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ON THE PATHWISE SOLUTIONS TO THE CAMASSA–HOLM EQUATION WITH MULTIPLICATIVE NOISE*

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Abstract. In this paper we consider the Camassa–Holm (CH) equation with multiplicative noise, which can be obtained when the nonhydrostatic pressure in the deterministic equation is subject to a turbulent velocity field. For the periodic boundary value problem for this SPDE, we establish the local existence and pathwise uniqueness of the pathwise solution in Sobolev spaces H^s with $s > 3/2$. For the linear noise case, conditions that lead to the global existence and the blow-up in finite time of the solution, and their associated probabilities, are also acquired. Finally, we study the pathwise dissipative effect of the linear noise on the periodic peakons to the deterministic CH equation.

Key words. stochastic Camassa–Holm equation, martingale solutions, pathwise solutions, global existence, blow up, periodic peakons

AMS subject classifications. Primary, 35Q53, 60H15; Secondary, 35A01, 35C07

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1. Introduction and main results.

In recent years, the well-known Camassa–Holm (CH) equation,

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

has been well studied and a series of achievements have been made. The CH equation was derived independently by Fokas and Fuchssteiner in [27] and by Camassa and Holm in [6]. Fokas and Fuchssteiner derived (1.1) in studying completely integrable generalizations of the KdV equation with bi-Hamiltonian structures, while Camassa and Holm proposed (1.1) to describe the unidirectional propagation of shallow water waves over a flat bottom, where $u(t, x)$ represents the fluid velocity at time t in the horizontal direction x. The bi-Hamoltonian structure of (1.1), which ensures the existence of an infinite number of conservation laws, was obtained in [6] and the integrability of CH (as an infinite-dimensional Hamiltonian system) was studied in [17, 10]. Moreover, the CH equation is such an equation that exhibits both phenomena of (peaked) soliton interaction and wave breaking (the solution remains bounded but its slope becomes unbounded in finite time; cf. [14]). In fact, wave breaking is one of the most intriguing long-standing problems of water wave theory [46]. The essential feature of (1.1) revealed in recent papers [12, 15, 16] is that the traveling waves with a peak at their crest are exactly like the waves of the greatest height solutions to the governing equations for water waves.

From the mathematical point of view, the deterministic CH equation is well studied. Constantin and Escher [9, 13] investigated the Cauchy problem for the periodic Camassa–Holm equation. Nonuniform continuity of the solution map was studied in [43, 44]. Constantin and Escher [13, 14] and McKean [37] studied the wave breaking of the Cauchy problem for the CH equation. Actually, wave breaking is the only way

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that singularities can occur in solutions [10]. Moreover, for a smooth initial profile, it is possible to predict (by establishing a necessary and sufficient condition) whether breaking wave occurs [14, 37]. As for the continuation of solutions after wave breaking, Bressan and Constantin developed a new approach to the analysis of the CH equation and proved the existence of the global conservative and dissipative solutions in [2, 3]. The orbital stability of the peakon

$$u(t,x) = ce^{-|x-ct|}, \quad x \in \mathbb{R}, \quad c > 0,$$

to the deterministic CH equation, was studied by Constantin and Strauss [18].

Notice that the CH equation (1.1) can be viewed as a model for shallow water waves with nonhydrostatic pressure,

$$\begin{cases} u_t + uu_x + P_x = 0, \\ \qquad P - P_{xx} = u^2 + \dfrac{1}{2}u_x^2, \end{cases} \tag{1.2}$$

where P is the (dimensionless) pressure. However, when we consider a physical system in the real world, we have to consider some influence of internal, external, or environmental noises. Besides, the whole background for the considered physical system may be difficult to describe deterministically. For example, for the system (1.2), there may be airflow around the surface or the bottom of a fluid may not be so flat. So we need to consider the randomness of the background movement, and this is one of the prevailing hypotheses on the onset of turbulence in fluid models [4, 21, 22, 38]. Here we remark that there has been a lot of recent work done on PDEs with random perturbations and we refer to [5, 19, 23, 24, 26, 30, 41] and references therein.

Motivated by this previous work, we assume that P in $(1.2)_2$ is perturbed under a turbulent velocity field $\int H(t,x)\mathrm{d}x \cdot \frac{\mathrm{d}W(\omega,t)}{\mathrm{d}t}$ for some function $H(t,x)$,

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2 - \int H(t,x)\mathrm{d}x \cdot \frac{\mathrm{d}W(\omega,t)}{\mathrm{d}t}, \tag{1.3}$$

where $W(\omega,t), \omega \in \Omega$ is a standard one-dimensional (1-D) Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ which consists of a probability measure $\mathbb{P}$ on Ω, a σ-algebra $\mathcal{F}$, and a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F})$ such that $\{\mathcal{F}_0\}$ contains all the $\mathbb{P}$-negligible subsets. Let $\sigma(t,x)$ be the function such that

$$\sigma(t,x) - \sigma_{xx}(t,x) = H(t,x), \tag{1.4}$$

then we arrive at $P_x = F(u) - \sigma(t,x)\frac{\mathrm{d}W}{\mathrm{d}t}$, where $F(u) = F_1(u) + F_2(u)$ and

$$F_1(u) = (1-\partial_x^2)^{-1}\partial_x\left(u^2\right), \quad F_2(u) = \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x\left(u_x^2\right). \tag{1.5}$$

Combining (1.2)–(1.5), we obtain a stochastic CH equation with noise

$$u_t + u\partial_x u + F(u) = \sigma(t,x)\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{1.6}$$

However, the external turbulent velocity field will interact with the fluid velocity u itself in many cases. That is to say, H will also depend on u itself. Hence we consider the case that $H(t,x) = H(u, u_x, u_{xx}, \ldots)$, then $\sigma(t,x) = \sigma(u, u_x, u_{xx}, \ldots)$ will be given by (1.4) correspondingly. For simplicity, we write $H(u, u_x, u_{xx}, \ldots) = H(u)$

and $\sigma(u, u_x, u_{xx}, \ldots) = \sigma(u)$. Then we obtain a stochastic CH equation with multiplicative noise,

$$u_t + u\partial_x u + F(u) = \sigma(u)\frac{\mathrm{d}W}{\mathrm{d}t}, \tag{1.7}$$

or equivalently,

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} + H(u)\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{1.8}$$

Example 1.1. One interesting example is $\sigma(u) = \beta u$ with $\beta \in \mathbb{R}$, i.e.,

$$u_t + u\partial_x u + F(u) = \beta u\frac{\mathrm{d}W}{\mathrm{d}t}, \quad \beta \in \mathbb{R}. \tag{1.9}$$

The above equation can also be expressed as

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} + \beta(u - u_{xx})\frac{\mathrm{d}W}{\mathrm{d}t}, \quad \beta \in \mathbb{R}. \tag{1.10}$$

It is easy to check that (1.10) has solutions $f(t)e^x$ and $f(t)e^{-x}$ for any differentiable function $f(t)$. Recall the weakly dissipative CH equation

$$u_t - u_{xxt} + 3uu_x + \lambda(u - u_{xx}) = 2u_x u_{xx} + uu_{xxx}, \ \lambda > 0, \tag{1.11}$$

where $\lambda(u - u_{xx})$ models the weak dissipation [36, 47]. Then, (1.10) can be explained as the CH equation with random dissipation $(-\beta)\frac{\mathrm{d}W}{\mathrm{d}t}(u - u_{xx})$, i.e.,

$$u_t - u_{xxt} + 3uu_x + (-\beta)\frac{\mathrm{d}W}{\mathrm{d}t}(u - u_{xx}) = 2u_x u_{xx} + uu_{xxx}. \tag{1.12}$$

However, $(-\beta)\frac{\mathrm{d}W}{\mathrm{d}t}$ may be negative in (1.12), while $\lambda > 0$ is required in (1.11). Particularly, for a modified Camassa–Holm equation with deterministic initial data and linear multiplicative noise, the existence of a pathwise solution can be found in Chen and Gao [7].

We remark that in this paper the SPDE (1.7) is understood as

$$\mathrm{d}u + [u\partial_x u + F(u)]\,\mathrm{d}t = \sigma(u)\mathrm{d}W, \tag{1.13}$$

and the others throughout the paper are also understood in a similar way.

The plan of this paper is as follows:

- We will consider the existence of the local $H^s (s > 3/2)$ pathwise solutions to the periodic boundary value problem of (1.13), i.e.,

  $$\begin{cases} \mathrm{d}u + [u\partial_x u + F(u)]\,\mathrm{d}t = \sigma(u)\mathrm{d}W, & x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \ t \in \mathbb{R}^+, \\ u(\omega, 0, x) = u_0(\omega, x), \ x \in \mathbb{T}; \end{cases} \tag{1.14}$$

  here the operator $(1 - \partial_x^2)^{-1}$ in $F(\cdot)$ is understood as

  (1.15)
  $$\left[(1 - \partial_x^2)^{-1} f\right](x) = [G_{\mathbb{T}} * f](x), \quad G_{\mathbb{T}} = \frac{\cosh(x - 2\pi\left[\frac{x}{2\pi}\right] - \pi)}{2\sinh(\pi)} \ \forall \ f \in L^2(\mathbb{T}),$$

  where $[x]$ stands for the integer part of x.
- When the noise in (1.13) is linear in u, i.e., $\sigma(u) = \beta u$ (in this case, (1.13) is equivalent to (1.9) or (1.10)), we will study the conditions that lead to the global existence and the blow-up in finite time of the solution, and then analyze the associated probabilities.
- We will also study how the linear noises in (1.10) impact the traveling waves of (1.1).

1.1. Notation and assumptions.

Notation. In this paper, $\mathcal{S}$ is a stochastic basis, that is,

$$\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W),$$

where $\mathbb{P}$ is a probability measure on Ω, $\mathcal{F}$ is a σ-algebra, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\{\mathcal{F}_0\}$ contains all the $\mathbb{P}$-negligible subsets, and $W(t) = W(\omega, t), \omega \in \Omega$ is a standard 1-D Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. For $t > 0$, $\sigma\{X(\tau), Y(\tau), Z(\tau)\}_{\tau\in[0,t]}$ stands for the completion of the union σ-algebra generated by $(X(\tau), Y(\tau), Z(\tau))$ with $\tau \in [0, t]$. All stochastic integrals are defined in the sense of Itô and $\mathbb{E}Y$ is the mathematical expectation of the stochastic process $Y = Y(\omega, t)$ with respect to $\mathbb{P}$. For any separable Banach space X, we use the symbol $\mathcal{B}(X)$ to denote its Borel sets and let $\mathcal{P}r(X)$ be the collection of Borel probability measures on X. For $E \subseteq X$, $\mathbf{1}_E$ is the indicator function on E, i.e., it is equal to 1 when $x \in E$ and zero otherwise.

$L^2(\mathbb{T})$ is the usual square integrable function space on $\mathbb{T}$ with the inner product $(f, g)_{L^2} = \int_{\mathbb{T}} f \cdot \overline{g}\ \mathrm{d}x$, where $\overline{g}$ indicates the complex conjugation of g. The Fourier transform of a periodic function $f(x) \in L^2(\mathbb{T})$ is defined by $\widehat{f}(k) = \mathcal{F}f(k) = \int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}xk} f(x)\ \mathrm{d}x,\ k \in \mathbb{Z}$. The inverse Fourier transform is given by $f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{\mathrm{i}xk}$. For any real number s, the operator $D^s = (1 - \partial_x^2)^{s/2}$ is defined by $\widehat{D^s f}(k) = (1 + k^2)^{s/2}\widehat{f}(k)$. Then the Sobolev space H^s on $\mathbb{T}$ can be defined as

$$H^s(\mathbb{T}) \triangleq \left\{ f \in L^2(\mathbb{T}) : \|f\|^2_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}} (1 + k^2)^s |\widehat{f}(k)|^2 < +\infty \right\},$$

and the inner product $(f, g)_{H^s}$ is defined as $(f, g)_{H^s} \triangleq \sum_{k\in\mathbb{Z}} (1 + k^2)^s \widehat{f}(k) \cdot \overline{\widehat{g}}(k) = (D^s f, D^s g)_{L^2}$. In the following, when the function spaces are over $\mathbb{T}$ we drop $\mathbb{T}$ if there is no ambiguity. We will use $\lesssim$ to denote estimates that hold up to some universal *deterministic* constants which may change from line to line but whose meaning is clear from the context. For linear operators A and B, we denote $[A, B] = AB - BA$.

Assumption 1.2. We shall make the following assumptions in this paper.

(I) When we consider the existence of a martingale solution (see Definition 1.3 below) to (1.14), we suppose that

$$\sigma(\cdot) : H^\rho \to H^\rho \ \text{ is continuous for } \ \rho > 3/2. \tag{1.16}$$

(II) We assume that there is an increasing and locally bounded (independent of t, x) function f with $f(0) = 0$ such that for all $u \in H^\rho, \rho > 3/2$, $\sigma(\cdot)$ satisfies

$$\|\sigma(u)\|_{H^\rho} \leq f(\|u\|_{W^{1,\infty}})(1 + \|u\|_{H^\rho}). \tag{1.17}$$

(III) For the existence of a pathwise solution (see Definitions 1.4 and 1.5 below) to (1.14), we need a condition stronger than (1.16), that is, for some increasing and locally bounded (independent of t, x) function g,

$$\|\sigma(u) - \sigma(v)\|_{H^\rho} \leq g(\|u\|_{W^{1,\infty}} + \|v\|_{W^{1,\infty}})\|u - v\|_{H^\rho} \quad \forall\ u, v \in H^\rho, \rho > 1/2. \tag{1.18}$$

(IV) When we consider the martingale solutions (see Definition 1.3 below), the stochastic basis $\mathcal{S}$ itself is an unknown part of the problem (1.14). In order to describe the initial condition u_0, which may be random in general and may only be regarded as an initial probability measure $\mu_0 \in \mathcal{P}r(H^s)$, we assume that μ_0 satisfies that for some $r > 4$,

$$\int_{H^s} \|u\|_{H^s}^r \mathrm{d}\mu_0(u) < \infty, \quad s > 3/2. \tag{1.19}$$

1.2. Definitions of the solutions. We now make precise the notions of the martingale and pathwise solution to the problem (1.14).

Definition 1.3 (martingale solutions). *Let $s > 3/2$ and $\mu_0 \in \mathcal{P}r(H^s)$ satisfy* (1.19). *A martingale solution to* (1.14) *is a triple $(\mathcal{S}, u, \tau)$ such that the following hold:*

1. *$\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ is a stochastic basis and τ is a stopping time relative to $\mathcal{F}_t$;*
2. *$u : \Omega\times[0,\infty) \to H^s$ is an $\mathcal{F}_t$ predictable H^s-valued process such that $\mu_0(Y) = \mathbb{P}\{u(0) \in Y\}$ $\forall$ $Y \in \mathcal{B}(H^s)$ and*

$$u(\cdot\wedge\tau) \in L^2\left(\Omega; C\left([0,\infty); H^s\right)\right) \quad \text{and} \quad u(\cdot\wedge\tau) \in C([0,\infty); H^s),\ \mathbb{P}\text{-}a.s. \tag{1.20}$$

3. *Let $F(\cdot)$ be given in* (1.5). *For every $t > 0$ and $v \in C^\infty(\mathbb{T})$,*

$$(u(t\wedge\tau), v)_{L^2} - (u(0), v)_{L^2} + \int_0^{t\wedge\tau} (u\partial_x u + F(u), v)_{L^2}\,\mathrm{d}t' = \int_0^{t\wedge\tau} (\sigma(u), v)_{L^2}\mathrm{d}W \tag{1.21}$$

almost surely.

4. *If $\tau = \infty, \mathbb{P}$-a.s., then we say the martingale solution is global.*

Definition 1.4 (pathwise solutions). *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ be a fixed stochastic basis. Let $s > 3/2$ and u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable* (*relative to $\mathcal{S}$*). *A local pathwise H^s solution to* (1.14) *is a pair (u, τ), where τ is a stopping time satisfying $\mathbb{P}\{\tau > 0\} = 1$ and $u : \Omega \times [0, \tau] \to H^s$ is an $\mathcal{F}_t$ predictable H^s-valued process satisfying* (1.20) *and* (1.21).

Definition 1.5 (pathwise uniqueness). *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ be a fixed stochastic basis. The local pathwise solutions are said to be pathwise unique if given any two pairs of local pathwise solutions (u_1, τ_1) and (u_2, τ_2) with $\mathbb{P}\{u_1(0) = u_2(0)\} = 1$, then*

$$\mathbb{P}\left\{u_1(t, x) = u_2(t, x)\ \forall\ (t, x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{T}\right\} = 1. \tag{1.22}$$

Definition 1.6 (maximal and global solutions). *Let $s > 3/2$. A maximal H^s solution to* (1.14) *is a triple $(u, \{\tau_n\}_{n\in\mathbb{N}}, \xi)$ such that for each $n \in \mathbb{N}$, (u, τ_n) is a pathwise H^s solution, $\tau_{n+1} \ge \tau_n$, $\lim_{n\to\infty} \tau_n = \xi$, and*

$$\sup_{t\in[0,\tau_n]} \|u\|_{H^s} \ge n,\ \mathbb{P}\text{-}a.s.,\ \text{on the set } \{\xi < \infty\}. \tag{1.23}$$

If $\xi = \infty$, $\mathbb{P}$-a.s., then we say that the pathwise solution exists globally.

1.3. Main results. We can now formulate our main results.

Theorem 1.7. *Let $s > 3/2$ and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ be a fixed stochastic basis. If u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|^2_{H^s} < \infty$ and if $\sigma(\cdot)$ satisfies the assumptions* (1.17), (1.18), *then* (1.14) *admits a unique maximal solution $(u, \{\tau_n\}_{n\in\mathbb{N}}, \xi)$, in the sense of Definitions* 1.4–1.6.

THEOREM 1.8 (global existence). *Let $s > 3$ and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ be a fixed stochastic basis. Assume that u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable satisfying $\mathbb{E}\|u_0\|^2_{H^s} < \infty$ and*

$$\mathbb{P}\left\{(1-\partial_x^2)u_0(x) > 0 \ \forall x \in \mathbb{T}\right\} = p_1, \quad \mathbb{P}\left\{(1-\partial_x^2)u_0(x) < 0 \ \forall x \in \mathbb{T}\right\} = p_2,$$

for some $p_1, p_2 \in [0,1]$; then the maximal solution $(u, \{\tau_n\}_{n\in\mathbb{N}}, \xi)$ to (1.9) *or to* (1.10) (*guaranteed by Theorem* 1.7) *satisfies*

$$\mathbb{P}\{\xi = \infty\} \geq p_1 + p_2, \tag{1.24}$$

or in other words, $\mathbb{P}\{u \text{ exists globally}\} \geq p_1 + p_2$.

THEOREM 1.9 (blow-up). *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ be a fixed stochastic basis and $s > 3$. If $u_0 = u_0(x) \in H^s$ is a* deterministic *function such that $u_0(x)$ is odd and $\partial_x u_0(0) < 0$, then the maximal solution $(u, \{\tau_n\}_{n\in\mathbb{N}}, \xi)$ to* (1.9) *or to* (1.10) (*guaranteed by Theorem* 1.7) *satisfies*

$$\mathbb{P}\left\{\xi \leq \frac{-2}{\partial_x u_0(0)}\right\} = 1. \tag{1.25}$$

That is to say, $\mathbb{P}\{u \text{ blows up in finite time}\} = 1$.

We also study the noise effect on the periodic peaked waves of the deterministic CH equation (1.1) and the results are stated in the following theorem.

THEOREM 1.10 (dissipative effect of the linear noise on periodic peakons). *Consider the deterministic CH equation* (1.1) *on $\mathbb{T}$ and let $u(x-ct)(t>0, c\in\mathbb{R}, x\in\mathbb{T})$ be the periodic peakons of* (1.1). *For a.e. $\omega \in \Omega$ and for the same initial profile $u(x)$, there is a pathwise* weak *solution $\widetilde{u}(\omega,t,x)$ to* (1.9) *in the sense that* (1.21) *is verified in the sense of distribution* (*i.e., the $(\cdot,\cdot)_{H^s}$ is replaced by the dual product for a distribution and a test function*) *and satisfies*

$$\mathbb{P}\{\widetilde{u}(\omega,0,x) = u(x);\ x\in\mathbb{T}\} = 1, \quad \mathbb{P}\left\{\lim_{t\to\infty}\widetilde{u}(\omega,t,x) = 0;\ x\in\mathbb{T}\right\} = 1. \tag{1.26}$$

Moreover, the "phase velocity" of $\widetilde{u}(\omega,t,x)$ (*the rate at which $\widetilde{u}$ propagates in the space*) *is a geometric Brownian motion $c\mathrm{e}^{(\beta W(t) - \frac{\beta^2}{2}t)}$, which tends to 0 when $t\to\infty$, almost surely.*

A few remarks concerning the main results are as follows.

Remark 1.11. Let us first briefly introduce the main ideas of the proof for Theorem 1.7 and the main difficulties we are confronted with. We first notice that since $F : H^s \to H^s$ reserves the regularity for $s > 3/2$ (see Lemma 2.3 below) and so does $\sigma(\cdot)$, $(1.14)_1$ can be viewed as a Burgers type transport equation with some nonlocal H^s-valued random perturbations as

$$u_t + uu_x = \text{nonlocal random perturbations.}$$

As pointed in [26], the generalization from linear to nonlinear stochastic transport equations, where the drift depends on u itself, turns out to be a difficult problem.

Step 1. *Approximations scheme.* The first difficulty for (1.14) comes from the choice of a suitable approximation scheme. Actually, when we consider the estimate for some kind of approximation solutions u_ε, we have

$$\mathbb{E}\|u_\varepsilon(t)\|_{H^s}^2 - \mathbb{E}\|u_\varepsilon(0)\|_{H^s}^2 \le C_s \int_0^t \mathbb{E}(\|u_\varepsilon\|_{H^s}^2 \|\partial_x u_\varepsilon\|_{L^\infty}) \mathrm{d}t' + \text{stochastic integral}.$$

Since the expectation $\mathbb{E}(\|\partial_x u_\varepsilon\|_{L^\infty}\|u_\varepsilon\|_{H^s}^2)$ *cannot* be split, which *prevents* us from closing the a priori $L^2(\Omega; H^s)$ estimate for u_ε, we will have to deal with a sequence of stopping times $\{\tau_\varepsilon\}$ with $\tau_\varepsilon = \inf\{t \ge 0 : \|\partial_x u_\varepsilon(t)\|_{L^\infty} > R\}$ such that

$$\int_0^{t\wedge\tau_\varepsilon} \mathbb{E}(\|u_\varepsilon\|_{H^s}^2 \|\partial_x u_\varepsilon\|_{L^\infty}) \mathrm{d}t' \le R \int_0^{t\wedge\tau_\varepsilon} \mathbb{E}\|u_\varepsilon\|_{H^s}^2 \mathrm{d}t'.$$

But how to prove $\mathbb{P}\{\inf_{0<\varepsilon<1} \tau_\varepsilon > 0\} = 1$ is generally not clear. Therefore we need a uniform $L^\infty(\Omega; W^{1,\infty})$ condition. Furthermore, we notice that the widely used Faedo–Galerkin method for many stochastic incompressible fluid models is hard to use here directly and this is because for the CH equation, we do *not* have the additional *cancellation property* (see, e.g., [24, 30]), which comes from the incompressibility and can be used to guarantee the global existence of the approximation solution u_N (see, e.g., [30, 31, 24]). Without this property, the global existence of the approximation solution will not be something we can know a priori. Then for the existence time τ_N of u_N, we need to find a uniform positive lower bound for τ_N, almost surely. Therefore a further approximation is needed after the mollifying and the cut-off, to guarantee that all the approximation solutions are global. In this paper, we will first add a *cut-off* function in (1.14), which provides the uniform $L^\infty(\Omega; W^{1,\infty})$ condition, then we *mollify* the transport term uu_x and approximate the resulted drift $G_{1,\varepsilon}(\cdot)$ and diffusion $G_2(\cdot)$ by using *Lipschitz approximations* $G_{1,\varepsilon,n}(\cdot)$ and $G_{2,n}(\cdot)$, respectively. Thus the existence theory of SDE in Hilbert space can be applied to obtain a sequence of global and unique approximation solutions $u_{\varepsilon,n}$. For more details on this approximation scheme, we refer to Remark 3.2. We believe that this method in our paper can be applicable to the study of well-posedness for other nonlinear stochastic transport type equations and therefore hold independent interest.

Step 2. *Limits of* $n \to \infty$. We will send $n \to \infty$ first to remove the Lipschitz approximation. We first consider an H^s initial data with $s > 3$ and take the limit of $n \to \infty$ to obtain an H^s pathwise solution with $s > 3$. When we consider the limit, we will be confronted with the essential difficulty in the stochastic setting, that is, the lack of compactness in the $\omega \in \Omega$. Generally speaking, we usually do not know if the embedding $L^2(\Omega; X) \hookrightarrow L^2(\Omega; Y)$ is compact, even if $X \hookrightarrow\hookrightarrow Y$. As a result, the usual compactness criteria, such as the standard Aubin or Arzelà-Ascoli type theorems, cannot be used directly. To overcome this difficulty, we will first use the Prokhorov theorem to find the tightness of the probability measures $\mu_{\varepsilon,n}$ defined by the approximate solutions $u_{\varepsilon,n}$ and then use the Skorokhod theorem to infer the existence of an almost sure convergent sequence $(\widetilde{u_{\varepsilon,n}}, \widetilde{W_{\varepsilon,n}})$, which is relative to a *different* probability space. Finally, using a refined martingale representation theorem, we can send $n \to \infty$ in $(\widetilde{u_{\varepsilon,n}}, \widetilde{W_{\varepsilon,n}})$ to build a global martingale solution in H^s with $s > 3$ to the mollified problem. We notice that the stochastic basis, as one part of the martingale solution, depends on ε. Before sending $\varepsilon \to 0$, we will prove the existence of a unique pathwise solution to the mollified problem, by combining pathwise uniqueness and the existence of a martingale solution, where the powerful Gÿongy–Krylov characterization of the convergence in probability (see Lemma 2.7 below) will be used.

Step 3. *Limit of* $\varepsilon \to 0$. Since we have obtained the pathwise solution to the mollified problem, we can basically repeat the above procedure to establish the existence of a global martingale solution in H^s with $s > 3$ to the cut-off problem. Then we prove the pathwise uniqueness and then use the Gÿongy–Krylov characterization again to obtain the existence of a unique pathwise solution to the cut-off problem.

Step 4. *Remove the cut-off.* We use a sequence of stopping times $\{\tau_k\}$ to remove the cut-off function and the $L^r(r > 4)$ integrability condition on u_0 and to guarantee the positivity of the final existence time simultaneously. Then we obtain the local pathwise solution to (1.14) in H^s with $s > 3$.

Step 5. *Extend the range of s to $s > 3/2$.* By mollifying the initial data, we obtain a sequence of approximation solutions $\{u_k\}_{k\in\mathbb{N}}$ in H^s with $s > 3$. Then we prove this sequence is a Cauchy sequence in H^s with $s > 3/2$ for some positive stopping time. Here the main difficulty is that, the estimate for $\mathbb{E}\|w_{m,k}(t')\|^2_{H^s}$, where $w_{m,k} = u_m - u_k$, is not closed and we need to show that $\mathbb{E}\left(\|w_{m,k}\|^2_{H^{s-1}}\|u_m\|^2_{H^{s+1}}\right) \sim o(1)$. This difficulty will be overcome by a series of estimates and the property of the mollifiers.

Remark 1.12. The existence and uniqueness result in Theorem 1.7 can be extended to the case that the driven process is a cylindrical Wiener process, which is H^s-valued. Of course we need to redefine $\sigma(\cdot)$ and impose certain assumptions similar to (1.16)–(1.18); see [30, 41], for example. We notice that in such a case, the Burkholder–Davis–Gundy inequality (see [30, 40] and the references therein) and the (generalized) martingale representation theorem (see Theorem A.1 in [34]) are also available.

Remark 1.13. The well-known global existence result for (1.1) is under the condition that $u_0 - \partial_{xx}u_0$ does not change sign (cf. [13]), which can be viewed as $(p_1, p_2) = (1, 0)$ or $(0, 1)$. Hence Theorem 1.8 covers the deterministic results. Similarly, Theorem 1.9 also covers the deterministic case; cf. [11, 13]. But for general random initial data, whether the solution blows up is still not clear.

Remark 1.14. From Theorem 1.10, we know that no matter how small $|\beta|$ is, all periodic peakons $u(t - cx)$ of the deterministic CH equation (1.1) are destroyed by the noise in the sense that, there is a "stochastic periodic peakon" $\widetilde{u}(\omega, t, x)$ with the same initial profile $u(x)$ satisfying that the amplitude of $\widetilde{u}$ and the rate $\widetilde{u}$ propagates in space tend to zero as $t \to \infty$, almost surely. We notice this phenomenon also happens for any $C^3(\mathbb{R})$ traveling waves, no matter what initial profile the traveling wave has (see Remark 6.2 for more details). Therefore one may conclude that the linear noise in (1.9) has a dissipative effect on the traveling waves of (1.1), in the sense of pathwise. However, roughly speaking, finding traveling waves in the deterministic case should be extended to finding some invariant measures in the stochastic setting. Therefore Theorem 1.10 implies that finding an invariant measure to (1.9) or (1.10) is very difficult. On the other hand, the randomness is one of the popular hypotheses on the onset of wave turbulence. Since wave turbulence in continuum mechanics is usually accompanied by energy dissipations and an external source of energy is needed to sustain the shapes and propagations of the waves. In this sense, the results in Theorem 1.10 are reasonable.

Now we outline the rest of the paper. In the next section, we briefly recall some relevant preliminaries. In section 3, we first establish the existence of the unique pathwise solution in sufficiently regular spaces. In section 4, we obtain the local existence and uniqueness of the pathwise solution in Sobolev spaces H^s with $s > 3/2$, which is Theorem 1.7. When the noise is linear, we consider the global existence and blow-up of the pathwise solution and prove Theorems 1.8 and 1.9 in section 5. Finally, we study the pathwise dissipative effect of the noise on the traveling waves in section 6.

2. Preliminaries. Now we briefly recall some relevant mathematical preliminaries from functional analysis and probability theory, which will be used later.

For each $\varepsilon \in (0,1]$, J_ε stands for the Friedrichs mollifier defined by

$$J_\varepsilon f(x) = j_\varepsilon * f(x), \tag{2.1}$$

where $*$ stands for the convolution. Here $j_\varepsilon(x) = \mathcal{F}^{-1}(\widehat{j}(\varepsilon\xi))$ and $j(x)$ is a Schwartz function satisfying $0 \le \widehat{j}(\xi) \le 1$ for all the $\xi \in \mathbb{R}$ and $\widehat{j}(\xi) = 1$ for any $\xi \in [-1,1]$. From this construction of j_ε, we see that $\widehat{j}_\varepsilon(\xi) = \widehat{j}(\varepsilon\xi)$ and for any $u \in H^s$ and $r \le s$,

$$\|u - J_\varepsilon u\|_{H^r} = o(\varepsilon^{s-r}). \tag{2.2}$$

In fact, since $\widehat{j}(\xi) = 1$ for any $\xi \in [-1,1]$ and $0 \le \widehat{j}(\xi) \le 1$, we have

$$\begin{aligned}\varepsilon^{2r-2s}\|u - J_\varepsilon u\|^2_{H^r} &= \sum_{k\in\mathbb{Z}} (1+k^2)^s \frac{\varepsilon^{2r-2s}}{(1+k^2)^{s-r}} \left|1 - \widehat{j}(\varepsilon k)\right|^2 |\widehat{u}(k)|^2 \\ &\lesssim \sum_{|k|>\frac{1}{\varepsilon}} (1+k^2)^s \frac{\varepsilon^{2r-2s}}{(\varepsilon^2)^{r-s}} \left|1 - \widehat{j}(\varepsilon k)\right|^2 |\widehat{u}(k)|^2 \lesssim \|u - J_\varepsilon u\|^2_{H^s} \sim o(1).\end{aligned}$$

J_ε also admits that for $u \in H^s$ and $r \ge s$,

$$\|J_\varepsilon u\|_{H^r} \lesssim \varepsilon^{s-r}\|u\|_{H^s}. \tag{2.3}$$

To see this, for $\varepsilon \in (0,1)$ and $r \ge s$, we consider

$$\|J_\varepsilon u\|^2_{H^r} = \sum_{k\in\mathbb{Z}} (1+k^2)^r |\widehat{j}(\varepsilon k)|^2 |\widehat{u}(k)|^2 \le \|u\|^2_{H^s} \left(\sup_{k\in\mathbb{Z}} \frac{(1+k^2)^r}{(1+k^2)^s} |\widehat{j}(\varepsilon k)|^2 \right).$$

By the construction of the $j_\varepsilon(x)$, there holds the following estimate:

$$\sup_{k\in\mathbb{Z}} \frac{(1+k^2)^r}{(1+k^2)^s} |\widehat{j}(\varepsilon k)|^2 = \sup_{\varepsilon k = m \in \mathbb{R}} \frac{(1+\left(\frac{m}{\varepsilon}\right)^2)^r}{(1+\left(\frac{m}{\varepsilon}\right)^2)^s} |\widehat{j}(m)|^2 = \varepsilon^{2s-2r} \sup_{m\in\mathbb{R}} \left(\varepsilon^2 + m^2\right)^{r-s} |\widehat{j}(m)|^2.$$

Since $(\varepsilon^2 + m^2)^{r-s}|\widehat{j}(m)|^2$ is bounded uniformly in $\varepsilon \in (0,1)$, we obtain (2.3). In addition,

$$D^s J_\varepsilon = J_\varepsilon D^s, \tag{2.4}$$

$$(J_\varepsilon f, g)_{L^2} = (f, J_\varepsilon g)_{L^2}, \tag{2.5}$$

$$\|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s}. \tag{2.6}$$

We first recall some commutator estimates.

Lemma 2.1 (see [35]). *If $f \in H^s \bigcap W^{1,\infty}, g \in H^{s-1} \bigcap L^\infty$ for $s > 0$, then*

$$\|\,[D^s, f]\,g\|_{L^2} \le C_s(\|D^s f\|_{L^2}\|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty}\|D^{s-1}g\|_{L^2}).$$

If $f, g \in H^s \bigcap L^\infty$, then

$$\|fg\|_{H^s} \le C_s(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}).$$

The following commutator estimate is also useful.

LEMMA 2.2 (Proposition 4.2 in [42]). *If $s > 3/2$ and $0 \le \eta + 1 \le s$, then for some $C > 0$,*

$$\|[D^\eta \partial_x, f]v\|_{L^2} \le C\|f\|_{H^s}\|v\|_{H^\eta} \quad \forall\ f \in H^s, v \in H^\eta.$$

A direct application of Lemma 2.1 gives the following estimates and we omit the details here.

LEMMA 2.3 (see [43]). *For the $F(\cdot)$ defined in (1.5) and for any $u, v \in H^s$ with $s > 1/2$, we have*

$$\|F(v)\|_{H^s} \lesssim (\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty})\, \|v\|_{H^s}, \quad s > 3/2, \tag{2.7}$$

$$\|F(v_1) - F(v_2)\|_{H^s} \lesssim (\|v_1\|_{H^s} + \|v_2\|_{H^s})\, \|v_1 - v_2\|_{H^s}, \quad s > 3/2, \tag{2.8}$$

$$\|F(v_1) - F(v_2)\|_{H^s} \lesssim (\|v_1\|_{H^{s+1}} + \|v_2\|_{H^{s+1}})\, \|v_1 - v_2\|_{H^s}, \quad 3/2 > s > 1/2. \tag{2.9}$$

LEMMA 2.4 (Prokhorov theorem, [39]). *Let X be a complete, separable metric space. A sequence of probability measures $\{\mu_n\} \subset \mathcal{P}r(X)$ is tight if and only if it is relatively compact, i.e., there is a subsequence $\{\mu_{n_k}\}$ which converges weakly to a probability measure μ on X.*

LEMMA 2.5 (Skorokhod theorem, [39]). *Let X be a complete, separable metric space. For an arbitrary sequence $\{\mu_n\} \subset \mathcal{P}r(X)$ such that $\{\mu_n\}$ is tight on $(X, \mathcal{B}(X))$, there exists a subsequence $\{\mu_{n_k}\}$ which converges weakly to a probability measure μ, and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with X valued Borel measurable random variables x_n and x, such that μ_n is the distribution of x_n, μ is the distribution of x, and $x_n \to x$, $\mathbb{P}$-a.s.*

LEMMA 2.6 (Vitali's convergence theorem, [8]). *Let $p \in [1, \infty)$, $X_n \in L^p$, and X_n converge to X in probability. Then the following are equivalent:*
1. $\lim_{n\to\infty} X_n = X$ in L^p;
2. $|X_n|^p$ is uniformly integrable;
3. $\lim_{n\to\infty} \mathbb{E}[|X_n|^p] = \mathbb{E}[|X|^p]$.

Particularly, if $\sup_n \mathbb{E}[|X_n|^q] < \infty$ for some $p < q < \infty$, or if there exists a $Y \in L^p$ such that $|X_n| < Y$ for all n, then the above properties hold true.

LEMMA 2.7 (Gyöngy–Krylov Lemma, [33]). *Let X be a Polish space equipped with the Borel σ-algebra. Let $\{Y_j\}_{j\ge 0}$ be a sequence of X valued random variables and $\{\mu_{j,l}\}_{j,l\ge 0}$ be the joint laws of $\{Y_j\}_{j\ge 0}$. Then $\{Y_j\}_{j\ge 0}$ converges in probability if and only if for every subsequence of $\{\mu_{j_k,l_k}\}_{k\ge 0}$, there exists a further subsequence which weakly converges to some $\mu \in \mathcal{P}r(X \times X)$ satisfying*

$$\mu\left(\{(u, v) \in X \times X,\ u = v\}\right) = 1.$$

3. Pathwise solution in sufficiently regular spaces.

We will prove the following result in this section.

THEOREM 3.1. *Let $s > 3$ and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ be a fixed stochastic basis. If u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$ and if $\sigma(\cdot)$ satisfies the assumptions (1.17), (1.18), then (1.14) admits a unique maximal solution $(u, \{\tau_n\}_{n\in\mathbb{N}}, \xi)$, in the sense of Definitions 1.4–1.6.*

The following subsections are devoted to the proof of Theorem 3.1.

3.1. Approximation scheme. We will construct the approximation scheme as follows.

Cut-off. For any $R > 1$, we let $\theta_R(x) : [0,\infty) \to [0,1]$ be a C^∞ function such that $\theta_R(x) = 1$ for $x \in [0,R]$ and $\theta_R(x) = 0$ for $x > 2R$. Then we consider the following problem by cutting the nonlinearities in (1.14):

$$\begin{cases} \mathrm{d}u + \theta_R(\|u\|_{W^{1,\infty}})\,[u\partial_x u + F(u)]\,\mathrm{d}t = \theta_R(\|u\|_{W^{1,\infty}})\sigma(u)\mathrm{d}W, \quad x \in \mathbb{T},\ t > 0, \\ u(\omega,0,x) = u_0(\omega,x) \in H^s,\ x \in \mathbb{T}. \end{cases} \tag{3.1}$$

Mollifying. From Lemma 2.3, we see that the nonlinear term $F(u)$ reserves the H^s regularity for any $s > 3/2$. However, to apply the theory of SDE in Hilbert space to (3.1), we will have to mollify the transport term uu_x since the product $u\partial_x u$ loses one regularity. Therefore we mollify (3.1) and consider

$$\begin{cases} \mathrm{d}u + G_{1,\varepsilon}(u)\mathrm{d}t = G_2(u)\mathrm{d}W, \quad x \in \mathbb{T},\ t > 0, \\ \qquad G_{1,\varepsilon}(u) = \theta_R(\|u\|_{W^{1,\infty}})\,[J_\varepsilon\,(J_\varepsilon u\partial_x J_\varepsilon u) + F(u)]\,, \\ \qquad G_2(u) = \theta_R(\|u\|_{W^{1,\infty}})\sigma(u), \\ \qquad u(\omega,0,x) = u_0(\omega,x) \in H^s(\mathbb{T}), \end{cases} \tag{3.2}$$

where J_ε is the Friedrichs mollifier defined in the previous section. From (2.3), Lemma 2.3, and (1.17), we see that for any $T > 0$ and $R > 1$, there exists an $l = l(R,\varepsilon)$ such that for all $u \in C([0,T];H^\rho), \rho > 3/2$, $G_{1,\varepsilon}(\cdot)$ and $G_2(\cdot)$ satisfy

$$\|G_{1,\varepsilon}(u)\|_{H^\rho} + \|G_2(u)\|_{H^\rho} \le l(1 + \|u\|_{H^\rho}). \tag{3.3}$$

Lipschitz approximations. We notice that for many stochastic incompressible fluid models, the incompressibility means that $((u\cdot\nabla)u,u)_{L^2} = 0$, then the drift term can be simplified, and then the global existence of the approximation solution can be obtained; cf. [24, 30]. Without such a property, here we need to approximate $G_{1,\varepsilon}$ and G_2 again. Following the steps as in Lemma 3.9 in [28] (see Theorem 3.12 in [29] also), we can construct two sequences of functions $G_{1,\varepsilon,n}(\cdot), G_{2,n}(\cdot) : C([0,T];H^\rho) \to C([0,T];H^\rho), \rho > 3/2$, such that for some $M = M(n) > 0, L = L(R,\varepsilon)$ and for all $u,v \in C([0,T];H^\rho)$,

$$\|G_{1,\varepsilon,n}(u)\|_{H^\rho} + \|G_{2,n}(u)\|_{H^\rho} \le L(1 + \|u\|_{H^\rho}), \tag{3.4}$$

$$\|G_{1,\varepsilon,n}(u) - G_{1,\varepsilon,n}(v)\|_{H^\rho} + \|G_{2,n}(u) - G_{2,n}(v)\|_{H^\rho} \le M\|u - v\|_{H^\rho}. \tag{3.5}$$

Moreover, we have

$$\sup_{t\in[0,T]} \|G_{1,\varepsilon,n}(\cdot) - G_{1,\varepsilon}(\cdot)\|_{H^\rho} + \sup_{t\in[0,T]} \|G_{2,n}(\cdot) - G_2(\cdot)\|_{H^\rho} \to 0 \tag{3.6}$$

uniformly on compact subsets of $C([0,T];H^\rho), \rho > 3/2$.

For $s > 3$, we choose $s_0 = s - 3/2 > 3/2$ and consider the approximations $G_{1,\varepsilon,n}(\cdot), G_{2,n}(\cdot) : C([0,T];H^{s_0}) \to C([0,T];H^{s_0})$. Then (3.6) is satisfied with $\rho = s_0$. Since (3.3) holds true for all $\rho > 3/2$, the restrictions of $G_{1,\varepsilon}(\cdot)$ and $G_2(\cdot)$ on H^s also enjoy the properties (3.3) with $\rho = s$. Since $H^s \hookrightarrow\hookrightarrow H^{s_0}$, one can repeat the process as in Theorem 3.12 in [28] to deduce that the restrictions of $G_{1,\varepsilon,n}(\cdot)$ and $G_{2,n}(\cdot)$ in H^s also enjoy (3.4) (3.5) with $\rho = s$.

Conclusion. Finally we consider the following approximation scheme of (3.1):

$$\begin{cases} \mathrm{d}u + G_{1,\varepsilon,n}(u)\mathrm{d}t = G_{2,n}(u)\mathrm{d}W, \quad x \in \mathbb{T},\ t > 0, \\ \qquad u(\omega,0,x) = u_0(\omega,x) \in H^s(\mathbb{T}), \end{cases} \tag{3.7}$$

where $G_{1,\varepsilon,n}(\cdot)$ and $G_{2,n}(\cdot)$ are the Lipschitz approximations of $G_{1,\varepsilon}(\cdot)$ and $G_2(\cdot)$, respectively. And for $s > 3$, $G_{1,\varepsilon,n}(\cdot)$ and $G_{2,n}(\cdot)$ satisfy

$$\|G_{1,\varepsilon,n}(u)\|_{H^s} + \|G_{2,n}(u)\|_{H^s} \le L(1 + \|u\|_{H^s}), \quad L = L(R,\varepsilon), \tag{3.8}$$

$$\|G_{1,\varepsilon,n}(u) - G_{1,\varepsilon,n}(v)\|_{H^s} + \|G_{2,n}(u) - G_{2,n}(v)\|_{H^s} \le M\|u - v\|_{H^s}, \quad M = M(n). \tag{3.9}$$

Moreover, let Y be any compact subset of $C([0,T]; H^{s_0})$, $s_0 = s - 3/2$, then

$$\|G_{1,\varepsilon,n}(\cdot) - G_{1,\varepsilon}(\cdot)\|_{C([0,T];H^{s_0})} + \|G_{2,n}(\cdot) - G_2(\cdot)\|_{C([0,T];H^{s_0})} \xrightarrow{\text{uniformly on } Y} 0. \tag{3.10}$$

Remark 3.2. After introducing the cut-off function, we need to guarantee that the cut-off makes sense. No matter when we consider the limits $n \to \infty$ and $\varepsilon \to 0$, or the uniqueness, at the first step we can only obtain the convergence $u_{\varepsilon,n} \to u_\varepsilon$ ($u_\varepsilon \to u$, resp.) and the uniqueness of u_ε (or u, resp.) in H^{s-1}; therefore we will have to consider (1.14) in sufficiently regular spaces H^s, such that $H^{s-1} \hookrightarrow W^{1,\infty}$ still holds true. Now we notice that $G_{1,\varepsilon}(\cdot)$ does not *uniformly* satisfy the linear growth condition, i.e., $l = l(R,\varepsilon) \propto \frac{R}{\varepsilon}$ in (3.3). Then it is difficult to obtain some estimates for the existence time τ_ε of u_ε uniformly in ε, and hence we do not a priori know that $\inf_{\varepsilon>0} \tau_\varepsilon > 0$, $\mathbb{P}$-a.s. To compensate for this difficulty, we are motivated to approximate (3.2) once more to obtain a uniform linear growth condition (3.8). In this paper, we use the *Lipschitz approximations* to achieve this. However, we remark that it is not a necessary step if we can obtain an almost sure positive lower bound for τ_ε, uniformly in ε. Besides, for each fixed $\varepsilon \in (0,1)$, only after cutting the nonlinearity can $G_{1,\varepsilon,n}(\cdot), G_{2,n}(\cdot)$ be constructed such that (3.8) is satisfied *uniformly* in n (see Theorem 3.12 in [28]). Moreover, as mentioned before, we will have to consider compactness in some regular space H^s, hence the Lipschitz approximations $G_{1,\varepsilon,n}(\cdot)$ and $G_{2,n}(\cdot)$ are constructed to satisfy (3.8) and (3.9) in H^s but satisfy (3.10) in H^{s_0} with $s_0 = s - 3/2$.

3.2. Some a priori estimates. Now we establish some a priori estimates for (3.2) and (3.7), respectively.

PROPOSITION 3.3. *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, W)$ be a fixed stochastic basis. Suppose that $\sigma(\cdot)$ satisfies the assumption* (1.17). *Let $s > 3$, $r > 4$, and $u_0 \in L^r(\Omega; H^s)$ be an H^s-valued $\mathcal{F}_0$ measurable random variable. For any $R > 1$, $\varepsilon \in (0,1)$ and $n \ge 1$, if for some $T > 0$, $u_{\varepsilon,n}$ solves* (3.7) *on $[0,T]$ and $u_{\varepsilon,n} \in C([0,T], H^s)$, $\mathbb{P}$-a.s., then for $0 < \alpha < \frac{r-4}{2r}$,*

$$u_{\varepsilon,n} \in L^r\left(\Omega; C\left([0,T]; H^s\right) \bigcap C^\alpha\left([0,T]; H^{s-1}\right)\right)$$

is bounded uniformly in n. *Moreover, for some $C_1 = C_1(R,\varepsilon,T,u_0,r,\alpha) > 0$,*

$$\sup_{n\ge1} \left\| \int_0^t G_{1,\varepsilon,n}(u_{\varepsilon,n})\mathrm{d}\tau \right\|_{L^r(\Omega; Lip([0,T];H^{s-1}))} \le C_1, \tag{3.11}$$

$$\sup_{n\ge1} \left\| \int_0^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W \right\|_{L^r(\Omega; C^\alpha([0,T];H^{s-1}))} \le C_1, \tag{3.12}$$

where $G_{1,\varepsilon,n}(\cdot)$ and $G_{2,n}(\cdot)$ are the Lipschitz approximations given in subsection 3.1.

Proof. To begin with, we apply $D^s (s > 3)$ to $(3.2)_1$ to find

$$\mathrm{d}D^s u_{\varepsilon,n} + D^s G_{1,\varepsilon,n}(u_{\varepsilon,n})\mathrm{d}t = D^s G_{2,n}(u_{\varepsilon,n})\mathrm{d}W. \tag{3.13}$$

Using the Itô formula, we deduce that

$$\begin{aligned}
\mathrm{d}\|u_{\varepsilon,n}\|_{H^s}^2 &= \mathrm{d}\left(D^s u_{\varepsilon,n}, D^s u_{\varepsilon,n}\right)_{L^2} \\
&= 2\left(\mathrm{d}D^s u_{\varepsilon}, D^s u_{\varepsilon}\right)_{L^2} + \|D^s G_{2,n}(u_{\varepsilon,n})\|_{L^2}^2 \mathrm{d}t \\
&= 2\left(D^s G_{2,n}(u_{\varepsilon,n}), D^s u_{\varepsilon,n}\right)_{L^2} \mathrm{d}W \\
&\quad - 2\left(D^s G_{1,\varepsilon,n}(u_{\varepsilon,n}), D^s u_{\varepsilon,n}\right)_{L^2} \mathrm{d}t \\
&\quad + \|D^s G_{2,n}(u_{\varepsilon,n})\|_{L^2}^2 \mathrm{d}t \\
&= I_1 \mathrm{d}W + \sum_{i=2}^{3} I_i \mathrm{d}t. \qquad (3.14)
\end{aligned}$$

Therefore for any $T > 0$ and $t \in [0, T]$,

$$\|u_{\varepsilon,n}(t)\|_{H^s}^2 - \|u_0\|_{H^s}^2 \le \left|\int_0^t I_1 \mathrm{d}W\right| + \sum_{i=2}^{3}\int_0^t |I_i|\mathrm{d}t'. \tag{3.15}$$

Taking a supremum over $t \in [0, T]$ and using the Burkholder–Davis–Gundy inequality yield that for some $C > 0$,

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}(t)\|_{H^s}^2 \le \mathbb{E}\|u_0\|_{H^s}^2 + C\mathbb{E}\left(\int_0^T |I_1|^2\mathrm{d}t\right)^{\frac{1}{2}} + \sum_{i=2}^{3}\int_0^T \mathbb{E}|I_i|\mathrm{d}t. \tag{3.16}$$

Employing the Cauchy–Schwarz inequality by using (3.8) leads to

$$\begin{aligned}
\mathbb{E}\left(\int_0^T |I_1|^2\mathrm{d}t\right)^{\frac{1}{2}} &\le 2\mathbb{E}\left(\sup_{t\in[0,T]}\|u_{\varepsilon,n}\|_{H^s}^2\int_0^T \|G_{2,n}(u_{\varepsilon,n})\|_{H^s}^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}\|_{H^s}^2 + CL^2\int_0^T \left(1 + \mathbb{E}\|u_{\varepsilon,n}\|_{H^s}^2\right)\mathrm{d}t, \qquad (3.17)
\end{aligned}$$

$$\mathbb{E}|I_2| \le 2\mathbb{E}\left\{\|G_{1,\varepsilon,n}(u_{\varepsilon,n})\|_{H^s}\|u_{\varepsilon,n}\|_{H^s}\right\} \le 2L(1 + \mathbb{E}\|u_{\varepsilon,n}\|_{H^s}^2), \tag{3.18}$$

and

$$\mathbb{E}|I_3| \le 2\mathbb{E}\|G_{2,n}(u_{\varepsilon,n})\|_{H^s}^2 \le 4L^2(1 + \mathbb{E}\|u_{\varepsilon,n}\|_{H^s}^2). \tag{3.19}$$

Combining (3.17)–(3.19), we may identify that for some $C = C(L(R, \varepsilon)) > 0$, $u_{\varepsilon,n}$ satisfies

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}(t)\|_{H^s}^2 \le 2\mathbb{E}\|u_0\|_{H^s}^2 + C\int_0^T \left(1 + \mathbb{E}\sup_{t'\in[0,t]}\|u_{\varepsilon,n}(t')\|_{H^s}^2\right)\mathrm{d}t,$$

which means

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}(t)\|_{H^s}^2 < C(L(R, \varepsilon), T, u_0). \tag{3.20}$$

For $r > 4$, since $\mathrm{d}\|u_{\varepsilon,n}\|^r_{H^s} = \mathrm{d}(\|u_{\varepsilon,n}\|^2_{H^s})^{\frac{r}{2}}$, we have

$$\mathrm{d}\|u_{\varepsilon,n}\|^r_{H^s} = \frac{r}{2}\|u_{\varepsilon,n}\|^{r-2}_{H^s}\left(I_1\mathrm{d}W + \sum_{i=2}^{3} I_i\mathrm{d}t\right) + \frac{r(r-2)}{8}\|u_{\varepsilon,n}\|^{r-4}_{H^s} I_1^2\mathrm{d}t, \tag{3.21}$$

which yields that for any $T > 0$ and $t \in [0, T]$,

$$\begin{aligned}\|u_{\varepsilon,n}(t)\|^r_{H^s} - \|u_{\varepsilon,n}(0)\|^r_{H^s} \le{}& C\int_0^t \|u_{\varepsilon,n}\|^{r-2}_{H^s}|I_1|\mathrm{d}W + \sum_{i=2}^{3} C\int_0^t \|u_{\varepsilon,n}\|^{r-2}_{H^s}|I_i|\mathrm{d}t' \\ &+ C\int_0^t \|u_{\varepsilon,n}\|^{r-4}_{H^s} I_1^2\mathrm{d}t', \quad C = C(r).\end{aligned} \tag{3.22}$$

Similarly, we use the Burkholder–Davis–Gundy inequality after taking a supremum over $t \in [0, T]$ to find that

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}(t)\|^r_{H^s} \le{}& \mathbb{E}\|u_0\|^r_{H^s} + C\mathbb{E}\left(\int_0^T \|u_{\varepsilon,n}\|^{2r-4}_{H^s}|I_1|^2\mathrm{d}t\right)^{\frac{1}{2}} \\ &+ C\sum_{i=2}^{4}\int_0^T \mathbb{E}(\|u_{\varepsilon,n}\|^{r-2}_{H^s}|I_i|)\mathrm{d}t + C\int_0^T \mathbb{E}(\|u_{\varepsilon,n}\|^{r-4}_{H^s}|I_1|^2)\mathrm{d}t, \quad C = C(r).\end{aligned} \tag{3.23}$$

Similar to the estimate as in (3.17), we have

$$\begin{aligned}\mathbb{E}\left(\int_0^T \|u_{\varepsilon,n}\|^{2r-4}_{H^s}|I_1|^2\mathrm{d}t\right)^{\frac{1}{2}} &\le CL^2\mathbb{E}\left(\sup_{t\in[0,T]}\|u_{\varepsilon,n}\|^r_{H^s}\int_0^T (1 + \|u_{\varepsilon,n}\|^r_{H^s})\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}\|^r_{H^s} + CL^2\int_0^T (1 + \mathbb{E}\|u_{\varepsilon,n}\|^r_{H^s})\,\mathrm{d}t.\end{aligned} \tag{3.24}$$

By a similar argument, we know that for some $C = C(r, L(R, \varepsilon)) > 0$,

$$\mathbb{E}\|u_{\varepsilon,n}\|^{r-2}_{H^s}|I_2|,\ \mathbb{E}\|u_{\varepsilon,n}\|^{r-2}_{H^s}|I_3|,\ \mathbb{E}\|u_{\varepsilon,n}\|^{r-4}_{H^s}|I_1|^2 \le C\left(1 + \mathbb{E}\|u_{\varepsilon,n}\|^r_{H^s}\right). \tag{3.25}$$

Therefore we identify that

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}(t)\|^r_{H^s} \le 2\mathbb{E}\|u_0\|^r_{H^s} + C\int_0^T\left(1 + \mathbb{E}\sup_{t'\in[0,t]}\|u_{\varepsilon,n}(t')\|^r_{H^s}\right)\mathrm{d}t,$$

and therefore for any $T > 0$,

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}(t)\|^r_{H^s} \le C(L(R,\varepsilon), T, u_0, r), \tag{3.26}$$

which means that $\{u_{\varepsilon,n}\} \subset L^r(\Omega; C([0,T]; H^s))$ is uniformly bounded.

Now we prove that for $0 < \alpha < \frac{r-4}{2r}$, $\{u_{\varepsilon,n}\} \subset L^r(\Omega; C^\alpha([0,T]; H^{s-1}))$ is also bounded uniformly in n. For any $[t', t] \subset [0, T]$, we first notice that from (3.7),

$$\|u_{\varepsilon,n}(t) - u_{\varepsilon,n}(t')\|_{H^{s-1}} \le \left\|\int_{t'}^t G_{1,\varepsilon,n}(u_{\varepsilon,n})\mathrm{d}\tau\right\|_{H^{s-1}} + \left\|\int_{t'}^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{H^{s-1}}. \tag{3.27}$$

Actually, on account of (3.8) and (3.26), we have that for any $[t',t]\subset[0,T]$,

$$\begin{aligned}\mathbb{E}\left(\left\|\int_{t'}^{t} G_{1,\varepsilon,n}(u_{\varepsilon,n})\mathrm{d}\tau\right\|_{H^{s-1}}\right)^{r} &\le |t-t'|^{r}\mathbb{E}\sup_{\tau\in[0,T]}\|G_{1,\varepsilon,n}(u_{\varepsilon,n})\|_{H^{s-1}}^{r}\\ &\le C|t-t'|^{r}(1+\mathbb{E}\sup_{\tau\in[0,T]}\|u_{\varepsilon,n}(\tau)\|_{H^{s}}^{r})\\ &\le C(L(R,\varepsilon),T,u_0,r)|t-t'|^{r}. \qquad (3.28)\end{aligned}$$

Let $\delta\in(\frac{1}{r},\frac{r-2}{2r})$. For a.e. $\omega\in\Omega$ and for any $\eta>0$, there exists a subinterval $[t_0',t_0]\subset[0,T]$ such that

$$\left(\sup_{t\neq t'}\frac{\left\|\int_{t'}^{t}G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{H^{s-1}}}{|t-t'|^{\delta}}\right)^{r}\le\left(\frac{\left\|\int_{t_0'}^{t_0}G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{H^{s-1}}}{|t_0-t_0'|^{\delta}}\right)^{r}+\eta.$$

It follows from the Burkholder–Davis–Gundy inequality, the Jensen inequality, and (3.8) that

$$\begin{aligned}\mathbb{E}\left(\sup_{t\neq t'}\frac{\left\|\int_{t'}^{t}G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{H^{s-1}}}{|t_0-t_0'|^{\delta}}\right)^{r} &\le\mathbb{E}\frac{\left(\int_{t_0'}^{t_0}\|G_{2,n}(u_{\varepsilon,n})\|_{H^{s-1}}^{2}\mathrm{d}\tau\right)^{\frac{r}{2}}}{|t_0-t_0'|^{\delta r}}+\eta\\ &\le\mathbb{E}\frac{|t_0-t_0'|^{\frac{r}{2}-1}\sup_{\tau\in[0,T]}\|G_{2,n}(u_{\varepsilon,n})\|_{H^{s-1}}^{r}}{|t_0-t_0'|^{\delta r}}+\eta\\ &\le CT^{\left(\frac{1}{2}-\delta\right)r-1}(1+\mathbb{E}\sup_{\tau\in[0,T]}\|u_{\varepsilon,n}(\tau)\|_{H^{s}}^{r})+\eta.\end{aligned}$$

Since $\left(\frac{1}{2}-\delta\right)r-1>0$ and $\eta>0$ is arbitrary, from (3.26), we have that for any $[t',t]\subset[0,T]$,

$$\mathbb{E}\left(\left\|\int_{t'}^{t}G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{H^{s-1}}^{r}\right)\le C(L(R,\varepsilon),T,u_0,r,\delta)|t-t'|^{r\delta}. \qquad (3.29)$$

Combining (3.28) and (3.29) into (3.27), we have

$$\mathbb{E}\|u_{\varepsilon,n}(t)-u_{\varepsilon,n}(t')\|_{H^{s-1}}^{r}\le 2^{r}C(L(R,\varepsilon),T,u_0,r,\delta)|t-t'|^{r\delta}.$$

Then the Kolmogorov's continuity theorem yields that for $\alpha\in(0,\frac{1}{2}-\frac{2}{r})$, $u_{\varepsilon,n}$ has a $C^{\alpha}\left([0,T];H^{s-1}\right)$ path almost surely and

$$\mathbb{E}\sup_{t\in[0,T]}\|u_{\varepsilon,n}(t')\|_{C^{\alpha}([0,T];H^{s-1})}^{r}\le C(L(R,\varepsilon),T,u_0,r,\alpha). \qquad (3.30)$$

Besides, from (3.28) and (3.29), we obtain (3.11) and (3.12). ∎

PROPOSITION 3.4. *Let $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be a stochastic basis fixed in advance. Let $s>3$, $r>4$, $R>1$, and $\varepsilon\in(0,1)$. Assume that $\sigma(\cdot)$ satisfies the assumption (1.17). Let $u_0\in L^r(\Omega;H^s)$ be an H^s-valued $\mathcal{F}_0$ measurable random variable. If for some $T>0$, u_ε solves (3.2) on $[0,T]$ and $u_\varepsilon\in C([0,T],H^{s'})$, $\mathbb{P}$-a.s. for some $s'\le s$, then for $0<\alpha<\frac{r-4}{2r}$,*

$$u_\varepsilon\in L^r\left(\Omega;L^{\infty}\left(0,T;H^s\right)\bigcap C^{\alpha}\left([0,T];H^{s-1}\right)\right)$$

is bounded uniformly in ε. *Besides, for some* $C_2 = C_2(R, T, u_0, r, \alpha) > 0$,

$$\sup_{0<\varepsilon<1} \left\| \int_0^t G_{1,\varepsilon}(u_\varepsilon)\mathrm{d}\tau \right\|_{L^r(\Omega; Lip([0,T]; H^{s-1}))} < C_2, \tag{3.31}$$

$$\sup_{0<\varepsilon<1} \left\| \int_0^t G_2(u_\varepsilon)\mathrm{d}W \right\|_{L^r(\Omega; C^\alpha([0,T]; H^{s-1}))} < C_2, \tag{3.32}$$

where $G_{1,\varepsilon}(\cdot)$ *and* $G_2(\cdot)$ *are defined in* (3.2).

Proof. Apply D^s with $s > 3$ to (3.2) to find

$$\begin{aligned} &\mathrm{d}D^s u_\varepsilon + \theta_R(\|u_\varepsilon\|_{W^{1,\infty}}) D^s J_\varepsilon \left[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon\right] \mathrm{d}t \\ &\qquad + \theta_R(\|u_\varepsilon\|_{W^{1,\infty}}) D^s F(u_\varepsilon)\mathrm{d}t = \theta_R(\|u_\varepsilon\|_{W^{1,\infty}}) D^s \sigma(u_\varepsilon)\mathrm{d}W. \end{aligned} \tag{3.33}$$

Repeated use of the Itô formula enables us to see that for $D^s u_\varepsilon$,

$$\begin{aligned} \mathrm{d}\|u_\varepsilon\|_{H^s}^2 &= 2\theta_R(\|u_\varepsilon\|_{W^{1,\infty}}) \left(D^s\sigma(u_\varepsilon), D^s u_\varepsilon\right)_{L^2} \mathrm{d}W \\ &\quad - 2\theta_R(\|u_\varepsilon\|_{W^{1,\infty}}) \left(D^s J_\varepsilon \left[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon\right], D^s u_\varepsilon\right)_{L^2} \mathrm{d}t \\ &\quad - 2\theta_R(\|u_\varepsilon\|_{W^{1,\infty}}) \left(D^s F(u_\varepsilon), D^s u_\varepsilon\right)_{L^2} \mathrm{d}t \\ &\quad + \theta_R^2(\|u_\varepsilon\|_{W^{1,\infty}}) \|D^s\sigma(u_\varepsilon)\|_{L^2}^2 \mathrm{d}t \\ &= J_1 \mathrm{d}W + \sum_{i=2}^4 J_i \mathrm{d}t. \end{aligned} \tag{3.34}$$

For $t \in [0, T]$, we have

$$\|u_\varepsilon(t)\|_{H^s}^2 - \|u_0\|_{H^s}^2 \le \left| \int_0^t J_1 \mathrm{d}W \right| + \sum_{i=2}^4 \int_0^t |J_i| \mathrm{d}t'. \tag{3.35}$$

Taking a supremum for $t \in [0, T]$ and using the Burkholder–Davis–Gundy inequality yield

$$\mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s}^2 \le \mathbb{E}\|u_0\|_{H^s}^2 + C\mathbb{E}\left(\int_0^T |J_1|^2 \mathrm{d}t\right)^{\frac{1}{2}} + \sum_{i=2}^4 \int_0^T \mathbb{E}|J_i| \mathrm{d}t, \tag{3.36}$$

and in the above equation,

$$\begin{aligned} \mathbb{E}\left(\int_0^T |J_1|^2 \mathrm{d}t\right)^{\frac{1}{2}} &\le 2\mathbb{E}\left(\int_0^T \|u_\varepsilon\|_{H^s}^2 \theta_R^2(\|u_\varepsilon\|_{W^{1,\infty}}) \|\sigma(u_\varepsilon)\|_{H^s}^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ &\le 2\mathbb{E}\left(\sup_{t\in[0,T]} \|u_\varepsilon\|_{H^s}^2 \int_0^T \theta_R^2(\|u_\varepsilon\|_{W^{1,\infty}}) \|\sigma(u_\varepsilon)\|_{H^s}^2 \mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \frac{1}{2}\mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon\|_{H^s}^2 + Cf^2(2R) \int_0^T \left(1 + \mathbb{E}\|u_\varepsilon\|_{H^s}^2\right) \mathrm{d}t. \end{aligned} \tag{3.37}$$

By first commuting J_ε and then commuting the operator D^s with $J_\varepsilon u_\varepsilon$, then applying the Cauchy–Schwarz inequality and Lemma 2.1, we see that

$$\begin{aligned}|J_2| &= \left|2\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})\int_{\mathbb{T}} D^s\left[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon\right]\cdot D^s J_\varepsilon u_\varepsilon \ \mathrm{d}x\right|\\ &\le 2\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})\int_{\mathbb{T}} \left|\left[D^s, J_\varepsilon u_\varepsilon\right]\partial_x J_\varepsilon u_\varepsilon \cdot D^s J_\varepsilon u_\varepsilon\right| \ \mathrm{d}x\\ &\quad + 2\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})\int_{\mathbb{T}} \left|J_\varepsilon u_\varepsilon \partial_x D^s J_\varepsilon u_\varepsilon \cdot D^s J_\varepsilon u_\varepsilon\right| \ \mathrm{d}x \le 4R\|u_\varepsilon\|_{H^s}^2.\end{aligned}$$

Therefore we find that for $t\in[0,T]$,

$$\mathbb{E}\int_0^t |J_2|\mathrm{d}t' \le 4R\int_0^t \mathbb{E}\|u_\varepsilon\|_{H^s}^2\mathrm{d}t'. \tag{3.38}$$

For J_3, we simply use the Cauchy–Schwarz inequality and Lemma 2.3 to deduce that

$$\mathbb{E}\int_0^t |J_3|\mathrm{d}t' \le 2\mathbb{E}\int_0^t \theta_R(\|u_\varepsilon\|_{W^{1,\infty}})\|u_\varepsilon\|_{W^{1,\infty}}\|u_\varepsilon\|_{H^s}^2\mathrm{d}t' \le 4R\int_0^t \mathbb{E}\|u_\varepsilon\|_{H^s}^2\mathrm{d}t'. \tag{3.39}$$

It follows from the assumptions of $\sigma(\cdot)$ that

$$\begin{aligned}\mathbb{E}\int_0^t |J_4|\mathrm{d}t' &\le 2\mathbb{E}\int_0^t \theta_R(\|u_\varepsilon\|_{W^{1,\infty}})f^2(\|u_\varepsilon\|_{W^{1,\infty}})\left(1+\|u_\varepsilon\|_{H^s}^2\right)\mathrm{d}t'\\ &\le 2f^2(2R)\int_0^t \left(1+\mathbb{E}\|u_\varepsilon\|_{H^s}^2\right)\mathrm{d}t'.\end{aligned} \tag{3.40}$$

Combining (3.37)–(3.40), we see that u_ε satisfies

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2 \le 2\mathbb{E}\|u_0\|_{H^s}^2 + C_R\int_0^T\left(1+\mathbb{E}\sup_{t'\in[0,t]}\|u_\varepsilon(t')\|_{H^s}^2\right)\mathrm{d}t,$$

which implies that for any $t\in[0,T]$,

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2 \le C(R,T,u_0). \tag{3.41}$$

Therefore $\{u_\varepsilon\}\subset L^2\left(\Omega;L^\infty\left(0,T;H^s\right)\right)$ is bounded uniformly in ε.

Given $r>4$, following the steps as in Proposition 3.3, we find that for some $C=C(r)>0$,

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^r &\le \mathbb{E}\|u_0\|_{H^s}^r + C\mathbb{E}\left(\int_0^T \|u_\varepsilon\|_{H^s}^{2r-4}|J_1|^2\mathrm{d}t\right)^{\frac12}\\ &\quad + C\sum_{i=2}^4\int_0^T \mathbb{E}(\|u_\varepsilon\|_{H^s}^{r-2}|J_i|)\mathrm{d}t + \int_0^T \mathbb{E}(\|u_\varepsilon\|_{H^s}^{r-4}|J_1|^2)\mathrm{d}t.\end{aligned} \tag{3.42}$$

Similarly,

$$\begin{aligned}&\mathbb{E}\left(\int_0^T \|u_\varepsilon\|_{H^s}^{2r-4}|J_1|^2\mathrm{d}t\right)^{\frac12}\\ &\quad\le C\mathbb{E}\left(\int_0^T \|u_\varepsilon\|_{H^s}^r\theta_R^2(\|u_\varepsilon\|_{W^{1,\infty}})f^2(\|u_\varepsilon\|_{W^{1,\infty}})\left(1+\|u_\varepsilon\|_{H^s}^r\right)\mathrm{d}t\right)^{\frac12}\end{aligned}$$

$$\leq C\mathbb{E}\left(\sup_{t\in[0,T]}\|u_\varepsilon\|_{H^s}^r\int_0^T\theta_R^2(\|u_\varepsilon\|_{W^{1,\infty}})f^2(\|u_\varepsilon\|_{W^{1,\infty}})\left(1+\|u_\varepsilon\|_{H^s}^r\right)\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon\|_{H^s}^r + Cf^2(2R)\int_0^T\left(1+\mathbb{E}\|u_\varepsilon\|_{H^s}^r\right)\mathrm{d}t. \tag{3.43}$$

Using estimates analogous to those in (3.38)–(3.40), we have

$$\sum_{i=2}^4\mathbb{E}\|u_\varepsilon\|_{H^s}^{r-2}|J_i|,\ \mathbb{E}\|u_\varepsilon\|_{H^s}^{r-4}|J_1|^2 \leq C_{r,R}\left(1+\mathbb{E}\|u_\varepsilon\|_{H^s}^r\right). \tag{3.44}$$

Combining the above estimates, we identify that

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^r \leq 2\mathbb{E}\|u_0\|_{H^s}^r + C_{r,R}\int_0^T\left(1+\mathbb{E}\sup_{t'\in[0,t]}\|u_\varepsilon(t')\|_{H^s}^r\right)\mathrm{d}t,$$

and therefore for any $T>0$,

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^r \leq C(R,T,u_0,r), \tag{3.45}$$

which means that $\{u_\varepsilon\}\subset L^r\left(\Omega;L^\infty\left(0,T;H^s\right)\right)$ is bounded uniformly in ε.

The proof for the uniform bound of $\{u_\varepsilon\}\subset L^r\left(\Omega;C^\alpha\left([0,T];H^{s-1}\right)\right)$ with $0<\alpha<\frac{r-4}{2r}$ is similar to the one in Proposition 3.3. We just notice that under the condition (3.45), the bounds

$$\mathbb{E}\sup_{\tau\in[0,T]}\|G_{1,\varepsilon}(u_\varepsilon)\|_{H^{s-1}}^r \leq CR\mathbb{E}\sup_{\tau\in[0,T]}\|u_\varepsilon(\tau)\|_{H^s}^r$$

and

$$\mathbb{E}\sup_{\tau\in[0,T]}\|G_2(u_\varepsilon)\|_{H^{s-1}}^r \leq C_{r,R}\left(1+\mathbb{E}\sup_{\tau\in[0,T]}\|u_\varepsilon(\tau)\|_{H^s}^r\right)$$

are also uniformly in ε. ∎

3.3. Approximation solutions. Let $\mu_0\in\mathcal{P}r(H^s)$ satisfy (1.19). Choose a stochastic basis $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0},W)$ such that u_0 is an $\mathcal{F}_0$ measurable random variable with the distribution μ_0 on H^s. For any $R>1$, $\varepsilon\in(0,1)$, and $n\geq 1$, (3.8) and (3.9) imply that the system (3.7) may be viewed as an SDE in H^s, with globally Lipschitz drift and diffusion. According to the existence theory of SDE in Hilbert space (see [28] or [39]), (3.7) admits a unique global in time solution $u_{\varepsilon,n}\in H^s$, which is continuous in time, that is, $u_{\varepsilon,n}\in C([0,\infty),H^s)$, $\mathbb{P}$-a.s.

3.4. Remove the Lipschitz approximation: The first limit. Now we take the first limit to prove that (3.2) has a global martingale solution. The precise statement is in Proposition 3.7 below.

Let $R>1$ and $0<\varepsilon<1$. Since u_0 is an $\mathcal{F}_0$ measurable random variable with the distribution μ_0 on H^s, we can infer from Proposition 3.3 that the approximate solution $\{u_{\varepsilon,n}\}$ is uniformly bounded in $L^r\left(\Omega;C\left([0,T];H^s\right)\bigcap C^\alpha\left([0,T];H^{s-1}\right)\right)$ for any $T>0$. Then we can define the sequence of probability measures $\mu_{\varepsilon,n}$ on X^s, where

$$X^s = X_u^s\times X_W,\quad X_u^s = C\left([0,T];H^s\right),\quad X_W = C\left([0,T];\mathbb{R}\right) \tag{3.46}$$

and

$$\mu_{\varepsilon,n}=\mu_{u_{\varepsilon,n}}\times\mu_W,\quad \mu_{u_{\varepsilon,n}}(\cdot)\triangleq\mathbb{P}(u_{\varepsilon,n}\in\cdot),\ \ \mu_W(\cdot)\triangleq\mathbb{P}(W\in\cdot). \tag{3.47}$$

For each fixed $0<\varepsilon<1$, we now prove $\{\mu_{\varepsilon,n}\}_{n\geq 1}$ is tight in $\mathcal{P}r(X^{s-1})$, where X^{s-1} is defined by (3.46).

Lemma 3.5. *Let $s>3$ and $0<\varepsilon<1$. The sequence of measures $\{\mu_{\varepsilon,n}\}_{n\geq 1}$ defined by* (3.47) *is tight in $\mathcal{P}r(X^{s-1})$, where X^{s-1} is defined by* (3.46).

Proof. We first notice that for any $r>4$ and $T>0$, when $t\in[0,T]$, $u_{\varepsilon,n}(t)$ satisfies

$$u_{\varepsilon,n}(t)-u_0+\int_0^t G_{1,\varepsilon,n}(u_{\varepsilon,n})\mathrm{d}t'=\int_0^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W,\quad u_0\in L^r(\Omega;H^s).$$

Via the proof for (3.11), we have actually obtained that for $0<\alpha<\frac{r-4}{2r}$,

$$\sup_{n\geq 1}\left\|u_{\varepsilon,n}(t)-\int_0^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{L^r(\Omega;C^\alpha([0,T];H^{s-1}))}\leq C(R,T,u_0,r,\varepsilon,\alpha). \tag{3.48}$$

For any $M>0$, let B_M^1 be the ball with radius M in $C\left([0,T];H^s\right)$ and B_M^2 be the ball with radius M in $C^\alpha\left([0,T];H^{s-1}\right)$. It follows from the Ascoli's theorem in a Banach space (see [20]) that, for $x\in\mathbb{T}$,

$$C\left([0,T];H^s\right)\bigcap C^\alpha\left([0,T];H^{s-1}\right)\hookrightarrow\hookrightarrow C\left([0,T];H^{s-1}\right), \tag{3.49}$$

and hence $B_M=B_M^1\bigcap B_M^2$ is precompact in X_u^{s-1}. Let $A=A_1\bigcap A_2\bigcap A_3\subset B_M$, where

$$\begin{aligned}
A_1&=\left\{u_{\varepsilon,n}:\|u_{\varepsilon,n}\|_{C([0,T];H^s)}<\frac{M}{3}\right\},\\
A_2&=\left\{u_{\varepsilon,n}:\left\|u_{\varepsilon,n}-\int_0^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{C^\alpha([0,T];H^{s-1})}<\frac{M}{3}\right\},\\
A_3&=\left\{u_{\varepsilon,n}:\left\|\int_0^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{C^\alpha([0,T];H^{s-1})}<\frac{M}{3}\right\}.
\end{aligned}$$

For any $\eta>0$, from the Chebyshev inequality, (3.26), (3.48), and (3.12), we may identify that for $M=\frac{9C}{\eta}$ with some C large enough,

$$\begin{aligned}
\mu_{u_{\varepsilon,n}}\left(\left(\overline{B_M}^{X_u^{s-1}}\right)^C\right)\leq{}&\mathbb{P}\left\{u_{\varepsilon,n}:\|u_{\varepsilon,n}\|_{C([0,T];H^s)}\geq\frac{3C}{\eta}\right\}\\
&+\mathbb{P}\left\{u_{\varepsilon,n}:\left\|u_{\varepsilon,n}-\int_0^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{C^\alpha([0,T];H^{s-1})}\geq\frac{3C}{\eta}\right\}\\
&+\mathbb{P}\left\{u_{\varepsilon,n}:\left\|\int_0^t G_{2,n}(u_{\varepsilon,n})\mathrm{d}W\right\|_{C^\alpha([0,T];H^{s-1})}\geq\frac{3C}{\eta}\right\}\leq\eta.
\end{aligned}$$

This means that for each fixed $0<\varepsilon<1$, $\mu_{u_{\varepsilon,n}}$ is tight on X_S. On the other hand, μ_W stays unchanged and hence it is trivially tight. We finish the proof. ∎

LEMMA 3.6. *Let $s>3$. For each $\varepsilon\in(0,1)$, we have the following properties:*

1. *There is a subsequence of $\{\mu_{\varepsilon,n}\}_{n\in\mathbb{N}}$, still denoted by $\mu_{\varepsilon,n}$, such that $\mu_{\varepsilon,n}\xrightarrow[n\to\infty]{}\mu_\varepsilon$ weakly for some μ_ε on X^{s-1}.*
2. *There exists a probability space $(\widetilde{\Omega_\varepsilon},\widetilde{\mathcal{F}_\varepsilon},\widetilde{\mathbb{P}_\varepsilon})$ on which there is a sequence of random variables $(\widetilde{u_{\varepsilon,n}},\widetilde{W_{\varepsilon,n}})$ converging almost surely in the topology of X^{s-1} to a random variable $(\widetilde{u_\varepsilon},\widetilde{W_\varepsilon})$. Moreover, $(\widetilde{u_{\varepsilon,n}},\widetilde{W_{\varepsilon,n}})$ and $(\widetilde{u_\varepsilon},\widetilde{W_\varepsilon})$ have the probability laws $\mu_{\varepsilon,n}$ and μ_ε, respectively.*
3. *$\widetilde{W_{\varepsilon,n}}$ is a Brownian motion, relative to $\widetilde{\mathcal{F}_t^{\varepsilon,n}}=\sigma\{\widetilde{u_{\varepsilon,n}}(\tau),\widetilde{W_{\varepsilon,n}}(\tau)\}_{\tau\in[0,t]}$ and $\widetilde{u_{\varepsilon,n}}$ satisfies* (3.7) *relative to the basis $\widetilde{\mathcal{S}_{\varepsilon,n}}=(\widetilde{\Omega_\varepsilon},\widetilde{\mathcal{F}_\varepsilon},\widetilde{\mathbb{P}_\varepsilon},\{\widetilde{\mathcal{F}_t^{\varepsilon,n}}\}_{t\ge0},\widetilde{W_{\varepsilon,n}})$.*
4. *Let $\widetilde{\mathcal{F}_t^\varepsilon}=\sigma\{\widetilde{u_\varepsilon}(\tau),\widetilde{W_\varepsilon}(\tau)\}_{\tau\in[0,t]}$; then $\widetilde{W_\varepsilon}$ is a Brownian motion relative to $\widetilde{\mathcal{F}_t^\varepsilon}$.*
5. *When $n\to\infty$,*

$$\begin{cases} G_{1,\varepsilon,n}(\widetilde{u_{\varepsilon,n}})\to G_{1,\varepsilon}(\widetilde{u_\varepsilon}) \ \text{ in } C\left([0,T];H^{s-3/2}\right),\ \widetilde{\mathbb{P}_\varepsilon}\text{-}a.s., \\ G_{2,n}(\widetilde{u_{\varepsilon,n}})\to G_2(\widetilde{u_\varepsilon}) \ \text{ in } C\left([0,T];H^{s-3/2}\right),\ \widetilde{\mathbb{P}_\varepsilon}\text{-}a.s. \end{cases} \tag{3.50}$$

Proof. Properties 1 and 2 come from Lemmas 2.4 and 2.5. For $t>0$, in a manner similar to previous works [1, 45], we can deduce 3 and 4. Now we prove (3.50). We notice that if $\mathbb{E}$ is replaced by $\widetilde{\mathbb{E}_\varepsilon}$, then the previous proof means that $\widetilde{u_{\varepsilon,n}}$ satisfies (3.20) and (3.30). Hence by Fatou's lemma, (3.20) and (3.30) also hold true for $\widetilde{u_\varepsilon}$ (corresponding to $\widetilde{\mathbb{E}_\varepsilon}$). Consequently, employing the Ascoli's theorem again yields that for a.e. $\omega\in\widetilde{\Omega_\varepsilon}$, $\overline{\{\widetilde{u_{\varepsilon,n}}\}}$ is a compact subset of $C([0,T];H^{s-3/2})$ and therefore it follows from (3.10) that for any $k\in\mathbb{N}^+$,

$$\begin{aligned} &G_{1,\varepsilon,n}(\widetilde{u_{\varepsilon,k}})\xrightarrow[n\to\infty]{} G_{1,\varepsilon}(\widetilde{u_{\varepsilon,k}}) \text{ and } G_{2,n}(\widetilde{u_{\varepsilon,k}})\xrightarrow[n\to\infty]{} G_2(\widetilde{u_{\varepsilon,k}}) \\ &\quad \text{in } C\left([0,T];H^{s-3/2}\right),\ \widetilde{\mathbb{P}_\varepsilon}\text{-a.s.} \end{aligned} \tag{3.51}$$

By using the diagonal argument, $(3.50)_1$, (3.51), and (3.10), we can finally obtain a subsequence, which is still labeled as n, such that

$$\begin{cases} G_{1,\varepsilon,n}(\widetilde{u_{\varepsilon,n}})\xrightarrow[n\to\infty]{} G_{1,\varepsilon}(\widetilde{u_\varepsilon}) \ \text{ in } C\left([0,T];H^{s-3/2}\right),\ \widetilde{\mathbb{P}_\varepsilon}\text{-a.s}, \\ G_{2,n}(\widetilde{u_{\varepsilon,n}})\xrightarrow[n\to\infty]{} G_2(\widetilde{u_\varepsilon}) \ \text{ in } C\left([0,T];H^{s-3/2}\right),\ \widetilde{\mathbb{P}_\varepsilon}\text{-a.s}, \end{cases} \tag{3.52}$$

which is (3.50). □

PROPOSITION 3.7. *Let $s>3$, $r>4$. Fix $\varepsilon\in(0,1)$. Assume that* (1.16), (1.17), *and* (1.19) *hold true. For any $R>1$,* (3.2) *has a global martingale solution in the sense of Definition* 1.3.

Proof.

Step 1 (existence). Now we prove that there is a process such that (1.21) holds true. For any fixed $\varepsilon\in(0,1)$, for all $n\ge1$ and $T>0$, define

$$\widetilde{M_{\varepsilon,n}}(t)=\widetilde{u_{\varepsilon,n}}(t)-\widetilde{u_{\varepsilon,n}}(0)+\int_0^t G_{1,\varepsilon,n}(\widetilde{u_{\varepsilon,n}})\mathrm{d}\tau,\quad t\in[0,T]. \tag{3.53}$$

It follows from 2 and 3 in Lemma 3.6 that $\widetilde{M_{\varepsilon,n}}(t)$ is an $H^{s-3/2}$-valued square integrable martingale under $\widetilde{\mathbb{P}_\varepsilon}$. Therefore for $\delta=s-3/2$, for all $\phi\in C^\infty(\mathbb{T})$

and $0 < t' < t < T$, for any bounded continuous $\widetilde{\mathcal{F}^{\varepsilon,n}_{t'}}$ measurable function φ on $C([0,t'],H^\delta)$, we have

$$\widetilde{\mathbb{E}_\varepsilon}\left[\left(\widetilde{M_{\varepsilon,n}}(t)-\widetilde{M_{\varepsilon,n}}(t'),\phi\right)_{H^\delta}\cdot\varphi\left(\widetilde{u_{\varepsilon,n}},\widetilde{W_{\varepsilon,n}}\right)\right]=0 \tag{3.54}$$

and

$$\widetilde{\mathbb{E}_\varepsilon}\left[\left(\left(\widetilde{M_{\varepsilon,n}}(t),\phi\right)^2_{H^\delta}-\left(\widetilde{M_{\varepsilon,n}}(t'),\phi\right)^2_{H^\delta}-\int_{t'}^t\left(G_{2,n}\left(\widetilde{u_{\varepsilon,n}}\right),\phi\right)_{H^\delta}\mathrm{d}\tau\right)\cdot\varphi\left(\widetilde{u_{\varepsilon,n}},\widetilde{W_{\varepsilon,n}}\right)\right]=0. \tag{3.55}$$

As $(\widetilde{u_{\varepsilon,n}},\widetilde{W_{\varepsilon,n}})$ satisfies (3.7) relative to $\widetilde{\mathcal{S}_n}$, we have that

$$\widetilde{M_{\varepsilon,n}}(t)=\int_0^t G_{2,n}(\widetilde{u_{\varepsilon,n}})\mathrm{d}\widetilde{W_{\varepsilon,n}},$$

from which we can use the Itô product rule to deduce that

$$\mathrm{d}\left(\widetilde{M_{\varepsilon,n}}\widetilde{W_{\varepsilon,n}}\right)=G_{2,n}(\widetilde{u_{\varepsilon,n}})\mathrm{d}t+\left(\widetilde{W_{\varepsilon,n}}G_{2,n}(\widetilde{u_{\varepsilon,n}})+\widetilde{M_{\varepsilon,n}}\right)\mathrm{d}\widetilde{W_{\varepsilon,n}},$$

and therefore

$$\begin{aligned}\widetilde{\mathbb{E}_\varepsilon}\Bigg[\Bigg(&\widetilde{W_{\varepsilon,n}}\left(\widetilde{M_{\varepsilon,n}},\phi\right)_{H^\delta}(t)-\widetilde{W_{\varepsilon,n}}\left(\widetilde{M_{\varepsilon,n}},\phi\right)_{H^\delta}(t')\\&-\int_{t'}^t\left(G_{2,n}\left(\widetilde{u_{\varepsilon,n}}\right),\phi\right)_{H^\delta}\mathrm{d}\tau\Bigg)\cdot\varphi\left(\widetilde{u_{\varepsilon,n}},\widetilde{W_{\varepsilon,n}}\right)\Bigg]=0.\end{aligned} \tag{3.56}$$

We also define

$$\widetilde{M_\varepsilon}(t)=\widetilde{u_\varepsilon}(t)-\widetilde{u_\varepsilon}(0)+\int_0^t G_{1,\varepsilon}(\widetilde{u_\varepsilon})\mathrm{d}\tau.$$

It follows from Lemma 3.6 that $\widetilde{M_{\varepsilon,n}}(t)$ converges to $\widetilde{M_\varepsilon}(t)$ in $C\left([0,T];H^\delta\right)$, $\widetilde{\mathbb{P}_\varepsilon}$-a.s. We notice that $\widetilde{M_\varepsilon}(t)$ is also a square integrable martingale. Actually, we first notice that for any bounded continuous $\widetilde{\mathcal{F}^{\varepsilon,n}_T}$ measurable function Φ on $C([0,T],H^\delta)$

$$\{f_{\varepsilon,n}(t)\}_n=\left\{\left(\widetilde{M_{\varepsilon,n}}(t),\phi\right)_{H^\delta}\cdot\Phi\left(\widetilde{u_{\varepsilon,n}},\widetilde{W_{\varepsilon,n}}\right)\right\}_n$$

is uniformly integrable. Indeed, since Proposition 3.3 also holds true for $\widetilde{u_{\varepsilon,n}}$ when $\mathbb{E}$ is replaced by $\widetilde{\mathbb{E}_\varepsilon}$, we can apply the Burkholder–Davis–Gundy inequality and (3.8) to find that for any $r>4$,

$$\widetilde{\mathbb{E}_\varepsilon}|f_{\varepsilon,n}|^r\le C\widetilde{\mathbb{E}_\varepsilon}\left(\int_0^T\|G_{2,n}(\widetilde{u_{\varepsilon,n}})\|^2_{H^\delta}\mathrm{d}\tau\right)^{\frac{r}{2}}\le C(R,T,u_0,r,\varepsilon).$$

By Lemmas 3.6 and 2.6, we can send $n\to\infty$ in (3.54), (3.55), and (3.56) to identify that

$$\widetilde{\mathbb{E}_\varepsilon}\left[(\widetilde{M_\varepsilon}(t)-\widetilde{M_\varepsilon}(t'),\phi)_{H^\delta}\cdot\varphi(\widetilde{u_\varepsilon},\widetilde{W_\varepsilon})\right]=0,$$

$$\widetilde{\mathbb{E}_\varepsilon}\left[\left(\left(\widetilde{M_\varepsilon}(t),\phi\right)^2_{H^\delta}-\left(\widetilde{M_\varepsilon}(t'),\phi\right)^2_{H^\delta}-\int_{t'}^t\left(G_2\left(\widetilde{u_\varepsilon}\right),\phi\right)_{H^\delta}\mathrm{d}\tau\right)\cdot\varphi\left(\widetilde{u_\varepsilon},\widetilde{W_\varepsilon}\right)\right]=0,$$

and

$$\widetilde{\mathbb{E}_\varepsilon}\left[\left(\widetilde{W_\varepsilon}\left(\widetilde{M_\varepsilon},\phi\right)_{H^s}(t)-\widetilde{W_\varepsilon}\left(\widetilde{M_\varepsilon},\phi\right)_{H^s}(t')-\int_{t'}^{t}\left(G_2\left(\widetilde{u_\varepsilon}\right),\phi\right)_{H^s}\mathrm{d}\tau\right)\cdot\varphi\left(\widetilde{u_\varepsilon},\widetilde{W_\varepsilon}\right)\right]=0.$$

Therefore $\widetilde{M_\varepsilon}(t)$ is a square integrable martingale with quadratic variation process

$$\langle\widetilde{M_\varepsilon}\rangle(t)=\int_0^t|G_2(\widetilde{u_\varepsilon})|^2\mathrm{d}\tau.$$

Applying the modified martingale representation theorem [34, Theorem A.1] to $\widetilde{M_\varepsilon}(t)$, we see that for any $T>0$, $\widetilde{M_\varepsilon}(t)$ can be represented as

$$\widetilde{M_\varepsilon}(t)=\int_0^t G_2(\widetilde{u_\varepsilon})\mathrm{d}\widetilde{W_\varepsilon},\ t\in[0,T],$$

which means that $\widetilde{u_\varepsilon}$ and $\widetilde{\mathcal{S}_\varepsilon}=(\widetilde{\Omega_\varepsilon},\widetilde{\mathcal{F}_\varepsilon},\widetilde{\mathbb{P}_\varepsilon},\{\widetilde{\mathcal{F}_t^\varepsilon}\}_{t\geq0},\widetilde{W_\varepsilon})$ satisfy

$$\widetilde{u_\varepsilon}(t)-\widetilde{u_\varepsilon}(0)+\int_0^t G_{1,\varepsilon}(\widetilde{u_\varepsilon})\mathrm{d}\tau=\int_0^t G_2(\widetilde{u_\varepsilon})\mathrm{d}\widetilde{W_\varepsilon},\quad t\in[0,T]. \tag{3.57}$$

Moreover, by 3 in Lemma 3.6, we see that $\widetilde{u_{\varepsilon,n}}(0)=u_{\varepsilon,n}(0)=u(0)$, $\widetilde{\mathbb{P}_\varepsilon}$-a.s., and therefore $\mu_0(Y)=\widetilde{\mathbb{P}_\varepsilon}\{\widetilde{u_\varepsilon}(0)\in Y\}$ for all $Y\in\mathcal{B}(H^s)$. Since $T>0$ is arbitrary, $\widetilde{u_\varepsilon}$ actually exists globally.

Step 2 (regularity). Now we prove (1.20) holds true for $\widetilde{u_\varepsilon}$. To this end, we only need to prove that for each fixed $\varepsilon\in(0,1)$, $\widetilde{u_\varepsilon}\in C([0,T];H^s)$, $\widetilde{\mathbb{P}_\varepsilon}$-a.s.; then Proposition 3.4 gives the desired result. Hence we only need to prove that for a.e. $\omega\in\widetilde{\Omega_\varepsilon}$ and for any $t\in[0,T]$, if $t_k\to t$ as $k\to\infty$, then

$$\lim_{k\to\infty}\|\widetilde{u_\varepsilon}(t_k)-\widetilde{u_\varepsilon}(t)\|_{H^s}^2=0. \tag{3.58}$$

Notice that

$$\|\widetilde{u_\varepsilon}(t_k)-\widetilde{u_\varepsilon}(t)\|_{H^s}^2=\|\widetilde{u_\varepsilon}(t_k)\|_{H^s}^2-2(\widetilde{u_\varepsilon}(t_k),\widetilde{u_\varepsilon}(t))_{H^s}+\|\widetilde{u_\varepsilon}(t)\|_{H^s}^2.$$

Then the original question of continuity can now be reduced to verifying that

$$\lim_{k\to\infty}(\widetilde{u_\varepsilon}(t_k),\widetilde{u_\varepsilon}(t))_{H^s}=\|\widetilde{u_\varepsilon}(t)\|_{H^s}^2,\ \widetilde{\mathbb{P}_\varepsilon}\text{-a.s}, \tag{3.59}$$

and the map $t\mapsto\|\widetilde{u_\varepsilon}(t)\|_{H^s}$ is continuous $\widetilde{\mathbb{P}_\varepsilon}$-a.s. We apply Proposition 3.4 to (3.57) to find

$$\widetilde{\mathbb{P}_\varepsilon}\left\{\|\widetilde{u_\varepsilon}(t)\|_{L^\infty(0,T;H^s)}+\|\widetilde{u_\varepsilon}(t)\|_{C^\alpha([0,T];H^{s-1})}<\infty\right\}=1, \tag{3.60}$$

which implies that u_ε is continuous in t with respect to the weak topology on H^s, and therefore (3.59) holds true. To prove $\|\widetilde{u_\varepsilon}(t)\|_{H^s}$ is continuous, similar to (3.34), we use the Itô formula for $D^s\widetilde{u_\varepsilon}$ to derive that for a.e. $\omega\in\widetilde{\Omega_\varepsilon}$ and for all $t_k<t$,

$$\|\widetilde{u_\varepsilon}(t)\|_{H^s}^2-\|\widetilde{u_\varepsilon}(t_k)\|_{H^s}^2\leq\left|\int_{t_k}^t J_1\mathrm{d}\widetilde{W}\right|+\sum_{i=2}^4\int_{t_k}^t|J_i|\mathrm{d}t', \tag{3.61}$$

where $J_i(i=1,2,3,4)$ are given in (3.34) with obvious replacements of u_ε by $\widetilde{u_\varepsilon}$. Then the continuity of $\|\widetilde{u_\varepsilon}(t)\|_{H^s}$ comes from the above estimate and hence we finish the proof. ∎

3.5. Pathwise solution to the mollified problem. We begin by establishing the pathwise uniqueness for the mollified problem (3.2).

LEMMA 3.8. *Let $R>1$, $0<\varepsilon<1$, $s>3$. Assume that $\sigma(\cdot)$ verifies the conditions* (1.17) *and* (1.18). *If $(\mathcal{S},u_\varepsilon,\infty)$ and $(\mathcal{S},v_\varepsilon,\infty)$, relative to the same stochastic basis $\mathcal{S}$, are two global martingale solutions to* (3.2) *satisfying*

$$\mathbb{P}\{u_\varepsilon(0)=v_\varepsilon(0)=u_0(x)\}=1,$$

where u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|^2_{H^s}<\infty$, then

$$\mathbb{P}\left\{u_\varepsilon(t,x)=v_\varepsilon(t,x)\ \forall\ (t,x)\in[0,\infty)\times\mathbb{T}\right\}=1. \tag{3.62}$$

Proof. We first assume that $\|u_0\|_{H^s}<M$, $\mathbb{P}$-a.s., for some deterministic $M>0$. Fix $0<\varepsilon<1$. Let $K>2M^2$ and define the stopping time ξ^ε_K as

$$\xi^\varepsilon_K:=\inf\left\{t>0:\|u_\varepsilon(t)\|^2_{H^s}+\|v_\varepsilon(t)\|^2_{H^s}>K\right\}. \tag{3.63}$$

One can first conclude from the boundedness of u_0 that $\mathbb{P}\{\xi^\varepsilon_K>0\}=1$ and second from $\mathbb{E}\|u_0\|^2_{H^s}<\infty$, Proposition 3.4, and Proposition 3.7 that

$$\mathbb{P}\{\xi^\varepsilon_K\xrightarrow{K\to\infty}\infty\}=1. \tag{3.64}$$

Let $w^\varepsilon=u_\varepsilon-v_\varepsilon$ and $h^\varepsilon=u_\varepsilon+v_\varepsilon$; then $w^\varepsilon(0)=0$, $\mathbb{P}$-a.s., and

$$\begin{aligned}
\mathrm{d}w^\varepsilon&+[\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})]\,[J_\varepsilon(J_\varepsilon u_\varepsilon\partial_x J_\varepsilon u_\varepsilon)+F(u_\varepsilon)]\,\mathrm{d}t\\
&+\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\left[\frac{1}{2}J_\varepsilon\partial_x(J_\varepsilon h^\varepsilon J_\varepsilon w^\varepsilon)\right]\mathrm{d}t\\
&+\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\,[F(u_\varepsilon)-F(v_\varepsilon)]\,\mathrm{d}t\\
&=[\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})\sigma(u_\varepsilon)-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\sigma(v_\varepsilon)]\,\mathrm{d}W.
\end{aligned}$$

Applying D^{s-1} with $s>3$ to the above equation yields

$$\begin{aligned}
&\mathrm{d}D^{s-1}w^\varepsilon+[\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})]\left[\frac{1}{2}D^{s-1}J_\varepsilon\partial_x((J_\varepsilon u_\varepsilon)^2)+D^{s-1}F(u_\varepsilon)\right]\mathrm{d}t\\
&\quad+\frac{1}{2}\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\left[D^{s-1}J_\varepsilon\partial_x(J_\varepsilon h^\varepsilon J_\varepsilon w^\varepsilon)\right]\mathrm{d}t+\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})D^{s-1}\left[F(u_\varepsilon)-F(v_\varepsilon)\right]\mathrm{d}t\\
&=\left[\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(u_\varepsilon)-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(v_\varepsilon)\right]\mathrm{d}W.
\end{aligned}$$

Via the Itô formula, we have

$$\begin{aligned}
&\mathrm{d}\|w^\varepsilon\|^2_{H^{s-1}}\\
&=2\left(\left[\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(u_\varepsilon)-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(v_\varepsilon)\right],D^{s-1}w^\varepsilon\right)_{L^2}\mathrm{d}W\\
&\quad-2\,[\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})]\left(\frac{1}{2}D^{s-1}J_\varepsilon\partial_x((J_\varepsilon u_\varepsilon)^2)+D^{s-1}F(u_\varepsilon),D^{s-1}w^\varepsilon\right)_{L^2}\mathrm{d}t\\
&\quad-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\left(\left[D^{s-1}J_\varepsilon\partial_x(J_\varepsilon h^\varepsilon J_\varepsilon w^\varepsilon)\right],D^{s-1}w^\varepsilon\right)_{L^2}\mathrm{d}t\\
&\quad-2\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\left(D^{s-1}\left[F(u_\varepsilon)-F(v_\varepsilon)\right],D^{s-1}w^\varepsilon\right)_{L^2}\mathrm{d}t\\
&\quad+\left\|\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(u_\varepsilon)-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(v_\varepsilon)\right\|^2_{L^2}\mathrm{d}t\\
&=N_1\mathrm{d}W+\sum_{i=2}^{5}N_i\mathrm{d}t.
\end{aligned}$$

We first integrate the above equation on $[0,t]$, then we take a supremum over $t\in[0,T\wedge\xi_K^\varepsilon]$ and use the Burkholder–Davis–Gundy inequality to find

$$\mathbb{E}\sup_{t\in[0,T\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t)\|_{H^{s-1}}^2\le C\mathbb{E}\left(\int_0^{T\wedge\xi_K^\varepsilon}|N_1|^2\mathrm{d}t\right)^{\frac12}+\sum_{i=2}^5\mathbb{E}\int_0^{T\wedge\xi_K^\varepsilon}|N_i|\mathrm{d}t. \tag{3.65}$$

Via the mean value theorem for $\theta_R(\cdot)$ and the embedding $H^{s-1}\hookrightarrow W^{1,\infty}$, we have

$$\begin{aligned}
&\left\|\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(u_\varepsilon)-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})D^{s-1}\sigma(v_\varepsilon)\right\|_{L^2}\\
&\quad\le\left\|\left(\theta_R(\|u_\varepsilon\|_{W^{1,\infty}})-\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\right)D^{s-1}\sigma(u_\varepsilon)\right\|_{L^2}\\
&\qquad+\left\|\theta_R(\|v_\varepsilon\|_{W^{1,\infty}})\left(D^{s-1}\sigma(u_\varepsilon)-D^{s-1}\sigma(v_\varepsilon)\right)\right\|_{L^2}\\
&\quad\le\|\theta_R'\|_{L^\infty}\|w\|_{W^{1,\infty}}\|\sigma(u_\varepsilon)\|_{H^{s-1}}+g(\|u_\varepsilon\|_{W^{1,\infty}}+\|v_\varepsilon\|_{W^{1,\infty}})\|w^\varepsilon\|_{H^{s-1}}\\
&\quad\le\left[C\|\sigma(u_\varepsilon)\|_{H^{s-1}}+g(\|u_\varepsilon\|_{W^{1,\infty}}+\|v_\varepsilon\|_{W^{1,\infty}})\right]\|w^\varepsilon\|_{H^{s-1}}.
\end{aligned}\tag{3.66}$$

By noticing (1.17), (3.63), and (3.66), for some $C=C(K)$, we arrive at

$$\mathbb{E}\int_0^{T\wedge\xi_K^\varepsilon}|N_5|\mathrm{d}t\le C\mathbb{E}\int_0^{T\wedge\xi_K^\varepsilon}\sup_{t'\in[0,t\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t)\|_{H^{s-1}}^2\mathrm{d}t\le C\int_0^T\mathbb{E}\sup_{t'\in[0,t\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t')\|_{H^{s-1}}^2\mathrm{d}t. \tag{3.67}$$

Similarly, the Cauchy–Schwarz inequality, (3.66), and (3.63) give rise to

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{T\wedge\xi_K^\varepsilon}|N_1|^2\mathrm{d}t\right)^{\frac12}\\
&\le\mathbb{E}\left(\sup_{t\in[0,T\wedge\xi_K^\varepsilon]}\|w^\varepsilon\|_{H^{s-1}}^2\cdot\int_0^T\left[C\|\sigma(u_\varepsilon)\|_{H^{s-1}}+g(\|u_\varepsilon\|_{W^{1,\infty}}+\|v_\varepsilon\|_{W^{1,\infty}})\right]^2\|w^\varepsilon\|_{H^{s-1}}^2\mathrm{d}t\right)^{\frac12}\\
&\le\frac12\mathbb{E}\sup_{t\in[0,T\wedge\xi_K^\varepsilon]}\|w^\varepsilon\|_{H^{s-1}}^2+C\int_0^T\mathbb{E}\sup_{t'\in[0,t\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t')\|_{H^{s-1}}^2\mathrm{d}t,\quad C=C(K).
\end{aligned}\tag{3.68}$$

By the mean value theorem for $\theta_R(\cdot)$, the Cauchy–Schwarz inequality, (2.7), and (3.63), we arrive at

$$\mathbb{E}\int_0^{T\wedge\xi_K^\varepsilon}|N_2|\mathrm{d}t\le C\int_0^T\mathbb{E}\sup_{t'\in[0,t\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t')\|_{H^{s-1}}^2\mathrm{d}t,\quad C=C(K). \tag{3.69}$$

We recall (2.4)–(2.6), commute $D^\sigma\partial_x$ with $J_\varepsilon h^\varepsilon$ via Lemma 2.2, and then integrate by parts to find that

$$\begin{aligned}
&\left|\left(\left[D^{s-1}J_\varepsilon\partial_x(J_\varepsilon h^\varepsilon J_\varepsilon w^\varepsilon)\right],D^{s-1}w^\varepsilon\right)_{L^2}\right|\\
&\quad=\left|\left(\left[D^{s-1}\partial_x(J_\varepsilon h^\varepsilon J_\varepsilon w^\varepsilon)\right],D^{s-1}J_\varepsilon w^\varepsilon\right)_{L^2}\right|\\
&\quad\le C\|h^\varepsilon\|_{H^s}\|w^\varepsilon\|_{H^{s-1}}^2+C\|(J_\varepsilon h^\varepsilon)_x\|_{L^\infty}\|w^\varepsilon\|_{H^{s-1}}^2\le 2C\|h^\varepsilon\|_{H^s}\|w^\varepsilon\|_{H^{s-1}}^2.
\end{aligned}$$

Therefore we have

$$\mathbb{E}\int_0^{T\wedge\xi_K^\varepsilon}|N_3|\mathrm{d}t\le C\int_0^T\mathbb{E}\sup_{t'\in[0,t\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t')\|_{H^{s-1}}^2\mathrm{d}t,\quad C=C(K). \tag{3.70}$$

For J_4, we use the Cauchy–Schwarz inequality and then apply Lemma 2.3 to find

$$\mathbb{E}\int_0^{T\wedge\xi_K^\varepsilon}|N_4|\mathrm{d}t \le C\int_0^T \mathbb{E}\sup_{t'\in[0,t\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t')\|_{H^{s-1}}^2\mathrm{d}t, \quad C=C(K). \tag{3.71}$$

Combining (3.67)–(3.71), we see that w^ε satisfies

$$\mathbb{E}\sup_{t\in[0,T\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t)\|_{H^{s-1}}^2 \le C\int_0^T \mathbb{E}\sup_{t'\in[0,t\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t')\|_{H^{s-1}}^2\mathrm{d}t,$$

which implies that $\mathbb{E}\sup_{t\in[0,T\wedge\xi_K^\varepsilon]}\|w^\varepsilon(t)\|_{H^{s-1}}^2=0$. Then the proof for (3.62) with bounded initial data can be completed by sending $K\to\infty$ using the monotone convergence theorem and (3.64) and noticing that $T>0$ is arbitrary. Now we remove the restrictions of the initial data. For general H^s-valued $\mathcal{F}_0$ measurable initial value with only $\mathbb{E}\|u_0\|_{H^s}^2<\infty$, we consider the decomposition

$$u_0(\omega,x)=\sum_{k\ge1}u_{0,k}(\omega,x)\triangleq\sum_{k\ge1}u_0(\omega,x)\mathbf{1}_{\Omega_k},$$
$$\Omega_k=\{k-1\le\|u_0\|_{H^s}<k\},\ k\in\mathbb{N},\ k\ge1,$$

and let $u_{\varepsilon,k}$ be the solution to (3.2) with initial data $u_{0,k}$. Since $\mathbb{E}\|u_0\|_{H^s}^2<\infty$, we have $1=\sum_{k\ge1}\mathbf{1}_{\Omega_k}$. For any fixed ε, for each $k\ge1$, we can take $K=K_k>2k^2$ to obtain that pathwise uniqueness holds true for $u_{\varepsilon,k}$. We first notice that $\mathbb{P}\{\bigcup_{k\ge1}\Omega_k\}=1$ implies $u_\varepsilon=u_\varepsilon\times1=u_\varepsilon\times(\sum_{k\ge1}\mathbf{1}_{\Omega_k})=\sum_{k\ge1}u_\varepsilon\mathbf{1}_{\Omega_k}$, $\mathbb{P}$-a.s. Second, it follows from $\Omega_k\bigcap\Omega_{k'}=\varnothing(k\ne k')$, $F(0)=0$, and $\sigma(0)=0$ (cf. (2.7) and (1.17)) that $u_\varepsilon\mathbf{1}_{\Omega_k}$ is a solution with initial data $u_{0,k}$. Then we can infer from the uniqueness for $u_{0,k}$ that $u_\varepsilon\mathbf{1}_{\Omega_k}=u_{\varepsilon,k}\mathbf{1}_{\Omega_k}$, $\mathbb{P}$-a.s. Similarly, $v_\varepsilon\mathbf{1}_{\Omega_k}=u_{\varepsilon,k}\mathbf{1}_{\Omega_k}$, $\mathbb{P}$-a.s., then

$$\mathbb{P}\{u_\varepsilon=v_\varepsilon\}\ge\mathbb{P}\left\{\bigcup_{k\in\mathbb{N},\ k\ge1}\Omega_k\right\}=\mathbb{P}\{\Omega\}=1,$$

which is (3.62). ◻

Then we can prove the existence and uniqueness of the pathwise solution to (3.2).

PROPOSITION 3.9. *Let $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be a fixed stochastic basis. Suppose that $\sigma(\cdot)$ satisfies the assumptions* (1.17), (1.18). *Let $s>3$, $r>4$ and u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|_{H^s}^r<\infty$. Then for any $0<\varepsilon<1$, the mollified problem* (3.2) *has a unique global pathwise solution in the sense of Definitions* 1.4–1.6.

Proof. The main target is to show that if $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ is given in advance, for the initial approximate solutions $u_{\varepsilon,n}$, we can first take limit $n\to\infty$, which is relative to the original probability space $(\Omega,\mathcal{F},\mathbb{P})$, to build a pathwise solution under the given basis $\mathcal{S}$.

Fix a $\varepsilon\in(0,1)$. For the given basis $\mathcal{S}$ and for the global pathwise solution $u_{\varepsilon,n}(t)$ to (3.7), we define the sequences of measures $\nu_{\varepsilon,n,m}$ and $\mu_{\varepsilon,n,m}$ as

$$\nu_{\varepsilon,n,m}(\cdot)=\mathbb{P}\left((u_{\varepsilon,n},u_{\varepsilon,m})\in\cdot\right),\quad \mu_{\varepsilon,n,m}(\cdot)=\mathbb{P}\left((u_{\varepsilon,n},u_{\varepsilon,m},W)\in\cdot\right) \tag{3.72}$$

on $X_u^s\times X_u^s$ and $X_u^s\times X_u^s\times X_W$, respectively, where $X_u^s=C([0,T];H^s)$, $X_W=C([0,T];\mathbb{R})$. Let us take any subsequence $\{\nu_{\varepsilon,n_k,m_k}\}_{k\in\mathbb{N}}$. With only minor modifications for Lemma 3.5, we see that $\{\mu_{\varepsilon,n_k,m_k}\}_{k\in\mathbb{N}}$, as a probability measure on

$X_u^{s-1} \times X_u^{s-1} \times X_W$, is tight and hence there exists a subsequence of $\{\mu_{\varepsilon,n_k,m_k}\}_{k\in\mathbb{N}}$, still denoted as $\{\mu_{\varepsilon,n_k,m_k}\}_{k\in\mathbb{N}}$, such that $\mu_{\varepsilon,n_k,m_k}$ converges to some μ_ε weakly. It follows from Lemma 2.5 that there is a probability space $(\widetilde{\Omega_\varepsilon}, \widetilde{\mathcal{F}_\varepsilon}, \widetilde{\mathbb{P}_\varepsilon})$ on which there exists a sequence of random variables $(\underline{u_{\varepsilon,n_k}}, \overline{u_{\varepsilon,m_k}}, \widetilde{W_{\varepsilon,k}})$ converging almost surely in $X_u^{s-1} \times X_u^{s-1} \times X_W$ to a random variable $(\underline{u_\varepsilon}, \overline{u_\varepsilon}, \widetilde{W_\varepsilon})$ and

$$\widetilde{\mathbb{P}_\varepsilon}\left(\left(\underline{u_{\varepsilon,n_k}}, \overline{u_{\varepsilon,m_k}}, \widetilde{W_{\varepsilon,k}}\right) \in \cdot\right) = \mu_{\varepsilon,n_k,m_k}(\cdot), \quad \widetilde{\mathbb{P}_\varepsilon}\left(\left(\underline{u_\varepsilon}, \overline{u_\varepsilon}, \widetilde{W_\varepsilon}\right) \in \cdot\right) = \mu_\varepsilon(\cdot).$$

Observe that in particular, $\nu_{\varepsilon,n_k,m_k}$ converges weakly to a measure ν_ε defined by

$$\nu_\varepsilon(\cdot) = \widetilde{\mathbb{P}_\varepsilon}\left((\underline{u_\varepsilon}, \overline{u_\varepsilon}) \in \cdot\right).$$

Let $\widetilde{\mathcal{S}_\varepsilon} = (\widetilde{\Omega_\varepsilon}, \widetilde{\mathcal{F}_\varepsilon}, \{\widetilde{\mathcal{F}_t^\varepsilon}\}_{t\geq 0}, \widetilde{\mathbb{P}_\varepsilon}, \widetilde{W_\varepsilon})$ with $\widetilde{\mathcal{F}_t^\varepsilon} = \sigma\{\underline{u_\varepsilon}(\tau), \overline{u_\varepsilon}(\tau), \widetilde{W}(\tau)\}_{\tau\in[0,t]}$. In the same manner as in Proposition 3.7, we infer that both $(\widetilde{\mathcal{S}_\varepsilon}, \underline{u_\varepsilon}, \infty)$ and $(\widetilde{\mathcal{S}_\varepsilon}, \overline{u_\varepsilon}, \infty)$ are the martingale solutions to (3.2). Since $\underline{u_\varepsilon}(0) = \overline{u_\varepsilon}(0)$, $\widetilde{\mathbb{P}_\varepsilon}$-a.s., then Lemma 3.8 shows that

$$\nu_\varepsilon\left(\left\{(\underline{u_\varepsilon}, \overline{u_\varepsilon}) \in X_u^{s-1} \times X_u^{s-1}, \underline{u_\varepsilon} = \overline{u_\varepsilon}\right\}\right) = 1.$$

It follows from Lemma 2.7 that the original sequence $u_{\varepsilon,n}$ defined on the initial probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has a subsequence that converges almost surely in the topology of X_u^{s-1} to a random variable u_ε. Using the method from Proposition 3.7 again, we see that (u_ε, ∞) is the unique global pathwise solution to (3.2). ∎

3.6. Remove the mollifier: The second limit.

PROPOSITION 3.10. *Let $s > 3$, $r > 4$, $R > 1$ and $\mu_0 \in \mathcal{P}r(H^s)$ satisfy* (1.19). *If $\sigma(\cdot)$ satisfies* (1.16) *and* (1.17)*, then the problem* (3.1) *has a global martingale solution in the sense of Definition* 1.3.

Proof. As the procedure here is quite similar to the one in the proof for Proposition 3.7, we only give a sketch.

First, for the pathwise solution $(\mathcal{S}, u_\varepsilon, \infty)$ to (3.2) obtained in Proposition 3.9, via Proposition 3.4 (remember that $u_\varepsilon \in C([0,T]; H^s)$, $\mathbb{P}$-a.s.) and some estimates analogous to those in Lemma 3.5, we see that for any $T > 0$, the sequence of measures

$$\{\mu_\varepsilon\} = \{\mu_{u_\varepsilon} \times \mu_W\} \quad \text{with} \quad \mu_{u_\varepsilon}(\cdot) \triangleq \mathbb{P}(u_\varepsilon \in \cdot) \text{ and } \mu_W(\cdot) \triangleq \mathbb{P}(W \in \cdot) \tag{3.73}$$

is also tight in $\mathcal{P}r(X^{s-1})$, where X^{s-1} is defined by (3.46). Then Lemma 2.4 implies that there is a subsequence of μ_ε (still denoted by μ_ε) such that $\mu_\varepsilon \to \underline{\mu}$ weakly for some $\underline{\mu}$ on X^{s-1} and Lemma 2.5 shows that there is a probability space $(\underline{\Omega}, \underline{\mathcal{F}}, \underline{\mathbb{P}})$ on which there is a sequence of random variables $(\underline{u_\varepsilon}, \underline{W_\varepsilon})$ and $(\underline{u}, \underline{W})$ such that $(\underline{u_\varepsilon}, \underline{W_\varepsilon})$ and $(\underline{u}, \underline{W})$ have the probability laws μ_ε and $\underline{\mu}$, respectively. Moreover,

$$\underline{u_\varepsilon} \to \underline{u} \text{ in } C\left([0,T]; H^{s-1}\right), \quad \underline{W_\varepsilon} \to \underline{W} \text{ in } C\left([0,T]; \mathbb{R}\right), \ \underline{\mathbb{P}}\text{-a.s.} \tag{3.74}$$

Using the arguments as in [1, 45] again, one can also prove that $\underline{u_\varepsilon}$ satisfies (3.2) with respect to the basis $\underline{\mathcal{S}_\varepsilon} = \left(\underline{\Omega}, \underline{\mathcal{F}}, \underline{\mathbb{P}}, \{\underline{\mathcal{F}_t^\varepsilon}\}_{t\geq 0}, \underline{W_\varepsilon}\right)$, where $\underline{\mathcal{F}_t^\varepsilon} = \sigma\{\underline{u_\varepsilon}(\tau), \underline{W_\varepsilon}(\tau)\}_{\tau\in[0,t]}$ and $\underline{W}$ is a Brownian motion relative to $\underline{\mathcal{F}_t} = \sigma\{\underline{u}(\tau), \underline{W}(\tau)\}_{\tau\in[0,t]}$. Proposition 3.4 means that $\underline{u_\varepsilon}$ enjoys (3.45) with $\mathbb{E}$ replaced by $\underline{\mathbb{E}}$ and hence by Fatou's lemma, (3.45) also holds true for $\underline{u}$ (corresponding to $\underline{\mathbb{E}}$). Using (3.74) in the explicit form of $G_{1,\varepsilon}(\cdot)$ and $G_2(\cdot)$, noticing (1.16) and (2.9), we have

$$\begin{cases} G_{1,\varepsilon}(\underline{u_\varepsilon}) \xrightarrow[\varepsilon\to 0]{} \theta_R(\|\underline{u}\|_{W^{1,\infty}})\left[\underline{u}(\partial_x\underline{u})+F(\underline{u})\right] & \text{in } C\left([0,T];H^{s-2}\right),\ \underline{\mathbb{P}}\text{-a.s.},\\ G_2(\underline{u_\varepsilon}) \xrightarrow[\varepsilon\to 0]{} \theta_R(\|\underline{u}\|_{W^{1,\infty}})\sigma(\underline{u}) & \text{in } C\left([0,T];H^{s-1}\right),\ \underline{\mathbb{P}}\text{-a.s.} \end{cases} \tag{3.75}$$

Second, as in the proof for Proposition 3.7, under the basis $\underline{\mathcal{S}_\varepsilon}$, one can construct a sequence of square integrable H^{s-2}-valued martingales. Then, by applying Proposition 3.4, Lemma 2.6, (3.74), and (3.75), we see that the limit of these martingales is also a martingale. Moreover, applying the martingale representation theorem [34] again, we see that the limit martingale can be represented as $\int_0^t \theta_R(\|\underline{u}\|_{W^{1,\infty}})\sigma(\underline{u})\mathrm{d}\underline{W}$. That is to say, the basis $\underline{\mathcal{S}} = (\underline{\Omega}, \underline{\mathcal{F}}, \underline{\mathbb{P}}, \{\underline{\mathcal{F}_t}\}_{t\geq 0}, \underline{W})$, where $\underline{\mathcal{F}_t} = \sigma\{\underline{u}(\tau), \underline{W}(\tau)\}_{\tau\in[0,t]}$ and the $\underline{\mathcal{F}_t}$-adapted process $\underline{u}$ satisfy (1.21).

Finally, as in step 2 in the proof for Proposition 3.7, by improving the regularity, we see that $(\underline{\mathcal{S}}, \underline{u}, \infty)$ is a global martingale solution to (3.1). ∎

3.7. Pathwise solution to the cut-off problem. Similar to Lemma 3.8, we consider the pathwise solution to cut-off problem (3.1).

LEMMA 3.11. *Let $R > 1$ and $s > 3$. Assume that $\sigma(\cdot)$ verifies the conditions* (1.17) *and* (1.18). *If $(\mathcal{S}, u_1, \infty)$ and $(\mathcal{S}, u_2, \infty)$ are two global martingale solutions to* (3.1) *obtained in Proposition* 3.10 *satisfying*

$$\mathbb{P}\{u_1(0) = u_2(0) = u_0(x)\} = 1,$$

where u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|^2_{H^s} < \infty$, then

$$\mathbb{P}\left\{u_1(t,x) = u_2(t,x)\ \forall\ (t,x)\in[0,\infty)\times\mathbb{T}\right\} = 1. \tag{3.76}$$

Proof. Recall that in the proof for Proposition 3.10, via the Fatou's property, we have proved that

$$\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|^2_{H^s} < C(R,T,u_0) < \infty.$$

Therefore we can basically repeat the proof for (3.62) to obtain (3.76). Here we omit the details. ∎

PROPOSITION 3.12. *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ be a fixed stochastic basis. Suppose that $\sigma(\cdot)$ satisfies the assumptions* (1.17), (1.18). *Let $s > 3$, $r > 4$ and u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|^r_{H^s} < \infty$. Then the cut-off problem* (3.1) *has a unique global pathwise solution in the sense of Definitions* 1.4–1.6.

Proof. Similar to Proposition 3.9, we need to show that we can take limit $\varepsilon \to 0$, which is relative to the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, to construct a pathwise solution to (3.1). Actually the process here is very similar to the one in Proposition 3.9 and so we just sketch it. Define $\nu_{\varepsilon^1,\varepsilon^2}$ and $\mu_{\varepsilon^1,\varepsilon^2}$ as

$$\nu_{\varepsilon^1,\varepsilon^2}(\cdot) = \mathbb{P}\left((u_{\varepsilon^1}, u_{\varepsilon^2})\in\cdot\right),\quad \mu_{\varepsilon^1,\varepsilon^2}(\cdot) = \mathbb{P}\left((u_{\varepsilon^1}, u_{\varepsilon^2}, W)\in\cdot\right) \tag{3.77}$$

on $X^s_u\times X^s_u$ and $X^s_u\times X^s_u\times X_W$ and take any subsequence $\{\nu_{\varepsilon^1_k,\varepsilon^2_k}\}_{k\in\mathbb{N}}$, where the subscript k satisfies that when $k\to\infty$, $\varepsilon^1_k, \varepsilon^2_k \to 0$. After establishing the tightness, one can find a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ on which there is a sequence of random variables $(\underline{u_{\varepsilon_k^1}}, \overline{u_{\varepsilon_k^2}}, \widetilde{W_k})$ converging almost surely in $X_u^{s-1} \times X_u^{s-1} \times X_W$ to a random variable $(\underline{u}, \overline{u}, \widetilde{W})$ and $\nu_{\varepsilon_k^1, \varepsilon_k^2}$ converges weakly to a measure ν on $X_u^{s-1} \times X_u^{s-1}$ defined by

$$\nu(\cdot) = \widetilde{\mathbb{P}}\left((\underline{u}, \overline{u}) \in \cdot\right).$$

Similar to Proposition 3.10, we see that both $(\widetilde{\mathcal{S}}, \underline{u}, \infty)$ and $(\widetilde{\mathcal{S}}, \overline{u}, \infty)$ are the martingale solutions to (3.1). As a consequence of Lemma 3.11,

$$\nu\left(\left\{(\underline{u}, \overline{u}) \in X_u^{s-1} \times X_u^{s-1}, \underline{u} = \overline{u}\right\}\right) = 1.$$

Lemma 2.7 shows that the original sequence u_ε defined on the initial probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has a subsequence that converges almost surely to a random variable u in X_S. Following the steps as listed in the proof for Proposition 3.10 with the given Brownian motion W, we may finally deduce that (u, ∞) is the unique global pathwise solution to (3.1). ∎

3.8. Remove the cut-off function. Now we remove the cut-off function and this will finish the proof for Theorem 3.1.

Proof for Theorem 3.1. According to Proposition 3.12, for $s > 3$ and $r > 4$, we let u be the pathwise unique global solution to the cut-off problem (3.1) with initial value u_0, where u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|_{H^s}^r < \infty$.

Step 1. Now we remove the cut-off function. We first suppose that for some deterministic $M > 0$, $\|u_0(\omega)\|_{H^s} < M$, $\mathbb{P}$-a.s., and let $D > 0$ be the embedding constant such that $\|\cdot\|_{W^{1,\infty}} \le D\|\cdot\|_{H^s}$ for $s > 3$. Let $R > DM$ and define

$$\tau = \inf\{t > 0 : \|u\|_{H^s} > M\}.$$

Then $\mathbb{P}\{\tau > 0\} = 1$ and $\|u\|_{W^{1,\infty}} \le D\|u\|_{H^s} \le DM < R$ on $[0, \tau]$. Hence $\theta_R(\|u\|_{W^{1,\infty}}) = 1$. Via Proposition 3.12, (u, τ) is a unique pathwise solution to (1.14).

Step 2. Similar to the technique used in Lemma 3.8, we can remove the restrictions of the initial data as follows. For general H^s-valued $\mathcal{F}_0$ measurable initial value with only $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we consider the decomposition

$$u_0(\omega, x) = \sum_{k \ge 1} u_{0,k}(\omega, x) = \sum_{k \ge 1} u_0(\omega, x)\mathbf{1}_{\Omega_k}, \quad \Omega_k = \{k-1 \le \|u_0\|_{H^s} < k\}, \ k \in \mathbb{N}, \ k \ge 1.$$

By the above step, for each $k \ge 1$, we take $R = R_k > Dk$ in $\theta_R(\cdot)$ to obtain a unique pathwise solution (u_k, τ_k) to (1.14) with initial value $u_{0,k}(\omega, x)$, where

$$\tau_k = \inf\{t > 0 : \|u_k\|_{H^s} > k\}, \quad \mathbb{P}\{\tau_k > 0\} = 1. \tag{3.78}$$

As one can verify, $\bigcup_{k \in \mathbb{N},\ k \ge 1} \Omega_k$ is a set of full measure and the nonlinear terms satisfy $F(0) = 0$ and $\sigma(0) = 0$ (cf. (2.7) and (1.17)), and therefore (u, τ) is the unique pathwise solution to (1.14) corresponding to the initial condition u_0, where

$$u = \sum_{k \ge 1} u_k \mathbf{1}_{k-1 \le \|u_0\|_{H^s} < k}, \quad \tau = \sum_{k \ge 1} \tau_k \mathbf{1}_{k-1 \le \|u_0\|_{H^s} < k}.$$

Step 3. Finally, the passage from (u, τ) to a maximal solution in the sense of Definition 1.6 is standard and may be carried out as in [30, 31, 41]. We omit the details here for simplicity.

4. Proof for Theorem 1.7. Now we are in position to prove Theorem 1.7. Let us first prove the pathwise uniqueness for the solutions in H^s with $s > 3/2$.

4.1. Pathwise uniqueness.

PROPOSITION 4.1. *Let $s > 3/2$ and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ be a fixed stochastic basis. Assume that $\sigma(\cdot)$ verifies the conditions* (1.17) *and* (1.18). *Let u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable (relative to $\mathcal{S}$) satisfying $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. Let $(\mathcal{S}, u_1, \tau_1)$ and $(\mathcal{S}, u_2, \tau_2)$ be two solutions, given in Definition* 1.4, *to* (1.14). *If $\mathbb{P}\{u_1(0) = u_2(0) = u_0(x)\} = 1$, then*

$$\mathbb{P}\left\{u_1(t,x) = u_2(t,x)\ \forall\ (t,x) \in [0, \tau_1 \wedge \tau_2) \times \mathbb{T}\right\} = 1.$$

Proof. Define the stopping time

$$\tau_K := \inf\left\{t \geq 0 : \|u_1(t)\|_{H^s} + \|u_2(t)\|_{H^s} > K\right\}. \tag{4.1}$$

We can first assume that $\mathbb{P}\{\tau_K > 0\} = 1$. Otherwise one can impose a boundedness condition on u_0 and then use the decomposition introduced in Lemma 3.8 to remove this. Since $u_i(\cdot \wedge \tau_i) \in C([0,\infty); H^s)(i = 1,2)$, $\mathbb{P}$-a.s., and $\mathbb{E}(\sum_{i=1}^2 \sup_{t\in[0,\tau_i]} \|u_i(t)\|^2_{H^s}) < \infty$, we have

$$\mathbb{P}\left\{\liminf_{K\to\infty} \tau_K > \tau_1 \wedge \tau_2\right\} = 1. \tag{4.2}$$

Otherwise, there is a sequence $K_n \to \infty$ as $n \to \infty$ such that for $\mathbb{P}\{\tau_{K_n} \leq \tau_1 \wedge \tau_2\ \forall n \in \mathbb{N}\} > 0$. Then we have

$$K_n \mathbb{P}\{\tau_{K_n} < \tau_1 \wedge \tau_2,\ \forall n \in \mathbb{N}\} < \mathbb{E}\left(\sup_{t\in[0,\tau_1]} \|u_1(t)\|_{H^s} + \sup_{t\in[0,\tau_2]} \|u_2(t)\|_{H^s}\right) < \infty,$$

which is a contradiction since $K_n \to \infty$. For any $T > 0$, we denote

$$\tau_K^T = \tau_K \wedge T, \tag{4.3}$$

and let $w = u_1 - u_2$, $h = u_1 + u_2$. Then $w(0) = 0$, $\mathbb{P}$-a.s., and w satisfies

$$\mathrm{d}w + \left[\frac{1}{2}\partial_x(hw) + F(u) - F(u)\right]\mathrm{d}t = (\sigma(u) - \sigma(u))\mathrm{d}W. \tag{4.4}$$

Applying D^{s-1} with $s > 3/2$ to the above equation and then uisng the Itô formula, we arrive at

$$\begin{aligned}
\mathrm{d}\|w\|^2_{H^{s-1}} &= \mathrm{d}\left(D^{s-1}w, D^{s-1}w\right)_{L^2}\\
&= 2\left(\mathrm{d}D^{s-1}w, D^{s-1}w\right)_{L^2} + \|D^{s-1}\left[\sigma(u_1) - \sigma(u_2)\right]\|^2_{L^2}\mathrm{d}t\\
&= 2\left(D^{s-1}\left[\sigma(u_1) - \sigma(u_2)\right], D^{s-1}w\right)_{L^2}\mathrm{d}W\\
&\quad - \left(D^{s-1}\partial_x(hw), D^{s-1}w\right)_{L^2}\mathrm{d}t\\
&\quad - 2\left(D^{s-1}\left[F(u_1) - F(u_2)\right], D^{s-1}w\right)_{L^2}\mathrm{d}t\\
&\quad + \|D^{s-1}\left[\sigma(u_1) - \sigma(u_1)\right]\|^2_{L^2}\mathrm{d}t\\
&= H_1\mathrm{d}W + \sum_{i=2}^4 H_i\mathrm{d}t.
\end{aligned} \tag{4.5}$$

Similar to the estimate for (3.65), we have

$$\mathbb{E}\sup_{t\in[0,\tau_K^T]}\|w(t)\|_{H^{s-1}}^2\leq C\mathbb{E}\left(\int_0^{\tau_K^T}|H_1|^2\mathrm{d}t\right)^{\frac{1}{2}}+\sum_{i=2}^5\mathbb{E}\int_0^{\tau_K^T}|H_1|\mathrm{d}t. \tag{4.6}$$

Using Lemmas 2.1 and 2.2 and integrating by parts, we have

$$|H_2|\lesssim\|h\|_{H^s}\|w\|_{H^{s-1}}^2+\|h_x\|_{L^\infty}\|w\|_{H^{s-1}}^2\lesssim\|h\|_{H^s}\|w\|_{H^{s-1}}^2.$$

From (2.9) in Lemma 2.3 and (1.18), we have that for some locally bounded function $H:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $H(\cdot)>g^2(\cdot)+1$,

$$\begin{aligned}|H_3|+|H_4|&\lesssim(\|u_1\|_{H^s}+\|u_2\|_{H^s})\|w\|_{H^{s-1}}^2+g^2(\|u_1\|_{H^s}+\|u_2\|_{H^s})\|w\|_{H^{s-1}}^2\\&\lesssim H(\|u_1\|_{H^s}+\|u_2\|_{H^s})\|w\|_{H^{s-1}}^2.\end{aligned}$$

Notice that the Cauchy–Schwarz inequality, (4.1), and (4.3) give rise to

$$\begin{aligned}\mathbb{E}\left(\int_0^{\tau_K^T}|H_1|^2\mathrm{d}t\right)^{\frac{1}{2}}&\leq\mathbb{E}\left(\sup_{t\in[0,\tau_K^T]}\|w\|_{H^{s-1}}^2\cdot\int_0^{\tau_K^T}\|\sigma(u_1)-\sigma(u_2)\|_{H^{s-1}}^2\mathrm{d}t\right)^{\frac{1}{2}}\\&\leq\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_K^T]}\|w\|_{H^{s-1}}^2+C\mathbb{E}\int_0^{\tau_K^T}H(\|u_1\|_{H^s}+\|u_2\|_{H^s})\|w\|_{H^{s-1}}^2\mathrm{d}t\\&\leq\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_K^T]}\|w\|_{H^{s-1}}^2+CH(2K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_K^t]}\|w(t')\|_{H^s}^2\mathrm{d}t.\end{aligned}$$

Consequently, we can combine the above estimates to find that for some $C=C(K,s)$,

$$\mathbb{E}\sup_{t\in[0,\tau_k^T]}\|w(t)\|_{H^{s-1}}^2\leq C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_K^t]}\|w(t')\|_{H^{s-1}}^2\mathrm{d}t.$$

Hence $\mathbb{E}\sup_{t\in[0,\tau_K]}\|w(t)\|_{H^{s-1}}^2=0$ and therefore $\mathbb{E}\sup_{t\in[0,\tau_K\wedge\tau_1\wedge\tau_2]}\|w(t)\|_{H^{s-1}}^2=0$. Sending $K\to\infty$ using the monotone convergence theorem and (4.2) yields

$$\mathbb{P}\{u_1(t,x)=u_2(t,x)\ \forall\ (t,x)\in[0,\tau_1\wedge\tau_2]\times\mathbb{T}\}=1,$$

which is the desired result. ∎

4.2. Approximation solution with bounded smooth initial data. To prove Theorem 1.7 using Theorem 3.1, when $s>3/2$, for an H^s-valued initial process $u_0(\omega,x)$, we first consider the following problem:

$$\begin{cases}\mathrm{d}u+[u(\partial_x u)+F(u)]\,\mathrm{d}t=\sigma(u)\mathrm{d}W,\quad x\in\mathbb{T},\ t>0,\\u(\omega,0,x)=J_\varepsilon u_0(\omega,x)\in H^\infty,\\\|u_0\|_{H^s}<M,\ \mathbb{P}\text{-a.s.}\end{cases} \tag{4.7}$$

Let $\varepsilon=\frac{1}{k}$ with $k\in\mathbb{N}$; Theorem 3.1 shows that for a given stochastic basis $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq0},W)$, (4.7) admits a unique maximal solution $(u_k,\{\tau_n\}_{n\in\mathbb{N}},\xi_k)$ such that for any $\eta>3$, $u_k\in C([0,\xi_k),H^\eta)$, $\mathbb{P}$-a.s.

Notice that with the help of the method in subsection 3.8, as long as we obtain the local existence of solutions for each fixed M, this boundedness assumption on u_0 can be relaxed to the general case.

Besides, we notice that (2.6) yields that

$$\sup_{0<\varepsilon<1}\|J_\varepsilon u_0\|_{H^s}\leq M. \tag{4.8}$$

4.3. Convergence of the approximation solutions. Motivated by [30, 32], we are going to show that u_k is a Cauchy sequence, as $k \to \infty$, in $C([0,\tau], H^s)$ for some stopping time τ and $s > 3/2$. To this end, we recall the following criterion.

LEMMA 4.2 (see [32, Lemma 5.1]). *For any $T > 0$, let*

$$\tau_k^T = \inf\left\{t \geq 0 : \|u_k\|_{H^s} > \|J_{\frac{1}{k}} u_0\|_{H^s} + 2\right\} \wedge T, \tag{4.9}$$

and for $k, m > 1$, define

$$\tau_{k,m}^T = \tau_k^T \wedge \tau_m^T. \tag{4.10}$$

If $\{u_k\}_{k\in\mathbb{N}}$ satisfies

$$\lim_{m\to\infty} \sup_{k\geq m} \mathbb{E} \sup_{t\in[0,\tau_{k,m}^T]} \|u_k - u_m\|_{H^s} = 0 \tag{4.11}$$

and

$$\lim_{K\to 0} \sup_{k\geq 1} \mathbb{P}\left\{\sup_{t\in[0,\tau_k^T\wedge K]} \|u_k\|_{H^s} \geq \|J_{\frac{1}{k}} u_0\|_{H^s} + 1\right\} = 0, \tag{4.12}$$

then there is a stopping time τ satisfying $\mathbb{P}\{0 < \tau \leq T\} = 1$ and a process $u \in C([0,\tau], H^s)$ such that for some subsequence k_n,

$$\lim_{n\to\infty} \sup_{t\in[0,\tau]} \|u_{k_n} - u\|_{H^s} = 0, \ \mathbb{P}\text{-a.s.} \tag{4.13}$$

Besides, $\sup_{t\in[0,\tau]} \|u\|_{H^s} \leq \sup_{k\in\mathbb{N}} \|J_{\frac{1}{k}} u_0\|_{H^s} + 2$, $\mathbb{P}$-a.s.

According to the above criterion, we need to prove that (4.11) and (4.12) hold true for $\{u_k\}$.

LEMMA 4.3. *$\{u_k\}$ satisfies (4.11).*

Proof. Let $w_{m,k} = u_m - u_k$. Then $w_{m,k}$ satisfies

$$\begin{cases} \mathrm{d}w_{m,k} + [w_{m,k}\partial_x u_m + u_k\partial_x w_{m,k} + F(u_m) - F(u_k)]\,\mathrm{d}t = (\sigma(u_m) - \sigma(u_k))\mathrm{d}W, \\ w_{m,k}(0) = J_{\frac{1}{m}} u_0 - J_{\frac{1}{k}} u_0 \in H^\infty. \end{cases} \tag{4.14}$$

Similar to the estimate for (4.5), we have

$$\begin{aligned} \mathrm{d}\|w_{m,k}\|_{H^s}^2 &= 2\left(D^s\left[\sigma(u_m) - \sigma(u_k)\right], D^s w_{m,k}\right)_{L^2} \mathrm{d}W \\ &\quad - 2\left(D^s(w_{m,k}\partial_x u_m), D^s w_{m,k}\right)_{L^2} \mathrm{d}t \\ &\quad - 2\left(D^s(u_k\partial_x w_{m,k}), D^s w_{m,k}\right)_{L^2} \mathrm{d}t \\ &\quad - 2\left(D^s\left[F(u_m) - F(u_k)\right], D^s w_{m,k}\right)_{L^2} \mathrm{d}t \\ &\quad + \|D^s\left[\sigma(u_m) - \sigma(u_k)\right]\|_{L^2}^2 \mathrm{d}t = A_{1,s}\mathrm{d}W + \sum_{i=2}^5 A_{i,s}\mathrm{d}t, \end{aligned} \tag{4.15}$$

and for the given $T > 0$ and $t \in [0, \tau_{k,m}^T]$,

$$\mathbb{E} \sup_{t\in[0,\tau_{k,m}^T]} \|w_{m,k}(t)\|_{H^s}^2 \leq \mathbb{E}\|w_{m,k}(0)\|_{H^s}^2 + C\mathbb{E}\left(\int_0^{\tau_{k,m}^T} |A_{1,s}|^2 \mathrm{d}t\right)^{\frac{1}{2}} + \sum_{i=2}^5 \mathbb{E}\int_0^{\tau_{k,m}^T} |A_{i,s}|\mathrm{d}t. \tag{4.16}$$

Using Lemma 2.1, $H^{s-1} \hookrightarrow L^\infty$ for $s > 3/2$ and integrating by parts gives rise to

$$\begin{aligned} |A_{2,s}| &\lesssim \|w_{m,k}\|_{H^s}\|\partial_x u_m\|_{L^\infty}\|w_{m,k}\|_{H^s} + \|w_{m,k}\|_{L^\infty}\|\partial_x u_m\|_{H^s}\|w_{m,k}\|_{H^s} \\ &\lesssim \|w_{m,k}\|_{H^s}^2\|u_m\|_{H^s} + \|w_{m,k}\|_{H^{s-1}}\|u_m\|_{H^{s+1}}\|w_{m,k}\|_{H^s} \end{aligned}$$

and

$$|A_{3,s}| \lesssim \|u_k\|_{H^s}\|\partial_x w_{m,k}\|_{L^\infty}\|w_{m,k}\|_{H^s} + \|\partial_x u_k\|_{L^\infty}\|w_{m,k}\|_{H^s}^2 \lesssim \|u_k\|_{H^s}\|w_{m,k}\|_{H^s}^2.$$

Therefore it follows from (4.8), (4.9), and (4.10) that

(4.17)
$$\begin{aligned} &\sum_{i=2}^{3}\mathbb{E}\int_0^{\tau_{k,m}^T}|A_{i,s}|\mathrm{d}t \\ &\le C\mathbb{E}\int_0^{\tau_{k,m}^T}(\|u_m\|_{H^s}+\|u_k\|_{H^s})\|w_{m,k}\|_{H^s}^2+(\|w_{m,k}\|_{H^{s-1}}\|u_m\|_{H^{s+1}})^2+\|w_{m,k}\|_{H^s}^2\mathrm{d}t \\ &\le C\mathbb{E}\int_0^{\tau_{k,m}^T}(\|u_m\|_{H^s}+\|u_k\|_{H^s}+1)\|w_{m,k}\|_{H^s}^2+\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2\mathrm{d}t \\ &\le C\mathbb{E}\int_0^{\tau_{k,m}^T}(2M+5)\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^s}^2+\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^{s-1}}^2\|u_m(t')\|_{H^{s+1}}^2\mathrm{d}t \\ &\le C(2M+5)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^s}^2+\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^{s-1}}^2\|u_m(t')\|_{H^{s+1}}^2\mathrm{d}t. \end{aligned}$$

Similar to the way we estimate $|H_4|+|H_5|$ in Proposition 4.1, we use (2.8) in Lemma 2.3, (1.18), (4.8), (4.9), and (4.10) to find that

$$\sum_{i=4}^{5}\mathbb{E}\int_0^{\tau_{k,m}^T}|A_{i,s}|\mathrm{d}t \le CH(2M+4)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^s}^2\mathrm{d}t. \tag{4.18}$$

Using (4.8), (4.9), and (4.10) leads to

(4.19)
$$\begin{aligned} \mathbb{E}\left(\int_0^{\tau_{k,m}^T}|A_{1,s}|^2\mathrm{d}t\right)^{\frac12} &\le \mathbb{E}\left(\sup_{t\in[0,\tau_{k,m}^T]}\|w_{m,k}\|_{H^s}^2\cdot\int_0^{\tau_{k,m}^T}\|\sigma(u_m)-\sigma(u_k)\|_{H^s}^2\mathrm{d}t\right)^{\frac12} \\ &\le \frac12\mathbb{E}\sup_{t\in[0,\tau_{k,m}^T]}\|w_{m,k}\|_{H^s}^2+CH(2M+4)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^s}^2\mathrm{d}t. \end{aligned}$$

Consequently, we find that for some $C = C(M, s)$,

$$\begin{aligned} \mathbb{E}\sup_{t\in[0,\tau_{k,m}^T]}\|w_{m,k}(t)\|_{H^s}^2 \le 2\mathbb{E}\|w_{m,k}(0)\|_{H^s}^2 + C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^s}^2 \\ + \mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^{s-1}}^2\|u_m(t')\|_{H^{s+1}}^2\mathrm{d}t. \end{aligned}$$

Hence we have that for some $C = C(M, s, T)$,

$$\mathbb{E}\sup_{t\in[0,\tau_{k,m}^T]}\|w_{m,k}(t)\|_{H^s}^2 \le C\mathbb{E}\|w_{m,k}(0)\|_{H^s}^2 + C\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^T]}\|w_{m,k}(t')\|_{H^{s-1}}^2\|u_m(t')\|_{H^{s+1}}^2. \tag{4.20}$$

Recall that J_ε is a bounded operator and satisfies (2.2). Hence we have

$$\lim_{m\to\infty}\sup_{k\ge m}\mathbb{E}\|w_{m,k}(0)\|_{H^s}^2 = 0. \tag{4.21}$$

Therefore to verify (4.11), it suffices to show that in (4.20),

$$\lim_{m\to\infty}\sup_{k\ge m}\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^T]}\|w_{m,k}(t')\|_{H^{s-1}}^2\|u_m(t')\|_{H^{s+1}}^2 = 0. \tag{4.22}$$

To show this, we use the Itô formula to (4.7) to find that for any $\rho > 0$,

$$\begin{aligned}
\mathrm{d}\|u_m\|_{H^\rho}^2 &= 2\left(D^\rho\sigma(u_m), D^\rho u_m\right)_{L^2}\mathrm{d}W - 2\left(D^\rho\left[u_m\partial_x u_m\right], D^\rho u_m\right)_{L^2}\mathrm{d}t\\
&\quad - 2\left(D^\rho F(u_m), D^\rho u_m\right)_{L^2}\mathrm{d}t + \|D^\rho\sigma(u_m)\|_{L^2}^2\mathrm{d}t\\
&= B_{1,\rho}\mathrm{d}W + \sum_{i=2}^4 B_{i,\rho}\mathrm{d}t.
\end{aligned} \tag{4.23}$$

As a result, by an application of the Itô product rule for (4.15) and (4.23) with $s > 3/2$, we have

$$\begin{aligned}
\mathrm{d}\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2 &= \left(\|w_{m,k}\|_{H^{s-1}}^2 B_{1,s+1} + \|u_m\|_{H^{s+1}}^2 A_{1,s-1}\right)\mathrm{d}W\\
&\quad + \sum_{i=2}^4 \|w_{m,k}\|_{H^{s-1}}^2 B_{i,s+1}\mathrm{d}t\\
&\quad + \sum_{i=2}^5 \|u_m\|_{H^{s+1}}^2 A_{i,s-1}\mathrm{d}t + A_{1,s-1}B_{1,s+1}\mathrm{d}t.
\end{aligned} \tag{4.24}$$

Therefore we have that for any $T > 0$ and $t \in [0, \tau_{k,m}^T]$,

$$\begin{aligned}
&\|w_{m,k}(t)\|_{H^{s-1}}^2\|u_m(t)\|_{H^{s+1}}^2 - \|w_{m,k}(0)\|_{H^{s-1}}^2\|u_m(0)\|_{H^{s+1}}^2\\
&\quad = \int_0^t \|w_{m,k}\|_{H^{s-1}}^2 B_{1,s+1}\mathrm{d}W + \int_0^t \|u_m\|_{H^{s+1}}^2 A_{1,s-1}\mathrm{d}W\\
&\qquad + \sum_{i=2}^4\int_0^t \|w_{m,k}\|_{H^{s-1}}^2 B_{i,s+1}\mathrm{d}t\\
&\qquad + \sum_{i=2}^5\int_0^t \|u_m\|_{H^{s+1}}^2 A_{i,s-1}\mathrm{d}t + \int_0^t A_{1,s-1}B_{1,s+1}\mathrm{d}t.
\end{aligned} \tag{4.25}$$

Using the Burkholder–Davis–Gundy inequality as before, we arrive at

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau_{k,m}^T]}\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2 - \mathbb{E}\|w_{m,k}(0)\|_{H^{s-1}}^2\|u_m(0)\|_{H^{s+1}}^2\\
&\quad \le C\mathbb{E}\left(\int_0^{\tau_{k,m}^T}\|w_{m,k}\|_{H^{s-1}}^4 B_{1,s+1}^2\mathrm{d}t\right)^{\frac12} + C\mathbb{E}\left(\int_0^{\tau_{k,m}^T}\|u_m\|_{H^{s+1}}^4 A_{1,s-1}^2\mathrm{d}t\right)^{\frac12}
\end{aligned} \tag{4.26}$$

$$+\sum_{i=2}^{4}\mathbb{E}\int_0^{\tau_{k,m}^T}\|w_{m,k}\|_{H^{s-1}}^2|B_{i,s+1}|\mathrm{d}t+\sum_{i=2}^{5}\mathbb{E}\int_0^{\tau_{k,m}^T}\|u_m\|_{H^{s+1}}^2|A_{i,s-1}|\mathrm{d}t$$
$$+\mathbb{E}\int_0^{\tau_{k,m}^T}|A_{1,s-1}B_{1,s+1}|\mathrm{d}t.$$

Via (1.17), we have

$$\begin{aligned}\|w_{m,k}\|_{H^{s-1}}^4B_{1,s+1}^2&\le 4\|w_{m,k}\|_{H^{s-1}}^4f^2(\|u_m\|_{W^{1,\infty}})(1+\|u_m\|_{H^{s+1}})^2\|u_m\|_{H^{s+1}}^2\\&\le 8\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2f^2(\|u_m\|_{W^{1,\infty}})\left(\|w_{m,k}\|_{H^{s-1}}^2+\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2\right).\end{aligned}$$

As a result, we can infer from (4.8), (4.9), and (4.10) that

$$\begin{aligned}&\mathbb{E}\left(\int_0^{\tau_{k,m}^T}\|w_{m,k}\|_{H^{s-1}}^4B_{1,s+1}^2\mathrm{d}t\right)^{\frac12}\\&\le Cf^2(M+2)\mathbb{E}\Bigg(\int_0^{\tau_{k,m}^T}\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2\\&\qquad\times\left(\|w_{m,k}\|_{H^{s-1}}^2+\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2\right)\mathrm{d}t\Bigg)^{\frac12}\\&\le Cf^2(M+2)\mathbb{E}\Bigg(\sup_{t\in[0,\tau_{k,m}^T]}\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2\\&\qquad\times\int_0^{\tau_{k,m}^T}(\|w_{m,k}\|_{H^{s-1}}^2+\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2)\mathrm{d}t\Bigg)^{\frac12}\\&\le\frac14\mathbb{E}\sup_{t\in[0,\tau_{k,m}^T]}\|w_{m,k}\|_{H^{s-1}}^2\|u_m\|_{H^{s+1}}^2+C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^{s-1}}^2\mathrm{d}t\\&\quad+C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^{s-1}}^2\|u_m(t')\|_{H^{s+1}}^2\mathrm{d}t,\quad C=C(M,s).\end{aligned}$$

After using Lemma 2.1, (2.7) in Lemma 2.3, (1.17), and the embedding of $H^s\hookrightarrow W^{1,\infty}$ for $s>3/2$, we can then apply (4.8), (4.9), and (4.10) to the resulting inequality to obtain that for some $C=C(M,s)$,

$$\begin{aligned}&\sum_{i=2}^{4}\mathbb{E}\int_0^{\tau_{k,m}^T}\|w_{m,k}\|_{H^{s-1}}^2|B_{i,s+1}|\mathrm{d}t\\&\le C\mathbb{E}\int_0^{\tau_{k,m}^T}\|w_{m,k}\|_{H^{s-1}}^2\left[\|u_m\|_{H^s}\|u_m\|_{H^{s+1}}^2+f^2(\|u_m\|_{H^s})(1+\|u_m\|_{H^{s+1}}^2)\right]\mathrm{d}t\\&\le C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^{s-1}}^2\|u_m(t')\|_{H^{s+1}}^2\mathrm{d}t+C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|w_{m,k}(t')\|_{H^{s-1}}^2\mathrm{d}t.\end{aligned}$$

We can repeat the procedure as in Proposition 4.1 to find that for some $C=C(M,s)$,

$$\sum_{i=2}^{5}\mathbb{E}\int_0^{\tau_{k,m}^T}\|u_m\|_{H^{s+1}}^2|A_{i,s-1}|\mathrm{d}t\le C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{k,m}^t]}\|u_m(t')\|_{H^{s+1}}^2\|w_{m,k}(t')\|_{H^{s-1}}^2\mathrm{d}t$$

and

$$\|u_m\|^4_{H^{s+1}} A^2_{1,s-1} \le 4\|u_m\|^4_{H^{s+1}} g^2(\|u_m\|_{H^s} + \|u_k\|_{H^s})\|w_{m,k}\|^4_{H^{s-1}},$$

which imply that

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{\tau^T_{k,m}} \|u_m\|^4_{H^{s+1}} A^2_{1,s-1}\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad\le Cg^2(2M+4)\mathbb{E}\left(\sup_{t\in[0,\tau^T_{k,m}]}\|w_{m,k}\|^2_{H^{s-1}}\|u_m\|^2_{H^{s+1}}\cdot\int_0^{\tau^T_{k,m}}\|w_{m,k}\|^2_{H^{s-1}}\|u_m\|^2_{H^{s+1}}\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad\le \frac{1}{4}\mathbb{E}\sup_{t\in[0,\tau^T_{k,m}]}\|w_{m,k}\|^2_{H^{s-1}}\|u_m\|^2_{H^{s+1}}\\
&\qquad + C\int_0^T \mathbb{E}\sup_{t'\in[0,\tau^t_{k,m}]}\|w_{m,k}(t')\|^2_{H^{s-1}}\|u_m(t')\|^2_{H^{s+1}}\mathrm{d}t, \quad C=C(M,s),
\end{aligned}$$

where (4.9) and (4.10) were used. Finally for $|A_{1,s-1}B_{1,s+1}|$, from (1.17) and (1.18), we have that for some locally bounded function $\widetilde{H}:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $\widetilde{H}(\cdot) > g(\cdot)f(\cdot)$,

$$\begin{aligned}
|A_{1,s-1}B_{1,s+1}| &\le g(\|u_m\|_{H^s}+\|u_k\|_{H^s})\|w_{m,k}\|^2_{H^{s-1}} f(\|u_m\|_{H^s})(1+\|u_m\|_{H^{s+1}})\|u_m\|_{H^{s+1}}\\
&\le \widetilde{H}(\|u_m\|_{H^s}+\|u_k\|_{H^s})\left(\|w_{m,k}\|^2_{H^{s-1}}\|u_m\|^2_{H^{s+1}} + \|w_{m,k}\|^2_{H^{s-1}}\right).
\end{aligned}$$

Consequently, as before, we have that for some $C=C(M,s)$

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau^T_{k,m}} |A_{1,s-1}B_{1,s+1}|\mathrm{d}t\\
&\quad\le C\mathbb{E}\int_0^{\tau^T_{k,m}} \sup_{t'\in[0,\tau^t_{k,m}]}\|w_{m,k}(t')\|^2_{H^{s-1}}\|u_m(t')\|^2_{H^{s+1}} + \sup_{t'\in[0,\tau^t_{k,m}]}\|w_{m,k}(t')\|^2_{H^{s-1}}\mathrm{d}t\\
&\quad\le C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{k,m}]}\|w_{m,k}(t')\|^2_{H^{s-1}}\|u_m(t')\|^2_{H^{s+1}}\mathrm{d}t + C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{k,m}]}\|w_{m,k}(t')\|^2_{H^{s-1}}\mathrm{d}t.
\end{aligned}$$

Combining the above estimates into (4.26), we have

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau^T_{k,m}]}\|w_{m,k}\|^2_{H^{s-1}}\|u_m\|^2_{H^{s+1}} &\le 2\mathbb{E}\|w_{m,k}(0)\|^2_{H^{s-1}}\|u_m(0)\|^2_{H^{s+1}}\\
&\quad + C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{k,m}]}\|w_{m,k}(t')\|^2_{H^{s-1}}\mathrm{d}t\\
&\quad + C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{k,m}]}\|w_{m,k}(t')\|^2_{H^{s-1}}\|u_m(t')\|^2_{H^{s+1}}\mathrm{d}t.
\end{aligned}$$

Then we see that for some $C=C(M,s,T)>0$,

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau^T_{k,m}]}\|w_{m,k}\|^2_{H^{s-1}}\|u_m\|^2_{H^{s+1}} &\le C\mathbb{E}\|w_{m,k}(0)\|^2_{H^{s-1}}\|u_m(0)\|^2_{H^{s+1}}\\
&\quad + C\mathbb{E}\sup_{t\in[0,\tau^T_{k,m}]}\|w_{m,k}(t)\|^2_{H^{s-1}}.
\end{aligned} \tag{4.27}$$

We first notice that the second term on the right side of (4.27) can be controlled by $C\mathbb{E}\|w_{m,k}(0)\|^2_{H^{s-1}}$, and this can be obtained in the same way in which we prove Proposition 4.1. Hence we use (2.2) to find that

$$\lim_{m\to\infty}\sup_{k\geq m}\mathbb{E}\sup_{t\in[0,\tau^T_{k,m}]}\|w_{m,k}(t)\|^2_{H^{s-1}}\leq C\lim_{m\to\infty}\sup_{k\geq m}\mathbb{E}\|w_{m,k}(0)\|^2_{H^{s-1}}=0. \tag{4.28}$$

Moreover, we can infer from (4.8), (2.2), and (2.3) that $\|w_{m,k}(0)\|_{H^{s-1}}\|u_m(0)\|_{H^{s+1}}\leq o\left(\frac{1}{m}\right)m=o(1)$, which gives

$$\lim_{m\to\infty}\sup_{k\geq m}\mathbb{E}\|w_{m,k}(0)\|^2_{H^{s-1}}\|u_m(0)\|^2_{H^{s+1}}=0. \tag{4.29}$$

Therefore we combine these two limits into (4.27) to obtain (4.22). Once we obtain (4.22), we can take the limit in (4.20) with noticing (4.21) to obtain (4.11). ∎

LEMMA 4.4. $\{u_k\}$ *satisfies* (4.12).

Proof. Recall (4.23). Then for any $K>0$, we have

$$\sup_{t\in[0,\tau^T_k\wedge K]}\|u_k(t)\|^2_{H^s}\leq\|J_{\frac{1}{k}}u_0\|^2_{H^s}+\sup_{t\in[0,\tau^T_k\wedge K]}\left|\int_0^t B_{1,s}\mathrm{d}W\right|+\sum_{i=2}^4\int_0^{\tau^T_k\wedge K}|B_{i,s}|\mathrm{d}t. \tag{4.30}$$

As a result, we have

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in[0,\tau^T_k\wedge K]}\|u_k(t)\|^2_{H^s}>\|J_{\frac{1}{k}}u_0\|^2_{H^s}+1\right\}\\ &\quad\leq\mathbb{P}\left\{\sup_{t\in[0,\tau^T_k\wedge K]}\left|\int_0^t B_{1,s}\mathrm{d}W\right|>\frac{1}{2}\right\}+\mathbb{P}\left\{\sum_{i=2}^4\int_0^{\tau^T_k\wedge K}|B_{i,s}|\mathrm{d}t>\frac{1}{2}\right\}.\end{aligned} \tag{4.31}$$

Using the Chebyshev inequality, Lemma 2.1, (2.7) in Lemma 2.3, (1.17), the embedding of $H^s\hookrightarrow W^{1,\infty}$ for $s>3/2$, (4.9), and (4.8), we have

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=2}^4\int_0^{\tau^T_k\wedge K}|B_{i,s}|\mathrm{d}t>\frac{1}{2}\right\}&\leq C\sum_{i=2}^4\mathbb{E}\int_0^{\tau^T_k\wedge K}|B_{i,s}|\mathrm{d}t\\ &\leq C\mathbb{E}\int_0^{\tau^T_k\wedge K}\left[\|u_k\|_{H^s}\|u_k\|^2_{H^s}+f^2(\|u_k\|_{H^s})(1+\|u_k\|^2_{H^s})\right]\mathrm{d}t\\ &\leq C\mathbb{E}\int_0^{\tau^T_k\wedge K}C(M,s)\mathrm{d}t\leq C(M,s)K.\end{aligned} \tag{4.32}$$

Similarly, for the stochastic integral, we use the Doob's maximal inequality, the Itô isometry, and the integral Minkowski inequality to obtain

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,\tau^T_k\wedge K]}\left|\int_0^t B_{1,s}\mathrm{d}W\right|>\frac{1}{2}\right\}&\leq 4\mathbb{E}\left(\int_0^{\tau^T_k\wedge K}|B_{1,s}|\mathrm{d}W\right)^2\\ &\leq C\mathbb{E}\int_0^{\tau^T_k\wedge K}\left[f^2(\|u_k\|_{W^{1,\infty}})(1+\|u_k\|_{H^s})^2\|u_k\|^2_{H^s}\right]\mathrm{d}t\\ &\leq C\mathbb{E}\int_0^{\tau^T_k\wedge K}C(M,s)\mathrm{d}t\leq C(M,s)K.\end{aligned} \tag{4.33}$$

Hence we have

$$\mathbb{P}\left\{\sup_{t\in[0,\tau_k^T\wedge K]}\|u_k(t)\|_{H^s}^2>\|J_{\frac{1}{k}}u_0\|_{H^s}^2+1\right\}\le C(M,s)K,$$

which gives (4.12). ∎

Proposition 4.5. *Let $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be a fixed stochastic basis. Suppose that $\sigma(\cdot)$ satisfies the assumptions* (1.17), (1.18). *Let $s>3/2$ and let u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\|u_0\|_{H^s}<M$, $\mathbb{P}$-a.s., for some deterministic $M>0$. Then* (1.14) *has a unique pathwise solution in the sense of Definition* 1.4.

Proof. Lemmas 4.2, 4.3, and 4.4 yield that for the solutions $\{u_k\}_{k\in\mathbb{N}}$ to (4.7) with $\varepsilon=\frac{1}{k}$, there is a stopping time τ satisfying $\mathbb{P}\{0<\tau\le T\}=1$ and a process $u\in C([0,\tau],H^s)$ such that for some subsequence $k_n\to\infty$,

$$\sup_{t\in[0,\tau]}\|u_{k_n}-u\|_{H^s}\to 0,\ \mathbb{P}\text{-a.s.} \tag{4.34}$$

With this almost sure convergence, we can repeat the method as in Propositions 3.7 and 3.10 to prove that (u,τ) is a pathwise solution, in the sense of Definition 1.4, to (1.14). Uniqueness comes from Proposition 4.1. ∎

4.4. Remove the boundedness assumption on initial data. Finally, we are in position to finish the proof for Theorem 1.7.

Proof for Theorem 1.7. The passage from Proposition 4.5 to Theorem 1.7 can be carried out as the passage from Proposition 3.12 to Theorem 3.1. More precisely, by using the cutting argument as employed in subsection 3.8, we can remove the almost sure bound on the initial data. Besides, one may pass from the case of local to maximal pathwise solutions as in [30, 31, 41]. Here the details are omitted for simplicity. ∎

5. Global existence and blow-up for the linear noise case.

Now we focus on the case that the noise is linear, that is, (1.9) (or equivalently, (1.10)). In this section, we study the conditions which will lead to the global existence and the blow-up in finite time of the solution to (1.9), and estimate the associated probabilities.

Motivated by [30, 41], we introduce the following Girsanov type transform:

$$v=\eta^{-1}(\omega,t)u,\quad \eta(\omega,t)=e^{\beta W(t)-\frac{\beta^2}{2}t}. \tag{5.1}$$

Proposition 5.1. *Let $s>3$ and $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be fixed in advance. Let $u_0(\omega,x)$ be an H^s-valued $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2<\infty$. If $(u,\{\tau_n\}_{n\in\mathbb{N}},\xi)$ is the unique maximal solution to* (1.9) (*or to* (1.10), *equivalently*), *then for $t\in[0,\xi)$, v defined by* (5.1) *solves the following problem, $\mathbb{P}$-a.s.:*

$$\begin{cases}\partial_t v+\eta v\partial_x v+\eta F(v)=0, \quad \eta(\omega,t)=e^{\beta W(t)-\frac{\beta^2}{2}t},\ x\in\mathbb{T},\\ v(\omega,0,x)=u_0(\omega,x),\ x\in\mathbb{T},\end{cases} \tag{5.2}$$

Moreover, $v\in C([0,\xi);H^s)\bigcap C^1([0,\xi);H^{s-1})$, $\mathbb{P}$-a.s.

Proof. Let $(u,\{\tau_n\}_{n\in\mathbb{N}},\xi)$ be the unique maximal solution to (1.9) with initial data u_0 and notice that $F(\cdot)$ does not act on t. By direct computation with using the Itô formula, we see that for a.e. $\omega\in\Omega$, v defined by (5.1) satisfies

$$\partial_t v+\eta v\partial_x v+\eta F(v)=0,\ x\in\mathbb{T},\ t\in[0,\xi). \tag{5.3}$$

Since $v(\omega,0,x){=}u_0(\omega,x)$, we see that v solves (5.2), $\mathbb{P}$-a.s. Moreover, from Theorem 1.7 and the equation itself we see that $v{\in}C([0,\xi);H^s)\bigcap C^1\left([0,\xi);H^{s-1}\right)$, $\mathbb{P}$-a.s. $\square$

5.1. Global existence. Let $(u,\{\tau_n\}_{n\in\mathbb{N}},\xi)$ be the maximal solution to (1.9). From Proposition 5.1, we see that for a.e. $\omega\in\Omega$, $v(\omega,t,x)$ defined by (5.1) solves (5.2) on $[0,\xi)$. Moreover, because $H^s\hookrightarrow C^1$ holds true for $s>3/2$, $v_x\in C\left([0,\xi)\times\mathbb{T}\right)$. Then we can conclude that for a.e. $\omega\in\Omega$, for any $x\in\mathbb{T}$, the particle curve equation,

$$\begin{cases}\dfrac{dq(\omega,t,x)}{dt}=\eta(\omega,t)v(\omega,t,q(\omega,t,x)), \quad t\in[0,\xi),\\ q(\omega,0,x)=x,\quad x\in\mathbb{T},\end{cases} \tag{5.4}$$

has a unique solution $q(\omega,t,x)$ such that for a.e. $\omega\in\Omega$, $q(\omega,t,x)\in C^1([0,\xi)\times\mathbb{T})$. Besides, for a.e. $\omega\in\Omega$, we differentiate (5.4) with respect to x to deduce that

$$\begin{cases}\dfrac{dq_x(\omega,t,x)}{dt}=\eta(\omega,t)v_x(\omega,t,q)q_x, \quad t\in[0,\xi),\\ q_x(\omega,0,x)=1,\quad x\in\mathbb{T}.\end{cases}$$

For a.e. $\omega\in\Omega$, the above equation has a solution $q_x(\omega,t,x)=\exp(\int_0^t\eta(t')v_x(\omega,t',q(\omega,t',x))\ \mathrm{d}t')$, which means $\mathbb{P}\left\{q_x(t,x)>0\ \forall(t,x)\in[0,\xi)\times\mathbb{T}\right\}=1$. Hence for a.e. $\omega\in\Omega$, for all $t\in[0,\xi)$, $q(\omega,t,\cdot)$ is an increasing diffeomorphism of $\mathbb{T}$.

The following lemma is very easy but it will play a key role in proving the global existence.

Lemma 5.2. *Let $V_0(\omega,x)=(1-\partial_x^2)u_0(\omega,x)$ and $V(\omega,t,x)=v(\omega,t,x)-v_{xx}(\omega,t,x)$, where $v(\omega,t,x)$ defined by* (5.1) *solves* (5.2) *on* $[0,\xi)$, *$\mathbb{P}$-a.s. Then for a.e. $\omega\in\Omega$, and for all $(t,x)\in[0,\xi)\times\mathbb{T}$,*

$$V(\omega,t,q(\omega,t,x))q_x^2(\omega,t,x)=V_0(\omega,x). \tag{5.5}$$

As a result, for a.e. $\omega\in\Omega$, and for all $(t,x)\in[0,\xi)\times\mathbb{T}$,

$$\operatorname{sign}(v)=\operatorname{sign}(V)=\operatorname{sign}(V_0). \tag{5.6}$$

Proof. We first notice that if v solves (5.2), $\mathbb{P}$-a.s., then

$$V_t+\eta V_xv+2\eta Vv_x=0,\ \mathbb{P}\text{-a.s.}$$

Thus for a.e. $\omega\in\Omega$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[V(\omega,t,q(\omega,t,x))q_x^2(\omega,t,x)\right]&=V_tq_x^2+V_xq_tq_x^2+2Vq_xq_{xt}\\&=q_x^2\left(V_t+\eta V_xv+2\eta Vv_x\right)=0.\end{aligned}$$

From the above equation with $q_x(\omega,0,x)=1$, we have (5.5) and therefore $\operatorname{sign}(V)=\operatorname{sign}(V_0)$, $\mathbb{P}$-a.s. Recalling (1.15), we see that $\operatorname{sign}(v)=\operatorname{sign}(V)$, $\mathbb{P}$-a.s., comes from the positivity of $G_{\mathbb{T}}$. $\square$

Lemma 5.3. *Let all the conditions be as in the statement of Proposition* 5.1. *If*

$$\mathbb{P}\{(1-\partial_x^2)u_0(\omega,x)>0\ \forall\ x\in\mathbb{T}\}=p_1,\quad \mathbb{P}\{(1-\partial_x^2)u_0(\omega,x)<0\ \forall\ x\in\mathbb{T}\}=p_2$$

for some $p_1,p_2\in[0,1]$, then the maximal solution u to (1.9) (*or to* (1.10), *equivalently*) *satisfies*

$$\mathbb{P}\left\{\|u_x(\omega,t)\|_{L^\infty}\le\|u(\omega,t)\|_{L^\infty}\lesssim\eta(\omega,t)\|u_0\|_{H^1}\ \forall\ t\in[0,\xi)\right\}\ge p_1+p_2.$$

Proof. Since $V = v - v_{xx}$, a direct computation shows that for a.e. $\omega \in \Omega$,

$$
\begin{aligned}
v(\omega,t,x) &= \frac{\mathrm{e}^{x-\pi}}{4\sinh(\pi)}\int_0^x \mathrm{e}^{-y}V(\omega,t,y)\mathrm{d}y + \frac{\mathrm{e}^{-x+\pi}}{4\sinh(\pi)}\int_0^x \mathrm{e}^{y}V(\omega,t,y)\mathrm{d}y\\
&\quad + \frac{\mathrm{e}^{x+\pi}}{4\sinh(\pi)}\int_x^{2\pi} \mathrm{e}^{-y}V(\omega,t,y)\mathrm{d}y + \frac{\mathrm{e}^{-x-\pi}}{4\sinh(\pi)}\int_x^{2\pi} \mathrm{e}^{y}V(\omega,t,y)\mathrm{d}y,\\
v_x(\omega,t,x) &= \frac{\mathrm{e}^{x-\pi}}{4\sinh(\pi)}\int_0^x \mathrm{e}^{-y}V(\omega,t,y)\mathrm{d}y - \frac{\mathrm{e}^{-x+\pi}}{4\sinh(\pi)}\int_0^x \mathrm{e}^{y}V(\omega,t,y)\mathrm{d}y\\
&\quad + \frac{\mathrm{e}^{x+\pi}}{4\sinh(\pi)}\int_x^{2\pi} \mathrm{e}^{-y}V(\omega,t,y)\mathrm{d}y - \frac{\mathrm{e}^{-x-\pi}}{4\sinh(\pi)}\int_x^{2\pi} \mathrm{e}^{y}V(\omega,t,y)\mathrm{d}y.
\end{aligned}
$$

Consequently, for a.e. $\omega \in \Omega$, and for all $(t,x) \in [0,\xi) \times \mathbb{T}$,

$$
[v + v_x](\omega,t,x) = \frac{1}{2\sinh(\pi)}\int_0^{2\pi} \mathrm{e}^{(x-y-2\pi[\frac{x-y}{2\pi}]-\pi)}V(\omega,t,y)\ \mathrm{d}y, \tag{5.7}
$$

$$
[v - v_x](\omega,t,x) = \frac{1}{2\sinh(\pi)}\int_0^{2\pi} \mathrm{e}^{(-x+y+2\pi[\frac{x-y}{2\pi}]+\pi)}V(\omega,t,y)\ \mathrm{d}y. \tag{5.8}
$$

Then one can employ (5.7), (5.8), and (5.6) to obtain that for a.e. $\omega \in \Omega$ and for all $(t,x) \in [0,\xi) \times \mathbb{T}$,

$$
-v(\omega,t,x) \le v_x(\omega,t,x) \le v(\omega,t,x) \ \text{ if } \ V_0(\omega,x) = (1-\partial_x^2)u_0(\omega,x) > 0, \tag{5.9}
$$

$$
v(\omega,t,x) \le v_x(\omega,t,x) \le -v(\omega,t,x) \ \text{ if } \ V_0(\omega,x) = (1-\partial_x^2)u_0(\omega,x) < 0. \tag{5.10}
$$

Therefore we find

$$
\mathbb{P}\left\{\|v_x(\omega,t,\cdot)\|_{L^\infty} \le \|v(\omega,t,\cdot)\|_{L^\infty}\ \forall\ t \in [0,\xi)\right\} \ge p_1 + p_2. \tag{5.11}
$$

Moreover, from $(5.2)_1$ we can deduce that for a.e. $\omega \in \Omega$, v satisfies

$$
v_t - v_{xxt} + 3\eta v v_x = 2\eta v_x v_{xx} + \eta v v_{xxx}. \tag{5.12}
$$

Multiplying both sides of the above equation by v and then integrating the resulting equation on $x \in \mathbb{T}$, we see that for a.e. $\omega \in \Omega$ and for all $t > 0$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \left(v^2 + v_x^2\right)\mathrm{d}x = 0,
$$

which implies that

$$
\mathbb{P}\left\{\|v(t)\|_{H^1} = \|v_0\|_{H^1} = \|u_0\|_{H^1}\ \forall\ t > 0\right\} = 1. \tag{5.13}
$$

By $H^1 \hookrightarrow L^\infty$, (5.13), and (5.11), we obtain that

$$
\mathbb{P}\left\{\|v_x(\omega,t)\|_{L^\infty} \le \|v(\omega,t)\|_{L^\infty} \lesssim \|v(\omega,t)\|_{H^1} = \|u_0\|_{H^1}\ \forall\ t \in [0,\xi)\right\} \ge p_1 + p_2.
$$

Via (5.1), we obtain the desired estimate. ∎

Proof for Theorem 1.8. Let $(u, \{\tau_n\}_{n\in\mathbb{N}}, \xi)$ be the maximal solution to (1.9) and recall that (1.9) is equivalent to $(5.2)_1$. We apply $D^s v \cdot D^s$ to $(5.2)_1$ and then integrate the resulting equation on $\mathbb{T}$ to find that for a.e. $\omega \in \Omega$,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{H^s}^2 = -\eta(\omega,t)\int_{\mathbb{T}} D^s v \cdot D^s\left[v\partial_x v\right]\ \mathrm{d}x - \eta(\omega,t)\int_{\mathbb{T}} D^s v \cdot D^s F(v)\ \mathrm{d}x. \tag{5.14}
$$

We commute the operator D^s with v to obtain that for a.e. $\omega \in \Omega$,

$$\int_{\mathbb{T}} D^s v \cdot D^s \left[v\partial_x v\right] \,\mathrm{d}x = \int_{\mathbb{T}} \left[D^s, v\right] \partial_x v \cdot D^s v \,\mathrm{d}x + \int_{\mathbb{T}} v\partial_x D^s v D^s v \,\mathrm{d}x.$$

Just like the estimates in Proposition 3.4, we find that for a.e. $\omega \in \Omega$,

$$\int_{\mathbb{T}} \left[D^s, v\right] \partial_x v \cdot D^s v \,\mathrm{d}x \leq C_s \|\partial_x v\|_{L^\infty} \|v\|_{H^s}^2,$$
$$\int_{\mathbb{T}} v\partial_x D^s v D^s v \,\mathrm{d}x \leq C_s \|\partial_x v\|_{L^\infty} \|v\|_{H^s}^2,$$

and

$$\int_{\mathbb{T}} D^s v \cdot D^s F(v) \,\mathrm{d}x \leq C_s \|v\|_{H^s} \|F(v)\|_{H^s} \leq C_s \left(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\right) \|v\|_{H^s}^2.$$

Combining these results we conclude that for a.e. $\omega \in \Omega$, for all $t > 0$, v satisfies

$$\|v(t\wedge\xi)\|_{H^s} - \|v(0)\|_{H^s} \leq C_s \int_0^{t\wedge\xi} \eta(t') \|v(t')\|_{W^{1,\infty}} \|v(t')\|_{H^s} \,\mathrm{d}t'. \tag{5.15}$$

Therefore for a.e. $\omega \in \Omega$ and for all $t > 0$,

$$\|u(t\wedge\xi)\|_{H^s} \leq C_s \eta(t\wedge\xi) \|u_0\|_{H^s} \exp\left(\int_0^{t\wedge\xi} \|u(\omega, t')\|_{W^{1,\infty}} \,\mathrm{d}t'\right). \tag{5.16}$$

From Lemma 5.3, $u(\omega, t, x)$ satisfies

$$\mathbb{P}\left\{\|u\|_{W^{1,\infty}} \lesssim 2\eta(\omega, t)\|u_0\|_{H^1} \ \forall\ t \in [0, \xi)\right\} \geq p_1 + p_2.$$

Notice that $A = A(\omega) = \sup_{t>0} \eta(\omega, t) < \infty$, $\mathbb{P}$-a.s.; then one can use (5.16), $H^1 \hookrightarrow L^\infty$, and the above estimate to find that

$$\mathbb{P}\left\{\forall\ T > 0, \sup_{t\in[0,T\wedge\xi]} \|u(t)\|_{H^s} \lesssim A\|u_0\|_{H^s} e^{AT\|u_0\|_{H^1}} < \infty\right\} \geq p_1 + p_2. \tag{5.17}$$

For this to happen, we must have $\mathbb{P}\{\xi = \infty\} \geq p_1 + p_2$. In other words,

$$\mathbb{P}\left\{u \text{ exists globally}\right\} \geq p_1 + p_2. \tag{5.18}$$

We finish the proof for Theorem 1.8. ∎

5.2. Blow-up in finite time. Now we focus on the proof for Theorem 1.9.

Proof for Theorem 1.9. If $(u, \{\tau_n\}_{n\in\mathbb{N}}, \xi)$ is the maximal solution to (1.9) (or to (1.10), equivalently) with deterministic initial data u_0, then Proposition 5.1 means that $v(\omega, t, x)$ defined by (5.1) solves (5.3) with the same initial data u_0. Using (1.5), (1.15), and (5.3), we identify that v_x satisfies

$$v_{tx} + \eta v v_{xx} = \eta v^2 - \frac{\eta}{2} v_x^2 - \eta \left[G_{\mathbb{T}} * \left(v^2 + \frac{1}{2} v_x^2\right)\right], \quad t \in [0, \xi),\ \mathbb{P}\text{-a.s.},$$

where $\eta = \eta(\omega, t)$ is given in (5.1). Notice that due to the positivity of $G_{\mathbb{T}}$ and η,

$$\eta \left[G_{\mathbb{T}} * \left(v^2 + \frac{1}{2} v_x^2 \right) \right] \geq 0, \ \mathbb{P}\text{-a.s.}$$

Therefore for a.e. $\omega \in \Omega$, $g(\omega, t) = \partial_x v(\omega, t, 0)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\omega, t) + \eta v(\omega, t, 0) v_{xx}(\omega, t, 0) \leq \eta v^2(\omega, t, 0) - \eta \frac{1}{2} g^2(\omega, t), \quad t \in [0, \xi). \tag{5.19}$$

Recall that (5.3) is equivalent to (5.12) and observe that $h(\omega, t, x) = -v(\omega, t, -x)$ also satisfies (5.12) on $[0, \xi)$, with $h(\omega, 0, x) = -u(\omega, 0, -x)$, $\mathbb{P}$-a.s. Then it follows from Lemma 4.1 that

$$\mathbb{P} \left\{ v(\omega, t, x) = h(\omega, t, x) \ \forall \ (t, x) \in [0, \xi) \times \mathbb{T} \right\} = 1.$$

As a result, $v(\omega, t, 0) = 0$ and therefore we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\omega, t) \leq -\eta \frac{1}{2} g^2(\omega, t) < 0, \quad t \in [0, \xi), \ \mathbb{P}\text{-a.s.} \tag{5.20}$$

That is to say,

$$\frac{-2}{\partial_x u_0(0)} \geq \int_0^t \eta(\omega, \tau) \mathrm{d}\tau, \quad t \in [0, \xi), \ \mathbb{P}\text{-a.s.} \tag{5.21}$$

Taking the mathematical expectation shows that for all $t \in [0, \xi)$,

$$\frac{-2}{\partial_x u_0(0)} \geq \int_0^t \left(\mathbb{E} \eta(\omega, \tau) \right) \mathrm{d}\tau = t, \ \mathbb{P}\text{-a.s.} \tag{5.22}$$

Therefore

$$\mathbb{P} \left\{ \frac{-2}{\partial_x u_0(0)} \geq \xi \right\} = 1.$$

We finish the proof. □

6. Dissipative effect of the noise on the periodic peakons. Now we consider the pathwise dissipative effect of the noise on the periodic peakons. We recall that

$$u(t, x) = \frac{c}{\cosh \pi} \cosh \left(x - ct - 2\pi \left[\frac{x - ct}{2\pi} \right] - \pi \right) \tag{6.1}$$

is a periodic peaked solution to (1.1) with velocity $c \in \mathbb{R}$; cf. [44].

Then we have the following proposition.

PROPOSITION 6.1. *Consider the deterministic CH equation* (1.1) *on the circle* $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. *For any* $\beta \in \mathbb{R}$, (1.9) *has infinitely many pathwise weak solutions* (*i.e.,* (1.21) *is verified with the* $(\cdot, \cdot)_{H^s}$ *replaced by the dual product for a distribution and a test function*) *given by*

$$\widetilde{u}(\omega, t, x) = \gamma e^{\beta W(t) - \frac{\beta^2}{2} t} \cosh \left(x - f(\omega, t) - 2\pi \left[\frac{x - f(\omega, t)}{2\pi} \right] - \pi \right), \quad \gamma \in \mathbb{R}, \gamma \neq 0, \tag{6.2}$$

where

$$f(\omega, t) = \gamma \cosh \pi \int_0^t e^{\beta W(\tau) - \frac{\beta^2}{2} \tau} \mathrm{d}\tau, \ t > 0. \tag{6.3}$$

Proof. In the following, we will use the following facts to compute (distributional) derivatives. Let $f \in L^1_{loc}(X)$, where $X \subset \mathbb{R}$ is an open set. If f' exists and is continuous except at single point $x_0 \in X$ and $f' \in L^1_{loc}(X)$, then the left-handed and right-handed limits $f(x_0\pm)$ exist, besides, $T'_f = T_{f'} + [f(x_0+) - f(x_0-)]\,\delta_{x_0}$, where T_f is the distribution associated to the function f and δ_{x_0} is the Dirac delta distribution centered at $x = x_0$.

For the linear noise case, recall that v defined by (5.1) is the solution to the random equation $(5.2)_1$. Using the operator $\left(1-\partial_x^2\right)$, $(5.2)_1$ becomes

$$\sum_{i=1}^{3} I_i = 0, \tag{6.4}$$

where $\eta(\omega,t) = e^{\beta W(t)-\frac{\beta^2}{2}t}$, $I_1 = v_t - v_{xxt}$, $I_2 = \frac{1}{2}\eta(\omega,t)\partial_x\left(1-\partial_x^2\right)v^2$, and $I_3 = \eta(\omega,t)\partial_x\left(\frac{1}{2}v_x^2 + v^2\right)$. By using (5.1), we need to prove that for a.e. $\omega \in \Omega$, for any $\gamma \neq 0$ and for the f given in (6.3),

$$v_\gamma(\omega,t,x) = \gamma\cosh\theta, \quad \theta = x - f(\omega,t) - 2\pi\left[\frac{x - f(\omega,t)}{2\pi}\right] - \pi$$

is a pathwise weak solution to (6.4). By direct computation, we have that for a.e. $\omega \in \Omega$,

$$v_x = \gamma\sinh\theta, \quad v_{xx} = \gamma\cosh\theta - 2\gamma\sinh\pi\cdot\delta_f, \tag{6.5}$$

where δ_f is the periodic Dirac delta function at $x = f(\omega,t)$ (mod 2π). Thus we have

$$I_1 = v_t - v_{xxt} = -2\gamma f_t(\omega,t)\sinh\pi\cdot\delta'_f.$$

Moreover, for a.e. $\omega \in \Omega$, direct computation shows that

$$\left(v^2\right)_{xx} = 2\gamma^2\left(2\cosh^2\theta - 1 - 2\cosh\pi\sinh\pi\cdot\delta_f\right)$$

and hence

$$I_2 = -3\eta(\omega,t)\gamma^2\cosh\theta\sinh\theta + 2\eta(\omega,t)\gamma^2\cosh\pi\sinh\pi\cdot\delta'_f.$$

Similarly, for a.e. $\omega \in \Omega$, using (6.5) yields

$$I_3 = 3\eta(\omega,t)\gamma^2\cosh\theta\sinh\theta.$$

On account of the above calculation, for a.e. $\omega \in \Omega$, we have

$$\sum_{i=1}^{3} I_i = 2\eta(\omega,t)\gamma^2\cosh\pi\sinh\pi\cdot\delta'_f - 2\gamma f_t(\omega,t),$$
$$\delta'_f = 2\gamma(\gamma\cosh\pi\eta(\omega,t) - f_t)\sinh\pi\cdot\delta'_f = 0.$$

Hence $v(\omega,t,x) = \gamma\cosh\theta$ is a periodic peaked solution to (1.9) if and only if

$$f(\omega,t) = \gamma\cosh\pi\int_0^t e^{\beta W(\tau)-\frac{\beta^2}{2}\tau}\mathrm{d}\tau \quad \forall\, t > 0,\ \mathbb{P}\text{-a.s.} \tag{6.6}$$

The proof is therefore completed by using (5.1). ∎

Proof for Theorem 1.10. For any $\beta \in \mathbb{R},\ c \in \mathbb{R}$ and for the process $\widetilde{u}(\omega,t,x)$ given by (6.2), we take $\gamma = \frac{c}{\cosh \pi}$ in (6.2) to see that for any $x \in \mathbb{T}$,

$$\widetilde{u}(\omega,0,x) = \frac{c}{\cosh \pi} \cosh\left(x - 2\pi\left[\frac{x}{2\pi}\right] - \pi\right) = u(x,0),\ \mathbb{P}\text{-a.s.}$$

Besides, we have

$$\lim_{t\to\infty} \widetilde{u}(\omega,t,x) \lesssim \lim_{t\to\infty} \mathrm{e}^{\left(\beta W(t) - \frac{\beta^2}{2}t\right)} = 0 \ \ \forall\ x \in \mathbb{T},\ \mathbb{P}\text{-a.s.}$$

Moreover, for a.e. $\omega \in \Omega$, the rate at which $\widetilde{u}(\omega,t,x)$ propagates in the x-direction at time $t > 0$ is $\partial_t f(\omega,t)$ with $f(\omega,t)$ given in (6.6) and $\gamma = \frac{c}{\cosh \pi}$. Notice that

$$\lim_{t\to\infty} \partial_t f(\omega,t) = \lim_{t\to\infty} c\mathrm{e}^{\left(\beta W(t) - \frac{\beta^2}{2}t\right)} = 0,\ \mathbb{P}\text{-a.s.}$$

Combining the above equations directly concludes the proof. ∎

Remark 6.2. Here we remark that the results in Proposition 6.1 can be extended to the traveling waves of the CH equation on $\mathbb{R}$. Actually, for any $C^3(\mathbb{R})$ traveling waves $g(x-ct)$ of the deterministic CH equation (1.1) with wave velocity $c \in \mathbb{R}$, one can check that for a.e. $\omega \in \Omega$,

$$\widetilde{g}(\omega,t,x) = e^{\beta W(t) - \frac{\beta^2}{2}t} g\left(x - c\int_0^t e^{\beta W(\tau) - \frac{\beta^2}{2}\tau} \mathrm{d}\tau\right) \tag{6.7}$$

is a pathwise solution to (1.9) or to (1.10) with $\widetilde{g}(\omega,0,x) = g(x)$, $\mathbb{P}$-a.s. And similar to the proof for Theorem 1.10, we see that the noise forces not only all the bounded $C^3(\mathbb{R})$ traveling waves to tend to the trivial stationary solution as $t \to \infty$ but also the velocities of the deterministic $C^3(\mathbb{R})$ traveling waves to tend to zero as $t \to \infty$, simultaneously.

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