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THREE-DIMENSIONAL FULL EULER FLOWS WITH NONTRIVIAL SWIRL IN AXISYMMETRIC NOZZLES*

XUEMEI DENG†, TIAN-YI WANG‡, AND WEI XIANG§

Abstract. We are concerned with the unique existence of three-dimensional steady compressible full Euler flows through arbitrary infinitely long axisymmetric and piecewise smooth nozzles with nontrivial swirl. We develop a new approach to prove the nondegeneracy of the axial velocity based on the observation of the potential flow. A modified argument is also employed to handle the stagnation at the corner points. It is the first result on the three-dimensional compressible Euler flow with more than one nonzero and large vorticity. In order to show it, one new stream-conserved quantity is constructed. Finally, the minimum flux limits and the incompressible limits are considered. Via the incompressible limit, we also establish the unique existence of incompressible Euler flows with nontrivial swirl. The methods and techniques developed in this paper are also helpful to other related problems.

Key words. full Euler equations, nontrivial swirl, axisymmetric nozzles, stream function

AMS subject classifications. 35Q31, 35M30, 35L65, 76N10, 76G25, 35B40, 35D30

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1. Introduction. We are concerned with the global existence of steady smooth swirl flows in infinitely long axisymmetric nozzles governed by the full Euler equations. The three-dimensional steady Euler equations are of the following form:

$$(\rho u_1)_{x_1} + (\rho u_2)_{x_2} + (\rho u_3)_{x_3} = 0, \tag{1}$$

$$(\rho u_1^2)_{x_1} + (\rho u_1 u_2)_{x_2} + (\rho u_1 u_3)_{x_3} + p_{x_1} = 0, \tag{2}$$

$$(\rho u_1 u_2)_{x_1} + (\rho u_2^2)_{x_2} + (\rho u_2 u_3)_{x_3} + p_{x_2} = 0, \tag{3}$$

$$(\rho u_1 u_3)_{x_1} + (\rho u_2 u_3)_{x_2} + (\rho u_3^2)_{x_3} + p_{x_3} = 0, \tag{4}$$

$$\Big(\rho u_1\Big(E+\frac{p}{\rho}\Big)\Big)_{x_1} + \Big(\rho u_2\Big(E+\frac{p}{\rho}\Big)\Big)_{x_2} + (\rho u_3\Big(E+\frac{p}{\rho}\Big)\Big)_{x_3} = 0. \tag{5}$$

Here ρ, (u_1, u_2, u_3), p and E denote the density, velocity, pressure, and total energy, respectively. Moreover, for the ideal polytropic gas,

$$E = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + \frac{p}{(\gamma-1)\rho} \tag{6}$$

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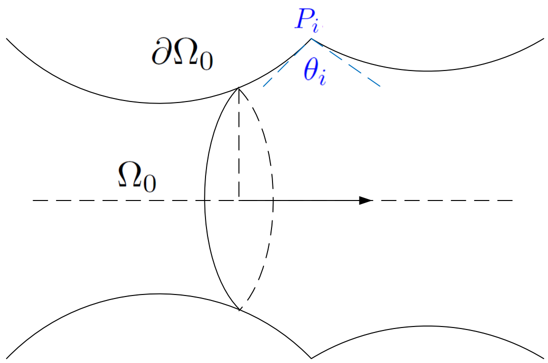

FIG. 1. *The infinitely long axisymmetric nozzle with corners.*

with the adiabatic exponent $\gamma > 1$. The sonic speed of the flow is

$$c = \sqrt{\frac{\gamma p}{\rho}} = ((\gamma - 1)S\rho^{\gamma-1})^{\frac{1}{2}},$$

where S is the entropy function satisfying that

$$S = \frac{\gamma p}{(\gamma - 1)\rho^{\gamma}}. \tag{7}$$

The Mach number is defined as

$$M = \frac{q}{c}, \tag{8}$$

where $q = (u_1^2 + u_2^2 + u_3^2)^{\frac{1}{2}}$ is the speed. We call the flow subsonic when $M < 1$, sonic when $M = 1$, and supersonic when $M > 1$.

Next as shown in Figure 1, the infinitely long axisymmetric nozzle is given by

$$\Omega_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \sqrt{x_2^2 + x_3^2} \in [0, f(x_1)),\ x_1 \in (-\infty, +\infty)\}$$

with the boundary that

$$\partial\Omega_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \sqrt{x_2^2 + x_3^2} = f(x_1),\ x_1 \in (-\infty, +\infty)\}.$$

The function $f(x_1)$ satisfies that

$$\lim_{x_1 \to -\infty} f(x_1) = 1, \ \lim_{x_1 \to +\infty} f(x_1) = a, \ \inf_{x_1 \in \mathbb{R}} f(x_1) > 0, \tag{9}$$

where the constant $a > 0$. Moreover, there exists a constant $\alpha \in (0, 1)$ such that

$$\|f\|_{\mathcal{C}^{2,\alpha}(\mathbb{R} \setminus \Pi)} \leq C \tag{10}$$

for some positive constant C. Here, $\Pi = \{P_i\}_{i=1}^m$ is the set consisting of all the corners $P_i = (x_{1,i}, f(x_{1,i}))$ on the boundary such that

(11) all the corner angles θ_i at P_i are strictly smaller than π.

Suppose that the nozzle wall is impermeable so that

$$(u_1, u_2, u_3) \cdot \overrightarrow{n}_0 = 0 \qquad \text{on } \partial\Omega_0 \backslash \Pi, \tag{12}$$

where $\overrightarrow{n}_0$ is the unit outward normal vector to $\partial\Omega_0\backslash\Pi$. Moreover, since the flow is axially symmetric, at the symmetric axis we have that

$$\lim_{(x_2^2+x_3^2)\to 0} \frac{x_2}{\sqrt{x_2^2+x_3^2}} u_2(x) + \frac{x_3}{\sqrt{x_2^2+x_3^2}} u_3(x) = 0. \tag{13}$$

From the equation of the conservation of mass (1) and impermeable boundary condition (12), we have the following mass conservative condition:

$$\int_\Sigma (\rho u_1, \rho u_2, \rho u_3) \cdot \overrightarrow{l} ds = m_0 \tag{14}$$

for $m_0 > 0$. Here Σ is any surface transversal to the x_1-axis direction, $\overrightarrow{l}$ is the unit normal of Σ in the positive x_1-axis direction.

In this paper, we will consider the unique existence of axially compressible smooth Euler flows with nontrivial swirl and large vorticity in Ω_0. Till now, all the results related to the axially symmetric subsonic flow are with zero swirl. Hence it is the first paper with nontrivial swirl, which brings several new features and difficulties. First of all, due to the nontrivial swirl, the vorticity has two nonzero components with respect to two different directions in the fluid field, which will make the structure of the flows more complicated. From the physical point of view, the swirl will deduce the centrifugal force, which may cause the bifurcation of the stream line at the axis of the symmetry $r = 0$. While from the mathematical point of view, since the product of the swirl and the distance to the axis of symmetry r is conserved along the streamline, the density is a function of the stream-conserved quantities as well as $1/r$ by Bernoulli's law. Then there is a new singularity due to the nonzero swirl when $r = 0$. Second, unlike the two dimensional case or the axially symmetric case with zero swirl, the elliptic condition (57) is no longer the criterion to ensure whether the flow is subsonic or not. Furthermore, the elliptic condition may still hold, when flow is supersonic. It is a new phenomenon when considering the minimum mass flux limit. For the above difficulties, we introduce a new streamline conserved quantity related to the swirl, which keeps the information concerning the additional vorticity. Then based on the quantity we can show that the flow will be elliptic when the mass flux m is sufficiently large. Next, due to the appearance of r in the coefficient b_2 of the lower order term in (76), it is not easy to follow the method developed in [4, 23] to control the terms coming from the derivative of ∂_r on this coefficient to show the nondegeneracy of the axial velocity. Therefore, we develop a new approach to prove the nondegeneracy of the axial velocity by an important observation on the nondegeneracy of the axial velocity for the potential flow. With a delicate analysis, the assumptions on the velocity at the inlet to guarantee the existence are either piecewise or pointwise small or signed, which is a generalization of the conditions in [14, 15]. Furthermore, it is noticeable that all the previous results on the axially symmetric subsonic flow are flows in a nozzle with a $C^{2,\alpha}$ wall. So this paper provides the first result of the Euler flows in the infinitely long nozzles with corners for the axially symmetric case. Finally, by introducing necessary quantities, the compactness frameworks in [5, 6] yield the existence of minimum flux limiting solutions and incompressible limiting solutions.

There is much literature related to the steady subsonic-sonic Euler flows. The problem of subsonic flows in an infinitely long nozzle governed by the steady Euler equations was introduced by Bers [1]. Up to now, there is a lot of literature. For the potential flows, Xie and Xin [21] established the existence and uniqueness of solutions of the subsonic flow in the two-dimensional case and the axially symmetric case in [22]. The general dimensional nozzles with uniform bound of the area of the cross-sections case was done by Du, Xin, and Yan [13], while the three-dimensional unbounded nozzle case was done by Liu and Yuan [20]. For the rotational flows, based on the assumptions on the vorticity, the study of the homeotropic flows and the full Euler flows can be divided into the small vorticity case and the signed vorticity case, which is the convexity assumption on the horizontal velocity at the inlet. For the homeotropic flows with small vorticity, it was first considered by Xin and Xie in [23] for the two-dimensional case. Later, Du and Duan [9] studied the axially symmetric case. Then the case that the boundary has corners was considered by Du and Xie [11] for the two-dimensional flows. For the homeotropic flow with signed vorticity, Du, Xie, and Xin showed the existence for the two-dimensional case in [12], while the axially symmetric case was studied in Du and Duan [10]. For the full Euler flows, the first result was done by Chen, Deng, and Xiang [4] for the two-dimensional case. The extension to the axially symmetric case was considered by [14]. The results on the full Euler flows with signed vorticity were addressed in [2, 15]. For the framework of the compensated compactness to the steady Euler equations, the first work was due to Morawetz [18, 19] for the irrotational case, which is under the assumption that the solutions are free of stagnation points and cavitation points when the flow angle is uniformly finite. Morawetz's result had been improved by Chen, Slemrod, and Wang [8] in which the approximate solutions away from cavitation ware constructed by a viscous perturbation. The compactness framework for the two-dimensional subsonic-sonic irrotational flows allowing stagnation points was due to Chen et al. [3] and Xie and Xin [21] by combining the conservation laws of mass and momentum, and the irrotational equations. Later, Xie and Xin [22] extended the result to the axisymmetric flow in an infinitely long nozzle. The multidimensional irrotational case was considered in Huang, Wang, and Wang [17], and the full Euler case was established in Chen, Huang, and Wang [5]. Moreover, the compactness framework from the steady compressible Euler flows to the steady incompressible Euler flows was constructed by Chen et al. [6] by allowing the adiabatic exponent γ to converge to ∞.

The rest of this paper is organized as follows. In section 2, we present the formulation of the axisymmetric flows and state the main theorem. In section 3, we further reformulate the problem, and give the rigorous definition of Problem 1, which is the main task of this paper. The existence and several important properties of the solutions of Problem 1 are proven in section 4. Finally, in section 5, the minimum flux limits and incompressible limits are considered via the compensated compactness.

2. Axisymmetric formulation and main theorem.

2.1. Axisymmetric formulation. Let us introduce the cylindrical coordinates

$$x = x_1, \qquad r = \sqrt{x_2^2 + x_3^2}, \qquad \theta = \arctan\left(\frac{x_2}{x_3}\right), \tag{15}$$

then

$$\Omega_0 = \{(x, r, \theta) | x \in (-\infty, +\infty), r \in [0, f(x)), \theta \in [0, 2\pi)\}. \tag{16}$$

For the axisymmetric flow in the new coordinates, we have

$$\rho(x,r,\theta)=\rho(x,r),\ (u_1,u_2,u_3)(x,r,\theta)=(u_1,u_2,u_3)(x,r),\ p(x,r,\theta)=p(x,r).$$

The decomposition of the velocity is as follows:

$$u_1=U(x,r),\ u_2=V(x,r)\frac{x_2}{r}+W(x,r)\frac{x_3}{r},\ u_3=V(x,r)\frac{x_3}{r}-W(x,r)\frac{x_2}{r}, \tag{17}$$

where U, V, and W are the axial, radial. and swirl velocities respectively. It is easy to see that

$$u_1^2+u_2^2+u_3^2=U^2+V^2+W^2.$$

Since the nozzle is axisymmetric, we can rewrite the nozzle to be

$$\Omega=\{(x,r)|x\in(-\infty,+\infty),r\in(0,f(x))\}$$

with the boundaries

$$\Gamma_1=\{(x,r)|x\in(-\infty,+\infty),r=0\},\ \Gamma_2=\{(x,r)|x\in(-\infty,+\infty),r=f(x)\}.$$

By (13), we have that

$$(U,V,W)\cdot\overrightarrow{n}=0\qquad\text{on }\Gamma_1, \tag{18}$$

where $\overrightarrow{n}$ is the unit vector perpendicular to the axis Γ_1. The slip condition (12) becomes

$$(U,V,W)\cdot\overrightarrow{n}=0\qquad\text{on }\Gamma_2\backslash\Pi, \tag{19}$$

where $\overrightarrow{n}$ is the unit outer normal to $\Gamma_2\backslash\Pi$ in the cylindrical coordinates. The mass flux condition (14) becomes

$$\int_\Sigma(r\rho U,r\rho V)\cdot\overrightarrow{l}\,dS=\frac{m_0}{2\pi} \tag{20}$$

in the (x,r)-coordinates, where Σ is a curve transverse to the x-axis direction and $\overrightarrow{l}$ is the corresponding unit normal. For simplicity, set

$$m=\frac{m_0}{2\pi}.$$

Then let us consider the Euler system in the new coordinates. It follows from (1) that

$$(r\rho U)_x+(r\rho V)_r=0. \tag{21}$$

From (2), one has

$$(r\rho U^2)_x+(r\rho UV)_r+rp_x=0. \tag{22}$$

Next, by $\frac{x_2}{r}\times(3)+\frac{x_3}{r}\times(4)$ with (21), one has

$$(r\rho UV)_x+(r\rho V^2)_r+rp_r=\rho W^2. \tag{23}$$

Similarly, by $\frac{x_3}{r}\times(3)-\frac{x_2}{r}\times(4)$ with (21), one also has

$$(r\rho U(rW))_x+(r\rho V(rW))_r=0. \tag{24}$$

It follows from (5) that for $B(x,r)=\frac{1}{2}(U^2+V^2+W^2)+\frac{\gamma p}{(\gamma-1)\rho}$, then

$$(r\rho UB)_x+(r\rho VB)_r=0. \tag{25}$$

Therefore, after the change of coordinates and variables, the Euler system in the (x,r)-coordinates is

$$\begin{cases}(r\rho U)_x+(r\rho V)_r=0,\\(r\rho U^2)_x+(r\rho UV)_r+rp_x=0,\\(r\rho UV)_x+(r\rho V^2)_r+rp_r=\rho W^2,\\(r\rho U(rW))_x+(r\rho V(rW))_r=0,\\(r\rho UB)_x+(r\rho VB)_r=0.\end{cases} \tag{26}$$

If the flow is away from the vacuum, then it follows from (26) that

$$(r\rho US)_x+(r\rho VS)_r=0. \tag{27}$$

2.2. Assumptions and the main results. For any given swirl velocity $W_-(r)$, horizontal velocity $U_-(r)$, and entropy function $S_-(r)$ at the inlet, assume that

$$\inf_{r\in[0,1]}S_-(r)>0,\ \inf_{r\in[0,1]}U_-(r)>0,\ U_-'(0)=S_-'(0)=W_-'(0)=0, \tag{28}$$

$$\left(r^2W_-^2S_-^{-1/\gamma}\right)'(1)\le 0, \tag{29}$$

and

$$\left((U_-^2+W_-^2)S_-^{-1/\gamma}\right)'(r)\ge 0,\qquad S_-'(r)\le 0\qquad \text{for } r\in[0,1]. \tag{30}$$

Then the main results of this paper are as follows.

THEOREM 1. *Suppose the nozzle walls satisfy* (9)–(11)*. Given the swirl velocity $W_-(r)$, the horizontal velocity $U_-(r)$, and the entropy function $S_-(r)$ in the upstream with the assumptions of* (28)–(30)*, there exists $\bar a\ge 1$, such that for any $0<a<\bar a$, there exist $\delta_0>0$ and $\underline{m}>0$ such that, for any $m\in(\underline{m},\infty)$ and $\delta\in[0,\delta_0)$, if*

$$\|S_-'\|_{C^{0,1}}\le\delta m^{1-\gamma}, \tag{31}$$

and for $r\in[0,1]$,

$$(U_-^2+W_-^2)''\ge\left(\ln(rU_-S_-^{-\frac{1}{\gamma}})\right)'(U_-^2+W_-^2)'-\delta m^{-\frac{1}{2}}, \tag{32}$$

$$(\ln(U_-S_-^{-\frac{1}{\gamma}}))'(r^2W_-^2)'\ge-\delta m^{-\frac{1}{2}}, \tag{33}$$

and

$$\left(r^2S_-^{-\frac{1}{\gamma}}W_-^2\right)'\le\delta m^{\frac{1-\gamma}{2}}r^2\left((U_-^2+W_-^2)S_-^{-1/\gamma}\right)', \tag{34}$$

then there exists a unique solution $(\rho, U, V, W, p) \in (C^{\alpha}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega))^5$ *to the Euler system* (26) *with the boundary condition* (18), (19), *the mass flux condition* (20), *such that the following hold.*

(i) *Ellipticity and the nondegeneracy of the flow*

$$U^2 + V^2 - c^2 < 0,\ U \geq 0 \qquad in\ \overline{\Omega}, \tag{35}$$

and

$$U > 0 \qquad in\ \overline{\Omega}\backslash\Pi. \tag{36}$$

(ii) *As* $x \to \pm\infty$,

$$\rho \to \rho_{\pm}, \qquad U \to U_{\pm}, \qquad V \to 0, \qquad W \to W_{\pm}, \qquad p \to p_{\pm}, \tag{37}$$

$$\nabla_{(x,r)}\rho \to (0, \rho'_{\pm}(r)), \qquad \nabla_{(x,r)}U \to (0, U'_{\pm}(r)), \qquad \nabla_{(x,r)}V \to (0,0), \tag{38}$$

and

$$\nabla_{(x,r)}W \to (0, W'_{\pm}(r)), \qquad \nabla_{(x,r)}p \to (0,0), \tag{39}$$

where p_- *and* p_+ *are positive constants, and* ρ_+, U_+, W_+, *and* p_+ *can be determined by* U_-, W_-, S_-, a, *and* m *uniquely.*

(iii) $\underline{m}$ *is the critical incoming mass flux for the existence of elliptic flow through the infinitely long axisymmetric nozzles in the following sense: either*

$$\sup_{\overline{\Omega}}(U^2 + V^2 - c^2) \to 0 \qquad as \qquad m \to \underline{m}$$

or there is no $\sigma > 0$, *such that for all* $m \in (\underline{m} - \sigma, \underline{m})$, *there are Euler flows satisfying* (20), (26), (35)–(39), *and*

$$\sup_{m \in (\underline{m}-\sigma, \underline{m})} \sup_{\overline{\Omega}}(U^2 + V^2 - c^2) < 0.$$

Remark 2. $a \in (0, \bar{a})$ is actually the sufficient and necessary condition to guarantee the existence of solutions in Theorem 1. The necessarity comes from the fact that when $a > \bar{a}$, then there is no asymptotic state $(\rho_+, U_+, V_+, W_+, p_+)$ at the outlet such that p_+ is a constant and $U_+ > 0$. The details will be discussed in Proposition 9.

Remark 3. The ellipticity condition in (35) does not mean that flow is subsonic, i.e., $U^2 + V^2 + W^2 < c^2$. Therefore, the solutions we constructed here could be supersonic.

Remark 4. Conditions (32) and (33), which prevent the bifurcation of the the stream line at the axis of the symmetry at $r = 0$, mean the corresponding derivatives are piecewise either small or signed. It is a generalization of the conditions in [14, 15].

Remark 5. Based on Theorem 1 and the compensated compactness framework developed in [5] and [6], the minimum flux limits and incompressible limits of the axisymmetric flows with nontrivial swirl will be considered in section 5.

Theorem 1 will be proved by the following two sections. In section 3, we will reformulate the problem to Problem 1, and then solve it in section 4.

3. Mathematical reformulation of the problem. In this section, we will introduce the stream function ψ and derive an equivalent problem for ψ in the nozzles.

3.1. Asymptotic behavior at the inlet, i.e., at $x = -\infty$. At the inlet of the nozzle, we can assume that $V_- \equiv 0$, which will be shown later. We assume that at the upstream for $r \in [0, 1]$,

$$S \to S_-(r) \qquad \text{as } x \to -\infty, \tag{40}$$

$$U \to U_-(r) \qquad \text{as } x \to -\infty, \tag{41}$$

and

$$W \to W_-(r) \qquad \text{as } x \to -\infty. \tag{42}$$

Then for any given $U_-(r)$, $S_-(r)$, $W_-(r)$, and m at the inlet, the asymptotic states p_-, $\rho_-(r)$, $B_-(r)$ at the inlet satisfy that

$$B_-(r) = \frac{(U_-(r))^2}{2} + \frac{(W_-(r))^2}{2} + \frac{\gamma p_-}{(\gamma-1)\rho_-(r)}, \tag{43}$$

$$S_-(r) = \frac{\gamma p_-}{(\gamma-1)(\rho_-(r))^\gamma}, \tag{44}$$

and

$$\int_0^1 s\rho_- U_-(s) ds = m. \tag{45}$$

For any given fixed constant p_-, the above relations imply

$$\rho_-(r, p_-) = \left(\frac{\gamma p_-}{(\gamma-1)S_-(r)}\right)^{\frac{1}{\gamma}} = d_\gamma p_-^{\frac{1}{\gamma}}(S_-(r))^{-\frac{1}{\gamma}}, \tag{46}$$

$$B_-(r, p_-) = \frac{(U_-(r))^2}{2} + \frac{(W_-(r))^2}{2} + \frac{\gamma}{(\gamma-1)d_\gamma} p_-^{\frac{\gamma-1}{\gamma}}(S_-(r))^{\frac{1}{\gamma}}, \tag{47}$$

and

$$\int_0^1 s d_\gamma p_-^{\frac{1}{\gamma}}(S_-(s))^{-\frac{1}{\gamma}} U_-(s) ds = m, \tag{48}$$

where $d_\gamma = (\frac{\gamma}{\gamma-1})^{\frac{1}{\gamma}}$. Therefore, we have that

$$p_- = \left(\frac{m}{d_\gamma \int_0^1 s(S_-(s))^{-\frac{1}{\gamma}} U_-(s) \mathrm{d}s}\right)^\gamma. \tag{49}$$

Plugging (49) into (46) and (47), we then have $\rho_-(r, p_-)$ and $B_-(r, p_-)$ at the inlet. Let

$$\underline{m} := (\gamma-1)^{-\frac{1}{\gamma-1}} \left(\max_{s\in(0,1)} (U_-^2 S_-^{-\frac{1}{\gamma}})(s)\right)^{\frac{1}{\gamma-1}} \int_0^1 s(S_-(s))^{-\frac{1}{\gamma}} U_-(s) \mathrm{d}s.$$

By (46) and (49), for any $m \in (\underline{m}, \infty)$, we have that the incoming flow is elliptic, i.e., $U_-^2 < c^2$.

In summary, we have the following property.

PROPOSITION 6. *For any given $U_-(r)$, $S_-(r)$, $W_-(r)$, and m at the inlet, there exists $\underline{m} > 0$, such that when $m \in (\underline{m}, \infty)$, the asymptotic state at the inlet p_-, $\rho_-(r)$, $B_-(r)$ are uniquely determined via (46)–(47) and (49) at the inlet. Moreover, the state at the inlet is elliptic, i.e., $U_-^2 < c^2$.*

3.2. Stream-conserved quantities. The stream-conserved quantities come from the linear degenerate parts of the system:

$$\begin{cases} (r\rho U)_x + (r\rho V)_r = 0, \\ (r\rho U(rW))_x + (r\rho V(rW))_r = 0, \\ (r\rho UB)_x + (r\rho VB)_r = 0, \\ (r\rho US)_x + (r\rho VS)_r = 0. \end{cases} \tag{50}$$

Form $(50)_1$, there exists a stream function ψ such that

$$\psi_x = -r\rho V, \qquad \psi_r = r\rho U. \tag{51}$$

For any $r \in (0,1)$, we have that

$$\psi_-(r) = \frac{m}{\int_0^1 s(S_-(s))^{-\frac{1}{\gamma}} U_-(s)ds} \int_0^r s(S_-(s))^{-\frac{1}{\gamma}} U_-(s)ds. \tag{52}$$

Let $\kappa := \psi_-^{-1}$. Obviously, κ is a one-to-one map from $[0,m]$ to $[0,1]$.

Form $(50)_2$–$(50)_4$, rW, B, and S are the stream-conserved qualities along the stream line. So we can define, for $0 \le \psi \le m$,

$$\mathcal{S}(\psi) = S_-(\kappa(\psi)), \ \mathcal{B}(\psi) = B_-(\kappa(\psi)), \text{ and } \mathcal{W}(\psi) = \kappa(\psi)W_-(\kappa(\psi)).$$

Therefore, we have that

$$\begin{cases} \psi_x = -r\rho V, \\ \psi_r = r\rho U, \\ \mathcal{B}(\psi) = \frac{1}{2}(U^2 + V^2 + W^2) + \frac{\gamma p}{(\gamma-1)\rho}, \\ \mathcal{S}(\psi) = \frac{\gamma p}{(\gamma-1)\rho^\gamma}, \\ \mathcal{W}(\psi) = rW(x,r). \end{cases} \tag{53}$$

3.3. Density $\boldsymbol{\rho}$. In this subsection, we will show that density ρ can be presented as a function of $|\nabla\psi|^2$, ψ, and r. First, we want to introduce:

$$H(\rho, |\nabla\psi|^2, \mathcal{W}, \mathcal{B}, \mathcal{S}, r) := \frac{1}{2r^2}[|\nabla_{(x,r)}\psi|^2 + \rho^2\mathcal{W}^2(\psi)] + \mathcal{S}(\psi)\rho^{\gamma+1} - \mathcal{B}(\psi)\rho^2, \tag{54}$$

of which Bernoulli's law implies $H(\rho, |\nabla\psi|^2, \mathcal{W}, \mathcal{B}, \mathcal{S}, r) = 0$. And, taking the derivative with respect to density ρ on (54), then

$$\begin{aligned} \frac{\partial}{\partial\rho} H(\rho, |\nabla\psi|^2, \mathcal{W}, \mathcal{B}, \mathcal{S}, r) &= \frac{1}{r^2}\rho\mathcal{W}^2(\psi) + (\gamma+1)\mathcal{S}(\psi)\rho^\gamma - 2\mathcal{B}(\psi)\rho \\ &= \frac{1}{\rho}\left((\gamma-1)\mathcal{S}(\psi)\rho^{\gamma+1} - \frac{1}{r^2}|\nabla_{(x,r)}\psi|^2\right). \end{aligned} \tag{55}$$

Hence, by the implicit theorem, the density function ρ can be represented as a function of r, ψ, and $|\nabla\psi|^2$ if and only if

$$(\gamma-1)\mathcal{S}(\psi)\rho^{\gamma+1} - \frac{1}{r^2}|\nabla_{(x,r)}\psi|^2 > 0. \tag{56}$$

By Bernoulli's law, it is equivalent to the inequality that

$$b_\gamma \left(\mathcal{B} - \frac{W^2}{2}\right)^{\frac{\gamma+1}{(\gamma-1)}} \mathcal{S}^{-\frac{2}{\gamma-1}} - \frac{1}{r^2}|\nabla_{(x,r)}\psi|^2 > 0, \tag{57}$$

where

$$b_\gamma = (\gamma - 1)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}. \tag{58}$$

We say the flow is elliptic if (57) holds. So the density function ρ can be represented by a function of r, ψ, and $|\nabla\psi|^2$ if the flow is elliptic.

Note that if the inequality

$$\mathcal{B} - \frac{W^2}{2} > 0 \tag{59}$$

holds uniformly in the nozzle, then $\rho > 0$ for the elliptic flows. Hence in order to find a solution away from the vacuum $\rho = 0$ for the elliptic flow, a sufficient condition is there exists a constant C such that in the nozzle

$$\mathcal{B} - \frac{W^2}{2} \geq C > 0. \tag{60}$$

Next, we will show that when m is large enough, condition (60) holds for the elliptic flows. In order to do it, we will introduce a stream-conserved quantity $\mathcal{T}(\psi)$ as a comparing function, such that

$$0 < C \leq \mathcal{B} - \frac{\mathcal{W}^2}{2\mathcal{T}} = \mathcal{B} - \frac{r^2W^2}{2\mathcal{T}} \leq \mathcal{B} - \frac{W^2}{2}. \tag{61}$$

The above inequalities hold if and only if

$$r^2 \geq \mathcal{T} \tag{62}$$

and

$$\mathcal{B} - \frac{\mathcal{W}^2}{2\mathcal{T}} \geq C > 0 \tag{63}$$

for all $\psi \in [0, m]$.

Let us consider (62) first. For any fixed x, we have that

$$\psi(x, r) = \int_0^r \partial_r\psi(x, s)ds = \int_0^r s\rho U(x, s)ds \leq \max_{(x,r)\in\bar{\Omega}}(\rho U)\int_0^r sds \leq \max_{(x,r)\in\bar{\Omega}}(\rho U)\frac{r^2}{2}. \tag{64}$$

By (57),

$$\begin{aligned}
\max_{(x,r)\in\bar{\Omega}}(\rho U) &\leq \max_{(x,r)\in\bar{\Omega}}\left(b_\gamma\left(\mathcal{B} - \frac{W^2}{2}\right)^{\frac{\gamma+1}{(\gamma-1)}}\mathcal{S}^{-\frac{2}{\gamma-1}}\right)^{\frac{1}{2}} \\
&\leq \max_{(x,r)\in\bar{\Omega}}\left(b_\gamma\left(\mathcal{B} - \frac{r^2W^2}{2\bar{f}^2}\right)^{\frac{\gamma+1}{(\gamma-1)}}\mathcal{S}^{-\frac{2}{\gamma-1}}\right)^{\frac{1}{2}} \\
&= \max_{\psi\in[0,m]}\left(b_\gamma\left(\mathcal{B} - \frac{\mathcal{W}^2}{2\bar{f}^2}\right)^{\frac{\gamma+1}{(\gamma-1)}}\mathcal{S}^{-\frac{2}{\gamma-1}}\right)^{\frac{1}{2}},
\end{aligned}$$

where $\bar{f} = \max_{x\in\mathbb{R}} f(z)$. Therefore, let us choose $\mathcal{T}$ to be

$$\mathcal{T} := \frac{2\psi}{\max_{\psi\in[0,m]}(b_\gamma(\mathcal{B} - \frac{\mathcal{W}^2}{2f^2})^{\frac{\gamma+1}{(\gamma-1)}}\mathcal{S}^{-\frac{2}{\gamma-1}})^{\frac{1}{2}}}. \tag{65}$$

Then by (62), we know that $\mathcal{B} - \frac{1}{2}W^2$ is uniformly positive by the following stream-conserved quantity:

$$\mathcal{B} - \frac{\mathcal{W}^2}{4\psi}\max_{\psi\in[0,m]}\left(b_\gamma\left(\mathcal{B} - \frac{\mathcal{W}^2}{2\bar{f}^2}\right)^{\frac{\gamma+1}{(\gamma-1)}}\mathcal{S}^{-\frac{2}{\gamma-1}}\right)^{\frac{1}{2}}. \tag{66}$$

Therefore, in order to show (63), we need to check the quantity (66) is bounded below at the inlet. Obviously, the condition that the quantity (66) is positively bounded below at the inlet is equivalent to the following condition:

$$B_-(r) - \frac{r^2W_-^2(r)}{4\int_0^r s\rho_-(s,p_-)U_-(s)ds}\max_{r\in(0,1)}(b_\gamma(B_-(r) - \frac{r^2W_-^2(r)}{2\bar{f}^2})^{\frac{\gamma+1}{(\gamma-1)}}S_-^{-\frac{2}{\gamma-1}}(r))^{\frac{1}{2}} > 0. \tag{67}$$

By (46)–(47) and (49), for any fixed U_-, W_-, and S_-, there exists $\underline{m} > 0$ such that when $m \in (\underline{m}, \infty)$, (67) holds. So the density ρ can be regarded as a function of r, ψ, and $|\nabla_{(x,r)}\psi|^2$ in the nozzle, i.e., $\rho = \rho(|\frac{\nabla\psi}{r}|^2, \psi, r)$.

3.4. Vorticity and the second order equation. From Bernoulli's law $(53)_3$, one can obtain

$$\rho\partial_x B = \rho UU_x + \rho VV_x + \rho WW_x + \frac{\gamma p_x}{\gamma-1} - \frac{\gamma p\rho_x}{(\gamma-1)\rho}.$$

Differentiating entropy function, one has

$$\partial_x S = \frac{\gamma p_x}{(\gamma-1)\rho^\gamma} - \frac{\gamma^2 p\rho_x}{(\gamma-1)\rho^{\gamma+1}}.$$

With $(26)_1$ and $(26)_2$, combining the above two equations gives

$$\rho\partial_x B - \frac{\rho^\gamma}{\gamma}\partial_x S = \rho V(V_x - U_r) + \rho WW_x. \tag{68}$$

Then by the definitions of $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{W}$ and (51), we have

$$(-r\rho V)\left(\rho\mathcal{B}' - \frac{\rho^\gamma}{\gamma}\mathcal{S}'\right) = \rho V(V_x - U_r) + \rho\frac{\mathcal{W}}{r^2}(-r\rho V)\mathcal{W}'.$$

So

$$U_r - V_x = \mathcal{K}\left(\left|\frac{\nabla_{(x,r)}\psi}{r}\right|^2, \psi, r\right) := r\left(\rho\mathcal{B}' - \frac{\rho^\gamma}{\gamma}\mathcal{S}'\right) - \frac{\rho\mathcal{W}\mathcal{W}'}{r}.$$

On the other hand, it follows from (51) that

$$U_r - V_x = \left(\frac{\psi_r}{r\rho}\right)_r + \left(\frac{\psi_x}{r\rho}\right)_x.$$

Therefore we have the following second order equation of ψ,

$$\operatorname{div}_{(x,r)}\left(\frac{\nabla_{(x,r)}\psi}{r\rho\left(\left|\frac{\nabla_{(x,r)}\psi}{r}\right|^2,\psi,r\right)}\right)=\mathcal{K}\left(\left|\frac{\nabla_{(x,r)}\psi}{r}\right|^2,\psi,r\right). \tag{69}$$

Since Γ_1 and Γ_2 are streamlines, ψ are constants along the walls. By (20), we have the following boundary conditions:

$$\psi=0 \text{ on } \Gamma_1 \qquad \text{and} \qquad \psi=m \text{ on } \Gamma_2.$$

Note that the other physical quantities are solved from ψ by

$$U=\frac{\psi_r}{r\rho},\ V=-\frac{\psi_x}{r\rho},\ W=\frac{\mathcal{W}(\psi)}{r}, \tag{70}$$

$$\rho=\rho(\mathcal{Q},\psi,r),\ p=\frac{\gamma-1}{\gamma}\mathcal{S}(\psi)\rho^\gamma,\ \mathcal{Q}=\left|\frac{\nabla_{(x,r)}\psi}{r}\right|^2. \tag{71}$$

Then the main task in the rest of the paper is to solve the following problem.

Problem 1. Find the solution of the boundary value problem

$$\begin{cases}\operatorname{div}_{(x,r)}\left(\frac{\nabla_{(x,r)}\psi}{r\rho(\mathcal{Q},\psi,r)}\right)=\mathcal{K}(\mathcal{Q},\psi,r) \text{ in } \Omega,\\ \psi=0 \text{ on } \Gamma_1,\ \psi=m \text{ on } \Gamma_2.\end{cases} \tag{72}$$

Moreover, the solutions satisfy the following additional properties:

$$\psi_r>0 \qquad \text{in } \bar{\Omega}\backslash\Pi, \tag{73}$$

$$\mathcal{Q}<c^2\rho^2 \qquad \text{in } \Omega, \tag{74}$$

$$0<\psi<m \qquad \text{in } \Omega^o, \tag{75}$$

where Ω^o is the set of the interior points of Ω.

Remark 7. For any solutions of Problem 1, the quantities determined by (70) and (71) actually give the Euler flow provided that the flow satisfies the Euler equations (26) at the inlet.

Finally, we will rewrite (69) into the nondivergent form. For notational simplicity, let $\partial_1:=\partial_x$ and $\partial_2:=\partial_r$. Multiplying the equation in (69) by $r\rho^2$, one can rewrite the equation in (72) into the following nondivergent form

$$a_{ij}\left(\frac{\nabla\psi}{r},\psi,r\right)\partial_{ij}\psi+b_2\partial_2\psi=\mathcal{F}(\mathcal{Q},\psi,r) \tag{76}$$

where

$$a_{ij}\left(\frac{\nabla\psi}{r},\psi,r\right)=\delta_{ij}(\rho^2c^2-\mathcal{Q})+\frac{\partial_i\psi\partial_j\psi}{r^2}, \tag{77}$$

$$b_2(\mathcal{Q},\psi,r)=-\frac{r^2\rho^2c^2+\rho^2\mathcal{W}^2}{r^3}, \tag{78}$$

$$\mathcal{F}(\mathcal{Q},\psi,r)=\rho^2\mathcal{Q}(r^2(\mathcal{B}'-\mathcal{S}'\rho^{\gamma-1})-\mathcal{W}\mathcal{W}')+r\rho(\rho^2c^2-\mathcal{Q})\mathcal{K}(\mathcal{Q},\psi,r). \tag{79}$$

It is easy to see that the equation is elliptic if (57) holds. The matrix consisting of the coefficients of the second order terms of (76) has two eigenvalues $\lambda=\rho^2c^2-\mathcal{Q}$ and $\Lambda=\rho^2c^2$. Thus, near the critical state that $\rho^2c^2=\mathcal{Q}$, the ellipticity of (76) degenerates.

4. Existence of solutions for Problem 1 and proof of Theorem 1. In this section, we will prove the existence of solutions to Problem 1 and finish the proof of Theorem 1.

4.1. Existence of solutions of the modified boundary value problem. Since the problem is nonlinear, in order to solve Problem 1, we further develop the method used in [4, 9, 21] to extend the functions $\mathcal{S}(\psi)$, $\mathcal{B}(\psi)$, and $\mathcal{W}(\psi)$ from the domain $[0,m]$ to $\mathbb{R}$, and introduce the elliptic cutoff to reformulate Problem 1 into Problem 2.

Let

$$\dot{\mathcal{S}}(s) = \begin{cases} \mathcal{S}'(s) & \text{if } 0 \le s \le m, \\ \mathcal{S}'(m)\dfrac{2m-s}{m} & \text{if } m \le s \le 2m, \\ \mathcal{S}'(0)\dfrac{s+m}{m} & \text{if } -m \le s \le 0, \\ 0 & \text{if } s \ge 2m, \text{ or } \sigma \le -m, \end{cases} \tag{80}$$

$$\dot{\mathcal{B}}(s) = \begin{cases} \left(\mathcal{B}\mathcal{S}^{-\frac{1}{\gamma}}\right)'(s) & \text{if } 0 \le s \le m, \\ \left(\mathcal{B}\mathcal{S}^{-\frac{1}{\gamma}}\right)'(m)\dfrac{2m-s}{m} & \text{if } m \le s \le 2m, \\ \left(\mathcal{B}\mathcal{S}^{-\frac{1}{\gamma}}\right)'(0)\dfrac{s+m}{m} & \text{if } -m \le s \le 0, \\ 0 & \text{if } s \ge 2m, \text{ or } s \le -m, \end{cases} \tag{81}$$

and

$$\dot{\mathcal{W}}(s) = \begin{cases} \left(\mathcal{W}\mathcal{S}^{-\frac{1}{2\gamma}}\right)'(s) & \text{if } 0 \le s \le m, \\ \left(\mathcal{W}\mathcal{S}^{-\frac{1}{2\gamma}}\right)'(m)\dfrac{2m-s}{m} & \text{if } m \le s \le 2m, \\ \left(\mathcal{W}\mathcal{S}^{-\frac{1}{2\gamma}}\right)'(0)\dfrac{s+m}{m} & \text{if } -m \le s \le 0, \\ 0 & \text{if } s \ge 2m, \text{ or } s \le -m. \end{cases} \tag{82}$$

Extend the functions as follows:

$$\begin{aligned} \tilde{\mathcal{S}}(s) &= \mathcal{S}(0) + \textstyle\int_0^s \dot{\mathcal{S}}(t)dt, \\ \tilde{\mathcal{B}}(s) &= \tilde{\mathcal{S}}^{\frac{1}{\gamma}}(s)\left(\mathcal{B}\mathcal{S}^{-\frac{1}{\gamma}}(0) + \textstyle\int_0^s \dot{\mathcal{B}}(t)dt\right), \\ \tilde{\mathcal{W}}(s) &= \tilde{\mathcal{S}}^{\frac{1}{2\gamma}}(s)\left(\mathcal{W}\mathcal{S}^{-\frac{1}{2\gamma}}(0) + \textstyle\int_0^s \dot{\mathcal{W}}(t)dt\right). \end{aligned} \tag{83}$$

Obviously, $\tilde{\mathcal{S}}(s),\ \tilde{\mathcal{B}}(s),\ \tilde{\mathcal{W}}(s) \in C^{1,1}(\mathbb{R})$. From the direct calculation, we have that, for $s \le 0$,

$$\left(\tilde{\mathcal{S}}^{-\frac{1}{\gamma}}\tilde{\mathcal{B}}\right)'(s) = 0, \qquad \left(\tilde{\mathcal{S}}^{-\frac{1}{\gamma}}\frac{\tilde{\mathcal{W}}^2}{2}\right)'(s) = 0,$$

and that for $s \ge m$,

$$\left(\tilde{\mathcal{S}}^{-\frac{1}{\gamma}}\tilde{\mathcal{B}}\right)'(s) \ge 0, \qquad \left(\tilde{\mathcal{S}}^{-\frac{1}{\gamma}}\frac{\tilde{\mathcal{W}}^2}{2}\right)'(s) \le 0.$$

Next, to make sure of the ellipticity of the nonlinear equation (72), for $\epsilon > 0$, let

$$\zeta_0(s) = \begin{cases} s & \text{if } s < 1-2\epsilon, \\ 1-\frac{3}{2}\epsilon & \text{if } s \geq 1-\epsilon \end{cases} \tag{84}$$

be a smooth increasing function such that $|\zeta_0'| \leq 1$. Define

$$\begin{aligned} &\tilde{\mathcal{Q}}(\mathcal{Q},\psi,r) \\ &\quad := \zeta_0\left(\frac{\mathcal{Q}}{\gamma-1}\left(\frac{2\tilde{\mathcal{B}}(\psi)-r^{-2}\tilde{\mathcal{W}}^2(\psi)}{\gamma+1}\right)^{-\frac{\gamma+1}{(\gamma-1)}}\tilde{\mathcal{S}}(\psi)^{\frac{2}{\gamma-1}}\right) \\ &\qquad\qquad \times\left(\frac{2\tilde{\mathcal{B}}(\psi)-r^{-2}\tilde{\mathcal{W}}^2(\psi)}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\frac{\gamma-1}{\tilde{\mathcal{S}}(\psi)^{\frac{2}{\gamma-1}}}, \end{aligned}$$

and $\tilde{\rho}$ is solved via

$$\frac{1}{2}\tilde{\mathcal{Q}}(\mathcal{Q},\psi,r)+\frac{1}{2r^2}\tilde{\rho}^2\tilde{\mathcal{W}}^2(\psi)+\tilde{\mathcal{S}}(\psi)\tilde{\rho}^{\gamma+1}=\tilde{\mathcal{B}}(\psi)\tilde{\rho}^2.$$

If we replace $\mathcal{Q}$, ρ, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{W}$ by $\tilde{\mathcal{Q}}$, $\tilde{\rho}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{S}}$, and $\tilde{\mathcal{W}}$, in the expression of a_{ij}, b_2, $\mathcal{K}$, and $\mathcal{F}$ in (76), and rewrite them as $\tilde{a}_{ij}$, $\tilde{b}_2$, $\tilde{\mathcal{K}}$, and $\tilde{\mathcal{F}}$, then we reformulate Problem 1 into Problem 2, in which the equation is always elliptic and is well-defined for any value of ψ.

Problem 2. Seek a solution of the following boundary value problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla\psi}{r\tilde{\rho}(|\frac{\nabla\psi}{r}|^2,\psi,r)}\right)=\tilde{\mathcal{K}}(\mathcal{Q},\psi,r) \text{ in } \Omega, \\ \psi=0 \text{ on } \Gamma_1,\ \ \psi=m \text{ on } \Gamma_2. \end{cases} \tag{85}$$

The equation in (85) can be rewritten in the nondivergent form as

$$\tilde{a}_{ij}\left(\frac{\nabla\psi}{r},\psi,r\right)\partial_{ij}\psi+\tilde{b}_2\partial_2\psi=\tilde{\mathcal{F}}(\mathcal{Q},\psi,r).$$

Obviously, there exist two positive constants $\lambda(\epsilon)$ and $\Lambda(\epsilon)$ such that

$$\lambda(\epsilon)|\xi|^2 \leq \tilde{a}_{ij}(q,z,r)\xi_i\xi_j \leq \Lambda(\epsilon)|\xi|^2 \tag{86}$$

for any $q\in\mathbb{R}^2$, $z\in\mathbb{R}$, $r\in\mathbb{R}^+$, and $\xi\in\mathbb{R}^2$.

Now, we have the following proposition for the existence of Problem 2.

PROPOSITION 8. *Suppose the hypotheses in Theorem* 1 *hold. Then there exists* $\underline{m}>0$*, such that if* $m\in(\underline{m},\infty)$*, then boundary value problem* (85) *has a solution* $\psi\in C^{2,\alpha}(\overline{\Omega}\backslash\Pi)\cap C^{1,\alpha}(\overline{\Omega})$ *satisfying*

$$\left|\frac{\nabla\psi}{r}\right|^2 \leq (1-2\epsilon)(\gamma-1)\left(\frac{2\tilde{\mathcal{B}}(\psi)-r^{-2}\tilde{\mathcal{W}}^2(\psi)}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\tilde{\mathcal{S}}(\psi)^{-\frac{2}{\gamma-1}} \tag{87}$$

for some $\epsilon>0$*, and* $0\leq\psi\leq m$*. Moreover,* ψ *satisfies* (74)*–*(75) *and*

$$\psi(x,r)\to\bar{\psi}(r)=\int_0^r s\rho_-(s)U_-(s)ds \text{ as } x\to-\infty. \tag{88}$$

Proof. The proof is divided into five steps.

Step 1. Following the technique developed in [9], we consider the following approximated problems in a sequence of bounded domains

$$\begin{cases} \tilde{a}_{ij}^{(k)}\partial_{ij}\psi_L^{(k)} + \tilde{b}_2^{(k)}\partial_r\psi_L^{(k)} = \tilde{\mathcal{F}}\left(\left|\frac{\nabla_{(x,r)}\psi_L^{(k)}}{r+k}\right|^2, \psi_L^{(k)}, r+k\right) \text{ in } \Omega_L, \\ \psi_L^{(k)} = \frac{mr^2}{f^2(x)} \quad \text{ on } \partial\Omega_L, \end{cases} \tag{89}$$

where

$$\{(x,r)|(x,r)\in\Omega,\ |x|<L\}\subset\Omega_L\subset\{(x,r)|(x,r)\in\Omega,\ |x|<4L\},$$

$0<k\ll 1$, and

$$\tilde{a}_{ij}^{(k)} = \tilde{a}_{ij}\left(\left|\frac{\nabla_{(x,r)}\psi^{(k)}}{r+k}\right|^2, \psi^{(k)}, r+k\right), \qquad \tilde{b}_2^{(k)} = \tilde{b}_2\left(\left|\frac{\nabla_{(x,r)}\psi^{(k)}}{r+k}\right|^2, \psi^{(k)}, r+k\right).$$

Here $\partial\Omega_L\backslash\Pi$ is C^{2,α_1}, $0<\alpha_1<\alpha<1$, and satisfies the uniform exterior sphere condition with uniform radius r_0.

Obviously, for any fixed k, there exist two positive constants λ and Λ, which do not depend on k, such that

$$\lambda|\xi|^2 \le \tilde{a}_{ij}^{(k)}\xi_i\xi_j \le \Lambda|\xi|^2$$

for any $\xi\in\mathbb{R}^2$. By the standard existence theory of elliptic equations, for fixed $k>0$ and $L>0$, there exists a solution $\psi_L^{(k)}\in C^{2,\mu}(\Omega_L)\cap C^0(\bar{\Omega}_L)$ to the problem (89) such that (cf. Theorems 3.7 and 12.5 in [16])

$$\left|\psi_L^{(k)}\right| \le \sup_{\Gamma_2}\left|\psi_L^{(k)}\right| + \frac{C}{\lambda_k}\sup_{\Omega_L}\tilde{\mathcal{F}}, \tag{90}$$

where $C=e^{\bar{f}}-1$. Thus

$$-\frac{C}{\lambda_k}|\tilde{\mathcal{F}}| \le \psi_L^{(k)} \le m + \frac{C}{\lambda_k}\left|\sup_{\Omega_L}\tilde{\mathcal{F}}\right|. \tag{91}$$

Next by the barrier function introduced in (122) later, we know that near the corner $P\in\Pi$, $|\psi_L^{(k)}-m|\le Cs^{\alpha+1}$, where s is the distance of the point to the corner and the constant C does not depend on k and L. Thanks to the weight Hölder space, by Lemma 6.20 in [16], we know that there exists $\mu=\mu(\frac{\Lambda_k}{\lambda_k})$, such that

$$[\psi_L^{(k)}]_{1,\mu;\Omega_L} \le C\left(\frac{\Lambda_k}{\lambda_k}\right)\left(1+m+\frac{\|\tilde{\mathcal{F}}\|_{0;\Omega_L}}{\lambda_k}\right) \tag{92}$$

and

$$\|\psi_L^{(k)}\|_{2,\alpha,\Omega_L}^{(-1-\alpha)} \le C\left(\frac{\|\tilde{\mathcal{F}}\|_{0;\Omega_L}}{\lambda_k} + \|\psi_L^{(k)}\|_{0;\Omega_L}\right) \tag{93}$$

for $0<\alpha<\mu$. For any compact set $K\subset\overline{\Omega}$, using the Arzela–Ascoli lemma and the diagonal procedure to select a subsequence of $\{\psi_L^{(k)}\}$ which is denoted as $\psi_{L_n}^{(k)}$, we can get the limit $\psi_{L_n}^{(k)} \to \psi^{(k)}$ in $C^{1,\alpha/2}(K)$ as $L_n \to \infty$ for any compact set $K \subset \bar{\Omega}$; and, $\psi^{(k)} \in C^{2,\alpha}(\Omega)$. Obviously, $\psi^{(k)}$ is a solution of (89).

Step 2. In order to show (75), we are now going to prove that

$$0 \le \psi_L^{(k)}(x,r) \le m \qquad \text{for } (x,r) \in \Omega_L. \tag{94}$$

For the simplicity, we only consider the upper bound to show that it is impossible that there exists a point $P_{\max} = (x_0, r_0)$, such that

$$\psi_L^{(k)}(x,r)(P_{\max}) = \max_{(x,r)\in\Omega_L} \psi_L^{(k)}(x,r)(x,r) \ge m + \epsilon.$$

Obviously, $\nabla\psi_L^{(k)}(x,r)(P_{\max}) = 0$ and $P_{\max}$ cannot be the point on the boundary $\partial\Omega_L$. Denote $\hat{\rho} = \tilde{\rho}(P_{\max})$ for the simplicity. Then at $P_{\max}$, by a straightforward calculation, we have

$$\begin{aligned}
&\tilde{a}_{ij}^{(k)}(0, \psi_L^{(k)}(x,r)(P_{\max}))\partial_{ij}\psi_L^{(k)}(x,r)(P_{\max}) \\
&= \hat{\rho}^4 c^2(\hat{\rho}) r^2 \left(\tilde{\mathcal{B}}'(\psi_L^{(k)}(P_{\max})) - \tilde{\mathcal{S}}'(\psi_L^{(k)}(P_{\max}))\frac{\hat{\rho}^{\gamma-1}}{\gamma} \right) \\
&\quad - \hat{\rho}^4 c^2(\hat{\rho}) \left(\frac{\tilde{\mathcal{W}}^2(\psi_L^{(k)}(P_{\max}))}{2} \right)' \\
&= \hat{\rho}^4 c^2(\hat{\rho}) r^2 \left(\tilde{\mathcal{B}}'(\psi_L^{(k)}(P_{\max})) - \frac{\tilde{\mathcal{S}}'(\psi_L^{(k)}(P_{\max}))\tilde{\mathcal{B}}(\psi_L^{(k)}(P_{\max}))}{\gamma\tilde{\mathcal{S}}(\psi_L^{(k)}(P_{\max}))} \right) \\
&\quad - \hat{\rho}^4 c^2(\hat{\rho}) \left(\left(\frac{\tilde{\mathcal{W}}^2(\psi_L^{(k)}(P_{\max}))}{2} \right)' - \frac{\tilde{\mathcal{W}}^2(\psi_L^{(k)}(P_{\max}))\tilde{\mathcal{S}}'(\psi_L^{(k)}(P_{\max}))}{2\gamma\tilde{\mathcal{S}}(\psi_L^{(k)}(P_{\max}))} \right) \\
&= \hat{\rho}^4 c^2(\hat{\rho}) r^2 \tilde{\mathcal{S}}^{\frac{1}{\gamma}}(\psi_L^{(k)}(P_{\max})) \left(\tilde{\mathcal{S}}^{-\frac{1}{\gamma}}\tilde{\mathcal{B}} \right)'(\psi_L^{(k)}(P_{\max})) \\
&\quad - \hat{\rho}^4 c^2(\hat{\rho}) \tilde{\mathcal{S}}^{\frac{1}{\gamma}}(\psi_L^{(k)}(P_{\max})) \left(\tilde{\mathcal{S}}^{-\frac{1}{\gamma}} \frac{\tilde{\mathcal{W}}^2}{2} \right)'(\psi_L^{(k)}(P_{\max})) \\
&\ge 0.
\end{aligned} \tag{95}$$

On the other hand, we also have that at $P_{\max}$, $\tilde{a}_{ij}^{(k)}(0, \psi_L^{(k)}(P_{\max}))\partial_{ij}\psi(P_{\max}) < 0$, which contradicts (95). Therefore $\psi_L^{(k)}(x,r) \le m + \epsilon$ for $(x,r) \in \Omega_L$. Then taking the limit $\epsilon \to 0$, we have (94). Finally, passing to the limit $L \to \infty$ again, we have (75). It means that $\tilde{\mathcal{S}} = \mathcal{S}$, $\tilde{\mathcal{B}} = \mathcal{B}$, and $\tilde{\mathcal{W}} = \mathcal{W}$.

Step 3. In this step, we are going to show that there is no singularity of the solutions on the axis $r = 0$. Set $\phi(r) = \frac{m}{b^2}(r+k)^2$ with $b = \inf_{x\in\mathbb{R}} f(x)$. Then at the point where $\nabla\psi = \nabla\phi = (0, \frac{2m}{b^2}(r+k))$, we have that

$$\begin{aligned}
&a_{ij}\left(\frac{\nabla\psi}{r+k}, \psi, r+k\right)\partial_{ij}\phi + b_2\left(\frac{\nabla\psi}{r+k}, \psi, r+k\right)\partial_2\phi \\
&= \tilde{\rho}^2\tilde{c}^2\frac{2m}{b^2} - \frac{(r+k)^2\tilde{\rho}^2\tilde{c}^2 + \tilde{\rho}^2\mathcal{W}^2}{(r+k)^3}\frac{2m}{b^2}(r+k) \\
&= -\frac{2m\tilde{\rho}^2\mathcal{W}^2}{(r+k)^2 b^2},
\end{aligned} \tag{96}$$

where $\mathcal{W}$ takes the value at ψ. Note that at the point where $\nabla\psi = \nabla\phi$, $\tilde{\rho}$ can be computed by Bernoulli's law, i.e.,

$$\mathcal{B}(\psi) = \frac{2m^2}{b^4\tilde{\rho}^2} + \frac{\mathcal{W}^2(\psi)}{2(r+k)^2} + \mathcal{S}(\psi)\tilde{\rho}^{\gamma-1}.$$

Then

$$\begin{aligned}
&a_{ij}\left(\frac{\nabla\psi}{r+k}, \psi, r+k\right)\partial_{ij}\psi + b_2\left(\frac{\nabla\psi}{r+k}, \psi, r+k\right)\partial_2\psi \\
&= \mathcal{F}\left(\left|\frac{\nabla\psi}{r+k}\right|^2, \psi, r+k\right) \\
&= (\gamma-1)(r+k)^2\tilde{\rho}^{\gamma+3}\left(\mathcal{S}\mathcal{B}' - \frac{1}{\gamma}\mathcal{S}'\mathcal{B} - \frac{\mathcal{S}\mathcal{W}\mathcal{W}'}{(r+k)^2} + \frac{1}{\gamma}\mathcal{S}'\frac{\mathcal{W}^2}{2(r+k)^2} - \frac{1}{\gamma}\mathcal{S}'\frac{2m^2}{b^2\tilde{\rho}^2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&a_{ij}\left(\frac{\nabla\psi}{r+k}, \psi, r+k)\partial_{ij}(\psi-\phi\right) + b_2\left(\frac{\nabla\psi}{r+k}, \psi, r+k\right)\partial_2(\psi-\phi) \\
&= (\gamma-1)(r+k)^2\tilde{\rho}^{\gamma+3}\left(\mathcal{S}\mathcal{B}' - \frac{1}{\gamma}\mathcal{S}'\mathcal{B} - \frac{\mathcal{S}\mathcal{W}\mathcal{W}'}{(r+k)^2} + \frac{1}{\gamma}\mathcal{S}'\frac{\mathcal{W}^2}{2(r+k)^2} - \frac{1}{\gamma}\mathcal{S}'\frac{2m^2}{b^2\tilde{\rho}^2}\right) \\
&\quad + \frac{2m\tilde{\rho}^2\mathcal{W}^2}{(r+k)^2b^2} \\
&= (\gamma-1)(r+k)^2\tilde{\rho}^{\gamma+3}\mathcal{S}^{\frac{\gamma+1}{\gamma}}\left((\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{B})' - \frac{(\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{W}^2)'}{2(r+k)^2}\right) \\
&\quad - \frac{2(\gamma-1)(r+k)^2\mathcal{S}'m^2\tilde{\rho}^{\gamma+1}}{\gamma b^2} + \frac{2m\tilde{\rho}^2\mathcal{W}^2}{(r+k)^2b^2}.
\end{aligned}$$

By (30), we know that $(\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{B})' \geq 0$ and $\mathcal{S}' \leq 0$. So if $(\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{W}^2)' \leq 0$, then

$$a_{ij}\left(\frac{\nabla\psi}{r+k}, \psi, r+k\right)\partial_{ij}(\psi-\phi) + b_2\left(\frac{\nabla\psi}{r+k}, \psi, r+k\right)\partial_2(\psi-\phi) \geq 0 \tag{97}$$

at the point where $\nabla\psi = \nabla\phi$. Next, if $(\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{W}^2)' \geq 0$, then by (62), we have that

$$(\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{B})' - \frac{(\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{W}^2)'}{2(r+k)^2} \geq (\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{B})' - \frac{(\mathcal{S}^{-\frac{1}{\gamma}}\mathcal{W}^2)'}{4\psi}\max_{\psi\in[0,m]}\left(b_\gamma\left(\mathcal{B} - \frac{\mathcal{W}^2}{2\bar{f}^2}\right)^{\frac{\gamma+1}{(\gamma-1)}}\mathcal{S}^{-\frac{2}{\gamma-1}}\right)^{\frac{1}{2}}.$$

Therefore by (34), we also have that (97) holds.

Obviously, on the boundary, we know that $\psi \leq \frac{m}{b^2}(r+k)^2$. Therefore, by the comparison principle and (94), we have that

$$0 \leq \psi \leq \frac{m}{b^2}(r+k)^2. \tag{98}$$

Now following the same arguments as was done in [9], we know that we can take the limit $k \to 0$, and there exists $b > 0$, such that $\frac{\nabla\psi}{r}$ is continuous up to the symmetry axis $r = 0$. Therefore, there exists some constant $C > 0$, such that

$$\left|\frac{\nabla\psi}{r}\right| \le Cm \qquad \text{for } 0 < r < \frac{b}{2}.$$

Step 4. In this step, we will show (87). Till now we have proved that the boundary problem (85) has a solution $\psi \in C^{2,\beta}(\Omega) \cap C^0(\overline{\Omega})$, $0 < \beta < \alpha$. Moreover, ψ satisfies

$$\left|\frac{\nabla\psi}{r}\right| \le C\left(1 + m + \frac{\tilde{\mathcal{F}}}{\lambda_0}\right), \tag{99}$$

where λ_0 is the smaller eigenvalue of the 2×2 matrix $[\tilde{a}_{ij}]$. Then, by the standard Schauder estimate, one can finally get the following higher order estimates

$$\|\psi\|_{1,\nu,\bar{\Omega}} \le C\left(\frac{\Lambda}{\lambda}, f, m)(1 + m + \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda_0}\right). \tag{100}$$

Similarly to [12], it is easy to prove that $C^{-1}m \le \frac{\|\tilde{\mathcal{F}}\|_0}{\lambda_0} \le Cm$. According to (100), there exists an $\tilde{m} > \tilde{m}_0$, such that (87) holds since

$$(\gamma - 1)\left(\frac{2\tilde{\mathcal{B}}(\psi) - r^{-2}\tilde{\mathcal{W}}^2(\psi)}{\gamma + 1}\right)^{\frac{\gamma+1}{(\gamma-1)}} \tilde{\mathcal{S}}(\psi)^{-\frac{2}{\gamma-1}} \ge C^{-1}m^{\frac{\gamma+1}{2}} > Cm.$$

Step 5. Finally, we will prove (88). The proof is standard (as in [14]), we only list the difference here. Letting ψ be a solution of Problem 2, one can prove that there exists a subsequence of $\psi^{(n)} = \psi\chi_{\{0<r<f(x-n)\}}$ with limit $\bar{\psi}$ in $C^{2,\vartheta}(K)$ for any compact set $K \Subset (-\infty, +\infty) \times [0, 1]$. Moreover, $\bar{\psi}$ is a solution of the following problem:

$$\begin{cases} \operatorname{div}\left(\frac{\bar{\psi}_r}{r\rho}\right) = \left[r\left(\mathcal{B}'\rho - \frac{1}{\gamma}\mathcal{S}'\rho^\gamma\right) - \frac{1}{r}\rho\mathcal{W}\mathcal{W}'\right] & \text{in } (-\infty, +\infty) \times (0, 1), \\ \bar{\psi} = 0 & \text{on } r = 0, \\ \bar{\psi} = m & \text{on } r = 1. \end{cases} \tag{101}$$

Set $w = \bar{\psi}_x$ and $\bar{\mathcal{Q}} = \left|\frac{\nabla\bar{\psi}}{r}\right|^2$. Define $\eta \in C_0^\infty(\mathbb{R})$ satisfying

$$\eta = 1 \text{ for } |s| < L,\ \eta = 0 \text{ for } |s| > L + 1, \text{ and } |\eta'(s)| \le 2. \tag{102}$$

Differentiating (101), multiplying it by $\eta^2(x)w$ and then integrating it on D, we obtain that

$$\begin{aligned}
&\iint_D \frac{\eta^2|\nabla w|^2}{r\rho}dxdr\\
&= -\iint_D \frac{\eta^2}{r\rho(\rho^2c^2-\bar{\mathcal{Q}})}\left|\frac{\nabla\bar{\psi}}{r}\cdot\nabla w\right|^2 dxdr - 2\iint_D \frac{\bar{a}_{ij}}{r\rho^2}\eta w\partial_j w\partial_i\eta dxdr\\
&\quad - 2\iint_D \frac{\rho(\mathcal{B}'-\mathcal{S}'\rho^{\gamma-1})\eta w^2}{\rho^2c^2-\bar{\mathcal{Q}}}\frac{\nabla\bar{\psi}}{r}\cdot\nabla\eta dxdr + 2\iint_D \frac{(\mathcal{B}'\rho-\mathcal{S}'\rho^{\gamma})\eta^2 w}{\rho^2c^2-\bar{\mathcal{Q}}}\frac{\nabla\bar{\psi}}{r}\cdot\nabla w dxdr\\
&\quad - \iint_D \frac{r\rho^3(\mathcal{B}'-\mathcal{S}'\rho^{\gamma-1})^2\eta^2w^2}{\rho^2c^2-\bar{\mathcal{Q}}}dxdr - \iint_D r(\rho\mathcal{B}''-\frac{1}{\gamma}\rho^{\gamma}\mathcal{S}'')\eta^2w^2dxdr\\
&\quad + \iint_D \frac{2\rho^3(\mathcal{B}'-\mathcal{S}'\rho^{\gamma-1})\mathcal{W}\mathcal{W}'\eta^2w^2}{r(\rho^2c^2-\bar{\mathcal{Q}})}dxdr + \iint_D \frac{2\rho\mathcal{W}\mathcal{W}'\eta w}{r^2(r^2c^2-\bar{\mathcal{Q}})}\frac{\nabla\psi}{r}\nabla\eta dxdr\\
&\quad - 2\iint_D \frac{\rho\mathcal{W}\mathcal{W}'\eta^2 w}{r^2(\rho^2c^2-\bar{\mathcal{Q}})}\nabla w\cdot\frac{\nabla\psi}{r}dxdr - \iint_D \frac{\rho^3(\mathcal{W}\mathcal{W}')^2\eta^2w^2}{r^3(\rho^2c^2-\bar{\mathcal{Q}})}dxdr\\
&\quad + \iint_D \frac{\rho[(\mathcal{W}')^2+\mathcal{W}\mathcal{W}'']\eta^2w^2}{r}dxdr\\
&:= \sum_{i=1}^{11} I_i,
\end{aligned}$$

where $\overline{a}_{ij}$ are uniformly bounded functions. It is easy to check that

$$\begin{aligned}
&I_1+I_4+I_5+I_7+I_9+I_{10}\\
&= -\iint \frac{\eta^2}{r\rho(\rho^2c^2-\bar{\mathcal{Q}})}\left\{\left|\frac{\nabla\psi}{r}\cdot\nabla w\right| - w\rho^2\left[(\mathcal{B}'-\mathcal{S}\rho^{\gamma-1}) - \frac{\mathcal{W}\mathcal{W}'}{r}\right]\right\}^2 dxdr \le 0.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathcal{B}'(\psi) &= \kappa' B'_-, & \mathcal{B}''(\psi) &= (\kappa')^2B''_- + \kappa'' B'_-,\\
\mathcal{S}'(\psi) &= \kappa' S'_-, & \mathcal{S}''(\psi) &= (\kappa')^2S''_- + \kappa'' S'_-,\\
\mathcal{W}'(\psi) &= \kappa' W_- + \kappa\kappa' W'_-, & \mathcal{W}''(\psi) &= \kappa(\kappa')^2W''_- + \kappa\kappa'' W'_- + 2(\kappa')^2W'_- + \kappa'' W_-,
\end{aligned}$$

and that by (49)–(52),

$$\kappa'' = \left(-\frac{1}{\kappa} - \frac{U'_-}{U_-} + \frac{S'_-}{\gamma S_-}\right)(\kappa')^2.$$

Then the direct computation yields that

$$\begin{aligned}
I_6+I_{11} &= \iint_D \frac{\rho\eta^2w^2(\kappa')^2r}{2}\left(-(U_-^2+W_-^2)'' + \left((\ln(U_-S_-^{-\frac{1}{\gamma}}))' + \frac{1}{\kappa}\right)(U_-^2+W_-^2)'\right)dxdr\\
&\quad - \iint_D \frac{\rho\eta^2w^2(\kappa')^2}{2r}\left(\ln(U_-S_-^{-\frac{1}{\gamma}})\right)'(\kappa^2W_-^2)'dxdr\\
&\quad + \iint_D \rho\eta^2w^2\kappa'(\psi)^2\left[\frac{1}{\gamma}(-rS''_- + rU'_-S'_- + S'_-)(\rho_-^{\gamma-1}-\rho^{\gamma-1})\right]dxdr\\
&\quad + \iint_D \frac{1}{r^2}S_-^{-1}S'_-\rho\eta^2w^2\kappa'(\psi)^2\left[(\gamma-2)rS'_-\rho_-^{\gamma-1} - \gamma(rU_-U'_- - W_-^2)\right]dxdr\\
&\le C\delta\iint_D \eta^2\frac{w^2}{r^3}dxdr,
\end{aligned}$$

where we use (32) and (33) for the first two rows and the fact that $C^{-1}m \le \rho(\bar{\mathcal{Q}}, \bar{\psi}) \le Cm$ and (31) for the third and fourth rows, to get the above inequality. Now let us extend w to the domain $r \ge 1$ where $w = 0$. Then $w \in W^{1,2}(\mathbb{R}^2_+)$, where $\mathbb{R}^2_+$ is the domain that $r \ge 0$. By the weighted Hardy inequality as in [24], where $r = 2$ and $s = -1$, we have that

$$\iint \eta^2 \frac{w^2}{r^3} dxdr = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\frac{\eta w}{r}\right)^2 r^{-1} dr dx$$
$$\le \int_{-\infty}^{\infty} \int_0^{\infty} |\nabla(\eta w)|^2 r^{-1} dr dx = \iint \frac{|\nabla(\eta w)|^2}{r} dxdr.$$

Then one can follow similar steps as in [9, 14] to estimate I_2 and I_3 and finally have that

$$\bar{\psi}_x = 0 \qquad \text{in } D.$$

Therefore, the assumption that $V_- \equiv 0$ in subsection 3.1 holds, and then following the argument there yields that the solution of the boundary value problem (101) at the inlet is

$$\bar{\psi}(r) = \int_0^r s\rho_-(s) U_-(s) ds.$$

This completes the proof. ∎

4.2. Asymptotic behavior at the outlet, i.e., $x = +\infty$. In this subsection, we will study the asymptotic behavior of solutions ψ at the outlet as follows.

PROPOSITION 9. *For any given $U_-(r)$, $S_-(r)$, $W_-(r)$, and m at the inlet, satisfying the assumptions in Theorem* 1, *there exists $\underline{m} > 0$ and $\bar{a} \ge 1$ such that when $m \in (\underline{m}, \infty)$ for any $a \in (0, \bar{a})$, the solutions obtained via Proposition* 8 *satisfy* (37)–(39) *at the outlet, where the asymptotic states at the outlet p_+, $\rho_+(r)$, and $U_+(r)$ are uniquely determined via* (105)–(107) *below.*

Remark 10. By the proof below, we know that if there exists a minimum point of $(S_-(s))^{-\frac{1}{\gamma}}(U_-^2(s) + \frac{r^2(s)-s^2}{r^2(s)} W_-^2(s))$ in $(0, 1)$, then $\bar{a} = \infty$. Moreover, for a given $r_0 \in (0, 1)$, if one defines for some $\beta > 0$

$$\delta_0' m^{-\beta} := \|U_-'\|_{C^{0,1}([0,r_0,]} + \|S_-'\|_{C^{0,1}([0,r_0,]} + \|W_-'\|_{C^{0,1}([0,r_0,]}, \tag{103}$$

then $\bar{a}$ is bigger when δ_0' is smaller or m is larger.

Proof. By following the same arguments as in Step 5 of the proof of Proposition 8, if the asymptotic states exist, then we must have $V_+ \equiv 0$ and

$$\psi(x, r) \to \bar{\psi}_+(r) = \int_0^r s\rho_+(s) U_+(s) ds \text{ as } x \to +\infty. \tag{104}$$

Therefore, we only need to consider the existence of the asymptotic states at the outlet with the assumption that $V_+ \equiv 0$. Let $r(s)$ be the point at the outlet connecting the point s at the inlet by the streamline, i.e., $r(s)$ is a one-to-one mapping from the inlet to the outlet as

$$r(s) : [0, 1] \to [0, a].$$

Define $\tilde{U}_+(s) = U_+(r(s))$ and $\tilde{\rho}_+(s) = \rho_+(r(s))$. Similarly to (43)–(45), we have

$$B_-(s) = B_+(r(s)) = \frac{(\tilde{U}_+(s))^2}{2} + \frac{\gamma p_+}{(\gamma-1)\tilde{\rho}_+(s)} + \frac{s^2 W_-^2(s)}{2r^2(s)}, \tag{105}$$

$$S_-(s) = S_+(r(s)) = \frac{\gamma p_+}{(\gamma-1)(\tilde{\rho}_+(s))^{\gamma}}, \tag{106}$$

and

$$\int_0^s t\rho_- U_-(t)dt = \int_0^{r(s)} r(t)\rho_+ U_+(t)dt. \tag{107}$$

By (105) and (106), we have that

$$\tilde{\rho}_+(s,p_+) = \left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}} p_+^{\frac{1}{\gamma}}(S_-(s))^{-\frac{1}{\gamma}}$$

and

$$\tilde{U}_+(s,p_+) = \left(2B_-(s) - \frac{s^2W_-^2(s)}{r^2(s)} - 2\left(\frac{\gamma}{\gamma-1}\right)^{1-\frac{1}{\gamma}} p_+^{\frac{\gamma-1}{\gamma}}(S_-(s))^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}.$$

Here, the ellipticity condition $U_+^2 < c^2$ and the nondegenerate streamline condition is

$$0 < \tilde{U}_+(s,p_+) < \left(2\frac{\gamma-1}{\gamma+1}(B_-(s) - \frac{s^2W_-^2(s)}{2r^2(s)})\right)^{\frac{1}{2}},$$

which implies

$$\underline{p}_+ < p_+ < \overline{p}_+, \tag{108}$$

where

$$\begin{aligned}\overline{p}_+ &:= \min_{s\in[0,1]}\left(\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}(S_-(s))^{-\frac{1}{\gamma}}(B_-(s) - \frac{s^2W_-^2(s)}{2r^2(s)})\right)^{\frac{\gamma}{\gamma-1}}\\ &= \left(\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}\min_{s\in[0,1]}(A(s)) + p_-^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}},\end{aligned}$$

$$\begin{aligned}\underline{p}_+ &:= \max_{s\in[0,1]}\left(\frac{2}{\gamma+1}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}(S_-(s))^{-\frac{1}{\gamma}}B_-(s)\right)^{\frac{\gamma}{\gamma-1}}\\ &= \left(\frac{1}{\gamma+1}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}\max_{s\in[0,1]}(A(s)) + \frac{2}{\gamma+1}p_-^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}},\end{aligned}$$

and

$$A(s) := (S_-(s))^{-\frac{1}{\gamma}}\left(U_-^2(s) + \frac{r^2(s)-s^2}{r^2(s)}W_-^2(s)\right). \tag{109}$$

By (62), we know that

$$\begin{aligned}&W_-^2(s) - \frac{s^2W_-^2(s)}{2\int_0^s t\rho_-(t,p_-)U_-(t)dt}\max_{s\in[0,1]}(b_\gamma(B_-(s) - \frac{s^2W_-^2(s)}{2a^2})^{\frac{\gamma+1}{(\gamma-1)}}S_-^{-\frac{2}{\gamma-1}}(s))^{\frac{1}{2}}\\ &\le \frac{r^2(s)-s^2}{r^2(s)}W_-^2(s)\\ &\le W_-^2(s).\end{aligned}$$

Then when p_- is sufficiently large, $\underline{p}_+$ and $\overline{p}_+$ exist. Moreover, taking $\frac{d}{ds}$ on both sides of (107), we have

$$\frac{dr(s)}{ds} = \frac{s\rho_- U_-}{r(s)\tilde{\rho}_+ \tilde{U}_+}(s). \tag{110}$$

It means that $r(s)$ satisfies that

$$\frac{1}{2}r^2(s) = \int_0^s \frac{s\rho_- U_-}{\tilde{\rho}_+ \tilde{U}_+}(s)ds. \tag{111}$$

So define the function

$$J(p_+;p_-) := \int_0^1 \frac{s\rho_- U_-}{\tilde{\rho}_+ \tilde{U}_+}(s)ds. \tag{112}$$

Therefore, the existence of the asymptotic states at the outlet is reduced to the problem to show that there exists a unique constant p_+ satisfying (108) such that $J(p_+;p_-) = \frac{a^2}{2}$, provided that p_- is sufficiently large. Of course, if it is true, the asymptotic states U_+ and ρ_+ can be uniquely determined by (105)–(107). By the straightforward calculation, one can easily see that J is an increasing function with respect to p_+, when $\underline{p}_+ < p_+ < \overline{p}_+$. Hence, the existence of p_+ is equivalent to the following inequality

$$J(\underline{p}_+;p_-) < \frac{a^2}{2} < J(\overline{p}_+;p_-).$$

By the straightforward calculation,

$$\frac{\rho_- U_-}{\tilde{\rho}_+ \tilde{U}_+} = \frac{2^{-\frac{1}{2}}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1-\gamma}{2\gamma}} U_-(s)(S_-(s))^{-\frac{1}{2\gamma}}}{(p_+p_-^{-1})^{\frac{1}{\gamma}}\left(\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}\left(B_-(s) - \frac{s^2W_-^2(s)}{2r^2(s)}\right)(S_-(s))^{-\frac{1}{\gamma}} - p_+^{\frac{\gamma-1}{\gamma}}\right)^{\frac{1}{2}}}.$$

First, we will show $J(\underline{p}_+;p_-) < \frac{a^2}{2}$. Note that

$$J(\underline{p}_+;p_-) = \int_0^1 \frac{2^{-\frac{1}{2}}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1-\gamma}{2\gamma}} sU_-(s)(S_-(s))^{-\frac{1}{2\gamma}}}{(\underline{p}_+p_-^{-1})^{\frac{1}{\gamma}}\left(\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}A(s) + p_-^{\frac{\gamma-1}{\gamma}} - \underline{p}_+^{\frac{\gamma-1}{\gamma}}\right)^{\frac{1}{2}}}ds.$$

Because

$$\begin{aligned}
&\frac{U_-(s)(S_-(s))^{-\frac{1}{2\gamma}}}{\left(\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}A(s) + p_-^{\frac{\gamma-1}{\gamma}} - \underline{p}_+^{\frac{\gamma-1}{\gamma}}\right)^{\frac{1}{2}}}\\
&= \frac{U_-(S_-)^{-\frac{1}{2\gamma}}}{\left(\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}\left(A(s) - \frac{2}{\gamma+1}\max_{s\in[0,1]}(A(s))\right) + \frac{\gamma-1}{\gamma+1}p_-^{\frac{\gamma-1}{\gamma}}\right)^{\frac{1}{2}}}\\
&= \left(\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1}\right.\\
&\quad\left.\times\left(1 - \frac{2}{\gamma+1}\frac{\max_{s\in[0,1]}(A(s))}{A(s)}\right)\left(1 + \frac{r^2(s)-s^2}{r^2(s)}\frac{W_-^2(s)}{U_-(s)}\right) + \frac{\gamma-1}{\gamma+1}\frac{p_-^{\frac{\gamma-1}{\gamma}}}{U_-^2(s)(S_-(s))^{-\frac{1}{\gamma}}}\right)^{-\frac{1}{2}}
\end{aligned}$$

and

$$\underline{p}_+ p_-^{-1} = \left(\frac{1}{\gamma+1} \left(\frac{\gamma}{\gamma-1} \right)^{\frac{1}{\gamma}-1} \frac{\max_{s\in[0,1]}(A(s))}{p_-^{\frac{\gamma}{\gamma-1}}} + \frac{2}{\gamma+1} \right)^{\frac{\gamma}{\gamma-1}},$$

then we know that $J(\underline{p}_+; p_-) < \frac{a^2}{2}$ when p_- is sufficiently large.

Let

$$\bar{a} := J(\overline{p}_+; p_-). \tag{113}$$

Then for any $a \in (0, \bar{a})$, there exists $\underline{m} > 0$ such that when $m \in (\underline{m}, \infty)$, p_+ exists. On the other hand, if $a \geq \bar{a}$, there is no p_+ satisfying (108) such that $J(p_+; p_-) = \frac{a^2}{2}$. Next we will divide the proof into two cases to check the value of $\bar{a}$.

Case I. Noting that $J(p_-; p_-) = \frac{1}{2} \geq \frac{a^2}{2}$, $p_- < \overline{p}_+$, and that J is monotonically increasing with respect to p_+, we have that $\bar{a} \geq 1$.

Case II. Similarly to the computation above, we have

$$\begin{aligned} &J(\overline{p}_+; p_-) \\ &= \int_0^1 \frac{2^{-\frac{1}{2}} \left(\frac{\gamma}{\gamma-1}\right)^{\frac{1-\gamma}{2\gamma}} sU_-(s)(S_-(s))^{-\frac{1}{2\gamma}}}{(\overline{p}_+ p_-^{-1})^{\frac{1}{\gamma}} \left(\frac{1}{2} \left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1} A(s) + p_-^{\frac{\gamma-1}{\gamma}} - \overline{p}_+^{\frac{\gamma-1}{\gamma}} \right)^{\frac{1}{2}}} ds \\ &= \int_0^1 \frac{2^{-\frac{1}{2}} sU_-(s)(S_-(s))^{-\frac{1}{2\gamma}}}{(\overline{p}_+ p_-^{-1})^{\frac{1}{\gamma}} \left(\frac{1}{2} A(s) - \frac{1}{2} \min_{s\in[0,1]}(A(s)) \right)^{\frac{1}{2}}} ds, \end{aligned}$$

where

$$\frac{U_-(s)(S_-(s))^{-\frac{1}{2\gamma}}}{\left(A(s) - \min_{s\in[0,1]}(A(s))\right)^{\frac{1}{2}}} = \frac{U_-(s)}{\left(1 - \frac{\min_{s\in[0,1]}(A(s))}{A(s)}\right)^{\frac{1}{2}} \left(U_-^2(s) + \frac{r^2(s)-s^2}{r^2(s)} W_-^2(s)\right)^{\frac{1}{2}}},$$

and

$$\overline{p}_+ p_-^{-1} = \left(\frac{1}{2} \left(\frac{\gamma}{\gamma-1} \right)^{\frac{1}{\gamma}-1} \frac{\min_{s\in[0,1]}(A(s))}{p_-^{\frac{\gamma}{\gamma-1}}} + 1 \right)^{\frac{\gamma}{\gamma-1}}.$$

If there exists a minimum point of $A(s)$ in $(0,1)$, from the $C^{1,1}$ regularity and the fact that the derivative must be zero, the integration on the neighborhood of the minimum point must be unbounded. It implies $\lim_{p_+ \to \overline{p}_+} J(p_+; p_-) = +\infty$. Hence in this case, we have

$$\bar{a} = \infty.$$

On the other hand, conditions (29) and (30) lead to $A'(1) \geq 0$. If the minimum of $A(s)$ is only attained at $s = 1$, then $A'(1) \leq 0$. So $A'(1) = 0$. By similar argument to the above case, we have $\bar{a} = \infty$.

When the minimal of $A(s)$ is only attained at $s = 0$, (103) implies that there exists $r_0 > 0$, and when in $s \in [0, r_0]$, we have

$$A(s) - A(0) \leq C\delta_0' m^{-\beta} s.$$

Then,

$$\begin{aligned}\bar{a} &= J(\overline{p}_+; p_-)\\ &\geq \int_0^{r_0} \frac{2^{-\frac{1}{2}}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1-\gamma}{2\gamma}} sU_-(s)(S_-(s))^{-\frac{1}{2\gamma}}}{(\overline{p}_+ p_-^{-1})^{\frac{1}{\gamma}}\left(\frac{1}{2}\left(\frac{\gamma}{\gamma-1}\right)^{\frac{1}{\gamma}-1} A(s) + p_-^{\frac{\gamma-1}{\gamma}} - \overline{p}_+^{\frac{\gamma-1}{\gamma}}\right)^{\frac{1}{2}}} ds\\ &= (\overline{p}_+ p_-^{-1})^{-\frac{1}{\gamma}} \int_0^{r_0} s \frac{U_-(s)}{\left(1 - \frac{\min_{s\in[0,1]}(A(s))}{A(s)}\right)^{\frac{1}{2}}\left(U_-^2(s) + \frac{r^2(s)-s^2}{r^2(s)} W_-^2(s)\right)^{\frac{1}{2}}} ds\\ &\geq C r_0^{\frac{3}{2}} (\delta_0')^{-\frac{1}{2}} m^{\frac{\beta}{2}}.\end{aligned}$$

Therefore, if m is larger or δ_0' is smaller, then $\bar{a}$ is larger. ∎

4.3. Positivity of ψ_r. In this section, we will focus on the proof of the positivity of ψ_r, i.e., to show (73). As we said in the introduction, it is not easy to use the techniques developed in [4] and [21] by taking derivatives on (72) with respect to r directly and then multiplying ψ_r on the both sides since the coefficients of (72) depend on r and the terms involving the derivative ∂_r on the coefficients cannot be controlled simply by the Poincaré inequality. Therefore, we have to find a new way to show it. The new way is based on the observation that if the flow is irrotational, then $U = \varphi_x$, where φ is the velocity potential. So we only need to take derivatives ∂_x on both sides to avoid the difficulty that the coefficients of (72) depend on r. Now we are going to show the following proposition.

PROPOSITION 11. *Suppose the assumptions in Proposition* 8 *hold, then in* $\overline{\Omega}\backslash\Pi$, $\psi_r > 0$ *Therefore, the solutions obtained in Proposition* 8 *are actually solutions of Problem* 1.

Proof. We divide the proof into four steps.

Step 1. We first consider the irrotational case without corners on the boundary. For the potential flow, the Bernoulli's constant $\underline{B}$, and the entropy $\underline{S}$ are constants, and the swirl velocity W must vanish. For the potential flow, the Euler equations (26) become

$$\begin{cases}(r\rho U)_x + (r\rho V)_r = 0,\\ U_r - V_x = 0,\end{cases} \tag{114}$$

with the Bernoulli law

$$\frac{1}{2}(U^2 + V^2) + \underline{S}\rho^{\gamma-1} = \underline{B}.$$

Due to the results in [21], we know that $U > 0$ in $\bar{\Omega}$ for the irrotational case, when Γ_2 is smooth.

Step 2. Next, let us consider the case of full Euler flow with nontrivial swirl which is a perturbation of the potential flow without corners on the boundary. In order to do it, we need to compare the solutions of these two cases. First for the potential flow, as said in Step 1, we denote the density by $\rho(|\frac{\nabla\bar{\psi}}{r}|^2, \underline{B}, \underline{S}, 0, r)$, then

$$\operatorname{div}\left(\frac{\nabla\bar{\psi}}{r\rho(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r)}\right) = 0. \tag{115}$$

For the Euler flow with nontrivial swirl, the density is $\rho(|\frac{\nabla\psi}{r}|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r)$, where ψ satisfies (69).

Let $\phi = \psi - \bar{\psi}$ and $\psi_\tau = \bar{\psi} + \tau(\psi - \bar{\psi})$ with $\tau \in (0,1)$. From (69) and (115), we have

$$
\begin{aligned}
&\operatorname{div}\left(\frac{\nabla\psi}{r\rho(\left|\frac{\nabla\psi}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r)} - \frac{\nabla\bar{\psi}}{r\rho(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r)}\right)\\
&= r\rho\left(\left|\frac{\nabla\psi}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)\left[\mathcal{B}' - \frac{\mathcal{S}'}{\gamma}\rho^\gamma\left(\left|\frac{\nabla\psi}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)\right]\\
&\quad - \frac{\rho\left(\left|\frac{\nabla\psi}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)\mathcal{W}\mathcal{W}'}{r}.
\end{aligned} \tag{116}
$$

Consider the terms in parentheses on the left-hand side,

$$
\begin{aligned}
&\frac{\nabla\psi}{r\rho\left(\left|\frac{\nabla\psi}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)} - \frac{\nabla\bar{\psi}}{r\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right)}\\
&= \left(\frac{\nabla\psi}{r\rho\left(\left|\frac{\nabla\psi}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)} - \frac{\nabla\bar{\psi}}{r\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)}\right)\\
&\quad + \left(\frac{\nabla\bar{\psi}}{r\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)} - \frac{\nabla\bar{\psi}}{r\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right)}\right)\\
&= \int_0^1 \frac{\nabla\phi}{r\rho\left(\left|\frac{\nabla\psi_\tau}{r}\right|^2, \mathcal{B}(\psi_\tau), \mathcal{S}(\psi_\tau), \mathcal{W}(\psi_\tau), r\right)} d\tau\\
&\quad - \int_0^1 \frac{\nabla\phi}{r\rho^2\left(\left|\frac{\nabla\psi_\tau}{r}\right|^2, \mathcal{B}(\psi_\tau), \mathcal{S}(\psi_\tau), \mathcal{W}(\psi_\tau), r\right)}\nabla\psi_\tau\\
&\quad \cdot \frac{\partial\rho\left(\left|\frac{\nabla\psi_\tau}{r}\right|^2, \mathcal{B}(\psi_\tau), \mathcal{S}(\psi_\tau), \mathcal{W}(\psi_\tau), r\right)}{\partial\nabla\psi} d\tau\\
&\quad + \frac{\nabla\bar{\psi}}{r}\left[\frac{1}{\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi), r\right)} - \frac{1}{\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right)}\right]\\
&:= \sum_{i=1}^3 J_i.
\end{aligned}
$$

For J_3, note that $\rho(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r)$ satisfies the following Bernoulli's laws,

$$\frac{1}{2}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2 + \frac{\mathcal{W}^2\rho^2(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r)}{r^2}\right) + \mathcal{S}\rho^{\gamma+1}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right) = \mathcal{B}\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right), \tag{117}$$

where we write $(\mathcal{B}(\psi), \mathcal{S}(\psi), \mathcal{W}(\psi))$ as $(\mathcal{B}, \mathcal{S}, \mathcal{W})$ for short. On the other hand,

$$\frac{1}{2}\left|\frac{\nabla\bar{\psi}}{r}\right|^2 + \underline{S}\rho^{\gamma+1}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0\right) = \underline{B}\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0\right). \tag{118}$$

From (53) and (118), we have

$$\begin{aligned}
&\frac{\mathcal{W}^2}{2r^2}\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right) + \mathcal{S}\rho^{\gamma+1}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right) \\
&- \underline{S}\rho^{\gamma+1}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right) \\
&= \mathcal{B}\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right) - \underline{B}\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right).
\end{aligned}$$

Then direct computation gives that

$$\begin{aligned}
&\mathcal{S}\left[\rho^{\gamma+1}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right) - \rho^{\gamma+1}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right)\right] \\
&- \mathcal{B}\left[\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right) - \rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right)\right] \\
&= (\mathcal{B} - \underline{B})\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right) - (\mathcal{S} - \underline{S})\rho^{\gamma+1}\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right) \\
&- \frac{\mathcal{W}^2}{2r^2}\rho^2\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right).
\end{aligned}$$

Using the intermediate value theorem of the integral form, we have

$$\mathcal{S}(\psi) - \underline{S} = \phi\int_0^1 \mathcal{S}'(\psi_\tau)d\tau + (\mathcal{S}(\bar{\psi}) - \underline{S}).$$

Therefore, there exist functions $a_i = a_i\left(\left|\frac{\nabla\psi_\tau}{r}\right|^2, \psi_\tau, r\right)$ $(i = 0, 1, 2, 3)$, such that

$$J_3 = \frac{\nabla\bar{\psi}[a_0\phi + a_1(\mathcal{B}(\bar{\psi}) - \underline{B}) + a_2(\mathcal{S}(\bar{\psi}) - \underline{S}) + a_3\mathcal{W}(\bar{\psi})]}{r\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \mathcal{B}, \mathcal{S}, \mathcal{W}, r\right)\rho\left(\left|\frac{\nabla\bar{\psi}}{r}\right|^2, \underline{B}, \underline{S}, 0, r\right)}. \tag{119}$$

Then ϕ satisfies

$$\tilde{a}_{ij}\partial_{ij}\phi+b_i\partial_i\phi+d_0\phi=(e_1^i+e_2^i\partial_i)\left[d_1(\mathcal{B}(\bar{\psi})-\underline{B}))+d_2(\mathcal{S}(\bar{\psi})-\underline{S})+d_3\mathcal{W}(\bar{\psi})\right] \tag{120}$$

with b_i, e_j^i $(i,j=1,2)$, and c depending on $\left|\frac{\phi_\tau}{r}\right|^2$, ϕ_τ, and r, where

$$\tilde{a}_{ij}=\int_0^1\frac{a_{ij}(\frac{\psi_\tau}{r},\psi_\tau,r)}{r\rho^2\left(\left|\frac{\psi_\tau}{r}\right|^2,\psi_\tau,r\right)}d\tau$$

and

$$b_i(\psi,\bar{\psi},r)=\partial_i(\tilde{a}_{1i}+\tilde{a}_{2i})+\frac{a_0\partial_i\bar{\psi}}{r\rho(|\frac{\nabla\bar{\psi}}{r}|^2,\mathcal{B},\mathcal{S},\mathcal{W},r)\rho(|\frac{\nabla\bar{\psi}}{r}|^2,\bar{B},\bar{S},\bar{W},r)}.$$

Here $a_0=-\int_0^1\mathcal{S}'(\psi_\tau)d\tau+\int_0^1\mathcal{B}'(\psi_\tau)d\tau-\frac{1}{r^2}\int_0^1\mathcal{W}\mathcal{W}'(\psi_\tau)d\tau$ and

$$d_0=\operatorname{div}\left(\frac{a_0\nabla\bar{\psi}}{r\rho(|\frac{\nabla\bar{\psi}}{r}|^2,\mathcal{B},\mathcal{S},\mathcal{W},r)\rho(|\frac{\nabla\bar{\psi}}{r}|^2,\bar{B},\bar{S},\bar{W},r)}\right).$$

Obviously, (120) is a second order elliptic equation. Thus, for any compact set $K\subset\Omega$, if $\|(\mathcal{S}',\mathcal{B}',\mathcal{W}')\|_{C^{0,1}([0,m])}\le\delta$, then

$$\begin{aligned}\|\phi\|_{C^{2,\alpha}(K)}&\le C\|\mathcal{B}(\bar{\psi})-\underline{B},\mathcal{S}(\bar{\psi})-\underline{S},\mathcal{W}(\bar{\psi})\|_{C^\alpha(K)}\\&\le C\|(\mathcal{S}',\mathcal{B}',\mathcal{W}')\|_{C^{0,1}([0,m])}\le C\delta.\end{aligned} \tag{121}$$

It follows from (121) that

$$\|\psi-\bar{\psi}\|_{C^{2,\alpha}(K)}\le C\delta.$$

Because of Step 1, $\bar{\psi}_r>0$. It means that if $\delta>0$ small enough, then we have that $\psi_r>0$.

Step 3. In this step, we will consider the smooth boundary case based on the modified Bers' skill method. By the arguments in [4, 21], if we know the solution satisfies $\psi_r\ge 0$, then $\psi_r>0$ via the strong maximum principle. Letting S_-, U_-, and W_- satisfy the assumptions in Proposition 8, then there exists a unique solution corresponding to them by Proposition 8. Let δ_0 be the largest positive number such that if $\delta<\delta_0$ and $\|(\mathcal{S}',\mathcal{B}',\mathcal{W}')\|_{C^{0,1}([0,m])}\le\delta$, then $\psi_r>0$. By Step 2, we know that if δ_0 exists, then $\delta_0>0$ and there exists $\|(\mathcal{S}',\mathcal{B}',\mathcal{W}')\|_{C^{0,1}([0,m])}=\delta_0$ such that the solution ψ obtained via Proposition 8 satisfies that $(\psi)_r\le 0$ at some points. By the definition of δ_0, there exists a sequence of $\mathcal{S}_n\to\mathcal{S}$, $\mathcal{B}_n\to\mathcal{B}$, and $\mathcal{W}_n\to\mathcal{W}$, and the corresponding solutions ψ_n satisfying that $(\psi_n)_r>0$. Then by the unique existence and the uniform $C^{2,\alpha}$-estimates, we know that $\psi_n\to\psi$ as $n\to\infty$. Therefore $\psi_r\ge 0$. Then the strong maximum principle given above yields that $\psi_r>0$, which contradicts the definition of δ_0. Hence we know that for any solutions obtained via Proposition 8, we have that $\psi_r>0$ if there is no corner on the boundary.

Step 4. Finally let us consider the case that there are corners on the boundary. In order to see that the solutions in Proposition 8 satisfy (73), we will introduce an approximate problem, show that the solution of the approximate problem is actually an approximation of the solution in Proposition 8, and then take the limit.

Let Ω^ε be the domain with $C^{2,\alpha}$ boundary $r = f^\varepsilon(x)$, where the regularity depends on ε. Assume that $\Omega^\varepsilon \subset \Omega$, and assume that $f^\varepsilon(x) \to f(x)$ in the C^0-sense as $\varepsilon \to 0$. Moreover, there exists $X > 0$ sufficiently large, such that for any $\varepsilon > 0$ and any $|x| \geq X$, we have $f^\varepsilon = f$. By Step 3, the solution ψ^ε corresponding to the nozzle Ω^ε satisfies that $\partial_r \psi^\varepsilon > 0$ in $\overline{\Omega}$. In order to take the limit $\varepsilon \to 0$, we need to consider the uniform estimates of ψ^ε which do not depend on ε. Without loss of generality, we consider the estimate only near a corner point P with the angle θ_0, i.e., in the domain $\Omega(R, \varepsilon) := \Omega^\varepsilon \cap B_R(P)$. Let Γ_+ (Γ_-) be the straight line starting at P and tangent to the wall of the nozzle in $B_R(P_0)$ from the right- (left-) hand side. Moreover, the sector bounded by $\Gamma_\pm$ contains $\Omega \cap B_R(x_0)$. Let (s, θ) be the polar coordinates centering at P with $\theta = 0$ on Γ_+, and the angle between Γ_+ and Γ_- be θ_0 with $\theta_0 \in (0, \pi)$, i.e., $\theta = \theta_0$ on Γ_-. Let $\theta_* = \frac{\pi - \theta_0}{2}$. Consider the barrier function

$$w = m - \frac{C_1 m}{R^{1+\beta}} s^{1+\beta} \sin(A\theta + \theta_*), \tag{122}$$

where $A \in (1, \frac{\pi+\theta_0}{2\theta_0})$, $C_1 = \max(\csc\theta_*, \csc(A\theta_0 + \theta_*))$, and $\beta \in (0, 1)$.

First, on $\Omega^\varepsilon \cap \partial B_R(x_0)$, $w \leq 0 \leq \psi^\varepsilon$. On $B_R(x_0) \cap \partial\Omega_\varepsilon$, $w \leq m \leq \psi^\varepsilon$. Next in $\Omega^\varepsilon \cap B_R(x_0)$,

$$\begin{aligned}
&\sum_{i,j} a_{ij}\partial_{ij} w + b_2 \partial_2 w \\
&= -\frac{C_1 m}{R^{1+\beta}}(a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta)(-A^2 + 1 + \beta)s^{\beta-1}\sin(A\theta + \theta_*) \\
&\quad - \frac{C_1 m}{R^{1+\beta}}(a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta)\beta(\beta+1)\sin(A\theta + \theta_*) \\
&\quad - \frac{2C_1 m}{R^{1+\beta}}((a_{22} - a_{11})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta))\beta A\cos(A\theta + \theta_*) \\
&\quad - \frac{C_1 m b_2}{R^{1+\beta}} s^\beta((1+\beta)\sin\theta\sin(A\theta + \theta_*) + A\cos\theta\cos(A\theta + \theta_*)).
\end{aligned}$$

Taking β sufficiently small and only depending on θ_0 and the given data, one can obtain

$$\sum_{i,j} a_{ij}\partial\psi^\varepsilon + b_2\psi^\varepsilon \leq Cs^{\beta-1} \leq \sum_{i,j} a_{ij}\partial_{ij} w + b_2\partial_2 w. \tag{123}$$

Then by the maximum principle, we know that, in $\Omega(R, \varepsilon)$

$$m - Cs^{1+\beta} \leq \psi^\varepsilon \leq m, \tag{124}$$

where the constant C does not depend on ε. By the standard scaling method and the Schauder estimates, we know that $\psi^\varepsilon \to \psi$ in the $C^{1,\beta}$-sense locally in Ω as $\varepsilon \to 0$, where ψ is the solution corresponding to the nozzle with corners, i.e., in Ω. Obviously, $\psi \in C^{1,\beta}$ and $|D\psi| \leq Cs^\beta$, so $D\psi = (0, 0)$ at the corners. So taking the limit $\varepsilon \to 0$ with the $C^{1,\beta}$-convergence of ψ^ε, we see that $\psi_r \geq 0$ in Ω. Then as in Step 3, by the strong maximum principle again, we know that (73) holds.

This completes the proof of this proposition. ∎

4.4. Proof of Theorem 1. Based on the obtained propositions, now we can prove Theorem 1 by following the standard Bers' skill.

Proof. First, following the same arguments as Step 5 in the proof of Proposition 8, one can show the uniqueness of ψ. Next, one can follow similar arguments as in [4, 9] to get the existence of the critical mass flux $\underline{m}$, such that for $m > \underline{m}$, there exists a unique solution ψ satisfying (73)–(75), and $\psi_x(x,r) \to 0$ as $x \to \pm\infty$. Furthermore, from (73) and (74), we know that (60) holds. ∎

5. Minimum flux limits and incompressible limits. For the minimum flux limits, let us first define Problem $3(m)$ as follows.

Problem $3(m)$. For given $B_-(r)$, $W_-(r)$, and $S_-(r)$, find (ρ, U, V, W, p) which satisfies the full Euler equations (26) with mass flux condition (20) and the far field behaviors (40)–(42) at the upstream.

From Theorem 1, we proved that for given functions $U_-(r)$,$W_-(r)$, and $S_-(r)$, there exists $\underline{m}$ such that, when $m > \underline{m}$, Problem $3(m)$ admits a unique solution (ρ, U, V, W, p) with the properties that in Ω, $U^2 + V^2 < c^2$, $U > 0$, and $B - \frac{W^2}{2} > 0$. And as $x_1 \to \pm\infty$, $U(x,r) \to U_\pm(r) > 0$. In order to employ the compactness framework in [5], we introduce the new nonnegative quantity that

$$\hat{B} := B - \frac{W^2}{2}.$$

Then we have the following theorem.

THEOREM 12. *Let $m^{(\epsilon)} > \underline{m}$ be a sequence of mass fluxes converging to $\underline{m}$. Let $(\rho^{(\epsilon)}, U^{(\epsilon)}, W^{(\epsilon)}, V^{(\epsilon)}, p^{(\epsilon)})(x,r)$ be the corresponding sequence of solutions to Problem $3(m)$. Then, as $m^{(\epsilon)} \to \underline{m}$, the solution sequence possesses a subsequence (still denoted as) $(\rho^{(\epsilon)}, U^{(\epsilon)}, V^{(\epsilon)}, W^{(\epsilon)}, p^{(\epsilon)})(x,r)$ converging strongly a.e. in Ω to a vector function $(\rho, U, V, W, p)(x,r)$ which is a weak solution of Problem $3(\underline{m})$. Furthermore, the limit solution $(\rho, U, V, W, p)(x,r)$ also satisfies* (26) *in the distributional sense and the boundary conditions $(r\rho U, r\rho V)\cdot\nu = 0$ on $\partial\Omega$ as the normal trace of the divergence-measure field $(r\rho U, r\rho V)$ on the boundary in the sense of Chen and Frid* [7].

Proof. We divide the proof into three steps.

Step 1. The approximate solutions satisfy the following stream-conserved equations:

$$\begin{cases} \partial_1(r\rho^\varepsilon U^\varepsilon) + \partial_2(r\rho^\varepsilon V^\varepsilon) = 0, \\ \partial_1(r\rho^\varepsilon U^\varepsilon(rW^\varepsilon)) + \partial_2(r\rho^\varepsilon V^\varepsilon(rW^\varepsilon)) = 0, \\ \partial_1(r\rho^\varepsilon U^\varepsilon B^\varepsilon) + \partial_2(r\rho^\varepsilon V^\varepsilon B^\varepsilon) = 0, \\ \partial_1(r\rho^\varepsilon U^\varepsilon S^\varepsilon) + \partial_2(r\rho^\varepsilon V^\varepsilon S^\varepsilon) = 0. \end{cases} \tag{125}$$

From $(125)_1$, we introduce ψ^ε as

$$\begin{cases} \partial_1\psi^\varepsilon = -r\rho^\varepsilon V^\varepsilon, \\ \partial_2\psi^\varepsilon = r\rho^\varepsilon U^\varepsilon. \end{cases} \tag{126}$$

From the far-field behavior of the Euler flows, we define $\psi_-^\varepsilon(r) := \lim_{x\to-\infty} \psi^\varepsilon(x,r)$. $\kappa^\varepsilon(x,r) := (\psi_-^\varepsilon)^{-1}\psi^\varepsilon(x,r)$ is a function from Ω to $[0,1]$. For fixed x, it can be regarded as a backward characteristic map with

$$\frac{\partial(\kappa^\varepsilon)^2}{\partial r} = \frac{2r\rho^\varepsilon U^\varepsilon}{\rho_-^\varepsilon U_-^\varepsilon} > 0.$$

The boundedness and positivity of $\rho_-^\varepsilon U_-^\varepsilon$ and $\rho^\varepsilon U^\varepsilon$ show that the map is not degenerate. Therefore, by $(125)_2$–$(125)_4$,

$$\begin{aligned} &B^\varepsilon(x,r) = B_-(\kappa^\varepsilon(x,r)),\\ &S^\varepsilon(x,r) = S_-(\kappa^\varepsilon(x,r)),\\ &W^\varepsilon(x,r) = \frac{\kappa^\varepsilon(x,r)}{r} W_-((\kappa^\varepsilon(x,r))). \end{aligned}$$

Therefore, we have

$$\begin{cases} \partial_1 B^\varepsilon = -r\rho^\varepsilon V^\varepsilon \frac{B_-'}{(\psi_-^\varepsilon)'},\\ \partial_2 B^\varepsilon = r\rho^\varepsilon U^\varepsilon \frac{B_-'}{(\psi_-^\varepsilon)'}, \end{cases} \tag{127}$$

where

$$\frac{B_-'}{(\psi_-^\varepsilon)'}(\kappa^\varepsilon) = \frac{B_-'(\kappa^\varepsilon)}{\kappa^\varepsilon \rho_-^\varepsilon U_-^\varepsilon(\kappa^\varepsilon)}. \tag{128}$$

Since $B_-'(0) = 0$ and $B_- \in C^{1,1}$, we conclude that $\frac{B_-'(s)}{s}$ is bounded. Then the sequence B^ε is uniformly bounded in BV, which implies its strong convergence. A similar argument can yield the strong convergence of S^ε and W^ε.

Step 2. Define

$$\hat{B}^\varepsilon := B^\varepsilon - \frac{(W^\varepsilon)^2}{2} = \frac{(U^\varepsilon)^2 + (V^\varepsilon)^2}{2} + \frac{\gamma p^\varepsilon}{(\gamma-1)\rho^\varepsilon}. \tag{129}$$

Obviously, $\hat{B}^\varepsilon$ is nonnegative in Ω by (57) and (60) and is strongly convergent by Step 1. Then a direct computation yields that

$$\partial_1 V^\varepsilon - \partial_2 U^\varepsilon = \frac{r}{(\psi_-^\varepsilon)'}\left(\frac{(\rho^\varepsilon)^\gamma S_-'}{\gamma} + \frac{1}{r^2}\rho^\varepsilon \kappa^\varepsilon W_-(W_- + \kappa^\varepsilon W_-') - \rho^\varepsilon B_-'\right), \tag{130}$$

which implies that $\partial_1 V^\varepsilon - \partial_2 U^\varepsilon$ is uniformly bounded in the bounded measure space.

Then, we have the following properties:

(A.1) $(U^\varepsilon)^2 + (V^\varepsilon)^2 < \frac{\gamma p^\varepsilon}{\rho^\varepsilon}$ a.e. in Ω;

(A.2) S^ε and $\hat{B}^\varepsilon$ are uniformly bounded, and for any compact set K, there exists a uniform constant $c(K)$ such that $\inf_{x\in K} S^\varepsilon(x,r) \geq c(K) > 0$. Moreover, $(S^\varepsilon, \hat{B}^\varepsilon) \to (\overline{S}, \overline{\hat{B}})$ a.e. in Ω;

(A.3) $\partial_1 V^\varepsilon - \partial_2 U^\varepsilon$ and $\partial_1(r\rho^\varepsilon U^\varepsilon) + \partial_2(r\rho^\varepsilon V^\varepsilon)$ are compact in $W_{loc}^{-1,p}$ for some $1 < p \leq 2$.

Then Theorem 2.4 in [5] implies that the solution sequence has a subsequence (still denoted by) $(\rho^\varepsilon, U^\varepsilon, V^\varepsilon, p^\varepsilon)(x,r)$ that converges a.e. in Ω to a vector function $(\rho, U, V, p)(x,r)$. Furthermore, it is easy to see that $(\rho, U, V, W, p)(x,r)$ satisfies (26).

Step 3. The boundary condition is satisfied in the sense of Chen and Frid [7], which implies

$$\begin{aligned} &\int_{\partial\Omega} \eta(w)(r\rho U, r\rho V)(w)\cdot \nu(w)\, d\mathcal{H}^1(w)\\ &= \int_\Omega (r\rho U, r\rho V)(x,r)\cdot \nabla\eta(x,r)\, dxdr + \langle \partial_1(r\rho U) + \partial_2(r\rho V)|_\Omega, \eta\rangle \end{aligned}$$

for $\eta \in \mathbf{C}_0^1$. From above, we can see $\langle (\partial_1(r\rho U) + \partial_2(r\rho V))|_\Omega, \eta\rangle = 0$. Furthermore, we have

$$\int_\Omega (r\rho U, r\rho V)(x,r)\cdot\nabla\eta(x,r)\,dxdr = \lim_{\varepsilon\to 0}\int_\Omega (r\rho^\varepsilon U^\varepsilon, r\rho^\varepsilon V^\varepsilon)(x,r)\cdot\nabla\eta(x,r)\,dxdr = 0,$$

which yields

$$\int_{\partial\Omega} \eta(w)(r\rho U, r\rho V)(w)\cdot\nu(w)\,d\mathcal{H}^1(w) = 0. \tag{131}$$

It is $(r\rho U, r\rho V)\cdot\nu = 0$ on $\partial\Omega$ in $\mathcal{D}'$.

This completes the proof. ∎

Next, we will consider the incompressible limits. Similarly to [6], let

$$\mathcal{G}(\psi) := \frac{\rho}{p^{\frac{1}{\gamma}}} = \left(\frac{\gamma-1}{\gamma}\mathcal{S}(\psi)\right)^{-\frac{1}{\gamma}}.$$

Before stating the incompressible limit, let us define Problem 4$(m,\,\gamma)$ first as follows.

Problem 4$(m,\,\gamma)$. For given functions $B_-(r)$, $W_-(r)$, and $G_-(r)$, find solution (ρ, U, V, W, p) which satisfies the full Euler equation (26) with mass flux condition (20) and the far-field behaviors at the upstream that satisfy (41)–(42) and

$$\frac{\rho}{p^{\frac{1}{\gamma}}} \to G_-(r) \text{ as } x \to -\infty. \tag{132}$$

By Theorem 1, we have that for the given $U_-(r)$, $W_-(r)$, and $G_-(r)$, there exists $m^{(\gamma)}$ such that, when $m > m^{(\gamma)}$, Problem 4 $(m,\,\gamma)$ has a unique solution (ρ, U, V, W, p) with the properties that in $\overline{\Omega}$, $U^2+V^2 < c^2$ and $B - \frac{W^2}{2} > 0$, and that in $\overline{\Omega}\backslash\Pi$, $U > 0$. And as $x_1 \to \pm\infty$, $U(x,r) \to U_\pm(r) > 0$.

Similarly to Theorem 12, we have the following.

THEOREM 13. *Let $(\rho^{(\gamma)}, U^{(\gamma)}, V^{(\gamma)}, W^{(\gamma)}, p^{(\gamma)})(x,r)$ be the sequence of solutions to Problem 4 $(m^{(\gamma)},\,\gamma)$ correspondingly with the properties that $\{m^{(\gamma)}\}$ is a bounded sequence. Then, as $\gamma\to\infty$, the sequence of solutions possesses a subsequence (still denoted as) $(\rho^{(\gamma)}, U^{(\gamma)}, V^{(\gamma)}, W^{(\gamma)}, p^{(\gamma)})(x,r)$ that converges strongly a.e. in Ω to a vector function $(\bar\rho, \bar U, \bar V, \bar W, \bar p)(x,r)$ which is a weak solution of*

$$\begin{cases} \partial_1(r\bar U) + \partial_2(r\bar V) = 0,\\ \partial_1(r\bar\rho\bar U) + \partial_2(r\bar\rho\bar V) = 0,\\ \partial_1(r\bar\rho\bar U^2) + \partial_2(r\bar\rho\bar U\bar V) + r\partial_1\bar p = 0,\\ \partial_1(r\bar\rho\bar U\bar V) + \partial_2(r\bar\rho\bar V^2)_r + r\partial_2\bar p = \bar\rho\bar W^2,\\ \partial_1[(r\bar\rho\bar U)r\bar W] + \partial_2[(r\bar\rho\bar V(r\bar W))] = 0. \end{cases} \tag{133}$$

Furthermore, the limit solution $(\bar\rho, \bar U, \bar V, \bar W, \bar p)(x,r)$ also satisfies the boundary condition $(r\bar U, r\bar V)\cdot\nu = 0$ as the normal trace of the divergence-measure field $(r\bar U, r\bar V)$ on the boundary in the sense of Chen and Frid [7].

Proof. Similarly to the proof of the minimum mass limit, we have that the approximate solutions satisfy the following stream-conserved equations:

$$\begin{cases} \partial_1(rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}) + \partial_2(rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}) = 0,\\ \partial_1(rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}(rW^{(\gamma)})) + \partial_2(rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}(rW^{(\gamma)})) = 0,\\ \partial_1(rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}B^{(\gamma)}) + \partial_2(rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}B^{(\gamma)}) = 0,\\ \partial_1(rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}G^{(\gamma)}) + \partial_2(rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}G^{(\gamma)}) = 0. \end{cases} \tag{134}$$

Then, similarly to the proof of Theorem 12 again, one can prove that the sequences $B^{(\gamma)}$, $G^{(\gamma)}$, and $W^{(\gamma)}$ are uniformly bounded in BV. It implies the strong convergence of the sequences. Then, similarly to (130), by presenting $\partial_1 V^{(\gamma)} - \partial_2 U^{(\gamma)}$ as functions of G'_-, B'_-, and W'_-, it is easy to prove that $\partial_1 V^{(\gamma)} - \partial_2 U^{(\gamma)}$ is bounded in the bounded measure space.

Then, we have

(B.1) $(U^{(\gamma)})^2 + (V^{(\gamma)})^2 < \frac{\gamma p^{(\gamma)}}{\rho^{(\gamma)}}$ a.e. in Ω;

(B.2) $p^{(\gamma)} \geq 0$ are uniformly bounded in $L^1_{loc}(\Omega)$;

(B.3) $\partial_1 V^{(\gamma)} - \partial_2 U^{(\gamma)}$ and $\partial_1(r\rho^{(\gamma)}U^{(\gamma)}) + \partial_2(r\rho^{(\gamma)}V^{(\gamma)})$ are in a compact set in $H^{-1}_{loc}(\Omega)$;

(B.4) as $\gamma \to \infty$, $\int_\Omega \ln\left(E^{(\gamma)}\right) dx = o(\gamma)$, while $E^{(\gamma)} = \frac{(U^{(\gamma)})^2+(V^{(\gamma)})^2}{2} + \frac{p^{(\gamma)}}{(\gamma-1)\rho^{(\gamma)}}$.

Then Proposition 2.4. in [6] implies that the solution sequence has a subsequence (still denoted as) $(\rho^{(\gamma)}, U^{(\gamma)}, V^{(\gamma)}, p^{(\gamma)})(x,r)$ that converges a.e. in Ω to a vector function $(\bar\rho, \bar U, \bar V, \bar p)(x,r)$, which satisfies (133).

The boundary condition is satisfied in the sense of Chen and Frid [7], which implies

$$\begin{aligned}&\int_{\partial\Omega} \eta(w)(r\bar U, r\bar V)(w)\cdot\nu(w)\, d\mathcal{H}^1(w)\\ &= \int_\Omega (r\bar U, r\bar V)(x,r)\cdot\nabla\eta(x,r)\, dxdr + \langle \partial_1(r\bar U) + \partial_2(r\bar U)|_\Omega, \eta\rangle\end{aligned}$$

for $\eta \in \mathbf{C}^1_0$. From above, we can see $\langle \partial_1(r\bar U) + \partial_2(r\bar U)|_\Omega, \eta\rangle = 0$. Furthermore, with $\lim_{\gamma\to\infty}(p^{(\gamma)})^{\frac{1}{\gamma}} = 1$, we have

$$\begin{aligned}&\int_\Omega (r\bar U, r\bar V)(x,r)\cdot\nabla\eta(x,r)\, dxdr\\ &= \lim_{\gamma\to\infty}\int_\Omega (rU^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}}, rV^{(\gamma)}(p^{(\gamma)})^{\frac{1}{\gamma}})(x,r)\cdot\nabla\eta(x,r)\, dxdr = 0,\end{aligned}$$

which yields

$$\int_{\partial\Omega} \eta(w)(r\bar U, r\bar V)(w)\cdot\nu(w)\, d\mathcal{H}^1(w) = 0, \tag{135}$$

that is, $(r\bar U, r\bar V)\cdot\nu = 0$ on $\partial\Omega$ in $\mathcal{D}'$.

This completes the proof. ∎

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