega \cap \{y \geq h_1\} \end{cases} \tag{2.8}$$

and

$$\Delta\phi_2 = 0 \ \text{ in } \ D_2 \ \text{ and } \ 0 < \phi_2 < m_0 \text{ in } \ D_2,$$

$$\phi_2(x,y) = \begin{cases} 0 & \text{if} \quad (x,y) \text{ lies below } I_b \cup N_2 \cup I_{h_2}, \\ m_0 & \text{if} \quad (x,y) \text{ lies above } N_1, \\ 0 & \text{if} \ \ (x,y) \in \Omega \cap \{(x,y) \mid x \geq 0, y \geq h_1\}. \end{cases} \tag{2.9}$$

Introduce a functional with two parameters λ and θ as follows:

$$J_{\lambda,\theta}(\psi) = \int_{\Omega} \left| \nabla\psi - \lambda e \chi_{\{0<\psi<m_0\}} \right|^2 dxdy, \tag{2.10}$$

where $e = (-\sin\theta, \cos\theta)$ and χ_E is the characteristic function of a set E.

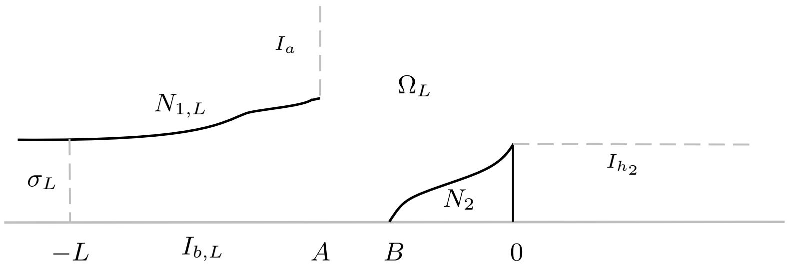

FIG. 10. *The truncated domain* Ω_L.

However, note that the functional $J_{\lambda,\theta}(\psi) = +\infty$ for any $\psi \in K$ so that we cannot work with the functional $J_{\lambda,\theta}(\psi)$ directly. Therefore, we consider the following truncated variational problem $(P_{\lambda,\theta,L})$ in a truncated domain. For any large number $L > -a > 0$, denote

$$N_{1,L} = N_1 \cap \{x \geq -L\}, \sigma_L = \{(-L, y) \mid 0 \leq y \leq f_1(-L)\}, I_{b,L} = I_b \cap \{x \geq -L\};$$

the truncated domain Ω_L is bounded by $N_{1,L}, N_2, \sigma_L, I_a, I_{h_2}$, and $I_{b,L}$ (see Figure 10), and let the functional $J_{\lambda,\theta,L}(\psi) = \int_{\Omega_L} |\nabla\psi - \lambda e\chi_{\{0<\psi<m_0\}}|^2 dxdy$.

Next, we define a variational problem in the truncated domain Ω_L.

The truncated variational problem $(P_{\lambda,\theta,L})$. Find a $\psi_{\lambda,\theta,L} \in K_L$ such that

$$J_{\lambda,\theta,L}(\psi_{\lambda,\theta,L}) = \min_{\psi \in K_L} J_{\lambda,\theta,L}(\psi),$$

where the admissible set $K_L = \{\psi \in K \mid \psi = \Psi_L(y) \text{ on } \sigma_L\}$ and $\Psi_L(y) = \frac{m_0}{f_1(-L)} y$.

By section 7.1, we have the following proposition.

PROPOSITION 2.1. *There exists a* $\psi_{\lambda,\theta,L} \in K_L$ *such that*

$$J_{\lambda,\theta,L}(\psi_{\lambda,\theta,L}) = \min_{\psi \in K_L} J_{\lambda,\theta,L}(\psi). \tag{2.11}$$

The minimizer $\psi_{\lambda,\theta,L}$ *satisfies that*

$$\Delta\psi_{\lambda,\theta,L} = 0 \quad \text{in} \quad \Omega_L \cap \{0 < \psi_{\lambda,\theta,L} < m_0\}, \tag{2.12}$$

and

$$\lim_{\varepsilon \to 0^+} \int_{G \cap \partial\{\varepsilon < \psi_{\lambda,\theta,L} < m_0 - \varepsilon\}} \left(|\nabla\psi_{\lambda,\theta,L}|^2 - \lambda^2\right) \zeta \cdot \nu dS = 0 \tag{2.13}$$

for any two-dimensional vector $\zeta \in C_0^1(G)$ *and any* $G \Subset \Omega_L$, *where* ν *is the normal vector.*

Moreover, $\psi_{\lambda,\theta,L}$ *is Lipschitz continuous in* G, *and* $\psi_{\lambda,\theta,L} \in C^{2,\alpha}(G_1)$ *in any compact subset* G_1 *of* $\Omega_L \cap \{0 < \psi_{\lambda,\theta,L} < m_0\}$.

The proof of Proposition 2.1 is long but standard, since its argument is similar to the one in Alt, Caffarelli, and Friedman [2, 4]. So we postpone the proof to the appendix, i.e., Lemmas 7.1, 7.3, and 7.5 and Propositions 7.2 and 7.4 for the self-contained and also for the reason that some of the arguments will be mimicked in the proof of several lemmas later.

Next, the monotonicity and uniqueness of the minimizer $\psi_{\lambda,\theta,L}$ to the truncated variational problem $(P_{\lambda,\theta,L})$ will be obtained in the following.

LEMMA 2.2. *The minimizer $\psi_{\lambda,\theta,L}$ to the truncated variational problem $(P_{\lambda,\theta,L})$ satisfies*

$$\psi_{\lambda,\theta,L}(x,y)\leq\Psi_L(y)\quad in\quad \Omega_L, \tag{2.14}$$

where $\Psi_L(y)=\frac{m_0}{f_1(-L)}y$.

Proof. Denote $\psi(x,y)=\psi_{\lambda,\theta,L}(x,y)$ for simplicity; it suffices to verify that

$$\psi(x,y)\leq\Psi_L(y)\ \text{ in }\quad G_1=\Omega_{L,R}\cap\{y\leq f_1(-L)\}. \tag{2.15}$$

It is easy to check that $\psi\leq\phi_1<m_0$ in G_1, where ψ_1 is defined in (2.8); it follows from similar arguments in Proposition 7.4 in the appendix; we can conclude that $\Delta\psi\geq 0$ in G_1 in the weak sense. Next, we consider two cases as follows.

Case 1. $f_1(-L)\leq h_2$. It is obvious that $\psi(x,y)\leq\Psi_L(y)$ on ∂G_1; then the inequality (2.15) can be obtained by using the maximum principle.

Case 2. $f_1(-L)>h_2$; then the domain G_1 is an unbounded domain. We need Phragmín–Lindeöf-type lemma (see Theorem 5.7 in [4]) to obtain (2.15), set $\omega(x,y)=\psi_{\lambda,\theta,L}(x,y)-\Psi_L(y)-\varepsilon(x+L)$, and extend $\psi_{\lambda,\theta,L}=0$ in $\{(x,y)\mid x\geq 0, 0\leq y\leq h_2\}$. Consider the function ω in the domain $D_\varepsilon=\{(x,y)\mid -L<x<N_\varepsilon-L, 0<y<f_1(-L)\}$, where $N_\varepsilon>\frac{m_0}{\varepsilon}$. It is easy to check that $\Delta\omega\geq 0$ in D_ε and $\omega\leq 0$ on ∂D_ε; by the maximum principle, one has $\omega\leq 0$ in D_ε, so we obtain that

$$\psi_{\lambda,\theta,L}(x,y)\leq\Psi_L(y)+\varepsilon(x+L)\quad\text{in}\quad D_\varepsilon. \tag{2.16}$$

For any fixed $(x_0,y_0)\in\Omega_L$, it follows from (2.16) that

$$\psi_{\lambda,\theta,L}(x_0,y_0)\leq\Psi_L(y_0)+\varepsilon(x_0+L)\quad\text{in}\quad D_\varepsilon, \tag{2.17}$$

for small $\varepsilon>0$. Taking $\varepsilon\to 0$ in (2.17), one has $\psi_{\lambda,\theta,L}(x_0,y_0)\leq\Psi_L(y_0)$. ∎

THEOREM 2.3. *The minimizer $\psi_{\lambda,\theta,L}$ to the truncated variational problem $(P_{\lambda,\theta,L})$ is decreasing with respect to x and increasing with respect to y. Furthermore, the minimizer $\psi_{\lambda,\theta,L}$ is unique for any fixed λ, θ, and L.*

Proof. Suppose that $\psi(x,y)=\psi_{\lambda,\theta,L}(x,y)$ and $\tilde\psi(x,y)=\tilde\psi_{\lambda,\theta,L}(x,y)$ are two minimizers to the truncated variational problem $(P_{\lambda,\theta,L})$. Define $\tilde\psi_\varepsilon(x,y)=\tilde\psi(x,y-\varepsilon)$, the corresponding functional $J^\varepsilon_{\lambda,\theta,L}$ with admissible set K^ε_L in truncated domain $\Omega^\varepsilon_L=\{(x,y)\mid(x,y-\varepsilon)\in\Omega_L\}$. Extend $\tilde\psi_\varepsilon(x,y)=0$ in $\Omega_L\setminus\Omega^\varepsilon_L$ and $\psi(x,y)=m_0$ in $\Omega^\varepsilon_L\setminus\Omega_L$, and set

$$\psi_1=\max\{\psi,\tilde\psi_\varepsilon\}\quad\text{and}\quad\psi_2=\min\{\psi,\tilde\psi_\varepsilon\};$$

it is clear that $\psi_1\in K_L$ and $\psi_2\in K^\varepsilon_L$. We claim that

$$J_{\lambda,\theta,L}(\psi_1)+J^\varepsilon_{\lambda,\theta,L}(\psi_2)=J_{\lambda,\theta,L}(\psi)+J^\varepsilon_{\lambda,\theta,L}(\tilde\psi_\varepsilon). \tag{2.18}$$

In fact, for any $R>0$, denote $\Omega_{L,R}=\Omega_L\cap\{(x,y)\mid x<R, y<R\}$ and $\Omega^\varepsilon_{L,R}=\Omega^\varepsilon_L\cap\{(x,y)\mid x<R,y<R\}$. Set $G=\Omega_{L,R}\cap\Omega^\varepsilon_{L,R}$, $G_1=\Omega_{L,R}\setminus\Omega^\varepsilon_{L,R}$, and $G_2=\Omega^\varepsilon_{L,R}\setminus\Omega_{L,R}$; one has

(2.19)

$$\begin{aligned}
&\int_{\Omega_{L,R}}\left|\nabla\psi_1-\lambda e\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy-\int_{\Omega_{L,R}}\left|\nabla\psi-\lambda e\chi_{\{0<\psi<m_0\}}\right|^2dxdy\\
&\quad=\int_{G\cup G_1}\left|\nabla\psi_1-\lambda e\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy-\int_{G\cup G_1}\left|\nabla\psi-\lambda e\chi_{\{0<\psi<m_0\}}\right|^2dxdy\\
&\quad=\int_{G\cap\{\psi<\tilde\psi_\varepsilon\}}\left|\nabla\tilde\psi_\varepsilon-\lambda e\chi_{\{0<\tilde\psi_\varepsilon<m_0\}}\right|^2-\left|\nabla\psi-\lambda e\chi_{\{0<\psi<m_0\}}\right|^2dxdy,
\end{aligned}$$

and

(2.20)
$$\begin{aligned}&\int_{\Omega^\varepsilon_{L,R}}\left|\nabla\psi_2-\lambda e\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy-\int_{\Omega^\varepsilon_{L,R}}\left|\nabla\tilde\psi_\varepsilon-\lambda e\chi_{\{0<\tilde\psi_\varepsilon<m_0\}}\right|^2dxdy\\&\quad=\int_{G\cup G_2}\left|\nabla\psi_2-\lambda e\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy-\int_{G\cup G_2}\left|\nabla\tilde\psi_\varepsilon-\lambda e\chi_{\{0<\tilde\psi_\varepsilon<m_0\}}\right|^2dxdy\\&\quad=-\int_{G\cap\{\psi<\tilde\psi_\varepsilon\}}\left|\nabla\tilde\psi_\varepsilon-\lambda e\chi_{\{0<\tilde\psi_\varepsilon<m_0\}}\right|^2-\left|\nabla\psi-\lambda e\chi_{\{0<\psi<m_0\}}\right|^2dxdy,\end{aligned}$$

where we have used the following facts:

$$\psi_1=\max\{\psi,0\}=\psi\ \text{ in }\ G_1,\quad\text{and}\quad\psi_2=\min\{m_0,\tilde\psi_\varepsilon\}=\tilde\psi_\varepsilon\ \text{ in }\ G_2.$$

Thus, (2.19) and (2.20) imply that

(2.21)
$$\begin{aligned}&\int_{\Omega_{L,R}}\left|\nabla\psi_1-\lambda e\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy+\int_{\Omega^\varepsilon_{L,R}}\left|\nabla\psi_2-\lambda e\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy\\&\quad=\int_{\Omega_{L,R}}\left|\nabla\psi-\lambda e\chi_{\{0<\psi<m_0\}}\right|^2dxdy+\int_{\Omega^\varepsilon_{L,R}}\left|\nabla\tilde\psi_\varepsilon-\lambda e\chi_{\{0<\tilde\psi_\varepsilon<m_0\}}\right|^2dxdy.\end{aligned}$$

Taking $R\to+\infty$ in (2.21) gives that (2.18) is valid.

However, one has

$$J_{\lambda,\theta,L}(\psi)\le J_{\lambda,\theta,L}(\psi_1)\ \text{ and }\ J^\varepsilon_{\lambda,\theta,L}(\tilde\psi_\varepsilon)\le J^\varepsilon_{\lambda,\theta,L}(\psi_2);\tag{2.22}$$

it follows from (2.18) and (2.22) that

$$J_{\lambda,\theta,L}(\psi)=J_{\lambda,\theta,L}(\psi_1)\ \text{ and }\ J^\varepsilon_{\lambda,\theta,L}(\tilde\psi_\varepsilon)=J^\varepsilon_{\lambda,\theta,L}(\psi_2).\tag{2.23}$$

Next, we claim that for any connected subset D of $\{0<\psi<m_0\}\cap\Omega_L$, if D touches σ_L, one has

$$\psi>\tilde\psi_\varepsilon\text{ in }D.\tag{2.24}$$

Suppose that the assertion (2.24) is not true. Note that $\psi_\varepsilon<\psi$ in an Ω_L-neighborhood of σ_L; there exists a point $X^0=(x_0,y_0)\in\Omega_L$ such that $0<\psi(X^0)=\tilde\psi_\varepsilon(X^0)<m_0$. The continuity of ψ and ψ_ε gives that there exists a disc $B_r(X^0)\subset\Omega_L$ such that $0<\psi<m_0$ and $0<\tilde\psi_\varepsilon<m_0$ in $B_r(X^0)$. Then, ψ_1 and ψ_2 solve the Dirichlet problem

$$\begin{cases}\Delta\psi_1=0,\quad\Delta\psi_2=0&\text{in }\ B_r(X^0),\\ \psi_2\le\psi_1&\text{on }\partial B_r(X^0).\end{cases}\tag{2.25}$$

In view of $\psi_1(X_0)=\psi_2(X_0)$, the strong maximum principle gives that $0<\psi_1\equiv\psi_2<m_0$ in $B_r(X^0)$. Furthermore, we can conclude that $0<\psi\equiv\psi_1\equiv\psi_2\equiv\psi_\varepsilon<m_0$ in D, which contradicts the fact that $\psi_\varepsilon<\psi$ on σ_L.

Now, we will show the monotonicity of ψ with respect to the variable y. Set $\psi=\tilde\psi$; it follows from the claim (2.24) that

$$\psi(x,y-\varepsilon)<\psi(x,y)\text{ in }D,\tag{2.26}$$

where D is a connected component of $\{0<\psi<m_0\}\cap\Omega_L$ and D touches σ_L. By the maximum principle and (2.26), we conclude that each component of $\{0<\psi<m_0\}$ must touch σ_L. Consequently,

$$\{0<\psi<m_0\} \text{ is a connected set, and } \psi(x,y-\varepsilon)\le\psi(x,y) \text{ in } \Omega_L.$$

Next, we will show that ψ is monotonic decreasing with respect to x. Define $\psi_\tau(x,y)=\psi(x+\tau,y)$ and the corresponding functional $J^\tau_{\lambda,\theta,L}$ with admissible set K^τ_L in truncated domain $\Omega^\tau_L=\{(x,y)\mid(x+\tau,y)\in\Omega_L\}$. Extend $\psi_\tau(x,y)=0$ in $\Omega_L\setminus\Omega^\tau_L$ and

$$\psi(x,y)=\begin{cases} m_0 & \text{for } (\Omega^\tau_L\setminus\Omega_L)\cap\{y\ge f_1(-L)\},\\ \Psi_L(y) & \text{for } 0\le y\le f_1(-L),\ -L-\tau\le x\le -L.\end{cases}\tag{2.27}$$

It follows from Lemma 2.2 that

$$\psi_\tau(x,y)\le\psi(x,y) \text{ in } \Omega_L\setminus\Omega^\tau_L, \text{ and } \psi_\tau(x,y)\le\psi(x,y) \text{ in } \Omega^\tau_L\setminus\Omega_L.\tag{2.28}$$

Set

$$\psi_1=\max\{\psi,\psi_\tau\} \quad\text{and}\quad \psi_2=\min\{\psi,\psi_\tau\};$$

it is clear that $\psi_1\in K_L$ and $\psi_2\in K^\tau_L$. Hence, we have $\psi(x+\tau,y)\le\psi(x,y)$ in Ω_L for any $\tau>0$ by using the previous arguments.

Finally, we will show the uniqueness of the minimizer to the truncated variational problem $(P_{\lambda,\theta,L})$. Since the set $\{0<\psi<m_0\}$ is connected, taking $\varepsilon\to0$ in (2.24) yields that

$$\tilde\psi(x,y)\le\psi(x,y) \quad \text{in } \Omega_L\cap\{0<\psi<m_0\}.\tag{2.29}$$

Similarly, one can show that $\tilde\psi\ge\psi$ in $\Omega_L\cap\{0<\tilde\psi<m_0\}$.

Hence, the uniqueness of the minimizer to the truncated variational problem $(P_{\lambda,\theta,L})$ is proved. ∎

2.3. Regularity of the free boundaries of the minimizer $\psi_{\lambda,\theta,L}$. In view of Theorem 8.4 in [1] and that the free boundaries are analytic curves in Ω_L, we will show that the free boundaries are y-graphs and every such graph can be described as a continuous function. Some preliminary lemmas established in the appendix are crucial to dealing with the properties of the free boundary.

Thanks to the monotonicity of $\psi_{\lambda,\theta,L}$ with respect to y, there exist two functions $k_{i,\lambda,\theta,L}(x)$ such that $\Gamma_i=\{(x,y)\in\Omega_L\mid y=k_{i,\lambda,\theta,L}(x)\}$, and we denote $k_i(x)=k_{1,\lambda,\theta,L}(x)$ $(i=1,2)$ in this subsection. The flow field can be denoted by

$$\begin{aligned}&\{0<\psi_{\lambda,\theta,L}<m_0\}\cap\{x>a\}\\ &\quad=\{(x,y)\mid k_2(x)<y<k_1(x) \text{ for } x>0, \tilde f_2(x)<y<k_1(x) \text{ for } a<x\le0\}.\end{aligned}\tag{2.30}$$

In fact, we will show that $k_1(x)$ is a continuous function in $(a,+\infty)$ and $k_2(x)$ is a continuous function in $(0,+\infty)$ for any λ,θ and L.

LEMMA 2.4. *The free boundary $k_1(x)$ is continuous in $(a,+\infty)$ and the free boundary $k_2(x)$ is continuous in $(0,+\infty)$.*

Proof. We only consider the continuity of the free boundary $y = k_1(x)$ in the following, since the continuity of the free boundary $y = k_2(x)$ can be obtained by similar arguments. Denote $\psi(x, y) = \psi_{\lambda,\theta,L}(x, y)$ for simplicity.

First, we will show that the one side limit of $k_1(x)$ exists as $x \to x_0^-$ or $x \to x_0^+$ for any $x_0 \in [a, +\infty)$.

For any fixed $x_0 \in [a, +\infty)$, and suppose that there are two limits α_1 and α_2 with $\alpha_2 < \alpha_1$ as $x \to x_0^+$, the free boundary $y = k_1(x)$ is oscillated as $x \to x_0^+$.

Due to the definition of the free boundary and the monotonicity $\psi(x, y)$ with respect to y, we can find two sequences $\{x_n\}_{n=1}^\infty$ and $\{\tilde{x}_n\}_{n=1}^\infty$ with $x_n < \tilde{x}_n < x_{n+1}$, such that $x_n \to x_0^+$, $\tilde{x}_n \to x_0^+$ and

$$\psi(x_n, y) = m_0 \quad \text{and} \quad \psi(\tilde{x}_n, y) < m_0, \tag{2.31}$$

for $\delta_1 < y < \delta_2$, where $\delta_1 = \frac{\alpha_1+3\alpha_2}{4}$ and $\delta_2 = \frac{3\alpha_1+\alpha_2}{4}$.

Denote $D_n \subset \Omega_L \cap \{\psi < m_0\}$ bounded by the arcs $y = \delta_1, y = \delta_2, \tilde{x} = \tilde{\gamma}_n(y)$, and $x = \gamma_n(y)$, where $(\gamma_n(y), y)$ and $(\tilde{\gamma}_n(y), y)$ are free boundary points of Γ_1 and $\gamma_n(y) < \tilde{\gamma}_n(y)$ with

$$\varepsilon_n = \sup_{\delta_1<y<\delta_2} \{\tilde{\gamma}_n(y) - \gamma_n(y)\} \to 0 \quad \text{as} \quad n \to \infty.$$

The existence of domains D_n follows from (2.31). The Lipschitz continuity of ψ gives that

$$\psi > 0 \quad \text{in } D_n \text{ for sufficiently large } n.$$

Therefore, by the nonoscillation lemma, Lemma 7.11 in D_n for sufficiently large n, one has

$$\frac{\alpha_1 - \alpha_2}{2} = \delta_2 - \delta_1 \le C\varepsilon_n \quad \text{for sufficiently large } n,$$

which leads to a contradiction.

Thus, we obtain that $\lim_{x\to x_0^+} k_1(x)$ exists for any $x_0 \in (a, +\infty)$. Similar arguments yield the existence of $\lim_{x\to a^+} k_1(x)$ denoted by $k_1(a)$, and the existence of $\lim_{x\to x_0^-} k_1(x)$ for any $x_0 \in (a, +\infty)$.

Next, we will show that $k_1(x)$ is a continuous function in $(a, +\infty)$. Denote

$$k_1(x_0 + 0) = \lim_{y\to x_0^+} k_1(x) \quad \text{and } k_1(x_0 - 0) = \lim_{x\to x_0^-} k_1(x),$$

and it suffices to show that

$$k_1(x_0 + 0) = k_1(x_0 - 0) = k_1(x_0) < +\infty \quad \text{for any} \quad x_0 \in (a, +\infty).$$

Suppose on the contrary that there exists a point $x_0 \in (a, +\infty)$ such that $k_1(x_0 + 0) \ne k_1(x_0)$, and without loss of generality we assume $k_1(x_0 + 0) > k_1(x_0)$. By virtue of the monotonicity of ψ with respect to y, one has $\sigma_0 = \{(x_0, y) \mid y_1 < y < y_2\} \subset \Gamma_1$ with $y_1 = \frac{3k_1(x_0+0)+k_1(x_0)}{4}$ and $y_2 = \frac{k_1(x_0+0)+3k_1(x_0)}{4}$; then

$$\frac{\partial\psi(x_0 + 0, y)}{\partial x} = -\lambda \quad \text{and} \quad \psi = m_0 \quad \text{on} \quad \sigma_0.$$

The continuity of ψ implies that there exists a small ε such that

$$0 < \psi < m_0 \quad \text{in } E_\varepsilon,$$

where $E_\varepsilon = \{(x,y) \mid x_0 < x < x_0+\varepsilon, y_1 < y < y_2\}$.

It follows from the Cauchy–Kovalevskaya theorem and unique continuation that

$$\psi(x,y) = -\lambda(x-x_0) + m_0 \text{ in } \tilde{E}_\varepsilon, \tag{2.32}$$

where $\tilde{E}_\varepsilon = \{x_0 < x < x_0+\varepsilon, -\infty < y < +\infty\} \cap \Omega_L$, which leads to a contradiction to the boundary condition. ∎

2.4. Uniform flows in the downstream. In this subsection, we define a translated solution and show that it converges to a constant flow with speed λ and direction $e^\perp = (\cos\theta, \sin\theta)$ in the downstream, such that we can show that the deflection angle of the Réthy flows satisfies the estimate (1.10) later.

First, we will obtain that the minimizer $\psi_{\lambda,\theta,L}$ converges to a constant flow with speed λ and direction $e^\perp = (\cos\theta, \sin\theta)$ with $\theta \in [0, \frac{\pi}{2}]$ in downstream. Taking $X_n = (x_n, y_n) \in \Gamma_2$ such that $|X_n| \to +\infty$ as $n \to +\infty$, the constant flow $S_\theta(x,y;x_0,y_0)$ with speed λ and direction $e^\perp = (\cos\theta, \sin\theta)$ is defined as

$$S_\theta(x,y;x_0,y_0) = \begin{cases} m_0 & \text{for } y\cos\theta - x\sin\theta + h_0 \geq \dfrac{m_0}{\lambda}, \\ \lambda(y\cos\theta - x\sin\theta + h_0) & \text{for } 0 \leq y\cos\theta - x\sin\theta + h_0 \leq \dfrac{m_0}{\lambda}, \\ 0 & \text{for } y\cos\theta - x\sin\theta + h_0 \leq 0, \end{cases} \tag{2.33}$$

where $h_0 = -y_0\cos\theta + x_0\sin\theta$.

Then we have the following lemma.

LEMMA 2.5. *Let $X_n = (x_n, y_n) \in \Gamma_2$ with $|X_n| \to +\infty$; then there exists a subsequence $\{\psi_n\}$ with $\psi_n(\tilde{x},\tilde{y}) = \psi_{\lambda,\theta,L}(x_n+\tilde{x}, y_n+\tilde{y})$ such that*

$$\psi_n(\tilde{x},\tilde{y}) \to S_\theta(\tilde{x},\tilde{y};0,0) \quad \textit{uniformly in any compact subset of } \mathbb{R}^2.$$

Moreover, the free boundaries of ψ_n converge to the free boundaries of S_θ in C^1. The same conclusion holds for $(x_n, y_n) \in \Gamma_1$.

Proof. For any $R > 0$, set

$$\begin{aligned} J_n &= \int_{\{|\tilde{X}|<R\}} \left|\nabla\psi_n - \lambda e \chi_{\{0<\psi_n<m_0\}}\right|^2 d\tilde{x}d\tilde{y} \\ &= \int_{\{|X_n|-R<|X|<|X_n|+R\}} \left|\nabla\psi_{\lambda,\theta,L} - \lambda e \chi_{\{0<\psi_{\lambda,\theta,L}<m_0\}}\right|^2 dxdy \end{aligned} \tag{2.34}$$

for sufficiently large $|X_n| > R$.

In view of Lemma 7.1 in the appendix, one has

$$\int_{\Omega_L} \left|\nabla\psi_{\lambda,\theta,L} - \lambda e \chi_{\{0<\psi_{\lambda,\theta,L}<m_0\}}\right|^2 dxdy \leq C,$$

which gives that

$$J_n \to 0 \quad \text{as } n \to +\infty, \quad \text{for any } R > 0. \tag{2.35}$$

Without loss of generality, we may assume that $\psi_n \rightharpoonup S$ in $H^1_{loc}(\mathbb{R}^2)$; it follows from the similar arguments in Proposition 7.2 in the appendix that

$$\int_{\{|\tilde{X}|<R\}} \left|\nabla S - \lambda e\chi_{\{0<S<m_0\}}\right|^2 d\tilde{x}d\tilde{y} \leq \liminf_{n\to+\infty} J_n = 0;$$

this implies that

$$\nabla S = \lambda e\chi_{\{0<S<m_0\}} \text{ a.e. in } \mathbb{R}^2. \tag{2.36}$$

Set $s = \tilde{y}\cos\theta - \tilde{x}\cos\theta$ and $t = \tilde{x}\cos\theta + \tilde{y}\sin\theta$, and define $\varphi(s,t) = S(\tilde{x},\tilde{y})$. Direct calculation yields that

$$\frac{\partial\varphi}{\partial s} = \lambda\chi_{\{0<S<m_0\}} \text{ and } \frac{\partial\varphi}{\partial t} = 0 \text{ a.e. in } \mathbb{R}^2; \tag{2.37}$$

thus $\varphi(s,t)$ depends only on the variation s and is increasing with respect to s. Next, we claim that

$$\varphi(s,t) \neq 0 \text{ in any neighborhood of } 0. \tag{2.38}$$

In fact, it follows from the bounded gradient lemma for $\psi_{\lambda,\theta,L}$ near Γ_2 that

$$|\nabla\psi_{\lambda,\theta,L}(\tilde{X}+X_n)| \leq \tilde{C} \quad \text{for } |\tilde{X}| \leq R \text{ and sufficiently large } n, \tag{2.39}$$

where the constant C is independent of R and n. By virtue of the compact embedding result $C^1 \hookrightarrow C^\alpha$ for $0<\alpha<1$, we can conclude that

$$\psi_n \to S \text{ uniformly in any compact subset of } \mathbb{R}^2, \tag{2.40}$$

and

$$\psi_{\lambda,\theta,L}(\tilde{X}+X_n) = \psi_{\lambda,\theta,L}(\tilde{X}+X_n) - \psi_{\lambda,\theta,L}(X_n) \leq |\nabla\psi_{\lambda,\theta,L}||\tilde{X}| \leq \tfrac{m_0}{2} \tag{2.41}$$

for $|\tilde{X}| \leq \frac{m_0}{2\tilde{C}}$.

In view of $\psi_{\lambda,\theta,L}(X_n) = 0$, it follows from the nondegeneracy lemma, Lemma 7.9, that

$$\frac{1}{r}\fint_{\partial B_r(0)} \psi_n dS = \frac{1}{r}\fint_{\partial B_r(X_n)} \psi_{\lambda,\theta,L} dS \geq \lambda C^* \quad \text{for} \quad r < \frac{m_0}{2\tilde{C}}.$$

The claim (2.38) can be obtained by taking $n \to +\infty$. It is easy to check that

$$\varphi(0,0) = 0 \quad \text{and} \quad 0 \leq \varphi \leq m_0. \tag{2.42}$$

It follows from (2.37) and (2.42) that

$$\varphi(s,t) = \begin{cases} m_0 & \text{for } s \geq \dfrac{m_0}{\lambda}, \\ \lambda s & \text{for } 0 \leq s \leq \dfrac{m_0}{\lambda}, \\ 0 & \text{for } s \leq 0, \end{cases} \tag{2.43}$$

which yields the first statement of the lemma, namely, $S(\tilde{x},\tilde{y}) = S_\theta(\tilde{x},\tilde{y};0,0)$.

Next, we will show the convergence of the free boundaries of ψ_n. For any $\tilde{X}_0 = (\tilde{x}_0, \tilde{y}_0)$ with $\tilde{y}_0 \cos\theta - \tilde{x}_0 \sin\theta > \frac{m_0}{\lambda}$, there exists a small $r > 0$ such that $S_\theta(\tilde{x}, \tilde{y}; 0, 0) = m_0$ in $B_r(\tilde{X}_0)$ and

$$\lim_{n\to+\infty} \frac{1}{r} \fint_{\partial B_r(\tilde{X}_0)} (m_0 - \psi_n) dS = 0, \quad \text{due to (2.40)}.$$

Furthermore, it follows from the nondegeneracy lemma, Lemma 7.8, that $\tilde{X}_0$ is a free boundary point for ψ_n for sufficiently large n.

Similarly, for any $\tilde{X}_0 = (\tilde{x}_0, \tilde{y}_0)$ with $\frac{m_0}{2\lambda} < \tilde{y}_0 \cos\theta - \tilde{x}_0 \sin\theta < \frac{m_0}{\lambda}$, there exists a small $r > 0$ such that $0 < S_\theta(\tilde{x}, \tilde{y}; 0, 0) < m_0$ in $B_r(\tilde{X}_0)$. It follows from (2.40) that

$$\lim_{n\to+\infty} \frac{1}{r} \fint_{\partial B_r(\tilde{X}_0)} (m_0 - \psi_n) dS \to +\infty \quad \text{as} \quad r \to 0.$$

Thus, Lemma 7.6 implies that $\tilde{X}_0$ is not in $\partial\{\psi_n < m_0\}$ for sufficiently large n.

Hence, we conclude that

$$\partial\{\psi_n < m_0\} \text{ converges to } \partial\{S_\theta < m_0\} \text{ locally in the Hausdorff metric} \tag{2.44}$$

as $n \to +\infty$.

Similarly, we can conclude that

$$\partial\{\psi_n > 0\} \text{ converges to } \partial\{S_\theta > 0\} \text{ locally in the Hausdorff metric} \tag{2.45}$$

as $n \to +\infty$.

For $\theta \in (0, \frac{\pi}{2})$, it follows from (2.44) and (2.45) that the free boundaries of ψ_n satisfy the flatness condition in section 7 in [1]. The flatness of the free boundaries of ψ_n implies that there exists a subsequence still labeled by $\{\psi_n\}$ such that the free boundaries of ψ_n converge to the boundaries of S_θ in $C^1(B_R(0))$ for any $R > 0$. Consequently,

$$k'_{1,\lambda,\theta,L}(\tilde{x} + x_n) \to \tan\theta \text{ and } k'_{2,\lambda,\theta,L}(x_n) \to \tan\theta \text{ for } |\tilde{x}| < R \text{ as } n \to +\infty. \tag{2.46}$$

Similarly, one has

$$k'_{1,\lambda,\theta,L}(\tilde{x} + x_n) \to +\infty \text{ and } k'_{2,\lambda,\theta,L}(x_n) \to +\infty \text{ for } |\tilde{x}| < R \text{ as } n \to +\infty, \tag{2.47}$$

for $\theta = \frac{\pi}{2}$, and

$$k'_{1,\lambda,\theta,L}(\tilde{x} + x_n) \to 0 \text{ and } k'_{2,\lambda,\theta,L}(x_n) \to 0 \text{ for } |\tilde{x}| < R \text{ as } n \to +\infty, \tag{2.48}$$

for $\theta = 0$. □

The following two lemmas tell us that the value of θ must be critical, namely, $\theta = \frac{\pi}{2}$ or $\theta = 0$, if Γ_1 and Γ_2 satisfy the following special properties.

LEMMA 2.6. *Let $(x_n, y_n) \in \Gamma_2$ with $x_n \to x_0 < +\infty$ and $y_n \to +\infty$, $x_0 > \frac{m_0}{\lambda} + a$; then there exists a subsequence $\{\psi_n\}$ with $\psi_n(\tilde{x}, \tilde{y}) = \psi_{\lambda,\theta,L}(x_n + \tilde{x}, y_n + \tilde{y})$ such that*

$$\psi_n(\tilde{x}, \tilde{y}) \to S_{\frac{\pi}{2}}(\tilde{x}, \tilde{y}; 0, 0) \quad \text{uniformly in any compact subset of } \{\tilde{x} > a - x_0\}.$$

The same conclusion holds for $(x_n, y_n) \in \Gamma_1$.

Proof. Along the lines of similar arguments in Lemma 2.5, there exists a subsequence $\{\psi_n\}$ with $\psi_n \to S_\theta(\tilde{x},\tilde{y};0,0)$ which is well defined in the strip $E_{x_0} = \{(\tilde{x},\tilde{y}) \mid \tilde{x} \geq a - x_0\}$. Similar to (2.38), we can conclude that $S_\theta(\tilde{x},\tilde{y};0,0) \neq 0$ in any E_{x_0}-neighborhood of $(0,0)$ and $S_\theta(0,0;0,0) = 0$. For any $R > 0$, one has

$$S_\theta(a - x_0, \tilde{y}; 0, 0) = \lim_{n\to+\infty} \psi_n(a - x_0, \tilde{y}) = \lim_{n\to+\infty} \psi_{\lambda,\theta,L}(x_n + a - x_0, y_n + \tilde{y}) = m_0 \tag{2.49}$$

for any $|\tilde{y}| < R$.

Next, we claim that the direction of the constant flow $S_\theta(\tilde{x},\tilde{y};0,0)$ is vertical, namely, $\theta = \frac{\pi}{2}$.

Suppose that $\theta < \frac{\pi}{2}$; it follows from similar arguments in Lemma 2.5 that

$$\tfrac{\partial \varphi}{\partial t} = 0 \ \text{ a.e. in } \mathbb{R}^2, \tag{2.50}$$

where $\varphi(s,t) = S_\theta(\tilde{x},\tilde{y};0,0)$ with $s = \tilde{y}\cos\theta - \tilde{x}\cos\theta$ and $t = \tilde{x}\cos\theta + \tilde{y}\sin\theta$.

With the aid of (2.49) and (2.50),

$$S_\theta(\tilde{x},\tilde{y};0,0) \equiv m_0 \quad \text{in} \quad E_{x_0},$$

which leads to a contradiction to the fact $S_\theta(\tilde{x},\tilde{y};0,0) \neq 0$ in any E_{x_0}-neighborhood of $(0,0)$ and $S_\theta(0,0;0,0) = 0$. We can now complete the proof of the lemma by using similar arguments in Lemma 2.5. ∎

LEMMA 2.7. *Let $(x_n, y_n) \in \Gamma_1$ with $y_n \to y_0 < +\infty$ and $x_n \to +\infty$, $y_0 > \frac{m_0}{\lambda} + h_2$; then there exists a subsequence $\{\psi_n\}$ with $\psi_n(\tilde{x},\tilde{y}) = \psi_{\lambda,\theta,L}(x_n + \tilde{x}, y_n + \tilde{y})$ such that*

$$\psi_n(\tilde{x},\tilde{y}) \to S_0\left(\tilde{x},\tilde{y};0,-\frac{m_0}{\lambda}\right) \quad \textit{uniformly in any compact subset of } \{\tilde{y} > h_2 - y_0\}.$$

The same conclusion holds for $(x_n, y_n) \in \Gamma_2$.

Proof. It follows from similar arguments in Lemma 2.6 that there exists a subsequence $\{\psi_n\} \to S_\theta(\tilde{x},\tilde{y};0,-\frac{m_0}{\lambda})$ which is well defined in the strip $E_{y_0} = \{(\tilde{x},\tilde{y}) \mid \tilde{x} \geq h_2 - y_0\}$, and

$$S_\theta\left(\tilde{x}, h_2 - y_0; 0, -\frac{m_0}{\lambda}\right) = 0 \ \text{ for any } |\tilde{x}| < R. \tag{2.51}$$

We claim that $\theta = 0$. Suppose that $\theta > 0$; one has

$$\frac{\partial \varphi}{\partial t} = 0 \ \text{ a.e. in } \mathbb{R}^2. \tag{2.52}$$

Then (2.51) and (2.52) yield that

$$S_\theta\left(\tilde{x},\tilde{y};0,-\frac{m_0}{\lambda}\right) \equiv 0 \quad \text{in} \quad E_{y_0},$$

which leads to a contradiction to the fact that $S_\theta(\tilde{x},\tilde{y};0,-\frac{m_0}{\lambda}) \neq m_0$ in any E_{y_0}-neighborhood of $(0,0)$ and $S_\theta(0,0;0,-\frac{m_0}{\lambda}) = m_0$. We can now proceed as in Lemma 2.5 to complete the proof. ∎

2.5. Properties of the free boundaries of the minimizer $\boldsymbol{\psi_{\lambda,\theta,L}}$. In this subsection, we will obtain some important properties of the free boundaries of the minimizer $\psi_{\lambda,\theta,L}$ to show the strict monotonicity of the free boundary.

LEMMA 2.8. *If $\theta \in (0, \frac{\pi}{2})$, then the free boundary Γ_1 is nonempty and the free boundary Γ_2 is nonempty .*

Proof. Suppose that Γ_1 is empty, namely, $\psi_{\lambda,\theta,L} < m_0$ in $\Omega_L \setminus D_1$, where D_1 is defined as in subsection 2.2. We claim that there exists an $r > 0$ such that

$$B_r(X_0) \cap \Gamma_2 \neq \varnothing \text{ for any disc } B_r(X_0) \subset \Omega_L \setminus D_1 \text{ with } 0 < \psi_{\lambda,\theta,L}(X_0) < m_0. \tag{2.53}$$

In fact, for any $r > 0$, if there exists a point $X_0 \in \Omega_L \setminus D_1$ such that $B_r(X_0) \cap \Gamma_2 = \varnothing$, it follows from the nondegeneracy lemma, Lemma 7.9, that

$$m_0 \geq \sup_{\partial B_r(X_0)} \psi_{\lambda,\theta,L} \geq c\lambda r,$$

which leads to a contradiction for large r.

The relation (2.53) gives that there exist a constant C and a sequence $X_n = (x_n, y_n) \in \Gamma_2$, such that $x_n \leq C$ and $y_n \to +\infty$ as $n \to +\infty$. Then there exists a subsequence still labeled by $\{X_n\}$ with $X_n = (x_n, y_n) \in \Gamma_1$ such that

$$x_n \to x_0 \leq C \quad \text{and} \quad y_n \to +\infty \quad \text{as} \quad n \to +\infty.$$

It follows from Lemma 2.6 that $\theta = \frac{\pi}{2}$, which leads to a contradiction for the assumption $\theta \in (0, \frac{\pi}{2})$.

Similarly, we can show that Γ_2 is nonempty for $\theta \in (0, \frac{\pi}{2})$. ∎

LEMMA 2.9. *If $\theta = \frac{\pi}{2}$, then at least one of the free boundaries of $\psi_{\lambda,\theta,L}$ is empty. Similarly, if $\theta = 0$, then at least one of the free boundaries of $\psi_{\lambda,\theta,L}$ is empty.*

Proof. If $\theta = \frac{\pi}{2}$, suppose that both of the free boundaries Γ_1 and Γ_2 are nonempty. We claim that

$$y_n \to +\infty, \quad \left|\frac{x_n}{y_n}\right| \to 0 \quad \text{as} \quad n \to +\infty, \tag{2.54}$$

for any sequence $X_n = (x_n, y_n) \in \Gamma_1$ with $|X_n| \to +\infty$ as $n \to +\infty$.

Suppose not, there exists a subsequence which is still labeled by $X_n = (x_n, y_n) \in \Gamma_1$ with $|X_n| \to +\infty$ as $n \to +\infty$, such that

$$y_n \to +\infty, \quad \left|\frac{x_n}{y_n}\right| \to \alpha > 0 \quad \text{as} \quad n \to +\infty.$$

Set $\psi_n(\tilde{x}, \tilde{y}) = \psi_{\lambda,\theta,L}(x_n + \tilde{x}, y_n + \tilde{y})$. It follows from the similar arguments in Lemma 2.5 that there exists a subsequence still be labeled by $X_n = (x_n, y_n) \in \Gamma_1$ with $|X_n| \to +\infty$ as $n \to +\infty$, such that

$$\psi_n(\tilde{x}, \tilde{y}) \to S_{\tilde{\theta}}\left(\tilde{x}, \tilde{y}; 0, -\frac{m_0}{\lambda}\right) \quad \text{uniformly in compact subsets of } \mathbb{R}^2,$$

where the direction $e^\perp = \frac{(1,\alpha)}{\sqrt{1+\alpha^2}} = (\cos\tilde{\theta}, \sin\tilde{\theta})$, which contradicts to our assumption $\theta = \frac{\pi}{2}$. By virtue of (2.54), there exists a straight line $l_\varepsilon : y = x\cot\varepsilon + \beta_\varepsilon$ for small $\varepsilon > 0$ (see Figure 11) such that

$$l_\varepsilon \text{ is tangent to the free boundary } \Gamma_1 \text{ at } X_\varepsilon \text{ and stays below it, and } \beta_\varepsilon < 0. \tag{2.55}$$

Consider a special flow $S_{\frac{\pi}{2}-\varepsilon}(x, y; \frac{m_0}{\lambda\cos\varepsilon}, \beta_\varepsilon)$ as in (2.33), and let

$$S_\tau(x, y) = S_{\frac{\pi}{2}-\varepsilon}\left(x, y + \tau; \frac{m_0}{\lambda\cos\varepsilon}, \beta_\varepsilon\right) \qquad \text{for} \quad \tau > 0,$$

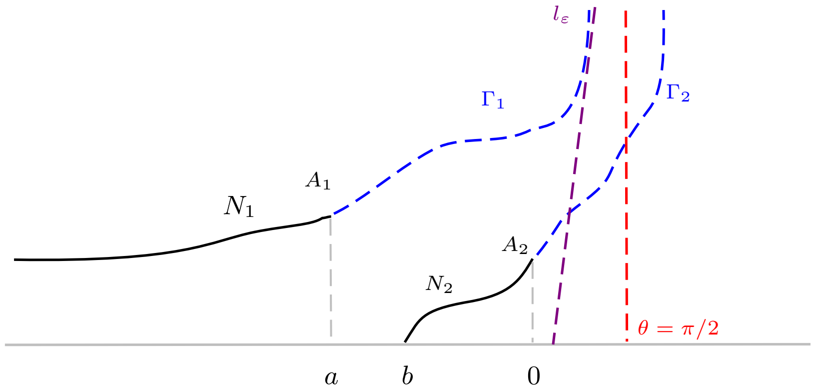

FIG. 11. *Critical direction:* $\theta = \pi/2$.

and choose the smallest τ_0 such that

(2.56)
$$\psi_{\lambda,\theta,L}(X) \le S_{\tau_0}(X) \text{ in } \Omega_L \setminus D_1, \text{ and } \psi_{\lambda,\theta,L}(X_0) = S_{\tau_0}(X_0) \text{ for some } X_0 \in \overline{\Omega_L \setminus D_1}.$$

Case 1. If $\tau_0 = 0$, we can take $X_0 = X_\varepsilon$; it follows from Hopf's lemma that

$$\lambda = \frac{\partial S_{\tau_0}}{\partial \nu} < \frac{\partial \psi_{\lambda,\theta,L}}{\partial \nu} = \lambda \quad \text{at} \quad X_0 = X_\varepsilon \in \Gamma_1,$$

which leads to a contradiction.

Case 2. If $\tau_0 > 0$, we claim that $X_0 \notin \{0 < \psi_{\lambda,\theta,L} < m_0\}$. Suppose that $0 < \psi_{\lambda,\theta,L}(X_0) = S_{\tau_0}(X_0) < m_0$; it follows from the continuity of ψ and S_{τ_0} that there exists a small $r > 0$, such that $0 < S_{\tau_0}(X) < m_0$ and $0 < \psi_{\lambda,\theta,L}(X) < m_0$ in $B_r(X_0)$. The strong maximum principle gives that $0 < h_{\tau_0}(X) = \psi_{\lambda,\theta,L}(X) < m_0$ in $B_r(X_0)$; furthermore, we can obtain a contradiction to the boundary value of $\psi_{\lambda,\theta,L}$ by using the strong maximum principle. So we conclude that $X_0 \in \overline{\Gamma_2} \cap \partial\{S_{\tau_0} > 0\}$.

The condition $\beta_\varepsilon < 0$ in (2.56) gives that $X_0 \notin \Gamma_2 \cap \{(0, y) \mid y \ge h_2\}$. In view of $\theta = \frac{\pi}{2}$, if $\overline{\Gamma_2} \cap I_{h_2}$ is nonempty, it follows from Theorem 9.1 in [4] or Theorem 7.9 in [2] that Γ_2 is C^1 at I_{h_2}, which combined with (2.56) yields that $X_0 \notin I_{h_2}$. In fact, if $X_0 \in I_{h_2}$, the previous arguments imply that $\psi_{\lambda,\theta,L}(X) > S_{\tau_0}(X) = 0$ in $B_r(X_0) \cap \{0 < \psi_{\lambda,\theta,L} < m_0\} \cap \{S_{\tau_0} = 0\}$ for small $r > 0$; this contradicts the definition of τ_0. Then $X_0 \in \Gamma_2 \cap \partial\{S_{\tau_0} > 0\}$. The fact $\tau_0 > 0$ implies that $|X_0| \le M$ and it follows from the Hopf's lemma that one has

$$-\lambda = \frac{\partial S_{\tau_0}}{\partial \nu} < \frac{\partial \psi_{\lambda,\theta,L}}{\partial \nu} = -\lambda \quad \text{at} \quad X_0 \in \Gamma_2 \cap \partial\{S_{\tau_0} > 0\}.$$

This leads to a contradiction. Hence, we conclude that Γ_1 is empty for $\theta = \frac{\pi}{2}$.

Similarly, we can show that at least one of free boundaries of $\psi_{\lambda,\theta,L}$ is empty for $\theta = 0$. ∎

With the aid of Lemmas 2.8 and 2.9, we can obtain the continuity of the free boundaries in Ω_L via the following two lemmas.

LEMMA 2.10. *If* $\theta \in (0, \frac{\pi}{2})$, *then* Γ_1 *is connected, and* $k_{1,\lambda,\theta,L}(x)$ *is finite-valued for all* $a < x < +\infty$. *Similarly, if* $\theta \in (0, \frac{\pi}{2})$, Γ_2 *is connected, and* $k_{2,\lambda,\theta,L}(x)$ *is finite-valued for all* $0 < x < +\infty$.

Proof. Consider the conclusion for Γ_1 and a similar one can be obtained for Γ_2.

It follows from Lemma 2.8 that Γ_1 and Γ_2 are nonempty for $\theta \in (0, \frac{\pi}{2})$. Set $I = (\alpha, \beta)$, which is a maximal interval such that $k_{1,\lambda,\theta,L}(x)$ is finite-valued. If $\beta < +\infty$, since Γ_1 is a smooth curve, one has

$$k_{1,\lambda,\theta,L}(x) \to +\infty \quad \text{as} \quad x \to \beta^-.$$

It follows from Lemma 2.9 that $\theta = \frac{\pi}{2}$, which leads to a contradiction. Due to the monotonicity of the $\psi_{\lambda,\theta,L}$ with respect to x and y, we can conclude that $\alpha = a$ and $\lim_{x \to a^+} k_{1,\lambda,\theta,L}(x)$ exists. ∎

Lemma 2.11. *If $\theta = 0$ and $\lambda(h_1 - h_2) < m_0$, then $g_{1,\lambda,\theta,L}(y)$ is finite-valued for all $h_1 < y < h_2 + \frac{m_0}{\lambda}$. If $\theta = 0$ and $\lambda(h_1 - h_2) > m_0$, then $g_{2,\lambda,\theta,L}(y)$ is finite-valued for all $h_2 < y < h_1 - \frac{m_0}{\lambda}$. Similarly, if $\theta = \frac{\pi}{2}$ and $-\lambda a < m_0$, then $k_{2,\lambda,\theta,L}(x)$ is finite-valued for all $0 < x < \frac{m_0}{\lambda} + a$. If $\theta = \frac{\pi}{2}$ and $-\lambda a > m_0$, then $k_{1,\lambda,\theta,L}(x)$ is finite-valued for all $a < x < -\frac{m_0}{\lambda}$.*

Proof. If $\theta = 0$ and $\lambda(h_1 - h_2) < m_0$, similar to the proof of Lemma 2.8, we can conclude that Γ_1 is nonempty. Set $I = (h_1, \alpha)$, which is a maximal interval such that $g_{1,\lambda,\theta,L}(y)$ is finite-valued. Lemma 2.9 implies that Γ_2 is empty. By virtue of similar arguments in Lemma 2.8, there exists an $r > 0$ such that

$$B_r(X_0) \cap \Gamma_1 \neq \varnothing \ \text{ for any disc } B_r(X_0) \subset \Omega_L \setminus D_1 \text{ with } 0 < \psi_{\lambda,\theta,L}(X_0) < m_0, \tag{2.57}$$

which implies that $\alpha < +\infty$.

Similarly, we can show that the remained part of the lemma. ∎

Finally, we will establish the strict monotonicity of $k_{1,\lambda,\theta,L}(x)$ and $k_{2,\lambda,\theta,L}(x)$.

Lemma 2.12. *The free boundary $y = k_{1,\lambda,\theta,L}(x)$ is strictly increasing in $(a, +\infty)$, and the free boundary $y = k_{2,\lambda,\theta,L}(x)$ is strictly increasing in $(0, +\infty)$.*

Proof. Suppose that $k_{1,\lambda,\theta,L}(x_1) = k_{1,\lambda,\theta,L}(x_2)$ for some $a < x_1 < x_2 < +\infty$. In view of Theorem 2.3, the $\psi_{\lambda,\theta,L}$ is monotonic decreasing with respect to x, which implies that $k_{1,\lambda,\theta,L}(x) = k_{1,\lambda,\theta,L}(x_1)$ for all $x_1 < x < x_2$. It follows from the similar arguments in the proof of Lemma 2.4 that

$$\frac{\partial \psi_{\lambda,\theta,L}(x, k_{1,\lambda,\theta,L}(x_1) - 0)}{\partial y} = \lambda \ \text{ and } \ \psi = m_0$$

on the segment $\sigma_0 = \{(x, k_{1,\lambda,\theta,L}(x_1)) \mid x_1 < x < x_2\}$. Then, again as in the arguments in the proof of Lemma 2.4, one has a contradiction to the boundary condition. Hence, $k_{1,\lambda,\theta,L}(x)$ is strictly increasing in $(a, +\infty)$.

Similarly, we can conclude that $k_{2,\lambda,\theta,L}(x)$ is strictly increasing in $(0, +\infty)$. ∎

It follows from similar arguments in Proposition 2.12 that the free boundaries are strictly monotonic increasing; then there exist two functions $g_{i,\lambda,\theta,L}(y)$ such that $\Gamma_i = \{(x, y) \in \Omega_L \mid x = g_{i,\lambda,\theta,L}(y)\}$ $(i = 1, 2)$.

Remark 2.13. In view of the monotonicity of the free boundaries Γ_1 and Γ_2 with respect to x and y, the initial point of Γ_1 may be on I_a or $\{(x, h_1) \mid x > a\}$. Similarly, the initial point of Γ_2 may be on I_{h_2} or $\{(0, y) \mid y \geq h_2\}$. So we use two functions $k_{i,\lambda,\theta,L}(x)$ and $g_{i,\lambda,\theta,L}(y)$ to describe the free boundary Γ_i for $i = 1, 2$.

2.6. Continuous dependence with respect to the parameters. In this section, we first show that the minimizer depends continuously on the parameters λ and θ. Furthermore, the free boundaries are also continuously dependent with respect to parameters λ and θ, such that we can show the existence of the Réthy flow satisfying the continuous fit condition and the estimate of the deflection angle θ, i.e., (1.10), in the next subsection.

LEMMA 2.14. *Let $\psi_n = \psi_{\lambda_n,\theta_n,L}$ be a minimizer to the variational problem $(P_{\lambda_n,\theta_n,L})$ with the admissible set K_L; then there exists a subsequence $\{\psi_n\}$ such that*

$$\psi_n \rightharpoonup \psi_{\lambda,\theta,L} \quad in \quad H^1_{loc}(\Omega_L) \text{ and } \psi_n \to \psi_{\lambda,\theta,L} \quad a.e. \text{ in } \quad \Omega_L,$$

as $\lambda_n \to \lambda$ and $\theta_n \to \theta$.

Proof. By virtue of similar arguments in Proposition 7.2 in the appendix, there exists a subsequence still labeled as $\{\psi_n\}$ such that

$$\psi_n \to \omega \quad \text{weakly in} \quad H^1_{loc}(\Omega_L) \text{ and } \psi_n \to \omega \text{ in } C^\alpha_{loc}(\Omega_L) \quad \text{for} \quad 0 < \alpha < 1. \tag{2.58}$$

Step 1. We will show $\partial\{0 < \psi_n < m_0\} \cap \Omega_L \to \partial\{0 < \omega < m_0\} \cap \Omega_L$ locally in the Hausdorff distance, where the Hausdorff distance $d(G, D)$ between two sets D and G is defined as follows:

$$d(G,D) = \inf\left\{\varepsilon > 0 \mid D \subset \bigcup_{X\in G} B_\varepsilon(X) \quad \text{and} \quad G \subset \bigcup_{X\in D} B_\varepsilon(X)\right\}.$$

First, let us consider any point X_0 which does not lie in the free boundary determined by ω. For any $X_0 = (x_0, y_0) \in \Omega_L$, if $X_0 \notin \partial\{0 < \omega < m_0\}$, then there exists a small $r > 0$ such that $B_r(X_0) \subset \Omega_L$ and $B_r(X_0) \cap \partial\{0 < \omega < m_0\} = \varnothing$. We consider three cases for the location of X_0 as follows.

Case 1. $0 < \omega < m_0$ in $B_r(X_0)$. It follows from (2.58) that

$$0 < \min_{X\in B_{\frac{r}{2}}(X_0)} \omega \le \max_{X\in B_{\frac{r}{2}}(X_0)} \omega < m_0, \text{ and } 0 < \psi_n < m_0 \text{ in } B_{\frac{r}{2}}(X_0)$$
$$\text{for sufficiently large } n,$$

which implies that $B_{\frac{r}{2}}(X_0) \cap \partial\{0 < \psi_n < m_0\} = \varnothing$ for sufficiently large n.

Case 2. $\omega = m_0$ in $B_r(X_0)$. It follows from (2.58) that for any $\varepsilon > 0$, there exists an N_ε such that

$$\psi_n > 0 \text{ and } |m_0 - \psi_n(X)| < \varepsilon \text{ in } B_{\frac{r}{2}}(X_0) \text{ for } n > N_\varepsilon.$$

Thanks to Lemma 7.8 for ψ_n, one has

$$\frac{2}{r}\fint_{\partial B_{\frac{r}{2}}(X_0)} (m_0 - \psi_n)dS < \frac{2\varepsilon}{r} \le \lambda_n c^* \text{ for sufficiently large } n,$$

which implies that $\psi_n \equiv m_0$ in $B_{\frac{r}{4}}(X_0)$ for sufficiently large n. Moreover, $B_{\frac{r}{4}}(X_0) \cap \partial\{0 < \psi_n < m_0\} = \varnothing$ for sufficiently large n.

Case 3. $\omega = 0$ in $B_r(X_0)$. It follows from (2.58) that for any $\varepsilon > 0$, there exists an N_ε such that

$$\psi_n(X) < \varepsilon \text{ in } B_{\frac{r}{2}}(X_0) \text{ for } n > N_\varepsilon.$$

Thanks to Lemma 7.8 for ψ_n, one has

$$\frac{2}{r}\fint_{\partial B_{\frac{r}{2}}(X_0)} \psi_n dS < \frac{2\varepsilon}{r} \leq \lambda_n c^* \text{ for sufficiently large } n,$$

which implies that $\psi_n \equiv 0$ in $B_{\frac{r}{4}}(X_0)$ for sufficiently large n. Moreover, $B_{\frac{r}{4}}(X_0) \cap \partial\{0 < \psi_n < m_0\} = \varnothing$ for sufficiently large n.

Reversely, let us consider the case when the point does not lie on the free boundary determined by ψ_n. For any $X_0 = (x_0, y_0) \in \Omega_L$, if $X_0 \notin \partial\{0 < \psi_n < m_0\}$, then there exists a small $r > 0$ such that $B_r(X_0) \cap \partial\{0 < \psi_n < m_0\} = \varnothing$. Similarly, let us consider the location of X_0 to two cases.

Case 1. $0 < \psi_n < m_0$ in $B_r(X_0)$ for a subsequence $\{\psi_n\}$, $n \in \mathbb{N}$.

Thanks to Proposition 7.4, one has

$$\Delta\psi_n = 0 \quad \text{in} \quad B_r(X_0).$$

It follows from Lemma 7.5 and (2.58) that

$$\Delta\omega = 0 \text{ in } B_r(X_0),\ 0 \leq \omega \leq m_0 \text{ in } B_r(X_0).$$

The strong maximum principle yields that

$$\text{either} \quad \omega \equiv m_0 \text{ or } \omega \equiv 0 \text{ or } \quad 0 < \omega < m_0 \quad \text{in } B_r(X_0),$$

which gives that $B_{\frac{r}{2}}(X_0) \cap \partial\{0 < \omega < m_0\} = \varnothing$.

Case 2. $\psi_n = m_0$ or $\psi_n = 0$ in $B_r(X_0)$ for a subsequence $\{\psi_n\}$, $n \in \mathbb{N}$. Then $\omega = m_0$ or $\omega = 0$ in $B_r(X_0)$; therefore, $B_{\frac{r}{2}}(X_0) \cap \partial\{0 < \omega < m_0\} = \varnothing$.

Hence, we obtain the convergence of the free boundary in the Hausdorff distance.

Step 2. In this step, we will prove $\chi_{\{0<\psi_n<m_0\}} \to \chi_{\{0<\omega<m_0\}}$ in $L^1_{loc}(\Omega_L)$.

With the aid of Lemmas 7.6 and 7.8 for ψ_n, one has

$$C^*\lambda_n \leq \frac{1}{r}\fint_{\partial B_r(X_n^0)} (m_0 - \psi_n) dS \leq c^*\lambda_n \quad \text{for} \quad X_n^0 \in \partial\{\psi_n < m_0\}$$

and

$$C^*\lambda_n \leq \frac{1}{r}\fint_{\partial B_r(X_n^0)} \psi_n dS \leq c^*\lambda_n \quad \text{for} \qquad X_n^0 \in \partial\{\psi_n > 0\},$$

where $r \in (0, r_0)$ is sufficiently small and fixed, the constants C^* and c^* are determined in Lemmas 7.6 and 7.8 independently of n and $c^* > C^* > 0$.

Then taking $n \to +\infty$, we have

$$C^*\lambda \leq \frac{1}{r}\fint_{\partial B_r(X_0)} (m_0 - \omega) dS \leq c^*\lambda \quad \text{for} \quad X_0 \in \partial\{\omega < m_0\}$$

and

$$C^*\lambda \leq \frac{1}{r}\fint_{\partial B_r(X_0)} \omega dS \leq c^*\lambda \quad \text{for} \qquad X_0 \in \partial\{\omega > 0\};$$

those together with Theorem 4.5 in [1] imply that

$$\mathcal{H}^1(\partial\{0 < \omega < m_0\} \cap \Omega_L \cap B_R) < +\infty \quad \text{for any } R > 0, \tag{2.59}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure on $\mathbb{R}^2$. Consequently,

$$\mathcal{L}^2(\partial\{0<\omega<m_0\}\cap\Omega_L\cap B_R)=0 \quad \text{for any } R>0,$$

where $\mathcal{L}^2$ is the two-dimensional Lebesgue measure on $\mathbb{R}^2$.

Let O_{ε_n} be an ε_n-neighborhood of $\partial\{0<\omega<m_0\}$ and denote $\mathcal{L}_{R,\varepsilon_n}=\mathcal{L}^2(\Omega_L\cap B_R\cap O_{\varepsilon_n})$ for any $R>0$. Obviously,

$$\mathcal{L}_{R,\varepsilon_n}\to 0 \quad \text{as } \varepsilon_n\to 0, \quad \text{for any } R>0. \tag{2.60}$$

Hence, for sufficiently large n, we have

$$\int_{\Omega_L\cap B_R}\left|\chi_{\{0<\psi_n<m_0\}}-\chi_{\{0<\omega<m_0\}}\right|dxdy\le\int_{\Omega_L\cap B_R\cap O_{\varepsilon_n}}1dxdy=\mathcal{L}_{R,\varepsilon_n} \tag{2.61}$$

for any $R>0$, which together with (2.60) gives that $\chi_{\{0<\psi_n<m_0\}}\to\chi_{\{0<\omega<m_0\}}$ in $L^1_{loc}(\Omega_L)$.

Step 3. In this step, we will show $\nabla\psi_n\to\nabla\omega$ a.e. locally in Ω_L.

Let D be any compact subset of $\Omega_L\cap(\{0<\omega<m_0\}\cup\text{int}\{\omega=m_0\}\cup\text{int}\{\omega=0\})$. The result in Step 1 implies that the minimizer ψ_n satisfies the equation

$$\Delta\psi_n=0 \ \text{ in } \ D \ \text{ for sufficiently large } n.$$

Thanks to the standard elliptic estimates for ψ_n, one has

$$\nabla\psi_n\to\nabla\omega$$

uniformly in any compact subset D of $\Omega_L\cap(\{0<\omega<m_0\}\cup\text{int}\{\omega=m_0\}\cup\text{int}\{\omega=0\})$, and furthermore recalling (2.59) implies that $\nabla\psi_n\to\nabla\omega$ a.e. locally in Ω_L.

Step 4. In this step, we will prove $\omega=\psi_{\lambda,\theta,L}$. Denote $\psi=\psi_{\lambda,\theta,L}$ in the proof for notational simplicity.

First, we will prove ω is a minimizer to the truncated variational problem $(P_{\lambda,\theta,L})$. It follows from $\omega\in K_L$ that

$$J_{\lambda,\theta,L}(\psi)\le J_{\lambda,\theta,L}(\omega).$$

We claim that

$$J_{\lambda,\theta,L}(\psi)\ge J_{\lambda,\theta,L}(\omega).$$

Suppose not; then we have

$$J_{\lambda,\theta,L}(\psi)<J_{\lambda,\theta,L}(\omega) \ \text{ and set } \delta=J_{\lambda,\theta,L}(\omega)-J_{\lambda,\theta,L}(\psi)>0.$$

Without loss of generality, we assume that $\theta\in(0,\frac{\pi}{2})$; it follows from Lemma 2.5 and Step 1 that

$$\psi(x_n+x,y_n+y)\rightharpoonup S_{\lambda,\theta}(x,y) \ \text{ weakly in } \ H^1_{loc}(\mathbb{R}^2),$$

and

$$\chi_{\{0<\psi(x_n+x,y_n+y)<m_0\}}\to\chi_{\{0<S_{\lambda,\theta}<m_0\}} \ \text{ in } L^1_{loc}(\mathbb{R}^2),$$

as $x_n^2+y_n^2\to+\infty$, where $S_{\lambda,\theta}(x,y)$ is the asymptotic behavior of ψ in downstream.

In view of the continuity of the free boundaries $k_{1,\lambda,\theta,L}(x)$ and $k_{2,\lambda,\theta,L}(x)$, then $M_2(R)\le k_{2,\lambda,\theta,L}(x)<k_{1,\lambda,\theta,L}(x)\le M_1(R)$ for $R\le x\le R+1$ and large $R>0$.

Furthermore, the domain $\Omega_{L,R} = \Omega_L \cap \{R \le x \le R+1\} \cap \{0 < \psi < m_0\}$ is a bounded domain.

Set $\tilde{\psi}_{R,n}(x,y) = \eta_R(x)\psi(x,y) + (1-\eta_R(x))S_{R,\lambda_n,\theta_n}(x,y)$ in $\Omega_{L,R}$, where

$$S_{R,\lambda_n,\theta_n}(x,y) = \min\{\max\{\lambda_n((y-k(R))\cos\theta_n - (x-R)\sin\theta_n), 0\}, m_0\},$$

and

$$\eta_R(x) = \min\{\max\{R+1-x, 0\}, 1\}$$

for the point $(R, k(R)) \in \partial\{S_{\lambda,\theta} > 0\} \cap \{x = R\}$.

Next, denote $\tilde{\psi}_n = \tilde{\psi}_{R_0,n}$ and $S_n = S_{R_0,\lambda_n,\theta_n}$ for notational simplicity. For $\varepsilon = \frac{\delta}{4} > 0$, there exists an R_0 such that

(2.62)
$$\int_{\Omega_{L,R_0}} \left|\nabla\psi - \lambda e\chi_{\{0<\psi<m_0\}}\right|^2 + |\psi - S_{\lambda,\theta}|^2 + \lambda^2|\chi_{\{0<\psi<m_0\}} - \chi_{\{0<S_{\lambda,\theta}<m_0\}}|dxdy \le \frac{\varepsilon}{12}.$$

For fixed R_0, in view of $\lambda_n \to \lambda$ and $\theta_n \to \theta$, there exists an N_1 such that $\lambda_n^2 \le 2\lambda^2$ and

(2.63)
$$\begin{aligned}
&\int_{\Omega_{L,R_0}} \left|\nabla\psi - \lambda_n e_n\chi_{\{0<\psi<m_0\}}\right|^2 + |\psi - S_n|^2 + \lambda_n^2|\chi_{\{0<\psi<m_0\}} - \chi_{\{0<S_n<m_0\}}|dxdy \\
&\quad \le 2\int_{\Omega_{L,R_0}} \left|\nabla\psi - \lambda e\chi_{\{0<\psi<m_0\}}\right|^2 + |\psi - S_n|^2 + \lambda^2|\chi_{\{0<\psi<m_0\}} - \chi_{\{0<S_{\lambda,\theta}<m_0\}}|dxdy \\
&\qquad + 2\int_{\Omega_{L,R_0}} |\lambda e - \lambda_n e_n|^2\chi_{\{0<\psi<m_0\}} + |S_{\lambda,\theta} - S_n|^2 + \lambda^2|\chi_{\{0<S_{\lambda,\theta}<m_0\}} - \chi_{\{0<S_n<m_0\}}|dxdy \\
&\quad \le \frac{\varepsilon}{3}
\end{aligned}$$

for any $n > N_1$.

Then, we have

(2.64)
$$\begin{aligned}
&\int_{\Omega_{L,R_0}} \left|\nabla\tilde{\psi}_n - \lambda_n e_n\chi_{\{0<\tilde{\psi}_n<m_0\}}\right|^2 dxdy \\
&\quad = \int_{\Omega_{L,R_0}\cap\{0<\tilde{\psi}_n<m_0\}} |\eta_{R_0}(\nabla\psi - \lambda_n e_n) + (1-\eta_{R_0})(\nabla S_n - \lambda_n e_n) + (\psi - S_n)\nabla\eta_{R_0}|^2 dxdy \\
&\quad \le 3\int_{\Omega_{L,R_0}\cap\{0<\tilde{\psi}_n<m_0\}} |\nabla\psi - \lambda_n e_n|^2 + |\nabla S_n - \lambda_n e_n|^2 + |\psi - S_n|^2 dxdy \\
&\quad \le 3\int_{\Omega_{L,R_0}\cap\{0<\psi<m_0\}} |\nabla\psi - \lambda_n e_n|^2 dxdy + 3\int_{\Omega_{L,R_0}\cap(\{0<S<m_0\}-\{0<\psi<m_0\})} \lambda_n^2 dxdy \\
&\qquad + 3\int_{\Omega_{L,R_0}\cap(\{0<\psi<m_0\}-\{0<S<m_0\})} \lambda_n^2 dxdy + 3\int_{\Omega_{L,R_0}} |\psi - S|^2 dxdy \\
&\quad \le 3\int_{\Omega_{L,R_0}} \left|\nabla\psi - \lambda_n e_n\chi_{\{0<\psi<m_0\}}\right|^2 + |\psi - S_n|^2 + \lambda_n^2|\chi_{\{0<\psi<m_0\}} - \chi_{\{0<S_n<m_0\}}|dxdy \\
&\quad \le \varepsilon,
\end{aligned}$$

for $n > N_1$, where we have used (2.63), $|\eta_{R_0}| \le 1$, $|\nabla\eta_{R_0}| = 1$, and

$$\int_{\Omega_{L,R_0}\cap\{0<S_n<m_0\}} |\nabla S_n - \lambda_n e_n|^2 dxdy = 0,$$

and

$$\Omega_{L,R_0} \cap \{0 < \tilde{\psi}_n < m_0\} = \Omega_{L,R_0} \cap (\{0 < \psi < m_0\} \cup \{0 < S_n < m_0\}).$$

For fixed $R_0 > 0$, in view of boundedness of $\Omega_0 = \Omega_L \cap \{0 < \psi < m_0\} \cap \{x \leq R_0\}$, there exists an N_2 such that

$$\begin{aligned}
&\int_{\Omega_0} |\nabla\psi - \lambda_n e_n|^2 \, dxdy \\
&\quad \leq \int_{\Omega_0} |\nabla\psi - \lambda e|^2 \, dxdy + \int_{\Omega_0} |\lambda e - \lambda_n e_n|^2 dxdy + 2\left(\int_{\Omega_0} |\nabla\psi - \lambda e|^2 \, dxdy\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{\Omega_0} |\lambda e - \lambda_n e_n|^2 dxdy\right)^{\frac{1}{2}} \\
&\quad \leq \int_{\Omega_0} |\nabla\psi - \lambda e|^2 \, dxdy + \varepsilon
\end{aligned} \tag{2.65}$$

for $n > N_2$, where we have used

$$\int_{\Omega_0} |\nabla\psi - \lambda e|^2 \, dxdy \leq C.$$

It follows from (2.58) that there exists an N_3 such that

$$\int_{\Omega_L} \left|\nabla\omega - \lambda e \chi_{\{0<\omega<m_0\}}\right|^2 dxdy \leq \int_{\Omega_L} \left|\nabla\psi_n - \lambda_n e_n \chi_{\{0<\psi_n<m_0\}}\right|^2 dxdy + \varepsilon \tag{2.66}$$

for $n > N_3$.

For $n > N = \max\{N_1, N_2, N_3\}$, one has

$$\begin{aligned}
&\int_{\Omega_L} \left|\nabla\tilde{\psi}_n - \lambda_n e_n \chi_{\{0<\tilde{\psi}_n<m_0\}}\right|^2 dxdy \\
&\quad = \int_{\Omega_0} |\nabla\psi - \lambda_n e_n|^2 \, dxdy + \int_{\Omega_{L,R_0}} \left|\nabla\tilde{\psi}_n - \lambda_n e_n \chi_{\{0<\psi<m_0\}}\right|^2 dxdy \\
&\quad \leq \int_{\Omega_L} \left|\nabla\omega - \lambda e \chi_{\{0<\omega<m_0\}}\right|^2 dxdy - \delta + 2\varepsilon \\
&\quad \leq \int_{\Omega_L} \left|\nabla\psi_n - \lambda_n e_n \chi_{\{0<\psi_n<m_0\}}\right|^2 dxdy - \frac{\delta}{4}.
\end{aligned} \tag{2.67}$$

It is easy to cheek that $\tilde{\psi}_n \in K_L$. Since ψ_n is the minimizer to the variational problem $(P_{\lambda_n,\theta_n,L})$, one has

$$J_{\lambda_n,\theta_n,L}(\psi_n) \leq J_{\lambda_n,\theta_n,L}(\tilde{\psi}_n) \leq J_{\lambda_n,\theta_n,L}(\psi_n) - \frac{\delta}{4}$$

for $\delta = J_{\lambda,\theta,L}(\omega) - J_{\lambda,\theta,L}(\psi) > 0$, which leads to a contradiction.

Hence,

$$J_{\lambda,\theta,L}(\psi) = J_{\lambda,\theta,L}(\omega).$$

Thanks to the uniqueness of the minimizer to the truncated variational problem $(P_{\lambda,\theta,L})$, we have $\omega = \psi = \psi_{\lambda,\theta,L}$. ∎

Next, we will prove the continuous dependence on the parameters λ and θ for the free boundaries.

PROPOSITION 2.15. *Let* $\Gamma_{1,n}: y = k_{1,n}(x) = k_{1,\lambda_n,\theta_n,L}(x)$ *and* $\Gamma_{2,n}: y = k_{2,n}(x) = k_{2,\lambda_n,\theta_n,L}(x)$ *be free boundaries of the minimizer* $\psi_n = \psi_{\lambda_n,\theta_n,L}$ *to the variational problem* $(P_{\lambda_n,\theta_n,L})$*; then*

$$k_{1,n}(x) \to k_{1,\lambda,\theta,L}(x) \quad \textit{for any} \quad x \in [a, +\infty),$$

and

$$k_{2,n}(x) \to k_{2,\lambda,\theta,L}(x) \quad \textit{for any} \quad x \in [0, +\infty),$$

as $\lambda_n \to \lambda$ *and* $\theta_n \to \theta$.

Proof. We only show the convergence of the free boundary $y = k_{1,n}(x)$ in $[a, +\infty)$; the convergence of the free boundary $y = k_{2,n}(x)$ in $[0, +\infty)$ can be obtained by similar arguments.

First, for any fixed $x_0 \in (a, +\infty)$, it follows from the continuity of $k_{1,n}(x)$ that there exists a subsequence still labeled by $k_{1,n}(x_0)$ such that

$$k_{1,n}(x_0) \to y_0 \quad \text{as} \quad n \to \infty,$$

where the point $X_n^0 = (x_0, k_{1,n}(x_0)) \in \Gamma_{1,n}$.

Then, it suffices to show that the limit point $X^0 = (x_0, y_0) \in \Gamma_1$. By virtue of the bounded gradient Lemma 7.13 for ψ_n near $\Gamma_{1,n}$, there exists a small $r_0 > 0$ such that

$$\psi_n > 0 \quad \text{in} \quad B_{2r_0}(X_n^0).$$

With the aid of Lemmas 7.6 and 7.8, we have

$$C^* \le \frac{1}{r} \fint_{\partial B_r(X_n^0)} (m_0 - \psi_n) dS \le c^* \quad \text{for some} \quad c^* > C^* > 0,$$

where $r \in (0, r_0)$ is sufficiently small and fixed, and the constants C^* and c^* are determined in Lemmas 7.6 and 7.8 and independently of n.

Taking $n \to \infty$ yields that

$$C^* \le \frac{1}{r} \fint_{\partial B_r(X^0)} (m_0 - \psi_{\lambda,\theta,L}) dS \le c^* \quad \text{for any } r \in (0, r_0),$$

which implies that $X^0 = (x_0, y_0)$ is a free boundary point of $\psi_{\lambda,\theta,L}$ due to Lemmas 7.6 and 7.8. Hence, by the monotonicity of the free boundaries, one has

$$k_{1,n}(x_0) \to y_0 = k_{1,\lambda,\theta,L}(x_0) \quad \text{for any} \quad x_0 \in (a, +\infty).$$

Finally, we will show that $k_{1,n}(a) \to k_{1,\lambda,\theta,L}(a)$ as $n \to +\infty$.

Suppose not; there exist subsequences still labeled by $\{\lambda_n\}$ and $\{\theta_n\}$ such that $k_{1,n}(a) \to k_{1,\lambda,\theta,L}(a) + \delta, \ \delta \ne 0$.

Case 1. $\delta < 0$. By the convergence of the free boundary of any relatively interior part and Theorem 3.1 in Chapter 2 in [26], we have

$$\frac{\partial \psi_{\lambda,\theta,L}(a+0, y)}{\partial x} = -\lambda \quad \text{if} \quad k_{1,\lambda,\theta,L}(a) + \delta < y < k_{1,\lambda,\theta,L}(a), \ x = a.$$

By virtue of the continuity of $\psi_{\lambda,\theta,L}$, there exists a small $\varepsilon > 0$ such that

$$0 < \psi_{\lambda,\theta,L} < m_0 \quad \text{in} \ \{(x, y) \mid a < x < a + \varepsilon, k_{1,\lambda,\theta,L}(a) + \delta < y < k_{1,\lambda,\theta,L}(a)\}.$$

It follows from the Cauchy–Kovalevskaya theorem and unique continuation that

$$\psi_{\lambda,\theta,L} = -\lambda(x-a) + m_0 \text{ in } \{(x,y) \mid a < x < a+\varepsilon, -\infty < y < +\infty\} \cap \Omega_L,$$

which is impossible.

Case 2. $\delta > 0$. The monotonicity of the free boundary and the results of Case 1 imply that there exists a small $\varepsilon_1 > 0$, such that

$$k_{1,n}(x) > k_{1,\lambda,\theta,L}(a) + \frac{\delta}{2} > k_{1,\lambda,\theta,L}(a) \text{ for sufficiently large } n \text{ and } a < x < a+\varepsilon_1. \tag{2.68}$$

The continuity of the free boundary gives that there exists a small $\varepsilon_2 > 0$ such that

$$k_{1,\lambda,\theta,L}(x) < k_{1,\lambda,\theta,L}(a) + \frac{\delta}{4} \text{ for } a < x < a+\varepsilon_2. \tag{2.69}$$

Combining (2.68) and (2.69), one has

$$k_{1,n}(x) > k_{1,\lambda,\theta,L}(x) + \frac{\delta}{4} \text{ for sufficiently large } n \text{ and } a + \frac{\varepsilon}{2} < x < a+\varepsilon,$$

where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, which leads to a contradiction to the results in Case 1 for sufficiently large n. □

2.7. Continuous fit conditions of the free boundaries. Let us introduce the continuous fit conditions of the free boundaries $\Gamma_1 : y = k_{1,\lambda,\theta,L}(x)$ and $\Gamma_2 : x = g_{2,\lambda,\theta,L}(y)$, namely,

$$k_{1,\lambda,\theta,L}(a) = h_1 \text{ and } g_{2,\lambda,\theta,L}(h_2) = 0 \quad \text{for some parameters } \lambda \text{ and } \theta. \tag{2.70}$$

To obtain the continuous fit conditions, define a set Σ_L as follows:

$$\Sigma_L = \left\{\lambda \mid \text{there exists a } \theta \in (0, \frac{\pi}{2}), \text{ such that } k_{1,\lambda,\theta,L}(a) > h_1 \text{ and } g_{2,\lambda,\theta,L}(h_2) > 0\right\},$$

and the following lemma implies that the set Σ_L is nonempty.

LEMMA 2.16. *For any sufficiently small $\lambda > 0$, the free boundary Γ_1 starts on I_a and the free boundary Γ_2 starts on I_{h_2}, namely, $k_{1,\lambda,\theta,L}(a) > h_1$ and $g_{2,\lambda,\theta,L}(h_2) > 0$.*

Proof. Suppose that the assertion is not true; without loss of generality we assume that the free boundary Γ_1 starts on $\{y = h_1\}$ for small enough $\lambda > 0$, namely, $g_{1,\lambda,\theta,L}(h_1) \geq a$.

Let B_0 be a ring centered at $(g_{1,\lambda,\theta,L}(h_1), h_1)$ with some suitable radius r_1 and r_2 (independent of λ) with $r_1 < r_2$, such that $\Gamma_1 \cap G$ and $(\Gamma_2 \cup I_{h_2}) \cap G$ are nonempty, where $G = \{0 < \psi_{\lambda,\theta,L} < m_0\} \cap \{x > a\} \cap B_0$.

It follows from the bounded gradient lemma, Lemma 7.13, and Lemma 5.2 in [2] that there exists a constant C, such that

$$|D\psi_{\lambda,\theta,L}| \leq C\lambda, \quad \text{in} \quad G.$$

Taking $X \in \Gamma_1 \cap G$, $X' \in (\Gamma_2 \cup I_{h_2}) \cap G$ with $|XF| = |X'F|$, $F = (g_{1,\lambda,\theta,L}(h_1), h_1)$, we have

$$m_0 = |\psi_{\lambda,\theta,L}(X) - \psi_{\lambda,\theta,L}(\bar{X})| \leq |D\psi_{\lambda,\theta,L}||X - \bar{X}| \leq C\lambda,$$

which is impossible for sufficiently small $\lambda > 0$. □

The uniform boundedness of the set Σ_L can be obtained for any $L > 0$ in the following.

LEMMA 2.17. *For any sufficiently large $\lambda > 0$, the free boundary Γ_1 starts on $\{(x, h_1) \mid x \geq a\}$, namely, $g_{1,\lambda,\theta,L}(h_1) \geq a$.*

Proof. Suppose that the free boundary Γ_1 starts on I_a for sufficiently large $\lambda > 0$, namely, $k_{1,\lambda,\theta,L}(a) > h_1$. For any $X_0 = (x_0, y_0) \in \Gamma_1$ with $y_0 > h_1$ and $x_0 < 0$, and a large fixed c, it follows from the continuity of $\psi_{\lambda,\theta,L}$ that there exists an $r = \frac{c}{\sqrt{\lambda}}$, such that $y_0 - 2r > h_1$ and $x_0 - 2r > 0$. Furthermore, $\psi_{\lambda,\theta,L} > 0$ in $B_r(X_0)$ for sufficiently large λ. Then we have

$$\frac{1}{r}\fint_{\partial B_r(X^0)} m_0 - \psi_{\lambda,\theta,L} dS \leq \frac{m_0}{r} = \frac{m_0}{c\sqrt{\lambda}}\lambda \leq c^*\lambda \quad \text{for sufficiently large } \lambda.$$

It follows from Lemma 7.8 that $\psi_{\lambda,\theta,L} \equiv m_0$ in $B_{\frac{r}{2}}(X_0)$, which contradicts the fact that $X_0 \in \Gamma_1$. ◻

Next, we will obtain the monotonicity of the solution $\psi_{\lambda,\theta,L}$ with respect to the parameter θ.

LEMMA 2.18. *Let θ_1 and $\theta_2 \in [0, \frac{\pi}{2}]$ with $\theta_1 \geq \theta_2$; then $\psi_{\lambda,\theta_1,L}(x,y) \leq \psi_{\lambda,\theta_2,L}(x,y)$ in Ω_L, where $\psi_{\lambda,\theta_1,L}$ and $\psi_{\lambda,\theta_2,L}$ are minimizers to the truncated variational problem $(P_{\lambda,\theta_1,L})$ and $(P_{\lambda,\theta_2,L})$, respectively.*

Proof. Denote $\psi_1 = \psi_{\lambda,\theta_1,L}$ and $\psi_2 = \psi_{\lambda,\theta_2,L}$; it is easy to check that

$$\phi_1 = \min\{\psi_1, \psi_2\} \in K_L \quad \text{and} \quad \phi_2 = \max\{\psi_1, \psi_2\} \in K_L.$$

We claim that

$$J_{\lambda,\theta_1,L}(\psi_1) + J_{\lambda,\theta_2,L}(\psi_2) = J_{\lambda,\theta_1,L}(\phi_1) + J_{\lambda,\theta_2,L}(\phi_2). \tag{2.71}$$

To prove the claim (2.71), we consider the following two cases.

Case 1. If $\theta_1 = \theta_2$, the claim (2.71) can be obtained by using the similar arguments in Theorem 2.3.

Case 2. If $\theta_1 > \theta_2$, by virtue of the results in Lemma 2.5 for θ_1 and θ_2, we conclude that $\psi_1 \leq \psi_2$ in downstream. Choose a straight line $l : y = (x - \frac{a}{2}) \tan \frac{\theta_1+\theta_2}{2} + \frac{h_1+h_2}{2}$ and a point $X_0 = (x_0, y_0) \in l$. Define two straight lines $l_1 : (y - y_0)\sin\theta_1 + (x - x_0)\cos\theta_1 = 0$ and $l_2 : (y - y_0)\sin\theta_2 + (x - x_0)\cos\theta_2 = 0$. Set $E_1 = \{(x, y) \mid (y-y_0)\sin\theta_1+(x-x_0)\cos\theta_1 \leq 0\}$ and $E_2 = \{(x, y) \mid (y-y_0)\sin\theta_2+(x-x_0)\cos\theta_2 \leq 0\}$ for sufficiently large $|X_0|$. It follows from Lemmas 2.5, 2.10, and 2.11 that

$$(\{0 < \psi_1 < m_0\} \cap G_0) \cap (\{0 < \psi_2 < m_0\} \cap G_0) = \varnothing \tag{2.72}$$

and

$$(\{0 < \psi_i < m_0\} \cap G \text{ is bounded and has positive distance to } l_j \tag{2.73}$$

for $i = 1, j = 2$ or $i = 2, j = 1$, where $G = \Omega_L \cap E_1 \cap E_2$ and $G_0 = \Omega_L \setminus G$.

It follows from (2.72) that

$$\phi_1 = \psi_1 \quad \text{and} \quad \phi_2 = \psi_2,$$

which implies that

(2.74)

$$\int_{G_0}\left|\nabla\phi_1-\lambda e_1\chi_{\{0<\phi_1<m_0\}}\right|^2dxdy+\int_{G_0}\left|\nabla\phi_2-\lambda e_2\chi_{\{0<\phi_2<m_0\}}\right|^2dxdy$$
$$=\int_{G_0}\left|\nabla\psi_1-\lambda e_1\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy+\int_{G_0}\left|\nabla\psi_2-\lambda e_2\chi_{\{0<\psi_2<m_0\}}\right|^2dxdy,$$

where $e_1=(-\sin\theta_1,\cos\theta_1)$ and $e_2=(-\sin\theta_2,\cos\theta_2)$.

Next, we will show

(2.75)

$$\int_{G}\left|\nabla\phi_1-\lambda e_1\chi_{\{0<\phi_1<m_0\}}\right|^2dxdy+\int_{G}\left|\nabla\phi_2-\lambda e_2\chi_{\{0<\phi_2<m_0\}}\right|^2dxdy$$
$$=\int_{G}\left|\nabla\psi_1-\lambda e_1\chi_{\{0<\psi_1<m_0\}}\right|^2dxdy+\int_{G}\left|\nabla\psi_2-\lambda e_2\chi_{\{0<\psi_2<m_0\}}\right|^2dxdy.$$

To obtain (2.75), it suffices to verify the following equalities:

$$\int_G|\nabla\psi_1|^2+|\nabla\psi_2|^2\,dxdy=\int_{\Omega_G}|\nabla\phi_1|^2+|\nabla\phi_2|^2\,dxdy, \tag{2.76}$$

and

$$\int_G\chi_{\{0<\psi_1<m_0\}}+\chi_{\{0<\psi_2<m_0\}}dxdy=\int_G\chi_{\{0<\phi_1<m_0\}}+\chi_{\{0<\phi_2<m_0\}}dxdy, \tag{2.77}$$

and

$$\begin{aligned}&\int_G\nabla\psi_1\cdot e_1\chi_{\{0<\psi_1<m_0\}}+\nabla\psi_2\cdot e_2\chi_{\{0<\psi_2<m_0\}}dxdy\\&\quad=\int_G\nabla\phi_1\cdot e_1\chi_{\{0<\phi_1<m_0\}}+\nabla\phi_2\cdot e_2\chi_{\{0<\phi_2<m_0\}}dxdy.\end{aligned} \tag{2.78}$$

The equalities (2.76) and (2.77) can be verified by using similar arguments in Theorem 2.3.

The facts (2.72) and (2.73) give that

$$\phi_1=\psi_1\ \text{ and }\ \phi_2=\psi_2\ \text{ on }\ \partial G. \tag{2.79}$$

Note that

$$\int_G\nabla\psi_i\cdot e_i\chi_{\{0<\psi_i<m_0\}}dxdy=\int_G\frac{\partial\psi_i}{\partial e_i}dxdy=\int_{\partial G}\Psi_i dS \tag{2.80}$$

and

$$\int_G\nabla\phi_i\cdot e_i\chi_{\{0<\phi_i<m_0\}}dxdy=\int_G\frac{\partial\phi_i}{\partial e_i}dxdy=\int_{\partial G}\Phi_i dS \tag{2.81}$$

for $i=1,2$. It follows from (2.79) that

$$\int_{\partial G}\Psi_i dS=\int_{\partial G}\Phi_i dS\ \text{ for }\ i=1,2,$$

which together with (2.80) and (2.81) give that (2.78) is valid.

The equalities (2.74) and (2.75) yield (2.71), and it follows from (2.71) that

$$J_{\lambda,\theta_1,L}(\psi_1) = J_{\lambda,\theta_1,L}(\phi_1) \quad \text{and} \quad J_{\lambda,\theta_2,L}(\psi_2) = J_{\lambda,\theta_2,L}(\phi_2). \tag{2.82}$$

In view of $\psi_1(x,y) \le \psi_2(x,y)$ in downstream, we can conclude that $\psi_1(x,y) \le \psi_2(x,y)$ in Ω_L along arguments similar to those in Theorem 2.3. ∎

Set

$$\lambda_L = \sup_{\lambda\in\Sigma_L} \lambda. \tag{2.83}$$

Now, we are going to show that for the parameter λ_L, we can find a θ_L such that $\psi_{\lambda_L,\theta_L,L}$ satisfy the continuous fit conditions (2.70).

LEMMA 2.19. *There exists a $\theta_L \in (0,\frac{\pi}{2})$ such that*

$$k_{1,\lambda_L,\theta_L,L}(a) = h_1 \quad \textit{and} \quad g_{2,\lambda_L,\theta_L,L}(h_2) = 0.$$

Proof. Step 1. Taking a sequence $\{\lambda_n\} \in \Sigma_L$ with $\lambda_n \uparrow \lambda_L$, it follows from the definition of λ_L in (2.83) that there exist two subsequences still labeled by $\{\lambda_n\}$ and $\{\theta_n\}$ such that

$$\lambda_n \to \lambda_L,\ \theta_n \to \theta_L,\ \psi_{\lambda_n,\theta_n,L} \rightharpoonup \psi_{\lambda_L,\theta_L,L} \ \text{ in } H^1_{loc}(\Omega_L) \text{ and a.e. in } \Omega_L,$$

as $n \to +\infty$.

By virtue of the similar arguments in Proposition 2.15, we can conclude that

$$\lim_{n\to+\infty} k_{1,\lambda_n,\theta_n,L}(x) = k_{1,\lambda_L,\theta_L,L}(x) \text{ for } x \in [a,+\infty),$$

and

$$\lim_{n\to+\infty} g_{2,\lambda_n,\theta_n,L}(y) = g_{2,\lambda_L,\theta_L,L}(y) \text{ for} y \in [h_2,+\infty), \tag{2.84}$$

which implies that

$$\lim_{n\to+\infty} k_{1,\lambda_n,\theta_n,L}(a) = k_{1,\lambda_L,\theta_L,L}(a) = \alpha \quad \text{and} \quad \lim_{n\to+\infty} g_{1,\lambda_n,\theta_n,L}(h_2) = g_{2,\lambda_L,\theta_L,L}(h_2) = \beta.$$

Furthermore, it follows from the definition of λ_L in (2.83) that

$$\alpha \ge h_1 \quad \text{and} \quad \beta \ge 0. \tag{2.85}$$

Step 2. In this step, we claim that

$$\theta_L \in \left(0, \frac{\pi}{2}\right).$$

In fact, suppose for instance that $\theta_L = 0$; then Lemma 2.11 gives that at least one of Γ_1 and Γ_2 is empty. We next consider the following two cases.

Case 1. Γ_2 is empty. Then we have that $g_{2,\lambda_L,\theta_L,L}(h_2) = +\infty$. Taking any $\theta > 0 = \theta_L$, Lemma 2.18 gives that

$$\psi_{\lambda_L,\theta,L} \le \psi_{\lambda_L,\theta_L,L} \ \text{ in } \Omega_L, \text{ and } k_{1,\lambda_L,\theta,L}(x) \ge k_{1,\lambda_L,\theta_L,L}(x) \ \text{ for } x \in [a,+\infty).$$

We first claim that

$$k_{1,\lambda_L,\theta,L}(x) > k_{1,\lambda_L,\theta_L,L}(x) \quad \text{for any } x \in (a,+\infty).$$

Suppose not; there exists an $x_0 \in (a,+\infty)$ such that $k_{1,\lambda_L,\theta,L}(x_0) = k_{1,\lambda_L,\theta_L,L}(x_0)$, the strong maximum principle gives that $\psi_{\lambda_L,\theta,L} < \psi_{\lambda_L,\theta_L,L}$ in $\Omega_L \cap \{0 < \psi_{\lambda_L,\theta,L} < m_0\}$, and Hopf's lemma gives that

$$\lambda_L = \frac{\partial \psi_{\lambda_L,\theta,L}}{\partial \nu} > \frac{\partial \psi_{\lambda_L,\theta_L,L}}{\partial \nu} = \lambda_L \quad \text{at } (x_0, k_{1,\lambda_L,\theta,L}(x_0)),$$

which leads to a contradiction.

We next claim that

$$k_{1,\lambda_L,\theta,L}(a) > k_{1,\lambda_L,\theta_L,L}(a). \tag{2.86}$$

Suppose that $k_{1,\lambda_L,\theta,L}(a) = k_{1,\lambda_L,\theta_L,L}(a)$; here the inner ball property at $P = (a, k_{1,\lambda_L,\theta,L}(a))$ which is needed for the previous arguments may be invalid, and thus Hopf's lemma cannot be applied directly. For a small ball $B_r(P)$, the continuity of $\psi_{\lambda_L,\theta,L}$ and $\psi_{\lambda_L,\theta_L,L}$ gives that

$$\psi_{\lambda_L,\theta,L} > 0 \quad \text{and} \quad \psi_{\lambda_L,\theta_L,L} > 0 \quad \text{in } B_r(P).$$

Denote $M = B_r(P) \cap \overline{\{0 < \psi_{\lambda_L,\theta,L} < m_0\}} \cap (N_1 \cup I_a)$; by using the strong maximum principle and Hopf's lemma, we have

$$\psi_{\lambda_L,\theta,L} < \psi_{\lambda_L,\theta_L,L} \text{ in } B_r(P) \cap \{\psi_{\lambda_L,\theta_L,L} < m_0\}, \quad \frac{\partial \psi_{\lambda_L,\theta,L}}{\partial \nu} > \frac{\partial \psi_{\lambda_L,\theta_L,L}}{\partial \nu} \quad \text{at } M.$$

This implies that there exists a small $\varepsilon > 0$ such that

$$m_0 - \psi_{\lambda_L,\theta,L} \geq (1+\varepsilon)(m_0 - \psi_{\lambda_L,\theta_L,L}) \quad \text{on } \partial B_r(P) \cap \{\psi_{\lambda_L,\theta_L,L} < m_0\}.$$

And then, it follows from maximum principle that

$$m_0 - \psi_{\lambda_L,\theta,L} > (1+\varepsilon)(m_0 - \psi_{\lambda_L,\theta_L,L}) \quad \text{in } B_r(P) \cap \{\psi_{\lambda_L,\theta_L,L} < m_0\}.$$

By similar arguments in the proof of Corollary 11.5 in Chapter 3 in [26] (see also the results in section 9 in [4]), we have

$$-\lambda_L = \frac{\partial(m_0 - \psi_{\lambda_L,\theta,L})}{\partial \nu} \leq (1+\varepsilon)\frac{\partial(m_0 - \psi_{\lambda_L,\theta_L,L})}{\partial \nu} = -(1+\varepsilon)\lambda_L \quad \text{at } P,$$

which leads to a contradiction.

Due to the continuity of the free boundaries with the parameters λ and θ in Proposition 2.15, it follows from $g_{2,\lambda_L,\theta_L,L}(h_2) = +\infty$ that

$$g_{2,\lambda_L,\tilde{\theta},L}(h_2) > 1 \quad \text{for sufficiently small } \tilde{\theta} > 0 = \theta_L. \tag{2.87}$$

Employing the continuity of the free boundaries with parameters λ and θ in Proposition 2.15 again, the inequalities (2.86) and (2.87) yield that

$$g_{2,\tilde{\lambda},\tilde{\theta},L}(h_2) > 0 \text{ and } k_{1,\tilde{\lambda},\tilde{\theta},L}(a) > k_{1,\lambda_L,\theta_L,L}(a) \geq h_1$$

for $\tilde{\lambda} > \lambda_L$, where $\tilde{\theta} > 0 = \theta_L$ is small enough and $\tilde{\lambda} - \lambda_L$ is sufficiently small, which leads to a contradiction to the definition λ_L in (2.83).

Case 2. Γ_1 is empty. Then $h_1 > h_2$ and the segment $l = \{(x, h_1) \mid x > a\}$ is the free boundary of $\psi_{\lambda_L,\theta_L,L}$. It follows from the convergence of the free boundary in Theorem 3.1 in [4] that

$$\frac{\partial\psi_{\lambda_L,\theta_L,L}}{\partial y} = \frac{\partial\psi_{\lambda_L,\theta_L,L}}{\partial\nu} = \lambda_L \quad \text{on} \quad l.$$

We can obtain a contradiction by using the Cauchy–Kovalevskaya theorem.

Step 3. Next, we will show that

$$\alpha = h_1 \quad \text{and} \quad \beta = 0. \tag{2.88}$$

Without loss of generality, we assume that $\alpha > h_1$. Along the previous arguments, one has

$$g_{2,\lambda_L,\tilde{\theta},L}(h_2) > g_{2,\lambda_L,\theta_L,L}(h_2) \geq 0 \quad \text{for } 0 < \tilde{\theta} < \theta_L \text{ and } \theta_L - \tilde{\theta} \text{ being sufficiently small.}$$

It follows from the continuity of the free boundaries with the parameters λ and θ in Proposition 2.15 that

$$k_{1,\lambda_L,\tilde{\theta},L}(a) > h_1 \quad \text{for } 0 < \tilde{\theta} < \theta_L \text{ and } \theta_L - \tilde{\theta} \text{ being sufficiently small.}$$

Similarly, taking $\tilde{\lambda} > \lambda_L$ and $0 < \tilde{\theta} < \theta_L$, where $\tilde{\lambda} - \lambda_L$ is sufficiently small and $\theta_L - \tilde{\theta}$ is sufficiently small, one has

$$g_{2,\tilde{\lambda},\tilde{\theta},L}(h_2) > 0 \text{ and } k_{1,\tilde{\lambda},\tilde{\theta},L}(a) > h_1,$$

which leads to a contradiction to the definition λ_L in (2.83).

Hence, we obtain that

$$\lim_{n\to+\infty} k_{1,\lambda_n,\theta_n,L}(a) = k_{1,\lambda_L,\theta_L,L}(a) = h_1 \text{ and}$$
$$\lim_{n\to+\infty} g_{2,\lambda_n,\theta_n,L}(h_2) = g_{1,\lambda_L,\theta_L,L}(h_2) = 0. \qquad \square$$

Lemma 2.19 also implies that the deflection angle of the Réthy flow satisfies the estimate (1.10).

3. The existence of the Réthy flow. In this section, we will give the existence of the Réthy flow problem based on the previous results. Let $\psi_{\lambda,\theta}$ be a local minimizer to the variational problem $(P_{\lambda,\theta})$; that is,

$$J_D(\psi_{\lambda,\theta}) \leq J_D(\psi) \quad \text{for any } \psi \in K, \quad \psi = \psi_{\lambda,\theta} \quad \text{on } \partial D, \tag{3.1}$$

for any bounded domain $D \subset \Omega$ with smooth boundary, where the functional $J_D(\psi) = \int_D |\nabla\psi - \lambda e\chi_{\{0<\psi<m_0\}}|^2 dxdy$, $e = (-\sin\theta, \cos\theta)$ with $\theta \in [0, \frac{\pi}{2}]$.

THEOREM 3.1. *Taking a sequence $\{L_n\}$ with $L_n \to +\infty$, there exist two corresponding subsequences $\{\lambda_{L_n}\}$ and $\{\theta_{L_n}\}$ as $n \to +\infty$, such that $\psi_{\lambda_{L_n},\theta_{L_n},L_n}$ is a minimizer to the variational problem $(P_{\lambda_{L_n},\theta_{L_n},L_n})$ and $k_{1,\lambda_{L_n},\theta_{L_n},L_n}(a) = h_1$ and $g_{2,\lambda_{L_n},\theta_{L_n},L_n}(h_2) = 0$. Then there exists a subsequence still labeled by $\{\psi_{\lambda_{L_n},\theta_{L_n},L_n}\}$ such that*

$$\psi_{\lambda_{L_n},\theta_{L_n},L_n} \to \psi_{\lambda,\theta} \text{ weakly in } H^1_{loc}(\Omega) \text{ and uniformly in any compact subset of } \Omega,$$

and

$$k_{1,\lambda,\theta}(a) = \lim_{n\to+\infty} k_{1,\lambda_{L_n},\theta_{L_n},L_n}(a) = h_1 \text{ and } g_{2,\lambda,\theta}(h_2) = \lim_{n\to+\infty} g_{2,\lambda_{L_n},\theta_{L_n},L_n}(h_2) = 0,$$

where $\psi_{\lambda,\theta}$ is a local minimizer to the variational problem $(P_{\lambda,\theta})$, and $y = k_{1,\lambda,\theta}(x)$ and $x = g_{2,\lambda,\theta}(y)$ are the free boundaries of $\psi_{\lambda,\theta}$, respectively.

Proof. For any bounded domain $D \subset \Omega$ with smooth boundary, there exists a subsequence still labeled by $\{L_n\}$ with $L_n \to +\infty$ as $n \to +\infty$, such that $D \subset \Omega_{L_n}$. It follows from similar arguments in Proposition 7.2 that there exists a subsequence still labeled by $\{\psi_n\}$ with $\psi_n = \psi_{\lambda_{L_n}}, \theta_{L_n}, L_n$ and $\lambda_n = \lambda_{L_n}$ such that

$$\psi_n \to \psi_{\lambda,\theta} \text{ weakly in } H^1_{loc}(\Omega) \text{ and uniformly in any compact subset of } \Omega,$$

and $\psi_n \to \psi_{\lambda,\theta}$ in the trace sense of ∂D.

Next, we will show that $\psi_{\lambda,\theta}$ is a local minimizer to the variational problem $(P_{\lambda,\theta})$. We first claim that

$$\int_D \left|\nabla\psi_n - \lambda_n e_n \chi_{\{0<\psi_n<m_0\}}\right|^2 dxdy \le \int_D \left|\nabla\tilde{\psi} - \lambda_n e_n \chi_{\{0<\tilde{\psi}<m_0\}}\right|^2 dxdy \tag{3.2}$$

for any $\tilde{\psi} \in K$ and $\tilde{\psi} = \psi_n$ on ∂D. In fact, extend $\tilde{\psi} = \psi_n$ outside of D such that $\tilde{\psi} = \Psi_{L_n}(y)$ on σ_{L_n}; then it implies that (3.2) is valid.

For any $\psi \in H^1(D)$ with $\psi = \psi_{\lambda,\theta}$ on ∂D, set

$$\tilde{\psi}_n = \psi + (1-\eta)(\psi_n - \psi_{\lambda,\theta}),$$

where $\eta \in C^1_0(D)$ and $0 \le \eta \le 1$. Obviously, $\tilde{\psi}_n \in K$ and $\tilde{\psi}_n = \psi_n$ on ∂D.

It follows from (2.87) that

$$\int_D \left|\nabla\psi_n - \lambda_n e_n \chi_{\{0<\psi_n<m_0\}}\right|^2 dxdy \le \int_D \left|\nabla\tilde{\psi}_n - \lambda_n e_n \chi_{\{0<\tilde{\psi}_n<m_0\}}\right|^2 dxdy. \tag{3.3}$$

Taking $n \to \infty$ and using the similar arguments in Step 4 in Proposition 2.14, one has

$$\int_D \left|\nabla\psi_{\lambda,\theta} - \lambda e \chi_{\{0<\psi_{\lambda,\theta}<m_0\}}\right|^2 dxdy \le \int_D \left|\nabla\psi - \lambda e \chi_{\{0<\psi<m_0\}}\right|^2 dxdy$$

for any $\psi \in H^1(D)$ with $\psi = \psi_{\lambda,\theta}$ on ∂D.

Furthermore, the local minimizer $\psi_{\lambda,\theta}$ is decreasing with respect to x and increasing with respect to y.

It follows from similar arguments in Proposition 2.15 that

$$k_{1,\lambda_{L_n},\theta_{L_n},L_n}(x) \to k_{1,\lambda,\theta}(x) \quad \text{for any} \quad x \in [a,+\infty),$$

and

$$g_{1,\lambda_{2n},\theta_{L_n},L_n}(y) \to g_{2,\lambda,\theta}(y) \quad \text{for any} \quad y \in [h_2,+\infty),$$

as $L_n \to +\infty$, and consequently,

$$k_{1,\lambda,\theta}(a) = h_1 \quad \text{and } g_{2,\lambda,\theta}(h_2) = 0.$$ □

Thanks to Theorem 3.1, we conclude that the continuous fit conditions are fulfilled for $\psi_{\lambda,\theta}$; the remaining smoothness assertions near the detachment points A_1 and A_2 follow immediately from Theorem 9.1 in [4] or Theorem 7.9 in [2]. Thus we can establish a solution $\psi_{\lambda,\theta}$ of the Réthy flow problem.

THEOREM 3.2. *The local minimizer $\psi_{\lambda,\theta}$ to the variational problem $(P_{\lambda,\theta})$ solves the Dirichlet problem* (2.6)*, satisfies the continuous fit conditions and smooth fit conditions, and furthermore, satisfies $\psi_{\lambda,\theta} \in C^{2,\alpha}$ in any compact subset of Ω_0, where Ω_0 is bounded by $N_1, N_2, I_b, \Gamma_1,$ and Γ_2.*

Proof. Taking any $\xi \in C_0^1(D)$ for any bounded open set $D \subset \Omega_0$, then $\psi_\lambda + \varepsilon\xi \in H^1(D)$. As in the similar arguments in Proposition 7.4 in the appendix, one has

$$0 = \lim_{\varepsilon \to 0} \frac{J_D(\psi_{\lambda,\theta} + \varepsilon\xi) - J_D(\psi_{\lambda,\theta})}{\varepsilon} = 2\int_D \nabla\psi_{\lambda,\theta} \cdot \nabla\xi dxdy,$$

which gives that $\Delta\psi_{\lambda,\theta} = 0$ in Ω_0 in the weak sense. It follows from Theorem 3.1 that $\psi_{\lambda,\theta}$ satisfies the boundary condition in the trace sense.

The smoothness near the detachment points A_1 and A_2 implies that $N_1 \cup \Gamma_1$ and $N_2 \cup \Gamma_2$ are C^1. By using the similar arguments in Proposition 7.4 in the appendix, we can conclude that $|\frac{\partial\psi_{\lambda,\theta}}{\partial\nu}| = |\nabla\psi_{\lambda,\theta}| = \lambda$ on $\Gamma_1 \cup \Gamma_2$, where ν is the outer normal vector to $\Gamma_1 \cup \Gamma_2$.

Furthermore, the standard interior Schauder estimates to the linear elliptic equation in Chapter 8 in [29] imply that

$$\psi_{\lambda,\theta} \in C^{2,\alpha}(D) \text{ for any bounded open set } D \subset \Omega \cap \{0 < \psi_{\lambda,\theta} < m_0\}. \qquad \square$$

Finally, we will obtain the positivity of the velocity in the fluid field.

PROPOSITION 3.3. *The horizontal velocity u and vertical velocity v satisfy*

$$u = \frac{\partial\psi_{\lambda,\theta}}{\partial y} > 0 \quad in \quad \overline{\Omega_0} \setminus B \quad and \quad v = -\frac{\partial\psi_{\lambda,\theta}}{\partial x} > 0 \quad in \quad \Omega_0. \tag{3.4}$$

Proof. By virtue of the monotonicity of $\psi_{\lambda,\theta}$ with respect to x and y, one has

$$\psi_{\lambda,\theta}(x_1, y) \le \psi_{\lambda,\theta}(x_2, y) \text{ for } x_1 > x_2, \text{ and } \psi_{\lambda,\theta}(x, y_1) \ge \psi_{\lambda,\theta}(x, y_2) \text{ for } y_1 > y_2,$$

which implies $\frac{\partial\psi_{\lambda,\theta}}{\partial y} \ge 0$ and $\frac{\partial\psi_{\lambda,\theta}}{\partial x} \le 0$ in Ω_0.

Consider $u = \frac{\partial\psi_{\lambda,\theta}}{\partial y}$ and $v = -\frac{\partial\psi_{\lambda,\theta}}{\partial x}$ in Ω_0, which still solves the Laplace equation

$$\Delta u = 0 \text{ and } \Delta v = 0 \text{ in } \Omega_0.$$

First, we claim that $u(x, y) > 0$ and $v(x, y) > 0$ in any compact subset of Ω_0.

For any compact subset $D \subset\subset \Omega_0$ with smooth boundary, we have

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = \dfrac{\partial\psi_{\lambda,\theta}}{\partial y} \ge 0 & \text{on } \partial D, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \Delta v = 0 & \text{in } D, \\ v = -\dfrac{\partial\psi_{\lambda,\theta}}{\partial x} \ge 0 & \text{on } \partial D. \end{cases} \tag{3.6}$$

The maximum principle gives that $u(x, y) > 0$ and $v(x, y) > 0$ in D.

Second, on the boundary, noticing that $\psi_{\lambda,\theta} = m_0$ along $N_1 \cup \Gamma_1$, the slip boundary condition (1.2) implies that

$$\partial_x\psi_{\lambda,\theta}(x, \tilde{f}_1(x)) + \partial_y\psi_{\lambda,\theta}(x, \tilde{f}_1(x))\tilde{f}_1'(x) = 0,$$

where $\tilde{f}_1(x) \in C^1$ and

$$\tilde{f}_1(x) = \begin{cases} f_1(x), & (x, f_1(x)) \in N_1, \\ k_{1,\lambda,\theta}(x), & (x, k_{\lambda,\theta}(x)) \in \Gamma_1. \end{cases}$$

Therefore, the outer normal derivative satisfies

$$\frac{\partial \psi_{\lambda,\theta}}{\partial \nu}\left(x, \tilde{f}_1(x)\right) = \partial_y \psi_{\lambda,\theta}\left(x, \tilde{f}_1(x)\right)\sqrt{1 + (\tilde{f}_1'(x))^2}.$$

On the other hand, $\psi_{\lambda,\theta}$ attains its maximum on $N_1 \cup \Gamma_1$, and thus it follows from Hopf's lemma that

$$u = \frac{\partial \psi_{\lambda,\theta}(x,y)}{\partial y} > 0 \quad \text{on} \quad (N_1 \cup \Gamma_1) \setminus A_1.$$

In view of the boundedness of $f'(x)$ at A_1, it follows from the continuity of u at A_1 that $u > 0$ at A_1.

Similarly, we can conclude that

$$u = \frac{\partial \psi_{\lambda,\theta}(x,y)}{\partial y} > 0 \quad \text{on} \quad (I_b \cup N_2) \setminus B.$$

Hence, we obtain the positivity of the horizontal velocity in $\overline{\Omega_0} \setminus B$ and the positivity of the vertical velocity in Ω_0. ∎

4. The uniqueness of the Réthy flow. In this section, we will consider the uniqueness of the solution to the Réthy flow problem and show Theorem 1.6.

Suppose that there exist two minimizers $\psi_{\lambda,\theta}$ and $\tilde{\psi}_{\lambda,\theta}$ to the variational problem $(P_{\lambda,\theta})$, such that

$$f_1(a) = k_{1,\lambda,\theta}(a) = \tilde{k}_{1,\lambda,\theta}(a) \text{ and } f_2(0) = k_{2,\lambda,\theta}(0) = \tilde{k}_{2,\lambda,\theta}(0). \tag{4.1}$$

Without loss of generality, we assume that $\lim_{x\to+\infty}(k_{1,\lambda,\theta}(x) - \tilde{k}_{1,\lambda,\theta}(x)) \geq 0$. Define $\psi^{\varepsilon}_{\lambda,\theta}(x,y) = \psi_{\lambda,\theta}(x, y-\varepsilon)$; then there exists an $\varepsilon_0 \geq 0$ such that

$$\psi^{\varepsilon_0}_{\lambda,\theta} \leq \tilde{\psi}_{\lambda,\theta} \text{ in } \Omega, \text{ and } \psi^{\varepsilon_0}_{\lambda,\theta}(X_0) = \tilde{\psi}_{\lambda,\theta}(X_0) \text{ for some } X_0 \in \bar{\Omega}.$$

We consider the following cases for ε_0.

Case 1. $\varepsilon_0 > 0$. We conclude that $X_0 \notin \{0 < \psi_{\lambda,\theta} < m_0\}$ by using the similar arguments in Lemma 2.9. Then X_0 is the free boundary point of $\psi^{\varepsilon_0}_{\lambda,\theta}$ and $\tilde{\psi}_{\lambda,\theta}$. Furthermore, $|X_0| < +\infty$. It follows from Hopf's lemma that

$$\lambda = \frac{\partial \psi^{\varepsilon_0}_{\lambda,\theta}}{\partial \nu} > \frac{\partial \tilde{\psi}_{\lambda,\theta}}{\partial \nu} = \lambda \quad \text{at } X_0 \in \Gamma^{\varepsilon_0}_1 \cap \tilde{\Gamma}_1,$$

or

$$-\lambda = \frac{\partial \psi^{\varepsilon_0}_{\lambda,\theta}}{\partial \nu} > \frac{\partial \tilde{\psi}_{\lambda,\theta}}{\partial \nu} = -\lambda \quad \text{at } X_0 \in \Gamma^{\varepsilon_0}_2 \cap \tilde{\Gamma}_2,$$

where $\Gamma^{\varepsilon_0}_i$ and $\tilde{\Gamma}_i$ $(i = 1, 2)$ are free boundaries of $\psi^{\varepsilon_0}_{\lambda,\theta}$ and $\tilde{\psi}_{\lambda,\theta}$, respectively. Those lead to a contradiction.

Case 2. $\varepsilon_0 = 0$. It follows from (4.1) that we can choose $X_0 = A_1$. By virtue of the proof of (2.86) in Lemma 2.19, we can take a small $\tau > 0$, such that

$$-\lambda = \frac{\partial(m_0 - \psi^{\varepsilon_0}_{\lambda,\theta})}{\partial\nu} \leq (1+\tau)\frac{\partial(m_0 - \tilde{\psi}_{\lambda,\theta})}{\partial\nu} = -(1+\tau)\lambda \quad \text{at } A_1,$$

which leads to a contradiction. Hence, we obtain the uniqueness of the local minimizer $\psi_{\lambda,\theta}$, which implies the uniqueness of the solution $(u, v, p, \Gamma_1, \Gamma_2)$ of Réthy flow problem immediately.

Next, we will show the second assertion of Theorem 1.6. Without loss of generality, we assume $\theta_1 > \theta_2$. It follows from similar arguments in Lemma 2.18 that

$$\psi_{\lambda,\theta_1}(x,y) \leq \psi_{\lambda,\theta_2}(x,y) \quad \text{in } \Omega.$$

The proof of (2.86) in Lemma 2.19 gives that

$$k_{1,\lambda,\theta_1}(a) > k_{1,\lambda,\theta_2}(a),$$

which leads to a contradiction to the continuous fit assumption, namely, $k_{1,\lambda,\theta_1}(a) = k_{1,\lambda,\theta_2}(a) = h_1$. Hence, $\theta_1 = \theta_2$.

5. The deflection angle estimate. In this section, we will give the estimate of the deflection angle for the special case $h_1 < h_2$ and complete the proof of Theorem 1.7.

Suppose that $\theta \leq \theta_0$; we first claim that

$$f_2'(0) \geq \tan\theta_0. \tag{5.1}$$

In fact, suppose that $f_2'(0) < \tan\theta_0$, and then it follows from the smooth fit condition (1.9) that there exists a small $r > 0$, such that

$$\tilde{f}_2'(x) < \tan\theta_0 \quad \text{for} \quad -r < x < r,$$

where

$$\tilde{f}_2(x) = \begin{cases} f_2(x), & -r < x \leq 0, \\ k_{2,\lambda,\theta}(x), & 0 \leq x < r. \end{cases}$$

Set $g(x) = f_2(x) - x\tan\theta_0 - h_2$ and $g(0) = 0$; we have

$$g(x) > g(0) = 0 \quad \text{for} \quad -r < x < 0,$$

which contradicts our assumption that the obstacle N_2 lies below $\overline{A_1A_2}$.

Now we are going to show the impossibility of $\theta \leq \theta_0$ for the following cases.

Case 1. $k_{2,\lambda,\theta}(x) \leq x\tan\theta_0 + h_2$ for any $x > 0$. We first claim that

$$f_2'(0) = \tan\theta_0. \tag{5.2}$$

In fact, in view of (5.1), we suppose that $f_2'(0) > \tan\theta_0$. It follows from the smooth fit conditions (1.9) and that there exists a small $r > 0$, such that

$$\tilde{f}_2'(x) > \tan\theta_0 \quad \text{for} \quad -r < x < r,$$

which implies that

$$k_{2,\lambda,\theta}(x) - x\tan\theta_0 - h_2 = g(x) > g(0) = 0 \quad \text{for} \quad 0 < x < r.$$

This contradicts our assumption in this case.

Define a constant flow $S_{\theta_0}(x, y; 0, h_2)$ as in (2.33). Set $S_\tau(x, y) = S_{\theta_0}(x, y - \tau; 0, h_2)$ for $\tau > 0$, and choose the smallest τ_0 such that

$$\text{(5.3)} \quad \psi_{\lambda,\theta}(X) \geq S_{\tau_0}(X) \quad \text{in } \Omega\backslash D_1, \text{ and } \psi_{\lambda,\theta}(X_0) = S_{\tau_0}(X_0) \text{ for some } X_0 \in \overline{\Omega \setminus D_1}.$$

Furthermore, we consider the following two subcases for τ_0.

Case 1.1. If $\tau_0 = 0$, we can take $X_0 = A_2$; the smooth fit conditions (1.9) and the claim (5.2) give that

$$\frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = -\lambda \quad \text{at} \quad A_2, \tag{5.4}$$

where $\nu = (\sin\theta_0, -\cos\theta_0)$ is the outer normal vector to $N_2 \cup \Gamma_2$ at A_2. In view of (5.3) and the smooth fit conditions (1.9), it is easy to check that the domain $\{S_{\theta_0}(x, y; 0, h_2) > 0\}$ satisfies the inner ball condition at A_2. Then, it follows from Hopf's lemma that

$$-\lambda = \frac{\partial S_{\theta_0}}{\partial \nu} > \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = -\lambda \quad \text{at} \quad A_2,$$

which leads to a contradiction.

Case 1.2. If $\tau_0 > 0$, we conclude that $X_0 \notin \{0 < \psi_{\lambda,\theta} < m_0\}$ along the similar arguments in Lemma 2.9, and then $X_0 \in \Gamma_1 \cap \partial\{S_{\tau_0} < m_0\}$. We claim that $|X_0| \leq M$. In fact, the width of the constant flow $S_{\tau_0}(x, y)$ is equal to the asymptotic width of Réthy flow in downstream, which together with our assumption in this case gives that

$$x\tan\theta_0 + h_2 + \frac{m_0}{\lambda\cos\theta_0} + \tau_0 > k_{1,\lambda,\theta}(x) \quad \text{for large } x.$$

This implies that $|X_0| < +\infty$, and then Hopf's lemma gives

$$\lambda = \frac{\partial S_{\tau_0}}{\partial \nu} > \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = \lambda \quad \text{at} \quad X_0 \in \Gamma_1 \cap \partial\{S_{\tau_0} < m_0\},$$

which leads to a contradiction.

Finally, we consider the second case as follows.

Case 2. $k_{2,\lambda,\theta}(x) > x\tan\theta_0 + h_2$ for some $x > 0$. We can choose a straight line $l_\varepsilon : y = x\tan\theta_\varepsilon + \beta_\varepsilon$ for $\theta_\varepsilon > \theta_0$ with sufficiently small $\theta_\varepsilon - \theta_0$, such that

(5.5)
l_ε is tangent to the free boundary Γ_2 at X_ε and stays above it, and $a\tan\theta_\varepsilon + \beta_\varepsilon > h_1$.

Likewise, consider the constant flow $S_\tau(x, y) = S_{\theta_\varepsilon}(x, y - \tau; 0, \beta_\varepsilon)$ as in (2.33) and choose the smallest τ_0 such that

$$\text{(5.6)} \quad \psi_{\lambda,\theta}(X) \geq S_{\tau_0}(X) \quad \text{in } \Omega\backslash D_1, \text{ and } \psi_{\lambda,\theta}(X_0) = S_{\tau_0}(X_0) \text{ for some } X_0 \in \overline{\Omega \setminus D_1}.$$

There are two subcases for τ_0 as follows.

Case 2.1. If $\tau_0 = 0$, we can take $X_0 = X_\varepsilon$. It follows from Hopf's lemma that

$$-\lambda = \frac{\partial S_{\theta_\varepsilon}}{\partial \nu} > \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = -\lambda \quad \text{at} \quad X_0 = X_\varepsilon \in \Gamma_2,$$

which leads to a contradiction.

Case 2.2. If $\tau_0 > 0$, we claim that $X_0 \notin \{0 < \psi_{\lambda,\theta} < m_0\}$ by using the similar arguments in Lemma 2.9. Then $X_0 \in \Gamma_1 \cap \partial\{S_{\tau_0} < m_0\}$, and the fact $\theta_\varepsilon > \theta_0$ implies that $|X_0| < +\infty$. Thanks to Hopf's lemma, one has

$$\lambda = \frac{\partial S_{\tau_0}}{\partial \nu} > \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = \lambda \quad \text{at} \quad X_0 \in \Gamma_1 \cap \partial\{S_{\tau_0} < m_0\},$$

which also leads to a contradiction.

6. The location and asymptotic behavior of the Réthy flow. In this subsection, we study the asymptotic behavior of the Réthy flow in upstream and downstream.

First, we will obtain the asymptotic behavior in downstream.

PROPOSITION 6.1. *There exists a unique β, such that*

$$k_{1,\lambda,\theta}(x) \to x\tan\theta + \beta + \frac{m_0}{\lambda\cos\theta} \quad \text{and} \quad k_{2,\lambda,\theta}(x) \to x\tan\theta + \beta - \frac{m_0}{\lambda\cos\theta}, \quad \text{as } x \to +\infty,$$

where $\beta \in [h_2 - \frac{m_0}{2\lambda\cos\theta}, h_1 - a\tan\theta + \frac{m_0}{2\lambda\cos\theta}]$.

Proof. In view of Lemma 2.10 and Theorem 1.6, there exists a unique β such that

$$k_{1,\lambda,\theta}(x) \to x\tan\theta + \beta + \frac{m_0}{2\lambda\cos\theta} \quad \text{and} \quad k_{2,\lambda,\theta}(x) \to x\tan\theta + \beta - \frac{m_0}{2\lambda\cos\theta}, \quad \text{as } x \to +\infty.$$

Furthermore, the free boundaries of $\psi_{\lambda,\theta}(X)$ satisfy the flatness condition in section 7 in [1] for sufficiently large $|X|$.

Next, we claim that

$$h_2 - \frac{m_0}{2\lambda\cos\theta} \le \beta \le h_1 - a\tan\theta + \frac{m_0}{2\lambda\cos\theta}.$$

In fact, without loss of generality we assume that $\beta < h_2 + \frac{m_0}{2\lambda\cos\theta}$. As in similar arguments in Lemma 2.9, there exists a straight line $l_\varepsilon : y = x\tan\theta_\varepsilon + \beta_\varepsilon$ for $\theta_\varepsilon < \theta$ with sufficiently small $\theta - \theta_\varepsilon > 0$, such that

$$l_\varepsilon \text{ is tangent to the free boundary } \Gamma_1 \text{ at } X_\varepsilon \text{ and stays below it.} \tag{6.1}$$

The assumption $\beta < h_2 + \frac{m_0}{2\lambda\cos\theta}$ implies that

$$\beta_\varepsilon < h_2 \quad \text{for sufficiently small } \theta - \theta_\varepsilon > 0. \tag{6.2}$$

Let $S_\tau(x,y) = S_{\theta_\varepsilon}(x, y+\tau; \frac{m_0}{\lambda\sin\theta_\varepsilon}, \beta_\varepsilon)$ with $\tau \ge 0$ a constant flow as in (2.33), and choose the smallest $\tau_0 \ge 0$ such that

$$\psi_{\lambda,\theta}(X) \le S_{\tau_0}(X) \ \text{ in } \Omega\backslash D_1, \ \text{ and } \psi_{\lambda,\theta}(X_0) = S_{\tau_0}(X_0) \text{ for some } X_0 \in \overline{\Omega \setminus D_1}. \tag{6.3}$$

Case 1. If $\tau_0 = 0$, we can take $X_0 = X_\varepsilon$. it follows from the strong maximum principle that

$$\lambda = \frac{\partial S_{\tau_0}}{\partial \nu} < \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = \lambda \quad \text{at} \quad X_0 = X_\varepsilon \in \Gamma_1,$$

which leads to a contradiction.

Case 2. If $\tau_0 > 0$, we claim that $X_0 \notin \{0 < \psi_{\lambda,\theta} < m_0\}$ by using the similar arguments in Lemma 2.9. $\beta_\varepsilon < h_2$ in (6.2) implies that $X_0 \notin \Gamma_2 \cap \{(0, y) \mid y \geq h_2\}$. If $\Gamma_2 \cap I_{h_2}$ is nonempty, we claim that $X_0 \notin \Gamma_2 \cap I_{h_2}$. In fact, suppose $X_0 \in \Gamma_2 \cap I_{h_2}$; it follows from Theorem 9.1 in [4] or Theorem 7.9 in [2] that Γ_2 is C^1 at X_0. The previous arguments imply that $\psi_{\lambda,\theta}(X) > S_{\tau_0}(X) = 0$ in $B_r(X_0) \cap \{0 < \psi_{\lambda,\theta} < m_0\} \cap \{S_{\tau_0} = 0\}$ for small $r > 0$; this contradicts the definition of τ_0. Then $X_0 \in \Gamma_2 \cap \partial\{S_{\tau_0} > 0\}$, the fact $\theta_\varepsilon < \theta$ implies that $|X_0| < +\infty$, and it follows from Hopf's lemma that one has

$$-\lambda = \frac{\partial S_{\tau_0}}{\partial \nu} < \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = -\lambda \quad \text{at} \quad X_0 \in \Gamma_2 \cap \partial\{S_{\tau_0} > 0\},$$

which also leads to a contradiction. ∎

The asymptotic behavior in upstream will be obtained in the following.

Proposition 6.2. *The Réthy flow satisfies the following asymptotic behavior in the far fields:*

$$(u(x,y), v(x,y), p(x,y)) \to (u_0, 0, p_1) \quad \textit{and} \quad \nabla(u,v,p) \to 0$$

uniformly in any compact subset of $(0, H)$ *as* $x \to -\infty$, *where* $u_0 = \frac{m_0}{H}$ *and* $p_1 = p_0 + \frac{\lambda^2}{2} - \frac{m_0^2}{2H^2}$.

Similarly, the Réthy flow satisfies the following asymptotic behavior in the downstream:

$$(u(x,y), v(x,y), p(x,y)) \to (\lambda\cos\theta, \lambda\sin\theta, p_0) \quad \textit{and} \quad \nabla(u,v,p) \to 0$$

as $x \to +\infty$.

Proof. The proof is based on the blow-up argument, and a similar idea has been applied in the compressible subsonic flows in infinitely long nozzles in [16, 20, 21, 22, 23, 36, 37]. For the convenience of the readers, we sketch the proof as follows.

Define the function $\psi^n_{\lambda,\theta}(x,y) = \psi_{\lambda,\theta}(x - n, y)$ for $x < \frac{n}{2}$ and n sufficiently large. Obviously, $\nabla\psi^n_{\lambda,\theta}(x,y) = \nabla\psi_{\lambda,\theta}(x + n, y)$. With the aid of the assumption of the nozzle N_1 in the inlet, it follows from the standard elliptic estimates that we have

$$\|\psi^n_{\lambda,\theta}\|_{C^{2,\alpha}(G)} \leq C(G) \quad \text{for sufficiently large } n, \quad 0 < \alpha < 1, \tag{6.4}$$

where G is any compact subset of $S = (-\infty, +\infty) \times (0, H)$.

It follows from the Arzela–Ascoli lemma that there exists a subsequence still labeled by $\psi^n_{\lambda,\theta}$, such that

$$\psi^n_{\lambda,\theta} \to \psi_0 \quad \text{uniformly in} \quad C^{2,\beta}(G) \tag{6.5}$$

for any $G \subset\subset S$ and $0 < \beta < \alpha$. Furthermore, ψ_0 satisfies

$$\begin{cases} \Delta\psi_0 = 0 \quad \text{in} \quad S, \\ \psi_0(x, 0) = 0, \ \psi_0(x, H) = m_0. \end{cases} \tag{6.6}$$

It is easy to check that the boundary value problem (6.6) possesses a unique solution

$$\psi_0 = \tfrac{m_0}{H} y \quad \text{in} \quad S. \tag{6.7}$$

Hence, this together with (6.5) yields

$$(-v(x,y),u(x,y)) = \nabla\psi_{\lambda,\theta}(x,y) \to \left(0, \frac{m_0}{H}\right), \tag{6.8}$$

and

$$\nabla(u,v) \to 0, \quad \text{uniformly in any compact subset of } (0,H) \quad \text{as } x \to -\infty,$$

which together with Bernoulli's law gives that $\nabla p \to 0$ as $x \to -\infty$.

Finally, we will obtain the asymptotic behavior in the downstream. Define the function $\varphi(s,t) = \psi_{\lambda,\theta}(x,y)$ with $s = y\cos\theta - x\sin\theta - \beta\cos\theta + \frac{m_0}{2\lambda}$ and $t = y\sin\theta + x\cos\theta$. Obviously, one has

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\psi_{\lambda,\theta}}{\partial x}\cos\theta + \frac{\partial\psi_{\lambda,\theta}}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial\varphi}{\partial s} = \frac{\partial\psi_{\lambda,\theta}}{\partial y}\cos\theta - \frac{\partial\psi_{\lambda,\theta}}{\partial x}\sin\theta.$$

Denote $\varphi_n(s,t) = \varphi(s+n,t)$; it follows from the flatness condition and the standard elliptic estimates that we have

$$\|\varphi_n(s,t)\|_{C^{2,\alpha}(G)} \le C(G) \quad \text{for sufficiently large } n, \quad 0<\alpha<1, \tag{6.9}$$

where G is any compact subset of $S = (0, \frac{m_0}{\lambda}) \times (-\infty,+\infty)$.

It follows from the Arzela–Ascoli lemma that there exists a subsequence still labeled by $\psi_{\lambda,\theta}^n$, such that

$$\varphi_n \to \psi_0 \quad \text{uniformly in} \quad C^{2,\beta}(G) \tag{6.10}$$

for any $G \subset\subset S$ and $0<\beta<\alpha$. Furthermore, ψ_0 satisfies

$$\begin{cases} \Delta\psi_0 = 0 \quad \text{in} \quad S, \\ \psi_0(0,t) = 0, \ \psi_0\left(\dfrac{m_0}{\lambda}, t\right) = m_0. \end{cases} \tag{6.11}$$

It is easy to check that the boundary value problem (6.11) possesses a unique solution

$$\psi_0 = \lambda s \quad \text{in} \quad S. \tag{6.12}$$

Hence, this together with (6.10) yields

$$(-v(x,y),u(x,y)) = \nabla\psi_{\lambda,\theta}(x,y) \to \left(-\frac{\partial\psi_0}{\partial s}\sin\theta, \frac{\partial\psi_0}{\partial s}\cos\theta\right) = (-\lambda\sin\theta, \lambda\cos\theta), \tag{6.13}$$

and

$$\nabla(u,v) \to 0 \quad \text{as} \quad x \to +\infty,$$

which together with Bernoulli's law gives that $\nabla p \to 0$ as $x \to +\infty$. □

7. Appendix.

7.1. Proof of Proposition 2.1. First, we will obtain the boundedness of the functional $J_{\lambda,\theta,L}(\psi)$ for some $\psi \in K_L$.

LEMMA 7.1. *The functional $J_{\lambda,\theta,L}(\psi) < +\infty$ for some $\psi \in K_L$.*

Proof. Let $\tilde{\psi}$ be a smooth function in Ω_L which coincides with $h_{\lambda,\theta}(x,y)$ in the domain $D = \{(x,y) \mid x > 2, y > \max\{h_1, h_2\}\}$, such that $\tilde{\psi} \in K_L$, where $h_{\lambda,\theta}(x,y) = \min\{m_0, \max\{\lambda(y\cos\theta - x\sin\theta), 0\}$. Thus, one has

$$
\begin{aligned}
J_{\lambda,\theta,L}(\tilde{\psi}) &= \int_{\Omega_L \setminus D} \left|\nabla\tilde{\psi} - \lambda e \chi_{\{0<\tilde{\psi}<m_0\}}\right|^2 dxdy \\
&\quad + \int_{\Omega_L \cap D} \left|\nabla h_{\lambda,\theta} - \lambda e \chi_{\{0<h_{\lambda,\theta}<m_0\}}\right|^2 dxdy \\
&= \int_{\Omega_L \setminus D} \left|\nabla\tilde{\psi} - \lambda e \chi_{\{0<\tilde{\psi}<m_0\}}\right|^2 dxdy \\
&< +\infty.
\end{aligned} \tag{7.1}
$$

□

Next, we give the existence of the minimizer to the truncated variational problem $(P_{\lambda,\theta,L})$.

PROPOSITION 7.2. *There exists a $\psi_{\lambda,\theta,L} \in K_L$ such that* (2.11) *holds.*

Proof. It follows from Lemma 7.1 that $\min_{\psi\in K_L} J_{\lambda,\theta,L}(\psi) < +\infty$, and set $M = \min_{\psi\in K_L} J_{\lambda,\theta,L}(\psi)$ and take a minimizing sequence $\{\psi_n\}$ for $J_{\lambda,\theta,L}$ such that

$$\{\psi_n\} \text{ is bounded in } H^1(\Omega_L \cap B_R) \text{ for any } R > 0.$$

Set $\Omega_{L,R} = \Omega_L \cap B_R(0)$ for any $R > 0$. The compact embedding $H^1(\Omega_{L,R_1}) \hookrightarrow L^p(\Omega_{L,R_1})$ for $R_1 > 0$ gives that there exists a subsequence $\{\psi_n\}$, such that

$$\psi_n \rightharpoonup \psi_{R_1} \text{ in } H^1(\Omega_{L,R_1}) \text{ and } \psi_n \to \psi_{R_1} \text{ in } L^p(\Omega_{L,R_1}) \text{ for any } 1 \le p < +\infty.$$

Take $R_2 > R_1$ and a subsequence still labeled by $\{\psi_n\}$, such that

$$\psi_n \rightharpoonup \psi_{R_1} \text{ in } H^1(\Omega_{L,R_1}), \text{ and } \psi_n \to \psi_{R_2} \text{ in } H^1(\Omega_{L,R_2}).$$

It follows from the uniqueness of the weak limit that

$$\psi_{R_1}(x,y) = \psi_{R_2}(x,y) \text{ a.e. in } \Omega_{L,R_1}.$$

Taking a sequence $\{R_k\}$ with $k \in \mathbb{N}$, such that $R_k \uparrow +\infty$ as $k \to +\infty$, and using a diagonalization argument, we can choose a subsequence $\{\psi_n\}$, such that

$$\psi_n \rightharpoonup \psi_{\lambda,\theta,L} \text{ in } H^1_{loc}(\Omega_L), \text{ and } \psi_{\lambda,\theta,L}(x,y) = \psi_{R_k}(x,y) \text{ in } \Omega_{L,R_k} \setminus \Omega_{L,R_{k-1}}.$$

Since $\chi_{\{0<\psi_n<m_0\}} \in L^\infty(\Omega_L)$, one has

$$\chi_{\{0<\psi_n<m_0\}} \rightharpoonup \gamma \text{ weakly star in } H^1(\Omega_{L,R}) \text{ for any } R > 0,$$

where $\gamma = 1$ a.e. in $\Omega_L \cap \{0 < \psi_{\lambda,\theta,L} < m_0\}$.

Then

$$|\nabla\psi_n - \lambda e \chi_{\{0<\psi_n<m_0\}}| \rightharpoonup |\nabla\psi_{\lambda,\theta,L} - \lambda\gamma e| \text{ in } L^2(\Omega_{L,R}) \text{ for any } R > 0.$$

Since $\gamma = 1$ a.e. in $\Omega_{L,R} \cap \{0 < \psi_{\lambda,\theta,L} < m_0\}$, we have

$$\int_{\Omega_R} |\nabla\psi_{\lambda,\theta,L} - \lambda\gamma e|^2 dxdy \le \liminf_{n\to+\infty} \int_{\Omega_R} |\nabla\psi_n - \lambda e\chi_{\{0<\psi_n<m_0\}}| dxdy \le M$$

for any $R > 0$.

Taking $R \to +\infty$, one has

$$J_{\lambda,\theta,L}(\psi_{\lambda,\theta,L}) \le M,$$

which gives that $\psi_{\lambda,\theta,L}$ is a minimizer to the variational problem $(P_{\lambda,\theta,L})$. ∎

Next, we give the Lipschitz continuity of $\psi_{\lambda,\theta,L}(x,y)$, and the proof can be obtained in Lemma 3.2 in [2].

LEMMA 7.3. *For any compact subset G of Ω_L, $\psi_{\lambda,\theta,L}$ is Lipschitz continuous in G. Furthermore,*

$$\sup_{(x,y)\in G} |\nabla\psi_{\lambda,\theta,L}(x,y)| \le C,$$

where C is positive constant depending on λ, θ, m_0, and G.

In the following, we will check that the minimizer $\psi_{\lambda,\theta,L}$ satisfies (2.3) in the flow field and $|\nabla\psi_{\lambda,\theta,L}| = \lambda$ on the free boundaries Γ_1 and Γ_2 in the weak sense.

PROPOSITION 7.4. *For any minimizer $\psi_{\lambda,\theta,L}$ to the variational problem $(P_{\lambda,\theta,L})$, we have* (2.12) *and* (2.13).

Proof. Denote $\psi(x,y) = \psi_{\lambda,\theta,L}(x,y)$ for simplicity. First, we will show that ψ satisfies $\Delta\psi = 0$ in $D = G \cap \{0 < \psi_{\lambda,\theta,L} < m_0\}$ in the weak sense for any minimizer $\psi \in K_L$.

Indeed, since ψ is continuous in any compact subset of Ω_L, D is open. For any $\xi \in C_0^1(D)$, then $d = \text{dist}(S, \partial D) > 0$ with $S = \text{supp}(\xi) \subset D$. Denote $M = \max_{(x,y)\in S} \xi(x,y)$ and $0 < m_1 = \min_{(x,y)\in S} \psi(x,y) \le \max_{(x,y)\in S} \psi(x,y) = M_1 < m_0$, and $\psi + \varepsilon\xi \in K_L$ for any $|\varepsilon| < \min\{\frac{m_0-M_1}{2M}, \frac{m_1}{2M}\}$ and $0 < \psi + \varepsilon\xi < m_0$ in D. One has

$$\begin{aligned}
0 &\le J_{\lambda,\theta,L}(\psi+\varepsilon\xi) - J_{\lambda,\theta,L}(\psi) \\
&= \int_D |\nabla(\psi+\varepsilon\xi) - \lambda e|^2 \, dxdy - \int_D |\nabla\psi - \lambda e|^2 \, dxdy \qquad (7.2) \\
&= \int_D \left(2\varepsilon\nabla\psi\cdot\nabla\xi + \varepsilon^2 |\nabla\xi|^2\right) dxdy,
\end{aligned}$$

which implies that

$$\int_D \nabla\psi\cdot\nabla\xi dxdy = 0 \quad \text{for any } \xi \in C_0^1(D),$$

that is, $\Delta\psi = 0$ in D in the weak sense.

Finally, we will prove the second part of this proposition. Let

$$\zeta(X) = \zeta(x,y) \in \left(C_0^1(G)\right)^2 \quad \text{and} \quad s_\tau(X) = X + \tau\zeta(X),$$

where τ is a real number and $|\tau| > 0$ is suitably small. Define $\psi_\tau(s_\tau(X)) = \psi(X)$, and it is easy to verify that $\psi_\tau \in K_L$ and

$$D(s_\tau(X))^{-1} = (I + \tau\nabla\cdot\zeta I - \tau D\zeta)(\text{det}Ds_\tau)^{-1} \text{ and } \text{det}Ds_\tau = 1 + \tau\nabla\cdot\zeta + o(\tau),$$

where I is the identity matrix.

Due to the fact that ψ is a minimizer to the problem $(P_{\lambda,\theta,L})$, one gets

$$
\begin{aligned}
0 &\leq J_{\lambda,\theta,L}(\psi_\tau) - J_{\lambda,\theta,L}(\psi)\\
&= \int_D |\nabla\psi(Ds_\tau)^{-1} - \lambda e|^2 \det Ds_\tau dxdy - \int_D |\nabla\psi - \lambda e|^2 dxdy\\
&= \tau\int_D |\nabla\psi|^2\nabla\cdot\zeta - 2\nabla\psi\cdot D\zeta\cdot\nabla\psi dxdy\\
&\quad +\tau\int_D \lambda^2\nabla\cdot\zeta - 2\lambda(e_2,-e_1)\cdot\nabla\zeta\cdot(\partial_y\psi,-\partial_x\psi)dxdy + o(\varepsilon)\\
&= \tau\int_D |\nabla\psi|^2\nabla\cdot\zeta - 2\nabla\psi\cdot D\zeta\cdot\nabla\psi + \lambda^2\nabla\cdot\zeta dxdy + o(\varepsilon).
\end{aligned}
\tag{7.3}
$$

In view of the arbitrariness of τ, the linear term of (7.3) in τ has to vanish, and then this gives that

$$
\begin{aligned}
0 &= \int_D |\nabla\psi|^2\nabla\cdot\zeta - 2\nabla\psi\cdot D\zeta\cdot\nabla\psi + \lambda^2\nabla\cdot\zeta dxdy\\
&= \int_D \nabla\cdot\big((|\nabla\psi|^2+\lambda^2)\zeta - 2(\zeta\cdot\nabla\psi)\nabla\psi\big)\,dxdy\\
&= \lim_{\varepsilon\to0^+}\int_{G\cap\partial\{\varepsilon<\psi<m_0-\varepsilon\}} \big((|\nabla\psi|^2+\lambda^2)\zeta - 2(\zeta\cdot\nabla\psi)\nabla\psi\big)\cdot\nu dS\\
&= \lim_{\varepsilon\to0^+}\int_{G\cap\partial\{\varepsilon<\psi<m_0-\varepsilon\}} \big(\lambda^2 - |\nabla\psi|^2\big)\,\zeta\cdot\nu dS,
\end{aligned}
\tag{7.4}
$$

where we have used the fact that $\nabla\psi \parallel \nu$ on the boundary $\partial\{\varepsilon<\psi<m_0+\varepsilon\}\cap\Omega_L$. ◻

The regularity of the minimizer to the truncated variational problem $(P_{\lambda,\theta,L})$ will be established in the following.

LEMMA 7.5. *The minimizer $\psi_{\lambda,\theta,L}\in C^{2,\alpha}(G)$ in any compact subset G of $\Omega_L\cap\{0<\psi_{\lambda,\theta,L}<m_0\}$.*

Proof. It follows from Proposition 7.4 that the minimizer $\psi_{\lambda,\theta,L}$ satisfies the equation

$$\Delta\psi_{\lambda,\theta,L}=0 \ \text{ in the weak sense in } \Omega_L\cap\{0<\psi_{\lambda,\theta,L}<m_0\}.$$

Thanks to the standard interior Schauder estimates to the linear elliptic equation in Chapter 8 in [29], one has

$$\psi_{\lambda,\theta,L}\in C^{2,\alpha}(G) \quad \text{in any compact subset of } G \text{ of } \Omega_L\cap\{0<\psi_{\lambda,\theta,L}<m_0\}. \qquad ◻$$

7.2. Some important lemmas on free boundaries. In this section, we recall some important lemmas which have been established by Alt, Caffarelli, and Friedman in [1, 4]. Let $\psi_{\lambda,\theta}$ be a local minimizer to the variational problem $(P_{\lambda,\theta})$, that is,

$$J_D(\psi_{\lambda,\theta}) \leq J_D(\psi) \ \text{ for any } \psi\in K, \quad \psi=\psi_{\lambda,\theta} \ \text{ on } \partial D, \tag{7.5}$$

for any bounded domain $D\subset\Omega$ with smooth boundary, where the functional $J_D(\psi)=\int_D|\nabla\psi-\lambda e\chi_{\{0<\psi<m_0\}}|^2dxdy$, $e=(-\sin\theta,\cos\theta)$ with $\theta\in[0,\frac{\pi}{2}]$.

First, we introduce the regularity of the minimizer $\psi_{\lambda,\theta}$ to the variational problem $(P_{\lambda,\theta})$.

LEMMA 7.6. *There exists a universal constant c^* such that, for any $X^0 = (x_0, y_0) \in \Omega$ with the disc $B_r(X^0) \subset \Omega$, satisfying*

$$\frac{1}{r}\fint_{\partial B_r(X^0)} \psi_{\lambda,\theta} dS \geq \lambda c^*, \quad \text{then } \psi_{\lambda,\theta} > 0 \text{ in } B_r(X^0).$$

Similarly,

$$\frac{1}{r}\fint_{\partial B_r(X^0)} (m_0 - \psi_{\lambda,\theta}) dS \geq \lambda c^*; \quad \text{then } \psi_{\lambda,\theta} < m_0 \text{ in } B_r(X^0).$$

Remark 7.7. If the disc $B_r(X^0)$ is not contained in Ω, provided that $B_r(X^0) \cap \partial\Omega$ is $C^{2,\alpha}$ and $\psi_{\lambda,\theta} = m_0$ or $\psi_{\lambda,\theta} = 0$ on $B_r(X^0) \cap \partial\Omega$, the assertion of Lemma 7.6 is still valid.

As an application of Lemma 7.6, we can conclude that

$$\text{the sets } \{\psi_{\lambda,\theta} = 0\} \text{ and } \{\psi_{\lambda,\theta} = m_0\} \text{ are closed subsets of } \Omega.$$

Indeed, consider a sequence $\{X^n\}$ in Ω which satisfies that

$$X^n \to X^0 = (x_0, y_0) \quad \text{and} \quad \psi_{\lambda,\theta}(X^n) = 0,$$

set $r_n = |X^n - X^0|$ for large n, and it follows from Lemma 7.6 that

$$\fint_{\partial B_{r_n}(X^0)} \psi_{\lambda,\theta} dxdy \leq \lambda c^* r_n \to 0 \quad \text{as} \quad n \to +\infty,$$

which gives that $\psi_{\lambda,\theta}(X^0) = 0$. Hence, we deduce that the set $\{\psi_{\lambda,\theta} = 0\}$ is a closed subset of Ω.

Similarly, we can show that the set $\{\psi_{\lambda,\theta} = m_0\}$ is a closed subset of Ω.

The continuity of the minimizer $\psi_{\lambda,\theta}$ implies that the free boundary $\Gamma_1 = \Omega \cap \partial\{\psi_{\lambda,\theta} < m_0\}$ and the free boundary $\Gamma_2 = \Omega \cap \partial\{\psi_{\lambda,\theta} > 0\}$ are disjoint sets.

Next, we introduce the nondegeneracy lemma, which plays an important role in investigating the properties of the free boundaries.

LEMMA 7.8. *There exists a universal positive constant C^*, such that for any disc $B_r(X^0) \subset \Omega$ with $X^0 \in \Omega \setminus D_2$, if*

$$\frac{1}{r}\fint_{\partial B_r(X^0)} \psi_{\lambda,\theta} dS \leq \lambda C^*, \quad \text{and} \quad \psi_{\lambda,\theta} < m_0 \quad \text{in} \quad B_r(X^0),$$

then $\psi_{\lambda,\theta} = 0$ in $B_{\frac{r}{2}}(X^0) \cap (\Omega \setminus D_2)$; similarly, if

$$\frac{1}{r}\fint_{\partial B_r(X^0)} (m_0 - \psi_{\lambda,\theta})\, dS \leq \lambda C^*, \quad \text{and} \quad \psi_{\lambda,\theta} > 0 \quad \text{in} \quad B_r(X^0),$$

then $\psi_{\lambda,\theta} = m_0$ in $B_{\frac{r}{2}}(X^0) \cap (\Omega \setminus D_1)$.

Lemma 7.8 implies the following nondegeneracy lemma.

LEMMA 7.9. *For any $X^0 \in \overline{\{\psi > 0\} \cap (\Omega \backslash D_1)}$, if $\psi_{\lambda,\theta} < m_0$ in $B_r(X^0)$ for some $r > 0$, then*

$$\frac{1}{r}\fint_{\partial B_r(X^0)} \psi_{\lambda,\theta} dS \geq \lambda C^*. \tag{7.6}$$

In particular,

$$\sup_{\partial B_r(X^0)} \psi_{\lambda,\theta} \geq \lambda C^* r. \tag{7.7}$$

Similarly, the result holds with $\psi_{\lambda,\theta}$ replaced by $m_0 - \psi_{\lambda,\theta}$.

Remark 7.10. The assertion of Lemma 7.8 is still valid, provided that the disc $B_r(X^0)$ is not contained in Ω and $\psi_{\lambda,\theta} = m_0$ or $\psi_{\lambda,\theta} = 0$ on $B_r(X^0) \cap \partial\Omega$.

Finally, we introduce the following nonoscillation lemma and the uniformly bounded gradient lemma.

In order to establish the continuity of free boundaries, first we should prove the following nonoscillation lemma, which implies that the free boundary cannot oscillate near the fixed boundary. Without loss of generality, we consider the upper free boundary Γ_1. Introduce a domain $E \subset \Omega \backslash D_1$ bounded by

$$y = y_0, \quad y = y_0 + \delta \ (h > 0),$$

and

$$\gamma_1 : X = X^1(t) = (x_1(t), y_1(t)), \quad \gamma_2 : X = X^2(t) = (x_2(t), y_2(t)),$$

where $0 \leq t \leq T$ and

$$y_0 < y_i < y_0 + \delta \ \text{ if } \ 0 < t < T, y_i(0) = y_0, y_i(T) = y_0 + \delta,$$

and

$$x_0 \leq x_i(t) < x_0 + \varepsilon, \quad i = 1, 2.$$

Furthermore, we assume that the arc γ_2 lies to the right of the arc γ_1. This implies that $x_2(0) > x_1(0)$, γ_1 and γ_2 do not intersect each other,

$$\gamma_2 \ \text{ is contained in } \Gamma_1,$$

and either

$$\text{Case 1. } \gamma_1 \ \text{ is contained in } \Gamma_1,$$

or

$$\text{Case 2. } \gamma_1 \ \text{ lies on } \{x = a, y \geq h_1\}, \text{ and then } \ y_1(0) \geq h_1, \quad x = a.$$

The set $\{\psi_{\lambda,\theta} < m_0\}$ is some E-neighborhood of γ_1. Furthermore, $\psi_{\lambda,\theta} > 0$ in E and $\text{dist}(E, \overline{A_1 A_2}) > c_0 > 0$.

Lemma 7.11 (nonoscillation). *Under the assumptions stated above, there exists a positive constant C depending only on λ, m_0, and c_0, such that*

$$\delta \leq C\varepsilon. \tag{7.8}$$

Remark 7.12. The nonoscillation lemma, Lemma 7.11, remains true if one of the arcs γ_2 is a line segment on I_a, provided that

$$\frac{\partial \psi_{\lambda,\theta}}{\partial \nu} \geq \lambda \ \text{ on } \ \gamma_2. \tag{7.9}$$

Lemma 7.3 implies that $|\nabla \psi_{\lambda,\theta}|$ is bounded; however, we can obtain that the bound of $\nabla \psi_{\lambda,\theta}$ is indeed independent of m_0 in the following bounded gradient lemma.

LEMMA 7.13. *Let $X^0 = (x_0, y_0)$ be a free boundary point in $\Omega \setminus D_1$, and let G be a compact subset of Ω and contain X^0. Then*

$$|\nabla\psi_{\lambda,\theta}(x,y)| \leq C \quad in \quad G,$$

where C depends only on R, λ, c_0, and G, but it is independent of m_0.

The regularity of the free boundary in the two-dimensional case is obtained in Theorem 8.4 in [1], which we state as follows.

LEMMA 7.14. *The free boundary $\partial\{0 < \psi_{\lambda,\theta} < m_0\}$ is analytic.*

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