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# GLOBAL SOLUTION FOR THE SPATIALLY INHOMOGENEOUS NON-CUTOFF KAC EQUATION*

TONG YANG† AND HONGJUN YU‡

**Abstract.** This paper is concerned with the Cauchy problem on the one-dimensional inhomogeneous non-cutoff Kac equation. Based on the analysis on the linearized operator obtained in [N. Lerner, Y. Morimoto, K. Pravda-Starov, and C.-J. Xu, *J. Funct. Anal.,* 269 (2015), pp. 459–535], we first prove the existence of a global solution to the equation around a global Maxwellian by combining two sets of macro-micro decomposition. Then by using the dissipative norm of the linearized operator in the fractional Hermite–Sobolev space and using the perturbation theory, the spectrum structure of the linearized Kac equation is given. Based on this, the optimal time decay estimate for the nonlinear Kac equation is obtained.

**Key words.** non-cutoff Kac equation, global existence, large time behavior

**AMS subject classifications.** 35Q20, 35B35

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**1. Introduction.** Consider the Cauchy problem for the spatially inhomogeneous non-cutoff Kac equation,

$$\partial_t F + v\partial_x F = Q(F,F), \qquad F(0,x,v) = F_0(x,v). \tag{1.1}$$

Here $F = F(t,x,v)$ is the density distribution of particles, which depends on time $t \geq 0$, position $x \in \mathbb{R}$, and velocity $v \in \mathbb{R}$. The Kac collision operator is given by

$$Q(F,G)(v) = \int_{\mathbb{R}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} B(\theta)\{F'_*G' - F_*G\} d\theta dv_*, \tag{1.2}$$

with $F'_* = F(t,x,v'_*)$, $F' = F(t,x,v')$, $F_* = F(t,x,v_*)$, and $F = F(t,x,v)$, where the pre- and postcollisional velocities satisfy

$$v' = v\cos\theta - v_*\sin\theta, \quad v'_* = v\sin\theta + v_*\cos\theta, \tag{1.3}$$

following from the conservation of the kinetic energy in the binary collision

$$v^2 + v_*^2 = v'^2 + v_*'^2. \tag{1.4}$$

In (1.2), the cross section is assumed to be an even nonnegative function satisfying

$$B(\theta) \geq 0, \quad B(\theta) \in L^1_{loc}(0,1), \quad B(\theta) = B(-\theta), \tag{1.5}$$

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with a nonintegrable singularity

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} B(\theta)d\theta = +\infty.$$

In particular, we consider a cross section of the form

$$B(\theta) \approx |\theta|^{-1-2s} \quad \text{when } \theta \to 0, \quad s \in (0,1). \tag{1.6}$$

Here and hereafter the notation $A \approx B$ is used to denote that there exist two generic positive constants $C_1 < C_2$ such that $C_1 B \leq A \leq C_2 B$. For simplicity of calculation, we make the following choice for the cross section:

$$B(\theta) = \frac{|\cos\frac{\theta}{2}|}{|\sin\frac{\theta}{2}|^{1+2s}}, \quad s \in (0,1), \quad |\theta| \leq \frac{\pi}{4}. \tag{1.7}$$

Note that this assumption on the cross section can be extended to a wider class of cross sections with the nonintegrable singularity stated in (1.5) and (1.6).

It follows from (1.2), (1.3), and (1.5) that the collision operator satisfies

$$\int_{\mathbb{R}} Q(h,h)dv = \int_{\mathbb{R}} v^2 Q(h,h)dv = 0, \tag{1.8}$$

and its nonnegative equilibrium is a local Maxwellian,

$$Q(h,h) = 0 \quad \Leftrightarrow h = M_{[\rho(t,x),\vartheta(t,x)]}(v) = \frac{\rho(t,x)}{\sqrt{2\pi\vartheta(t,x)}} \exp\Big(-\frac{|v|^2}{2\vartheta(t,x)}\Big), \tag{1.9}$$

where $\rho$ and $\vartheta$ are the density and the temperature given by

$$\rho(t,x) = \int_{\mathbb{R}} M_{[\rho(t,x),\vartheta(t,x)]}(v)dv, \quad p(t,x) = \rho(t,x)\vartheta(t,x) = \int_{\mathbb{R}} v^2 M_{[\rho(t,x),\vartheta(t,x)]}(v)dv. \tag{1.10}$$

Define an inner product in $v \in \mathbb{R}$ with respect to a Maxwellian $\widetilde{M}(v)$ as

$$\langle g_1, g_2 \rangle_{\widetilde{M}} = \int_{\mathbb{R}} \frac{g_1(v)g_2(v)}{\widetilde{M}(v)} dv.$$

If $\widetilde{M}(v)$ is the local Maxwellian $M(v)$ defined in (1.9), then the macroscopic space is spanned by the following orthonormal base:

$$\chi_0(v) = \frac{1}{\sqrt{\rho}} M, \quad \chi_2(v) = \frac{1}{\sqrt{2\rho}}\Big(\frac{v^2}{\vartheta} - 1\Big)M, \quad \langle \chi_0, \chi_2 \rangle_M = 0. \tag{1.11}$$

The macroscopic projection $P_0$ and the microscopic projection $P_1$ are

$$P_0 h = \sum_{i=0,2} \langle h, \chi_i \rangle_M \chi_i, \quad P_1 h = h - P_0 h. \tag{1.12}$$

Then a function $h(v)$ is called microscopic if

$$\int_{\mathbb{R}} h(v)dv = \int_{\mathbb{R}} v^2 h(v)dv = 0. \tag{1.13}$$

With these projections, the solution of the Kac equation $F(t,x,v)$ can be decomposed into the macroscopic component, i.e., the local Maxwellian $M = M_{[\rho(t,x),\vartheta(t,x)]}(v)$ defined in (1.9), and the microscopic component $\sqrt{\mu}g = \sqrt{\mu}g(t,x,v)$, which satisfies (1.13). That is,

$$F(t,x,v) = M + \sqrt{\mu}g, \quad P_0F = M, \quad P_1F = \sqrt{\mu}g.$$

Here $\mu = M_{[1,1]}(v)$ is a global Maxwellian. Then the Kac equation (1.1) becomes

$$\partial_t(M+\sqrt{\mu}g) + v\partial_x(M+\sqrt{\mu}g) = Q(M+\sqrt{\mu}g, M+\sqrt{\mu}g). \tag{1.14}$$

By integrating the product of (1.14) and the collision invariants $(1, v^2)$ with respect to $v$ over $\mathbb{R}$, one has the following system for the macroscopic variables $(\rho, p)$:

$$\begin{cases} \partial_t\rho + \partial_x \displaystyle\int_{\mathbb{R}} v\sqrt{\mu}g dv = 0, \\ \partial_t p + \partial_x \displaystyle\int_{\mathbb{R}} v^3\sqrt{\mu}g dv = 0. \end{cases} \tag{1.15}$$

Here we have used (1.8), (1.10), and the fact that $\sqrt{\mu}\partial_t g$ and $v\partial_x M$ are microscopic.

Noting that $Q(M,M) = 0$, we apply the projection operator $P_1$ to (1.14) to get

$$\begin{aligned} \sqrt{\mu}\partial_t g + P_1 v\partial_x(M+\sqrt{\mu}g) &= Q(M+\sqrt{\mu}g, M+\sqrt{\mu}g) \\ &= Q(M,\sqrt{\mu}g) + Q(\sqrt{\mu}g, M) + Q(\sqrt{\mu}g,\sqrt{\mu}g). \end{aligned} \tag{1.16}$$

Noting that $P_0 v\partial_x M = 0$, we have from (1.16) that

$$\partial_t g + v\partial_x g + \mathcal{L}g = \Gamma(g,g) + \Gamma\left(\frac{M-\mu}{\sqrt{\mu}}, g\right) + \Gamma\left(g, \frac{M-\mu}{\sqrt{\mu}}\right) - \frac{v\partial_x M}{\sqrt{\mu}} + \frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}. \tag{1.17}$$

Here the linearized Kac operator with respect to the global Maxwellian $\mu = M_{[1,1]}(v)$, denoted by $\mathcal{L}g$, and the nonlinear operator $\Gamma(g,g)$ are

$$\mathcal{L}g = -\frac{1}{\sqrt{\mu}}\{Q(\mu,\sqrt{\mu}g) + Q(\sqrt{\mu}g,\mu)\}, \quad \Gamma(g_1,g_2) = \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g_1,\sqrt{\mu}g_2). \tag{1.18}$$

To present the results, the following notation is needed. We use $\langle\cdot,\cdot\rangle$ to denote the standard $L^2$ inner product in $\mathbb{R}_v$ with its corresponding $L^2$ norm $|\cdot|_2$. Denote by $(\cdot,\cdot)$ the $L^2$ inner products in $\mathbb{R}_x$ or $\mathbb{R}_x\times\mathbb{R}_v$ with its corresponding $L^2$ norm $\|\cdot\|$. Let $\gamma$ and $\beta$ be nonnegative integers, and let $C_\beta^{\bar{\beta}}$ be the usual binomial coefficient. Denote $\langle v\rangle = \sqrt{1+v^2}$. Throughout the paper, generic positive constants are denoted by either $c$ or $C$. For the time decay rate estimates, the space $Z^q = L^2(\mathbb{R}_v; L^q(\mathbb{R}_x))$ is used with its norm defined by

$$\|f\|_{Z^q} = \Big(\int_{\mathbb{R}_v}\Big(\int_{\mathbb{R}_x}|f(x,v)|^q dx\Big)^{\frac{2}{q}} dv\Big)^{\frac{1}{2}}.$$

As shown in [28, 27] and further discussed in section 5, the operator $\mathcal{L}$ is a nonnegative and self-adjoint operator on $L^2(\mathbb{R}_v)$ with null space $\mathcal{N} = \text{span}\{\psi_0(v), \psi_2(v)\}$. Moreover, there exist constants $\delta > 0$ and $\delta_1 > 0$ such that for any function $h\in\mathcal{N}^\perp$,

$$\langle\mathcal{L}h, h\rangle \geq \delta|\mathcal{H}_v^{\frac{s}{2}}h|_2^2 \geq \delta_1|h|_2^2, \tag{1.19}$$

where $s \in (0,1)$ and the operator $\mathcal{H}_v = -\Delta_v + \frac{v^2}{4}$ is the harmonic oscillator operator. The operator $\mathcal{L}^{-1}$ exists from $\mathcal{N}^\perp$ to itself. Since the microscopic component $\sqrt{\mu}g$ satisfies (1.13), the function $g$ is in the space $\mathcal{N}^\perp$. It follows from (1.17) that

$$\begin{aligned} g =& -\mathcal{L}^{-1}\Big(\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}\Big) + \mathcal{L}^{-1}\Big( - \partial_t g - \mathbf{P}_1 v\partial_x g + \Gamma(g,g) + \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big) \\ &+\Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big) + \mathbf{P}_1\frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}\Big) = -\mathcal{L}^{-1}\Big(\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}\Big) + \mathcal{L}^{-1}\Theta. \end{aligned} \tag{1.20}$$

Here $\mathbf{P}_1$ is as defined in (5.7). Substituting (1.20) into (1.15) yields the dissipative macroscopic system

$$\begin{cases} \partial_t \rho - \partial_x \displaystyle\int_{\mathbb{R}} v\sqrt{\mu}\mathcal{L}^{-1}\Big(\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}\Big)dv = -\partial_x \int_{\mathbb{R}} v\sqrt{\mu}\mathcal{L}^{-1}\Theta dv, \\ \partial_t p - \partial_x \displaystyle\int_{\mathbb{R}} v^3\sqrt{\mu}\mathcal{L}^{-1}\Big(\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}\Big)dv = -\partial_x \int_{\mathbb{R}} v^3\sqrt{\mu}\mathcal{L}^{-1}\Theta dv. \end{cases} \tag{1.21}$$

Define the following perturbation variables:

$$\widetilde{\rho} = \rho - 1, \qquad \widetilde{p} = p - 1. \tag{1.22}$$

For the later energy estimates, the instant energy functional $\widetilde{\mathcal{E}}(t)$ is defined as

$$\widetilde{\mathcal{E}}(t) \approx \mathcal{E}(t) = \sum_{|\gamma|\le 3} (\|\partial_x^\gamma \widetilde{\rho}\|^2 + \|\partial_x^\gamma \widetilde{p}\|^2) + \sum_{|\gamma|+|\beta|\le 3} \|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} g\|^2. \tag{1.23}$$

In addition, the corresponding dissipation functional denoted by $\widetilde{\mathcal{D}}(t)$ satisfies

$$\widetilde{\mathcal{D}}(t) \approx \mathcal{D}(t) = \sum_{|\gamma|\le 2} (\|\partial_x^\gamma \widetilde{\rho}_x\|^2 + \|\partial_x^\gamma \widetilde{p}_x\|^2) + \sum_{|\gamma|+|\beta|\le 3} \|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2. \tag{1.24}$$

The first main result on global existence is given in the following theorem.

THEOREM 1.1. *If $\mathcal{E}(0) < \varepsilon$ for some $\varepsilon > 0$ small enough, then the Cauchy problem of Kac equation* (1.1) *has a unique global solution $F(t,x,v) \ge 0$ which satisfies for any $t > 0$*

$$\mathcal{E}(t) + C\int_0^t \mathcal{D}(s)ds \le C\mathcal{E}(0). \tag{1.25}$$

Next we study the large time behavior of the global solution obtained in Theorem 1.1. For this, define the perturbation $f(t,x,v)$ by $F = \mu + \sqrt{\mu}f$. Then (1.1) for $f(t,x,v)$ is

$$\partial_t f + v\partial_x f + \mathcal{L}f = \Gamma(f,f). \tag{1.26}$$

Here $\mathcal{L}f$ and $\Gamma(f,f)$ are given in (1.18). The second result on the time decay rates is given as follows.

THEOREM 1.2. *Suppose that $F(t,x,v)$ is the global solution of Kac equation* (1.1) *obtained in Theorem* 1.1. *Assume further that $F = \mu + \sqrt{\mu}f$ and $\|f_0\|_{Z^1} < \varepsilon$ for some small $\varepsilon > 0$. Then global solution $f(t,x,v)$ has the following decay estimates:*

$$\|f\|^2 = \|\mathbf{P}_0 f\|^2 + \|\mathbf{P}_1 f\|^2 \le C\varepsilon(1+t)^{-\frac{1}{2}}, \tag{1.27}$$

$$\sum_{1\leq|\gamma|\leq 3}\|\partial_x^\gamma \mathbf{P}_0 f\|^2+\sum_{|\gamma|+|\beta|\leq 3}\|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1 f\|^2\leq C\varepsilon(1+t)^{-\frac{3}{2}}. \tag{1.28}$$

*If* $\mathbf{P}_0 f_0=0$*, then*

$$\|f\|^2=\|\mathbf{P}_0 f\|^2+\|\mathbf{P}_1 f\|^2\leq C\varepsilon(1+t)^{-\frac{3}{2}}, \tag{1.29}$$

$$\sum_{1\leq|\gamma|\leq 3}\|\partial_x^\gamma \mathbf{P}_0 f\|^2+\sum_{|\gamma|+|\beta|\leq 3}\|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1 f\|^2\leq C\varepsilon(1+t)^{-\frac{5}{2}}. \tag{1.30}$$

*Furthermore, if we assume that there exist some constants* $k_0>0$ *and* $d_0>0$ *such that* $\inf_{|k|\leq k_0}|\langle \hat{f}_0,\psi_j(v)\rangle|\geq d_0$ *for* $j=0,\,2$*, then for* $t>0$ *large enough,*

$$C_1(1+t)^{-\frac{1}{4}}\leq\|f\|\leq C_2(1+t)^{-\frac{1}{4}}. \tag{1.31}$$

Now let us review some works related to the study in this paper. In 1956, Kac proposed the Kac equation, with $B(\theta)$ being a constant to model the one-dimensional Boltzmann equation in [24], and Mckean studied large time behavior of global solution to the space homogeneous Kac equation in [33]. Some detail on the physics background can be found in [45] and the references therein. Using the method of moments, Desvillettes proved a partial result of convergence to the Maxwellian equilibrium for the space homogeneous Kac equation in [12]. In 1999, Carlen, Gabetta, and Toscani proved the exponential convergence to equilibrium in some Sobolev norms in [10]. Desvillettes later introduced the non-cutoff assumption as (1.5) and (1.6) for the Kac equation and studied the existence and the regularizing properties for the solution of the spatially homogeneous non-cutoff Kac equation in [13]. On the other hand, Graham and Méléard obtained a stochastic representation of the solution to the non-cutoff Kac equation in [21] as the density of a Poisson-driven nonlinear stochastic differential equation. Fournier proved the strict positivity of the solutions to the non-cutoff Kac equation [17, 18]. Toscani studied the grazing collision limit of the non-cutoff Kac equation in [42], and the similar limit of the inelastic Kac equation was also studied in [20]. Fournier and Godinho also studied the asymptotics of grazing collision and particle approximation for the non-cutoff Kac equation in [19].

The propagation of Gevrey regularity of the solution to the spatially homogeneous non-cutoff Boltzmann and Kac equations was proved in [15]. Global solution and time decay rate of the spatially inhomogeneous cutoff Kac equation around a global Maxwellian were obtained in [36]. The asymptotic preserving scheme for the spatially inhomogeneous cutoff Kac equation in the diffusion limit was studied in [7] by Bennoune, Lemou, and Mieussens. Recently, in [28] Lerner et al. constructed the local existence of the solution to the Cauchy problem for the spatially inhomogeneous non-cutoff Kac equation (1.1) around a global Maxwellian and then proved that this local solution enjoys the Gelfand–Shilov regularizing properties with respect to the velocity variable $v$ and the Gevrey regularizing properties in the position variable $x$. In addition to the works mentioned above, there are other related works; cf. [4, 5, 14, 15, 27, 29, 43, 45] and references therein.

Recently, there appeared a series of important works about the existence, smoothing effect, and large time behavior of global solutions to the three-dimensional inhomogeneous non-cutoff Boltzmann equation around a global Maxwellian; cf. [1, 2, 3, 22, 37, 39]. Motivated by these works, in this paper we first prove the existence of a global solution to the one-dimensional inhomogeneous non-cutoff Kac equation near a global Maxwellian. We also analyze the spectrum structure of the linearized equation precisely. This precise spectrum structure of the linearized equation is then used to prove the time decay rate of global solution to the one-dimensional nonlinear Kac equation.

In physics, it has been justified by the H-theorem that the solution to the Boltzmann equation converges to an equilibrium in the form of a Maxwellian. In mathematics, the H-theorem yields the dissipation estimate of the linearized collision operator on the microscopic component of the solution, while the dissipation on the macroscopic component comes from the viscous and heat conducting effects in higher order of the Knudsen number.

By using the classical macro-micro decomposition around the global Maxwellian, the linearized collision operator of the Kac collision operator has an intrinsic structure like that of a fractional harmonic operator. Hence, its expansion in Hermite basis has a precise formula with an asymptotic estimate on the eigenvalues; cf. [28]. However, unlike the Boltzmann equation in three space dimensions, the classical macro-micro decomposition in one space dimension does not capture the pointwise behavior of the macroscopic component, and thus its derivation from the state in time at infinity does not yield a higher order estimate in the nonlinear coupling through the collision operator. For this, we use the macro-micro decomposition around the local Maxwellian (cf. [31, 30]) to overcome this difficulty.

Note that the Sobolev imbedding theorem depends on dimension; the energy method developed in [23, 47, 1, 2, 3, 22] in three dimensions cannot be applied to the one-dimensional non-cutoff Kac equation (1.1). For example, the nonlinear term $\Gamma(\mathbf{P}_0 f, \mathbf{P}_0 f)$ does not vanish by (4.18) and (4.21), and the term $(\Gamma(\mathbf{P}_0 f, \mathbf{P}_0 f), \mathbf{P}_1 f)$ in (4.22) cannot be controlled by $\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t)$ in one dimension, even though it holds in three dimensions. To overcome this, motivated by the decomposition introduced in [30, 31] in which the macroscopic component is the local Maxwellian $M$, we combine the decomposition $F = M + \sqrt{\mu} g$ and the classical decomposition $F = \mu + \sqrt{\mu} f$ to use the precise structure of the linearized and nonlinear operators.

In addition, in estimating the velocity derivative of the solution, due to the loss of the conservation of momentum for the Kac equation, it would be difficult to apply the weighted derivatives directly as for the Boltzmann equation; cf. [1, 2, 3, 22]. By observing the properties of the linearized operator $\mathcal{L}$, we make use of the integer order harmonic oscillator operator $\mathcal{H}_v^{\frac{\beta}{2}}$ on the solution so that the solution is in the Hermite–Sobolev space $\mathcal{H}^3(\mathbb{R})$ in the velocity variable. Some properties of this space can be found in [6, 9, 35, 38, 40, 41].

Finally, due to the slow decay rate of $t^{-1/4}$ of the linearized equation in one dimension, direct energy estimates [47, 2, 37, 39] as for the Boltzmann equation in three dimensions cannot be applied for the optimal time decay estimates for the nonlinear problem. For this, we first study the spectrum structure of its corresponding linearized operator by adopting the method used in [46] for the Boltzmann equation without angular cutoff.

The rest of the paper will be arranged as follows. In the next section, we will perform the energy estimates by combining two sets of macro-micro decomposition. The spectrum structure will be given in section 3, and the time decay estimates will be given in section 4. In the appendix, we will recall the properties of the linearized Kac operator without cutoff obtained in [28], give an estimate of the harmonic oscillator operator in the fractional order Hermite–Sobolev space $\mathcal{H}^s(\mathbb{R})$ with $s \in (0, 1)$, and give some technical estimates of the linearized and nonlinear operators.

**2. Energy estimates.** In this section, we will establish a uniform energy estimate for the global existence of the Cauchy problem (1.1). For this, we will derive some energy estimates of macroscopic component $(\rho, p)$ and microscopic component $g$. Thus, we will reformulate the system (1.21) in terms of the perturbation $(\widetilde{\rho}, \widetilde{p})$. To obtain energy estimates of macroscopic component $(\rho, p)$, we first reformulate the system for the macroscopic component $(\rho, p)$. To simplify the terms involving $\mathcal{L}^{-1}$ in (1.21), we have from (5.2) that

$$\begin{cases} \psi_0(v) = 2^{-\frac14}(\sqrt{\pi})^{-\frac12}e^{-\frac{v^2}{4}}, & \psi_1(v) = 2^{-\frac14}(2\sqrt{\pi})^{-\frac12}e^{-\frac{v^2}{4}}\cdot\sqrt{2}v, \\ \psi_2(v)=2^{-\frac14}(8\sqrt{\pi})^{-\frac12}e^{-\frac{v^2}{4}}\cdot(2v^2-2), & \psi_3(v)=2^{-\frac14}(48\sqrt{\pi})^{-\frac12}e^{-\frac{v^2}{4}}\cdot(2\sqrt{2}v^3-6\sqrt{2}v). \end{cases} \tag{2.1}$$

By using (1.19), (2.1), and (5.9), we deduce

$$\mathcal{L}\psi_1 = \lambda_1\psi_1 \Leftrightarrow \mathcal{L}^{-1}(v\sqrt{\mu}) = \frac{v\sqrt{\mu}}{\lambda_1}. \tag{2.2}$$

Similarly, we can also obtain

$$\mathcal{L}\psi_3=\lambda_3\psi_3 \Leftrightarrow \mathcal{L}(v^3\sqrt{\mu}-3v\sqrt{\mu})=\lambda_3(v^3\sqrt{\mu}-3v\sqrt{\mu}) \Leftrightarrow \mathcal{L}^{-1}(v^3\sqrt{\mu})=3d_1(v\sqrt{\mu})+d_2(v^3\sqrt{\mu}). \tag{2.3}$$

Here, we have used the notation

$$d_1 = \frac{1}{\lambda_1} - \frac{1}{\lambda_3} > 0, \qquad d_2 = \frac{1}{\lambda_3} > 0, \tag{2.4}$$

by using (5.12). It follows from (1.9) that

$$\partial_x M = \Big(\frac{\partial_x\rho}{\rho} - \frac{\partial_x\vartheta}{2\vartheta}\Big)M + \frac{\partial_x\vartheta}{2\vartheta^2}v^2M. \tag{2.5}$$

Noting that $\mathcal{L}^{-1}$ is self-adjoint and $(\frac{v\partial_x M}{\sqrt{\mu}}) \in \mathcal{N}^{\perp}$, we have from (2.3) and (2.5) that

$$\begin{aligned} \int_{\mathbb{R}} v^3\sqrt{\mu}\mathcal{L}^{-1}\Big(\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}\Big)dv &= \int_{\mathbb{R}} \mathcal{L}^{-1}(v^3\sqrt{\mu})\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}dv \\ &= \Big(\frac{\partial_x\rho}{\rho} - \frac{\partial_x\vartheta}{2\vartheta}\Big)\Big(3d_1\int_{\mathbb{R}} v^2Mdv + d_2\int_{\mathbb{R}} v^4Mdv\Big) \\ &\quad + \frac{\partial_x\vartheta}{2\vartheta^2}\Big(3d_1\int_{\mathbb{R}} v^4Mdv + d_2\int_{\mathbb{R}} v^6Mdv\Big). \end{aligned} \tag{2.6}$$

It holds that

$$\int_{\mathbb{R}} v^2Mdv = \rho\vartheta, \quad \int_{\mathbb{R}} v^4Mdv = 3\rho\vartheta^2, \quad \int_{\mathbb{R}} v^6Mdv = 15\rho\vartheta^3.$$

Thus, we have that

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{L}^{-1}(v^3\sqrt{\mu})\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}dv &= \Big(\frac{\partial_x\rho}{\rho} - \frac{\partial_x\vartheta}{2\vartheta}\Big)(3d_1\rho\vartheta + 3d_2\rho\vartheta^2) + \frac{\partial_x\vartheta}{2\vartheta^2}(9d_1\rho\vartheta^2 + 15d_2\rho\vartheta^3) \\ &= \vartheta\partial_x\rho(3d_1 + 3d_2\vartheta) + \rho\partial_x\vartheta(3d_1 + 6d_2\vartheta) \\ &= \partial_x(\rho\vartheta)(3d_1 + 3d_2\vartheta) + 3d_2\rho\vartheta\partial_x\vartheta. \end{aligned} \tag{2.7}$$

By using the fact that $p = \rho\vartheta$, one has from (2.7) that

$$\int_{\mathbb{R}} \mathcal{L}^{-1}(v^3\sqrt{\mu})\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}dv = \Big(3d_1 + \frac{6d_2 p}{\rho}\Big)\partial_x p - \frac{3d_2p^2}{\rho^2}\partial_x\rho. \tag{2.8}$$

Hence, $(1.21)_2$ can be written as

$$\partial_t p - \partial_x\Big(\Big(3d_1 + \frac{6d_2p}{\rho}\Big)\partial_x p\Big) + \partial_x\Big(\frac{3d_2p^2}{\rho^2}\partial_x\rho\Big) = -\partial_x\int_{\mathbb{R}} v^3\sqrt{\mu}\mathcal{L}^{-1}\Theta dv. \tag{2.9}$$

Similarly, we also have from (2.2) that

$$\begin{aligned}\int_{\mathbb{R}} v\sqrt{\mu}\mathcal{L}^{-1}\Big(\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}\Big)dv &= \int_{\mathbb{R}}\mathcal{L}^{-1}(v\sqrt{\mu})\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}dv\\ &= \frac{1}{\lambda_1}\int_{\mathbb{R}}(v\sqrt{\mu})\mathbf{P}_1\frac{v\partial_x M}{\sqrt{\mu}}dv\\ &= \frac{1}{\lambda_1}\Big(\frac{\partial_x\rho}{\rho} - \frac{\partial_x\vartheta}{2\vartheta}\Big)\int_{\mathbb{R}} v^2 M dv + \frac{1}{\lambda_1}\frac{\partial_x\vartheta}{2\vartheta^2}\int_{\mathbb{R}} v^4 M dv\\ &= \frac{1}{\lambda_1}\Big(\vartheta\partial_x\rho - \frac{\rho\partial_x\vartheta}{2}\Big) + \frac{3}{2\lambda_1}\rho\partial_x\vartheta = \frac{1}{\lambda_1}(\rho\vartheta)_x.\end{aligned} \tag{2.10}$$

Thus, one has from $(1.21)_1$ that

$$\partial_t\rho - \frac{1}{\lambda_1}\partial_{xx}p = -\partial_x\int_{\mathbb{R}} v\sqrt{\mu}\mathcal{L}^{-1}\Theta dv. \tag{2.11}$$

It follows from (2.9) and (2.11) that

$$\begin{cases}\partial_t\rho - \dfrac{1}{\lambda_1}\partial_{xx}p = -\partial_x\displaystyle\int_{\mathbb{R}} v\sqrt{\mu}\mathcal{L}^{-1}\Theta dv,\\ \partial_t p - \partial_x\Big(\Big(3d_1 + \dfrac{6d_2p}{\rho}\Big)\partial_x p\Big) + \partial_x\Big(\dfrac{3d_2p^2}{\rho^2}\partial_x\rho\Big) = -\partial_x\displaystyle\int_{\mathbb{R}} v^3\sqrt{\mu}\mathcal{L}^{-1}\Theta dv.\end{cases} \tag{2.12}$$

By using the perturbation variables

$$\widetilde{\rho} = \rho - 1, \qquad \widetilde{p} = p - 1,$$

the macroscopic system for $(\widetilde{\rho}, \widetilde{p})$ becomes

$$\begin{cases}\partial_t\widetilde{\rho} - \dfrac{1}{\lambda_1}\partial_{xx}\widetilde{p} = Q_1,\\ \partial_t\widetilde{p} - (3d_1 + 6d_2)\partial_{xx}\widetilde{p} + 3d_2\partial_{xx}\widetilde{\rho} = Q_2.\end{cases} \tag{2.13}$$

Here $\lambda_1$, $d_1$, and $d_2$ are positive constants defined in (2.2) and (2.4), and the source terms $Q_1$ and $Q_2$ are given by

$$Q_1 = -\partial_x\int_{\mathbb{R}} v\sqrt{\mu}\mathcal{L}^{-1}\Theta dv, \tag{2.14}$$

$$Q_2 = 6d_2\partial_x\Big(\Big(\frac{p}{\rho} - 1\Big)\partial_x\widetilde{p}\Big) - 3d_2\partial_x\Big(\Big(\frac{p^2}{\rho^2} - 1\Big)\partial_x\widetilde{\rho}\Big) - \partial_x\int_{\mathbb{R}} v^3\sqrt{\mu}\mathcal{L}^{-1}\Theta dv. \tag{2.15}$$

Before we perform energy estimates on the macroscopic component, we first consider the estimate on nonlinear term $\Gamma(g, g)$.

LEMMA 2.1. *Let $F = M + \sqrt{\mu} g$ and $|\gamma| + |\beta| \le 3$. It holds that, for any $\epsilon > 0$,*

$$|(\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} \Gamma(g,g), h)| \le \epsilon \|\mathcal{H}_v^{\frac{s}{2}} h\|^2 + C_\epsilon \mathcal{E}(t)\mathcal{D}(t). \tag{2.16}$$

*Proof.* By using Lemma 5.3, one has

$$\begin{aligned}|(\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} \Gamma(g,g), h)| &= \Big| \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma_1} (\mathcal{H}_v^{\frac{\beta}{2}} \Gamma(\partial_x^{\gamma_1} g, \partial_x^{\gamma-\gamma_1} g), h) \Big| \\ &\le \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma_1} \int_{\mathbb{R}_x} |\partial_x^{\gamma_1} g|_2 |\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{\beta+s}{2}} g|_2 |\mathcal{H}_v^{\frac{s}{2}} h|_2 dx \\ &\quad + \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma_1} \int_{\mathbb{R}_x} |\partial_x^{\gamma_1} \mathcal{H}_v^{\frac{\beta}{2}} g|_2 |\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{s}{2}} g|_2 |\mathcal{H}_v^{\frac{s}{2}} h|_2 dx. \end{aligned} \tag{2.17}$$

We only consider the first term in (2.17) because the second term can be estimated similarly. If $|\gamma - \gamma_1| + |\beta| \le 1$, for any $\epsilon > 0$, we get

$$\begin{aligned}\int_{\mathbb{R}_x} |\partial_x^{\gamma_1} g|_2 |\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{\beta+s}{2}} g|_2 |\mathcal{H}_v^{\frac{s}{2}} h|_2 dx &\le \epsilon \|\mathcal{H}_v^{\frac{s}{2}} h\|^2 + C_\epsilon \sup_{x\in\mathbb{R}} |\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{\beta+s}{2}} g|_2^2 \|\partial_x^{\gamma_1} g\|^2 \\ &\le \epsilon \|\mathcal{H}_v^{\frac{s}{2}} h\|^2 + C_\epsilon \left( \sum_{|\gamma_2|\le 1} \|\partial_x^{\gamma-\gamma_1+\gamma_2} \mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 \right) \|\partial_x^{\gamma_1} g\|^2.\end{aligned}$$

Here, we have used the fact that $L^\infty(\mathbb{R}) \subset H^1(\mathbb{R})$. If $|\gamma - \gamma_1| + |\beta| \ge 2$, for any $\epsilon > 0$, we also get

$$\begin{aligned}\int_{\mathbb{R}_x} |\partial_x^{\gamma_1} g|_2 |\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{\beta+s}{2}} g|_2 |\mathcal{H}_v^{\frac{s}{2}} h|_2 dx &\le \epsilon \|\mathcal{H}_v^{\frac{s}{2}} h\|^2 + C_\epsilon \sup_{x\in\mathbb{R}} |\partial_x^{\gamma_1} g|_2^2 \|\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 \\ &\le \epsilon \|\mathcal{H}_v^{\frac{s}{2}} h\|^2 + C_\epsilon \left( \sum_{|\gamma_2|\le 1} \|\partial_x^{\gamma_1+\gamma_2} g\|^2 \right) \|\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2.\end{aligned}$$

The above two estimates are both bounded by the right-hand side of (2.16). Since the second term in (2.17) can be estimated similarly, we complete the proof of this lemma. ∎

The next lemma is for the estimate on the linear terms $\Gamma(\frac{M-\mu}{\sqrt{\mu}}, g)$ and $\Gamma(\frac{M-\mu}{\sqrt{\mu}}, g)$.

LEMMA 2.2. *If $|\gamma| + |\beta| \le 3$ and $\mathcal{E}(t) < \varepsilon$ for some small $\varepsilon > 0$, we have that, for any $\epsilon > 0$,*

$$\begin{aligned}&\Big| \Big( \partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big), h \Big) \Big| + \Big| \Big( \partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} \Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big), h \Big) \Big| \\ &\le C(\epsilon + \sqrt{\varepsilon}) \|\mathcal{H}_v^{\frac{s}{2}} h\|^2 + C(\epsilon + \sqrt{\varepsilon}) \|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 + C_\epsilon \mathcal{E}(t)\mathcal{D}(t).\end{aligned} \tag{2.18}$$

*Proof.* By using Lemma 5.3, we can obtain

$$\begin{aligned}&\Big| \Big( \partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big), h \Big) \Big| = \Big| \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma_1} \Big( \mathcal{H}_v^{\frac{\beta}{2}} \Gamma\Big( \frac{\partial_x^{\gamma_1}(M-\mu)}{\sqrt{\mu}}, \partial_x^{\gamma-\gamma_1} g \Big), h \Big) \Big| \\ &\le \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma_1} \int_{\mathbb{R}_x} \Big| \frac{\partial_x^{\gamma_1}(M-\mu)}{\sqrt{\mu}} \Big|_2 |\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{\beta+s}{2}} g|_2 |\mathcal{H}_v^{\frac{s}{2}} h|_2 dx \\ &\quad + \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma_1} \int_{\mathbb{R}_x} \Big| \mathcal{H}_v^{\frac{\beta}{2}} \Big( \frac{\partial_x^{\gamma_1}(M-\mu)}{\sqrt{\mu}} \Big) \Big|_2 |\partial_x^{\gamma-\gamma_1} \mathcal{H}_v^{\frac{s}{2}} g|_2 |\mathcal{H}_v^{\frac{s}{2}} h|_2 dx.\end{aligned} \tag{2.19}$$

By using the imbedding theorem and the fact that $\mathcal{E}(t) < \varepsilon$ for some small $\varepsilon > 0$, one has that, for any $|\gamma| \leq 2$,

$$|\partial^\gamma(\rho - 1)|^2 + |\partial^\gamma(\vartheta - 1)|^2 \leq C\varepsilon. \tag{2.20}$$

Note that $M = M_{[\rho,\vartheta]}(v)$ and $\mu = M_{[1,1]}(v)$. We have from (2.20) that, for any $|\beta| \geq 0$, there exists $R > 0$ small enough such that

$$\int_{|v|\geq R} \frac{\langle v\rangle^{2m_0}|\partial_v^\beta(M-\mu)|^2}{\mu}dv \leq C\varepsilon. \tag{2.21}$$

Here $m_0 > 0$ large enough. We have from (2.20) and the mean value theorem that

$$\int_{|v|\leq R} \frac{\langle v\rangle^{2m_0}|\partial_v^\beta(M-\mu)|^2}{\mu}dv \leq C\varepsilon. \tag{2.22}$$

One has from (5.3), (2.21), and (2.22) that

$$\sum_{|\beta|\leq 3}\Big|\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{(M-\mu)}{\sqrt{\mu}}\Big)\Big|_2 \leq C\sum_{|\beta|\leq 3}\Big|\frac{\langle v\rangle^{m_0}\partial_v^\beta(M-\mu)}{\sqrt{\mu}}\Big|_2 \leq C\sqrt{\varepsilon}. \tag{2.23}$$

If $|\gamma_1| = 0$, we have from (2.19), (2.23), and (5.17) that

$$\Big(\mathcal{H}_v^{\frac{\beta}{2}}\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, \partial_x^\gamma g\Big), h\Big) \leq C\sqrt{\varepsilon}\|\mathcal{H}_v^{\frac{s}{2}}h\|^2 + C\sqrt{\varepsilon}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}}g\|^2.$$

If $1 \leq |\gamma_1| \leq |\gamma|$, one has

$$\partial_x^{\gamma_1}M = \Big(\frac{\partial_x^{\gamma_1}\rho}{\rho} - \frac{\partial_x^{\gamma_1}\vartheta}{2\vartheta}\Big)M + \frac{\partial_x^{\gamma_1}\vartheta}{2\vartheta^2}v^2M + O(1)(|\partial_x^{\gamma_1-1}\rho||\partial_x\rho| + |\partial_x^{\gamma_1-1}p||\partial_xp| + \cdots)\zeta(v)M. \tag{2.24}$$

Here $\zeta(v)$ is a polynomial of $v$, and when $|\gamma_1| = 1$, the last term vanishes.

By using (5.16), (5.3), (2.23), and the exponential decay of $M$ and $\mu$ in $v$, for any $a > 0$, we can obtain

$$\Big|\mathcal{H}_v^{\frac{a}{2}}\sqrt{\mu}\Big|_2 \leq C, \quad \Big|\mathcal{H}_v^{\frac{a}{2}}\Big(\frac{\langle v\rangle^{m_0}(M-\mu)}{\sqrt{\mu}}\Big)\Big|_2 \leq C\sqrt{\varepsilon}. \tag{2.25}$$

If $\mathcal{E}(t) < \varepsilon$ for some $\varepsilon > 0$ small enough, we have that

$$\Big|\frac{\partial_x^{\gamma_1}(M-\mu)}{\sqrt{\mu}}\Big|_2 + \Big|\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{\partial_x^{\gamma_1}(M-\mu)}{\sqrt{\mu}}\Big)\Big|_2 \leq C\sum_{1\leq|\gamma_2|\leq|\gamma_1|}(|\partial_x^{\gamma_2}\rho| + |\partial_x^{\gamma_2}p|).$$

For the second term in (2.19), if $|\gamma_1| + |\beta| \leq 1$ and $1 \leq |\gamma_1| \leq |\gamma|$, for any $\epsilon > 0$, one has

$$\begin{aligned}
&\int_{\mathbb{R}_x}|\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{\partial_x^{\gamma_1}(M-\mu)}{\sqrt{\mu}}\Big)|_2|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{s}{2}}g|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2dx\\
&\quad\leq C\sum_{1\leq|\gamma_2|\leq|\gamma_1|}\int_{\mathbb{R}_x}(|\partial_x^{\gamma_2}\rho| + |\partial_x^{\gamma_2}p|)|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{s}{2}}g|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2dx\\
&\quad\leq \epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2 + C_\epsilon\sum_{1\leq|\gamma_2|\leq|\gamma_1|}\sup_{x\in\mathbb{R}}(|\partial_x^{\gamma_2}\rho|^2 + |\partial_x^{\gamma_2}p|^2)\|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{s}{2}}g\|^2\\
&\quad\leq \epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2 + C_\epsilon\sum_{1\leq|\gamma_2|\leq|\gamma_1|}\sum_{|\gamma_3|\leq 1}(\|\partial_x^{\gamma_2+\gamma_3}\rho\|^2 + \|\partial_x^{\gamma_2+\gamma_3}p\|^2)\|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{s}{2}}g\|^2.
\end{aligned}$$

If $|\gamma_1|+|\beta|\geq 2$ and $1\leq|\gamma_1|\leq|\gamma|$, for any $\epsilon>0$, similarly, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_x}\Big|\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{\partial_x^{\gamma_1}(M-\mu)}{\sqrt{\mu}}\Big)\Big|_2|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{s}{2}}g|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2dx\\
&\quad\leq C\sum_{1\leq|\gamma_2|\leq|\gamma_1|}\int_{\mathbb{R}_x}(|\partial_x^{\gamma_2}\rho|+|\partial_x^{\gamma_2}p|)|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{s}{2}}g|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2dx\\
&\quad\leq \epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2+C_\epsilon\sum_{1\leq|\gamma_2|\leq|\gamma_1|}(\|\partial_x^{\gamma_2}\rho\|^2+\|\partial_x^{\gamma_2}p\|^2)\sum_{|\gamma_3|\leq 1}\|\partial_x^{\gamma-\gamma_1+\gamma_3}\mathcal{H}_v^{\frac{s}{2}}g\|^2.
\end{aligned}$$

The above two estimates are both bounded by the right-hand side of (2.18). By a similar argument, we can handle the first part of (2.19). The second term on the left-hand side of (2.18) can be treated similarly, so we omit the details for brevity. Thus, this completes the proof of this lemma. ∎

The terms involving $Q_1$ and $Q_2$ in (2.14) and (2.15) will be handled in the following lemma.

Lemma 2.3. *If $|\gamma|\leq 2$ and $\mathcal{E}(t)<\varepsilon$ for some small $\varepsilon>0$, there exist functions $\xi_1(v)$ and $\xi_2(v)$, which are exponential decay in $v$, such that for any $\epsilon>0$,*

$$\begin{aligned}
&\frac{d}{dt}[(\partial_x^\gamma g_x,\partial_x^\gamma\widetilde{\rho}\xi_1(v))\\
&\qquad+(\partial_x^\gamma g_x,\partial_x^\gamma\widetilde{p}\xi_2(v))]+(\partial_x^\gamma Q_2,\partial_x^\gamma\widetilde{p})+3d_2\lambda_1(\partial_x^\gamma Q_1,\partial_x^\gamma\widetilde{\rho})\\
&\qquad-\kappa(\partial_x^\gamma Q_1,\partial_x^\gamma\widetilde{p})-\kappa(\partial_x^\gamma Q_2,\partial_x^\gamma\widetilde{\rho})\\
&\quad\leq C_\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}g_x\|^2+C(\epsilon+\sqrt{\varepsilon})(\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}g\|^2+\|\partial_x^\gamma\widetilde{p}_x\|^2+\|\partial_x^\gamma\widetilde{\rho}_x\|^2)+C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
\end{aligned}$$

*Proof.* We first estimate the term $(\partial_x^\gamma Q_2,\partial_x^\gamma\widetilde{p})$. Recalling (2.15), we have that

$$\begin{aligned}
(\partial_x^\gamma Q_2,\partial_x^\gamma\widetilde{p})&=6d_2\Big(\partial_x\partial_x^\gamma\Big(\Big(\frac{p}{\rho}-1\Big)\partial_x\widetilde{p}\Big),\partial_x^\gamma\widetilde{p}\Big)\\
&\quad-3d_2\Big(\partial_x\partial_x^\gamma\Big(\Big(\frac{p^2}{\rho^2}-1\Big)\partial_x\widetilde{\rho}\Big),\partial_x^\gamma\widetilde{p}\Big)-\Big(\partial_x\partial_x^\gamma\int_{\mathbb{R}}v^3\sqrt{\mu}\mathcal{L}^{-1}\Theta dv,\partial_x^\gamma\widetilde{p}\Big).
\end{aligned}\tag{2.26}$$

For the first term in (2.27), one has that

$$\begin{aligned}
\Big(\partial_x\partial_x^\gamma\Big(\Big(\frac{p}{\rho}-1\Big)\partial_x\widetilde{p}\Big),\partial_x^\gamma\widetilde{p}\Big)&=-\Big(\partial_x^\gamma\Big(\frac{\widetilde{p}-\widetilde{\rho}}{\rho}\partial_x\widetilde{p}\Big),\partial_x^\gamma\widetilde{p}_x\Big)\\
&=-\sum_{\gamma_1\leq\gamma}C_\gamma^{\gamma_1}\Big(\partial_x^{\gamma_1}\Big(\frac{\widetilde{p}-\widetilde{\rho}}{\rho}\Big)\partial_x^{\gamma-\gamma_1}\widetilde{p}_x,\partial_x^\gamma\widetilde{p}_x\Big).
\end{aligned}$$

If $|\gamma_1|\leq\frac{|\gamma|}{2}$, it holds that

$$\begin{aligned}
&\Big|\Big(\partial_x^{\gamma_1}\Big(\frac{\widetilde{p}-\widetilde{\rho}}{\rho}\Big)\partial_x^{\gamma-\gamma_1}\widetilde{p}_x,\partial_x^\gamma\widetilde{p}_x\Big)\Big|\\
&\quad\leq C\Big\|\partial_x^{\gamma_1}\Big(\frac{\widetilde{p}-\widetilde{\rho}}{\rho}\Big)\Big\|_{L^\infty(\mathbb{R}_x)}\|\partial_x^{\gamma-\gamma_1}\widetilde{p}_x\|\|\partial_x^\gamma\widetilde{p}_x\|\leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
\end{aligned}$$

Here, we have used $\mathcal{E}(t)<\varepsilon$ and the imbedding theorem that $L^\infty(\mathbb{R})\subset H^1(\mathbb{R})$.

If $|\gamma-\gamma_1|<\frac{|\gamma|}{2}$, it follows that

$$\begin{aligned}
&\Big|\Big(\partial_x^{\gamma_1}\Big(\frac{\widetilde{p}-\widetilde{\rho}}{\rho}\Big)\partial_x^{\gamma-\gamma_1}\widetilde{p}_x,\partial_x^\gamma\widetilde{p}_x\Big)\Big|\\
&\quad\leq C\|\partial_x^{\gamma-\gamma_1}\widetilde{p}_x\|_{L^\infty(\mathbb{R}_x)}\Big\|\partial_x^{\gamma_1}\Big(\frac{\widetilde{p}-\widetilde{\rho}}{\rho}\Big)\Big\|\|\partial_x^\gamma\widetilde{p}_x\|\leq C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
\end{aligned}$$

We thus have the estimate of the first term in (2.27) as

$$\Big|\Big(\partial_x\partial_x^\gamma\Big(\Big(\frac{p}{\rho}-1\Big)\partial_x\widetilde{p}\Big),\partial_x^\gamma\widetilde{p}\Big)\Big|\le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.27}$$

Similarly, we have the following estimate for the second term in (2.27):

$$\Big|\Big(\partial_x\partial_x^\gamma\Big(\Big(\frac{p^2}{\rho^2}-1\Big)\partial_x\widetilde{\rho}\Big),\partial_x^\gamma\widetilde{p}\Big)\Big|\le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.28}$$

For the last term, one has

$$-\Big(\partial_x\partial_x^\gamma\int_{\mathbb{R}}v^3\sqrt{\mu}\mathcal{L}^{-1}\Theta dv,\partial_x^\gamma\widetilde{p}\Big)=-\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x\partial_x^\gamma\Theta,\partial_x^\gamma\widetilde{p}\Big). \tag{2.29}$$

It follows from (1.20) that

$$\Theta=-\partial_t g-\mathbf{P}_1 v\partial_x g+\Gamma(g,g)+\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},g\Big)+\Gamma\Big(g,\frac{M-\mu}{\sqrt{\mu}}\Big)+\mathbf{P}_1\frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}. \tag{2.30}$$

One has from (1.15) and (2.3) that

$$\begin{aligned}
-\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma\partial_t g_x,\partial_x^\gamma\widetilde{p}\Big)&=-\frac{d}{dt}\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma g_x,\partial_x^\gamma\widetilde{p}\Big)-\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma g_x,\partial_x^\gamma\widetilde{p}_t\Big)\\
&=-\frac{d}{dt}\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma g_x,\partial_x^\gamma\widetilde{p}\Big)+\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma g_x,\int_{\mathbb{R}}v\sqrt{\mu}\partial_x^\gamma g_x dv\Big)\\
&\le-\frac{d}{dt}\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma g_x,\partial_x^\gamma\widetilde{p}\Big)+C\|\partial_x^\gamma g_x\|^2.
\end{aligned} \tag{2.31}$$

It follows from (2.3) and (5.7) that

$$\begin{aligned}
&\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x\partial_x^\gamma\mathbf{P}_1 v\partial_x g,\partial_x^\gamma\widetilde{p}\Big)\\
&\quad=-\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma\mathbf{P}_1 v\partial_x g,\partial_x^\gamma\widetilde{p}_x\Big)\le C_\epsilon\|\partial_x^\gamma g_x\|^2+\epsilon\|\partial_x^\gamma\widetilde{p}_x\|^2.
\end{aligned} \tag{2.32}$$

By using (2.3), Lemma 2.1, and the estimate (2.25), one has that

$$\begin{aligned}
&\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x\partial_x^\gamma\Gamma(g,g),\partial_x^\gamma\widetilde{p}\Big)=-\Big(\partial_x^\gamma\Gamma(g,g),\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma\widetilde{p}_x\Big)\\
&\le C_\epsilon\mathcal{E}(t)\mathcal{D}(t)+\epsilon\|\partial_x^\gamma\widetilde{p}_x\mathcal{H}_v^{\frac{s}{2}}\mathcal{L}^{-1}(v^3\sqrt{\mu})\|^2\le C_\epsilon\mathcal{E}(t)\mathcal{D}(t)+C\epsilon\|\partial_x^\gamma\widetilde{p}_x\|^2.
\end{aligned} \tag{2.33}$$

It follows from (2.3), Lemma 2.2, and the estimate (2.25) that

$$\begin{aligned}
\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x\partial_x^\gamma\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},g\Big),\partial_x^\gamma\widetilde{p}\Big)&=-\Big(\partial_x^\gamma\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},g\Big),\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^\gamma\widetilde{p}_x\Big)\\
&\le C(\epsilon+\sqrt{\varepsilon})\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}g\|^2+C(\epsilon+\sqrt{\varepsilon})\|\partial_x^\gamma\widetilde{p}_x\|^2+C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
\end{aligned} \tag{2.34}$$

Similarly, we have the following estimate on the fifth term in (2.30):

$$\begin{aligned}
&\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x\partial_x^\gamma\Gamma\Big(g,\frac{M-\mu}{\sqrt{\mu}}\Big),\partial_x^\gamma\widetilde{p}\Big)\\
&\qquad\le C(\epsilon+\sqrt{\varepsilon})\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}g\|^2+C(\epsilon+\sqrt{\varepsilon})\|\partial_x^\gamma\widetilde{p}_x\|^2+C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
\end{aligned} \tag{2.35}$$

Noting that $\mathcal{L}^{-1}(v^3\sqrt{\mu}) \in \mathcal{N}^{\perp}$, we have from (2.3) that

$$\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x\partial_x^{\gamma}\mathbf{P}_1\frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}, \partial_x^{\gamma}\widetilde{p}\Big) = -\Big(\frac{\partial_x^{\gamma}(P_0 v\sqrt{\mu}\partial_x g)}{\sqrt{\mu}}, \mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^{\gamma}\widetilde{p}_x\Big)$$
$$= -\Big(\partial_x^{\gamma}(P_0 v\sqrt{\mu}\partial_x g)(3d_1 v + d_2 v^3), \partial_x^{\gamma}\widetilde{p}_x\Big). \tag{2.36}$$

Suppose that $|\gamma| \le 2$. It follows from (1.11) and (1.12) that

$$\partial_x^{\gamma}(P_0 v\sqrt{\mu}\partial_x g) = \sum_{j=0,2}\sum_{\gamma_0+\gamma_1+\gamma_2=\gamma}\int_{\mathbb{R}} v\sqrt{\mu}(\partial_x\partial_x^{\gamma_0}g)\partial_x^{\gamma_1}\Big(\frac{\chi_j}{M}\Big)dv(\partial_x^{\gamma_2}\chi_j). \tag{2.37}$$

By this and the imbedding theorem, we can obtain

$$\|\partial_x^{\gamma}(P_0 v\sqrt{\mu}\partial_x g)(3d_1 v + d_2 v^3)\|^2 \le C\|\partial_x^{\gamma}g_x\|^2 + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.38}$$

With the help of (2.36), (2.37), and (2.38), we deduce that

$$\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x\partial_x^{\gamma}\mathbf{P}_1\frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}, \partial_x^{\gamma}\widetilde{p}\Big) \le \epsilon\|\partial_x^{\gamma}\widetilde{p}_x\|^2 + C_{\epsilon}\|\partial_x^{\gamma}g_x\|^2 + C_{\epsilon}\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.39}$$

Collecting the above estimates, by using (5.17), we can obtain

$$-\Big(\partial_x\partial_x^{\gamma}\int_{\mathbb{R}} v^3\sqrt{\mu}\mathcal{L}^{-1}\Theta dv, \partial_x^{\gamma}\widetilde{p}\Big) \le -\frac{d}{dt}\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^{\gamma}g_x, \partial_x^{\gamma}\widetilde{p}\Big)$$
$$+ C_{\epsilon}\|\partial_x^{\gamma}\mathcal{H}_v^{\frac{s}{2}}g_x\|^2 + C(\epsilon+\sqrt{\varepsilon})(\|\partial_x^{\gamma}\mathcal{H}_v^{\frac{s}{2}}g\|^2 + \|\partial_x^{\gamma}\widetilde{p}_x\|^2) + C_{\epsilon}\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.40}$$

By using (2.27)–(2.29) and (2.41), we can obtain

$$(\partial_x^{\gamma}Q_2, \partial_x^{\gamma}\widetilde{p}) \le -\frac{d}{dt}\Big(\mathcal{L}^{-1}(v^3\sqrt{\mu})\partial_x^{\gamma}g_x, \partial_x^{\gamma}\widetilde{p}\Big)$$
$$+C_{\epsilon}\|\partial_x^{\gamma}\mathcal{H}_v^{\frac{s}{2}}g_x\|^2 + C(\epsilon+\sqrt{\varepsilon})(\|\partial_x^{\gamma}\mathcal{H}_v^{\frac{s}{2}}g\|^2 + \|\partial_x^{\gamma}\widetilde{p}_x\|^2) + C_{\epsilon}\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.41}$$

By (2.2) and (2.3), the functions $\mathcal{L}^{-1}(v^3\sqrt{\mu})$ and $\mathcal{L}^{-1}(v\sqrt{\mu})$ are exponential decay in $v$. By (2.41) and an argument similar to (2.41), we can prove that the estimate in this lemma holds. □

Now we perform energy analysis on the macroscopic component. It follows from $(2.13)_2$ that

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^{\gamma}\widetilde{p}\|^2 + (3d_1+6d_2)\|\partial_x^{\gamma}\widetilde{p}_x\|^2 - 3d_2(\partial_x^{\gamma}\widetilde{\rho}_x, \partial_x^{\gamma}\widetilde{p}_x) = (\partial_x^{\gamma}Q_2, \partial_x^{\gamma}\widetilde{p}).$$

One has from $(2.13)_1$ that

$$\frac{3d_2\lambda_1}{2}\frac{d}{dt}\|\partial_x^{\gamma}\widetilde{\rho}\|^2 + 3d_2(\partial_x^{\gamma}\widetilde{p}_x, \partial_x^{\gamma}\widetilde{\rho}_x) = 3d_2\lambda_1(\partial_x^{\gamma}Q_1, \partial_x^{\gamma}\widetilde{\rho}).$$

Combining the above two inequalities, we can obtain

$$\frac{d}{dt}\Big(\frac{3d_2\lambda_1}{2}\|\partial_x^{\gamma}\widetilde{\rho}\|^2 + \frac{1}{2}\|\partial_x^{\gamma}\widetilde{p}\|^2\Big) + (3d_1+6d_2)\|\partial_x^{\gamma}\widetilde{p}_x\|^2 = (\partial_x^{\gamma}Q_2, \partial_x^{\gamma}\widetilde{p}) + 3d_2\lambda_1(\partial_x^{\gamma}Q_1, \partial_x^{\gamma}\widetilde{\rho}). \tag{2.42}$$

It follows from $(2.13)_2$ that

$$3d_2\|\partial_x^{\gamma}\widetilde{\rho}_x\|^2 + (\partial_x^{\gamma}Q_2, \partial_x^{\gamma}\widetilde{\rho}) = (\partial_t\partial_x^{\gamma}\widetilde{p}, \partial_x^{\gamma}\widetilde{\rho}) + (3d_1+6d_2)(\partial_x^{\gamma}\widetilde{p}_x, \partial_x^{\gamma}\widetilde{\rho}_x).$$

One has from $(2.13)_1$ that

$$(\partial_x^\gamma Q_1, \partial_x^\gamma \widetilde{p}) = (\partial_t \partial_x^\gamma \widetilde{\rho}, \partial_x^\gamma \widetilde{p}) + \frac{1}{\lambda_1}\|\partial_x^\gamma \widetilde{p}_x\|^2.$$

By using the above two equations, for any constants $\kappa > 0$ and $\epsilon_1 > 0$, we obtain

$$\begin{aligned} 3d_2\kappa\|\partial_x^\gamma \widetilde{\rho}_x\|^2 &= \kappa\frac{d}{dt}(\partial_x^\gamma \widetilde{\rho}, \partial_x^\gamma \widetilde{p}) + \frac{\kappa}{\lambda_1}\|\partial_x^\gamma \widetilde{p}_x\|^2 + \kappa(3d_1+6d_2)(\partial_x^\gamma \widetilde{p}_x, \partial_x^\gamma \widetilde{\rho}_x) \\ &\quad -\kappa(\partial_x^\gamma Q_1, \partial_x^\gamma \widetilde{p}) - \kappa(\partial_x^\gamma Q_2, \partial_x^\gamma \widetilde{\rho}) \\ &\le \kappa\frac{d}{dt}(\partial_x^\gamma \widetilde{\rho}, \partial_x^\gamma \widetilde{p}) + C_{\epsilon_1}\kappa\|\partial_x^\gamma \widetilde{p}_x\|^2 + C\kappa\epsilon_1\|\partial_x^\gamma \widetilde{\rho}_x\|^2 \\ &\quad -\kappa(\partial_x^\gamma Q_1, \partial_x^\gamma \widetilde{p}) - \kappa(\partial_x^\gamma Q_2, \partial_x^\gamma \widetilde{\rho}). \end{aligned} \tag{2.43}$$

It follows from (2.42) and (2.43) that

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{3d_2\lambda_1}{2}\|\partial_x^\gamma \widetilde{\rho}\|^2 + \frac{1}{2}\|\partial_x^\gamma \widetilde{p}\|^2 - \kappa(\partial_x^\gamma \widetilde{\rho}, \partial_x^\gamma \widetilde{p})\Big) + (3d_1+6d_2)\|\partial_x^\gamma \widetilde{p}_x\|^2 + 3d_2\kappa\|\partial_x^\gamma \widetilde{\rho}_x\|^2 \\ &\quad \le C_{\epsilon_1}\kappa\|\partial_x^\gamma \widetilde{p}_x\|^2 + C\kappa\epsilon_1\|\partial_x^\gamma \widetilde{\rho}_x\|^2 + (\partial_x^\gamma Q_2, \partial_x^\gamma \widetilde{p}) + 3d_2\lambda_1(\partial_x^\gamma Q_1, \partial_x^\gamma \widetilde{\rho} \\ &\qquad -\kappa(\partial_x^\gamma Q_1, \partial_x^\gamma \widetilde{p}) - \kappa(\partial_x^\gamma Q_2, \partial_x^\gamma \widetilde{\rho}). \end{aligned}$$

By choosing $\epsilon_1 > 0$ and $\kappa > 0$ to be small enough, we can obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{3d_2\lambda_1}{2}\|\partial_x^\gamma \widetilde{\rho}\|^2 + \frac{1}{2}\|\partial_x^\gamma \widetilde{p}\|^2 - \kappa(\partial_x^\gamma \widetilde{\rho}, \partial_x^\gamma \widetilde{p})\Big) + C_1\|\partial_x^\gamma \widetilde{p}_x\|^2 + C_2\kappa\|\partial_x^\gamma \widetilde{\rho}_x\|^2 \\ &\quad \le (\partial_x^\gamma Q_2, \partial_x^\gamma \widetilde{p}) + 3d_2\lambda_1(\partial_x^\gamma Q_1, \partial_x^\gamma \widetilde{\rho}) - \kappa(\partial_x^\gamma Q_1, \partial_x^\gamma \widetilde{p}) - \kappa(\partial_x^\gamma Q_2, \partial_x^\gamma \widetilde{\rho}). \end{aligned}$$

By using Lemma 2.3, we have that

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{3d_2\lambda_1}{2}\|\partial_x^\gamma \widetilde{\rho}\|^2 + \frac{1}{2}\|\partial_x^\gamma \widetilde{p}\|^2 - \kappa(\partial_x^\gamma \widetilde{\rho}, \partial_x^\gamma \widetilde{p}) + (\partial_x^\gamma g_x, \partial_x^\gamma \widetilde{\rho}\xi_1(v)) + (\partial_x^\gamma g_x, \partial_x^\gamma \widetilde{p}\xi_2(v))\Big) \\ &\quad +C_1\|\partial_x^\gamma \widetilde{p}_x\|^2 + C_2\kappa\|\partial_x^\gamma \widetilde{\rho}_x\|^2 \le C_\epsilon\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g_x\|^2 + C(\epsilon+\sqrt{\varepsilon})(\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 \\ &\qquad +\|\partial_x^\gamma \widetilde{p}_x\|^2 + \|\partial_x^\gamma \widetilde{\rho}_x\|^2) + C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \end{aligned}$$

If we further choose $\epsilon > 0$ small enough, we can obtain

$$\begin{aligned} &\sum_{|\gamma|\le 2}\frac{d}{dt}\Big(\frac{3d_2\lambda_1}{2}\|\partial_x^\gamma \widetilde{\rho}\|^2 + \frac{1}{2}\|\partial_x^\gamma \widetilde{p}\|^2 - \kappa(\partial_x^\gamma \widetilde{\rho}, \partial_x^\gamma \widetilde{p}) + (\partial_x^\gamma g_x, \partial_x^\gamma \widetilde{\rho}\xi_1(v)) + (\partial_x^\gamma g_x, \partial_x^\gamma \widetilde{p}\xi_2(v))\Big) \\ &+C\sum_{|\gamma|\le 2}[\|\partial_x^\gamma \widetilde{p}_x\|^2 + \|\partial_x^\gamma \widetilde{\rho}_x\|^2] \le C_\epsilon\sum_{|\gamma|\le 2}\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g_x\|^2 + C\epsilon\|\mathcal{H}_v^{\frac{s}{2}} g\|^2 + C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t), \end{aligned} \tag{2.44}$$

where $\kappa > 0$ small enough.

Next we estimate the spatial derivative of the microscopic component. By applying $\partial_x^\gamma$ with $|\gamma| \le 2$ to (1.17) and integrating its product with $\partial_x^\gamma g$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_x^\gamma g\|^2 + (\mathcal{L}\partial_x^\gamma g, \partial_x^\gamma g) = (\partial_x^\gamma \Gamma(g,g), \partial_x^\gamma g) + \Big(\partial_x^\gamma \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big), \partial_x^\gamma g\Big) \\ &\quad +\Big(\partial_x^\gamma \Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big), \partial_x^\gamma g\Big) - \Big(\frac{v\partial_x\partial_x^\gamma M}{\sqrt{\mu}}, \partial_x^\gamma g\Big) + \Big(\partial_x^\gamma\Big(\frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}\Big), \partial_x^\gamma g\Big). \end{aligned} \tag{2.45}$$

We now estimate (2.45) term by term. Noting that $g \in \mathcal{N}^{\perp}$, for the second term on the left side of (2.45), we have from (1.19) that

$$(\mathcal{L}\partial_x^\gamma g, \partial_x^\gamma g) \geq \delta\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2.$$

It follows from Lemma 2.1 and 2.2 that

$$|(\partial_x^\gamma \Gamma(g,g), \partial_x^\gamma g)| \leq \epsilon\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\| + C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t)$$

and

$$\Big|\Big(\partial_x^\gamma \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big) + \partial_x^\gamma \Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big), \partial_x^\gamma g\Big)\Big| \leq C(\epsilon+\sqrt{\varepsilon})\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 + C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

Note that, for $|\gamma_0| \geq 1$,

$$\begin{aligned}\partial_x^{\gamma_0} M = \Big(\frac{\partial_x^{\gamma_0}\rho}{\rho} - \frac{\partial_x^{\gamma_0}\vartheta}{2\vartheta}\Big)M + \frac{\partial_x^{\gamma_0}\vartheta}{2\vartheta^2}v^2 M + O(1)(|\partial_x^{\gamma_0-1}\rho||\partial_x\rho| \\ +|\partial_x^{\gamma_0-1}p||\partial_x p| + \cdots)\zeta(v)M := I_1 + I_2 + I_3.\end{aligned} \tag{2.46}$$

Here $\zeta(v)$ is a polynomial in $v$, and if $|\gamma_0| = 1$, the last term vanishes. By Hölder's inequality and the imbedding theorem, one has from (2.46) that

$$\Big|\Big(\frac{v\partial_x\partial_x^\gamma M}{\sqrt{\mu}}, \partial_x^\gamma g\Big)| \leq C_\epsilon(\|\partial_x^\gamma \rho_x\|^2 + \|\partial_x^\gamma p_x\|^2) + \epsilon\|\partial_x^\gamma g\|^2 + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

By (2.37) and (2.38), an argument similar to (2.39) implies that

$$\Big|\Big(\partial_x^\gamma\Big(\frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}\Big), \partial_x^\gamma g\Big)\Big| \leq C_\epsilon\|\partial_x^\gamma g_x\|^2 + \epsilon\|\partial_x^\gamma g\|^2 + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

Thus, we have from (5.17), (2.45), and the above estimates that

$$\begin{aligned}&\frac{d}{dt}\Big[\sum_{|\gamma|\leq 2} C_\gamma\|\partial_x^\gamma g\|^2\Big] + \delta_1\sum_{|\gamma|\leq 2}\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 \\ &\leq C\sum_{|\gamma|\leq 2}(\|\partial_x^\gamma \widetilde{\rho}_x\|^2 + \|\partial_x^\gamma \widetilde{p}_x\|^2) + C\sum_{|\gamma|=2}\|\partial_x^\gamma g_x\|^2 + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).\end{aligned} \tag{2.47}$$

Next, we consider the high order spatial derivative of the microscopic component. Recalling $F = M + \sqrt{\mu}g$, rewrite (1.1) as

$$\partial_t\Big(\frac{F}{\sqrt{\mu}}\Big) + v\partial_x\Big(\frac{F}{\sqrt{\mu}}\Big) + \mathcal{L}g = \Gamma(g,g) + \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big) + \Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big). \tag{2.48}$$

By applying $\partial_x^\gamma$ with $1 \leq |\gamma| \leq 3$ to (2.48) and integrating its product with $\frac{\partial_x^\gamma F}{\sqrt{\mu}}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big\|^2 + (\mathcal{L}\partial_x^\gamma g, \partial_x^\gamma g) + \Big(\mathcal{L}\partial_x^\gamma g, \frac{\partial_x^\gamma M}{\sqrt{\mu}}\Big) \\ &= \Big(\partial_x^\gamma\Gamma(g,g), \frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big) + \Big(\partial_x^\gamma\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big) + \partial_x^\gamma\Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big), \frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big).\end{aligned} \tag{2.49}$$

We now estimate (2.49) term by term. It follows from (1.19) that

$$(\mathcal{L}\partial_x^\gamma g, \partial_x^\gamma g) \geq \delta \|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2. \tag{2.50}$$

Recalling (2.46), we have that

$$\Big(\mathcal{L}\partial_x^\gamma g, \frac{\partial_x^\gamma \vartheta}{2\vartheta^2}\frac{v^2 M}{\sqrt{\mu}}\Big) = (\mathcal{L}\partial_x^\gamma g, \frac{\partial_x^\gamma \vartheta}{2\vartheta^2} v^2\sqrt{\mu}) + \Big(\mathcal{L}\partial_x^\gamma g, \frac{\partial_x^\gamma \vartheta}{2\vartheta^2}\frac{v^2(M-\mu)}{\sqrt{\mu}}\Big). \tag{2.51}$$

Due to the fact that $v^2\sqrt{\mu} \in \mathcal{N}$, the first part of (2.51) vanishes. Since $\mathcal{L}g = -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu})$, we have from (2.25), (5.17), and Lemma 5.3 that

$$\begin{aligned}\Big(\mathcal{L}\partial_x^\gamma g, \frac{\partial_x^\gamma \vartheta}{2\vartheta^2}\frac{v^2(M-\mu)}{\sqrt{\mu}}\Big) &\leq C\int_{\mathbb{R}} \Big(\big|\sqrt{\mu}\big|_2 \big|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\big|_2 + \big|\partial_x^\gamma g\big|_2 \big|\mathcal{H}_v^{\frac{s}{2}}\sqrt{\mu}\big|_2\Big) \\ &\qquad \times \Big|\frac{\partial_x^\gamma \vartheta}{2\vartheta^2}\mathcal{H}_v^{\frac{s}{2}}\Big(\frac{v^2(M-\mu)}{\sqrt{\mu}}\Big)\Big|_2 dx \\ &\leq C\sqrt{\varepsilon}\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 + C\sqrt{\varepsilon}\|\partial_x^\gamma \vartheta\|^2.\end{aligned}$$

One has from this and (2.51) that

$$\Big(\mathcal{L}\partial_x^\gamma g, \frac{\partial_x^\gamma \vartheta}{2\vartheta^2}\frac{v^2 M}{\sqrt{\mu}}\Big) \leq C\sqrt{\varepsilon}\|\partial_x^\gamma \widetilde{\rho}\|^2 + C\sqrt{\varepsilon}\|\partial_x^\gamma \widetilde{p}\|^2 + C\sqrt{\varepsilon}\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2. \tag{2.52}$$

By an argument similar to (2.52), the first term in (2.46) shares the same estimates as (2.52). The last term is controlled by $C\sqrt{\mathcal{E}(t)}\mathcal{D}(t)$ by using $\mathcal{E}(t) < \varepsilon$ and the imbedding theorem. Thus, we can obtain

$$\Big(\mathcal{L}\partial_x^\gamma g, \frac{\partial_x^\gamma M}{\sqrt{\mu}}\Big) \leq C\sqrt{\varepsilon}\|\partial_x^\gamma \widetilde{\rho}\|^2 + C\sqrt{\varepsilon}\|\partial_x^\gamma \widetilde{p}\|^2 + C\sqrt{\varepsilon}\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t). \tag{2.53}$$

It follows from Lemma 2.1 that

$$\Big(\partial_x^\gamma \Gamma(g,g), \frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big) \leq C_\epsilon \sqrt{\mathcal{E}(t)}\mathcal{D}(t) + C\epsilon\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 + C\epsilon\|\mathcal{H}_v^{\frac{s}{2}}\Big(\frac{\partial_x^\gamma M}{\sqrt{\mu}}\Big)\|^2.$$

One has from Lemma 2.2 that

$$\begin{aligned}&\Big(\partial_x^\gamma \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big) + \partial_x^\gamma \Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big), \frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big) \\ &\leq C_\epsilon \sqrt{\mathcal{E}(t)}\mathcal{D}(t) + C(\epsilon + \sqrt{\varepsilon})\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 + C(\epsilon + \sqrt{\varepsilon})\|\mathcal{H}_v^{\frac{s}{2}}\Big(\frac{\partial_x^\gamma M}{\sqrt{\mu}}\Big)\|^2.\end{aligned}$$

By using (2.20), (2.46), and (5.16), we can obtain

$$\|\mathcal{H}_v^{\frac{s}{2}}\Big(\frac{\partial_x^\gamma M}{\sqrt{\mu}}\Big)\|^2 \leq C\|\partial_x^\gamma \widetilde{\rho}\|^2 + C\|\partial_x^\gamma \widetilde{p}\|^2 + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

The right-hand side of (2.49) can be bounded by

$$\begin{aligned}&\Big(\partial_x^\gamma \Gamma(g,g), \frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big) + \Big(\partial_x^\gamma \Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big) + \partial_x^\gamma \Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big), \frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big) \\ &\leq C_\epsilon \sqrt{\mathcal{E}(t)}\mathcal{D}(t) + C(\epsilon + \sqrt{\varepsilon})\|\partial_x^\gamma \mathcal{H}_v^{\frac{s}{2}} g\|^2 + C(\epsilon + \sqrt{\varepsilon})(\|\partial_x^\gamma \widetilde{\rho}\|^2 + \|\partial_x^\gamma \widetilde{p}\|^2).\end{aligned}$$

By using this, (2.50), and (2.53), if we choose $\epsilon > 0$ small enough, we have from (2.49) that

(2.54)
$$\sum_{1\le|\gamma|\le 3}\Big[\frac{d}{dt}\|\frac{\partial_x^\gamma F}{\sqrt{\mu}}\|^2+C\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}g\|^2\Big]\le C(\epsilon+\sqrt{\varepsilon})\sum_{1\le|\gamma|\le 3}(\|\partial_x^\gamma\widetilde{\rho}\|^2+\|\partial_x^\gamma\widetilde{p}\|^2)+C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t).$$

Next, we estimate the velocity derivatives of the microscopic component $g$. Taking $\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}$ with $|\gamma|+|\beta|\le 3$ ( $1\le|\beta|\le 3$ ) over (1.17) and multiplying the resulting equation by $\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g$, and then integrating the resulting equality over $\mathbb{R}^2$, one can get

(2.55)
$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g\|^2+(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(v\partial_xg),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g)+(\mathcal{H}_v^{\frac{\beta}{2}}\mathcal{L}\partial_x^\gamma g,\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g)\\&=(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(g,g),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g)+\Big(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}},g\Big)+\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma\Big(g,\frac{M-\mu}{\sqrt{\mu}}\Big),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g\Big)\\&\qquad-\Big(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{v\partial_xM}{\sqrt{\mu}}\Big),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g\Big)+\Big(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{P_0v\sqrt{\mu}\partial_xg}{\sqrt{\mu}}\Big),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}g\Big).\end{aligned}$$

It requires a more careful estimation, rather than the standard weight energy estimate on the second term on the left-hand side in (2.56), because the operator $\mathcal{H}_v^{\frac{\beta}{2}}$ involves the mixing of the $|\beta|$ order velocity operator and a velocity weight. Note that, for $|\beta|=1$,

$$(\mathcal{H}_v^{\frac{1}{2}}(v\partial_xg),\mathcal{H}_v^{\frac{1}{2}}g)=(\mathcal{H}_v^1(v\partial_xg),g)=\Big(\Big\{-\Delta_v+\frac{|v|^2}{4}\Big\}(v\partial_xg),g\Big)=(\partial_xg,\partial_vg)\tag{2.56}$$

and

$$\|\mathcal{H}_v^{\frac{1}{2}}g\|^2=\Big(\Big\{-\Delta_v+\frac{|v|^2}{4}\Big\}g,g\Big)=\|\partial_vg\|^2+\frac{1}{4}\|vg\|^2.\tag{2.57}$$

This implies that

$$|(\partial_x^\gamma\mathcal{H}_v^{\frac{1}{2}}(v\partial_xg),\partial_x^\gamma\mathcal{H}_v^{\frac{1}{2}}g)|=|(\partial_x\partial_x^\gamma g,\partial_v\partial_x^\gamma g)|\le C_\epsilon\|\partial_x\partial_x^\gamma g\|^2+C\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{1}{2}}g\|^2.\tag{2.58}$$

For $|\beta|=2$, one has

$$\mathcal{H}_v^1(v\partial_xg)=-\partial_{vv}(v\partial_xg)+v\partial_x\Big(\frac{|v|^2}{4}g\Big)=-2\partial_{xv}g+v\partial_x\mathcal{H}_v^1g.\tag{2.59}$$

By this and (2.57) we can obtain

$$\begin{aligned}|(\partial_x^\gamma\mathcal{H}_v^1(v\partial_xg),\partial_x^\gamma\mathcal{H}_v^1g)|&=|(-2\partial_{xv}\partial_x^\gamma g+v\partial_x\partial_x^\gamma\mathcal{H}_v^1g,\partial_x^\gamma\mathcal{H}_v^1g)|\\&=|(-2\partial_{xv}\partial_x^\gamma g,\partial_x^\gamma\mathcal{H}_v^1g)|\le C_\epsilon\|\partial_x\partial_x^\gamma\mathcal{H}_v^{\frac{1}{2}}g\|^2+C\epsilon\|\partial_x^\gamma\mathcal{H}_v^1g\|^2.\end{aligned}\tag{2.60}$$

For $|\beta|=3$, one has from (2.59) that

$$(\mathcal{H}_v^{\frac{3}{2}}(v\partial_xg),\mathcal{H}_v^{\frac{3}{2}}g)=(\mathcal{H}_v^{\frac{1}{2}}(\mathcal{H}_v^1(v\partial_xg)),\mathcal{H}_v^{\frac{3}{2}}g)=(\mathcal{H}_v^{\frac{1}{2}}(-2\partial_{xv}g+v\partial_x\mathcal{H}_v^1g),\mathcal{H}_v^{\frac{3}{2}}g).$$

It follows from (2.56) and (2.57) that

$$\begin{aligned}|(\mathcal{H}_v^{\frac{1}{2}}(v\partial_x\mathcal{H}_v^1g),\mathcal{H}_v^{\frac{1}{2}}(\mathcal{H}_v^1g))|&=|(\partial_x\mathcal{H}_v^1g,\partial_v\mathcal{H}_v^1g)|\\&\le\epsilon\|\partial_v\mathcal{H}_v^1g\|^2+C_\epsilon\|\partial_x\mathcal{H}_v^1g\|^2\le\epsilon\|\mathcal{H}_v^{\frac{3}{2}}g\|^2+C_\epsilon\|\partial_x\mathcal{H}_v^1g\|^2.\end{aligned}$$

One has from (2.57) that

$$
\begin{aligned}
|(\mathcal{H}_v^{\frac{1}{2}}(-2\partial_{xv}g), \mathcal{H}_v^{\frac{1}{2}}(\mathcal{H}_v^1 g))| = 2|(\mathcal{H}_v^1\partial_v g, \partial_x\mathcal{H}_v^1 g)| &= 2\Big|\Big(\partial_v\mathcal{H}_v^1 g - \frac{v}{2}g, \partial_x\mathcal{H}_v^1 g\Big)\Big| \\
&\le C\epsilon\|\partial_v\mathcal{H}_v^1 g\|^2 + C\epsilon\|vg\|^2 + C_\epsilon\|\partial_x\mathcal{H}_v^1 g\|^2 \\
&\le C\epsilon\|\mathcal{H}_v^{\frac{3}{2}} g\|^2 + C\epsilon\|\mathcal{H}_v^{\frac{1}{2}} g\|^2 + C_\epsilon\|\partial_x\mathcal{H}_v^1 g\|^2.
\end{aligned}
$$

Thus, for $|\beta| = 3$, we have from (5.16) and the above relations that

$$
|(\mathcal{H}_v^{\frac{3}{2}}(v\partial_x g), \mathcal{H}_v^{\frac{3}{2}} g)| \le C\epsilon\|\mathcal{H}_v^{\frac{3}{2}} g\|^2 + C_\epsilon\|\partial_x\mathcal{H}_v^1 g\|^2. \tag{2.61}
$$

With the help of (2.58), (2.60), and (2.61), if $|\gamma| + |\beta| \le 3$ and $1 \le |\beta| \le 3$, we deduce

$$
|(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(v\partial_x g), \partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g)| \le C_\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta-1}{2}} g_x\|^2 + C\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\|^2. \tag{2.62}
$$

For the third term on the left-hand side, one has from Lemma 5.2 that

$$
(\mathcal{H}_v^{\frac{\beta}{2}}\mathcal{L}\partial_x^\gamma g, \partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g) \ge \delta_1\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 - C\|\partial_x^\gamma g\|^2.
$$

It follows from Lemma 2.1 that

$$
(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(g,g), \partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g) \le C\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 + C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
$$

We have from Lemma 2.2 that

$$
\begin{aligned}
&\Big(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma\Big(\frac{M-\mu}{\sqrt{\mu}}, g\Big) + \partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma\Big(g, \frac{M-\mu}{\sqrt{\mu}}\Big), \partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\Big) \\
&\qquad \le C(\epsilon + \sqrt{\varepsilon})\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 + C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
\end{aligned}
$$

If $|\gamma| + |\beta| \le 3$ and $1 \le |\beta| \le 3$, we deduce from (2.20), (2.25), and (2.46) that

$$
\Big(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{v\partial_x M}{\sqrt{\mu}}\Big), \partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\Big) \le C\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\|^2 + C_\epsilon(\|\partial_x^\gamma\widetilde{\rho}_x\|^2 + \|\partial_x^\gamma\widetilde{p}_x\|^2) + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
$$

By (2.37) and (2.38), an argument similar to (2.39) implies that

$$
\Big|\Big(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{P_0 v\sqrt{\mu}\partial_x g}{\sqrt{\mu}}\Big), \partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\Big)\Big| \le C_\epsilon\|\partial_x^\gamma g_x\|^2 + \epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\|^2 + C\sqrt{\mathcal{E}(t)}\mathcal{D}(t).
$$

Thus, we have from (5.17), (2.56), and the above estimates that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\|^2 + \delta_1'\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 &\le C_\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta-1}{2}} g_x\|^2 + C\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 + C_\epsilon\sqrt{\mathcal{E}(t)}\mathcal{D}(t) \\
&\quad + C_\epsilon(\|\partial_x^\gamma\widetilde{\rho}_x\|^2 + \|\partial_x^\gamma\widetilde{p}_x\|^2 + \|\partial_x^\gamma g\|^2 + \|\partial_x^\gamma g_x\|^2).
\end{aligned} \tag{2.63}
$$

For any $1 \le |\beta| \le 3$, for any $\epsilon > 0$ small enough, the summation of (2.63) over $|\gamma| + |\beta| \le 3$ by a suitable linear combination gives

$$
\begin{aligned}
&\frac{d}{dt}\Bigg[\sum_{1\le|\beta|,|\gamma|+|\beta|\le 3} C_\beta\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\|^2\Bigg] + \delta_2'\sum_{1\le|\beta|,|\gamma|+|\beta|\le 3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}} g\|^2 \\
&\le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t) + C\sum_{|\gamma|\le 2}(\|\partial_x^\gamma\widetilde{\rho}_x\|^2 + \|\partial_x^\gamma\widetilde{p}_x\|^2 + \|\partial_x^\gamma g\|^2 + \|\partial_x^\gamma g_x\|^2).
\end{aligned} \tag{2.64}
$$

After we obtain the estimates on the solution, we will make use of these estimates to prove the existence of global solution to (1.1).

*Proof of Theorem* 1.1. By choosing $0 < \widetilde{C}_1 \ll \widetilde{C}_2 \ll \widetilde{C}_3$ with $\widetilde{C}_1$, $\widetilde{C}_2$, and $\widetilde{C}_3$ large enough, we define a functional

$$\begin{aligned}\widetilde{\mathcal{E}}(t) = \widetilde{C}_1 \sum_{|\gamma|\le 2} \Big(&\frac{3d_2\lambda_1}{2}\|\partial_x^\gamma \widetilde{\rho}\|^2 + \frac{1}{2}\|\partial_x^\gamma \widetilde{p}\|^2 - \kappa(\partial_x^\gamma\widetilde{\rho}, \partial_x^\gamma \widetilde{p})\\ &+ (\xi_1(v)\partial_x^\gamma g_x, \partial_x^\gamma\widetilde{\rho}) + (\xi_2(v)\partial_x^\gamma g_x, \partial_x^\gamma \widetilde{p})\Big)\\ &+ \sum_{|\gamma|\le 2} C_\gamma \|\partial_x^\gamma g\|^2 + \widetilde{C}_2 \sum_{1\le|\gamma|\le 3} \|\frac{\partial_x^\gamma F}{\sqrt{\mu}}\|^2 + \frac{1}{\widetilde{C}_3}\sum_{1\le|\beta|, |\gamma|+|\beta|\le 3} C_\beta \|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} g\|^2.\end{aligned} \tag{2.65}$$

We also define $\widetilde{\mathcal{D}}(t)$ as (1.24). If we further choose $\epsilon > 0$ small enough, we deduce from a suitable linear combination of (2.45), (2.47), (2.54), and (2.64) that

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \widetilde{\mathcal{D}}(t) \le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t) \le C\sqrt{\mathcal{E}(t)}\widetilde{\mathcal{D}}(t). \tag{2.66}$$

If $\mathcal{E}(t) < \varepsilon$, we choose $\kappa > 0$ and $\varepsilon > 0$ small enough, and we claim that

$$\widetilde{\mathcal{E}}(t) \approx \mathcal{E}(t) = \sum_{|\gamma|\le 3} (\|\partial_x^\gamma\widetilde{\rho}\|^2 + \|\partial_x^\gamma \widetilde{p}\|^2) + \sum_{|\gamma|+|\beta|\le 3} \|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} g\|^2. \tag{2.67}$$

The relation (2.67) can be proved as follows. Recalling $F = M + \sqrt{\mu} g$, one has that, for $1 \le |\gamma| \le 3$,

$$\Big\|\frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big\|^2 = \Big\|\frac{\partial_x^\gamma M}{\sqrt{\mu}}\Big\|^2 + \|\partial_x^\gamma g\|^2 + 2\Big(\frac{\partial_x^\gamma M}{\mu}, \sqrt{\mu}\partial_x^\gamma g\Big). \tag{2.68}$$

We first consider the third term on the right-hand side in (2.68). Note that

$$\Big(\frac{\partial_x^\gamma M}{\mu}, \sqrt{\mu}\partial_x^\gamma g\Big) = \Big(\frac{\partial_x^\gamma M}{M}, \sqrt{\mu}\partial_x^\gamma g\Big) + \Big(\partial_x^\gamma M\Big(\frac{1}{\mu} - \frac{1}{M}\Big), \sqrt{\mu}\partial_x^\gamma g\Big).$$

Recalling (2.46), for the first two terms in (2.46), we have that $(\frac{I_1+I_2}{M}, \sqrt{\mu}\partial_x^\gamma g) = 0$. Note that $I_3 = 0$ in (2.46) if $|\gamma| = 1$. Thus for $2 \le |\gamma| \le 3$, by using the imbedding theorem and the fact that $\mathcal{E}(t) < \varepsilon$, one has

$$\begin{aligned}\Big|\Big(\frac{I_3}{M}, \sqrt{\mu}\partial_x^\gamma g\Big)\Big| &\le C\|\partial_x^\gamma g\|\Big(\sup_{x\in\mathbb{R}}|\partial_x\rho|\|\partial_x^{\gamma-1}\rho\| + \sup_{x\in\mathbb{R}}|\partial_x p|\|\partial_x^{\gamma-1}p\| + \cdots\Big)\\ &\le C\sqrt{\varepsilon}\|\partial_x^\gamma g\|^2 + C\sqrt{\varepsilon}\sum_{1\le|\gamma_1|\le|\gamma|-1}(\|\partial_x^{\gamma_1}\rho\|^2 + \|\partial_x^{\gamma_1}p\|^2).\end{aligned}$$

Thus, we can obtain

$$\Big|\Big(\frac{\partial_x^\gamma M}{M}, \sqrt{\mu}\partial_x^\gamma g\Big)| \le C\sqrt{\varepsilon}\|\partial_x^\gamma g\|^2 + C\sqrt{\varepsilon}\sum_{1\le|\gamma_1|\le|\gamma|-1}(\|\partial_x^{\gamma_1}\rho\|^2 + \|\partial_x^{\gamma_1}p\|^2).$$

For some $m_0 > 0$ large enough, one has from (2.20), (2.23), and (2.46) that

$$\begin{aligned}\Big|(\partial_x^\gamma M\Big(\frac{1}{\mu} - \frac{1}{M}\Big), \sqrt{\mu}\partial_x^\gamma g)\Big| &\le C\|\partial_x^\gamma g\|\Big(\int_{\mathbb{R}_x}\sup_{v\in\mathbb{R}}\frac{\langle v\rangle^{-m_0}|\partial_x^\gamma M|^2}{M^2}\int_{\mathbb{R}}\langle v\rangle^{m_0}\frac{|M-\mu|^2}{\mu}dvdx\Big)^{1/2}\\ &\le C\sqrt{\varepsilon}\|\partial_x^\gamma g\|^2 + C\sqrt{\varepsilon}\sum_{1\le|\gamma_1|\le|\gamma|}(\|\partial_x^{\gamma_1}\widetilde{\rho}\|^2 + \|\partial_x^{\gamma_1}\widetilde{p}\|^2).\end{aligned}$$

Here, we have also used the imbedding theorem and the fact that $\mathcal{E}(t) < \varepsilon$.

By the above two estimates, the third term on the right-hand side in (2.68) can be estimated by

$$2\Big|\Big(\frac{\partial_x^\gamma M}{\mu}, \sqrt{\mu}\partial_x^\gamma g\Big)\Big| \leq C\sqrt{\varepsilon}\|\partial_x^\gamma g\|_2^2 + C\sqrt{\varepsilon}\sum_{1\leq|\gamma_1|\leq|\gamma|}(\|\partial_x^{\gamma_1}\rho\|^2 + \|\partial_x^{\gamma_1}p\|^2). \tag{2.69}$$

Now we estimate the first term of the right-hand side in (2.68). Setting $|\gamma| = 1$, we have from (2.46) and (2.20) that

$$\Big|\frac{(\frac{\partial_x^\gamma\rho}{\rho} - \frac{\partial_x^\gamma\vartheta}{2\vartheta})M + \frac{\partial_x^\gamma\vartheta}{2\vartheta^2}v^2M}{\sqrt{M}}\Big|_2^2 = \frac{(\partial_x^\gamma\rho)^2}{\rho} + \frac{\rho(\partial_x^\gamma\vartheta)^2}{2\vartheta^2} \approx (\partial_x^\gamma\widetilde{\rho})^2 + (\partial_x^\gamma\widetilde{p})^2.$$

One has from (2.20) and (2.46) that for $2 \leq |\gamma| \leq 3$,

$$\Big|\frac{(|\partial_x^{\gamma-1}\rho||\partial_x\rho| + |\partial_x^{\gamma-1}\vartheta||\partial_x\vartheta| + \cdots)\zeta(v)M}{\sqrt{M}}\Big|_2^2 \leq C\varepsilon\sum_{1\leq|\gamma_1|\leq|\gamma|}((\partial_x^{\gamma_1}\widetilde{\rho})^2 + (\partial_x^{\gamma_1}\widetilde{p})^2).$$

Similarly, for $1 \leq |\gamma| \leq 3$, we also have that

$$\begin{aligned}\sum_{1\leq|\gamma|\leq3}\Big|\frac{(\frac{\partial_x^\gamma\rho}{\rho} - \frac{\partial_x^\gamma\vartheta}{2\vartheta})M + \frac{\partial_x^\gamma\vartheta}{2\vartheta^2}v^2M}{\sqrt{M}}\Big|_2^2 &= \sum_{1\leq|\gamma|\leq3}\Big(\frac{(\partial_x^\gamma\rho)^2}{\rho} + \frac{\rho(\partial_x^\gamma\vartheta)^2}{2\vartheta^2}\Big)\\ &\approx \sum_{1\leq|\gamma|\leq3}((\partial_x^\gamma\widetilde{\rho})^2 + (\partial_x^\gamma\widetilde{p})^2).\end{aligned}$$

By the fact that $\mathcal{E}(t) < \varepsilon$, we have from (2.46) and the above two estimates that

$$\sum_{1\leq|\gamma|\leq3}\Big\|\frac{\partial_x^\gamma M}{\sqrt{M}}\Big\|^2 \approx \sum_{1\leq|\gamma|\leq3}(\|\partial_x^\gamma\widetilde{\rho}\|^2 + \|\partial_x^\gamma\widetilde{p}\|^2).$$

It follows from (2.46), (2.20), and (2.23) that, for $1 \leq |\gamma| \leq 3$,

$$\begin{aligned}\int|\partial_x^\gamma M|^2\Big|\frac{1}{\mu} - \frac{1}{M}\Big|dv &\leq \Big(\int\frac{|\partial_x^\gamma M|^4}{M^2\mu}dv\Big)^{1/2}\Big(\int\frac{|M-\mu|^2}{\mu}dv\Big)^{1/2}\\ &\leq C\sqrt{\varepsilon}\sum_{1\leq|\gamma_1|\leq|\gamma|}((\partial_x^{\gamma_1}\widetilde{\rho})^2 + (\partial_x^{\gamma_1}\widetilde{p})^2).\end{aligned}$$

We deduce from the above two estimates that

$$\sum_{1\leq|\gamma|\leq3}\Big\|\frac{\partial_x^\gamma M}{\sqrt{\mu}}\Big\|^2 \approx \sum_{1\leq|\gamma|\leq3}(\|\partial_x^\gamma\widetilde{\rho}\|^2 + \|\partial_x^\gamma\widetilde{p}\|^2). \tag{2.70}$$

If both $\epsilon > 0$ and $\varepsilon > 0$ are small enough, one has from (2.68), (2.69), and (2.70) that

$$\sum_{1\leq|\gamma|\leq3}\Big\|\frac{\partial_x^\gamma F}{\sqrt{\mu}}\Big\|^2 \approx \sum_{1\leq|\gamma|\leq3}(\|\partial_x^\gamma\widetilde{\rho}\|^2 + \|\partial_x^\gamma\widetilde{p}\|^2 + \|\partial_x^\gamma g\|^2).$$

This and (2.65) imply that (2.67) holds. Thus, we have from (2.66) that

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \widetilde{\mathcal{D}}(t) \leq C\sqrt{\widetilde{\mathcal{E}}(t)}\widetilde{\mathcal{D}}(t). \tag{2.71}$$

If $\mathcal{E}(0) < \varepsilon$ for any $\varepsilon > 0$ small enough, similarly to the arguments used in [1, 2] or [28], the existence of a nonnegative local solution to (1.1) can be constructed by using Lemmas 5.2 and 5.3. We thus omit the details for brevity. In fact, by the a priori assumption that $\mathcal{E}(t) < 2\varepsilon$, one has that $\widetilde{\mathcal{E}}(t) \le C\mathcal{E}(t) < 2C\varepsilon$. It follows from this and (2.71) that

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + C\widetilde{\mathcal{D}}(t) \le 0$$

for some constant $C > 0$. Thus, we have

$$\widetilde{\mathcal{E}}(t) + C\int_0^t \widetilde{\mathcal{D}}(s)ds \le \widetilde{\mathcal{E}}(0).$$

This implies $\widetilde{\mathcal{E}}(t) \le \widetilde{\mathcal{E}}(0) < C\varepsilon$ so that the a priori estimate is closed. By this and the local existence, the standard continuity argument gives the existence of global solution to (1.1).

**3. Spectrum analysis.** In this section, we will study the spectrum structure of the linearized Kac equation and then obtain the time decay rate of the solution to the linearized equation. We use the sign $\sigma(A)$ to denote the spectrum of the operator $A$. The discrete spectrum of $A$, denoted by $\sigma_d(A)$, is the set of all isolated eigenvalues with finite multiplicity. The essential spectrum of $A$, $\sigma_{ess}(A)$, is the set $\sigma(A) \cap \sigma_d^c(A)$. We denote by $\mathrm{Res}(A)$ the resolvent set of the operator $A$ and denote by $D(A)$ and $R(A)$ the domain and the range of the operator $A$. For any complex number $\lambda \in \mathbb{C}$, $\mathrm{Re}\lambda$ and $\mathrm{Im}\lambda$ are real and imaginary parts of $\lambda$, respectively.

As usual, the Fourier transform of any $f(x)$ function can be defined by

$$\hat{f}(k) = \mathcal{F}f(x) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} e^{-ixk}f(x)dx, \quad i = \sqrt{-1}.$$

By (5.13), for any $g \in D(\mathcal{L})$, there is a constant $\overline{C} > 0$ such that

$$\langle \mathcal{L}g, g\rangle \ge \delta_0|\mathcal{H}_v^{\frac{s}{2}}g|_2^2 - \overline{C}|g|_2^2. \tag{3.1}$$

Now we decompose the linearized Kac operator $-\mathcal{L} = -A + K$ as

$$Ag = (\mathcal{L}g + \overline{C}g) - \overline{C}\mathbf{I}_{|v|\ge R}g, \qquad Kg = \overline{C}\mathbf{I}_{|v|\le R}g. \tag{3.2}$$

Here $R > 0$ is chosen to be large enough, and $\mathbf{I}_{|v|\ge R}$ is the characteristic function of the domain $|v| \ge R$. It is easy to see that the operators $A$ and $K$ are self-adjoint. It holds that

$$|\langle \overline{C}\mathbf{I}_{|v|\ge R}g, g\rangle| \le CR^{-2s}|\langle v\rangle^s g|_2^2 \le CR^{-2s}|\mathcal{H}_v^{\frac{s}{2}}g|_2^2.$$

Here, we have used the fact from Lemma 5.1 that

$$|\mathcal{H}_v^{\frac{s}{2}}g|_2^2 \approx |\langle v\rangle^s g|_2^2 + |(\mathbf{1} - \Delta_v)^{\frac{s}{2}}g|_2^2. \tag{3.3}$$

With the help of this and (3.1), for some constant $\nu_0 > 0$ and $\nu_1 > 0$, one has that

$$\langle Ag, g\rangle \ge \nu_1|\mathcal{H}_v^{\frac{s}{2}}g|_2^2 \ge \nu_0|g|_2^2. \tag{3.4}$$

As usual, we use the following notation:

$$A(k)f = (-A - ikv)f, \quad B(k)f = (-\mathcal{L} - ikv)f. \tag{3.5}$$

Here, $f \in D(A(k)) = \{f \in L^2(\mathbb{R}_v); A(k)f \in L^2(\mathbb{R}_v)\}$ and $D(A(k)) = D(B(k))$.

We first show that $B(k)$ generates a continuous contraction semigroup on $L^2(\mathbb{R}_v)$.

LEMMA 3.1. *The operator $B(k)$ generates a continuous contraction semigroup on $L^2(\mathbb{R}_v)$. Moreover, $\{\lambda;\ \mathrm{Re}\lambda > 0\} \subset \mathrm{Res}(B(k))$.*

*Proof.* Since $\mathcal{L}$ is a nonnegative self-adjoint operator on $L^2(\mathbb{R}_v)$, for any $g \in D(B(k))$, we have that

$$\mathrm{Re}\langle B(k)g, g\rangle = -\langle \mathcal{L}g, g\rangle \le 0, \tag{3.6}$$

which shows that $B(k)$ is a dissipative operator. Since $C_0^\infty(\mathbb{R}) \subset D(B(k))$, $D(B(k))$ is dense in $L^2(\mathbb{R}_v)$ for any $k \in \mathbb{R}$. Then $B(k)$ is a closed operator in $L^2(\mathbb{R}_v)$ (cf. [34]) because the adjoint operator $B(-k)$ of $B(k)$ is densely defined in $L^2(\mathbb{R}_v)$. By (3.6) its adjoint operator $B(-k)$ is also dissipative. Thus, $B(k)$ generates a continuous contraction semigroup $e^{tB(k)}$ on $L^2(\mathbb{R}_v)$; cf. [34]. By this and (3.6), we know that $\{\lambda;\ \mathrm{Re}\lambda > 0\} \subset \mathrm{Res}(B(k))$. ◻

In what follows we will show the distribution profile of the spectrum of the operator $B(k)$.

LEMMA 3.2. *It holds that $\sigma_{ess}(B(k)) \subset \{\lambda;\ \mathrm{Re}\lambda \le -\nu_0\}$ and $\sigma(B(k)) \cap \{\lambda;\ -\nu_0 < \mathrm{Re}\lambda \le 0\} \subset \sigma_d(B(k))$. Here the constant $\nu_0 > 0$ is given in* (3.4).

*Proof.* For any sequences with both $\{u_n\}$ and $\{A(k)u_n\}$ bounded in $L^2(\mathbb{R}_v)$, we have from (3.3) and (3.4) that

$$\langle Au_n, u_n\rangle \ge C_1(|(\mathbf{1}-\Delta_v)^{\frac{s}{2}}u_n|_2^2 + |\langle v\rangle^s u_n|_2^2)$$

for some constant $C_1 > 0$, and

$$\langle Au_n, u_n\rangle \le |-\langle Au_n, u_n\rangle - i\langle (k\cdot v)u_n, u_n\rangle| = |\langle A(k)u_n, u_n\rangle| \le |A(k)u_n|_2 |u_n|_2 \le C.$$

One has from the above estimates that $\|u_n\|_{H^s(|v|\le R)} \le C(R)$ for some constant $C(R) > 0$. Since the imbedding operator from $H^s(|v| \le R)$ to $L^2(|v| \le R)$ is compact and $K$ is bounded from $L^2(|v| \le R)$ to itself, by (3.2), $\{Ku_n\}$ contains a convergent subsequence. Then for any $k \in \mathbb{R}$, the operator $K$ is compact with respect to the operator $A(k)$.

With the help of (3.4) and (3.5), an argument similar to Lemma 3.1 implies that $A(k)$ generates a continuous contraction semigroup on $L^2(\mathbb{R}_v)$ and $\{\lambda;\ \mathrm{Re}\lambda > -\nu_0\} \subset \mathrm{Res}(A(k))$. Thus, $\sigma_{ess}(A(k)) \subset \sigma(A(k)) \subset \{\lambda;\ \mathrm{Re}\lambda \le -\nu_0\}$. Since $A(k)$ and $B(k)$ are closed operators in $L^2(\mathbb{R}_v)$ and $K$ is an $A(k)$-compact operator, by Theorem 5.35 in [25, page 244], one has that $\sigma_{ess}(B(k)) = \sigma_{ess}(A(k)) \subset \{\lambda;\ \mathrm{Re}\lambda \le -\nu_0\}$. By Lemma 3.1, $\{\lambda;\ \mathrm{Re}\lambda > 0\} \subset \mathrm{Res}(B(k))$. Thus, we have $\sigma(B(k)) \cap \{\lambda;\ -\nu_0 < \mathrm{Re}\lambda \le 0\} \subset \sigma_d(B(k))$. ◻

Next we consider the spectrum structure of $B(k)$ when $|k| > 0$. To this end, we first prove some operator estimates. For this, for any $\epsilon \in (0, 10^{-5})$ small enough, set

$$\varepsilon(\mathrm{Im}\lambda, k) = (1+|k|^2+|\mathrm{Im}\lambda|^2)^{-\epsilon}, \qquad \sigma(\mathrm{Im}\lambda, k) = (1+|k|^2+|\mathrm{Im}\lambda|^2)^{\frac{1}{2}-100\epsilon}. \tag{3.7}$$

LEMMA 3.3. *Let $R > 1$ be a fixed large constant. For any $\lambda_0 > 0$, if $|\mathrm{Im}\lambda| \le \lambda_0$ and $|k|$ large enough, we have*

$$\mathcal{J} = \Big(\int_{|\eta|\le 2R} \frac{1}{1+(\frac{\mathrm{Im}\lambda}{\sigma}+\frac{k}{\sigma}\eta)^2} d\eta\Big)^{1/2} \le C(R)\Big(\frac{\sigma}{|k|}\Big)^{1/3}. \tag{3.8}$$

*For any $k_0 > 0$, if $|k| \le k_0$ and $|\mathrm{Im}\lambda| \ge 4k_0R$, we have*

$$\mathcal{J} = \Big(\int_{|\eta|\le 2R} \frac{1}{1+(\frac{\mathrm{Im}\lambda}{\sigma}+\frac{k}{\sigma}\eta)^2}d\eta\Big)^{1/2} \le C(R)\Big(\frac{\sigma}{|Im\lambda|}\Big). \tag{3.9}$$

*Proof.* As in [44], for any $\varrho > 0$, we set

$$\Sigma_1 = \{\eta\in\mathbb{R}|\ \ |\eta|\le 2R,\ \ |\mathrm{Im}\lambda + k\eta| \le \varrho|k|\},\ \ \Sigma_2 = \{\eta\in\mathbb{R}|\ \ |\eta|\le 2R\} - \Sigma_1.$$

In $\Sigma_1$, if $|\mathrm{Im}\lambda| \le \lambda_0$, it holds that

$$|\eta| \le \frac{|\mathrm{Im}\lambda|}{|k|} + \varrho \le \frac{\lambda_0}{|k|} + \varrho.$$

Hence, if $|\mathrm{Im}\lambda| \le \lambda_0$, one has

$$\mathcal{J}^2 \le C(R)\Big(\frac{\lambda_0}{|k|} + \varrho + \Big(\varrho\frac{|k|}{\sigma}\Big)^{-2}\Big).$$

If we choose $\varrho = (\sigma/|k|)^{2/3}$, we have that

$$\mathcal{J}^2 \le C(R)\Big(\frac{\lambda_0}{|k|} + \Big(\frac{\sigma}{|k|}\Big)^{2/3}\Big).$$

If the positive constant $\epsilon$ in (3.7) is small enough, when $|k|$ large enough, it holds that $\frac{\lambda_0}{|k|} \le C(\frac{\sigma}{|k|})^{2/3}$. By using this and the above estimates, we can obtain (3.8).

If $|k| \le k_0$, $|\eta| \le 2R$, and $|\mathrm{Im}\lambda| \ge 4k_0R$, we have

$$|\mathrm{Im}\lambda + k\eta| \ge |\mathrm{Im}\lambda| - |k||\eta| \ge |\mathrm{Im}\lambda| - 2k_0R \ge |\mathrm{Im}\lambda|/2.$$

It follows that

$$\mathcal{J}^2 \le C(R)\Big(1+\Big(\frac{|\mathrm{Im}\lambda|}{2\sigma}\Big)^2\Big)^{-1} \le C(R)\Big(\frac{\sigma}{|\mathrm{Im}\lambda|}\Big)^2.$$

This concludes the proof of Lemma 3.3. ◻

By an argument similar to Lemma 3.3, we can also prove, if $|\mathrm{Im}\lambda| \le \lambda_0$ and $|k|$ large enough, that

$$\mathcal{J}_1 = \Big(\int_{|\eta|\le 2R} \frac{1}{1+(\frac{\mathrm{Im}\lambda}{\sigma}+\frac{k}{\sigma}\eta)^4}d\eta\Big)^{1/2} \le C(R)\Big(\frac{\sigma}{|k|}\Big)^{2/5}. \tag{3.10}$$

If $|k| \le k_0$ and $|\mathrm{Im}\lambda| \ge 4k_0R$, we have

$$\mathcal{J}_1 = \Big(\int_{|\eta|\le 2R} \frac{1}{1+(\frac{\mathrm{Im}\lambda}{\sigma}+\frac{k}{\sigma}\eta)^4}d\eta\Big)^{1/2} \le C(R)\Big(\frac{\sigma}{|\mathrm{Im}\lambda|}\Big)^2. \tag{3.11}$$

The following lemma is a crucial operator estimate, which is used to prove the spectrum structure of the operator $B(k)$ when $|k| > 0$.

LEMMA 3.4. *For any $\epsilon > 0$, if $\mathrm{Re}\lambda \ge -\nu_0 + \epsilon$, we have*

$$|K(\lambda - A(k))^{-1}|_2 \to 0 \quad \text{as} \quad |\mathrm{Im}\lambda| + |k| \to \infty.$$

*Here $|\cdot|_2$ denotes the operator norm of $L^2(\mathbb{R}_v)$.*

*Proof.* In the proof of Lemma 3.2, one has that $\{\lambda;\ \mathrm{Re}\lambda > -\nu_0\} \subset \mathrm{Res}(A(k))$. Thus if $\mathrm{Re}\lambda > -\nu_0$, then $(\lambda - A(k))^{-1}$ exists. Recalling that $Ku = \overline{C}\mathbf{I}_{|v|\le R}u$, one has that for any $u \in L^2(\mathbb{R}_v)$,

$$|K\mathbf{I}_{|v|>R}(\lambda - A(k))^{-1}u|_2 = 0. \tag{3.12}$$

Choose a smooth cutoff function $\chi(v) \in [0,1]$ such that $\chi(v) = 1$ if $|v| \le R$ and $\chi(|v|) = 0$ if $|v| > R+1$. Since $K$ is bounded in $L^2(\mathbb{R}_v)$, we have

$$|K\mathbf{I}_{|v|\le R}(\lambda - A(k))^{-1}u|_2 \le C|\mathbf{I}_{|v|\le R}(\lambda - A(k))^{-1}u|_2 \le C|\chi(v)(\lambda - A(k))^{-1}u|_2. \tag{3.13}$$

Letting $\varphi = (\lambda - A(k))^{-1}u$, we deduce that

$$i(\mathrm{Im}\lambda + kv)\varphi = u - (\mathrm{Re}\lambda + A)\varphi. \tag{3.14}$$

If $\mathrm{Re}\lambda \ge -\nu_0 + \epsilon$, we claim that

$$\langle A\varphi, \varphi\rangle \le C(\epsilon)|u|_2^2. \tag{3.15}$$

Equation (3.15) can be proved as follows. It follows from (3.14) and (3.4) that

$$(\mathrm{Re}\lambda + \nu_0)|\varphi|_2^2 \le \langle A\varphi, \varphi\rangle + \mathrm{Re}\lambda|\varphi|_2^2 = \mathrm{Re}\langle u, \varphi\rangle.$$

By using this and the fact that $\mathrm{Re}\lambda \ge -\nu_0 + \epsilon$, we get

$$|\varphi|_2 \le \frac{1}{\mathrm{Re}\lambda + \nu_0}|u|_2 \le \frac{1}{\epsilon}|u|_2.$$

Hence, we can obtain

$$\langle A\varphi, \varphi\rangle \le \langle A\varphi, \varphi\rangle + (\mathrm{Re}\lambda + \nu_0)|\varphi|_2^2 \le |u|_2|\varphi|_2 + \nu_0|\varphi|_2^2 \le \frac{\nu_0 + \epsilon}{\epsilon^2}|u|_2^2 \le C(\epsilon)|u|_2^2.$$

This completes the proof of (3.15). Recalling (3.7) and setting $\varepsilon = \varepsilon(\mathrm{Im}\lambda, k)$, we choose a mollifier

$$\rho_\varepsilon(v) = \frac{1}{\varepsilon}\rho_1\Big(\frac{v}{\varepsilon}\Big), \quad \rho_1 \in C_0^\infty(\mathbb{R}), \quad \int \rho_1 = 1.$$

Since $\varphi$ depends on $\mathrm{Im}\lambda$ and $k$, we denote $\varphi = \varphi(\mathrm{Im}\lambda, k, v)$. It is easy to see that

$$\chi\varphi(\mathrm{Im}\lambda, k, v) = \rho_\varepsilon *_v (\chi\varphi(\mathrm{Im}\lambda, k, v)) + (\chi\varphi(\mathrm{Im}\lambda, k, v) - \rho_\varepsilon *_v (\chi\varphi(\mathrm{Im}\lambda, k, v))). \tag{3.16}$$

For the second term in the above equality, we deduce that

$$\begin{aligned}
&|\chi\varphi(\mathrm{Im}\lambda, k, v) - \rho_\varepsilon *_v (\chi\varphi(\mathrm{Im}\lambda, k, v))|_2 \\
&\quad = \Big|\int [\chi\varphi(\mathrm{Im}\lambda, k, v) - \chi\varphi(\mathrm{Im}\lambda, k, v-u)]\rho_\varepsilon(u)du\Big|_2 \\
&\quad \le \int |[\chi\varphi(\mathrm{Im}, k, v) - \chi\varphi(\mathrm{Im}, k, v-u)]\rho_\varepsilon(u)|_2 du \\
&\quad \le \Big(\int \frac{|\chi\varphi(\mathrm{Im}\lambda, k, v) - \chi\varphi(\mathrm{Im}\lambda, k, v-u)|^2}{|u|^{1+2s}}dudv\Big)^{1/2}\Big(\int |\rho_\varepsilon(u)|^2|u|^{1+2s}du\Big)^{1/2} \\
&\quad \le C\varepsilon^s|D_v^s(\chi\varphi(\mathrm{Im}\lambda, k, v))|_2.
\end{aligned} \tag{3.17}$$

With the help of (3.3), (3.4), and (3.15), we deduce that if $\mathrm{Re}\lambda \geq -\nu_0 + \epsilon$,

$$C(|\langle v\rangle^s \varphi|_2 + |(\mathbf{1} - \Delta_v)^{\frac{s}{2}}\varphi|_2) \leq C|\mathcal{H}_v^{\frac{s}{2}}\varphi|_2 \leq \sqrt{\langle A\varphi, \varphi\rangle} \leq C(\epsilon)|u|_2. \tag{3.18}$$

It follows from (3.18), (3.18), and the property of $\chi(v)$ that

$$|\chi\varphi(\mathrm{Im}\lambda, k, v) - \rho_\varepsilon *_v (\chi\varphi(\mathrm{Im}\lambda, k, v))|_2 \leq C\varepsilon^s |D_v^s(\chi\varphi)|_2 \leq C(R,\epsilon)\varepsilon^s |u|_2. \tag{3.19}$$

By using (3.2), we can rewrite (3.14) for some parameter $\sigma > 0$ as

$$\sigma\chi\varphi + i(\mathrm{Im}\lambda + kv)\chi\varphi = \sigma\chi\varphi + \chi(u - \mathrm{Re}\lambda\varphi) - \chi A\varphi \equiv \sigma\chi\varphi + \chi G - \chi\mathcal{L}\varphi. \tag{3.20}$$

Here $G = u - \mathrm{Re}\lambda\varphi - \overline{C}\mathbf{I}_{|v|\leq R}\varphi$. It is easy to see that

$$\chi\varphi = \frac{\sigma\chi\varphi + \chi G - \chi\mathcal{L}\varphi}{\sigma + i(\mathrm{Im}\lambda + kv)},$$

which implies that

$$\rho_\varepsilon *_v (\chi\varphi(\mathrm{Im}\lambda, k, v)) = \int \frac{\sigma\chi\varphi + \chi G - \chi\mathcal{L}\varphi}{\sigma + i(\mathrm{Im}\lambda + k\eta)}\rho_\varepsilon(v-\eta)d\eta. \tag{3.21}$$

We will estimate each term in (3.21). We first estimate the first term,

$$\begin{aligned}
&\Big|\int \frac{\sigma\chi\varphi}{\sigma + i(\mathrm{Im}\lambda + k\eta)}\rho_\varepsilon(v-\eta)d\eta\Big| = \Big|\int \frac{\chi\varphi}{1 + i(\mathrm{Im}\lambda + k\eta)/\sigma}\rho_\varepsilon(v-\eta)d\eta\Big| \\
&\leq \Big|\chi^{1/2}\varphi|\rho_\varepsilon(v-\eta)|^{1/2}\Big|_2 \Big(\int \frac{|\chi\rho_\varepsilon(v-\eta)|}{|1 + i(\mathrm{Im}\lambda + k\eta)/\sigma|^2}d\eta\Big)^{1/2} \\
&\leq \frac{C(R)}{\varepsilon^{1/2}}\Big|\chi^{1/2}\varphi|\rho_\varepsilon(v-\eta)|^{1/2}\Big|_2 \Big(\int_{|\eta|\leq 2R} \frac{1}{1 + (\frac{\mathrm{Im}\lambda}{\sigma} + \frac{k}{\sigma}\eta)^2}d\eta\Big)^{1/2}.
\end{aligned} \tag{3.22}$$

One has from (3.18) that

$$\Big|\Big|\chi^{1/2}\varphi|\rho_\varepsilon(v-\eta)|^{1/2}\Big|_2\Big|_2 \leq C\Big|\chi^{1/2}\varphi\Big|_2 \leq C(\epsilon)|u|_2.$$

By this, (3.8), and (3.23), we deduce that

$$\Big|\int \frac{\sigma\chi\varphi}{\sigma + i(\mathrm{Im}\lambda + k\eta)}\rho_\varepsilon(v-\eta)d\eta\Big|_2 \leq \frac{C(R,\epsilon)}{\varepsilon^{1/2}}|u|_2\mathcal{J}. \tag{3.23}$$

By an argument similar to (3.23), the second term of (3.21) can be bounded by

$$\begin{aligned}
\Big|\int \frac{\chi G}{\sigma + i(\mathrm{Im}\lambda + k\eta)}\rho_\varepsilon(v-\eta)d\eta\Big| &\leq \frac{1}{\sigma\varepsilon^{1/2}}\Big|\chi^{1/2}G|\rho_\varepsilon(v-\eta)|^{1/2}\Big|_2 \\
&\quad\times\Big(\int_{|\eta|\leq 2R} \frac{1}{1 + (\frac{\mathrm{Im}\lambda}{\sigma} + \frac{k}{\sigma}\eta)^2}d\eta\Big)^{1/2}.
\end{aligned} \tag{3.24}$$

Recalling $G = u - \mathrm{Re}\lambda\varphi - \overline{C}\mathbf{I}_{|v|\leq R}\varphi$, we have from (3.18) that

$$\Big|\chi^{1/2}G\Big|_2 \leq \Big|\chi^{1/2}u\Big|_2 + \Big|\chi^{1/2}\mathrm{Re}\lambda\varphi\Big|_2 + \Big|\chi^{1/2}\mathbf{I}_{|v|\leq R}\varphi\Big|_2 \leq C(\mathrm{Re}\lambda, R, \epsilon)|u|_2.$$

By using this and (3.24), for the second term of (3.21), we can obtain

$$\Big|\int \frac{\chi G}{\sigma+i(\mathrm{Im}\lambda+k\eta)}\rho_\varepsilon(v-\eta)d\eta\Big|_2 \le \frac{C(\mathrm{Re}\lambda,R,\epsilon)}{\sigma\varepsilon^{1/2}}|u|_2\mathcal{J}. \tag{3.25}$$

In what follows, we consider the last term of (3.21). Note that $H^1(\mathbb{R}_v)\subset H^s(\mathbb{R}_v)$ with $s\in(0,1)$ and $\mathcal{L}g=-\Gamma(\sqrt{\mu},\varphi)-\Gamma(\varphi,\sqrt{\mu})$. By using the property of $\chi(v)$, it follows from (3.3) and Lemma 5.3 that

$$\begin{aligned}\Big|\int \frac{-\chi\mathcal{L}\varphi}{\sigma+i(\mathrm{Im}\lambda+k\eta)}\rho_\varepsilon(v-\eta)d\eta\Big| &= \Big|\Big\langle\Gamma(\sqrt{\mu},\varphi)+\Gamma(\varphi,\sqrt{\mu}),\frac{\chi\rho_\varepsilon(v-\eta)}{\sigma+i(\mathrm{Im}\lambda+k\eta)}\Big\rangle\Big|\\ &\le C\Big|\mathcal{H}_\eta^{\frac{s}{2}}\varphi\Big|_2\Big|\mathcal{H}_\eta^{\frac{s}{2}}\Big(\frac{\chi\rho_\varepsilon(v-\eta)}{\sigma+i(\mathrm{Im}\lambda+k\eta)}\Big)\Big|_2 \le C(R)\Big|\mathcal{H}_\eta^{\frac{s}{2}}\varphi\Big|_2\Big|\frac{\chi\rho_\varepsilon(v-\eta)}{\sigma+i(\mathrm{Im}\lambda+k\eta)}\Big|_{H^1_\eta}.\end{aligned} \tag{3.26}$$

By the property of $\chi(v)$, there exists a constant $C'(R)>0$ such that

$$\begin{aligned}\frac{1}{C'(R)}\Big|\frac{\chi\rho_\varepsilon(v-\eta)}{\sigma+i(\mathrm{Im}\lambda+k\eta)}\Big|^2_{H^1_\eta} \le& \int\frac{|\chi\rho_\varepsilon(v-\eta)|^2}{\sigma^2+|(\mathrm{Im}\lambda+k\cdot\eta)|^2}d\eta+\int\frac{|\partial_\eta\chi\rho_\varepsilon(v-\eta)|^2}{\sigma^2+|(\mathrm{Im}\lambda+k\eta)|^2}d\eta\\ &+\frac{1}{\varepsilon^2}\int\frac{|\chi\partial_\eta\rho_\varepsilon(v-\eta)|^2}{\sigma^2+|(\mathrm{Im}\lambda+k\eta)|^2}d\eta+|k|^2\int\frac{|\chi\rho_\varepsilon(v-\eta)|^2}{\sigma^4+|(\mathrm{Im}\lambda+k\eta)|^4}d\eta.\end{aligned} \tag{3.27}$$

For the first terms of (3.28), one has from (3.8) that

$$\begin{aligned}\int\int\frac{|\chi\rho_\varepsilon(v-\eta)|^2}{\sigma^2+|(\mathrm{Im}\lambda+k\eta)|^2}d\eta dv &= \frac{1}{\sigma^2}\int\int\frac{|\chi\rho_\varepsilon(v-\eta)|^2}{1+|(\frac{\mathrm{Im}\lambda}{\sigma}+\frac{k}{\sigma}\eta)|^2}d\eta dv\\ &\le \frac{C(R)}{\sigma^2\varepsilon}\int_{|\eta|\le 2R}\frac{1}{1+(\frac{\mathrm{Im}\lambda}{\sigma}+\frac{k}{\sigma}\eta)^2}d\eta \le \frac{C(R)}{\sigma^2\varepsilon}\mathcal{J}^2.\end{aligned}$$

Similarly, we also have

$$\int\int\frac{|\partial_\eta\chi\rho_\varepsilon(v-\eta)|^2}{\sigma^2+|(\mathrm{Im}\lambda+k\eta)|^2}d\eta dv+\frac{1}{\varepsilon^2}\int\int\frac{|\chi\partial_\eta\rho_\varepsilon(v-\eta)|^2}{\sigma^2+|(\mathrm{Im}\lambda+k\eta)|^2}d\eta dv \le \frac{C(R)}{\sigma^2\varepsilon}\mathcal{J}^2+\frac{C(R)}{\sigma^2\varepsilon^3}\mathcal{J}^2.$$

By (3.10), the fourth term of (3.28) can bounded by

$$\begin{aligned}&|k|^2\int\int\frac{|\chi\rho_\varepsilon(v-\eta)|^2}{\sigma^4+|(\mathrm{Im}\lambda+k\eta)|^4}d\eta dv\\ &\qquad\le \frac{C(R)|k|^2}{\sigma^4\varepsilon}\int_{|\eta|\le 2R}\frac{1}{1+(\frac{\mathrm{Im}\lambda}{\sigma}+\frac{k}{\sigma}\eta)^4}d\eta \le \frac{C(R)|k|^2}{\sigma^4\varepsilon}\mathcal{J}_1^2.\end{aligned}$$

By using (3.27), (3.28), and the above estimates, we deduce from (3.18) that

$$\Big|\int\frac{-\chi\mathcal{L}\varphi}{\sigma+i(\mathrm{Im}\lambda+k\eta)}\rho_\varepsilon(v-\eta)d\eta\Big|_2 \le C(R,\epsilon)|u|_2\Big\{\Big(\frac{1}{\sigma\varepsilon^{1/2}}+\frac{1}{\sigma\varepsilon^{3/2}}\Big)\mathcal{J}+\frac{|k|}{\sigma^2\varepsilon^{1/2}}\mathcal{J}_1\Big\}. \tag{3.28}$$

With help of the estimates (3.23), (3.25), and (3.28), we have from (3.21) that

$$|\rho_\varepsilon *_v(\chi\varphi(\mathrm{Im}\lambda,k,v))|_2 \le C(\mathrm{Re}\lambda,R,\epsilon)|u|_2\Big\{\Big(\frac{1}{\varepsilon^{1/2}}+\frac{1}{\sigma\varepsilon^{1/2}}+\frac{1}{\sigma\varepsilon^{3/2}}\Big)\mathcal{J}+\frac{|k|}{\sigma^2\varepsilon^{1/2}}\mathcal{J}_1\Big\}. \tag{3.29}$$

If $|k| > 0$ large enough and $|\text{Im}\lambda| \le \lambda_0$ for any $\lambda_0 > 0$, we deduce from (3.7), (3.8), (3.10), and (3.29) that

$$|\rho_\varepsilon *_v (\chi\varphi(\text{Im}\lambda, k, v))|_2 \to 0 \quad \text{as} \quad |k| \to \infty.$$

By using this, (3.16), and (3.19), one has from (3.13) that

$$|K\mathbf{I}_{|v|\le R}(\lambda - A(k))^{-1}u|_2 \le C|\chi\varphi(\text{Im}\lambda, k, v)|_2 \to 0 \quad \text{as} \quad |k| \to \infty.$$

This and (3.12) imply that, if $\text{Re}\lambda \ge -\nu_0 + \epsilon$ and $|\text{Im}\lambda| \le \lambda_0$,

$$|K(\lambda - A(k))^{-1}u|_2 \to 0 \quad \text{as} \quad |k| \to \infty. \tag{3.30}$$

If $|k| \le k_0$ and $|\text{Im}\lambda| \ge 4k_0R$ large enough for any $k_0 > 0$, we deduce from (3.7), (3.9), (3.11), and (3.29) that

$$|\rho_\varepsilon *_v (\chi\varphi(\text{Im}\lambda, k, v))|_2 \to 0 \quad \text{as} \quad |\text{Im}\lambda| \to \infty.$$

By using this, (3.16), and (3.19), one has from (3.13) that

$$|K\mathbf{I}_{|v|\le R}(\lambda - A(k))^{-1}u|_2 \le C|\chi\varphi(\text{Im}\lambda, k, v)|_2 \to 0 \quad \text{as} \quad |\text{Im}\lambda| \to \infty.$$

This and (3.12) imply that, if $\text{Re}\lambda \ge -\nu_0 + \epsilon$ and $|k| \le k_0$,

$$|K(\lambda - A(k))^{-1}u|_2 \to 0 \quad \text{as} \quad |\text{Im}\lambda| \to \infty. \tag{3.31}$$

The estimates (3.30) and (3.31) imply that the lemma holds. ☐

With the help of Lemma 3.1, 3.2, and 3.4, an argument similar to Proposition 2.2.8 in [44] implies that the following lemma holds. For completeness, we give its proof.

LEMMA 3.5. *For any $\epsilon > 0$, there exists a constant $\tau_1 > 0$ such that the following holds for all $k \in \mathbb{R}$:*

(I) $\{\lambda; \text{Re}\lambda > 0\} \cup \{\lambda;\ \text{Re}\lambda > -\nu_0 + \epsilon, |\text{Im}\lambda| \ge \tau_1\} \subset \text{Res}(B(k))$.

(II) *There are only finite eigenvalues of $B(k)$ in $\sigma(B(k)) \cap \{\lambda;\ -\nu_0 + \epsilon < \text{Re}\lambda \le 0\}$.*

(III) *If $\lambda$ is an eigenvalue of $B(k)$, then $\text{Re}\lambda \le 0$. When $k \ne 0$, $B(k)$ has no eigenvalue on the imaginary axis. If $\text{Re}\lambda = 0$, $\text{Im}\lambda = 0$ if and only if $k = 0$.*

*Proof.* For any $\epsilon > 0$, $\{\lambda;\ -\nu_0 + \epsilon < \text{Re}\lambda \le 0\} \subset \text{Res}(A(k))$. By Lemma 3.4, if $\text{Re}\lambda \ge -\nu_0 + \epsilon$, we have $|K(\lambda - A(k))^{-1}|_2 \to 0$ as $|\text{Im}\lambda| \to \infty$. Then there exists $\tau_1 > 0$ such that for any $k$ and $|\text{Im}\lambda| \ge \tau_1$, $|[I - K(\lambda - A(k))^{-1}]^{-1}|_2 \le 2$. If $\text{Re}\lambda \ge -\nu_0 + \epsilon$ and $|\text{Im}\lambda| \ge \tau_1$, by the second resolvent equation

$$(\lambda - B(k))^{-1} = (\lambda - A(k))^{-1} + (\lambda - A(k))^{-1}[I - K(\lambda - A(k))^{-1}]^{-1}K(\lambda - A(k))^{-1}, \tag{3.32}$$

we deduce that $\lambda \in \text{Res}(B(k))$. This and Lemma 3.1 imply that (I) holds.

In view of Lemma 3.2, one has that $\sigma(B(k)) \cap \{\lambda;\ -\nu_0 < \text{Re}\lambda\}$ consists of discrete eigenvalues with possible accumulation points on the boundary of $\{\lambda;\ -\nu_0 < \text{Re}\lambda\}$. Since

$$\sigma(B(k)) \cap \overline{\{\lambda;\ -\nu_0 + \epsilon < \text{Re}\lambda \le 0\}} \subset \{\lambda;\ -\nu_0 + \epsilon \le \text{Re}\lambda \le 0,\ |\text{Im}\lambda| \le \tau_1\}$$

is a compact set in $\{\lambda;\ -\nu_0 < \text{Re}\lambda\}$ and does not touch the boundary of $\{\lambda;\ -\nu_0 < \text{Re}\lambda\}$, the number of eigenvalues in it is finite. Thus (II) holds.

Let $\phi \neq 0$ such that $B(k)\phi = \lambda\phi$. Then we have

$$\lambda|\phi|_2^2 = \langle B(k)\phi, \phi\rangle = \langle -\mathcal{L}\phi, \phi\rangle - i(kv\phi, \phi).$$

Noting that $\mathcal{L}$ is self-adjoint and nonnegative, we have $\mathrm{Re}\lambda \leq 0$. If $k \neq 0$ and $\mathrm{Re}\lambda = 0$, we have $\mathcal{L}\phi = 0$, which implies that $\phi = \mathbf{P}_0\phi$. The eigenvalue equation is reduced to $(\mathrm{Im}\lambda + kv)\phi = 0$. By this we deduce $k = 0$ and $\mathrm{Im}\lambda = 0$. This is a contradiction and implies that $B(k)$ has no eigenvalues on the imaginary axis for $k \neq 0$. This also shows that $\lambda = 0$ if and only if $k = 0$. ∎

By using the above results we will prove the spectrum structure of $B(k)$ for $|k|$ bounded away from zero.

THEOREM 3.6. *For any $k_0 > 0$, there exists $\tau = \tau(k_0) > 0$ such that for all $|k| > k_0$,*

$$\sigma(B(k)) \subset \{\lambda : \mathrm{Re}\lambda < -\tau\}.$$

*Proof.* By Lemma 3.4, if $\mathrm{Re}\lambda \geq -\nu_0 + \epsilon$ with any $\epsilon > 0$ and for any $|k| \geq k_1$ with some $k_1 > 0$ large enough, one has that $|[I - K(\lambda - A(k))^{-1}]^{-1}|_2 \leq 2$. Thus, if $\mathrm{Re}\lambda \geq -\nu_0 + \epsilon$ and $|k| \geq k_1$, we deduce from (3.32) that $\lambda \in \mathrm{Res}(B(k))$. Thus, for any $|k| \geq k_1$, one has

$$\sigma(B(k)) \subset \{\lambda : \mathrm{Re}\lambda < -\nu_0 + \epsilon\}. \tag{3.33}$$

For any $k_0 > 0$, we claim that there exists $\sigma_1 > 0$ such that for all $|k| \in (k_0, k_1]$,

$$\sigma_d(B(k)) \subset \{\lambda : \mathrm{Re}\lambda \leq -\sigma_1\}. \tag{3.34}$$

By Lemma 3.2, $\sigma_{ess}(B(k)) \subset \{\lambda : \mathrm{Re}\lambda \leq -\nu_0\}$. If we take $\tau = \min\{\nu_0 - \epsilon, \sigma_1\}$, this completes the proof of Theorem 3.6.

The relation (3.34) can be proved by contradiction. Suppose (3.34) is not true; there exists $\lambda_n \in \mathbb{C}$, $k_n \in \mathbb{R}$, and functions $\varphi_n \in D(B(k_n)) \subset L^2(\mathbb{R})$ such that

$$B(k_n)\varphi_n = (-\mathcal{L} - ik_n v)\varphi_n = \lambda_n\varphi_n, \quad |\varphi_n|_2 = 1, \quad n = 1, 2, 3, \ldots, \tag{3.35}$$

and

$$-\frac{1}{n} \leq \mathrm{Re}\lambda_n \leq 0; \quad k_0 < |k_n| \leq k_1. \tag{3.36}$$

If we assume that $\mathrm{Im}\lambda_n \to \infty$ by (3.32) and Lemma 3.4, then $\lim_{n\to\infty} |K(\lambda_n - A(k_n))^{-1}|_2 = 0$ and $\lambda_n \in \mathrm{Res}(B(k_n))$ for $n$ large enough, which contradicts (3.35). Thus, $\{\lambda_n\}$ is a bounded sequence.

It follows from (3.35) and (1.19) that

$$\mathrm{Re}\lambda_n = -\langle \mathcal{L}\varphi_n, \varphi_n\rangle = -\langle \mathcal{L}\mathbf{P}_1\varphi_n, \mathbf{P}_1\varphi_n\rangle \leq -\delta_1|\mathbf{P}_1\varphi_n|_2^2.$$

By this and (3.36), $\lim_{n\to\infty} |\mathbf{P}_1\varphi_n|_2^2 = 0$, which implies that $\lim_{n\to\infty} |\mathbf{P}_0\varphi_n|_2^2 = 1$.

By (5.6), set $\mathbf{P}_0\varphi_n = a_{0n}\psi_0 + a_{2n}\psi_2$, and then $|\mathbf{P}_0\varphi_n|_2^2 = |a_{0n}|^2 + |a_{2n}|^2 \to 1$ as $n \to \infty$. Hence, we can extract a subsequence $(a_{0n}, a_{2n}) \to (a_0, a_2)$ as $n \to \infty$. Then we have $\mathbf{P}_0\varphi_n \to \varphi_0 = a_0\psi_0 + a_2\psi_2 \in \mathcal{N}$ with $|\varphi_0|_2 = 1$. Finally, we have $\varphi_n \to \varphi_0$ in $L^2(\mathbb{R})$ as $n \to \infty$. Up to a subsequence, one has that $(k_n, \lambda_n,) \to (\bar{k}, i\bar{\lambda})$ as $n \to \infty$, where $\bar{\lambda}$ is a real number and $\kappa_0 < |\bar{k}| \leq \kappa_1$. We deduce from (3.35) that $B(\bar{k})\varphi_0 = i\bar{\lambda}\varphi_0$. It follows from (III) in Lemma 3.5 that $\bar{\lambda} = \bar{k} = 0$. This contradicts the fact that $k_0 \leq |\bar{k}| \leq k_1$. This completes the proof of (3.34) and the theorem. ∎

Next we will consider the eigenvalues of the operator $B(k)$ for $|k|$ near zero.

LEMMA 3.7. *Let $\lambda$ be an eigenvalue of $B(k)$ satisfying* $\mathrm{Re}\lambda \geq -\delta_1 + \epsilon$ *for any $\epsilon > 0$, where $\delta_1 > 0$ is the constant in* (1.19)*; then $|\lambda| \leq C|k|\sqrt{\delta_1/\epsilon}$.*

*Proof.* Let $\lambda$ be an eigenvalue of $B(k)$ and $\varphi \in D(B(k))$ such that

$$\lambda\varphi = B(k)\varphi = -\mathcal{L}\varphi - ikv\varphi, \quad |\varphi|_2 = 1. \tag{3.37}$$

Taking the inner product successively by $\varphi$ and $\mathbf{P}_0\varphi$, we obtain

$$-\langle\mathcal{L}\varphi,\varphi\rangle - i\langle kv\varphi,\varphi\rangle = \lambda \ \text{ and } \ -i\langle kv\varphi,\mathbf{P}_0\varphi\rangle = \lambda\langle\varphi,\mathbf{P}_0\varphi\rangle. \tag{3.38}$$

One has from (3.38) and (1.19) that

$$\mathrm{Re}\lambda = -\langle\mathcal{L}\varphi,\varphi\rangle \leq -\delta_1|\mathbf{P}_1\varphi|_2^2 \tag{3.39}$$

and

$$|\lambda||\mathbf{P}_0\varphi|_2^2 \leq |k||\varphi|_2|v\mathbf{P}_0\varphi|_2 \leq C|k||\mathbf{P}_0\varphi|_2. \tag{3.40}$$

Note that $\mathrm{Re}\lambda \geq -\delta_1 + \epsilon$. It follows from (3.37) and (3.39) that

$$\epsilon \leq \delta_1 + \mathrm{Re}\lambda \leq \delta_1|\mathbf{P}_0\varphi|_2^2. \tag{3.41}$$

Then the desired estimate can be deduced from (3.40) and (3.41). ◻

Since the eigenvalue 0 is a discrete eigenvalue of $-\mathcal{L}$ with two eigenfunctions, we can expect $B(k)$ to have two smooth eigenvalues branches when $|k| \approx 0$.

THEOREM 3.8. *For any $k_0 > 0$ small enough, there exists $\tau > 0$ such that for any $|k| \leq k_0$ the following hold:*

(I) $\sigma(B(k)) \cap \{\lambda : \mathrm{Re}\lambda \geq -\tau\} = \{\lambda_0(k), \quad \lambda_2(k)\}$.

(II) *The eigenvalue $\lambda_j(k)$ and the eigenfunction $\mathrm{e}_j(k)$ with $j = 0,\ 2$ are $C^\infty$ in $k$ for $|k| \leq \kappa_0$, which have the asymptotic expansions*

$$\lambda_j(k) = \lambda_j^{(2)}k^2 + O(|k|^3), \quad (|k| \to 0),$$

$$\mathrm{e}_j(k,v) = \mathrm{e}_j^{(0)}(v) + k\mathrm{e}_j^{(1)}(v) + k^2\mathrm{e}_j^{(2)}(v) + O(|k|^3), \quad (|k| \to 0),$$

*with the coefficient*

$$\lambda_j^{(2)} = -\langle\mathcal{L}^{-1}\mathbf{P}_1 v\mathrm{e}_j^{(0)}, \mathbf{P}_1 v\mathrm{e}_j^{(0)}\rangle < 0.$$

*Here $\mathrm{e}_j^{(n)}(v)$ with $n \geq 0$ are in $L^2(\mathbb{R})$, and $\{\mathrm{e}_j(k,v)\}_{j=0,2}$ can be normalized by*

$$\langle\mathrm{e}_j(k,v), \mathrm{e}_j(-k,v)\rangle = \delta_{ij}, \quad i,j = 0,2.$$

(III) *Denote the eigen-projection and eigen-nilpotent corresponding to the eigenvalue $\lambda_j(k)$ by $P_j(k)$ and $Q_j(k)$, respectively. It holds that, for $j = 0,\ 2$,*

$$P_j(k) = P_j^{(0)}(k) + |k|P_j^{(1)}(k), \qquad Q_j(k) = 0,$$

*where $P_j^{(0)}(k)$ and $P_j^{(1)}(k)$ are defined in* (3.57) *with*

$$\mathbf{P}_0 = \sum_{j=0,2} P_j^{(0)}(k).$$

*Moreover, the operator norm $|P_j^{(1)}(k)|_2$ is uniformly bounded for $|k| \leq \kappa_0$.*

*Proof.* We will prove the existence of the eigenvalues of $B(k)$ by using the implicit function theorem as in [26, 32]. Apply $\mathbf{P}_0$ and $\mathbf{P}_1$ to the eigenvalue problem

$$B(k)\mathrm{e}(k,v) = \big(-\mathcal{L} - ikv\big)\mathrm{e}(k,v) = \lambda(k)\mathrm{e}(k,v).$$

Letting $\lambda(k) = -ik\sigma(k)$, one has

$$\mathbf{P}_0 v\Big(\mathbf{P}_0\mathrm{e} + \mathbf{P}_1\mathrm{e}\Big) = \sigma(k)\mathbf{P}_0\mathrm{e}, \tag{3.42}$$

$$-\mathcal{L}\mathbf{P}_1\mathrm{e} - ik\mathbf{P}_1 v\Big(\mathbf{P}_0\mathrm{e} + \mathbf{P}_1\mathrm{e}\Big) = -ik\sigma(k)\mathbf{P}_1\mathrm{e}. \tag{3.43}$$

For any $k \in \mathbb{R}$ and $g \in \mathcal{N}(L)^\perp$, one has from (1.19) that

$$\mathrm{Re}\langle(-\mathcal{L} - ik\mathbf{P}_1 v)g, g\rangle = \langle -\mathcal{L}g, g\rangle \le -\delta_1 |g|_2^2.$$

Then for any $k \in \mathbb{R}$, the operator $(-\mathcal{L} - ik\mathbf{P}_1 v)$ is invertible on $\mathcal{N}(L)^\perp$. Thus, when $k \in \mathbb{R} \cap \{|k|, |\sigma| \ll 1\}$, the operator $(-\mathcal{L} - ik\mathbf{P}_1 v + ik\sigma)$ is also invertible on $\mathcal{N}(L)^\perp$. It follows from (3.42) and (3.43) that

$$\mathbf{P}_1\mathrm{e} = ik[-\mathcal{L} - ik\mathbf{P}_1 v + ik\sigma]^{-1}\mathbf{P}_1 v\mathbf{P}_0\mathrm{e}. \tag{3.44}$$

By using (3.44), we can rewrite (3.42) as

$$\Big(\mathbf{P}_0 v + ik\mathbf{P}_0 v[-\mathcal{L} - ik\mathbf{P}_1 v + ik\sigma]^{-1}\mathbf{P}_1 v\Big)\mathbf{P}_0\mathrm{e} = \sigma\mathbf{P}_0\mathrm{e}. \tag{3.45}$$

For simplicity of notation, put

$$\Im(\sigma,k) = \mathbf{P}_0 v\mathbf{P}_0 + ik\mathbf{P}_0 v[-\mathcal{L} - ik\mathbf{P}_1 v + ik\sigma]^{-1}\mathbf{P}_1 v\mathbf{P}_0 - \sigma\mathbf{P}_0.$$

Recalling (5.6), we denote

$$\mathbf{A}_{ij}(\sigma,k) = \langle \mathbf{P}_0 v[-\mathcal{L} - ik\mathbf{P}_1 v + ik\sigma]^{-1}\mathbf{P}_1 v\mathbf{P}_0\psi_i, \psi_j\rangle, \quad i,j = 0,2.$$

The operator $\Im(\sigma,k)$ can be represented by a matrix in the basis of $\psi_0$ and $\psi_2$. This matrix is

$$D(\sigma,k) = -\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} + ik\mathbf{A}_{ij}(\sigma,k).$$

Note that (3.45) has a solution if and only if we have $\det D(\sigma,k) = 0$. Since $[-\mathcal{L} - ik\mathbf{P}_1 v + ik\sigma]^{-1}$ exists for $k \in \mathbb{R} \cap \{|k|, |\sigma| \ll 1\}$, $\mathbf{A}_{ij}(\sigma,k)$ is smooth in this set. By the implicit function theorem, for any $k \in \mathbb{R} \cap \{|k|, |\sigma| \ll 1\}$, there exists a unique $C^\infty$ function $\sigma_j = \sigma_j(k)$ satisfying $\sigma_j(0) = 0$ with $j = 0,2$. This implies that there exist two eigenvalues $\lambda_0(k)$ and $\lambda_2(k)$ of $B(k)$ when $|k| \le k_0$.

Next we will compute the Taylor expansions of the functions $\{\lambda_j(k), \mathrm{e}_j(k,v)\}_{j=0,2}$ similarly to [16, 11] for the linearized Boltzmann and Landau equations. Set

$$\lambda_j(k) \cong \sum_{n=0}^{\infty} \lambda_j^{(n)} k^n, \qquad \mathrm{e}_j(k,v) \cong \sum_{n=0}^{\infty} \mathrm{e}_j^{(n)} k^n.$$

Comparing terms of the same order in $k$, we get

$$-\mathcal{L}\mathrm{e}_j^{(0)} = \lambda_j^{(0)}\mathrm{e}_j^{(0)}, \quad (j = 0, 2), \tag{3.46}$$

$$-\mathcal{L}\mathrm{e}_j^{(1)} = (\lambda_j^{(1)} + iv)\mathrm{e}_j^{(0)}, \quad (j = 0, 2), \tag{3.47}$$

$$-\mathcal{L}\mathrm{e}_j^{(n)} = (\lambda_j^{(1)} + iv)\mathrm{e}_j^{(n-1)} + \sum_{l=2}^{n}\lambda_j^{(l)}\mathrm{e}_j^{(n-l)}, \quad (n \geq 2, \quad j = 0, 2). \tag{3.48}$$

By (3.46) and the fact that $\lambda_j(0) = 0$, one has that $\lambda_j^{(0)} = 0$ and $\mathrm{e}_j^{(0)} = \psi_j(v)$ with $j = 0$, 2, which are as in (5.6). We take the inner product of (3.47) with $\mathrm{e}_j^{(0)}$ to get

$$\lambda_j^{(1)} = -i\langle v\mathrm{e}_j^{(0)}, \mathrm{e}_j^{(0)}\rangle = 0, \quad j = 0, 2. \tag{3.49}$$

Thus, one has from (3.47) that $\mathrm{e}_j^{(1)} = -i\mathcal{L}^{-1}\mathbf{P}_1 v\mathrm{e}_j^{(0)} + e_j^{''}$, where $e_j^{''} \in \mathcal{N}(\mathcal{L})$ and $\mathcal{L}^{-1}$ denotes the inverse of $\mathcal{L}$ restricted to $\mathcal{N}(\mathcal{L})^{\perp}$. We have from (3.48) that

$$-\mathcal{L}\mathrm{e}_j^{(2)} = (\lambda_j^{(1)} + iv)\mathrm{e}_j^{(1)} + \lambda_j^{(2)}\mathrm{e}_j^{(0)}. \tag{3.50}$$

We multiply $\mathrm{e}_j^{(0)}$ by (3.50) and integrate over $\mathbb{R}$ to deduce

$$0 = -\langle(\lambda_j^{(1)} + iv)i\mathcal{L}^{-1}\mathbf{P}_1 v\mathrm{e}_j^{(0)}, \mathrm{e}_j^{(0)}\rangle + \langle(\lambda_j^{(1)} + iv)e_j^{''}, \mathrm{e}_j^{(0)}\rangle + \lambda_j^{(2)}. \tag{3.51}$$

Noting that $e_j^{''} \in \mathcal{N}(\mathcal{L})$ and $\lambda_j^{(1)} = 0$, we see that

$$\langle(\lambda_j^{(1)} + iv)e_j^{''}, \mathrm{e}_j^{(0)}\rangle = \lambda_j^{(1)}\langle e_j^{''}, \mathrm{e}_j^{(0)}\rangle + i\langle ve_j^{''}, \mathrm{e}_j^{(0)}\rangle = 0.$$

By this, (3.51), (1.19), and (2.1), we have that

$$\lambda_j^{(2)} = -\langle\mathcal{L}^{-1}\mathbf{P}_1 v\mathrm{e}_j^{(0)}, \mathbf{P}_1 v\mathrm{e}_j^{(0)}\rangle < 0. \tag{3.52}$$

Recalling $\mathrm{e}_j^{(0)} = \psi_j(v)$ with $j = 0$, 2, without loss of generality, we assume that

$$\mathrm{e}_j^{(1)} = -i\mathcal{L}^{-1}\mathbf{P}_1 v\mathrm{e}_j^{(0)} + \sum_{m=0,2} B_{jm}^{(1)}\psi_m(v). \tag{3.53}$$

Here we can choose the constants $B_{jm}^{(1)}$ such that

$$\langle\mathrm{e}_0^{(1)}, \mathrm{e}_2^{(1)}\rangle = 0. \tag{3.54}$$

It follow from (3.50) and (3.53) that

$$\mathrm{e}_j^{(2)} = -i\mathcal{L}^{-1}\mathbf{P}_1 v\mathrm{e}_j^{(1)} + \sum_{m=0,2} B_{jm}^{(2)}\psi_m(v) = \mathcal{L}^{-1}\mathbf{P}_1 v\mathcal{L}^{-1}\mathbf{P}_1 v\mathrm{e}_j^{(0)} + \sum_{m=0,2} B_{jm}^{(2)}\psi_m(v). \tag{3.55}$$

Here we can choose the constants $B_{jm}^{(2)}$ such that

$$\langle\mathrm{e}_0^{(2)}, \mathrm{e}_2^{(2)}\rangle = 0. \tag{3.56}$$

By using (3.48) again, we can obtain any order expansions of eigenvalue and eigenfunction. Since $e^{(n)}(v)$ with $n \geq 1$ is some linear combination of $(\mathcal{L}^{-1}\mathbf{P}_1 v)^n \mathrm{e}_j^{(0)}(v)$ plus the combination of $\psi_0(v)$ and $\psi_2(v)$, by using (1.19), (2.1), and the induction, we can prove that $e^{(n)}(v) \in L^2(\mathbb{R})$.

If $\mathrm{e}_j(k,v)$ is an eigenfunction corresponding to the eigenvalue $\lambda_j(k)$ of $B(k)$, then $\mathrm{e}_j(-k,v)$ is an eigenfunction corresponding to the eigenvalue $\lambda_j(-k)$ of $B(-k)$. Hence, for any $i,j=0,2$,

$$\lambda_j(k)\langle \mathrm{e}_j(k,v), \mathrm{e}_i(-k,v)\rangle = \langle B(k)\mathrm{e}_j(k,v), \mathrm{e}_i(-k,v)\rangle = \langle \mathrm{e}_j(k,v), B(-k)\mathrm{e}_i(-k,v)\rangle$$

$$= \langle \mathrm{e}_j(k,v), \lambda_i(-k)\mathrm{e}_i(-k,v)\rangle = \lambda_i(k)\langle \mathrm{e}_j(k,v), \mathrm{e}_i(-k,v)\rangle.$$

Thus, $\langle \mathrm{e}_j(k,v), \mathrm{e}_i(-k,v)\rangle = 0$ when $i \neq j$. The normalization can be determined by choosing $\langle \mathrm{e}_j(k,v), \mathrm{e}_j(-k,v)\rangle = 1$. Therefore, $\{\mathrm{e}_j(k,v)\}_{j=0,2}$ can be normalized as

$$\langle \mathrm{e}_j(k,v), \mathrm{e}_j(-k,v)\rangle = \delta_{ij}, \quad j = 0, 2.$$

This and the property of the operator $B(k)$ imply that the eigen-nilpotent $Q_j(k) = 0$ for $j = 0$, 2. For any fixed $j$, let $C_j$ be a small circle which encloses $\lambda_j(k)$ but encloses no other eigenvalue. Then for any $f \in L^2(\mathbb{R})$, we have from the expression of $\mathrm{e}_j(k,v)$ that

$$P_j(k)f = \frac{1}{2\pi i}\oint_{C_j}(\lambda - B(k))^{-1} f d\lambda = \langle f, \mathrm{e}_j(-k,v)\rangle \mathrm{e}_j(k,v)$$

$$= \langle f, \mathrm{e}_j^{(0)}\rangle \mathrm{e}_j^{(0)} + |k|(\langle f, \mathrm{e}_j^{(0)}\rangle \mathrm{e}_j^{(1)} + \langle f, \mathrm{e}_j^{(1)}\rangle \mathrm{e}_j^{(0)} + O(k)).$$

Denote that

$$P_j^{(0)}(k)f = \langle f, \mathrm{e}_j^{(0)}\rangle \mathrm{e}_j^{(0)}, \quad P_j^{(1)}(k)f = \langle f, \mathrm{e}_j^{(0)}\rangle \mathrm{e}_j^{(1)} + \langle f, \mathrm{e}_j^{(1)}\rangle \mathrm{e}_j^{(0)} + O(k). \tag{3.57}$$

Then by the fact that $\mathbf{P}_0 f = \sum_{j=0,2}\langle f, \mathrm{e}_j^{(0)}\rangle \mathrm{e}_j^{(0)}$, we see that (III) holds. This completes the proof of the theorem. ◻

As an application of the spectrum structure obtained above, we will have the time decay rates of the solutions to the following linearized Kac equation:

$$\partial_t f + v\partial_x f + \mathcal{L}f = 0. \tag{3.58}$$

By Lemma 3.1 the operator $B(k)$ generates a continuous semigroup on $L^2(\mathbb{R}_v)$. By using Theorems 3.6 and 3.8 and an argument similar to Theorem 2.2.14 in [44] or Lemma 5.1 in [46], we can obtain the following properties of the semigroup $\Phi(t,k) = e^{tB(k)}$. For completeness, we give the proof.

Theorem 3.9. *For the semigroup $\Phi(t,k) = e^{tB(k)}$, we have*

$$\Phi(t,k)f = \Phi_1(t,k)f + \Phi_2(t,k)f, \quad f \in L^2(\mathbb{R}_v), \quad t \geq 0,$$

*where*

$$\Phi_1(t,k)f = \sum_{j=0,2} e^{t\lambda_j(k)}\langle f, \overline{e_j(k,v)}\rangle e_j(k,v)\mathbf{I}_{|k|\leq \kappa_0},$$

*and for some constant $\alpha_0 > 0$,*

$$|\Phi_2(t,k)|_2 \leq Ce^{-\alpha_0 t}.$$

*Proof.* For any $\lambda \in \mathrm{Res}(B(k)) \cap \mathrm{Res}(A(k))$, we have

$$(\lambda - B(k))^{-1} = (\lambda - A(k))^{-1} + (\lambda - A(k))^{-1}[I - K(\lambda - A(k))^{-1}]^{-1}K(\lambda - A(k))^{-1}$$
$$\equiv (\lambda - A(k))^{-1} + Z(\lambda, k). \quad (3.59)$$

By Lemma 3.5 and the Laplace inverse transform, we have for any $\sigma > 0$ and $t > 0$ that

$$e^{tB(k)}g = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t}(\lambda - B(k))^{-1} g(k, v)d\lambda.$$

It follows from this and (3.59) that

$$e^{tB(k)}g = e^{tA(k)}g + \lim_{a\to\infty}\frac{1}{2\pi i}\int_{\sigma - ia}^{\sigma + ia} e^{\lambda t} Z(\lambda, k)g(k, v)d\lambda. \quad (3.60)$$

We have from (3.4) that

$$|e^{tA(k)}g|_2 \le e^{-\nu_0 t}|g|_2. \quad (3.61)$$

We shift the integral path of (3.60) from $\mathrm{Re}\lambda = \sigma$ to $\mathrm{Re}\lambda = -\tau$, where $\tau > 0$ is given in Theorem 3.8. We can choose $\tau$ such that

$$\frac{1}{2\pi i}\int_{\sigma - ia}^{\sigma + ia} e^{\lambda t} Z(\lambda, k)g(k, v)d\lambda = \sum_{j=0,2} \mathcal{R}es\{e^{\lambda t} Z(\lambda, k)g;\ \lambda = \lambda_j(k)\}$$
$$+ \frac{1}{2\pi i}\int_{-\tau - ia}^{-\tau + ia} e^{\lambda t} Z(\lambda, k)g(k, v)d\lambda$$
$$+ \frac{1}{2\pi i}\int_{-\tau + ia}^{\sigma + ia} e^{\lambda t} Z(\lambda, k)g(k, v)d\lambda + \frac{1}{2\pi i}\int_{\sigma - ia}^{-\tau - ia} e^{\lambda t} Z(\lambda, k)g(k, v)d\lambda. \quad (3.62)$$

Here, $\mathcal{R}es$ means the residue. Since $\lambda_j(k) \in \mathrm{Res}(A(k))$, we know that

$$\mathcal{R}es\{e^{\lambda t} Z(\lambda, k);\ \lambda = \lambda_j(k)\} = \mathcal{R}es\{e^{\lambda t}(\lambda - B(k))^{-1};$$
$$\lambda = \lambda_j(k)\} = e^{\lambda_j(k)t}P_j(k) \quad (3.63)$$

for any $|k| \le \kappa_0$; otherwise, it is 0. It follows from (3.59) and Lemma 3.4 that

$$\Big|\int_{-\tau + ia}^{\sigma + ia} e^{\lambda t} Z(\lambda, k)d\lambda\Big|_2 + \Big|\int_{\sigma - ia}^{-\tau - ia} e^{\lambda t} Z(\lambda, k)d\lambda\Big|_2 \to 0 \quad \text{as} \quad a \to \infty. \quad (3.64)$$

Next we consider the second term of (3.62). By Lemma 3.4, one has $|[I - K(\lambda - A(k))^{-1}]^{-1}|_2 \le C$ for $|\mathrm{Im}\lambda|$ large enough. Since $\{\lambda; \mathrm{Re}\lambda = -\tau\} \subset \mathrm{Res}(A(k))$ and $B(k)$ has no eigenvalue on the line $\mathrm{Re}\lambda = -\tau$ for $|\mathrm{Im}\lambda|$ bounded, it follows from (3.59) that $|[I - K(\lambda - A(k))^{-1}]^{-1}|_2 \le C$, where $C$ is a constant independent of $k$. Thus it holds for any $g$, $h \in L^2$ that

$$\Big|\Big\langle \int_{-\tau - ia}^{-\tau + ia} e^{\lambda t} Z(\lambda, k)g(k, v)d\lambda, h\Big\rangle\Big| = \Big|\Big\langle \int_{-a}^{a} e^{(-\tau + is)t} Z(-\tau + is, k)g(k, v)ds, h\Big\rangle\Big|$$

$$\le C|K|_2 e^{-\tau t}\int_{-a}^{a} |(-\tau + is - A(k))^{-1}g|_2 |(-\tau + is - A(-k))^{-1}h|_2 ds, \quad (3.65)$$

which is less than $Ce^{-\tau t}|g|_2|h|_2$. In view of (3.60), (3.62), and (3.63), we arrive at

$$\Phi_1(t,k)f = \sum_{j=0,2} e^{t\lambda_j(k)}\langle f, \overline{\mathrm{e}_j(k,v)}\rangle \mathrm{e}_j(k,v)\mathbf{I}_{|k|\le\kappa_0}.$$

With the help of (3.60), (3.61), (3.62), and (3.65), we put $e^{tA(k)}$ and the other terms in (3.62) into $\Phi_2(t,k)$. Thus, for some constant $\alpha_0>0$, we can obtain

$$|\Phi_2(t,k)|_2 \le Ce^{-\alpha_0 t}.$$

We conclude the proof of the lemma. ∎

For any $f_0\in H^N_x(L^2_v)$ with $N\ge 1$, we denote $e^{tB}f_0$ by $e^{tB}f_0 = (\mathcal{F}^{-1}e^{tB(k)}\mathcal{F})f_0$. We see that

$$\|e^{tB}f_0\|^2_{H^N_x(L^2_v)} = \int_{\mathbb{R}}(1+|k|^2)^N|e^{tB(k)}\hat f_0|_2^2dk \le C\int_{\mathbb{R}}(1+|k|^2)^N|\hat f_0|_2^2dk = C\|f_0\|^2_{H^N_x(L^2_v)}.$$

Then the operator $B$ generates a continuous semigroup $e^{tB}$ in $H^N_x(L^2_v)$, and $f(t,x,v)=e^{tB}f_0$ is a global solution to (3.58) with the initial data $f(0,x,v)=f_0(x,v)$. Next we prove the time decay rates of the solution $f(t,x,v)$.

Theorem 3.10. *Assume that $f_0\in H^N_x\cap Z^q$ with $N\ge 1$ and $q\in[1,2]$. The global solutions $f(t,x,v)$ to (3.58) satisfies, for any $\gamma$, $\gamma'\in\mathbb{N}$ with $|\gamma|\le N$, $\gamma'\le\gamma$, and $m=|\gamma-\gamma'|$, that*

$$\|\partial_x^\gamma \mathbf{P}_0 f\| \le C(1+t)^{-\frac12(\frac1q-\frac12)-\frac m2}(\|\partial_x^\gamma f_0\| + \|\partial_x^{\gamma'}f_0\|_{Z^q}), \tag{3.66}$$

$$\|\partial_x^\gamma \mathbf{P}_1 f\| \le C(1+t)^{-\frac12(\frac1q-\frac12)-\frac{m+1}2}(\|\partial_x^\gamma f_0\| + \|\partial_x^{\gamma'}f_0\|_{Z^q}). \tag{3.67}$$

*If $\mathbf{P}_0 f_0 = 0$, we have the faster time decay rates*

$$\|\partial_x^\gamma \mathbf{P}_0 f\| \le C(1+t)^{-\frac12(\frac1q-\frac12)-\frac{m+1}2}(\|\partial_x^\gamma f_0\| + \|\partial_x^{\gamma'}f_0\|_{Z^q}), \tag{3.68}$$

$$\|\partial_x^\gamma \mathbf{P}_1 f\| \le C(1+t)^{-\frac12(\frac1q-\frac12)-\frac{m+2}2}(\|\partial_x^\gamma f_0\| + \|\partial_x^{\gamma'}f_0\|_{Z^q}). \tag{3.69}$$

*Furthermore, if one assumes that $f_0\in Z^1$ and there exist some constants $k_0>0$ and $d_0>0$ such that $\inf_{|k|\le k_0}|\langle \hat f_0,\psi_j(v)\rangle|\ge d_0$ for $j=0$, $2$, then it holds that for $t>0$ large enough, $|\gamma|=m$, and some constants $0<C_1\le C_2$,*

$$C_1(1+t)^{-\frac14-\frac m2} \le \|\partial_x^\gamma f\| \le C_2(1+t)^{-\frac14-\frac m2}. \tag{3.70}$$

*Proof.* By Theorem 3.9, we have that

$$\|k^\gamma\Phi_2(t,k)\hat f_0\|^2_{L^2_{k,v}} \le C\int_{\mathbb{R}} e^{-2\alpha_0 t}|k^\gamma|^2|\hat f_0|_2^2dk \le Ce^{-2\alpha_0 t}\|\partial_x^\gamma f_0\|^2. \tag{3.71}$$

It follows from Theorems 3.9 and 3.8 that for any $|k|\le k_0$,

$$\begin{aligned}\Phi_1(t,k)\hat f_0 &= \sum_{j=0,2} e^{t\lambda_j(k)}\langle \hat f_0, \overline{\mathrm{e}_j(k,v)}\rangle \mathrm{e}_j(k,v) = \sum_{j=0,2} e^{t\lambda_j(k)}\Big\{\langle \hat f_0,\psi_j(v)\rangle\psi_j(v)\\ &\quad + k\Big(\langle \hat f_0,\psi_j(v)\rangle \mathrm{e}_j^{(1)}(v) + \langle \hat f_0, \overline{\mathrm{e}_j^{(1)}(v)}\rangle\psi_j(v)\Big) + k^2\widetilde\Phi_j(k)\hat f_0\Big\}.\end{aligned} \tag{3.72}$$

Here we have used the fact that $\mathrm{e}_j(k,v)=\psi_j(v)+k\mathrm{e}_j^{(1)}+O(|k|^2)$ as $|k|\to 0$, and by Theorem 3.8, $\widetilde{\Phi}_j(k)$ is a uniformly bounded linear operator in $L^2(\mathbb{R}_v)$ for any $|k|\le\kappa_0$. For any $l\in\{0,2\}$, one has from (3.72) that

$$\langle\Phi_1(t,k)\hat{f}_0,\psi_l(v)\rangle=e^{t\lambda_l(k)}\langle\hat{f}_0,\psi_l(v)\rangle+k\sum_{j=0,2}e^{t\lambda_j(k)}\langle\widetilde{\Phi}_3^j(k)\hat{f}_0,\psi_l(v)\rangle \tag{3.73}$$

and

$$\mathbf{P}_1(\Phi_1(t,k)\hat{f}_0)=k\sum_{j=0,2}e^{t\lambda_j(k)}\mathbf{P}_1\widetilde{\Phi}_4^j(k)\hat{f}_0. \tag{3.74}$$

Here, by Theorem 3.8, $\widetilde{\Phi}_3^j(k)$ and $\widetilde{\Phi}_4^j(k)$ are uniformly bounded linear operators in $L^2(\mathbb{R}_v)$ for any $|k|\le k_0$.

Note that $\mathrm{Re}\lambda_j(k)=\lambda_j^{(2)}k^2+O(|k|^3)\le-\alpha_1|k|^2$ for any $|k|\le k_0$, where $\max(\lambda_0^{(2)},\lambda_2^{(2)})<-\alpha_1$ for some $\alpha_1>0$. We have from (3.73) and (3.74) that

$$|\langle\Phi_1(t,k)\hat{f}_0,\psi_l(v)\rangle|^2\le Ce^{-2\alpha_1|k|^2t}(|\langle\hat{f}_0,\psi_l(v)\rangle|^2+|k|^2|\hat{f}_0|_2^2)\le Ce^{-2\alpha_1|k|^2t}|\hat{f}_0|_2^2 \tag{3.75}$$

and

$$|\mathbf{P}_1(\Phi_1(t,k)\hat{f}_0)|_2^2\le Ce^{-2\alpha_1|k|^2t}|k|^2|\hat{f}_0|_2^2. \tag{3.76}$$

One has from (3.75) that

$$\begin{aligned}\|k^\gamma\langle\Phi_1(t,k)\hat{f}_0,\psi_l(v)\rangle\|_{L_k^2}^2&\le C\int_{|k|\le k_0}|k^\gamma|^2e^{-2\alpha_1|k|^2}|\hat{f}_0|_2^2dk\\&\le C\Big(\int_{|k|\le k_0}|k|^{2p'm}e^{-2p'\alpha_1|k|^2t}dk\Big)^{1/p'}\Big(\int_{|k|\le k_0}|k^{\gamma'}\hat{f}_0|_2^{2q'}dk\Big)^{1/q'}\end{aligned} \tag{3.77}$$

with $\frac{1}{p'}+\frac{1}{q'}=1$ and $m=|\gamma-\gamma'|$. Note that

$$\int_{|k|\le k_0}|k|^{2p'm}e^{-2p'\alpha_1|k|^2t}dk\le C(1+t)^{-1/2-p'm}.$$

For $\frac{1}{q}+\frac{1}{2q'}=1$, one has from the Hausdorff–Young inequality that

$$\Big(\int_{|k|\le k_0}|k^{\gamma'}\hat{f}_0|_2^{2q'}dk\Big)^{1/q'}\le\int_{\mathbb{R}_v}\Big(\int_{|k|\le k_0}|k^{\gamma'}\hat{f}_0|^{2q'}dk\Big)^{1/q'}dv\le C\|\partial^{\gamma'}f_0\|_{Z^q}^2.$$

Then we have from (3.71) and (3.77) that

$$\begin{aligned}\|\partial_x^\gamma\mathbf{P}_0f\|&\le C\sum_{l=0,2}\|\langle\partial_x^\gamma f,\psi_l(v)\rangle\|\\&\le C\sum_{l=0,2}\|\langle k^\gamma\Phi_1(t,k)\hat{f}_0,\psi_l(v)\rangle\|_{L_k^2}+C\|k^\gamma\Phi_2(t,k)\hat{f}_0\|_{L_{k,v}^2}\\&\le C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{2})-\frac{m}{2}}\|\partial^{\gamma'}f_0\|_{Z^q}+Ce^{-\alpha_0t}\|\partial_x^\gamma f_0\|.\end{aligned} \tag{3.78}$$

Thus, we can obtain (3.66) from (3.79). By (3.76) and (3.71), an argument similar to (3.79) implies that (3.67) holds.

Next we will prove (3.68) and (3.69). If $\mathbf{P}_0\hat{f}_0 = 0$, the expression (3.72) is reduced to

$$\Phi_1(t,k)\hat{f}_0 = \sum_{j=0,2}\Big\{ke^{t\lambda_j(k)}\langle \hat{f}_0, \overline{\mathrm{e}_j^{(1)}(v)}\rangle\psi_j(v) + k^2\widetilde{\Phi}_j(k)\hat{f}_0\Big\}. \tag{3.79}$$

Noting that $\mathrm{e}_j(k,v) = \psi_j(v) + k\mathrm{e}_j^{(1)} + k^2\mathrm{e}_j^{(2)} + O(|k|^3)$ as $|k| \to 0$, we have that for any $|k| \le k_0$,

$$\begin{aligned}
\Phi_1(t,k)\hat{f}_0 &= \sum_{j=0,2} e^{t\lambda_j(k)}\langle \hat{f}_0, \overline{\mathrm{e}_j(k,v)}\rangle \mathrm{e}_j(k,v)\\
&= \sum_{j=0,2} e^{t\lambda_j(k)}\Big\{\langle \hat{f}_0, \psi_j(v)\rangle\psi_j(v) + k\Big(\langle \hat{f}_0, \psi_j(v)\rangle \mathrm{e}_j^{(1)}(v) + \langle \hat{f}_0, \overline{\mathrm{e}_j^{(1)}(v)}\rangle\psi_j(v)\Big)\\
&\quad + k^2\Big(\langle \hat{f}_0, \psi_j(v)\rangle \mathrm{e}_j^{(2)}(v) + \langle \hat{f}_0, \overline{\mathrm{e}_j^{(1)}(v)}\rangle \mathrm{e}_j^{(1)}(v) + \langle \hat{f}_0, \overline{\mathrm{e}_j^{(2)}(v)}\rangle\psi_j(v)\Big) + k^3\widetilde{\Psi}_j(k)\hat{f}_0\Big\}.
\end{aligned} \tag{3.80}$$

Here, by Theorem 3.8, $\widetilde{\Psi}_j(k)$ is a uniformly bounded linear operator in $L^2(\mathbb{R}_v)$ for any $|k| \le \kappa_0$.

If $\mathbf{P}_0\hat{f}_0 = 0$, it follows from (3.80) that

$$\mathbf{P}_1\Phi_1(t,k)\hat{f}_0 = \sum_{j=0,2}\Big\{k^2e^{t\lambda_j(k)}\langle \hat{f}_0, \overline{\mathrm{e}_j^{(1)}(v)}\rangle\mathbf{P}_1\mathrm{e}_j^{(1)}(v) + k^3\mathbf{P}_1\widetilde{\Psi}_j(k)\hat{f}_0\Big\}. \tag{3.81}$$

Similarly to (3.71)–(3.79), with (3.79) and (3.81), the time decay rates in (3.68) and (3.69) hold.

Now we will prove the lower bound of the time decay rates of (3.70). Note that $\mathrm{Re}\lambda_j(k) = \lambda_j^{(2)}k^2 + O(|k|^3) \ge -\alpha_2 k^2$ for any $|k| \le k_0$, where $\min(\lambda_0^{(2)}, \lambda_2^{(2)}) > -\alpha_2$ for some $\alpha_2 > 0$.

By (3.73), for any $l \in \{0,2\}$ and $|\gamma| = m$, we have

$$\begin{aligned}
|\langle k^\gamma\Phi_1(t,k)\hat{f}_0, \psi_l(v)\rangle|^2 &= |k^\gamma e^{t\lambda_l(k)}\langle \hat{f}_0, \psi_l(v)\rangle + k^\gamma\sum_{j=0,2} e^{t\lambda_j(k)}k\langle\widetilde{\Phi}_3^j(k)\hat{f}_0, \psi_l(v)\rangle|^2\\
&\ge Ce^{-2\alpha_2|k|^2t}|k|^{2m}|\langle \hat{f}_0, \psi_l(v)\rangle|^2 - C|k|^{2m+2}e^{-2\alpha_1|k|^2t}|\hat{f}_0|_2^2.
\end{aligned}$$

Then, for $t \ge 1$, we have from this and (3.77) that

$$\begin{aligned}
\|\langle k^\gamma\Phi_1(t,k)\hat{f}_0, \psi_l(v)\rangle\|_{L_k^2}^2 &\ge C\int_{|k|\le k_0} e^{-2\alpha_2|k|^2t}|k|^{2m}|\langle \hat{f}_0, \psi_l(v)\rangle|^2dk\\
&\quad - C\int_{|k|\le k_0}|k|^{2m+2}e^{-2\alpha_1|k|^2t}|\hat{f}_0|_2^2dk\\
&\ge C_1\inf_{|k|\le k_0}|\langle \hat{f}_0, \psi_l(v)\rangle|^2 t^{-\frac{2m+1}{2}}\int_0^{\sqrt{2\alpha_2}k_0} e^{-y^2}y^{2m}dy\\
&\quad - C_2t^{-\frac{2m+3}{2}}\|f_0\|_{Z^1}^2.
\end{aligned}$$

Since $f_0 \in Z^1$ and $\inf_{|k|\le k_0}|\langle \hat{f}_0, \psi_j(v)\rangle| \ge d_0$ for $j = 0,\ 2$, when $t \ge 1$ large enough, we have that

$$\|\langle k^\gamma\Phi_1(t,k)\hat{f}_0, \psi_l(v)\rangle\|_{L_k^2}^2 \ge C_3d_0^2t^{-\frac{2m+1}{2}} - C_4t^{-\frac{2m+3}{2}} \ge C_5(1+t)^{-\frac{2m+1}{2}}.$$

It follows from (3.71) and the above estimates that, when $t \geq 1$ large enough, the following holds:

$$
\begin{aligned}
\|\partial_x^\gamma \mathbf{P}_0 f\|^2 &= \sum_{j=0,2} \|\langle \partial_x^\gamma f, \psi_j(v)\rangle\|^2 = \sum_{j=0,2} \|\langle k^\gamma \hat{f}, \psi_j(v)\rangle\|^2_{L^2_k} \\
&\geq C \sum_{j=0,2} \|\langle k^\gamma \Phi_1(t,k)\hat{f}_0, \psi_j(v)\rangle\|^2_{L^2_k} - C\|k^\gamma \Phi_2(t,k)\hat{f}_0\|^2_{L^2_{k,v}} \\
&\geq C_6(1+t)^{-\frac{2m+1}{2}} - Ce^{-2\alpha_0 t}\|\partial_x^\gamma f_0\|^2. \qquad (3.82)
\end{aligned}
$$

When $t \geq 1$ large enough, it follows from (3.82) and (5.7) that

$$
\|\partial_x^\gamma f\|^2 = \|\partial_x^\gamma \mathbf{P}_0 f\|^2 + \|\partial_x^\gamma \mathbf{P}_1 f\|^2 \geq C_7(1+t)^{-\frac{2m+1}{2}}. \qquad (3.83)
$$

This, (3.66), and (3.67) imply that (3.70) holds. This completes the proof of Theorem 3.10. ∎

**4. Time decay rate of the nonlinear Kac equation.** In this section, we will first deduce some energy estimates for the nonlinear Kac equation (1.26) and then prove the time decay rate of global solutions constructed in Theorem 1.1. To this end, we first define the instant energy function $\widetilde{\mathfrak{E}}(t)$ for the solution to (1.26) as

$$
\widetilde{\mathfrak{E}}(t) \approx \mathfrak{E}(t) = \sum_{|\gamma|+|\beta|\leq 3} \|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}} f\|^2. \qquad (4.1)
$$

In addition, the corresponding dissipation functional denoted by $\widetilde{\mathfrak{D}}(t)$ satisfies

$$
\widetilde{\mathfrak{D}}(t) \approx \mathfrak{D}(t) = \sum_{1\leq|\gamma|\leq 3} \|\partial_x^\gamma \mathbf{P}_0 f\|^2 + \sum_{|\gamma|+|\beta|\leq 3} \|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta+s}{2}} \mathbf{P}_1 f\|^2. \qquad (4.2)
$$

Here the projections $\mathbf{P}_0$ and $\mathbf{P}_1$ are defined as in (5.7). By using (5.7), we rewrite (1.26) as

$$
[\partial_t + v\partial_x]\mathbf{P}_0 f = -[\partial_t + v\partial_x]\mathbf{P}_1 f - \mathcal{L}f + \Gamma(f,f). \qquad (4.3)
$$

With the help of (5.6) and (5.7), we can write

$$
\mathbf{P}_0 f = a_0\psi_0 + a_2\psi_2. \qquad (4.4)
$$

Recalling (2.1), we have from (4.3) and (4.4) that

$$
\begin{cases}
\partial_x a_0 + \sqrt{2}\partial_x a_2 = \langle -[\partial_t + v\partial_x]\mathbf{P}_1 f - \mathcal{L}f + \Gamma(f,f), v\psi_0\rangle, \\
\sqrt{2}\partial_x a_0 + 5\partial_x a_2 = \langle -[\partial_t + v\partial_x]\mathbf{P}_1 f - \mathcal{L}f + \Gamma(f,f), v\psi_2\rangle.
\end{cases}
$$

It follows that

$$
\begin{cases}
\partial_x \partial_x^\gamma a_0 = \langle -[\partial_t + v\partial_x]\partial_x^\gamma \mathbf{P}_1 f - \mathcal{L}\partial_x^\gamma f + \partial_x^\gamma \Gamma(f,f), \frac{1}{3}v(5\psi_0 - \sqrt{2}\psi_2)\rangle, \\
\partial_x \partial_x^\gamma a_2 = \langle -[\partial_t + v\partial_x]\partial_x^\gamma \mathbf{P}_1 f - \mathcal{L}\partial_x^\gamma f + \partial_x^\gamma \Gamma(f,f), \frac{1}{3}v(\psi_2 - \sqrt{2}\psi_0)\rangle.
\end{cases} \qquad (4.5)
$$

We will make use of the system (4.5) to estimate the term $\|\partial_x \partial_x^\gamma \mathbf{P}_0 f\|^2$ with $|\gamma| \leq 2$.

LEMMA 4.1. *If $\mathfrak{E}(t) < \varepsilon_1$ for some $\varepsilon_1 > 0$ small enough, there exist functions $\bar{\xi}_3(v)$ and $\bar{\xi}_4(v)$, which are exponential decay rates in $v$ such that, for any $|\gamma| \le 2$,*

$$\sum_{|\gamma|\le 2} \frac{d}{dt}[(\partial_x^\gamma \mathbf{P}_1 f, \partial_x \partial_x^\gamma a_0 \bar{\xi}_3(v)) + (\partial_x^\gamma \mathbf{P}_1 f, \partial_x\partial_x^\gamma a_2 \bar{\xi}_4(v))] + \sum_{|\gamma|\le 2} \|\partial_x \partial_x^\gamma \mathbf{P}_0 f\|^2$$

$$\le C \sum_{|\gamma|\le 3} \|\partial_x^\gamma \mathbf{P}_1 f\|^2 + C \sup_{x\in\mathbb{R}} |\mathbf{P}_0 f|_2^2 \|\mathbf{P}_0 f\|^2 + C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t). \tag{4.6}$$

*Proof.* By (2.1), there are two constants $c_1$ and $c_2$ such that

$$\frac{1}{3} v(5\psi_0 - \sqrt{2}\psi_2) = c_1\psi_3(v) + c_2\psi_1(v) = \xi_3(v). \tag{4.7}$$

By using this and $(4.5)_1$, we have that for $|\gamma| \le 2$,

$$\|\partial_x \partial_x^\gamma a_0\|^2 = (-[\partial_t + v\partial_x]\partial_x^\gamma \mathbf{P}_1 f - \mathcal{L}\partial_x^\gamma f + \partial_x^\gamma \Gamma(f,f), \partial_x\partial_x^\gamma a_0 \xi_3(v)). \tag{4.8}$$

Now we estimate the right-hand side in (4.8). It holds that

$$\begin{aligned}(-\partial_t\partial_x^\gamma \mathbf{P}_1 f, \partial_x\partial_x^\gamma a_0\xi_3(v)) &= -\frac{d}{dt}(\partial_x^\gamma \mathbf{P}_1 f, \partial_x\partial_x^\gamma a_0 \xi_3(v)) - (\partial_x\partial_x^\gamma \mathbf{P}_1 f, \partial_x^\gamma a_{0t}\xi_3(v))\\ &\le -\frac{d}{dt}(\partial_x^\gamma \mathbf{P}_1 f, \partial_x\partial_x^\gamma a_0\xi_3(v)) + C_\epsilon \|\partial_x\partial_x^\gamma \mathbf{P}_1 f\|^2 + \epsilon\|\partial_x^\gamma a_{0t}\|^2.\end{aligned}$$

Note that

$$(-v\partial_x\partial_x^\gamma \mathbf{P}_1 f, \partial_x\partial_x^\gamma a_0\xi_3(v)) \le C_\epsilon\|\partial_x\partial_x^\gamma \mathbf{P}_1 f\|^2 + \epsilon\|\partial_x\partial_x^\gamma a_0\|^2.$$

It follows from (2.2) and (2.3) that

$$\begin{aligned}(-\mathcal{L}\partial_x^\gamma f, \partial_x\partial_x^\gamma a_0 \xi_3(v)) &= (-\partial_x^\gamma \mathbf{P}_1 f, \partial_x\partial_x^\gamma a_0 \mathcal{L}\xi_3(v))\\ &= (-\partial_x^\gamma \mathbf{P}_1 f, \partial_x\partial_x^\gamma a_0(c_1\lambda_3\psi_3 + c_2\lambda_1\psi_1))\\ &\le C_\epsilon\|\partial_x^\gamma \mathbf{P}_1 f\|^2 + \epsilon\|\partial_x\partial_x^\gamma a_0\|^2.\end{aligned}$$

For the last term, we deduce from (5.65) and (5.53) that

$$\begin{aligned}(\partial_x^\gamma\Gamma(f,f), \partial_x\partial_x^\gamma a_0\psi_3(v)) &= \sum_{\gamma_1\le\gamma}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} C_\gamma^{\gamma_1}(\alpha_{k,l}\partial_x^{\gamma_1} f_k \partial_x^{\gamma-\gamma_1} f_l \psi_{k+l}(v), \partial_x\partial_x^\gamma a_0\psi_3(v))\\ &= \sum_{\gamma_1\le\gamma}\sum_{k+l=3} C_\gamma^{\gamma_1}(\alpha_{k,l}\partial_x^{\gamma_1} f_k\partial_x^{\gamma-\gamma_1} f_l\psi_{k+l}(v), \partial_x\partial_x^\gamma a_0).\end{aligned}$$

By using (1.7), (5.54), and (5.55), we have that $\alpha_{k,l}$ with $k+l=3$ is bounded. For any $|\gamma| \le 2$, by the imbedding theorem, one has from (5.7), (4.1), (4.2), and (5.17) that

$$\begin{aligned}(\partial_x^\gamma\Gamma(f,f), \partial_x\partial_x^\gamma a_0\psi_3(v)) &= \sum_{\gamma_1\le\gamma}\sum_{k+l=3} C_\gamma^{\gamma_1}(\alpha_{k,l}\partial_x^{\gamma_1} f_k\partial_x^{\gamma-\gamma_1} f_l\psi_{k+l}(v), \partial_x\partial_x^\gamma a_0)\\ &\le C\sum_{\gamma_1\le\gamma}(|\partial_x^{\gamma_1} f|_2 |\partial_x^{\gamma-\gamma_1} f|_2, |\partial_x\partial_x^\gamma a_0|)\\ &\le C_\epsilon \sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2 + \epsilon\|\partial_x\partial_x^\gamma a_0\|^2 + C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\end{aligned}$$

By this and (4.7) we have that

$$(\partial_x^\gamma\Gamma(f,f),\partial_x\partial_x^\gamma a_0\xi_3(v))\le C_\epsilon\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2+\epsilon\|\partial_x\partial_x^\gamma a_0\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).$$

Thus, for $|\gamma|\le 2$, we have from (4.8) and the above estimates that

$$\begin{aligned}\|\partial_x\partial_x^\gamma a_0\|^2\le&-\frac{d}{dt}(\partial_x^\gamma\mathbf{P}_1f,\partial_x\partial_x^\gamma a_0\xi_3(v))+C_\epsilon\|\partial_x\partial_x^\gamma\mathbf{P}_1f\|^2+C_\epsilon\|\partial_x^\gamma\mathbf{P}_1f\|^2\\&+\epsilon\|\partial_x^\gamma a_{0t}\|^2+\epsilon\|\partial_x\partial_x^\gamma a_0\|^2+C\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\end{aligned}\tag{4.9}$$

Similarly, for the term $\partial_x\partial_x^\gamma a_2$ and some function $\xi_4(v)$, we also have

$$\begin{aligned}\|\partial_x\partial_x^\gamma a_2\|^2\le&-\frac{d}{dt}(\partial_x^\gamma\mathbf{P}_1f,\partial_x\partial_x^\gamma a_2\xi_4(v))+C_\epsilon\|\partial_x\partial_x^\gamma\mathbf{P}_1f\|^2+C_\epsilon\|\partial_x^\gamma\mathbf{P}_1f\|^2\\&+\epsilon\|\partial_x^\gamma a_{2t}\|^2+\epsilon\|\partial_x\partial_x^\gamma a_2\|^2+C\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\end{aligned}\tag{4.10}$$

It follows from (4.4) and (1.26) that

$$\begin{aligned}\|\partial_x^\gamma a_{0t}\|^2+\|\partial_x^\gamma a_{2t}\|^2\approx\|\partial_t\partial_x^\gamma\mathbf{P}_0f\|^2&=\|\partial_x^\gamma\mathbf{P}_0\partial_tf\|^2=\|\partial_x^\gamma\mathbf{P}_0(-v\partial_xf-\mathcal{L}f+\Gamma(f,f))\|^2\\&\le C\|\partial_x\partial_x^\gamma\mathbf{P}_0f\|^2+C\|\partial_x\partial_x^\gamma\mathbf{P}_1f\|^2.\end{aligned}\tag{4.11}$$

Here, we have used the facts that

$$\mathbf{P}_0\mathcal{L}f=0,\qquad \mathbf{P}_0\Gamma(f,f)=0.$$

If we choose $\epsilon>0$ small enough, we can have from (4.9), (4.10), and (4.11) that

$$\begin{aligned}\|\partial_x\partial_x^\gamma a_0\|^2+\|\partial_x\partial_x^\gamma a_2\|^2&\approx\|\partial_x\partial_x^\gamma\mathbf{P}_0f\|^2\\&\le-\frac{d}{dt}[(\partial_x^\gamma\mathbf{P}_1f,\partial_x\partial_x^\gamma a_0\bar{\xi}_3(v))+(\partial_x^\gamma\mathbf{P}_1f,\partial_x\partial_x^\gamma a_2\bar{\xi}_4(v))]\\&\quad+C\|\partial_x^\gamma\mathbf{P}_1f\|^2+C\|\partial_x\partial_x^\gamma\mathbf{P}_1f\|^2\\&\quad+C\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\end{aligned}\tag{4.12}$$

This implies that (4.6) holds, which concludes the proof of Lemma 4.1. ∎

Next we will consider the estimate of the nonlinear term, which will be used to get the estimate of the derivatives of the solution to (1.26).

LEMMA 4.2. *Assume that $|\gamma|+|\beta|\le 3$ and $\mathfrak{E}(t)<\varepsilon_1$ for some $\varepsilon_1>0$ small enough. For any $\epsilon>0$, one has*

$$|(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(\Gamma(f,f)-\Gamma(\mathbf{P}_0f,\mathbf{P}_0f)),h)|\le\epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2+C_\epsilon\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\tag{4.13}$$

*Proof.* By using the decomposition (5.7), we have

$$\Gamma(f,f)-\Gamma(\mathbf{P}_0f,\mathbf{P}_0f)=\Gamma(\mathbf{P}_0f,\mathbf{P}_1f)+\Gamma(\mathbf{P}_1f,\mathbf{P}_0f)+\Gamma(\mathbf{P}_1f,\mathbf{P}_1f).\tag{4.14}$$

We only consider the first term of the right-hand side in (4.14), while the other terms can be estimated similarly. It follows from Lemma 5.3 that

$$\begin{aligned}|(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(\mathbf{P}_0f,\mathbf{P}_1f),h)|\le&C\sum_{\gamma_1\le\gamma}\int_{\mathbb{R}_x}(|\partial_x^{\gamma_1}\mathbf{P}_0f|_2|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1f|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2\\&+|\partial_x^{\gamma_1}\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_0f|_2|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1f|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2)dx.\end{aligned}\tag{4.15}$$

We only estimate the first term in (4.15) because the second term can be estimated similarly.

If $|\gamma-\gamma_1|+|\beta|\le 1$, for any $\epsilon>0$, we get

$$\begin{aligned}\int_{\mathbb{R}_x} &|\partial_x^{\gamma_1}\mathbf{P}_0 f|_2|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1 f|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2dx\\ &\le \epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2+C_\epsilon\sup_{x\in\mathbb{R}}|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1 f|_2^2\|\partial_x^{\gamma_1}\mathbf{P}_0 f\|^2\\ &\le \epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2+C_\epsilon\left(\sum_{|\gamma_2|\le 1}\|\partial_x^{\gamma-\gamma_1+\gamma_2}\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1 f\|^2\right)\|\partial_x^{\gamma_1}\mathbf{P}_0 f\|^2.\end{aligned}$$

If $|\gamma-\gamma_1|+|\beta|\ge 2$, one has

$$\begin{aligned}\int_{\mathbb{R}_x} &|\partial_x^{\gamma_1}\mathbf{P}_0 f|_2|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1 f|_2|\mathcal{H}_v^{\frac{s}{2}}h|_2dx\\ &\le \epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2+C_\epsilon\sup_{x\in\mathbb{R}}|\partial_x^{\gamma_1}\mathbf{P}_0 f|_2^2\|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1 f\|^2\\ &\le \epsilon\|\mathcal{H}_v^{\frac{s}{2}}h\|^2+C_\epsilon\left(\sum_{|\gamma_2|\le 1}\|\partial_x^{\gamma_1+\gamma_2}\mathbf{P}_0 f\|^2\right)\|\partial_x^{\gamma-\gamma_1}\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1 f\|^2.\end{aligned}$$

The above two estimates are both bounded by the right-hand side of (4.13). Since the second part in (4.15) can be estimated similarly, we complete the proof of this lemma. ∎

LEMMA 4.3. *If $\mathfrak{E}(t)<\varepsilon_1$ for some $\varepsilon_1>0$ small enough, for any $\epsilon>0$ small enough, one has*

$$\begin{aligned}(4.16)\quad &\frac{d}{dt}\Big[\|\mathbf{P}_1 f\|^2+\sum_{1\le|\gamma|\le 3}\|\partial_x^\gamma f\|^2\Big]+\delta_2\sum_{|\gamma|\le 3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1 f\|^2\\ &\le C\sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2+C\|\partial_x\mathbf{P}_0 f\|^2+C\epsilon\sum_{1\le|\gamma|\le 3}\|\partial_x^\gamma\mathbf{P}_0 f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t)\end{aligned}$$

*and*

$$\begin{aligned}&\sum_{|\gamma|\le 3}\frac{d}{dt}\|\partial_x^\gamma f\|^2+\delta_2'\sum_{|\gamma|\le 3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1 f\|^2\\ (4.17)\quad &\le C\sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2+C\epsilon\sum_{1\le|\gamma|\le 3}\|\partial_x^\gamma\mathbf{P}_0 f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\end{aligned}$$

*Proof.* We apply $\partial_x^\gamma$ with $1\le|\gamma|\le 3$ to (1.26) and take the inner product with $\partial_x^\gamma f$ to get

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\gamma f\|^2+(\mathcal{L}\partial_x^\gamma f,\partial_x^\gamma f)=(\partial_x^\gamma\Gamma(f,f),\partial_x^\gamma f).$$

It follows from (1.19) that

$$(\mathcal{L}\partial_x^\gamma f,\partial_x^\gamma f)\ge\delta\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1 f\|^2.$$

By using Lemma 4.2 and the fact that $1\le|\gamma|\le 3$, for any $\epsilon>0$ small enough, we have that

$$(\partial_x^\gamma(\Gamma(f,f)-\Gamma(\mathbf{P}_0 f,\mathbf{P}_0 f)),\partial_x^\gamma f)\le\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}f\|^2+C_\epsilon\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).$$

By using (4.4) and (5.53)–(5.55), one has that

(4.18)
$$\Gamma(\mathbf{P}_0 f,\mathbf{P}_0 f)=a_0^2\alpha_{0,0}\psi_0(v)+a_0a_2(\alpha_{2,0}+\alpha_{0,2})\psi_2(v)+a_2^2\alpha_{2,2}\psi_4(v)=a_2^2\alpha_{2,2}\psi_4(v).$$

Thus, we have from (5.60) and the imbedding theorem that

$$(\partial_x^\gamma\Gamma(\mathbf{P}_0 f,\mathbf{P}_0 f),\partial_x^\gamma f)=(\partial_x^\gamma a_2^2\alpha_{2,2}\psi_4(v),\partial_x^\gamma f)\le C\|\partial_x^\gamma a_2^2\|\|\partial_x^\gamma f\|\le C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).$$

Combining the above estimates, by (5.7) and (5.17), we can obtain

(4.19)
$$\sum_{1\le|\gamma|\le 3}\frac{d}{dt}\|\partial_x^\gamma f\|^2+\delta'\sum_{1\le|\gamma|\le 3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1 f\|^2\le C\epsilon\sum_{1\le|\gamma|\le 3}\|\partial_x^\gamma\mathbf{P}_0 f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).$$

We see from (4.3) that

$$\partial_t\mathbf{P}_1 f+v\partial_x\mathbf{P}_1 f+\mathcal{L}\mathbf{P}_1 f=-\partial_t\mathbf{P}_0 f-v\partial_x\mathbf{P}_0 f+\Gamma(f,f). \tag{4.20}$$

It follows from (4.20) and (4.4) that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{P}_1 f\|^2+(\mathcal{L}\mathbf{P}_1 f,\mathbf{P}_1 f)=(-v\partial_x\mathbf{P}_0 f,\mathbf{P}_1 f)+(\Gamma(f,f),\mathbf{P}_1 f).$$

We have from Lemma 4.2 that

$$((\Gamma(f,f)-\Gamma(\mathbf{P}_0 f,\mathbf{P}_0 f)),\mathbf{P}_1 f)\le C\epsilon\|\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1 f\|^2+C_\epsilon\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).$$

Note that $\alpha_{2,2}\neq 0$ in (4.18). In fact, for any $\epsilon_0>0$ small enough, we have from (1.7) and (5.54) that

$$\begin{aligned}\alpha_{2,2}=\sqrt{6}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}B(\theta)(\sin\theta)^2(\cos\theta)^2d\theta &\ge C\int_0^{\epsilon_0}(\sin\theta)^{1-2s}(\cos\theta)^2d\theta\\ &\approx\int_0^{\epsilon_0}(\sin\theta)^{1-2s}(\cos\theta)d\theta\neq 0.\end{aligned} \tag{4.21}$$

One has from the imbedding theorem that

$$(\Gamma(\mathbf{P}_0 f,\mathbf{P}_0 f),\mathbf{P}_1 f)=(a_2^2\alpha_{2,2}\psi_4(v),\mathbf{P}_1 f)\le C_\epsilon\sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2+\epsilon\|\mathbf{P}_1 f\|^2. \tag{4.22}$$

Thus, by using (1.19), (5.17), and the above estimates and choosing $\epsilon>0$ small enough, we can obtain

$$\frac{d}{dt}\|\mathbf{P}_1 f\|^2+\frac{\delta}{2}\|\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1 f\|^2\le C\sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2+C\|\partial_x\mathbf{P}_0 f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t). \tag{4.23}$$

A suitable linear combination of (4.19) and (4.23) yields (4.17). Similarly, we also can obtain

$$\frac{d}{dt}\|f\|^2+\frac{\delta}{2}\|\mathcal{H}_v^{\frac{s}{2}}\mathbf{P}_1 f\|^2\le C\sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t). \tag{4.24}$$

Here, we have used the fact that $(\Gamma(\mathbf{P}_0 f,\mathbf{P}_0 f),\mathbf{P}_0 f)=0$ by (1.8).

A suitable linear combination of (4.19) and (4.24) yields (4.17). This completes the proof of Lemma 4.3. ∎

LEMMA 4.4. *Assume that $F(t,x,v)$ is the solution of* (1.1) *constructed in Theorem* 1.1. *Then one has*

$$\frac{d}{dt}\widetilde{\mathfrak{E}}(t)+\widetilde{\mathfrak{D}}(t)\le C\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2 \tag{4.25}$$

*and*

$$\frac{d}{dt}\overline{\mathfrak{E}}(t)+\widetilde{\mathfrak{D}}(t)\le C\|\partial_x\mathbf{P}_0f\|^2+C\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2. \tag{4.26}$$

*Here $\overline{\mathfrak{E}}(t)$ is defined as*

$$\overline{\mathfrak{E}}(t)\approx\sum_{1\le|\gamma|\le3}\|\partial_x^\gamma\mathbf{P}_0f\|^2+\sum_{|\gamma|+|\beta|\le3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f\|^2. \tag{4.27}$$

*Proof.* By applying $\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}$ with $|\gamma|+|\beta|\le3$ and $1\le|\beta|\le3$ on (4.20), we obtain

$$\begin{aligned}&\partial_t\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f+\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(v\partial_x\mathbf{P}_1f)+\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathcal{L}\mathbf{P}_1f\\&\quad=\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(f,f)-\partial_t\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_0f-\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(v\partial_x\mathbf{P}_0f).\end{aligned} \tag{4.28}$$

We take the inner product of (4.28) over $\mathbb{R}^2_{x,v}$ with $\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f$, and we estimate it term by term. The first term on the left-hand side is $\frac{1}{2}\frac{d}{dt}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f\|^2$. An argument similar to (2.62) implies that

$$(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(v\partial_x\mathbf{P}_1f),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f)\le C\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f\|^2+C_\epsilon\|\partial_x\partial_x^\gamma\mathcal{H}_v^{\frac{\beta-1}{2}}\mathbf{P}_1f\|^2.$$

It follows from Lemma 5.2 that

$$(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathcal{L}\mathbf{P}_1f,\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f)\ge\delta_1\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1f\|^2-C\|\partial_x^\gamma\mathbf{P}_1f\|^2.$$

One has from Lemma 4.2 that

$$(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(\Gamma(f,f)-\Gamma(\mathbf{P}_0f,\mathbf{P}_0f)),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f)\le\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).$$

By using (4.18), for $|\gamma|\le2$, we can obtain

$$\begin{aligned}&(\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(\mathbf{P}_0f,\mathbf{P}_0f),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f)=(\partial_x^\gamma a_2^2\alpha_{2,2}\mathcal{H}_v^{\frac{\beta}{2}}\psi_4(v),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f)\\&\le\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f\|^2+C_\epsilon\|\partial_x^\gamma a_2^2\|^2\le\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f\|^2+C\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\end{aligned}$$

By using (4.4) and (4.11), for any $\epsilon>0$, one has that

$$\begin{aligned}&(-\partial_t\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_0f-\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}(v\partial_x\mathbf{P}_0f),\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f)\\&\le C\epsilon\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1f\|^2+C_\epsilon\|\partial_x\partial_x^\gamma\mathbf{P}_0f\|^2+C_\epsilon\|\partial_x\partial_x^\gamma\mathbf{P}_1f\|^2.\end{aligned}$$

Thus, if we choose $\epsilon>0$ small enough, we can obtain

$$\begin{aligned}&\frac{d}{dt}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1f\|^2+\delta_1'\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1f\|^2\le C\|\partial_x\partial_x^\gamma\mathcal{H}_v^{\frac{\beta-1}{2}}\mathbf{P}_1f\|^2+C\|\partial_x^\gamma\mathbf{P}_1f\|^2\\&\quad+C\|\partial_x\partial_x^\gamma\mathbf{P}_1f\|^2+C\|\partial_x\partial_x^\gamma\mathbf{P}_0f\|^2+C\sup_{x\in\mathbb{R}}|\mathbf{P}_0f|_2^2\|\mathbf{P}_0f\|^2+C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).\end{aligned} \tag{4.29}$$

For any $1 \le |\beta| \le 3$, by (5.17), the summation of (4.30) over $|\gamma| + |\beta| \le 3$ and a suitable linear combination give

(4.30)
$$\frac{d}{dt}\Big[\sum_{1\le|\beta|,|\gamma|+|\beta|\le3} C_\beta\|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1 f\|^2\Big] + \delta_2 \sum_{1\le|\beta|,|\gamma|+|\beta|\le3}\|\partial_x^\gamma \mathcal{H}_v^{\frac{\beta+s}{2}}\mathbf{P}_1 f\|^2$$
$$\le C\sum_{|\gamma|\le2}\|\partial_x\partial_x^\gamma \mathbf{P}_0 f\|^2 + C\sum_{|\gamma|\le3}\|\partial_x^\gamma\mathbf{P}_1 f\|^2 + C\sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2 + C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t).$$

For the two sufficiently large positive constants $\widetilde{C}_1 \ll \widetilde{C}_2$, we define the function $\widetilde{\mathfrak{E}}(t)$ as

(4.31)
$$\widetilde{\mathfrak{E}}(t) = \widetilde{C}_2\Big(\widetilde{C}_1\sum_{|\gamma|\le3}\|\partial_x^\gamma f\|^2 + \sum_{|\gamma|\le2}[(\partial_x^\gamma\mathbf{P}_1 f, \partial_x\partial_x^\gamma a_0\bar{\xi}_3(v)) + (\partial_x^\gamma\mathbf{P}_1 f, \partial_x\partial_x^\gamma a_2\bar{\xi}_4(v))]\Big)$$
$$+ \sum_{1\le|\beta|,|\gamma|+|\beta|\le3} C_\beta\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\mathbf{P}_1 f\|^2 \approx \mathfrak{E}(t) = \sum_{|\gamma|+|\beta|\le3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} f\|^2.$$

We also define $\widetilde{\mathfrak{D}}(t)$ as in (4.2). For any $\epsilon > 0$ small enough, a suitable linear combination of (4.6), (4.17), (4.31), and (4.32) yields

$$\frac{d}{dt}\widetilde{\mathfrak{E}}(t) + \widetilde{\mathfrak{D}}(t) \le C\sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2 + C\sqrt{\mathfrak{E}(t)}\mathfrak{D}(t). \tag{4.32}$$

Noting that $F = \mu + \sqrt{\mu}f = M + \sqrt{\mu}g$ in (1.1), we have from (2.23), (2.24), and Theorem 1.1 that

$$\sum_{|\gamma|+|\beta|\le3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} f\|^2 \le C\sum_{|\gamma|+|\beta|\le3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}}\Big(\frac{M-\mu}{\sqrt{\mu}}\Big)\|^2$$
$$+ C\sum_{|\gamma|+|\beta|\le3}\|\partial_x^\gamma\mathcal{H}_v^{\frac{\beta}{2}} g\|^2 \le C\mathcal{E}(t) + C\varepsilon \le C\varepsilon.$$

Then we also have that

$$\widetilde{\mathfrak{E}}(t) \le C\mathfrak{E}(t) \le C\varepsilon. \tag{4.33}$$

The inequality (4.25) follows from this and (4.32). An argument similar to (4.25) implies that (4.26) holds with the help of (4.6), (4.17), (4.31), and (4.27). ∎

In order to prove the time decay rate of global solution to (1.1), we first prove the following lemma.

LEMMA 4.5. *It holds that*

$$\|\Gamma(f,f)\|_{Z^1} \le C\|\mathbf{P}_0 f\|^2 + C\|\mathcal{H}_v^s\mathbf{P}_1 f\|^2, \qquad \|\partial_x\Gamma(f,f)\|_{Z^1} \le C\|\partial_x\mathcal{H}_v^s f\|\|\mathcal{H}_v^s f\|$$

*and*

$$\|\Gamma(f,f)\| \le C\|\partial_x\mathcal{H}_v^s f\|^{\frac12}\|\mathcal{H}_v^s f\|^{\frac32}, \qquad \|\partial_x\Gamma(f,f)\| \le C\|\partial_x\mathcal{H}_v^s f\|^{\frac32}\|\mathcal{H}_v^s f\|^{\frac12}.$$

*Proof.* For any suitable function $h$, we have from Lemma 5.3 that

$$\langle \Gamma(f,g), h\rangle = \langle \mathcal{H}_v^{\frac{s}{2}}\Gamma(f,g), \mathcal{H}_v^{-\frac{s}{2}}h\rangle \le C|f|_2|\mathcal{H}_v^s g|_2|h|_2 + C|\mathcal{H}_v^{\frac{s}{2}}f|_2|\mathcal{H}_v^{\frac{s}{2}}g|_2|h|_2.$$

By this and (5.16), one has that

$$|\Gamma(f,g)|_2 \le C|f|_2|\mathcal{H}_v^s g|_2 + C|\mathcal{H}_v^{\frac{s}{2}}f|_2|\mathcal{H}_v^{\frac{s}{2}}g|_2 \le C|\mathcal{H}_v^s f|_2|\mathcal{H}_v^s g|_2.$$

Hence, we can obtain

$$\|\Gamma(f,f)\|_{Z^1} = \||\Gamma(f,f)|_2\|_{L^1_x} \le C\|\mathcal{H}_v^s f\|^2 \le C\|\mathbf{P}_0 f\|^2 + C\|\mathcal{H}_v^s\mathbf{P}_1 f\|^2$$

and

$$\|\partial_x\Gamma(f,f)\|_{Z^1} = \||\partial_x\Gamma(f,f)|_2\|_{L^1_x} \le C\|\partial_x\mathcal{H}_v^s f\|\|\mathcal{H}_v^s f\|.$$

By the imbedding theorem we also have

$$\|\Gamma(f,f)\| = \||\Gamma(f,f)|_2\| \le C\||\mathcal{H}_v^s f|_2\|_{L^\infty_x}\|\mathcal{H}_v^s f\| \le C\|\partial_x\mathcal{H}_v^s f\|^{\frac{1}{2}}\|\mathcal{H}_v^s f\|^{\frac{3}{2}},$$

$$\|\partial_x\Gamma(f,f)\| = \||\partial_x\Gamma(f,f)|_2\| \le C\||\mathcal{H}_v^s f|_2\|_{L^\infty_x}\|\partial_x\mathcal{H}_v^s f\| \le C\|\partial_x\mathcal{H}_v^s f\|^{\frac{3}{2}}\|\mathcal{H}_v^s f\|^{\frac{1}{2}}.$$

In what follows, we will prove the time decay rates of global solution to the Cauchy problem for the Kac equation (1.26) with the help of the asymptotical behavior of the linearized problem established in section 3.

*Proof of Theorem* 1.2. Let $f$ be global solution to (1.26) obtained in Theorem 1.1. For $t>0$, we can represent this solution in terms of the semigroup $e^{tB}$ as

$$f(t) = e^{tB}f_0 + \int_0^t e^{(t-s)B}\Gamma(f,f)ds. \tag{4.34}$$

For this global solution $f$, we define the functional $M(t)$ for any $t>0$ as

$$M(t) = \sup_{0\le\tau\le t}\Big\{(1+\tau)^{\frac{1}{4}}\|\mathbf{P}_0 f(\tau)\| + (1+\tau)^{\frac{3}{4}}\sqrt{\overline{\mathfrak{E}}(\tau)}\Big\}. \tag{4.35}$$

Here $\overline{\mathfrak{E}}(t)$ is defined as in (4.27). Since the nonlinear term $\Gamma(f,f)$ satisfies $\mathbf{P}_0\Gamma(f,f)=0$, one has from (3.66), (3.68), and Lemma 4.5 that

$$\begin{aligned}\|\mathbf{P}_0 f(t)\| &\le C(1+t)^{-\frac{1}{4}}(\|f_0\| + \|f_0\|_{Z^1})\\ &\quad + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|\Gamma(f,f)\| + \|\Gamma(f,f)\|_{Z^1})d\tau\\ &\le C\sqrt{\varepsilon}(1+t)^{-\frac{1}{4}} + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{1}{2}}M^2(\tau)d\tau\\ &\le C\sqrt{\varepsilon}(1+t)^{-\frac{1}{4}} + C(1+t)^{-\frac{1}{4}}M^2(t)\end{aligned} \tag{4.36}$$

and

$$
\begin{aligned}
\|\partial_x \mathbf{P}_0 f(t)\| \le\, & C(1+t)^{-\frac{3}{4}}(\|\partial_x f_0\| + \|f_0\|_{Z^1}) \\
& + C\int_0^{t/2}(1+t-\tau)^{-\frac{5}{4}}(\|\partial_x \Gamma(f,f)\| + \|\Gamma(f,f)\|_{Z^1})d\tau \\
& + C\int_{t/2}^{t}(1+t-\tau)^{-\frac{3}{4}}(\|\partial_x \Gamma(f,f)\| + \|\partial_x\Gamma(f,f)\|_{Z^1})d\tau \\
\le\, & C\sqrt{\varepsilon}(1+t)^{-\frac{3}{4}} + C\int_0^{t/2}(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{1}{2}}M^2(\tau)d\tau \\
& + C\int_{t/2}^{t}(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-1}M^2(\tau)d\tau \\
\le\, & C\sqrt{\varepsilon}(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}M^2(t).
\end{aligned} \tag{4.37}
$$

It follows from (4.35) and the imbedding theorem that

$$
\sup_{x\in\mathbb{R}} |\mathbf{P}_0 f(\tau)|_2^2 \|\mathbf{P}_0 f(\tau)\|^2 \le C\|\mathbf{P}_0 f(\tau)\|^3 \|\partial_x \mathbf{P}_0 f(\tau)\| \le C(1+\tau)^{-\frac{3}{2}}M^4(\tau). \tag{4.38}
$$

By (4.2) and (4.27), it holds that $c\overline{\mathfrak{E}}(t) \le \widetilde{\mathfrak{D}}(t)$. One has from (4.26), (4.37), and (4.38) that

$$
\begin{aligned}
\overline{\mathfrak{E}}(t) \le\, & e^{-ct}\overline{\mathfrak{E}}(0) + C\int_0^t e^{-c(t-\tau)}\Big(\|\partial_x \mathbf{P}_0 f\|^2 + \sup_{x\in\mathbb{R}}|\mathbf{P}_0 f|_2^2\|\mathbf{P}_0 f\|^2\Big)d\tau \\
\le\, & C\varepsilon e^{-ct} + C(1+t)^{-\frac{3}{2}}(\sqrt{\varepsilon} + M^2(t))^2.
\end{aligned}
$$

This, together with (4.36), leads to

$$
M(t) \le C\sqrt{\varepsilon} + CM^2(t).
$$

This implies that $M(t) \le C\sqrt{\varepsilon}$ for $\varepsilon > 0$ small enough. This fact and (4.35) imply that (1.27) and (1.28) hold.

Assume that $\mathbf{P}_0 f_0 = 0$. For the global solution $f$ of (1.26), we define the functional $\overline{M}(t)$ for any $t > 0$ as

$$
\overline{M}(t) = \sup_{0\le\tau\le t}\Big\{(1+\tau)^{\frac{3}{4}}\|\mathbf{P}_0 f(\tau)\| + (1+\tau)^{\frac{5}{4}}\sqrt{\overline{\mathcal{E}}(\tau)}\Big\}. \tag{4.39}
$$

Since $\mathbf{P}_0 f_0 = 0$, we have from (3.68) and Lemma 4.5 that

$$
\begin{aligned}
\|\mathbf{P}_0 f(t)\| \le\, & C(1+t)^{-\frac{3}{4}}(\|f_0\| + \|f_0\|_{Z^1}) \\
& + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|\Gamma(f,f)\| + \|\Gamma(f,f)\|_{Z^1})d\tau \\
\le\, & C\sqrt{\varepsilon}(1+t)^{-\frac{3}{4}} + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}}\overline{M}^2(\tau)d\tau \\
\le\, & C\sqrt{\varepsilon}(1+t)^{-\frac{3}{4}} + C(1+t)^{-\frac{3}{4}}\overline{M}^2(t)
\end{aligned} \tag{4.40}
$$

and

$$
\begin{aligned}
\|\partial_x \mathbf{P}_0 f(t)\| \leq & C(1+t)^{-\frac{5}{4}}(\|\partial_x f_0\| + \|f_0\|_{Z^1}) \\
& + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}(\|\partial_x \Gamma(f,f)\| + \|\Gamma(f,f)\|_{Z^1})d\tau \\
& \leq C\sqrt{\varepsilon}(1+t)^{-\frac{5}{4}} + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{2}}\overline{M}^2(\tau)d\tau \\
& \leq C\sqrt{\varepsilon}(1+t)^{-\frac{5}{4}} + C(1+t)^{-\frac{5}{4}}\overline{M}^2(t).
\end{aligned} \tag{4.41}
$$

It follows from (4.39) and the imbedding theorem that

$$
\sup_{x\in\mathbb{R}} |\mathbf{P}_0 f(\tau)|_2^2 \|\mathbf{P}_0 f(\tau)\|^2 \leq C\|\mathbf{P}_0 f(\tau)\|^3 \|\partial_x \mathbf{P}_0 f(\tau)\| \leq C(1+\tau)^{-\frac{7}{2}} M^4(\tau). \tag{4.42}
$$

By (4.2) and (4.27), it holds that $c\overline{\mathfrak{E}}(t) \leq \widetilde{\mathfrak{D}}(t)$. One has from (4.26), (4.41), and (4.42) that

$$
\begin{aligned}
\overline{\mathfrak{E}}(t) \leq & e^{-ct}\overline{\mathfrak{E}}(0) + C\int_0^t e^{-c(t-\tau)}\Big(\|\partial_x \mathbf{P}_0 f\|^2 + \sup_{x\in\mathbb{R}} |\mathbf{P}_0 f|_2^2 \|\mathbf{P}_0 f\|^2\Big)d\tau \\
& \leq C\varepsilon e^{-ct} + C(1+t)^{-\frac{5}{2}}(\sqrt{\varepsilon} + \overline{M}^2(t))^2.
\end{aligned}
$$

This, together with (4.41), leads to

$$
\overline{M}(t) \leq C\sqrt{\varepsilon} + C\overline{M}^2(t).
$$

This implies that $\overline{M}(t) \leq C\sqrt{\varepsilon}$ for $\varepsilon > 0$ small enough. This fact and (4.39) imply that (1.29) and (1.30) hold.

Note that $f_0 \in Z^1$ and $\inf_{|k|\leq k_0} |\langle \hat{f}_0, \psi_j(v)\rangle| \geq d_0$ with some constants $k_0 > 0$ and $d_0 > 0$ for $j = 0, 2$. For $t > 0$ large enough and $\varepsilon > 0$ small enough, we have from (3.70) and (4.36) that

$$
\|f\| \geq \|e^{tB} f_0\| - \int_0^t \|e^{(t-s)B}\Gamma(f,f)\|ds \geq C_5(1+t)^{-\frac{1}{4}} - C\varepsilon(1+t)^{-\frac{1}{4}} \geq C(1+t)^{-\frac{1}{4}}.
$$

The lower bound of the time decay rate of (1.31) is obtained, and the upper bound can be obtained in (1.27). This completes the proof of Theorem 1.2. ∎

**5. Appendix.** In this appendix, we first recall some facts about the Hermite functions in [28, 35] and some basic properties of the linearized Kac operator $\mathcal{L}$ obtained in [28]. Then we prove the property of the norm of the fractional Hermite–Sobolev space, and we will prove some estimates of the linearized Kac operator $\mathcal{L}$ and nonlinear term $\Gamma(g, g)$, which have been used in the energy estimates in the previous sections.

The standard Hermite functions $(\phi_n(v))_{n\in\mathbb{N}}$ are defined for $v \in \mathbb{R}$ as

$$
\phi_n(v) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}} e^{\frac{v^2}{2}} \frac{d^n}{dv^n}(e^{-v^2}) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}}\Big(v - \frac{d}{dv}\Big)^n (e^{-\frac{v^2}{2}}). \tag{5.1}
$$

The family $(\phi_n(v))_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}_v)$. Set for $n \in \mathbb{N}$, $v \in \mathbb{R}$,

$$
\psi_n(v) = 2^{-1/4}\phi_n(2^{-1/2}v), \quad \psi_n(v) = \frac{1}{\sqrt{n!}}\Big(\frac{v}{2} - \frac{d}{dv}\Big)^n \phi_0. \tag{5.2}
$$

The family $(\psi_n(v))_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}_v)$ composed of the eigenfunctions of the harmonic oscillator

$$\mathcal{H}_v = -\Delta_v + \frac{v^2}{4} = \sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)\mathbb{P}_n, \quad \mathbf{1} = \sum_{n=0}^{+\infty}\mathbb{P}_n, \tag{5.3}$$

where $\Delta_v = \partial_{vv}$ and $\mathbb{P}_n$ stands for the orthogonal projection

$$\mathbb{P}_n f = \langle f, \psi_n\rangle \psi_n. \tag{5.4}$$

Note that $\mathcal{L}$ is the linearized Kac operator with respect to the global Maxwellian $\mu = M_{[1,1]}(v)$:

$$\mathcal{L}g = -\frac{1}{\sqrt{\mu}}\{Q(\mu,\sqrt{\mu}g) + Q(\sqrt{\mu}g,\mu)\}, \quad \Gamma(g_1,g_2) = \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g_1,\sqrt{\mu}g_2). \tag{5.5}$$

The linearized Kac operator was studied in [28, 27]. We recall from [28, page 463] that $\mathcal{L}$ is a nonnegative unbounded operator on $L^2(\mathbb{R}_v)$ with domain

$$D(\mathcal{L}) = \Big\{f\in L^2(\mathbb{R}_v),\ \ \sum_{n=0}^{+\infty} n^{2s}|\mathbb{P}_n f|_2^2 < +\infty\Big\} = \Big\{f\in L^2(\mathbb{R}_v),\ \ \mathcal{H}_v^s f\in L^2(\mathbb{R}_v)\Big\},$$

and the null space of $\mathcal{L}$ is given by

$$\mathcal{N} = \mathrm{span}\{\psi_0(v),\psi_2(v)\}. \tag{5.6}$$

Define $\mathbf{P}_0$ as the orthogonal projection in $L^2(\mathbb{R}_v)$ to the null space $\mathcal{N}$ and $\mathbf{P}_1 = \mathbf{1}-\mathbf{P}_0$. $\mathcal{N}^\perp$ denotes the orthogonal complement of the null space $\mathcal{N}$. For any fixed $(t,x)$, any function $f(t,x,v)$ can be decomposed into

$$f(t,x,v) = \mathbf{P}_0 f(t,x,v) + \mathbf{P}_1 f(t,x,v). \tag{5.7}$$

For any $a\geq 0$, the power harmonic oscillator

$$\mathcal{H}_v^a = \sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)^a\mathbb{P}_n \tag{5.8}$$

is defined. The linearized Kac operator $\mathcal{L}$ is diagonal in the Hermite basis,

$$\mathcal{L} = \sum_{n=1}^{+\infty}\lambda_n\mathbb{P}_n, \tag{5.9}$$

with a spectrum composed only of the nonnegative eigenvalues

$$\lambda_{2n+1} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} B(\theta)(1-(\cos\theta)^{2n+1})d\theta, \quad n\geq 0 \tag{5.10}$$

and

$$\lambda_{2n} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} B(\theta)(1-(\cos\theta)^{2n}-(\sin\theta)^{2n})d\theta, \quad n\geq 1, \tag{5.11}$$

satisfying the asymptotic estimates

$$\lambda_n \approx n^s \quad \text{as} \quad n \to +\infty.$$

We note that

$$0 = \lambda_2 \le \lambda_{2n} < \lambda_{2l}, \quad 0 < \lambda_1 < \lambda_{2n+1} < \lambda_{2l+1}, \tag{5.12}$$

when $1 \le n < l$, and that $\lambda_1$ is the lowest positive eigenvalue.

In the following we show the coercivity of the linearized Kac operator $\mathcal{L}$. For any $N > 0$ large enough and any function $f$, one has from (5.8) that

$$C\sum_{n=N}^{+\infty} \lambda_n |\mathbb{P}_n f|_2^2 \ge \sum_{n=N}^{+\infty} n^s |\mathbb{P}_n f|_2^2 \ge \left(\frac{1}{2}\right)^s \sum_{n=N}^{+\infty} \left(n+\frac{1}{2}\right)^s |\mathbb{P}_n f|_2^2$$
$$\ge \left(\frac{1}{2}\right)^s \sum_{n=0}^{+\infty} \left(n+\frac{1}{2}\right)^s |\mathbb{P}_n f|_2^2 - C_{N,s}\sum_{n=0}^{N-1} |\mathbb{P}_n f|_2^2 \ge \left(\frac{1}{2}\right)^s |\mathcal{H}_v^{\frac{s}{2}} f|_2^2 - C_{N,s}|f|_2^2.$$

By this and (5.9), there exist $\delta_0 > 0$ and $\overline{C} > 0$ such that for any $N > 0$ large enough and any $f \in D(\mathcal{L})$,

$$\langle \mathcal{L}f, f\rangle = \sum_{n=N}^{+\infty} \lambda_n |\mathbb{P}_n f|_2^2 + \sum_{n=1}^{N-1} \lambda_n |\mathbb{P}_n f|_2^2 \ge \delta_0 |\mathcal{H}_v^{\frac{s}{2}} f|_2^2 - \overline{C}|f|_2^2. \tag{5.13}$$

For any function $f \in \mathcal{N}^{\perp}$, we have $\mathbf{P}_0 f = 0$ , $\mathbb{P}_0 f = 0$, and $\mathbb{P}_2 f = 0$. One has from (5.12) that

$$\langle \mathcal{L}f, f\rangle = \sum_{n=1}^{+\infty} \lambda_n |\mathbb{P}_n f|_2^2 \ge \lambda_1 |\mathbb{P}_1 f|_2^2 + \sum_{n=3}^{+\infty} \lambda_n |\mathbb{P}_n f|_2^2 \ge \lambda_1 \sum_{n=0}^{+\infty} |\mathbb{P}_n f|_2^2 = \lambda_1 |f|_2^2. \tag{5.14}$$

Thus, from (5.13) and (5.14), there is a $\delta > 0$ such that for any function $f \in D(\mathcal{L})$,

$$\langle \mathcal{L}f, f\rangle \ge \delta |\mathcal{H}_v^{\frac{s}{2}} \mathbf{P}_1 f|_2^2. \tag{5.15}$$

By using (5.3), (5.4), and (5.8), for any $a, b \in [0, \infty)$ with $a < b$, one has

$$|\mathcal{H}_v^a f|_2^2 = \sum_{n=0}^{+\infty} \left(n+\frac{1}{2}\right)^{2a} |\mathbb{P}_n f|_2^2 \le C(a,b) \sum_{n=0}^{+\infty} \left(n+\frac{1}{2}\right)^{2b} |\mathbb{P}_n f|_2^2 = |\mathcal{H}_v^b f|_2^2. \tag{5.16}$$

In particular, for any $b > 0$,

$$|f|_2^2 \le C_b \sum_{n=0}^{+\infty} \left(n+\frac{1}{2}\right)^{2b} |\mathbb{P}_n f|_2^2 = |\mathcal{H}_v^b f|_2^2. \tag{5.17}$$

For any $a > 0$, the Hermite–Sobolev space with regularity $a$ can be defined by

$$\mathcal{H}^a(\mathbb{R}) = \{g \in L^2(\mathbb{R}), \quad |\mathcal{H}_v^{\frac{a}{2}} g|_2 < +\infty\}. \tag{5.18}$$

The Hermite–Sobolev space $\mathcal{H}^a(\mathbb{R})$ has been studied in [6, 9, 35, 38, 40, 41]. It is shown in [6, Proposition 4] and [9] that the integer order Hermite–Sobolev space $\mathcal{H}^a(\mathbb{R})$ with the integer $a$ has the following properties:

$$g \in \mathcal{H}^a(\mathbb{R}) \Leftrightarrow \ f \text{ admits derivatives up to order } a \text{ satisfying} \quad \partial_v^{\ell} f, v^{a-\ell}\partial_v^{\ell} f \in L^2(\mathbb{R}), \quad \ell = 0, \dots, a. \tag{5.19}$$

However, the fractional order Hermite–Sobolev space $\mathcal{H}^s(\mathbb{R})$ with $s \in (0,1)$ is only studied partially in [9, 38, 40, 41]. Since the dissipative norm $|\mathcal{H}_v^{\frac{s}{2}} g|_2$ with $0 < s < 1$ plays an essential role in spectrum analysis of the linearized Kac equation, we provide the following lemma on the fractional order Hermite–Sobolev space for easy reference.

LEMMA 5.1. *There exists constant $C > 1$ such that, for any $s \in (0,1)$ and any function $g(v)$ in Schwartz class $\mathcal{S}$,*

$$\frac{1}{C}(|\langle v\rangle^s g|_2^2 + |(\mathbf{1}-\Delta_v)^{\frac{s}{2}} g|_2^2) \le |\mathcal{H}_v^{\frac{s}{2}} g|_2^2 \le C(|\langle v\rangle^s g|_2^2 + |(\mathbf{1}-\Delta_v)^{\frac{s}{2}} g|_2^2). \tag{5.20}$$

*Proof.* Inspired by the papers [38, 40], for any function $g(v) \in \mathcal{S}$ and $s \in (0,1)$, by using (5.8) and [38, Lemma 5.5], we give the fractional harmonic oscillator $\mathcal{H}_v^s$ by the classical formula

$$\mathcal{H}_v^s g(v) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\mathcal{H}_v} g(v) - g(v)) \frac{dt}{t^{1+s}}, \tag{5.21}$$

where $u(v,t) = e^{-t\mathcal{H}_v} g(v)$ is the solution of the heat-diffusion equation $\partial_t u + \mathcal{H}_v u = 0$ in $\mathbb{R}_v \times (0,\infty)$, with initial datum $u(v,0) = g(v)$ on $\mathbb{R}_v$. As in [38], define

$$F_s(v,z) = \frac{1}{-\Gamma(-s)} \int_0^\infty G_t(v,z) \frac{dt}{t^{1+s}}, \qquad B_s(v) = \frac{1}{-\Gamma(-s)} \int_0^\infty (1 - e^{-t\mathcal{H}_v} 1(v)) \frac{dt}{t^{1+s}}, \tag{5.22}$$

and $G_t(v,z)$ is the kernel of the heat-diffusion semigroup generated by $\mathcal{H}_v$, which is given in [8, page 141] by

$$G_t(v,z) = \left(\frac{1}{4\pi \sinh t}\right)^{1/2} e^{-\frac{1}{4}[\coth t(v^2+z^2) - 2\mathrm{cosech}\ t(vz)]}. \tag{5.23}$$

Next, we first consider the term $B_s(v)$ in (5.22) as in [38]. It follows from (5.23) that

$$e^{-t\mathcal{H}_v} 1(v) = \int_{\mathbb{R}} G_t(v,z) dz = \frac{1}{(\cosh t)^{1/2}} e^{-\frac{\tanh t}{4} v^2}. \tag{5.24}$$

Consider the change of the parameter

$$t = t(\varrho) = \ln \frac{1+\varrho}{1-\varrho}, \quad t \in (0,\infty), \quad \varrho \in (0,1), \tag{5.25}$$

which produces

$$\frac{dt}{t^{1+s}} = d\mu_s(\varrho) = \frac{2 d\varrho}{(1-\varrho^2)(\ln \frac{1+\varrho}{1-\varrho})^{1+s}} \quad t \in (0,\infty), \quad \varrho \in (0,1). \tag{5.26}$$

Thus, we can obtain

$$e^{-t\mathcal{H}_v} 1(v) = \frac{1}{(\cosh t)^{1/2}} e^{-\frac{\tanh t}{4} v^2} = \left(\frac{1-\varrho^2}{1+\varrho^2}\right)^{1/2} e^{-\frac{\varrho}{2(1+\varrho^2)} v^2} \le 1. \tag{5.27}$$

Then, by using (5.22), we can write, up to the positive factor $\frac{1}{-\Gamma(-s)}$,

$$B_s(v) = \int_0^1 \left[1 - \left(\frac{1-\varrho^2}{1+\varrho^2}\right)^{1/2}\right] e^{-\frac{\varrho}{2(1+\varrho^2)} v^2} d\mu_s(\varrho) + \int_0^1 \left(1 - e^{-\frac{\varrho}{2(1+\varrho^2)} v^2}\right) d\mu_s(\varrho) = I + II. \tag{5.28}$$

Observe in (5.26) that

$$d\mu_s(\varrho) \approx \frac{d\varrho}{\varrho^{1+s}}, \quad \varrho \approx 0, \quad d\mu_s(\varrho) \approx \frac{d\varrho}{(1-\varrho)(-\ln(1-\varrho))^{1+s}}, \quad \varrho \approx 1. \tag{5.29}$$

Fix a constant $M > 1$ large enough and then take a constant $\epsilon_0 > 0$ small enough. For the term $I$ in (5.28), one has from (5.29) that

$$0 \le I \le C\int_0^{\epsilon_0} \Big[1 - \Big(\frac{1-\varrho^2}{1+\varrho^2}\Big)^{1/2}\Big]\frac{d\varrho}{\varrho^{1+s}} + C\int_{\epsilon_0}^1 d\mu_s(\varrho) \le C\int_0^{\epsilon_0} \varrho^2 \frac{d\varrho}{\varrho^{1+s}} + C = C. \tag{5.30}$$

For any $\rho \in (0,\epsilon_0)$ and $v^2 \le M$, there exists $\xi \in (-\frac{\varrho v^2}{2(1+\varrho^2)}, 0) \subset (-\epsilon_0 M, 0)$ by the mean value theorem such that

$$1 - e^{-\frac{\varrho}{2(1+\varrho^2)}v^2} = \frac{\varrho v^2}{2(1+\varrho^2)} e^{\xi} \approx \varrho v^2. \tag{5.31}$$

For the term $II$ of (5.28), one has from (5.31) that, for $v^2 \le M$,

$$\begin{aligned}\mathbf{1}_{v^2\le M}\int_0^{\epsilon_0}\Big(1 - e^{-\frac{\varrho}{2(1+\varrho^2)}v^2}\Big)d\mu_s(\varrho) &\approx \mathbf{1}_{v^2\le M}\int_0^{\epsilon_0}\Big(1 - e^{-\frac{\varrho}{2(1+\varrho^2)}v^2}\Big)\frac{d\varrho}{\varrho^{1+s}} \\ &\approx \mathbf{1}_{v^2\le M}\int_0^{\epsilon_0} v^2\varrho\frac{d\varrho}{\varrho^{1+s}} = C\mathbf{1}_{v^2\le M}v^2.\end{aligned} \tag{5.32}$$

Thus for $v^2 \le M$, by using (5.26), (5.28), and (5.32),

$$II(v)\mathbf{1}_{v^2\le M} \le C\mathbf{1}_{v^2\le M}v^2 + \int_{\epsilon_0}^1 d\mu_s(\varrho) \le C. \tag{5.33}$$

For $v^2 > M$, we see that

$$II(v) = \int_0^{1/v^2}\Big(1 - e^{-\frac{\varrho}{2(1+\varrho^2)}v^2}\Big)d\mu_s(\varrho) + \int_{1/v^2}^1\Big(1 - e^{-\frac{\varrho}{2(1+\varrho^2)}v^2}\Big)d\mu_s(\varrho). \tag{5.34}$$

For any $\rho \in (0, 1/v^2)$ and $v^2 > M$, there exists $\xi \in (-\frac{\varrho v^2}{2(1+\varrho^2)}, 0) \subset (-1, 0)$ such that (5.31) holds. Thus, for the first term, one has

$$\begin{aligned}\mathbf{1}_{v^2>M}\int_0^{1/v^2}\Big(1 - e^{-\frac{\varrho}{2(1+\varrho^2)}v^2}\Big)d\mu_s(\varrho) &\approx \mathbf{1}_{v^2>M}\int_0^{1/v^2}\Big(1 - e^{-\frac{\varrho}{2(1+\varrho^2)}v^2}\Big)\frac{d\varrho}{\varrho^{1+s}} \\ &\approx \mathbf{1}_{v^2>M}\int_0^{1/v^2} v^2\varrho\frac{d\varrho}{\varrho^{1+s}} = C\mathbf{1}_{v^2>M}|v|^{2s}.\end{aligned} \tag{5.35}$$

Then for $v^2 \ge M$, one has from (5.34) and (5.35) that

$$\begin{aligned}II(v)\mathbf{1}_{v^2>M} &\le C\mathbf{1}_{v^2>M}|v|^{2s} \\ &\quad + C\mathbf{1}_{v^2>M}\int_{1/v^2}^{1-(1/v^2)} d\mu_s(\varrho) + C\mathbf{1}_{v^2>M}\int_{1-(1/v^2)}^1 \frac{d\varrho}{(1-\varrho)(-\ln(1-\varrho))^{1+s}} \\ &\le C\mathbf{1}_{v^2>M}|v|^{2s} + C\mathbf{1}_{v^2>M} + C\mathbf{1}_{v^2>M}[\ln(v^2)]^{-s} \le C\mathbf{1}_{v^2>M}(1+|v|^{2s}).\end{aligned} \tag{5.36}$$

By using (5.30), (5.33), and (5.36), we have from (5.28) that

$$B_s(v) \le C(1+|v|^{2s}) \le C\langle v\rangle^{2s}. \tag{5.37}$$

Since both terms of $II(v)$ in (5.34) and the first term of $B_s(v)$ in (5.28) are nonnegative, we have from (5.34) and (5.35) that

$$B_s(v)\mathbf{1}_{v^2>M} \geq II(v)\mathbf{1}_{v^2>M} \geq C\mathbf{1}_{v^2>M}|v|^{2s}.$$

If $v^2 \leq M$, one has from (5.28) that

$$B_s(v) \geq \int_{1/100}^{1/10} \Big[1-\Big(\frac{1-\varrho^2}{1+\varrho^2}\Big)^{1/2}\Big] e^{-\frac{\varrho}{2(1+\varrho^2)}v^2} d\mu_s(\varrho) \geq C\int_{1/100}^{1/10} e^{-\frac{\varrho M}{2(1+\varrho^2)}} d\varrho = C_1.$$

By using the above two estimates, one has that

$$B_s(v) \geq C\mathbf{1}_{v^2>M}|v|^{2s} + C_1\mathbf{1}_{v^2\leq M} \geq C_2\langle v\rangle^{2s}. \tag{5.38}$$

It follows from (5.37) and (5.38) that

$$C_2\langle v\rangle^{2s} \leq B_s(v) \leq C_1\langle v\rangle^{2s}. \tag{5.39}$$

The function $B_s(v)$ is differentiable since the gradient of the integrand in its definition of (5.22) and (5.27) is bounded by

$$|\partial_v(1-e^{-t\mathcal{H}_v}1(v))| = |v|\frac{\varrho}{(1+\varrho^2)}\Big(\frac{1-\varrho^2}{1+\varrho^2}\Big)^{1/2} e^{-\frac{\varrho}{2(1+\varrho^2)}v^2} \leq C|v|\varrho \in L^1((0,1);d\mu_s(\varrho)),$$

and thus we have that

$$|\partial_v B_s(v)| = \Big|v\int_0^1 \frac{\varrho}{(1+\varrho^2)}\Big(\frac{1-\varrho^2}{1+\varrho^2}\Big)^{1/2} e^{-\frac{\varrho}{2(1+\varrho^2)}v^2} d\mu_s(\varrho)\Big| \leq C.$$

For higher order derivatives we can proceed similarly. Then $B_s(v) \in C^\infty(\mathbb{R})$.

Next we will estimate $F_s(v,z)$ in (5.22). It follows from (5.25) that

$$\frac{1}{4\pi\sinh t} = \frac{1-\varrho^2}{8\pi\varrho}, \qquad \coth t = \frac{1}{2}\Big(\varrho+\frac{1}{\varrho}\Big), \qquad \operatorname{cosech} t = \frac{1}{2}\Big(\frac{1}{\varrho}-\varrho\Big).$$

Then we have from (5.23) that

$$G_t(v,z) = \Big(\frac{1-\varrho^2}{8\pi\varrho}\Big)^{1/2} e^{-\frac{1}{8}[\varrho(v+z)^2+\frac{1}{\varrho}(v-z)^2]}. \tag{5.40}$$

By this we can obtain, for $\varrho \in (0,1)$,

$$\begin{aligned} G_t(v,z) &\leq C\Big(\frac{1-\varrho}{\varrho}\Big)^{1/2} e^{-\frac{1}{16\varrho}(v-z)^2} e^{-\frac{1}{16}[\varrho(v+z)^2+\frac{1}{\varrho}(v-z)^2]} \\ &\leq C\Big(\frac{1-\varrho}{\varrho}\Big)^{1/2} e^{-\frac{1}{16\varrho}(v-z)^2} e^{-\frac{1}{16}|v+z||v-z|}. \end{aligned}$$

If $vz>0$, then $|v+z| \geq |v|$, which gives

$$e^{-\frac{1}{16\varrho}(v-z)^2}e^{-\frac{1}{16}|v+z||v-z|} \leq e^{-\frac{1}{32\varrho}(v-z)^2}e^{-\frac{1}{16}|v||v-z|} \leq e^{-\frac{1}{32\varrho}(v-z)^2}e^{-\frac{1}{32}|v||v-z|}.$$

If $vz \leq 0$, then $|v-z| \geq |v|$, and in this case

$$e^{-\frac{1}{16\varrho}(v-z)^2}e^{-\frac{1}{16}|v+z||v-z|} \leq e^{-\frac{1}{32\varrho}(v-z)^2}e^{-\frac{1}{32\varrho}|v||v-z|} \leq e^{-\frac{1}{32\varrho}(v-z)^2}e^{-\frac{1}{32}|v||v-z|}.$$

Thus, we have that, for $\varrho \in (0,1)$,

$$\begin{aligned} G_t(v,z) &\le C\Big(\frac{1-\varrho}{\varrho}\Big)^{1/2} e^{-\frac{1}{32\varrho}(v-z)^2} e^{-\frac{1}{32}|v||v-z|} \\ &\le C\Big(\frac{1-\varrho}{\varrho}\Big)^{1/2} e^{-\frac{1}{64\varrho}(v-z)^2} e^{-\frac{1}{64}(v-z)^2} e^{-\frac{1}{32}|v||v-z|}. \end{aligned}$$

By using the fact that $\sup_{x>0} x^{1/2}e^{-x} \le C$, we have that

$$G_t(v,z) \le \frac{C}{|v-z|}(1-\varrho)^{1/2} e^{-\frac{|v-z|^2}{C\varrho}} e^{-\frac{|v-z|^2}{C}} e^{-\frac{|v||v-z|}{C}}. \tag{5.41}$$

By using (5.22) and (5.26), we have from (5.41) that

$$\begin{aligned} F_s(v,z) &= \frac{1}{-\Gamma(-s)}\int_0^\infty G_t(v,z)\frac{dt}{t^{1+s}} = \frac{1}{-\Gamma(-s)}\int_0^1 G_t(v,z)d\mu_s(\varrho) \\ &\le \frac{C}{|v-z|} e^{-\frac{|v-z|^2}{C}} e^{-\frac{|v||v-z|}{C}} \int_0^1 (1-\varrho)^{1/2} e^{-\frac{|v-z|^2}{C\varrho}} d\mu_s(\varrho). \end{aligned} \tag{5.42}$$

For any $\epsilon_0 > 0$ small enough, one has from (5.29) that

$$\begin{aligned} \int_0^{\epsilon_0} (1-\varrho)^{1/2} e^{-\frac{|v-z|^2}{C\varrho}} d\mu_s(\varrho) &\le C\int_0^{\epsilon_0} e^{-\frac{|v-z|^2}{C\varrho}} \frac{d\varrho}{\varrho^{1+s}} \\ &= \frac{C_1}{|v-z|^{2s}} \int_{\frac{|v-z|^2}{C\epsilon_0}}^\infty y^{s-1}e^{-y}dy \le \frac{C_2}{|v-z|^{2s}}, \end{aligned}$$

$$\int_{\epsilon_0}^{1-\epsilon_0} (1-\varrho)^{1/2} e^{-\frac{|v-z|^2}{C\varrho}} d\mu_s(\varrho) \le Ce^{-\frac{|v-z|^2}{C}} \int_{\epsilon_0}^{1-\epsilon_0} d\mu_s(\varrho) \le Ce^{-\frac{|v-z|^2}{C}},$$

and

$$\int_{1-\epsilon_0}^1 (1-\varrho)^{1/2} e^{-\frac{|v-z|^2}{C\varrho}} d\mu_s(\varrho) \le Ce^{-\frac{|v-z|^2}{C}} \int_{1-\epsilon_0}^1 \frac{d\varrho}{(1-\varrho)(-\ln(1-\varrho))^{1+s}} \le Ce^{-\frac{|v-z|^2}{C}}.$$

It follows from (5.42) and the above estimates that

$$F_s(v,z) \le \frac{C}{|v-z|^{1+2s}} e^{-\frac{|v-z|^2}{C}} e^{-\frac{|v||v-z|}{C}} \le \frac{C}{|v-z|^{1+2s}}. \tag{5.43}$$

As in [38], we deduce from (5.21), (5.23),, and (5.24) that

$$\begin{aligned} \mathcal{H}_v^s g(v) &= \frac{1}{\Gamma(-s)}\int_0^\infty (e^{-t\mathcal{H}_v}g(v) - g(v))\frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)}\int_0^\infty \Big(\int_{\mathbb{R}} G_t(v,z)g(z)dz - g(v)\Big)\frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)}\int_0^\infty \Big[\int_{\mathbb{R}} G_t(v,z)(g(z)-g(v))dz + g(v)\Big(\int_{\mathbb{R}} G_t(v,z)dz - 1\Big)\Big]\frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)}\int_0^\infty \Big(\int_{\mathbb{R}} G_t(v,z)g(z)dz - g(v)\Big)\frac{dt}{t^{1+s}} + \frac{g(v)}{\Gamma(-s)}\int_0^\infty (e^{-t\mathcal{H}_v}1(v)-1)\frac{dt}{t^{1+s}} \\ &= \frac{1}{-\Gamma(-s)}\int_0^\infty\int_{\mathbb{R}} G_t(v,z)(g(v)-g(z))dz\frac{dt}{t^{1+s}} + \frac{g(v)}{-\Gamma(-s)}\int_0^\infty (1-e^{-t\mathcal{H}_v}1(v))\frac{dt}{t^{1+s}}. \end{aligned} \tag{5.44}$$

By using this, (5.21), and (5.22), an argument similar to Theorem 5.7 in [38] implies that

$$\mathcal{H}_v^s g(v) = \int_{\mathbb{R}} F_s(v,z)(g(v)-g(z))dz + g(v)B_s(v), \quad v\in\mathbb{R}, \quad g\in\mathcal{S}.$$

Here $F_s(v,z)$ and $B_s(v)$ are defined as in (5.22). Hence, we can obtain

$$|\mathcal{H}_v^{\frac{s}{2}} g|_2^2 = \langle \mathcal{H}_v^s g(v), g(v)\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} F_s(v,z)(g(v)-g(z))g(v)dzdv + \langle B_s(v)g(v), g(v)\rangle. \tag{5.45}$$

By using (5.39), we can obtain

$$C_1|\langle v\rangle^s g|_2^2 \le \langle B_s(v)g(v), g(v)\rangle \le C_2|\langle v\rangle^s g|_2^2. \tag{5.46}$$

It follows from (5.22), (5.40), (5.42), and (5.43) that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} F_s(v,z)(g(v)-g(z))g(v)dzdv \\ &= \frac{1}{-2\Gamma(-s)}\int_0^\infty \int_{\mathbb{R}}\int_{\mathbb{R}} G_t(v,z)(g(v)-g(z))^2 dzdv\frac{dt}{t^{1+s}} \\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} F_s(v,z)(g(v)-g(z))^2dzdv \le C\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(g(v)-g(z))^2}{|v-z|^{1+2s}}dzdv.\end{aligned} \tag{5.47}$$

By using (5.46) and (5.48), we have from (5.45) that

$$\begin{aligned}|\mathcal{H}_v^{\frac{s}{2}} g|_2^2 = \langle \mathcal{H}_v^s g(v), g(v)\rangle &\le C\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(g(v)-g(z))^2}{|v-z|^{1+2s}}dzdv + C|\langle v\rangle^s g|_2^2 \\ &\le C(|\langle v\rangle^s g|_2^2 + |(\mathbf{1}-\Delta_v)^{\frac{s}{2}} g|_2^2).\end{aligned} \tag{5.48}$$

This completes the proof of the right-hand side in (5.20).

Note that $(-\Delta_v + \frac{v^2}{4})^{-s/2}$ is a pseudodifferential operator with symbol $a(v,\xi)$ satisfying

$$|\partial_\xi^\gamma \partial_v^\beta a(v,\xi)| \le C(1+|v|+|\xi|)^{-s-|\gamma|-|\beta|},$$

where $C>0$ is independent of $v$. Therefore, it follows that $(\mathbf{1}-\Delta_v)^{\frac{s}{2}}(-\Delta_v+\frac{v^2}{4})^{-s/2}$ is a pseudodifferential operator with symbol $S^0_{1,0}$ and hence is bounded on $L^2(\mathbb{R}_v)$; cf. [40, 41]. Thus, we have that

$$|(\mathbf{1}-\Delta_v)^{\frac{s}{2}} g|_2^2 \le C|\mathcal{H}_v^{\frac{s}{2}} g|_2^2. \tag{5.49}$$

By (5.48) and (5.23), the first term of (5.45) is nonnegative. Thus, it follows from (5.45), (5.46), and (5.49) that

$$|\langle v\rangle^s g|_2^2 + |(\mathbf{1}-\Delta_v)^{\frac{s}{2}} g|_2^2 \le C|\mathcal{H}_v^{\frac{s}{2}} g|_2^2.$$

This and (5.48) complete the proof of this lemma. ∎

Next we prove the high order derivative estimate of the linearized operator $\mathcal{L}$.

LEMMA 5.2. *There exist constants $\delta_1>0$ and $C>0$ such that, for any $|\beta|>0$,*

$$\langle \mathcal{H}_v^{\frac{\beta}{2}}\mathcal{L}g, \mathcal{H}_v^{\frac{\beta}{2}} g\rangle \ge \delta_1|\mathcal{H}_v^{\frac{\beta+s}{2}} g|_2^2 - C|g|_2^2. \tag{5.50}$$

*Proof.* Decompose the function $g(v)$ into the Hermite basis

$$g=\sum_{n=0}^{+\infty} g_n\psi_n(v),\quad g_n=\langle g,\psi_n\rangle. \tag{5.51}$$

We see that

$$|g|_2^2=\sum_{n=0}^{+\infty}|g_n|^2,\qquad \Big|\mathcal{H}^m g\Big|_2^2=\sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)^{2m}\Big|g_n\Big|^2. \tag{5.52}$$

By using (5.5) and Lemma 2.1 in [28], we have that

$$\Gamma(\psi_k,\psi_l)=\alpha_{k,l}\psi_{k+l},\quad k,l\geq 0, \tag{5.53}$$

with

$$\alpha_{2n,m}=\sqrt{C_{2n+m}^{2n}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}B(\theta)(\sin\theta)^{2n}(\cos\theta)^m d\theta,\quad n\geq 1,\quad m\geq 0, \tag{5.54}$$

$$\alpha_{0,m}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}B(\theta)((\cos\theta)^m-1)d\theta,\quad m\geq 1,\quad \alpha_{0,0}=\alpha_{2n+1,m}=0,\quad m,n\geq 0. \tag{5.55}$$

Recall that $\psi_0(v)=\sqrt{\mu}$ and $\mathcal{L}g=-\Gamma(\sqrt{\mu},g)-\Gamma(g,\sqrt{\mu})$. It follows from (5.8), (5.51), and (5.53) that

$$\begin{aligned}\Big\langle\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(\sqrt{\mu},g),\mathcal{H}_v^{\frac{\beta}{2}}g\Big\rangle&=\Big\langle\mathcal{H}_v^{\frac{\beta}{2}}\Big(\sum_{n=0}^{+\infty}\alpha_{0,n}g_n\psi_n(v)\Big),\mathcal{H}_v^{\frac{\beta}{2}}g\Big\rangle\\&=\Big\langle\sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{0,n}g_n\psi_n(v),\sum_{m=0}^{+\infty}\Big(m+\frac{1}{2}\Big)^{\frac{\beta}{2}}g_m\psi_m(v)\Big\rangle\\&=\sum_{n=0}^{+\infty}(n+\frac{1}{2})^{\beta}\alpha_{0,n}|g_n|^2.\end{aligned} \tag{5.56}$$

It follows from (5.55) and the estimate in [28, page 478] that

$$-\alpha_{0,n}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}B(\theta)(1-(\cos\theta)^n)d\theta\approx\frac{2^{1+s}}{s}\Gamma(1-s)n^s \tag{5.57}$$

when $n\to+\infty$, where $\Gamma$ denotes the Gamma function. Thus, we have from (5.56) and (5.57) that

$$\begin{aligned}-\Big\langle\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(\sqrt{\mu},g),\mathcal{H}_v^{\frac{\beta}{2}}g\Big\rangle&=-\sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)^{\beta}\alpha_{0,n}|g_n|^2\\&\geq C\sum_{n=N}^{+\infty}\Big(n+\frac{1}{2}\Big)^{\beta}n^s|g_n|^2\geq C_s\sum_{n=N}^{+\infty}\Big(n+\frac{1}{2}\Big)^{\beta+s}|g_n|^2\\&\geq C_s\sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)^{\beta+s}|g_n|^2-C\sum_{n=0}^{N-1}\Big(n+\frac{1}{2}\Big)^{\beta+s}|g_n|^2\geq C|\mathcal{H}_v^{\frac{\beta+s}{2}}g|_2^2-C|g|_2^2.\end{aligned} \tag{5.58}$$

Here the integer $N > 0$ is large enough. It follows from (5.53), (5.54), and (5.55) that

$$\begin{aligned}\Big\langle \mathcal{H}_v^{\frac{\beta}{2}}\Gamma(g,\sqrt{\mu}), \mathcal{H}_v^{\frac{\beta}{2}} g\Big\rangle &= \Big\langle \mathcal{H}_v^{\frac{\beta}{2}}\Big(\sum_{n=0}^{+\infty}\alpha_{n,0}g_n\psi_n(v)\Big), \mathcal{H}_v^{\frac{\beta}{2}} g\Big\rangle \\ &= \Big\langle \sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{n,0}g_n\psi_n(v), \sum_{m=0}^{+\infty}\Big(m+\frac{1}{2}\Big)^{\frac{\beta}{2}}g_m\psi_m(v)\Big\rangle \\ &= \sum_{n=0}^{+\infty}\Big(n+\frac{1}{2}\Big)^{\beta}\alpha_{n,0}|g_n|^2 = \sum_{m=1}^{+\infty}\Big(2m+\frac{1}{2}\Big)^{\beta}\alpha_{2m,0}|g_{2m}|^2. \end{aligned} \tag{5.59}$$

Under the assumption (1.7), we recall from Lemma 2.2 in [28] that

$$0 \le \alpha_{2n,m} = \sqrt{C_{2n+m}^{2n}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} B(\theta)(\sin\theta)^{2n}(\cos\theta)^m d\theta \le \frac{C}{n^{\frac{3}{4}}}\widetilde{\mu}_{n,m}, \quad n \ge 1, \quad m \ge 0, \tag{5.60}$$

where

$$\widetilde{\mu}_{n,m} = \Big(1+\frac{m}{n}\Big)^{s}\Big(1+\frac{n}{m+1}\Big)^{\frac{1}{4}}. \tag{5.61}$$

With the help of (5.59), (5.60), and (5.61), for any large $N > 0$ and small $\epsilon > 0$, we have that

$$\begin{aligned}\Big|\Big\langle \mathcal{H}_v^{\frac{\beta}{2}}\Gamma(g,\sqrt{\mu}), \mathcal{H}_v^{\frac{\beta}{2}} g\Big\rangle\Big| &= \sum_{m=1}^{+\infty}\Big(2m+\frac{1}{2}\Big)^{\beta}\alpha_{2m,0}|g_{2m}|^2 \\ &\le C\sum_{m=1}^{+\infty}\Big(2m+\frac{1}{2}\Big)^{\beta}\frac{(1+m)^{\frac{1}{4}}}{m^{\frac{3}{4}}}|g_{2m}|^2 \\ &\le C_\beta\sum_{m=N}^{+\infty}\Big(m+\frac{1}{4}\Big)^{\beta}\frac{(1+m)^{\frac{1}{4}}}{m^{\frac{3}{4}}}|g_{2m}|^2 + C_{N,\beta}\sum_{m=1}^{N-1}|g_{2m}|^2 \\ &\le C_\beta\sum_{m=N}^{+\infty}\Big(m+\frac{1}{2}\Big)^{\beta+s}\Big(m+\frac{1}{2}\Big)^{-\frac{1}{2}-s}|g_m|^2 + C_{N,\beta}\sum_{m=1}^{N-1}|g_{2m}|^2 \\ &\le C_\beta\epsilon\sum_{m=N}^{+\infty}(m+\frac{1}{2})^{\beta+s}|g_m|^2 + C_{N,\beta}\sum_{m=1}^{N-1}|g_m|^2 \\ &\le C\epsilon|\mathcal{H}_v^{\frac{\beta+s}{2}} g|_2^2 + C|g|_2^2. \end{aligned} \tag{5.62}$$

By choosing $\epsilon > 0$ small enough, (5.50) follows from (5.59) and (5.62). ∎

The last lemma is for the estimate on the nonlinear collision term.

LEMMA 5.3. *There exists a constant $C > 0$ such that, for any $|\beta| \ge 0$,*

$$|\langle \mathcal{H}_v^{\frac{\beta}{2}}\Gamma(f,g), h\rangle| \le C|f|_2|\mathcal{H}_v^{\frac{\beta+s}{2}} g|_2|\mathcal{H}_v^{\frac{s}{2}} h|_2 + C|\mathcal{H}_v^{\frac{\beta}{2}} f|_2|\mathcal{H}_v^{\frac{s}{2}} g|_2|\mathcal{H}_v^{\frac{s}{2}} h|_2. \tag{5.63}$$

*Proof.* We expand these functions in the Hermite basis in the velocity variable

$$f = \sum_{n=0}^{+\infty} f_n\psi_n(v), \qquad g = \sum_{n=0}^{+\infty} g_n\psi_n(v), \qquad h = \sum_{n=0}^{+\infty} h_n\psi_n(v), \tag{5.64}$$

with

$$f_n = \langle f, \psi_n\rangle, \quad g_n = \langle g, \psi_n\rangle, \quad h_n = \langle h, \psi_n\rangle. \tag{5.65}$$

By using (5.53), (5.54), and (5.55), we have from (5.64) that

$$\begin{aligned}\Big\langle \mathcal{H}_v^{\frac{\beta}{2}}\Gamma(f,g), h\Big\rangle &= \Big\langle \mathcal{H}_v^{\frac{\beta}{2}}\Big(\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\alpha_{k,l}f_k g_l\psi_{k+l}(v)\Big), h\Big\rangle\\ &= \Big\langle \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\Big(k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{k,l}f_k g_l\psi_{k+l}(v), \sum_{m=0}^{+\infty}h_m\psi_m(v)\Big\rangle\\ &= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\Big(k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{k,l}f_k g_l h_{k+l}.\end{aligned} \tag{5.66}$$

Due to the fact that $\alpha_{2k+1,l} = 0$ by (5.55), one has that

$$\begin{aligned}\Big\langle \mathcal{H}_v^{\frac{\beta}{2}}\Gamma(f,g), h\Big\rangle &= \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\Big(k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{k,l}f_k g_l h_{k+l}\\ &= \sum_{k=0}^{+\infty}\sum_{l\geq k}\Big(k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{k,l}f_k g_l h_{k+l} + \sum_{l=0}^{+\infty}\sum_{k>l}\Big(k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{k,l}f_k g_l h_{k+l}\\ &= \sum_{k=0}^{+\infty}\sum_{l\geq 2k}\Big(2k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{2k,l}f_{2k} g_l h_{2k+l}\\ &\quad + \sum_{l=0}^{+\infty}\sum_{2k>l}\Big(2k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{2k,l}f_{2k} g_l h_{2k+l} := \Delta_1 + \Delta_2.\end{aligned} \tag{5.67}$$

For the first term $\Delta_1$ of (5.68), we have that

$$\Delta_1 = \sum_{l\geq 0}\Big(l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{0,l}f_0 g_l h_l + \sum_{k=1}^{+\infty}\sum_{l\geq 2k}\Big(2k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{2k,l}f_{2k}g_l h_{2k+l} := \Delta_{11} + \Delta_{12}. \tag{5.68}$$

For the term $\Delta_{11}$, we have from (5.17), (5.52), (5.55), and (5.57) that

$$\begin{aligned}|\Delta_{11}| &\leq |f_0|\sum_{l=0}^{N-1}\Big(l+\frac{1}{2}\Big)^{\frac{\beta}{2}}|\alpha_{0,l}||g_l h_l| + |f_0|\sum_{l=N}^{+\infty}\Big(l+\frac{1}{2}\Big)^{\frac{\beta}{2}}|\alpha_{0,l}||g_l h_l|\\ &\leq C_{N,\beta}|f_0|\sum_{l=0}^{N-1}|g_l h_l| + C|f_0|\sum_{l=N}^{+\infty}\Big(l+\frac{1}{2}\Big)^{\frac{\beta}{2}}l^s|g_l h_l|\\ &\leq C_{N,\beta}|f_0|\Big(\sum_{l=0}^{+\infty}|g_l|^2\Big)^{\frac{1}{2}}\Big(\sum_{l=0}^{+\infty}|h_l|^2\Big)^{\frac{1}{2}}\\ &\quad + C_s|f_0|\Big(\sum_{l=0}^{+\infty}\Big(l+\frac{1}{2}\Big)^{\beta+s}|g_l|^2\Big)^{\frac{1}{2}}\Big(\sum_{l=0}^{+\infty}\Big(l+\frac{1}{2}\Big)^{s}|h_l|^2\Big)^{\frac{1}{2}}\\ &\leq C_{N,\beta}|f_0||g|_2|h|_2 + C_s|f_0|\Big|\mathcal{H}_v^{\frac{\beta+s}{2}}g\Big|_2\Big|\mathcal{H}_v^{\frac{s}{2}}h\Big|_2 \leq C|f|_2\Big|\mathcal{H}_v^{\frac{\beta+s}{2}}g\Big|_2\Big|\mathcal{H}_v^{\frac{s}{2}}h\Big|_2.\end{aligned} \tag{5.69}$$

Here, $N > 0$ is large enough, and we have used the fact that $|\alpha_{0,l}| \le C$ with $0 \le l \le N-1$.

For the term $\Delta_{12}$, we deduce from (5.60) and (5.61) that

$$
\begin{aligned}
|\Delta_{12}| &\le \sum_{k=1}^{+\infty}\sum_{l\ge 2k}\Big(2k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}|\alpha_{2k,l}f_{2k}g_lh_{2k+l}| \\
&\le C\sum_{k=1}^{+\infty}\sum_{l\ge 2k}\Big(2k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\frac{\widetilde{\mu}_{k,l}}{k^{\frac{3}{4}}}|f_{2k}g_lh_{2k+l}| \\
&\le C\sum_{k=1}^{+\infty}\sum_{l\ge 2k}\Big(2l+\frac{1}{2}\Big)^{\frac{\beta}{2}}(l+1)^s\frac{1}{k^{\frac{3}{4}}}|f_{2k}g_lh_{2k+l}| \\
&\le C_{\beta,s}\sum_{k=1}^{+\infty}\sum_{l\ge 2k}\Big(l+\frac{1}{2}\Big)^{\frac{\beta}{2}+s}\frac{1}{k^{\frac{3}{4}}}|f_{2k}g_lh_{2k+l}|.
\end{aligned}
\tag{5.70}
$$

It follows from (5.71) that

$$
\begin{aligned}
|\Delta_{12}| &\le C\sum_{k=1}^{+\infty}\frac{1}{k^{\frac{3}{4}}}|f_{2k}|\Big(\sum_{l\ge 2k}\Big(l+\frac{1}{2}\Big)^{\beta+s}|g_l|^2\Big)^{\frac{1}{2}}\Big(\sum_{l\ge 2k}\Big(l+\frac{1}{2}\Big)^{s}|h_{2k+l}|^2\Big)^{\frac{1}{2}} \\
&\le C\Big(\sum_{k=1}^{+\infty}\frac{1}{k^{\frac{3}{2}}}\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{+\infty}|f_{2k}|^2\Big)^{\frac{1}{2}}\Big(\sum_{l=0}^{+\infty}\Big(l+\frac{1}{2}\Big)^{\beta+s}|g_l|^2\Big)^{\frac{1}{2}}\Big(\sum_{l=0}^{+\infty}\Big(l+\frac{1}{2}\Big)^{s}|h_l|^2\Big)^{\frac{1}{2}} \\
&\le C|f|_2\Big|\mathcal{H}_v^{\frac{\beta+s}{2}}g\Big|_2\Big|\mathcal{H}_v^{\frac{s}{2}}h\Big|_2.
\end{aligned}
\tag{5.71}
$$

We deduce from (5.68), (5.70), and (5.71) that

$$
|\Delta_1| \le C|f|_2\Big|\mathcal{H}_v^{\frac{\beta+s}{2}}g\Big|_2\Big|\mathcal{H}_v^{\frac{s}{2}}h\Big|_2.
\tag{5.72}
$$

For the second term $\Delta_2$ of (5.68), one has that

$$
\Delta_2 = \sum_{2k>0}\Big(2k+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{2k,0}f_{2k}g_0h_{2k}+\sum_{l=1}^{+\infty}\sum_{2k>l}\Big(2k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{2k,l}f_{2k}g_lh_{2k+l} := \Delta_{21}+\Delta_{22}.
\tag{5.73}
$$

It follows from (1.7) and (5.54) that

$$
0 \le \alpha_{2k,0} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} B(\theta)(\sin\theta)^{2k}d\theta = 2\int_0^{\frac{\pi}{4}} B(\theta)(\sin\theta)^{2k}d\theta \le 2\int_0^{\frac{\pi}{4}} B(\theta)(\sin\theta)^{2}d\theta = C.
$$

By using this and (5.73), we can obtain

$$
\begin{aligned}
|\Delta_{21}| &= \Big|\sum_{2k>0}\Big(2k+\frac{1}{2}\Big)^{\frac{\beta}{2}}\alpha_{2k,0}f_{2k}g_0h_{2k}\Big| \\
&\le C|g_0|\Big(\sum_{l\ge 0}\Big(l+\frac{1}{2}\Big)^{\beta}|f_l|^2\Big)^{\frac{1}{2}}\Big(\sum_{l\ge 0}|h_l|^2\Big)^{\frac{1}{2}} \le C|g|_2\Big|\mathcal{H}_v^{\frac{\beta}{2}}f\Big|_2\Big|h\Big|_2.
\end{aligned}
\tag{5.74}
$$

For the second term $\Delta_{22}$ of (5.73), we deduce from (5.60) and (5.61) that

$$|\Delta_{22}| \le \sum_{l=1}^{+\infty}\sum_{2k>l}\Big(2k+l+\frac{1}{2}\Big)^{\frac{\beta}{2}}|\alpha_{2k,l}||f_{2k}g_l h_{2k+l}| \le C\sum_{l=1}^{+\infty}\sum_{2k>l}\Big(2k+\frac{1}{2}\Big)^{\frac{\beta}{2}}\frac{\widetilde{\mu}_{k,l}}{k^{\frac{3}{4}}}|f_{2k}g_l h_{2k+l}|$$

$$\le C\sum_{l=1}^{+\infty}\sum_{2k>l}\Big(2k+\frac{1}{2}\Big)^{\frac{\beta}{2}}\frac{1}{k^{\frac{1}{2}}}|f_{2k}g_l h_{2k+l}|. \tag{5.75}$$

It follows from (5.75) that

(5.76)

$$\begin{aligned}|\Delta_{22}| &\le C\sum_{l=1}^{+\infty}\frac{1}{l^{\frac{1}{2}}}|g_l|\Big(\sum_{2k>l}\Big(2k+\frac{1}{2}\Big)^{\beta}|f_{2k}|^2\Big)^{\frac{1}{2}}\Big(\sum_{2k>l}|h_{2k+l}|^2\Big)^{\frac{1}{2}}\\ &\le C\Big(\sum_{l=1}^{+\infty}\frac{1}{l^{1+s}}\Big)^{\frac{1}{2}}\Big(\sum_{l=1}^{+\infty}\Big(l+\frac{1}{2}\Big)^{s}|g_l|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=0}^{+\infty}\Big(m+\frac{1}{2}\Big)^{\beta}|f_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=0}^{+\infty}|h_m|^2\Big)^{\frac{1}{2}}\\ &\le C|\mathcal{H}_v^{\frac{\beta}{2}}f|_2\Big|\mathcal{H}_v^{\frac{s}{2}}g\Big|_2\Big|h\Big|_2.\end{aligned}$$

By using (5.17), one has from (5.73), (5.74), and (5.77) that

$$|\Delta_2| \le C|\mathcal{H}_v^{\frac{\beta}{2}}f|_2\Big|\mathcal{H}_v^{\frac{s}{2}}g\Big|_2\Big|h\Big|_2. \tag{5.77}$$

It follows from (5.68), (5.72), and (5.77) that

$$\Big|\Big\langle\mathcal{H}_v^{\frac{\beta}{2}}\Gamma(f,g),h\Big\rangle\Big| \le C|f|_2\Big|\mathcal{H}_v^{\frac{\beta+s}{2}}g\Big|_2\Big|\mathcal{H}_v^{\frac{s}{2}}h\Big|_2 + C|\mathcal{H}_v^{\frac{\beta}{2}}f|_2\Big|\mathcal{H}_v^{\frac{s}{2}}g\Big|_2\Big|h\Big|_2.$$

This and (5.17) complete the proof of Lemma 5.3. ∎

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