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ANALYSIS OF GALERKIN FEMS FOR MIXED FORMULATION OF TIME-DEPENDENT GINZBURG–LANDAU EQUATIONS UNDER TEMPORAL GAUGE*

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Abstract. The paper focuses on analysis of linearized Galerkin FEMs for a mixed formulation of the time-dependent Ginzburg–Landau equations under the temporal gauge. We provide optimal error estimates in L^2-norm for the order parameter ψ_h and the magnetic field σ_h unconditionally, although the accuracy of the numerical magnetic potential $\mathbf{A}_h$ is one-order lower than the optimal one due to the degeneracy of the magnetic potential equation. Since the states of superconductors are determined by the order parameter ψ_h (or the density of the superconducting electron pairs $|\psi_h|$), the accuracy of ψ_h is more important for the vortex simulation in superconditors. Our analysis is based on a nonclassical Ritz projection, which may reduce the pollution of inaccuracy of the numerical magnetic potential in analysis. Numerical experiments confirm our theoretical analysis.

Key words. nonclassical Ritz projection, Ginzburg–Landau equations, Galerkin FEMs, temporal gauge, optimal error estimate

AMS subject classifications. 65N12, 65N30, 35K61

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1. Introduction.

The time-dependent Ginzburg–Landau (GL) equations are defined by

$$\eta\frac{\partial\psi}{\partial t} + i\eta\kappa\Phi\psi + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi + (|\psi|^2-1)\psi = 0 \qquad \text{in } \Omega\times(0,T] \tag{1.1}$$

$$\frac{\partial\mathbf{A}}{\partial t} + \nabla\Phi + \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A} = 0 \qquad \text{in } \Omega\times(0,T] \tag{1.2}$$

with the boundary and initial conditions

$$\left(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\right)\cdot\mathbf{n} = 0, \quad \mathbf{curl}\,\mathbf{A}\times\mathbf{n} = \mathbf{H}_e\times\mathbf{n} \qquad \text{on } \partial\Omega\times[0,T] \tag{1.3}$$

$$\psi(x,0) = \psi_0(x), \quad \mathbf{A}(x,0) = \mathbf{A}_0(x) \qquad \text{in } \Omega \tag{1.4}$$

where $\mathbf{A}_0\cdot\mathbf{n}|_{\partial\Omega} = 0$. The complex scalar function ψ is the order parameter, the real vector-valued function $\mathbf{A}$ is the magnetic potential, the real scalar function Φ is the electric potential, and ψ^* denotes the complex conjugate of the function ψ. In the GL equations (1.1)–(1.4), $|\psi|^2$ denotes the density of the superconducting electron pairs. $|\psi|^2 = 1$ and $|\psi|^2 = 0$ represent the perfectly superconducting state and the normal state, respectively, while $0 < |\psi|^2 < 1$ represents a mixed (vortex) state. The real vector-valued function $\mathbf{H}_e$ is the external magnetic field, κ is the GL (positive) parameter, and η is a dimensionless constant. In this paper, we set $\eta = 1$ for the sake of simplicity. The GL equations were first deduced by Gor'kov and Kopnin in [18] from

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the microscopic Bardeen–Cooper–Schrieffer theory of superconductivity. In the past three decades, numerical studies of the GL equations have attracted much attention. In particular, numerical solution of the system plays a key role in the study of the transient behavior and the vortex motion of the type II superconductors under an external applied magnetic field. For detailed physical description and mathematical modeling of the superconductivity phenomena, we refer the reader to the review articles [5, 10, 11]. Theoretical analysis for the GL equations can be found in [6, 12, 35, 36].

It should be remarked that the GL equations are gauge invariant [6, 13]. There are several typical gauges of interest in practice, such as the temporal gauge, the Lorentz gauge, and the Coulomb gauge. It is easy to verify that the GL equations under different gauges produce the same magnitude of the order parameter $|\psi|$ and the magnetic field $\sigma := \mathbf{curl}\,\mathbf{A}$, although the magnetic potential $\mathbf{A}$ may be different. Among these gauges, the temporal gauge is the most popular one in physical and engineering community, and numerical simulations on a variety of practical models have been done under the gauge, such as [1, 8, 19, 20, 21, 31, 32, 33, 34, 38]. By taking $\Phi = 0$, we obtain the GL equations under the temporal gauge

$$\frac{\partial \psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi + (|\psi|^2 - 1)\psi = 0 \qquad \text{in } \Omega \times (0,T] \tag{1.5}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} + \frac{i}{2\kappa}(\psi^* \nabla \psi - \psi \nabla \psi^*) + |\psi|^2 \mathbf{A} = 0 \qquad \text{in } \Omega \times (0,T]\,. \tag{1.6}$$

Clearly, the temporal gauge has certain advantages in computation, particularly vortex simulations in complex geometries and geometries with defects, such as a nonconvex polygon. The convergence of conventional Galerkin FEMs for the problem in a polygon with the reentrant corner was observed by several researchers [25, 26, 14, 15, 16], while in this case, Lagrange FEMs for the GL equations under the Lorentz gauge may converge to a spurious vortex pattern. An apparent drawback of the temporal gauge is the degeneracy of the magnetic potential equation (1.6). In the viewpoint of numerical analysis, numerical methods for such a degenerate parabolic equation usually provide a solution of one-order lower than optimal accuracy. The degeneracy may also lead to certain difficulties in solving linear systems at each time step [16]. A suboptimal error analysis of Galerkin FEMs for the GL equations under the temporal gauge was presented in [39], and numerical results confirm the reduced accuracy of the numerical magnetic potential $\mathbf{A}_h$. To overcome the drawback, a corresponding perturbed problem with an extra term $-\epsilon\nabla(\mathrm{div}\,\mathbf{A})$ was investigated by several authors. Du [9] proposed a nonlinear backward Euler scheme with Lagrange finite element approximations to the perturbed GL equations. A decoupled alternating Crank–Nicolson–Galerkin method with a weakly nonlinear scheme in the temporal direction was proposed and analyzed by Mu and Huang [29] for the perturbed GL equations, in which the two discrete systems for ψ and $\mathbf{A}$ are decoupled and can be solved simultaneously at each time step. An optimal L^2 error estimate was presented in [29] under the time step restrictive condition $\tau = O(h^{\frac{11}{12}})$ for the two-dimensional (2D) model and $\tau = O(h^2)$ for the three-dimensional (3D) model. The error bound depends upon the perturbed parameter ϵ. Later, a linearized Crank–Nicolson-type scheme was proposed in [30], and a numerical investigation was made there.

It is noted that $\mathbf{A}$ denotes the magnetic potential, a nonphysical component, and ψ (order parameter) or more precisely $|\psi|$ (the density of the superconducting electron pairs) plays a more important role in the vortex simulation, from which the superconducting state is determined. The question is whether Galerkin FEMs can provide numerical order parameter ψ_h (or numerical density $|\psi_h|$) with optimal-order accuracy although the numerical magnetic potential has been polluted due to the degeneracy of its equation. In a recent work [16], a mixed formulation of GL (MGL) equations was proposed under the temporal gauge, defined by

$$\frac{\partial \psi}{\partial t} + \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi + (|\psi|^2 - 1)\psi = 0 \tag{1.7}$$

$$\frac{\partial \sigma}{\partial t} - \Delta\,\sigma + \frac{i}{2\kappa}(\mathbf{curl}\,\psi \cdot \nabla\psi^* - \mathbf{curl}\,\psi^* \cdot \nabla\psi) + |\psi|^2\sigma - \mathbf{A}\cdot\mathbf{curl}\,|\psi|^2 = 0 \tag{1.8}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{curl}\,\sigma + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A} = 0 \tag{1.9}$$

for $x \in \Omega$, $t \in (0,T]$ with boundary and initial conditions [24]

$$\nabla\psi\cdot\mathbf{n} = 0, \quad \sigma\times\mathbf{n} = \mathbf{H}_e\times\mathbf{n}, \qquad \mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega\times[0,T] \tag{1.10}$$

$$\psi(x,0) = \psi_0(x), \quad \sigma(x,0) = \sigma_0(x) := \mathrm{curl}\,\mathbf{A}_0(x) \qquad \text{in } \Omega \tag{1.11}$$

where $\sigma = \mathbf{curl}\,\mathbf{A}$. The second equation (1.8) is obtained by taking **curl** operator on both sides of the equation (1.6). Systematic numerical simulations were presented in [16] in comparison with commonly used Galerkin methods for the GL equations under both the Lorentz gauge and the temporal gauge. Numerical results show clearly that Galerkin FEMs for the mixed formulation produce the numerical ψ_h and σ_h with optimal-order accuracy, although the accuracy of the numerical magnetic potential $\mathbf{A}_h$ is one-order lower. Also, methods for the mixed formulation are more effective in computation mainly because the mixed GL system is regular parabolic. On the other hand, since no spatial derivative of $\mathbf{A}$ is involved in (1.9), classical finite element approximations are convergent under the weaker regularity assumption $\mathbf{A} \in \mathbf{H}^s$, $0 < s < 1$, while for a parabolic equation, the regularity of the solution in H^{1+s} is required for Lagrange-type FEMs in general [7]. Therefore, the mixed formulation is applicable for the problem in nonconvex polygons for which $\mathbf{A} \in H^s$ [25]; however, no analysis has been given in [16]. In this paper, we provide theoretical analysis of a class of linearized backward Euler–Galerkin FEMs for the above MGL system under the temporal gauge. Our analysis confirms the numerical observations given in [16] that numerical methods provide optimal accuracy for these two physical components, ψ_h and σ_h. Moreover, the analysis is given unconditionally, i.e., when $h \le h_0$ and $\tau \le \tau_0$ for some $h_0, \tau_0 > 0$. The key to our analysis is a reformulation of the GL system into a regular nonlinear parabolic system with an extra diffusion of memory. A nonclassical Ritz projection is introduced, which may help us to clean up the effect of the pollution of $\mathbf{A}_h$ in analysis. It is noted that a similar nonclassical H^1 projection was proposed in [4, 27] for a Volterra integro-differential equation of parabolic type.

The paper is organized as follows. In section 2, we present a class of fully discrete linearized backward Euler–Galerkin FEMs, which only requires the solution of two uncoupled discrete linearized parabolic equations at each time step. Following an error splitting technique, in section 3 we introduce a time-discrete system. The fully discrete Galerkin finite element solutions can be viewed as the spatial discrete finite element solutions of the time-discrete system. The regularity of the solution of the time-discrete system is presented, and then the boundedness of the solution of the fully discrete system is established in terms of the splitting technique. Moreover, to deal with the degeneracy of the magnetic potential equation, we introduce a nonclassical Ritz projection which is defined by a discrete elliptic system of memory. In section

4, we present optimal error estimates of numerical order parameter and numerical magnetic field unconditionally, although the accuracy of numerical magnetic potential is one-order lower than the optimal one. In section 5, we present numerical results to confirm our analysis and show the efficiency of proposed methods.

2. Fully discrete Galerkin methods. For simplicity, we only focus on the MGL equations in a convex domain in 2D space, $d = 2$, for which div, ∇, curl, and **curl** are defined by

$$\mathrm{div}\mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y}, \nabla\psi = \left[\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right]^T, \mathrm{curl}\mathbf{A} = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}, \mathbf{curl}\,\psi = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right]^T.$$

Analysis and formulas given here are valid for $d = 3$ with classical 3D notations of these operators.

Let $W^{k,p}(\Omega)$ be the conventional Sobolev space defined on Ω and $H^k(\Omega) := W^{k,2}(\Omega)$. We denote by $\mathcal{H}^k(\Omega) = \{u + iv | u, v \in H^k(\Omega)\}$ a Sobolev space for the complex-valued functions. The variational formulation of the MGL equations (1.7)–(1.8) is to find $\psi \in L^2(0,T;\mathcal{H}^1(\Omega))$ with $\frac{\partial \psi}{\partial t} \in L^2(0,T;\mathcal{H}^{-1}(\Omega))$, $\sigma \in L^2(0,T;H^1(\Omega))$, $\frac{\partial \sigma}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$, and $\sigma = H_e$ on $\partial\Omega$ such that

$$\begin{aligned}&\left(\frac{\partial \psi}{\partial t}, \omega\right) + \left(\left(\frac{i}{\kappa}\nabla + \mathbf{A}(\psi,\sigma)\right)\psi, \left(\frac{i}{\kappa}\nabla + \mathbf{A}(\psi,\sigma)\right)\omega\right)\\ &\quad + ((|\psi|^2 - 1)\psi, \omega) = 0 \qquad \forall\, \omega \in \mathcal{H}^1(\Omega)\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\left(\frac{\partial \sigma}{\partial t}, \theta\right) + (\nabla\sigma\,, \nabla\theta) + \frac{1}{\kappa}(\mathrm{Im}(\mathbf{curl}\,\psi^* \cdot \nabla\psi), \theta) + (|\psi|^2\sigma, \theta)\\ &\qquad - (\mathbf{A}(\psi,\sigma)\cdot \mathbf{curl}\,|\psi|^2, \theta) = 0 \qquad \forall\, \theta \in H_0^1(\Omega)\end{aligned} \tag{2.2}$$

with $\psi(x,0) = \psi_0(x)$ and $\sigma(x,0) = \sigma_0(x)$. Since (1.9) is an ordinary differential equation, we have the analytic formula

$$\mathbf{A} = \frac{1}{\widetilde{\psi}}\left(\mathbf{A}_0 + \int_0^t \widetilde{\psi}(s)\boldsymbol{g}(s)ds\right),$$

where

$$\begin{aligned}&\widetilde{\psi}(x,t) = exp\left(\int_0^t |\psi(s)|^2 ds\right)\\ &\mathbf{g} = -\frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \mathbf{curl}\,\sigma.\end{aligned} \tag{2.3}$$

It is noted that (2.1)–(2.2) defines a nonclassical parabolic system in which an extra diffusion of memory is added through the function $\mathbf{A}$.

Before we introduce a fully discrete Galerkin scheme, we present an approximation to $\mathbf{A}(\psi,\sigma)$. Let $\{t_n\}_{n=0}^N$ be a uniform partition in the time direction with the step size $\tau = \frac{T}{N}$, and let $u^n = u(\cdot, n\tau)$. For a sequence of functions $\{U^n\}_{n=0}^N$ defined on Ω, we denote

$$D_\tau U^n = \frac{U^n - U^{n-1}}{\tau} \quad \text{for } n = 1, 2, \ldots, N.$$

By Taylor expansion, we have the following approximations:

$$\widetilde{\psi}^n = \widetilde{\psi}^{n-1}\exp\left(\int_{t_{n-1}}^{t_n} |\psi|^2\, ds\right) = \widetilde{\psi}^{n-1}(1 + \tau|\psi^n|^2) + O(\tau^2),$$

$$
\begin{aligned}
\mathbf{A}^n &= \frac{1}{\widetilde{\psi}^n}\left(\mathbf{A}_0(x) + \int_0^{t_n} \widetilde{\psi}(s)\mathbf{g}(s)\,ds\right) \\
&= \frac{1}{\widetilde{\psi}^n}\left(\mathbf{A}_0(x) + \int_0^{t_{n-1}} \widetilde{\psi}(s)\mathbf{g}(s)\,ds + \int_{t_{n-1}}^{t_n} \widetilde{\psi}(s)\mathbf{g}(s)\,ds\right) \\
&= (1-\tau|\psi^n|^2)\mathbf{A}^{n-1} + \tau\mathbf{g}^n + O(\tau^2), \qquad (2.4)
\end{aligned}
$$

where $\psi^0 = \psi_0$ and $\mathbf{A}^0 = \mathbf{A}_0(x)$.

Let $\mathcal{T}_h$ be a regular partition of Ω with $\Omega = \cup_K \Omega_K$ and the mesh size $h = \max_{\Omega_K \in \mathcal{T}_h}\{\operatorname{diam}\Omega_K\}$. For a given partition $\mathcal{T}_h$, we denote by V_h^r, $\mathring{V}_h^r$ and $\mathcal{V}_h^r$ the rth order Lagrange finite element subspaces of $H^1(\Omega)$, $H_0^1(\Omega)$ and $\mathcal{H}^1(\Omega)$, respectively. We denote by I_h the commonly used Lagrange nodal interpolation operator on V_h^r and $\mathcal{V}_h^r$ and define $A_h^n := A(\psi_h^n, \sigma_h^n)$.

With the above notations, a linearized backward-Euler FEM for the MGL equations under temporal gauge is to find $\psi_h^n \in \mathcal{V}_h^r$ and $\sigma_h^n \in V_h^r$, with $\sigma_h^n|_{\partial\Omega} = I_h H_e^n|_{\partial\Omega}$, such that for $n = 1, 2, \ldots, N$

$$
(D_\tau \psi_h^n, \omega_h) + \frac{1}{\kappa^2}(\nabla\psi_h^n\,, \nabla\omega_h) = b_1(\psi_h^{n-1}, \sigma_h^{n-1}, \omega_h) \quad \forall\, \omega_h \in \mathcal{V}_h^r \qquad (2.5)
$$

$$
(D_\tau \sigma_h^n, \theta_h) + (\nabla\sigma_h^n\,, \nabla\theta_h) = b_2(\psi_h^{n-1}, \sigma_h^{n-1}, \theta_h) \quad \forall\, \theta_h \in \mathring{V}_h^r \qquad (2.6)
$$

where $\psi_h^0 = I_h\psi_0$, $\sigma_h^0 = I_h\sigma_0$,

$$
\begin{aligned}
b_1(\psi_h^{n-1}, \sigma_h^{n-1}, \omega_h) =& -\frac{i}{\kappa}(\nabla\psi_h^{n-1}\,, \mathbf{A}_h^{n-1}\omega_h) + \frac{i}{\kappa}(\mathbf{A}_h^{n-1}\psi_h^{n-1}\,, \nabla\omega_h) \\
& - ((|\mathbf{A}_h^{n-1}|^2 + |\psi_h^{n-1}|^2 - 1)\psi_h^{n-1}, \omega_h)\,, \\
b_2(\psi_h^{n-1}, \sigma_h^{n-1}, \theta_h) =& -\frac{1}{\kappa}(\operatorname{Im}(\mathbf{curl}\,(\psi_h^{n-1})^* \cdot \nabla(\psi_h^{n-1})),\, \theta_h) \\
& - (|\psi_h^{n-1}|^2\sigma_h^{n-1}, \theta_h) + (\mathbf{curl}\,|\psi_h^{n-1}|^2 \cdot \mathbf{A}_h^{n-1}, \theta_h),
\end{aligned}
$$

and

$$
\mathbf{A}_h^n = (1-\tau|\psi_h^n|^2)\mathbf{A}_h^{n-1} + \tau\boldsymbol{g}_h^n \qquad (2.7)
$$

$$
\boldsymbol{g}_h^n = \frac{1}{\kappa}\operatorname{Im}((\psi_h^n)^*\nabla\psi_h^n) - \mathbf{curl}\,\sigma_h^n\,. \qquad (2.8)
$$

Clearly, the above scheme is fully linearized and decoupled. At each time step, one only needs to solve a linear system for each variable. A slight different scheme was introduced in [16]. A systematic numerical simulation was made, while no analysis was presented there. Since the coefficient matrixes are symmetric positive definite, the existence of numerical solution follows immediately.

Here we assume the solution to (1.7)–(1.11) satisfies the regularity

$$
\begin{aligned}
&\|\psi\|_{L^2((0,T);\mathcal{W}^{2,4}\cap\mathcal{W}^{r+1,3})} + \|\psi\|_{H^1((0,T);H^{r+1})} + \|\psi\|_{H^2((0,T);L^2)} \\
&+ \|\sigma\|_{L^2((0,T);W^{2,4}\cap W^{r+1,3})} + \|\sigma\|_{H^1((0,T);H^{r+1})} + \|\sigma\|_{H^2((0,T);L^2)} \\
&+ \|\psi_0\|_{\mathcal{W}^{2,4}\cap\mathcal{W}^{r+1,3}} + \|\mathbf{A}_0\|_{\boldsymbol{W}^{1,4}\cap\boldsymbol{W}^{r,3}} + \|\sigma_0\|_{W^{2,4}\cap W^{r+1,3}} \le K\,. \qquad (2.9)
\end{aligned}
$$

The main result is given in the following theorem.

THEOREM 2.1. *Suppose that the system* (1.7)–(1.11) *has a unique solution* (ψ, σ) *satisfying the condition* (2.9). *Then there exist positive constants* h_0 *and* τ_0 *such that when* $\tau \le \tau_0$ *and* $h \le h_0$, *the finite element system* (2.5)–(2.6) *admits a unique solution* (ψ_h^n, σ_h^n), $0 \le n \le N$, *satisfying*

$$\|\psi^n - \psi_h^n\|_{L^2}^2 + \|\sigma^n - \sigma_h^n\|_{L^2}^2 \le C_0(\tau^2 + h^{2r+2}),$$

where C_0 *is a constant independent of* h, n, *and* τ *and dependent on* K *and* T.

For the simplicity of notations, we denote by C a generic positive constant involved in some classical inequalities, such as Gagliardo–Nirenberg interpolation inequality, inverse inequalities, and inequalities for classical Lagrange interpolation and Ritz projection, which may depend upon the domain Ω and the partition $\mathcal{T}_h$. Also, we denote by C_K a generic positive constant independent of n, h, τ, and C_0 in the above theorem, which may depend upon K in (2.9), T, C, and the physical parameter κ. Both C and C_K could be different in different places.

3. Boundedness of numerical solution. In this section, we establish the boundedness of the numerical solution in certain norm by an error splitting technique proposed in [22, 23]. Our proof will often use the classical Gagliardo–Nirenburg interpolation inequality,

$$\|\partial^j u\|_{L^p} \le C\|\partial^m u\|_{L^\gamma}^a \, \|u\|_{L^q}^{1-a} + C\|u\|_{L^q}, \tag{3.1}$$

for $0 \le j < m$ and $\frac{j}{m} \le a \le 1$ with $\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{\gamma} - \frac{m}{d}\right) + (1-a)\frac{1}{q}$, unless $1 < \gamma < \infty$ and $m - j - \frac{n}{\gamma}$ is a nonnegative integer, in which case the above estimate holds only for $\frac{j}{m} \le a < 1$.

3.1. Time-discrete solution. We define a corresponding time-discrete and spatial continuous system by

$$D_\tau \psi_\tau^n - \frac{1}{\kappa^2}\Delta\psi_\tau^n = g_1(\psi_\tau^{n-1}, \sigma_\tau^{n-1}) \tag{3.2}$$

$$D_\tau \sigma_\tau^n - \Delta\sigma_\tau^n = g_2(\psi_\tau^{n-1}, \sigma_\tau^{n-1}) \tag{3.3}$$

with the boundary and initial conditions

$$\nabla\psi_\tau \cdot \mathbf{n} = 0, \quad \sigma_\tau = H_e \qquad \text{on } \partial\Omega \times [0, T] \tag{3.4}$$

$$\psi_\tau(x,0) = \psi_0(x), \qquad \sigma_\tau(x,0) = \sigma_0(x), \quad \mathbf{A}_\tau(x,0) = \mathbf{A}_0(x) \quad \text{in } \Omega \tag{3.5}$$

where

$$\begin{aligned} g_1(\psi_\tau^{n-1}, \sigma_\tau^{n-1}) &= -\frac{i}{\kappa}\nabla\psi_\tau^{n-1} \cdot \mathbf{A}_\tau^{n-1} - \frac{i}{\kappa}\mathrm{div}(\mathbf{A}_\tau^{n-1}\psi_\tau^{n-1}) \\ &\quad - (|\mathbf{A}_\tau^{n-1}|^2 + |\psi_\tau^{n-1}|^2 - 1)\psi_\tau^{n-1} \\ g_2(\psi_\tau^{n-1}, \sigma_\tau^{n-1}) &= -\frac{1}{\kappa}\mathrm{Im}(\mathbf{curl}\,(\psi_\tau^{n-1})^* \cdot \nabla\psi_\tau^{n-1}) - |\psi_\tau^{n-1}|^2\sigma_\tau^{n-1} \\ &\quad + \mathbf{curl}\,|\psi_\tau^{n-1}|^2 \cdot \mathbf{A}_\tau^{n-1} \end{aligned}$$

and

$$\mathbf{A}_\tau^n = (1 - \tau|\psi_\tau^n|^2)\mathbf{A}_\tau^{n-1} + \tau\boldsymbol{g}_\tau^n \tag{3.6}$$

$$\boldsymbol{g}_\tau^n = \frac{1}{\kappa}\mathrm{Im}((\psi_\tau^n)^*\nabla\psi_\tau^n) - \mathbf{curl}\,\sigma_\tau^n\,. \tag{3.7}$$

From (3.6) we can see that $\mathbf{A}_\tau^n \cdot \mathbf{n}|_{\partial\Omega} = 0$. Then we have the following lemma.

LEMMA 3.1. *Suppose that the system in* (1.7)–(1.11) *has a unique solution* (ψ, σ) *satisfying the condition* (2.9). *Then there exists a positive constant* τ_0 *such that when* $\tau \le \tau_0$*, the system* (3.2)–(3.5) *admits a unique solution* $(\psi_\tau^n, \sigma_\tau^n)$, $0 \le n \le N$, *satisfying*

$$\|\psi^n - \psi_\tau^n\|_{H^1} + \|\sigma^n - \sigma_\tau^n\|_{H^1} \le C_0^* \tau \tag{3.8}$$

and

$$\|\psi_\tau^n\|_{W^{2,4}}^2 + \|\sigma_\tau^n\|_{W^{2,4}}^2 + \tau \sum_{k=1}^{n} \left(\|D_\tau \psi_\tau^k\|_{H^2}^2 + \|D_\tau \sigma_\tau^k\|_{H^2}^2 \right) \le C_K, \tag{3.9}$$

where C_0^* *is a positive constant independent of* τ *and dependent on* K *and* T.

The proof of the lemma will be given in the appendix. From Lemma 3.1, (3.6), and (3.7), we see further

$$\|\mathbf{A}_\tau^n\|_{W^{1,4}} \le C_K. \tag{3.10}$$

Moreover, we define a classical Ritz projection R_h from H^1 to V_h^r by

$$(\nabla \sigma_\tau^n, \nabla \theta_h) = (\nabla R_h \sigma_\tau^n, \nabla \theta_h) \quad \forall\, \theta_h \in \mathring{V}_h^r \tag{3.11}$$

with $R_h \sigma_\tau^n = H_e$ on the $\partial\Omega$. Since the system (2.1)–(2.2) is nonclassical parabolic, we define a nonclassical Ritz projection $\widetilde{R}_h$ by

(3.12)
$$(\nabla(\psi_\tau^n - \widetilde{R}_h \psi_\tau^n),\, \nabla \omega_h) = i\tau \sum_{j=0}^{n-1} \left(\mathrm{Im}((\psi_\tau^j)^* \nabla(\psi_\tau^j - \widetilde{R}_h \psi_\tau^j)) \Psi^{j+1,n-1} \psi_\tau^{n-1},\, \nabla \omega_h \right)$$

for all $\omega_h \in \mathcal{V}_h^r$ and $n \ge 1$, where

$$\Psi^{j+1,n-1} = \begin{cases} 1 & \text{for } n \ge 2, j = n-1 \\ \displaystyle\prod_{l=j+1}^{n-1} (1 - \tau|\psi_h^l|^2) & \text{for } n \ge 3, 1 \le j \le n-2 \\ 0 & \text{for } j = 0 \end{cases} \tag{3.13}$$

and $\widetilde{R}_h \psi_\tau^0 = I_h \psi_\tau^0 = I_h \psi_0$ and $R_h \sigma_\tau^0 = I_h \sigma_\tau^0 = I_h \sigma_0$. Let

$$\widetilde{\rho}_{\psi_\tau}^n := \psi_\tau^n - \widetilde{R}_h \psi_\tau^n, \qquad \rho_{\sigma_\tau}^n := \sigma_\tau^n - R_h \sigma_\tau^n.$$

Following classical finite element theory [2, 37] and [3, 4, Lemma 3.1], we have the estimates

$$\begin{cases} \|\rho_{\sigma_\tau}^n\|_{L^p} + h\|\rho_{\sigma_\tau}^n\|_{W^{1,p}} \le Ch^2 \|\sigma_\tau^n\|_{W^{2,p}} \\ \displaystyle\sum_{k=1}^{n} \tau \|D_\tau \rho_{\sigma_\tau}^k\|_{L^2} \le Ch^2 \sum_{k=1}^{n} \tau \|D_\tau \sigma_\tau^k\|_{H^2} \\ \|\widetilde{\rho}_{\psi_\tau}^n\|_{L^p} + h\|\widetilde{\rho}_{\psi_\tau}^n\|_{W^{1,p}} \le Ch^2 \left(\|\psi_\tau^n\|_{W^{2,p}} + \tau \displaystyle\sum_{k=0}^{n-1} \|\psi_\tau^k\|_{W^{2,p}} \right) \\ \displaystyle\sum_{k=1}^{n} \tau \|D_\tau \widetilde{\rho}_{\psi_\tau}^k\|_{L^2} \le Ch^2 \sum_{k=1}^{n} \tau \left(\|D_\tau \psi_\tau^k\|_{H^2} + \tau \sum_{l=1}^{k} \|D_\tau \psi_\tau^l\|_{H^2} \right) \end{cases} \tag{3.14}$$

for $2 \le p \le 4$. By an inverse inequality and Lemma 3.1, we have

$$\|\widetilde{\rho}^n_{\psi_\tau}\|_{W^{1,\infty}} + \|\rho^n_{\sigma_\tau}\|_{W^{1,\infty}} \le C_K. \tag{3.15}$$

One can see clearly from (3.11)–(3.12) the difference between the classical Ritz projection R_h and the nonclassical projection $\widetilde{R}_h$. If only the classical Ritz projection is used as usual, the right-hand side of (3.12) will appear in error equations and pollute the error function $\widetilde{e}_{\psi_\tau}$ (or $\widetilde{e}_\psi$) eventually.

3.2. A primary spatial error estimate. It is noted that the fully-discrete finite element solution (ψ^n_h, σ^n_h) can be viewed as the finite element solution of the time-discrete system (3.2)–(3.3). Let

$$\widetilde{e}^n_{\psi_\tau} := \widetilde{R}_h \psi^n_\tau - \psi^n_h, \quad e^n_{\sigma_\tau} := R_h \sigma^n_\tau - \sigma^n_h\,. \tag{3.16}$$

Subtracting the fully discrete system (2.5)–(2.6) from the time-discrete system (3.2)–(3.3), respectively, the error functions $\widetilde{e}^n_{\psi_\tau}$ and $e^n_{\sigma_\tau}$ satisfy

(3.17)

$$\begin{aligned}
&(D_\tau \widetilde{e}^n_{\psi_\tau}, \widetilde{e}^n_{\psi_\tau}) + \frac{1}{\kappa^2}\|\nabla \widetilde{e}^n_{\psi_\tau}\|^2_{L^2} \\
&= -\,(D_\tau \widetilde{\rho}^n_{\psi_\tau}, \widetilde{e}^n_{\psi_\tau}) - \frac{1}{\kappa^2}(\nabla \widetilde{\rho}^n_{\psi_\tau}, \nabla \widetilde{e}^n_{\psi_\tau}) + b_1(\psi^{n-1}_\tau, \sigma^{n-1}_\tau, \widetilde{e}^n_{\psi_\tau}) - b_1(\psi^{n-1}_h, \sigma^{n-1}_h, \widetilde{e}^n_{\psi_\tau}) \\
&= -\,(D_\tau \widetilde{\rho}^n_{\psi_\tau}, \widetilde{e}^n_{\psi_\tau}) - \left[\frac{1}{\kappa^2}(\nabla \widetilde{\rho}^n_{\psi_\tau}, \nabla \widetilde{e}^n_{\psi_\tau}) - \frac{i}{\kappa}(\mathbf{A}^{n-1}_\tau \psi^{n-1}_\tau, \nabla \widetilde{e}^n_{\psi_\tau}) + \frac{i}{\kappa}(\mathbf{A}^{n-1}_h \psi^{n-1}_h, \nabla \widetilde{e}^n_{\psi_\tau})\right] \\
&\quad - \left[\frac{i}{\kappa}(\nabla \psi^{n-1}_\tau, \mathbf{A}^{n-1}_\tau \widetilde{e}^n_{\psi_\tau}) - \frac{i}{\kappa}\left(\nabla \psi^{n-1}_h, \mathbf{A}^{n-1}_h \widetilde{e}^n_{\psi_\tau}\right)\right] \\
&\quad - \left[((|\mathbf{A}^{n-1}_\tau|^2 + |\psi^{n-1}_\tau|^2 - 1)\psi^{n-1}_\tau, \widetilde{e}^n_{\psi_\tau}) - \left((|\mathbf{A}^{n-1}_h|^2 + |\psi^{n-1}_h|^2 - 1)\psi^{n-1}_h, \widetilde{e}^n_{\psi_\tau}\right)\right] \\
&:= -\,(D_\tau \widetilde{\rho}^n_{\psi_\tau}, \widetilde{e}^n_{\psi_\tau}) + J_1 + J_2 + J_3
\end{aligned}$$

and

(3.18)

$$\begin{aligned}
&(D_\tau e^n_{\sigma_\tau}, e^n_{\sigma_\tau}) + \|\nabla e^n_{\sigma_\tau}\|^2_{L^2} \\
&= -\,(D_\tau \rho^n_{\sigma_\tau}, e^n_{\sigma_\tau}) + b_2(\psi^{n-1}_\tau, \sigma^{n-1}_\tau, e^n_{\sigma_\tau}) - b_2(\psi^{n-1}_h, \sigma^{n-1}_h, e^n_{\sigma_\tau}) \\
&= -\,(D_\tau \rho^n_{\sigma_\tau}, e^n_{\sigma_\tau}) + [(\mathrm{Im}(\mathbf{curl}\,(\psi^{n-1}_\tau)^* \cdot \nabla(\psi^{n-1}_\tau)), e^n_{\sigma_\tau}) \\
&\quad - (\mathrm{Im}(\mathbf{curl}\,(\psi^{n-1}_h)^* \cdot \nabla(\psi^{n-1}_h)), e^n_{\sigma_\tau})] - [(|\psi^{n-1}_\tau|^2 \sigma^{n-1}_\tau, e^n_{\sigma_\tau}) - (|\psi^{n-1}_h|^2 \sigma^{n-1}_h, e^n_{\sigma_\tau})] \\
&\quad + [(\mathbf{curl}\,|\psi^{n-1}_\tau|^2 \cdot \mathbf{A}^{n-1}_\tau, e^n_{\sigma_\tau}) - (\mathbf{curl}\,|\psi^{n-1}_h|^2 \cdot \mathbf{A}^{n-1}_h, e^n_{\sigma_\tau})] \\
&:= -\,(D_\tau \rho^n_{\sigma_\tau}, e^n_{\sigma_\tau}) + J_4 + J_5 + J_6\,.
\end{aligned}$$

Before we prove the boundedness of numerical solution, we present a primary estimate of the error function $(\widetilde{e}^n_{\psi_\tau}, e^n_{\sigma_\tau})$ in the following lemma.

LEMMA 3.2. *Under the assumptions of Theorem* 2.1*, there exists a positive constant* h_0 *such that when* $\tau \le \tau_0$ *and* $h \le h_0$*,*

$$\|\widetilde{e}^n_{\psi_\tau}\|^2_{L^2} + \|e^n_{\sigma_\tau}\|^2_{L^2} + \tau \sum_{k=0}^{n} (\|\nabla \widetilde{e}^k_{\psi_\tau}\|^2_{L^2} + \|\nabla e^k_{\sigma_\tau}\|^2_{L^2}) \le h^{11/3}, \tag{3.19}$$

where τ_0 *is defined in Lemma* 3.1*.*

Proof. We prove (3.19) by mathematics induction. Since $\|\widetilde{e}^0_{\psi_\tau}\|_{L^2} = \|e^0_{\sigma_\tau}\|_{L^2} = \|\nabla\widetilde{e}^0_{\psi_\tau}\|_{L^2} = \|\nabla e^0_{\sigma_\tau}\|_{L^2} = 0$, (3.19) holds for $n = 0$. We assume that (3.19) holds for $0 \le n \le m-1$ for some integer $m \ge 1$, and below we prove that it holds for $n = m$. By an inverse inequality, Lemma 3.1, and (3.14), there exists $h_1 > 0$ such that when $h \le h_1$,

$$\|\psi_h^n\|_{L^\infty} + \|\sigma_h^n\|_{L^\infty} + \|\psi_h^n\|_{W^{1,4}} + \|\sigma_h^n\|_{W^{1,4}} \le C_K, \quad n \le m-1, \tag{3.20}$$

which, with (2.7), leads to

$$\|\mathbf{A}_h^{m-1}\|_{L^p} \le C_K, \quad 2 \le p \le 4\,. \tag{3.21}$$

Let $e^m_{\mathbf{A}_\tau} := \mathbf{A}^m_\tau - \mathbf{A}^m_h$. From (2.7) and (3.6), we obtain

$$\begin{aligned} e^{m-1}_{\mathbf{A}_\tau} &= (1-\tau|\psi_h^{m-1}|^2)e^{m-2}_{\mathbf{A}_\tau} - \tau(|\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2)\mathbf{A}_\tau^{m-2} + \tau(\mathbf{g}_\tau^{m-1} - \mathbf{g}_h^{m-1}) \\ &= (1-\tau|\psi_h^{m-1}|^2)e^{m-2}_{\mathbf{A}_\tau} - \tau(|\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2)\mathbf{A}_\tau^{m-2} \\ &\quad + \frac{\tau}{\kappa}\left[\mathrm{Im}((\widetilde{\rho}^{m-1}_{\psi_\tau} + \widetilde{e}^{m-1}_{\psi_\tau})^*\nabla\psi_h^{m-1}) + \mathrm{Im}((\psi_\tau^{m-1})^*\nabla(\widetilde{\rho}^{m-1}_{\psi_\tau} + \widetilde{e}^{m-1}_{\psi_\tau}))\right] \\ &\quad - \tau\,\mathbf{curl}\,(\rho^{m-1}_{\sigma_\tau} + e^{m-1}_{\sigma_\tau}) \\ &= (1-\tau|\psi_h^{m-1}|^2)e^{m-2}_{\mathbf{A}_\tau} + \frac{\tau}{\kappa}\mathrm{Im}((\psi_\tau^{m-1})^*\nabla\widetilde{\rho}^{m-1}_{\psi_\tau}) - \tau\,\mathbf{curl}\,\rho^{m-1}_{\sigma_\tau} + \tau G_\tau^{m-1}, \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} G_\tau^{m-1} = &-(|\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2)\mathbf{A}_\tau^{m-2} + \frac{1}{\kappa}\left[\mathrm{Im}((\psi_\tau^{m-1} - \psi_h^{m-1})^*\nabla\psi_h^{m-1})\right. \\ &\left. + \mathrm{Im}((\psi_\tau^{m-1})^*\nabla\widetilde{e}^{m-1}_{\psi_\tau})\right] - \mathbf{curl}\, e^{m-1}_{\sigma_\tau}\,. \end{aligned}$$

It follows that

$$e^{m-1}_{\mathbf{A}_\tau} = \frac{\tau}{\kappa}\sum_{k=0}^{m-1}\mathrm{Im}((\psi_\tau^k)^*\nabla\widetilde{\rho}^k_{\psi_\tau})\Psi^{k+1,m-1} + \tau\sum_{k=0}^{m-1}(G_\tau^k - \mathbf{curl}\,\rho^k_{\sigma_\tau})\Psi^{k+1,m-1},$$

and for $2 \le q \le 4$ and $0 \le k \le m-1$,

$$\begin{aligned} \|G_\tau^k\|_{L^p} \le & C_K\|\psi_\tau^k - \psi_h^k\|_{L^p}\|(\psi_\tau^k + \psi_h^k)\mathbf{A}_\tau^{k-1}\|_{L^\infty} \\ &+ C_K\|\psi_\tau^k\|_{L^\infty}\|\nabla\widetilde{e}^k_{\psi_\tau}\|_{L^p} + C_K\|\nabla\widetilde{e}^k_{\sigma_\tau}\|_{L^p} \\ &+ C_K\|\psi_\tau^k - \psi_h^k\|_{L^\infty}\|\nabla\widetilde{e}^k_{\psi_\tau}\|_{L^p} + C_K\|\psi_\tau^k - \psi_h^k\|_{L^p}\|\nabla\widetilde{R}_h\psi_\tau^k\|_{L^\infty} \\ \le & C_K(\|\nabla\widetilde{e}^k_{\psi_\tau}\|_{L^p} + \|\nabla e^k_{\sigma_\tau}\|_{L^p} + \|\widetilde{e}^k_{\psi_\tau}\|_{L^p} + \|\widetilde{\rho}^k_{\psi_\tau}\|_{L^p})\,. \end{aligned} \tag{3.23}$$

By noting (3.12), we have

(3.24)
$$(\nabla\widetilde{\rho}^m_{\psi_\tau}, \nabla\omega_h) - i\kappa(e^{m-1}_{\mathbf{A}_\tau}\psi_\tau^{m-1}, \nabla\omega_h) = \left(\tau\kappa\psi_\tau^{m-1}\sum_{k=0}^{m-1}(G_\tau^k - \mathbf{curl}\,\rho^k_{\sigma_\tau})\Psi^{k+1,m-1}, \nabla\omega_h\right)$$

for $\omega_h \in \mathcal{V}_h^r$. By Lemma 3.1, the induction assumption, and an inverse inequality, we have further

$$\|e_{\mathbf{A}_\tau}^{m-1}\|_{L^p} \le C_K\tau \sum_{k=0}^{m-1} \|\nabla \widetilde{\rho}_{\psi_\tau}^k\|_{L^p} + \tau \sum_{k=0}^{m-1} \|(G_\tau^k - \mathbf{curl}\,\rho_{\sigma_\tau}^k)\|_{L^p}$$

$$\le C_K(h + h^{2/p-1} \sum_{k=0}^{m-1} \tau(\|\nabla \widetilde{e}_{\psi_\tau}^k\|_{L^2} + \|\nabla e_{\sigma_\tau}^k\|_{L^2})) \le C_K h \tag{3.25}$$

for $2 \le p \le 4$ and

$$\begin{aligned}
|(e_{\mathbf{A}_\tau}^{m-1}, \xi)| \le{}& \frac{\tau}{\kappa} \sum_{k=0}^{m-1} \left|(\mathrm{Im}((\psi_\tau^k)^* \nabla \widetilde{\rho}_{\psi_\tau}^k) \Psi^{k+1,m-1}, \xi)\right| \\
&+ \tau \sum_{k=0}^{m-1} \left|((G_\tau^k - \mathbf{curl}\,\rho_{\sigma_\tau}^k) \Psi^{k+1,m-1}, \xi)\right| \\
\le{}& C_K\tau \sum_{k=0}^{m-1} (\|\widetilde{\rho}_{\psi_\tau}^k\|_{L^2} + \|\rho_{\sigma_\tau}^k\|_{L^2})(\|\Psi^{k+1,m-1}\|_{W^{1,4}}\|\xi\|_{L^4} \\
&+ \|\Psi^{k+1,m-1}\|_{L^\infty}\|\xi\|_{H^1}) \\
&+ C_K\tau \sum_{k=0}^{m-1} |(G_\tau^k \Psi^{k+1,m-1}, \xi)| \\
\le{}& C_K\tau \sum_{k=0}^{m-1} \big(|((|\psi_\tau^k|^2 - |\psi_h^k|^2)\mathbf{A}_\tau^{k-1}, \Psi^{k+1,m-1}\xi)| \\
&+ |((\widetilde{\rho}_{\psi_\tau}^k + \widetilde{e}_{\psi_\tau}^k)^* \nabla \psi_h^k, \Psi^{k+1,m-1}\xi)|\big) \\
&+ C_K\tau \sum_{k=0}^{m-1} \left|((\psi_\tau^k)^* \nabla \widetilde{e}_{\psi_\tau}^k - \mathbf{curl}\, e_{\sigma_\tau}^k, \Psi^{k+1,m-1}\xi)\right| + C_K h^2 \|\xi\|_{H^1}
\end{aligned}$$

$$\le C_K \sum_{k=0}^{m-1} \tau(h^2 + \|\widetilde{e}_{\psi_\tau}^k\|_{L^2} + \|e_{\sigma_\tau}^k\|_{L^2})\|\xi\|_{H^1} \tag{3.26}$$

for $\xi \in \mathbf{H}_0^1(\Omega) := \{\mathbf{u} \in \mathbf{H}_0^1(\Omega),\ \mathbf{u}\cdot\mathbf{n} = 0, \text{ on } \partial\Omega\}$, which leads to

$$\|e_{\mathbf{A}_\tau}^{m-1}\|_{H^{-1}} := \sup_{\xi \in \mathbf{H}_0^1} \frac{(e_{\mathbf{A}_\tau}^{m-1}, \xi)}{\|\xi\|_{H^1}} \le C_K \sum_{k=0}^{m-1} \tau(h^2 + \|\widetilde{e}_{\psi_\tau}^k\|_{L^2} + \|e_{\sigma_\tau}^k\|_{L^2}). \tag{3.27}$$

With the above inequalities, we are able to estimate these terms in the right-hand sides of (3.17)–(3.18). It is easy to see that

$$\begin{aligned}
|(D_\tau \widetilde{\rho}_{\psi_\tau}^m, \widetilde{e}_{\psi_\tau}^m)| &\le Ch^4 \|D_\tau \psi_\tau^m\|_{H^2}^2 + \|\widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 \\
|(D_\tau \rho_{\sigma_\tau}^m, e_{\sigma_\tau}^m)| &\le Ch^4 \|D_\tau \sigma_\tau^m\|_{H^2}^2 + \|e_{\sigma_\tau}^m\|_{L^2}^2 .
\end{aligned}$$

By using (3.20)–(3.21), (3.25)–(3.27), and Lemma 3.1 and noting the fact $\mathbf{A}_\tau^{m-1}\cdot\mathbf{n} = 0$ on boundary, we get

$$\begin{aligned}
|J_3| \le{}& |((|\mathbf{A}_\tau^{m-1}|^2 + |\psi_\tau^{m-1}|^2 - |\mathbf{A}_h^{m-1}|^2 - |\psi_h^{m-1}|^2)\psi_\tau^{m-1}, \widetilde{e}_{\psi_\tau}^m)| \\
&+ |((|\mathbf{A}_h^{m-1}|^2 + |\psi_h^{m-1}|^2 - 1)(\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}), \widetilde{e}_{\psi_\tau}^m)| \\
\le{}& C_K |(e_{\mathbf{A}_\tau}^{m-1}(2\mathbf{A}_\tau^{m-1} - e_{\mathbf{A}_\tau}^{m-1})\psi_\tau^{m-1}, \widetilde{e}_{\psi_\tau}^m)| + \||\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2\|_{L^2}\|\widetilde{e}_{\psi_\tau}^m\|_{L^2} \\
&+ \|\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}\|\widetilde{e}_{\psi_\tau}^m\|_{L^6}
\end{aligned}$$

$$
\begin{aligned}
&\le C_K \|e_{\mathbf{A}_\tau}^{m-1}\|_{H^{-1}} \|\widetilde{e}_{\psi_\tau}^m\|_{H^1} + \|e_{\mathbf{A}_\tau}^{m-1}\|_{L^3}^2 \|\widetilde{e}_{\psi_\tau}^m\|_{L^3} + \|\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2} \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2} \\
&\le \frac{1}{8\kappa^2} \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 + C_K \left(\sum_{k=0}^{m-1} \tau (\|\widetilde{e}_{\psi_\tau}^k\|_{L^2}^2 + \|e_{\sigma_\tau}^k\|_{L^2}^2) + \|\widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 + \|\widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + h^4 \right),
\end{aligned}
$$

$$
\begin{aligned}
|J_4| &\le \frac{1}{\kappa} |(\mathrm{Im}((\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1})^* \nabla (\psi_h^{m-1})), \mathbf{curl}\, e_{\sigma_\tau}^m)| \\
&\quad + \frac{1}{\kappa} |(\mathrm{Im}(\mathbf{curl}\, (\psi_\tau^{m-1})^* (\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1})), \nabla e_{\sigma_\tau}^m)| \\
&\le \frac{1}{8\kappa^2} \|\nabla \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + \frac{1}{4} \|\nabla e_{\sigma_\tau}^m\|_{L^2}^2 + C_K (\|\widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + h^4)
\end{aligned}
$$

and

$$
\begin{aligned}
|J_5| &\le |((|\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2) \sigma_\tau^{m-1}, e_{\sigma_\tau}^m)| + |(|\psi_h^{m-1}|^2 (\rho_{\sigma_\tau}^{m-1} + e_{\sigma_\tau}^{m-1}), e_{\sigma_\tau}^m)| \\
&\le C_K (\|\widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + \|e_{\sigma_\tau}^{m-1}\|_{L^2}^2 + \|e_{\sigma_\tau}^m\|_{L^2}^2 + h^4),
\end{aligned}
$$

where we have used classical Gagliardo–Nirenberg interpolation inequality (3.1). Moreover, using integration by part and (3.25)–(3.27),

$$
\begin{aligned}
|J_2| &= \frac{1}{\kappa} |(\nabla (\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}), \mathbf{A}_\tau^{m-1} \widetilde{e}_{\psi_\tau}^m) + (\nabla \psi_\tau^{m-1}, e_{\mathbf{A}_\tau}^{m-1} \widetilde{e}_{\psi_\tau}^m) \\
&\quad - (\nabla (\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}), e_{\mathbf{A}_\tau}^{m-1} \widetilde{e}_{\psi_\tau}^m)| \\
&\le C_K (\|\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2} \|\widetilde{e}_{\psi_\tau}^m\|_{H^1} + \|e_{\mathbf{A}_\tau}^{m-1}\|_{H^{-1}} \|\widetilde{e}_{\psi_\tau}^m\|_{H^1} \\
&\quad + \|\nabla (\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi}^{m-1})\|_{L^2} \|e_{\mathbf{A}_\tau}^{m-1}\|_{L^3} \|\widetilde{e}_{\psi_\tau}^m\|_{L^6}) \\
&\le \frac{1}{8\kappa^2} (\|\nabla \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2}^2) \\
&\quad + C_K \left(\sum_{k=0}^{m-1} \tau (\|\widetilde{e}_{\psi_\tau}^k\|_{L^2}^2 + \|e_{\sigma_\tau}^k\|_{L^2}^2) + \|\widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + h^4 \right)
\end{aligned}
$$

and

$$
\begin{aligned}
|J_6| &\le |(\mathbf{curl}\, (|\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2) \cdot \mathbf{A}_\tau^{m-1}, e_{\sigma_\tau}^m)| + |(\mathbf{curl}\, |\psi_\tau^{m-1}|^2 \cdot e_{\mathbf{A}_\tau}^{m-1}, e_{\sigma_\tau}^m)| \\
&\quad + |(\mathbf{curl}\, (|\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2) \cdot e_{\mathbf{A}_\tau}^{m-1}, e_{\sigma_\tau}^m)| \\
&\le C_K \| |\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2 \|_{L^2} \|\mathbf{curl}\, e_{\sigma_\tau}^m\|_{L^2} + C_K \|e_{\mathbf{A}_\tau}^{m-1}\|_{H^{-1}} \|e_{\sigma_\tau}^m\|_{H^1} \\
&\quad + \|\mathbf{curl}\, (|\psi_\tau^{m-1}|^2 - |\psi_h^{m-1}|^2)\|_{L^2} \|e_{\mathbf{A}_\tau}^{m-1}\|_{L^3} \|e_{\sigma_\tau}^m\|_{L^6} \\
&\le \frac{1}{8\kappa^2} \|\nabla \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + \frac{1}{4} \|\nabla e_{\sigma_\tau}^m\|_{L^2}^2 \\
&\quad + C_K \left(\sum_{k=0}^{m-1} \tau (\|\widetilde{e}_{\psi_\tau}^k\|_{L^2}^2 + \|e_{\sigma_\tau}^k\|_{L^2}^2) + \|\widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + h^4 \right),
\end{aligned}
$$

where we have noted $\nabla \psi_\tau^{m-1} \cdot n = e_{\sigma_\tau}^m = 0$ on boundary. Finally, by (3.21), (3.24), (3.23), and Lemma 3.1, we have

$$
\begin{aligned}
|J_1| &\le \left| \frac{1}{\kappa^2} (\nabla \widetilde{\rho}_{\psi_\tau}^m, \nabla \widetilde{e}_{\psi_\tau}^m) - \frac{i}{\kappa} \left(e_{\mathbf{A}_\tau}^{m-1} \psi_\tau^{m-1}, \nabla \widetilde{e}_{\psi_\tau}^m \right) \right| + \left| \frac{i}{\kappa} \left(\mathbf{A}_h^{m-1} (\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}), \nabla \widetilde{e}_{\psi_\tau}^m \right) \right| \\
&\le C_K \left(\left| \sum_{k=0}^{m-1} \tau ((G_\tau^l - \mathbf{curl}\, \rho_{\sigma_\tau}^k) \psi_\tau^{m-1}, \nabla \widetilde{e}_{\psi_\tau}^m) \right| + \left| (\mathbf{A}_h^{m-1} (\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}), \nabla \widetilde{e}_{\psi_\tau}^m) \right| \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_K \sum_{k=0}^{m-1} \tau \|G_\tau^k\|_{L^2} \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2} + C_K \sum_{k=0}^{m-1} \tau \|\rho_{\sigma_\tau}^k\|_{L^4} \|\mathbf{curl}\, \psi_\tau^{m-1}\|_{L^4} \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2} \\
&\quad + C_K \|\widetilde{\rho}_{\psi_\tau}^{m-1} + \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^4} \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2} \\
&\leq \frac{1}{8\kappa^2} \left(\|\nabla \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 \right) \\
&\quad + C_K \left(\sum_{k=0}^{m-1} \tau \left(\|\nabla \widetilde{e}_{\psi_\tau}^k\|_{L^2}^2 + \|\nabla e_{\sigma_\tau}^k\|_{L^2}^2 \right) + \|\widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + h^4 \right),
\end{aligned}
$$

where we have noted the fact that

$$
(\mathbf{curl}\, (\rho_{\sigma_\tau}^k \psi_\tau^{m-1}), \nabla \widetilde{e}_{\psi_\tau}^m) = 0 \,. \tag{3.28}
$$

With the above estimates, adding (3.17) and (3.18) together gives

$$
\begin{aligned}
&D_\tau \|e_{\sigma_\tau}^m\|_{L^2}^2 + D_\tau \|\widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 + \|\nabla e_{\sigma_\tau}^m\|_{L^2}^2 + \frac{1}{\kappa^2} \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 \\
&\quad \leq C_K \left(\sum_{k=0}^{m-1} (\|\nabla e_{\sigma_\tau}^k\|_{L^2}^2 + \|\nabla \widetilde{e}_{\psi_\tau}^k\|_{L^2}^2) + \|\widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + \|e_{\sigma_\tau}^{m-1}\|_{L^2}^2 + h^4 \right) \\
&\qquad + \frac{1}{2\kappa^2} \|\nabla \widetilde{e}_{\psi_\tau}^{m-1}\|_{L^2}^2 + \frac{3}{8\kappa^2} \|\nabla \widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 + \frac{1}{2} \|\nabla e_{\sigma_\tau}^m\|_{L^2}^2,
\end{aligned}
$$

which, with Gronwall's inequality, leads to

$$
\|\widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 + \|e_{\sigma_\tau}^m\|_{L^2}^2 + \tau \sum_{k=1}^{m} \left(\|\nabla \widetilde{e}_{\psi_\tau}^k\|_{L^2}^2 + \|\nabla e_{\sigma_\tau}^k\|_{L^2}^2 \right) \leq C_K h^4 \,.
$$

Thus, when $h \leq h_0 = \min\{1/C_K^3, h_1\}$,

$$
\|\widetilde{e}_{\psi_\tau}^m\|_{L^2}^2 + \|e_{\sigma_\tau}^m\|_{L^2}^2 + \tau \sum_{k=1}^{m} \left(\|\nabla \widetilde{e}_{\psi_\tau}^k\|_{L^2}^2 + \|\nabla e_{\sigma_\tau}^k\|_{L^2}^2 \right) \leq h^{11/3} \,.
$$

The mathematics induction is closed, and therefore we have proved that (3.19) holds for $0 \leq n \leq N$. ∎

From the above proof, we also see that the boundedness of numerical solution in (3.20)–(3.21) holds for $1 \leq m \leq N + 1$.

4. Proof of Theorem 2.1.

Let

$$
\begin{aligned}
&\widetilde{\rho}_\psi^n := \psi^n - \widetilde{R}_h \psi^n, \quad \rho_\sigma^n := \sigma^n - R_h \sigma^n \\
&\widetilde{e}_\psi^n := \widetilde{R}_h \psi^n - \psi_h^n, \quad e_\sigma^n := R_h \sigma^n - \sigma_h^n \,.
\end{aligned}
$$

Similarly, by the regularity assumption (2.9) and classical finite element theory [2, 37] and following [3, 4, Lemma 3.1], we can get

$$
\|\widetilde{\rho}_\psi^n\|_{L^p} + h \|\widetilde{\rho}_\psi^n\|_{W^{1,p}} + \|\rho_\sigma^n\|_{L^p} + h \|\nabla \rho_\sigma^n\|_{L^p} \leq C h^{r+1} \tag{4.1}
$$

$$
\sum_{k=1}^{n} \tau (\|D_\tau \widetilde{\rho}_\psi^k\|_{L^2} + \|D_\tau \rho_\sigma^k\|_{L^2}) \leq C h^{r+1} \tag{4.2}
$$

and

$$\|\widetilde{\rho}_\psi^n\|_{W^{1,\infty}} + \|\rho_\sigma^n\|_{W^{1,\infty}} \le C_K \tag{4.3}$$

for $2 \le p \le 4$ when $r = 1$ and $2 \le p \le 3$ when $r \ge 2$.

From the fully discrete system (2.5)–(2.6) and the variational formulation (2.1)–(2.2), we see that $\widetilde{e}_\psi^n$ and e_σ^n satisfy

$$\begin{aligned}
&(D_\tau \widetilde{e}_\psi^n, \widetilde{e}_\psi^n) + \frac{1}{\kappa^2}(\nabla \widetilde{e}_\psi^n, \nabla \widetilde{e}_\psi^n) \\
&\quad = (T_\psi^n, \widetilde{e}_\psi^n) - (D_\tau \widetilde{\rho}_\psi^n, \widetilde{e}_\psi^n) - \frac{1}{\kappa^2}(\nabla \widetilde{\rho}_\psi^n, \nabla \widetilde{e}_\psi^n) \\
&\qquad + b_1(\psi^{n-1}, \sigma^{n-1}, \widetilde{e}_\psi^n) - b_1(\psi_h^{n-1}, \sigma_h^{n-1}, \widetilde{e}_\psi^n) \\
&\quad = (T_\psi^n, \widetilde{e}_\psi^n) - (D_\tau \widetilde{\rho}_\psi^n, \widetilde{e}_\psi^n) \\
&\qquad - \left[\frac{1}{\kappa^2}(\nabla \widetilde{\rho}_\psi^n, \nabla \widetilde{e}_\psi^n) - \frac{i}{\kappa}(\mathbf{A}^{n-1}\psi^{n-1}, \nabla \widetilde{e}_\psi^n) + \frac{i}{\kappa}(\mathbf{A}_h^{n-1}\psi_h^{n-1}, \nabla \widetilde{e}_\psi^n)\right] \\
&\qquad - \left[\frac{i}{\kappa}(\nabla \psi^{n-1}, \mathbf{A}^{n-1}\widetilde{e}_\psi^n) - \frac{i}{\kappa}(\nabla \psi_h^{n-1}, \mathbf{A}_h^{n-1}\widetilde{e}_\psi^n)\right] \\
&\qquad - \left[((|\mathbf{A}^{n-1}|^2 + |\psi^{n-1}|^2 - 1)\psi^{n-1}, \widetilde{e}_\psi^n) - ((|\mathbf{A}_h^{n-1}|^2 + |\psi_h^{n-1}|^2 - 1)\psi_h^{n-1}, \widetilde{e}_\psi^n)\right] \\
&\quad := (T_\psi^n, \widetilde{e}_\psi^n) - (D_\tau \widetilde{\rho}_\psi^n, \widetilde{e}_\psi^n) + \widetilde{J}_1 + \widetilde{J}_2 + \widetilde{J}_3
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&(D_\tau e_\sigma^n, e_\sigma^n) + (\nabla e_\sigma^n, \nabla e_\sigma^n) \\
&\quad = (T_\sigma^n, e_\sigma^n) - (D_\tau \rho_\sigma^n, e_\sigma^n) + b_2(\psi^{n-1}, \sigma^{n-1}, e_\sigma^n) - b_2(\psi_h^{n-1}, \sigma_h^{n-1}, e_\sigma^n) \\
&\quad = (T_\sigma^n, e_\sigma^n) - (D_\tau \rho_\sigma^n, e_\sigma^n) - \frac{1}{\kappa}[(\mathrm{Im}(\mathbf{curl}\,(\psi^{n-1})^* \cdot \nabla(\psi^{n-1})), e_\sigma^n) \\
&\qquad - (\mathrm{Im}(\mathbf{curl}\,(\psi_h^{n-1})^* \cdot \nabla(\psi_h^{n-1})), e_\sigma^n)] - [(|\psi^{n-1}|^2\sigma^{n-1}, e_\sigma^n) - (|\psi_h^{n-1}|^2\sigma_h^{n-1}, e_\sigma^n)] \\
&\qquad + [(\mathbf{curl}\,|\psi^{n-1}|^2 \cdot \mathbf{A}^{n-1}, e_\sigma^n) - (\mathbf{curl}\,|\psi_h^{n-1}|^2 \cdot \mathbf{A}_h^{n-1}, e_\sigma^n)] \\
&\quad := (T_\sigma^n, e_\sigma^n) - (D_\tau \rho_\sigma^n, e_\sigma^n) + \widetilde{J}_4 + \widetilde{J}_5 + \widetilde{J}_6,
\end{aligned} \tag{4.5}$$

where T_ψ^n and T_σ^n denote the truncation errors. By the regularity assumption (2.9), we have

$$\sum_{k=0}^{N} \tau(\|T_\psi^k\|_{L^2}^2 + \|T_\sigma^k\|_{L^2}^2) \le C\tau^2\,.$$

An optimal estimate of the error function $(\widetilde{e}_\psi^n, e_\sigma^n)$ is given in the following lemma.

LEMMA 4.1. *Under the assumptions of Theorem* 2.1*, when* $\tau \le \tau_0$ *and* $h \le h_0$*,*

$$\|\widetilde{e}_\psi^n\|_{L^2}^2 + \|e_\sigma^n\|_{L^2}^2 + \tau\sum_{k=1}^{n}(\|\nabla \widetilde{e}_\psi^k\|_{L^2}^2 + \|\nabla e_\sigma^k\|_{L^2}^2) \le \widehat{C}_0(\tau^2 + h^{2(r+1)}), \tag{4.6}$$

where τ_0 *and* h_0 *are defined in Lemmas* 3.1 *and* 3.2*, respectively, and* $\widehat{C}_0$ *is a positive constant, independent of* n*,* h*, and* τ *and dependent upon* K *and* T*.*

Proof. Let $e_{\mathbf{A}}^n := \mathbf{A}^n - \mathbf{A}_h^n$. It follows from (2.4) and (2.7) that

$$\begin{aligned} &D_\tau e_{\mathbf{A}}^n + |\psi_h^n|^2 e_{\mathbf{A}}^{n-1} \\ =&(|\psi_h^n|^2 - |\psi^n|^2)\mathbf{A}^n - (\mathbf{A}^n - \mathbf{A}^{n-1})|\psi_h^n|^2 + D_\tau \mathbf{A}^n - \frac{\partial \mathbf{A}^n}{\partial t} + g^n - g_h^n \\ =&\left(D_\tau \mathbf{A}^n - \frac{\partial \mathbf{A}^n}{\partial t} - (\mathbf{A}^n - \mathbf{A}^{n-1})|\psi_h^n|^2\right) - (|\psi^n|^2 - |\psi_h^n|^2)\mathbf{A}^n \\ &+ \frac{1}{\kappa}\mathrm{Im}((\widetilde{\rho}_\psi^n + \widetilde{e}_\psi^n)^* \nabla \psi_h^n) + \frac{1}{\kappa}\mathrm{Im}((\psi^n)^* \nabla(\widetilde{\rho}_\psi^n + \widetilde{e}_\psi^n)) - \mathbf{curl}\,(\rho_\sigma^n + e_\sigma^n) \\ :=&\frac{1}{\kappa}\mathrm{Im}((\psi^n)^*\nabla\widetilde{\rho}_\psi^n) + G^n - \mathbf{curl}\,\rho_\sigma^n + T_{\mathbf{A}}^n, \end{aligned}$$

where $T_{\mathbf{A}}^n$ denotes the truncation error and

$$G^n = -(|\psi^n|^2 - |\psi_h^n|^2)\mathbf{A}^n + \frac{1}{\kappa}\mathrm{Im}((\widetilde{\rho}_\psi^n + \widetilde{e}_\psi^n)^* \nabla \psi_h^n) + \frac{1}{\kappa}\mathrm{Im}((\psi^n)^* \nabla \widetilde{e}_\psi^n) - \mathbf{curl}\, e_\sigma^n\,.$$

Therefore,

$$e_{\mathbf{A}}^n = \frac{\tau}{\kappa}\sum_{k=0}^{n} \mathrm{Im}((\psi^k)^* \nabla \widetilde{\rho}_\psi^k)\Psi^{k+1,n} + \tau \sum_{k=0}^{n}(G^k - \mathbf{curl}\,\rho_\sigma^k + T_{\mathbf{A}}^k)\Psi^{k+1,n}\,. \tag{4.7}$$

By the regularity assumption (2.9) and Taylor expansion, we have

$$\sum_{k=0}^{N} \tau \|T_{\mathbf{A}}^k\|_{L^2}^2 \le C_K \tau^2\,. \tag{4.8}$$

By noting (3.12) and (4.7), we get

(4.9)

$$(\nabla \widetilde{\rho}_\psi^n, \nabla\omega_h) - i\kappa(e_{\mathbf{A}}^{n-1}\psi^{n-1}, \nabla\omega_h) = \tau\kappa(\psi^{n-1}\sum_{k=0}^{n-1}(G^k - \mathbf{curl}\,\rho_\sigma^k + T_{\mathbf{A}}^k)\Psi^{k+1,n-1}, \nabla\omega_h)$$

for $\omega_h \in \mathcal{V}_h^r$, and by (3.20)–(3.21), with similar proof to (3.23), (3.25), and (3.27), we can get

$$\|G^k\|_{L^p} \le C_K(\|\nabla \widetilde{e}_\psi^k\|_{L^p} + \|\nabla e_\sigma^k\|_{L^p} + \|\widetilde{e}_\psi^k\|_{L^p} + \|\widetilde{\rho}_\psi^k\|_{L^p}), \quad 0 \le k \le n-1 \tag{4.10}$$

$$\|e_{\mathbf{A}}^n\|_{L^p} \le C_K(h^r + h^{2/p-1}\sum_{k=0}^{n}\tau(\|\nabla \widetilde{e}_\psi^k\|_{L^2} + \|\nabla e_\sigma^k\|_{L^2})) + C_K\tau \tag{4.11}$$

$$\|e_{\mathbf{A}}^n\|_{H^{-1}} := \sup_{\xi\in \mathbf{H}_0^1}\frac{(e_{\mathbf{A}}^n, \xi)}{\|\xi\|_{H^1}} \le C_K\sum_{k=0}^{n}\tau(\|\widetilde{e}_\psi^k\|_{L^2} + \|e_\sigma^k\|_{L^2}) + C_K(\tau + h^{r+1}) \tag{4.12}$$

for $2 \le p \le 3$.

We estimate below these terms in the right-hand sides of (4.4)–(4.5). By (4.1)–(4.3),

$$\begin{aligned} |(D_\tau \widetilde{\rho}_\psi^n, \widetilde{e}_\psi^n)| &\le C_K h^{2r+2} + \|\widetilde{e}_\psi^n\|_{L^2}^2 \\ |(D_\tau \rho_\sigma^n, e_\sigma^n)| &\le C_K h^{2r+2} + \|e_\sigma^n\|_{L^2}^2\,. \end{aligned}$$

By noting (3.20)–(3.21) and (4.11)–(4.12), we have

$$
\begin{aligned}
|\widetilde{J_3}| &\le |((|\mathbf{A}^{n-1}|^2 + |\psi^{n-1}|^2 - |\mathbf{A}_h^{n-1}|^2 - |\psi_h^{n-1}|^2)\psi^{n-1}, \widetilde{e}_\psi^n)| \\
&\quad + |((|\mathbf{A}_h^{n-1}|^2 + |\psi_h^{n-1}|^2 - 1)(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}), \widetilde{e}_\psi^n)| \\
&\le \frac{1}{8\kappa^2}\|\nabla \widetilde{e}_\psi^n\|_{L^2}^2 + C_K(\|\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \sum_{k=0}^{n-1}\tau(\|\nabla \widetilde{e}_\psi^k\|_{L^2}^2 + \|\nabla e_\sigma^k\|_{L^2}^2) + \tau^2 + h^{2r+2}),
\end{aligned}
$$

$$
\begin{aligned}
|\widetilde{J_4}| \le& \frac{1}{\kappa}|(\mathrm{Im}((\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1})^* \nabla \psi_\tau^{n-1}), \mathbf{curl}\, e_\sigma^n)| \\
&+ \frac{1}{\kappa}|(\mathrm{Im}((\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1})^* \nabla \widetilde{\rho}_{\psi_\tau}^{n-1}), \mathbf{curl}\, e_\sigma^n)| \\
&+ \frac{1}{\kappa}|(\mathrm{Im}(\nabla(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1})^* \widetilde{e}_{\psi_\tau}^{n-1}), \mathbf{curl}\, e_\sigma^n)| \\
&+ \frac{1}{\kappa}|(\mathrm{Im}(\mathbf{curl}\,(\psi^{n-1})^*(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1})), \nabla e_\sigma^n)| \\
\le& C_K\|\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}\|_{L^2}\|\mathbf{curl}\, e_\sigma^n\|_{L^2} + C_K\|\widetilde{e}_{\psi_\tau}^{n-1}\|_{L^2}\|\mathbf{curl}\, e_\sigma^n\|_{L^2} \\
&+ C_K\|\nabla \widetilde{e}_\psi^{n-1}\|_{L^2}\|\widetilde{e}_{\psi_\tau}^{n-1}\|_{L^\infty}\|\mathbf{curl}\, e_\sigma^n\|_{L^2} + C_K\|\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}\|_{L^2}\|\nabla e_\sigma^n\|_{L^2} \\
\le& \frac{1}{8\kappa^2}\|\nabla \widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \frac{1}{4}\|\nabla e_\sigma^n\|_{L^2}^2 + C_K(\|\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \tau^2 + h^{2r+2}) \\
&+ C_K\|\nabla \widetilde{e}_{\psi_\tau}^{n-1}\|_{L^6}^2(\|\nabla \widetilde{e}_\psi^{n-1}\|_{L^2}^2 + h^{2r+2})
\end{aligned}
$$

$$
\begin{aligned}
|\widetilde{J_5}| &\le |((|\psi^{n-1}|^2 - |\psi_h^{n-1}|^2)\sigma^{n-1}, e_\sigma^n)| + |(|\psi_h^{n-1}|^2(\rho_\sigma^{n-1} + e_\sigma^{n-1}), e_\sigma^n)| \\
&\le C_K(\|\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \|e_\sigma^{n-1}\|_{L^2}^2 + \|e_\sigma^n\|_{L^2}^2 + h^{2r+2}).
\end{aligned}
$$

Using integration by part, we further get

$$
\begin{aligned}
|\widetilde{J_2}| &= \frac{1}{\kappa}|(\nabla(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}), \mathbf{A}_\tau^{n-1}\widetilde{e}_\psi^n) + (\nabla\psi^{n-1}, e_{\mathbf{A}}^{n-1}\widetilde{e}_\psi^n) - (\nabla(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}), e_{\mathbf{A}_\tau}^{n-1}\widetilde{e}_\psi^n)| \\
&\le C_K(\|\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}\|_{L^2}\|\nabla \widetilde{e}_\psi^n\|_{L^2} + \|e_{\mathbf{A}}^{n-1}\|_{H^{-1}}\|\widetilde{e}_\psi^n\|_{H^1} \\
&\quad + \|\nabla(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1})\|_{L^3}\|e_{\mathbf{A}_\tau}^{n-1}\|_{L^3}\|\widetilde{e}_\psi^n\|_{L^6}) \\
&\le \frac{1}{8\kappa^2}(\|\nabla \widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \|\nabla \widetilde{e}_\psi^n\|_{L^2}^2) + C_K(\|\widetilde{e}_\psi^{n-1}\|_{L^2}^2) \\
&\quad + C_K\sum_{k=0}^{n-1}\tau(\|\nabla \widetilde{e}_\psi^k\|_{L^2}^2 + \|\nabla e_\sigma^k\|_{L^2}^2) + \tau^2 + h^{2r+2}
\end{aligned}
$$

$$
\begin{aligned}
|\widetilde{J_6}| &\le |(\mathbf{curl}\,(|\psi^{n-1}|^2 - |\psi_h^{n-1}|^2)\cdot \mathbf{A}_\tau^{n-1}, e_\sigma^n)| + |(\mathbf{curl}\,|\psi^{n-1}|^2 \cdot e_{\mathbf{A}}^{n-1}, e_\sigma^n)| \\
&\quad + |(\mathbf{curl}\,(|\psi^{n-1}|^2 - |\psi_h^{n-1}|^2)\cdot e_{\mathbf{A}_\tau}^{n-1}, e_\sigma^n)| \\
&\le C_K(\||\psi^{n-1}|^2 - |\psi_h^{n-1}|^2\|_{L^2}\|\mathbf{curl}\, e_\sigma^n\|_{L^2} + \|e_{\mathbf{A}}^{n-1}\|_{H^{-1}}\|e_\sigma^n\|_{H^1} \\
&\quad + \|\mathbf{curl}\,(|\psi^{n-1}|^2 - |\psi_h^{n-1}|^2)\|_{L^2}\|e_{\mathbf{A}_\tau}^{n-1}\|_{L^3}\|e_\sigma^n\|_{L^6}) \\
&\le \frac{1}{8\kappa^2}\|\nabla \widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \frac{1}{4}\|\nabla e_\sigma^n\|_{L^2}^2 \\
&\quad + C_K(\|\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \sum_{k=0}^{n-1}\tau(\|\nabla \widetilde{e}_\psi^k\|_{L^2}^2 + \|\nabla e_\sigma^k\|_{L^2}^2) + \tau^2 + h^{2r+2}),
\end{aligned}
$$

where we have used (4.11)–(4.12). Moreover, by noting (4.7) and (4.9),

$$
\begin{aligned}
|\widetilde{J_1}| &\le |-\frac{1}{\kappa^2}(\nabla\widetilde{\rho}_\psi^n, \nabla\widetilde{e}_\psi^n) + \frac{i}{\kappa}(e_{\mathbf{A}}^{n-1}\psi^{n-1}, \nabla\widetilde{e}_\psi^n)| + |\frac{i}{\kappa}(\mathbf{A}_h^{n-1}(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}), \nabla\widetilde{e}_\psi^n)| \\
&\le C_K \left(|\sum_{k=0}^{n-1} \tau((G^k - \mathbf{curl}\,\rho_\sigma^k)\psi^{n-1}, \nabla\widetilde{e}_\psi^n)| \right. \\
&\qquad \left. + |(\mathbf{A}_h^{n-1}(\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}), \nabla\widetilde{e}_\psi^n)| + \tau\|\nabla\widetilde{e}_\psi^n\|_{L^2} \right) \\
&\le C_K \left(\sum_{k=0}^{n-1} \tau(\|G^k\|_{L^2} + \|\rho_\sigma^k\|_{L^3})\|\nabla\widetilde{e}_\psi^n\|_{L^2} \right. \\
&\qquad \left. + \|\widetilde{\rho}_\psi^{n-1} + \widetilde{e}_\psi^{n-1}\|_{L^2}\|\nabla\widetilde{e}_\psi^n\|_{L^2} + \tau\|\nabla\widetilde{e}_\psi^n\|_{L^2} \right) \\
&\le \frac{1}{8\kappa^2}(\|\nabla\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \|\nabla\widetilde{e}_\psi^n\|_{L^2}^2) \\
&\quad + C_K \left(\|\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \sum_{k=0}^{n-1} \tau(\|\nabla\widetilde{e}_\psi^k\|_{L^2}^2 + \|\nabla e_\sigma^k\|_{L^2}^2) + \tau^2 + h^{2r+2} \right),
\end{aligned}
$$

where we have also noted $(\mathbf{curl}\,(\rho_\sigma^k\psi^{n-1}), \nabla\widetilde{e}_\psi^n) = 0$. Finally, adding (4.4) and (4.5) together and using the above estimates, we arrive at

$$
\begin{aligned}
&(D_\tau e_\sigma^n, e_\sigma^n) + (D_\tau\widetilde{e}_\psi^n, \widetilde{e}_\psi^n) + \|\nabla e_\sigma^n\|_{L^2}^2 + \frac{1}{\kappa^2}\|\nabla\widetilde{e}_\psi^n\|_{L^2}^2 \\
\le& C_K \left[\|\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \|e_\sigma^{n-1}\|_{L^2}^2 + \sum_{k=0}^{n-1} \tau(\|\nabla\widetilde{e}_\psi^k\|_{L^2}^2 + \|\nabla e_\sigma^k\|_{L^2}^2)) + \tau^2 + h^{2r+2} \right] \\
&+ \frac{1}{2\kappa^2}\|\nabla\widetilde{e}_\psi^{n-1}\|_{L^2}^2 + \frac{3}{8\kappa^2}\|\nabla\widetilde{e}_\psi^n\|_{L^2}^2 + \frac{1}{2}\|\nabla e_\sigma^n\|_{L^2}^2 . \qquad (4.13)
\end{aligned}
$$

By the discrete Gronwalls inequality, we get

$$
\|e_\sigma^n\|_{L^2}^2 + \|\widetilde{e}_\psi^n\|_{L^2}^2 + \tau\sum_{k=1}^{n}(\|\nabla e_\sigma^k\|_{L^2}^2 + \|\nabla\widetilde{e}_\psi^k\|_{L^2}^2) \le \widehat{C}_0(\tau^2 + h^{2r+2}), \qquad (4.14)
$$

where $\widehat{C}_0 = \exp(8TC_K)$. The proof of Lemma 4.1 is complete. ∎

Theorem 2.1 follows immediately from Lemma 4.1 and the projection errors (4.1)–(4.3).

5. Numerical experiments. In this section, we provide two numerical examples in 2D and 3D spaces, respectively, to confirm our theoretical analysis and show the efficiency of the linearized Galerkin FEMs for the mixed formulation. The computations are carried out with the free blackPython package FEniCS [28].

Example 5.1. In this example, we consider an artificial 2D problem

$$
\frac{\partial\psi}{\partial t} + (\frac{i}{\kappa}\nabla + \mathbf{A})^2\psi + (|\psi|^2 - 1)\psi = g \qquad (5.1)
$$

$$
\frac{\partial\mathbf{A}}{\partial t} + \mathbf{curl}\,\mathrm{curl}\mathbf{A} + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A} = \mathbf{f} \qquad (5.2)
$$

for $t \in (0,T]$, $x \in \Omega$, with boundary and initial conditions

$$\frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \text{curl}\mathbf{A} = H_e, \quad \mathbf{A}\cdot\mathbf{n} = 0 \qquad \text{on} \quad \partial\Omega,$$
$$\psi(x,0) = \psi_0(x), \quad \mathbf{A}(x,0) = \mathbf{A}_0(x) \qquad \text{in} \quad \Omega,$$

where $\Omega = (0,1)\times(0,1)$ and $\kappa = 1$. The functions $\mathbf{f}$, g, ψ_0, and $\mathbf{A}_0$ are chosen correspondingly to the exact solution

$$\psi = \exp(-t)(\cos(4\pi x) + i\cos(2\pi y)), \quad \mathbf{A} = \begin{bmatrix} \exp(t-y)\sin(2\pi x) \\ \exp(t-x)\sin(4\pi y) \end{bmatrix}$$

with $H_e = -\exp(t-x)\sin(4\pi y) + \exp(t-y)\sin(2\pi x)$. We set the terminal time T $=$ 1.0 in this example.

A uniform triangular mesh with $M+1$ vertices in each direction is used in finite element approximations, where $h = \frac{\sqrt{2}}{M}$ (see Figure 1 for the illustration with $M = 8$). We solve the system (5.1)–(5.2) by the proposed method in (2.5)–(2.6) with $r = 1,2,3$, respectively. As the expected optimal convergence rate is $O(\tau + h^{r+1})$ in L^2-norm, we set $\tau = (\frac{1}{M})^{r+1}$ in our computation. The L^2-norm errors of ψ_h, σ_h, and $\mathbf{A}_h$ are presented in Table 1, where Err_u denotes the error $\|u_h^N - u(\cdot, N\tau)\|_{L^2}$. From Table 1, we can see that the convergence rates for ψ_h and σ_h are optimal with the order $O(h^{r+1})$ and the convergence rate for $\mathbf{A}_h$ is one-order lower as usual, while the numerical solutions ψ_h and σ_h are of higher interest in physics.

To test the stability of the proposed method, we solve the system (5.1)–(5.2) by our scheme (2.5)–(2.6) with a linear finite element approximation on gradually refined meshes with $M = 8, 16, 32, 64, 128, 256$, where three fixed time steps $\tau = 0.1, 0.01$, and 0.0025 are used. The L^2 errors of ψ_h and σ_h are shown in Figure 2. One can see from Figure 2 that for each fixed τ, when the mesh is refined gradually, each L^2 error function converges to a small constant of order $O(\tau)$. This shows that the proposed method is unconditionally stable; i.e., the method does not require the mesh ratio condition $\tau \le Ch^\alpha$ for certain $\alpha > 0$. Therefore, the proposed method is robust, and large time steps can be used for practical computation in a long time period.

Example 5.2. We study an example in 3D space, defined in (5.1)–(5.2) with those 3D notations of classical differential operators and the boundary and initial conditions

$$\frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \mathbf{curl A} \times \mathbf{n} = \mathbf{H_e} \times \mathbf{n}, \quad \mathbf{A}\cdot\mathbf{n} = \mathbf{0} \qquad \text{on} \quad \partial\Omega,$$
$$\psi(x,0) = \psi_0(x), \quad \mathbf{A}(x,0) = \mathbf{A}_0(x) \qquad \text{in} \quad \Omega,$$

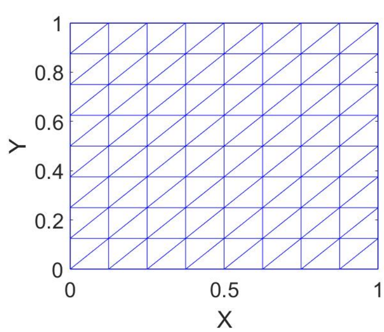

FIG. 1. *A uniform triangular mesh on the unit square domain with* $M = 8$.

TABLE 1
L^2-norm errors of ψ, $\mathbf{A}$, and σ on the unit square (Example 5.1).

$\tau = \frac{1}{M^2}(r=1)$	Err_ψ	Err_σ	$Err_{\mathbf{A}}$
M = 16	1.1998e-1	6.8852e-2	5.3889e-1
M = 32	2.8072e-2	1.7164e-2	1.3908e-1
M = 64	6.3821e-3	4.2860e-3	3.7346e-2
M = 128	1.9612e-3	1.0715e-3	1.0645e-2
order	1.9942	2.0019	1.8882
$\tau = \frac{1}{M^3}(r=2)$	Err_ψ	Err_σ	$Err_{\mathbf{A}}$
M = 8	1.4653e-2	2.7300e-2	5.8080e-1
M = 16	1.9173e-3	3.5300e-3	1.9185e-1
M = 32	2.4500e-4	4.4500e-4	5.2452e-2
M = 64	3.0900e-5	5.5800e-5	1.3437e-2
order	2.9636	2.9791	1.8172
$\tau = \frac{1}{M^4}(r=3)$	Err_ψ	Err_σ	$Err_{\mathbf{A}}$
M = 8	1.3125e-3	2.4265e-3	8.0614e-2
M = 16	8.3800e-5	1.5400e-4	8.2775e-3
M = 32	5.2700e-6	9.6700e-6	9.6400e-4
order	3.9801	3.9856	3.1929

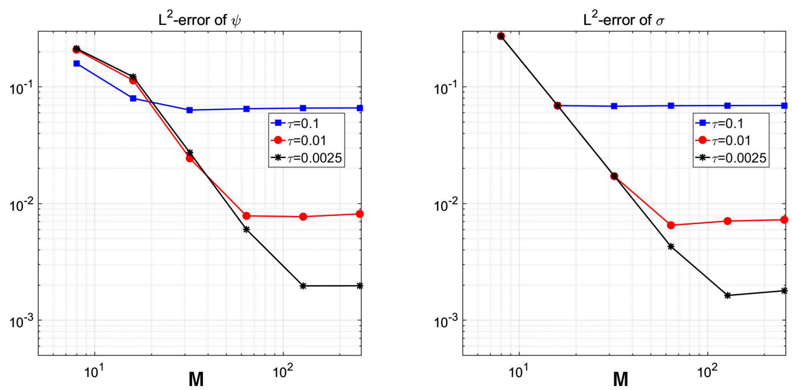

FIG. 2. *L^2 errors of ψ and σ on gradually refined meshes with fixed τ (Example 5.1).*

where $\Omega = (0,1)\times(0,1)\times(0,1)$ and $\kappa = 1$. The functions $\mathbf{f}$, g, ψ_0, and $\mathbf{A}_0$ are chosen correspondingly to the exact solution

$$\psi = \exp(-t)(\cos(\pi x)cos(\pi z) + i\cos(\pi y)cos(\pi z)), \quad \mathbf{A} = \begin{bmatrix} \exp(-t)\sin(2\pi x)\sin(2\pi y) \\ \exp(-t)\sin(2\pi y)\sin(2\pi z) \\ \exp(-t)\sin(2\pi z) \end{bmatrix}$$

with

$$\mathbf{H_e} = \begin{bmatrix} -2\pi\exp(-t)\sin(2\pi y)\sin(2\pi z) \\ 0 \\ -2\pi\exp(-t)\sin(2\pi x)\cos(2\pi y) \end{bmatrix}.$$

Also, we set the terminal time $T = 1.0$ in this example, and a uniform tetrahedral mesh with M+1 vertices in each direction of the cube is used in our computation, where $h = \frac{\sqrt{3}}{M}$. We solve the system by the Galerkin FEM scheme (2.5)–(2.6) with

TABLE 2
L^2-norm errors of ψ, $\mathbf{A}$, and σ on the unit cube (Example 5.2).

$\tau = \frac{1}{M^2}(r=1)$	Err_ψ	Err_σ	$Err_{\mathbf{A}}$
M = 4	0.5770e+0	0.8952e+0	0.1015e+2
M = 8	0.1196e+0	0.2275e+0	0.4371e+1
M = 16	0.3080e-1	0.5181e-1	0.1713e+1
order	2.11	2.05	1.28

$r = 1$. As the expected optimal convergence rate is $O(\tau + h^{r+1})$ in L^2-norm, we set $\tau = (\frac{1}{M})^2$ in our computation. We present in Table 2 the L^2-norm errors of ψ_h, σ_h, and $\mathbf{A}_h$. We can see that the L^2-norm errors for both ψ_h and σ_h are in the order of $O(h^2)$ and that the L^2-norm error for $\mathbf{A}_h$ is in the order of $O(h)$.

Appendix. *Proof of Lemma 3.1.* Since the system (3.2)–(3.3) is an iterated elliptic system and at each time step it is a regular elliptic system, the existence follows immediately.

Let $\delta_\psi^n = \psi^n - \psi_\tau^n$, $\delta_\sigma^n = \sigma^n - \sigma_\tau^n$, and $\delta_{\mathbf{A}}^n = \mathbf{A}^n - \mathbf{A}_\tau^n$. From (1.7)–(1.8) and (3.2)–(3.3), we can see

$$
\begin{aligned}
D_\tau \delta_\psi^n - \frac{1}{\kappa^2}\Delta\delta_\psi^n &= D_\tau\psi^n - \frac{\partial\psi^n}{\partial\tau} + g_1(\psi^n,\sigma^n) - g_1(\psi^{n-1},\sigma^{n-1}) \\
&\quad + g_1(\psi^{n-1},\sigma^{n-1}) - g_1(\psi_\tau^{n-1},\sigma_\tau^{n-1}) D_\tau\delta_\sigma^n - \Delta\delta_\sigma^n \qquad \text{(A.3)}\\
&= D_\tau\sigma^n - \frac{\partial\sigma^n}{\partial\tau} + g_2(\psi^n,\sigma^n) - g_2(\psi^{n-1},\sigma^{n-1}) \\
&\quad + g_2(\psi^{n-1},\sigma^{n-1}) - g_2(\psi_\tau^{n-1},\sigma_\tau^{n-1}) \qquad \text{(A.4)}
\end{aligned}
$$

and, from (2.4) and (3.6),

$$\delta_{\mathbf{A}}^n = \sum_{k=0}^{n} \tau(\boldsymbol{g}^k - \boldsymbol{g}_\tau^k + (|\psi_\tau^n|^2 - |\psi^n|^2)\mathbf{A}^{n-1})\Psi_\tau^{k+1,n} + \widetilde{T}_{\mathbf{A}}^n, \qquad \text{(A.5)}$$

where $\widetilde{T}_{\mathbf{A}}^n$ denotes the truncation error and $\Psi_\tau^{k+1,n}$ is defined by ψ_τ^n in a similar way to $\Psi^{k+1,n}$. By the regularity assumption (2.9) and Taylor expansion, we have

$$\|\widetilde{T}_{\mathbf{A}}^n\|_{L^2} \leq C_K\tau^2. \qquad \text{(A.6)}$$

To prove Lemma 3.1, first we show the following primary estimate by mathematics induction

$$\|\psi_\tau^n\|_{H^2} + \|\sigma_\tau^n\|_{H^2} \leq K + 1\,. \qquad \text{(A.7)}$$

By the regularity assumption (2.9), we see that (A.7) holds for $n = 0$. We assume that (A.7) holds for $0 \leq n \leq m-1$ for some integer $m \geq 1$, and we prove that it also holds for $n = m$ below. By noting (2.3) and (3.7), from (A.5) we have the estimate

$$\|\delta_{\mathbf{A}}^{m-1}\|_{H^1} \leq C_K\tau \sum_{k=0}^{m-1} (\|\Delta\delta_\psi^k\|_{L^2} + \|\Delta\delta_\sigma^k\|_{L^2})\,.$$

By noting the regularity assumption (2.9) and the induction assumption (A.7), we see that

$$\begin{aligned}
&\|g_1(\psi^{m-1},\sigma^{m-1}) - g_1(\psi_\tau^{m-1},\sigma_\tau^{m-1})\|_{L^2} \\
&\quad \le C_K(\|\nabla\delta_\psi^{m-1}\|_{L^2} + \|\delta_{\mathbf{A}}^{m-1}\|_{H^1}) \\
&\quad \le C_K(\|\nabla\delta_\psi^{m-1}\|_{L^2} + \tau\sum_{k=0}^{m-1}(\|\Delta\delta_\psi^k\|_{L^2} + \|\Delta\delta_\sigma^k\|_{L^2}))
\end{aligned}$$

and

$$\begin{aligned}
&\|g_2(\psi^{m-1},\sigma^{m-1}) - g_2(\psi_\tau^{m-1},\sigma_\tau^{m-1})\|_{L^2} \\
&\quad \le C_K(\|\delta_\psi^{m-1}\|_{H^1} + \|\delta_\sigma^{m-1}\|_{H^1} + \|\delta_\psi^{m-1}\|_{W^{1,4}}^2 + \|\delta_{\mathbf{A}}^{m-1}\|_{H^1}) \\
&\quad \le C_K(\|\delta_\psi^{m-1}\|_{H^1} + \|\delta_\sigma^{m-1}\|_{H^1} + \|\delta_\psi^{m-1}\|_{H^2}^{7/8}\|\delta_\psi^{m-1}\|_{L^2}^{1/8} + \|\delta_{\mathbf{A}}^{m-1}\|_{H^1}) \\
&\quad \le \frac{1}{4\kappa^2}\|\Delta\delta_\psi^{m-1}\|_{L^2} + C_K(\|\nabla\delta_\psi^{m-1}\|_{L^2} + \|\nabla\delta_\sigma^{m-1}\|_{L^2} \\
&\qquad + \tau\sum_{k=0}^{m-1}(\|\Delta\delta_\psi^k\|_{L^2} + \|\Delta\delta_\sigma^k\|_{L^2}))\,.
\end{aligned}$$

Then (A.3)–(A.4) reduces to

$$D_\tau\|\nabla\delta_\psi^m\|_{L^2}^2 + \|\Delta\delta_\psi^m\|_{L^2}^2 \le C_K(\|\nabla\delta_\psi^{m-1}\|_{L^2}^2 + \tau\sum_{k=0}^{m-1}(\|\Delta\delta_\psi^k\|_{L^2}^2 + \|\Delta\delta_\sigma^k\|_{L^2}^2) + \tau^2)$$

and

$$\begin{aligned}
D_\tau\|\nabla\delta_\sigma^m\|_{L^2}^2 + \|\Delta\delta_\sigma^m\|_{L^2}^2 \le& \frac{1}{2}\|\Delta\delta_\psi^{m-1}\|_{L^2}^2 + C_K(\|\nabla\delta_\psi^{m-1}\|_{L^2}^2 + \|\nabla\delta_\sigma^{m-1}\|_{L^2}^2 \\
&+ \tau\sum_{k=0}^{m-1}(\|\Delta\delta_\psi^k\|_{L^2}^2 + \|\Delta\delta_\sigma^k\|_{L^2}^2) + \tau^2)\,.
\end{aligned}$$

Adding the last two inequalities together and by Gronwall's inequality, we obtain

$$\|\nabla\delta_\psi^m\|_{L^2}^2 + \|\nabla\delta_\sigma^m\|_{L^2}^2 + \tau\sum_{k=1}^{m}(\|\Delta\delta_\psi^k\|_{L^2}^2 + \|\Delta\delta_\sigma^k\|_{L^2}^2) \le C_K\tau^2, \tag{A.8}$$

which further shows

$$\begin{aligned}
\|\nabla\delta_\psi^m\|_{L^2}^2 + \|\nabla\delta_\sigma^m\|_{L^2}^2 \le C_K\tau^2 \le \frac{1}{2} \\
\|\Delta\delta_\psi^j\|_{L^2}^2 + \|\Delta\delta_\sigma^j\|_{L^2}^2) \le C_K\tau \le \frac{1}{2}
\end{aligned}$$

when $\tau \le \tau_0 = \min\{\frac{1}{2C_K}, \frac{1}{\sqrt{2C_K}}\}$. Finally, we get

$$\|\psi_\tau^m\|_{H^2} + \|\sigma_\tau^m\|_{H^2} \le \|\psi^m\|_{H^2} + \|\sigma^m\|_{H^2} + \|\delta_\psi^m\|_{H^2} + \|\delta_\sigma^m\|_{H^2} \le K+1\,.$$

Thus, the mathematics induction is closed, and we have proved that (A.7) holds for $n \le N$. (3.8) follows from (A.8) immediately.

Second, from (A.8), we further get

$$\|D_\tau\delta_\psi^n\|_{L^p} + \|D_\tau\delta_\sigma^n\|_{L^p} \le \|D_\tau\nabla\delta_\psi^n\|_{L^2} + \|D_\tau\nabla\delta_\sigma^n\|_{L^2} \le C_K,$$

and therefore, by noting the regularity assumption (2.9),

$$\|D_\tau \psi_\tau^n\|_{L^p} + \|D_\tau \sigma_\tau^n\|_{L^p} \le \|D_\tau \psi^n\|_{L^p} + \|D_\tau \sigma^n\|_{L^p} + \|D_\tau \delta_\psi^n\|_{L^p} + \|D_\tau \delta_\sigma^n\|_{L^p} \le C_K\,.$$

Moreover, from (3.2)–(3.3), we have

$$\begin{aligned}&\|\psi_\tau^n\|_{W^{2,4}} + \|\sigma_\tau^n\|_{W^{2,4}}\\ \le&\|g_1(\psi_\tau^{n-1},\sigma_\tau^{n-1})\|_{L^4} + \|g_2(\psi_\tau^{n-1},\sigma_\tau^{n-1})\|_{L^4} + \|D_\tau \psi_\tau^n\|_{L^4} + \|D_\tau \sigma_\tau^n\|_{L^4} \le C_K\end{aligned}$$

and

$$\begin{aligned}&\tau\sum_{k=1}^n(\|D_\tau \psi_\tau^k\|_{H^2}^2 + \|D_\tau \sigma_\tau^k\|_{H^2}^2)\\ \le&\tau\sum_{k=1}^n(\|D_\tau \psi^k\|_{H^2}^2 + \|D_\tau \sigma^k\|_{H^2}^2 + \|D_\tau e_\psi^k\|_{H^2}^2 + \|D_\tau e_\sigma^k\|_{H^2}^2) \le C_K,\end{aligned}$$

which further shows that

$$\|\psi_\tau^n\|_{W^{2,4}}^2 + \|\sigma_\tau^n\|_{W^{2,4}}^2 + \tau\sum_{k=1}^n(\|D_\tau \psi_\tau^k\|_{H^2}^2 + \|D_\tau \sigma_\tau^k\|_{H^2}^2) \le C_K\,.$$

The proof is completed.

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