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WELL-POSEDNESS OF THE MHD BOUNDARY LAYER SYSTEM IN GEVREY FUNCTION SPACE WITHOUT STRUCTURAL ASSUMPTION*

WEI-XI LI[†] AND TONG YANG[‡]

Abstract. We establish the well-posedness of the MHD boundary layer system in Gevrey function space without any structural assumption. Compared to the classical Prandtl equation, the loss of tangential derivative comes from both the velocity and magnetic fields that are coupled with each other. By observing a new type of cancellation mechanism in the system for overcoming the loss derivative degeneracy, we show that the MHD boundary layer system is well-posed with Gevrey index up to $3/2$ in both two- and three-dimensional spaces.

Key words. MHD boundary layer, nonstructural assumption, cancellation, well-posedness theory, Gevrey class

AMS subject classifications. 35Q35, 76W05

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1. Introduction. Magnetohydrodynamics (MHD) is concerned with the motion of conducting fluid under the influence of the self-induced magnetic field. In the incompressible framework, the governing equations are

$$(1.1) \quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla P - \frac{1}{\text{Re}} \Delta \mathbf{u} = 0, \\ \partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{1}{\text{Rm}} \Delta \mathbf{B} = 0, \\ \nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{B}|_{t=0} = \mathbf{B}_0, \end{cases}$$

where Re and Rm stand for the hydrodynamic and magnetic Reynolds numbers, respectively. The MHD system is well-explored when the fluid region is the whole space; see, for instance, the survey paper [25] and the references therein. Here we assume that the fluid is in the half-space $\mathbb{R}_+^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_d > 0\}$ with $d = 2$ or $d = 3$, and the system (1.1) is equipped with the no-slip boundary condition on the velocity field and perfectly conducting boundary condition on the magnetic field, that is,

$$\mathbf{u}|_{x_d=0} = \mathbf{0}, \quad (\partial_{x_d} \mathbf{B}_h, \mathbf{B}_{x_d})|_{x_d=0} = \mathbf{0},$$

where $\mathbf{B}_h, \mathbf{B}_{x_d}$ represent the tangential and normal components of \mathbf{B} , respectively. In this work we will investigate the well-posedness of the MHD boundary layer system derived from the high Reynolds numbers limit of the MHD system (1.1). More

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precisely, when the hydrodynamic and magnetic Reynolds numbers are of the same order, i.e., $1/\text{Re} = \nu\varepsilon$ and $1/\text{Rm} = \mu\varepsilon$ for $\varepsilon \ll 1$, the following MHD boundary layer system was derived in [31] (cf. the work [10] for the derivation with the insulating boundary condition on magnetic field):

$$(1.2) \quad \begin{cases} (\partial_t + \vec{u} \cdot \nabla - \nu \partial_z^2) u_h - (\vec{f} \cdot \nabla) f_h + \nabla_h p = 0, \\ \partial_t \vec{f} - \nabla \times (\vec{u} \times \vec{f}) - \mu \partial_z^2 \vec{f} = 0, \\ \text{div } \vec{u} = \text{div } \vec{f} = 0, \\ \vec{u}|_{z=0} = (\partial_z f_h, f_z)|_{z=0} = \mathbf{0}, \quad (u_h, f_h)|_{z \rightarrow +\infty} = (\mathbf{U}, \mathbf{F}), \\ u_h|_{t=0} = u_{h,0}, \quad f_h|_{t=0} = f_{h,0}, \end{cases}$$

where $x_h \in \mathbb{R}^{d-1}$ is the tangential component of $(x_h, z) \in \mathbb{R}_+^d$ and we use the notation $\nabla = (\nabla_h, \partial_z)$ with $\nabla_h = \partial_{x_h} = (\partial_{x_1}, \dots, \partial_{x_{d-1}})$, and denote by $\vec{u} = (u_h, u_z)$ and $\vec{f} = (f_h, f_z)$ the velocity and magnetic fields, respectively, with the tangential components u_h, f_h and the normal components u_z, f_z . Here p, \mathbf{U} , and \mathbf{F} are given functions in (t, x_h) variables satisfying the Bernoulli's law

$$\begin{cases} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_h) \mathbf{U} - (\mathbf{F} \cdot \nabla_h) \mathbf{F} + \nabla_h p = 0, \\ \partial_t \mathbf{F} + (\mathbf{U} \cdot \nabla_h) \mathbf{F} - (\mathbf{F} \cdot \nabla_h) \mathbf{U} = 0. \end{cases}$$

In view of the divergence-free and boundary conditions we can write the normal components u_z and f_z as

$$u_z(t, x_h, z) = - \int_0^z \nabla_h \cdot u_h(t, x_h, \tilde{z}) d\tilde{z}, \quad f_z(t, x_h, z) = - \int_0^z \nabla_h \cdot f_h(t, x_h, \tilde{z}) d\tilde{z}.$$

Thus the MHD boundary layer system (1.2) is a degenerate system with the loss of tangential derivatives in f_z and u_z as nonlocal terms. Note the equation for f_z in (1.2) is just an immediate consequence of those for f_h , in view of the representation of f_z given above. The degeneracy coupled with the nonlocal property is the main difficulty in studying the well-posedness of this system.

In the absence of a magnetic field, the MHD system is reduced to the classical incompressible Navier–Stokes equations, and the corresponding boundary layer system (1.2) is the classical Prandtl equation derived by Prandtl in 1904. The mathematical study on the Prandtl boundary layer has a long history, and there have been extensive works concerning its well/ill-posedness theories. So far the two-dimensional (2D) Prandtl equation is well-explored in various function spaces; see, e.g., [1, 2, 3, 4, 6, 13, 15, 18, 19, 20, 22, 32, 42, 43, 44] and the references therein. Among these works we can see that there are basically two main settings based on whether or not the structural assumption is imposed. One refers to Oleinik's monotonicity condition and another one to the analytic or Gevrey class. Under Oleinik's monotonicity, the well-posedness in function space with finite order of regularity was first achieved by Oleinik (see, e.g., [37]) by using the Crocco transformation and was recently proved by two research groups [1, 35] independently with new understanding of the cancellation mechanism through an energy method. Hence, the loss of a one order tangential derivative can be overcome by using either Crocco transformation or a cancellation mechanism with the monotonicity condition.

Without any structural assumption, it is natural to introduce the analytic function space to overcome the loss of a one order derivative by shrinking the radius of analyticity in time; cf. [39] and the later improvement in [19, 32] that holds in both two and three dimensions. Recently some new idea of cancellation was observed in [3] to establish the well-posedness in Gevrey function space with index up to 2 rather than in an analytic setting for the 2D Prandtl equation. Compared to the 2D case, much less is known for the 3D Prandtl equation outside the analytic framework. Here, we refer to [28] for the existence of classical solutions based on some structural assumption such that the secondary flow does not appear and to the work [33] about the mild solution to this system. Recently, the well-posedness in Gevrey space with the Gevrey index up to 2 was obtained in [21] for 3D Prandtl equations without any structural assumption, inspired by the work [3] for two dimensions. Note that the Gevrey index 2 in [3, 21] is optimal in view of the ill-posedness theory in [6, 27].

Finally, let us also mention the work [42] on the global existence of weak solutions under an additional favorable pressure condition and the work [38] on the existence of global solutions in analytic function space for small initial data. All these results are in fact related to the high Reynolds number limit for the purely hydrodynamic flow with physical boundary conditions, and to show that the Navier–Stokes equations can be approximated by the Euler equation away from boundary and by the Prandtl equation near the boundary. The mathematically rigorous justification of the limit was obtained by [40, 41] in the analytic function space without any structural assumption. And there is a significant improvement to the Gevrey setting in [8, 9] with some kind of concave condition on the Prandtl boundary layer profile. If the initial vorticity is supported away from the boundary, then the limit in L^∞ norm was established in [34] and [5], respectively, for 2D and 3D cases. The aforementioned works on the inviscid limit are concerned with the time dependent problem. On the other hand, for steady flow we refer to [7, 11, 12, 14, 16, 17] and references therein for the study of the inviscid limit in a Sobolev or an L^∞ setting.

Back to the MHD system, we have a new difficulty caused by the additional loss of a tangential derivative in the magnetic field. With some structural condition, the stabilizing effect of the magnetic field on the boundary layer has been observed; see, e.g., [10, 26, 30, 31]. Precisely, under the assumption on the nondegenerate tangential magnetic field, the well-posedness of the MHD boundary layer in Sobolev space together with the justification of the high Reynolds numbers limit was obtained in [30, 31] without Oleinik’s monotonicity condition on the velocity field. On the other hand, the magnetic field may act as a destabilizing factor and lead to the boundary layer separation; cf. [36]. Inspired by the well-posedness theory established in [3, 21] for the Prandtl equation, this paper aims to investigate the well-posedness of the MHD boundary layer without any structural assumption in the Gevrey function space. For this, we need to explore an other intrinsic cancellation mechanism to overcome the additional loss of tangential derivative in the magnetic field coupled with the velocity field.

To simply the argument we will assume without loss of generality that $(\mathbf{U}, \mathbf{F}) \equiv \mathbf{0}$ in the system (1.2) because the result holds in the general case if we use some kind of the nontrivial weighted functions similar to those used in [3] for the Prandtl equation. Hence, we consider

$$(1.3) \quad \begin{cases} (\partial_t + \vec{u} \cdot \nabla - \nu \partial_z^2)u_h - (\vec{f} \cdot \nabla)f_h = 0, \\ \partial_t \vec{f} - \nabla \times (\vec{u} \times \vec{f}) - \mu \partial_z^2 \vec{f} = 0, \\ \operatorname{div} \vec{u} = \operatorname{div} \vec{f} = 0, \\ \vec{u}|_{z=0} = (\partial_z f_h, f_z)|_{z=0} = \mathbf{0}, \quad (u_h, f_h)|_{z \rightarrow +\infty} = \mathbf{0}, \\ u_h|_{t=0} = u_{h,0}, \quad f_h|_{t=0} = f_{h,0}. \end{cases}$$

For clear presentation, let us first introduce the Gevrey function spaces used in this paper.

DEFINITION 1.1. *Let $\ell \geq 1$ be a given number. With a given integer $N \geq 0$ and a pair (ρ, σ) , $\rho > 0$ and $\sigma \geq 1$, a Banach space $X_{\rho, \sigma, N}$ consists of all smooth vector-valued functions $\mathbf{A} = \mathbf{A}(x_h, z)$ with $(x_h, z) \in \mathbb{R}_+^d$ such that the Gevrey norm $\|\mathbf{A}\|_{\rho, \sigma, N} < +\infty$, where $\|\cdot\|_{\rho, \sigma, N}$ is defined below. Denote $\partial_{x_h}^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{d-1}}^{\alpha_{d-1}}$ and define*

$$\begin{aligned} \|\mathbf{A}\|_{\rho, \sigma, N} &= \sup_{\substack{0 \leq j \leq N \\ |\alpha|+j \geq 7}} \frac{\rho^{|\alpha|+j-7}}{[(|\alpha|+j-7)!]^\sigma} \|\langle z \rangle^{\ell+j} \partial_{x_h}^\alpha \partial_z^j \mathbf{A}\|_{L^2(\mathbb{R}_+^d)} \\ &+ \sup_{\substack{0 \leq j \leq N \\ |\alpha|+j \leq 6}} \|\langle z \rangle^{\ell+j} \partial_{x_h}^\alpha \partial_z^j \mathbf{A}\|_{L^2(\mathbb{R}_+^d)}, \end{aligned}$$

where $\langle z \rangle = (1 + |z|^2)^{1/2}$ and

$$\|\mathbf{A}\|_{L^2(\mathbb{R}_+^d)} \stackrel{\text{def}}{=} \left(\sum_{1 \leq j \leq k} \|A_j\|_{L^2(\mathbb{R}_+^d)}^2 \right)^{1/2}$$

for $\mathbf{A} = (A_1, \dots, A_k)$. Here, σ is the Gevrey index.

The main theorem in this paper can be stated as follows.

THEOREM 1.2. *Let the dimension $d = 2$ or 3 . Suppose the initial data $(u_{h,0}, f_{h,0})$ in the system (1.3) belong to $X_{2\rho_0, \sigma, 8}$ for some $1 < \sigma \leq 3/2$ and some $0 < \rho_0 \leq 1$, compatible with the boundary condition. Then the system (1.3) admits a unique solution $(u_h, f_h) \in L^\infty([0, T]; X_{\rho, \sigma, 4})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$.*

Note that the instability result in [29] suggests $\sigma = 2$ may be the optimal Gevrey index for the well-posedness theory of the MHD boundary layer without structural assumption similar to the classical Prandtl equation. Hence, it remains an interesting problem to establish a well-posedness theory in Gevrey function space with optimal index.

The rest of the paper is organized as follows. For clear presentation, we will prove in section 2 the well-posedness of the 2D MHD boundary layer system. The discussion on 3D MHD will be given in section 3 by pointing out the difference.

2. 2D MHD boundary layer. For the 2D MHD boundary layer system, we will use (u, w) and (f, h) to denote the velocity and magnetic fields, respectively, and denote by $(x, z) \in \mathbb{R}_+^2$ the spatial variable. Then the MHD boundary layer system (1.3) is

$$\begin{cases} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)u - (f\partial_x + h\partial_z)f = 0, \\ \partial_t f + \partial_z(wf - uh) - \mu\partial_z^2 f = 0, \\ \partial_t h - \partial_x(wf - uh) - \mu\partial_z^2 h = 0, \end{cases}$$

with the divergence-free and initial-boundary conditions

$$(2.1) \quad \begin{cases} \partial_x u + \partial_z w = \partial_x f + \partial_z h = 0, \\ (u, w)|_{z=0} = (\partial_z f, h)|_{z=0} = (0, 0), \quad (u, f)|_{z \rightarrow +\infty} = (0, 0), \\ (u, f)|_{t=0} = (u_0, f_0). \end{cases}$$

By (2.1), we can rewrite (1.3) as

$$(2.2) \quad \begin{cases} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)u = \xi, \\ (\partial_t + u\partial_x + w\partial_z - \mu\partial_z^2)f = \eta, \\ (\partial_t + u\partial_x + w\partial_z - \mu\partial_z^2)h = f\partial_x w - h\partial_x u, \end{cases}$$

where

$$(2.3) \quad \xi = (f\partial_x + h\partial_z)f, \quad \eta = (f\partial_x + h\partial_z)u.$$

Note

$$w(t, x, z) = -\int_0^z \partial_x u(t, x, \tilde{z})d\tilde{z}, \quad h(t, x, z) = -\int_0^z \partial_x f(t, x, \tilde{z})d\tilde{z}.$$

We remark that the equation for h in (2.2) can be derived from the one for f and the main difficulty in analysis is the loss of x -derivatives in the two terms w and h .

The existence and uniqueness theory for (2.2) can be stated as follows.

THEOREM 2.1. *Suppose $(u_0, f_0) \in X_{2\rho_0, \sigma, 8}$ for some $1 < \sigma \leq 3/2$ and $0 < \rho_0 \leq 1$, compatible with the boundary condition in (2.1). Then the system (2.2) with the condition (2.1) admits a unique solution $(u, f) \in L^\infty([0, T]; X_{\rho, \sigma, 4})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$.*

The main part of the proof of Theorem 2.1 will be given in subsections 2.2–2.7 for proving the a priori estimate stated in subsection 2.1.

Notation. Throughout this section we will use $\|\cdot\|_{L^2}$ and $(\cdot, \cdot)_{L^2}$ to denote the norm and inner product of $L^2 = L^2(\mathbb{R}_+^2)$ and use the notation $\|\cdot\|_{L^2(\mathbb{R}_x)}$ and $(\cdot, \cdot)_{L^2(\mathbb{R}_x)}$ when the variable is specified. Similar notation will be used for L^∞ . Moreover, we use $L_x^p(L_z^q) = L^p(\mathbb{R}; L^q(\mathbb{R}_+))$ for the classical Sobolev space.

2.1. A priori estimate. Let $(u, f) \in L^\infty([0, T]; X_{\rho_0, \sigma, 4})$ be a solution to the boundary layer system (2.2) with initial datum $(u_0, f_0) \in X_{2\rho_0, \sigma, 8}$ for some $0 < \rho_0 \leq 1$ and $1 < \sigma \leq 3/2$, recalling $X_{\rho, \sigma, N}$ is the Gevrey function space given in Definition 1.1. Moreover, suppose $(\partial_t^i u, \partial_t^i f) \in L^\infty([0, T]; X_{\rho_0, \sigma, 4-i})$ for each $i \leq 4$. This subsection together with the following subsections 2.2–2.7 aims to close the a priori estimate on u and f . For this, we first introduce some auxiliary functions defined below. Some of these functions were given in [21] for the study on Prandtl equation inspired by the work [3].

Let \mathcal{U} be a solution to the linear initial-boundary problem

$$(2.4) \quad \begin{cases} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2) \int_0^z \mathcal{U}d\tilde{z} = -\partial_x w, \\ \mathcal{U}|_{t=0} = 0, \quad \partial_z \mathcal{U}|_{z=0} = \mathcal{U}|_{z \rightarrow +\infty} = 0, \end{cases}$$

where $\int_0^z \mathcal{U}d\tilde{z} = \int_0^z \mathcal{U}(t, x, \tilde{z})d\tilde{z}$. In addition, we define

$$(2.5) \quad \lambda = \partial_x u - (\partial_z u) \int_0^z \mathcal{U}d\tilde{z}, \quad \delta = \partial_x f - (\partial_z f) \int_0^z \mathcal{U}d\tilde{z}.$$

Note the existence of a solution to the initial-boundary problem (2.4) follows from the standard parabolic theory. With these functions and ξ, η defined in (2.3), set

$$\vec{a} = (u, f, \mathcal{U}, \lambda, \delta, \xi, \eta).$$

And then we define the following Gevrey norm on \vec{a} .

DEFINITION 2.2. Letting \vec{a} be given above, define (2.6)

$$\begin{aligned} |\vec{a}|_{\rho, \sigma} &= \sup_{\substack{i+j \leq 4 \\ m+i+j \geq 7}} \frac{\rho^{m+i+j-7}}{[(m+i+j-7)!]^\sigma} (\|\partial_t^i \partial_x^m \partial_z^j u\|_{L^2} + \|\partial_t^i \partial_x^m \partial_z^j f\|_{L^2}) \\ &+ \sup_{\substack{i \leq 4 \\ m+i \geq 6}} \frac{\rho^{m+i-6}}{[(m+i-6)!]^\sigma} \|\partial_t^i \partial_x^m \mathcal{U}\|_{L^2} \\ &+ \sup_{\substack{i \leq 4 \\ m+i \geq 6}} \frac{\rho^{m+i-6}}{[(m+i-6)!]^\sigma} (m^{1/2} \|\partial_t^i \partial_x^m \lambda\|_{L^2} + m^{1/2} \|\partial_t^i \partial_x^m \delta\|_{L^2}) \\ &+ \sup_{\substack{i \leq 4 \\ m+i \geq 6}} \frac{\rho^{m+i-6}}{[(m+i-6)!]^\sigma} (m \| \langle z \rangle^\ell \partial_t^i \partial_x^m \xi \|_{L^2} + m \| \langle z \rangle^\ell \partial_t^i \partial_x^m \eta \|_{L^2}) \\ &+ \sup_{\substack{i+j \leq 4 \\ m+i+j \leq 6}} (\|\partial_t^i \partial_x^m \partial_z^j u\|_{L^2} + \|\partial_t^i \partial_x^m \partial_z^j f\|_{L^2}) + \sup_{\substack{i \leq 4 \\ m+i \leq 5}} \|\partial_t^i \partial_x^m \mathcal{U}\|_{L^2} \\ &+ \sup_{\substack{i \leq 4 \\ m+i \leq 5}} (\|\partial_t^i \partial_x^m \lambda\|_{L^2} + \|\partial_t^i \partial_x^m \delta\|_{L^2} + \| \langle z \rangle^\ell \partial_t^i \partial_x^m \xi \|_{L^2} + \| \langle z \rangle^\ell \partial_t^i \partial_x^m \eta \|_{L^2}), \end{aligned}$$

where the number ℓ is given in Definition 1.1.

Remark 2.3. Note that we have different powers of m for the L^2 norms of the m th order derivatives $\partial_x^m \mathcal{U}, \partial_x^m \lambda$, and $\partial_x^m \xi$. This is motivated by the following relations between these functions:

$$(2.7) \quad \begin{cases} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\mathcal{U} = \partial_x \lambda + \text{l.o.t.}, \\ (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\lambda = \partial_x \xi + \text{l.o.t.}, \end{cases}$$

where l.o.t. refers to lower order terms. Formally, there is one order derivative loss in both equations of (2.7). However, if ξ can be estimated, then we lose only 2/3 order rather than one order derivative by treating λ and ξ as $-1/3$ and $-2/3$ order derivatives of \mathcal{U} . This corresponds to the different powers of m before these auxiliary functions in the definition (2.6).

Remark 2.4. As in [21] the auxiliary function \mathcal{U} is used to overcome the loss of derivatives in w . And to overcome the loss of derivatives in h , we observe a new cancellation mechanism for the magnetic convection term $\xi = (f\partial_x + h\partial_z)f$ and this enables us to close the a priori estimate.

Now we state the main result concerning the a priori estimate. Without loss of generality we only consider the case when the Gevrey index $\sigma = 3/2$, and the argument works with slight modification for $1 < \sigma < 3/2$ (see subsection 2.8).

Assumption 2.5. Let $X_{\rho, \sigma}$ be the Gevrey function space given in Definition 1.1. Suppose $(u, f) \in L^\infty([0, T]; X_{\rho_0, \sigma, 4})$ with some $0 < \rho_0 \leq 1$ and $\sigma = 3/2$ is a solution to the boundary layer system (2.2) equipped with the condition (2.1), where the

initial datum $(u_0, f_0) \in X_{2\rho_0, \sigma, 8}$ which is compatible with the boundary condition in (2.1). Without loss of generality we may assume $T \leq 1$. Moreover, we suppose $(\partial_t^i u, \partial_t^i f) \in L^\infty([0, T]; X_{\rho_0, \sigma, 4-i})$ for $1 \leq i \leq 4$ and there exists a constant C_* such that, for any $t \in [0, T]$,

$$(2.8) \quad \sup_{\substack{i+j \leq 4 \\ k+i+j \leq 10}} \left(\| \langle z \rangle^{\ell+j} \partial_t^i \partial_x^k \partial_z^j u(t) \|_{L^2} + \| \langle z \rangle^{\ell+j} \partial_t^i \partial_x^k \partial_z^j f(t) \|_{L^2} \right) \leq C_*,$$

where the constant $C_* \geq 1$ depends only on $\|(u_0, f_0)\|_{2\rho_0, \sigma, 8}$, the Sobolev embedding constants, and the numbers ρ_0, σ, ℓ that are given in Definition 1.1.

THEOREM 2.6. *Let $|\vec{a}|_{\rho, \sigma}$ be given in (2.6). Under Assumption 2.5, there exist two constants $C_1, C_2 \geq 1$, such that for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, the estimate*

$$(2.9) \quad \begin{aligned} |\vec{a}(t)|_{\rho, \sigma}^2 &\leq C_1 \left(\|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 + \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^4 \right) \\ &+ e^{C_2 C_*^2} \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right) \end{aligned}$$

holds for any $t \in [0, T]$, where the constants C_1 and C_2 depend only on the Sobolev embedding constants and the numbers ρ_0, σ, ℓ given in Definition 1.1. Both C_1 and C_2 are independent of the constant C_* given in (2.8).

The rest of this section is devoted to the proof of this a priori estimate. We will proceed in the following subsections 2.2–2.7 to derive the estimates on the terms involved in the definition (2.6) of $|\vec{a}|_{\rho, \sigma}$.

To simplify the notation, from now on the capital letter C denotes some generic constant that may vary from line to line and that depends only on the Sobolev embedding constants and the numbers ρ_0, σ, ℓ given in Definition 1.1 but is independent of the constant C_* in (2.8) and the order of differentiation denoted by m .

2.2. Tangential derivatives of \mathcal{U} . For the tangential derivatives of \mathcal{U} defined in (2.4), we have the following estimate.

PROPOSITION 2.7. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned} &\sup_{m \geq 6} \frac{\rho^{2(m-6)}}{[(m-6)!]^{2\sigma}} \|\partial_x^m \mathcal{U}(t)\|_{L^2}^2 + \sup_{m \leq 5} \|\partial_x^m \mathcal{U}(t)\|_{L^2}^2 \\ &\leq C C_* \left(\int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where $C_* \geq 1$ is the constant given in (2.8).

Proof. We apply ∂_z to (2.4) and then use the representation (2.5) of λ to get

$$(\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\mathcal{U} = \partial_x \lambda + (\partial_x \partial_z u) \int_0^z \mathcal{U} d\tilde{z} + (\partial_x u)\mathcal{U}.$$

Then applying ∂_x^m to the above equation yields

$$\begin{aligned} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\partial_x^m \mathcal{U} &= \partial_x^{m+1} \lambda - \sum_{j=1}^m \binom{m}{j} \left[(\partial_x^j u) \partial_x^{m-j+1} \mathcal{U} + (\partial_x^j w) \partial_x^{m-j} \partial_z \mathcal{U} \right] \\ &+ \partial_x^m \left[(\partial_x \partial_z u) \int_0^z \mathcal{U} d\tilde{z} + (\partial_x u)\mathcal{U} \right]. \end{aligned}$$

We take the scalar product with $\partial_x^m \mathcal{U}$ on both sides of this equation. Since $\mathcal{U}|_{t=0} = \partial_z \mathcal{U}|_{z=0} = 0$, it holds that

$$\begin{aligned}
 & \frac{1}{2} \|\partial_x^m \mathcal{U}(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_z \partial_x^m \mathcal{U}(s)\|_{L^2}^2 ds = \int_0^t \left(\partial_x^{m+1} \lambda, \partial_x^m \mathcal{U} \right)_{L^2} ds \\
 (2.10) \quad & - \int_0^t \left(\sum_{j=1}^m \binom{m}{j} \left[(\partial_x^j u) \partial_x^{m-j+1} \mathcal{U} + (\partial_x^j w) \partial_x^{m-j} \partial_z \mathcal{U} \right], \partial_x^m \mathcal{U} \right)_{L^2} ds \\
 & + \int_0^t \left(\partial_x^m \left[(\partial_x \partial_z u) \int_0^z \mathcal{U} d\tilde{z} + (\partial_x u) \mathcal{U} \right], \partial_x^m \mathcal{U} \right)_{L^2} ds.
 \end{aligned}$$

It remains to derive the upper bound for the terms on the right side in the above equation. From definition (2.6) of $|\vec{a}|_{\rho, \sigma}$, it follows that, for any $0 < r \leq \rho_0$ and any $j \geq 6$,

$$(2.11) \quad \|\partial_x^j \mathcal{U}\|_{L^2} + j^{1/2} (\|\partial_x^j \lambda\|_{L^2} + \|\partial_x^j \delta\|_{L^2}) \leq \frac{[(j-6)!]^\sigma}{r^{(j-6)}} |\vec{a}|_{r, \sigma}.$$

When $\sigma = 3/2$, we have

$$\begin{aligned}
 \int_0^t \left(\partial_x^{m+1} \lambda, \partial_x^m \mathcal{U} \right)_{L^2} ds & \leq \int_0^t m^{-1/2} \frac{[(m-5)!]^\sigma}{\tilde{\rho}^{(m-5)}} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{(m-6)}} |\vec{a}(s)|_{\tilde{\rho}, \sigma}^2 ds \\
 & \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{m^{\sigma-1/2}}{\tilde{\rho}} \left(\frac{\rho}{\tilde{\rho}} \right)^{2(m-6)} |\vec{a}(s)|_{\tilde{\rho}, \sigma}^2 ds \\
 & \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds,
 \end{aligned}$$

where in the last inequality we have used the fact that for any integer $k \geq 1$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq 1$,

$$(2.12) \quad k \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{k}{\tilde{\rho}} \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{1}{\tilde{\rho} - \rho}.$$

On the other hand, the following two estimates are proved respectively in Lemmas 3.3 and 3.4 in [21]:

$$\begin{aligned}
 & - \int_0^t \left(\sum_{j=1}^m \binom{m}{j} \left[(\partial_x^j u) \partial_x^{m-j+1} \mathcal{U} + (\partial_x^j w) \partial_x^{m-j} \partial_z \mathcal{U} \right], \partial_x^m \mathcal{U} \right)_{L^2} ds \\
 & \leq \frac{\nu}{2} \int_0^t \|\partial_z \partial_x^m \mathcal{U}\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^3 + |\vec{a}(s)|_{\rho, \sigma}^4) ds \right) \\
 & \quad + CC_* \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^t \left(\partial_x^m \left[(\partial_x \partial_z u) \int_0^z \mathcal{U} d\tilde{z} + (\partial_x u) \mathcal{U} \right], \partial_x^m \mathcal{U} \right)_{L^2} ds \\
 & \leq \frac{\nu}{2} \int_0^t \|\partial_z \partial_x^m \mathcal{U}\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t (|\vec{a}(s)|_{\rho, \sigma}^3 + |\vec{a}(s)|_{\rho, \sigma}^4) ds
 \end{aligned}$$

with $C_* \geq 1$ the constant in (2.8). Then we combine the above inequalities with (2.10) to obtain, for any $m \geq 6$,

$$\frac{\rho^{2(m-6)}}{[(m-6)!]^{2\sigma}} \|\partial_x^m \mathcal{U}(t)\|_{L^2}^2 \leq C \int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + CC_* \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds.$$

The estimate for $m \leq 5$ is straightforward. Thus the proof of Proposition 2.7 is completed. \square

2.3. Tangential derivatives of u and f . For the tangential derivatives of u, f , we have the following estimate.

PROPOSITION 2.8. *Under Assumption 2.5, for any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$, we have*

$$\begin{aligned} & \sup_{m \geq 7} \frac{\rho^{2(m-7)}}{[(m-7)!]^{2\sigma}} \|\langle z \rangle^\ell \partial_x^m u(t)\|_{L^2}^2 + \sup_{m \leq 6} \|\langle z \rangle^\ell \partial_x^m u(t)\|_{L^2}^2 \\ & + \sup_{m \geq 7} \frac{\rho^{2(m-7)}}{[(m-7)!]^{2\sigma}} \int_0^t \|\langle z \rangle^\ell \partial_z \partial_x^m u(s)\|_{L^2}^2 ds + \sup_{m \leq 6} \int_0^t \|\langle z \rangle^\ell \partial_z \partial_x^m u(s)\|_{L^2}^2 ds \\ & \leq C \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 + CC_*^3 \left(\int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where $C_* \geq 1$ is the constant given in (2.8). Similarly, the same upper bound holds when $\partial_x^m u$ is replaced by $\partial_x^m f$.

Proof. Applying ∂_x^m to the first equation in (2.2) gives

$$(2.13) \quad (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2) \partial_x^m u = -(\partial_x^m w) \partial_z u + \partial_x^m \xi + F_m$$

with

$$F_m = -\sum_{j=1}^m \binom{m}{j} (\partial_x^j u) \partial_x^{m-j+1} u - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \partial_z u.$$

On the other hand, applying $(\partial_z u) \partial_x^{m-1}$ to (2.4) yields

$$(2.14) \quad \begin{aligned} & (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2) (\partial_z u) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z} \\ & = -(\partial_x^m w) \partial_z u + L_m + (\partial_z \xi) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z} \end{aligned}$$

with

$$L_m = -(\partial_z u) \sum_{j=1}^{m-1} \binom{m-1}{j} \left[(\partial_x^j u) \int_0^z \partial_x^{m-j} \mathcal{U} d\tilde{z} + (\partial_x^j w) \partial_x^{m-1-j} \mathcal{U} \right] - 2\nu (\partial_z^2 u) \partial_x^{m-1} \mathcal{U}.$$

Subtract (2.14) by (2.13) to eliminate the highest order term $(\partial_x^m w) \partial_z u$ and this gives the equation for

$$(2.15) \quad \psi_m \stackrel{\text{def}}{=} \partial_x^m u - (\partial_z u) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z}.$$

That is,

$$(\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\psi_m = \partial_x^m \xi + F_m - L_m - (\partial_z \xi) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z},$$

and thus

$$\begin{aligned} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2) \langle z \rangle^\ell \psi_m &= \langle z \rangle^\ell \partial_x^m \xi - \langle z \rangle^\ell (\partial_z \xi) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z} \\ &+ \langle z \rangle^\ell F_m - \langle z \rangle^\ell L_m + w(\partial_z \langle z \rangle^\ell) \psi_m - \nu(\partial_z^2 \langle z \rangle^\ell) \psi_m - 2\nu(\partial_z \langle z \rangle^\ell) \partial_z \psi_m. \end{aligned}$$

Then we take the scalar product with $\langle z \rangle^\ell \psi_m$ on both sides of the above equation and observe $\langle z \rangle^\ell \psi_m|_{z=0} = 0$ to obtain

$$\begin{aligned} (2.16) \quad & \frac{1}{2} \|\langle z \rangle^\ell \psi_m(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_z (\langle z \rangle^\ell \psi_m)\|_{L^2}^2 ds \\ &= \frac{1}{2} \|\langle z \rangle^\ell \psi_m(0)\|_{L^2}^2 + \int_0^t (\langle z \rangle^\ell \partial_x^m \xi, \langle z \rangle^\ell \psi_m)_{L^2} ds \\ &- \int_0^t \left(\langle z \rangle^\ell (\partial_z \xi) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z}, \langle z \rangle^\ell \psi_m \right)_{L^2} ds \\ &+ \int_0^t \left(\langle z \rangle^\ell F_m - \langle z \rangle^\ell L_m, \langle z \rangle^\ell \psi_m \right)_{L^2} ds \\ &+ \int_0^t \left(w(\partial_z \langle z \rangle^\ell) \psi_m - \nu(\partial_z^2 \langle z \rangle^\ell) \psi_m - 2\nu(\partial_z \langle z \rangle^\ell) \partial_z \psi_m, \langle z \rangle^\ell \psi_m \right)_{L^2} ds. \end{aligned}$$

As for the first term on the right side, since $\langle z \rangle^\ell \psi_m|_{t=0} = \langle z \rangle^\ell \partial_x^m u_0$, we have

$$\|\langle z \rangle^\ell \psi_m(0)\|_{L^2}^2 \leq \frac{[(m-7)!]^{2\sigma}}{(2\rho_0)^{2(m-7)}} \|(u_0, f_0)\|_{2\rho_0, \sigma}^2 \leq \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \|(u_0, f_0)\|_{2\rho_0, \sigma}^2.$$

The upper bound for the last three terms on the right side of (2.16) was obtained in [21] (see the proof of [21, Lemma 4.2]); that is,

$$\begin{aligned} & \int_0^t \left(\langle z \rangle^\ell F_m - \langle z \rangle^\ell L_m, \langle z \rangle^\ell \psi_m \right)_{L^2} ds \\ &+ \int_0^t \left(w(\partial_z \langle z \rangle^\ell) \psi_m - \nu(\partial_z^2 \langle z \rangle^\ell) \psi_m - 2\nu(\partial_z \langle z \rangle^\ell) \partial_z \psi_m, \langle z \rangle^\ell \psi_m \right)_{L^2} ds \\ &\leq CC_*^3 \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \left(\int_0^t (|\tilde{a}(s)|_{\rho, \sigma}^2 + |\tilde{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\tilde{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

We omit the details and refer to the argument in [21, Lemma 4.2]. It remains to estimate the second and third terms on the right of (2.16). From the definition of $|\tilde{a}|_{r, \sigma}$ it follows that, for any $0 < r \leq \rho_0$ and any $j \geq 7$,

$$(2.17) \quad \|\langle z \rangle^\ell \partial_x^j u\|_{L^2} + \|\langle z \rangle^\ell \partial_x^j f\|_{L^2} \leq \frac{[(j-7)!]^\sigma}{r^{(j-7)}} |\tilde{a}|_{r, \sigma}$$

and

$$(2.18) \quad \left\| \langle z \rangle^{-1} \int_0^z \partial_x^j \mathcal{U} d\tilde{z} \right\|_{L_x^2(L_z^\infty)} \leq C \|\partial_x^j \mathcal{U}\|_{L^2} \leq \frac{C[(j-6)!]^\sigma}{r^{j-6}} |\tilde{a}|_{r, \sigma}.$$

Then we use the definition (2.15) of ψ_m and the condition (2.8) to obtain, for any $0 < r \leq \rho_0$ and any $m \geq 7$,

$$(2.19) \quad \|\langle z \rangle^\ell \psi_m\|_{L^2} \leq \|\langle z \rangle^\ell \partial_x^m u\|_{L^2} + CC_* \|\partial_x^{m-1} \mathcal{U}\|_{L^2} \leq CC_* \frac{[(m-7)!]^\sigma}{r^{m-7}} |\bar{a}|_{r,\sigma}.$$

Moreover, note that

$$(2.20) \quad \forall 0 < r \leq \rho_0, \forall j \geq 6, j \|\langle z \rangle^\ell \partial_x^j \xi\|_{L^2} + j \|\langle z \rangle^\ell \partial_x^j \eta\|_{L^2} \leq \frac{[(j-6)!]^\sigma}{r^{(j-6)}} |\bar{a}|_{r,\sigma}$$

from the definition of $|\bar{a}|_{r,\sigma}$. The above two inequalities give

$$\begin{aligned} \int_0^t \left(\langle z \rangle^\ell \partial_x^m \xi, \langle z \rangle^\ell \psi_m \right)_{L^2} ds &\leq CC_* \int_0^t \frac{1}{m} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}} \frac{[(m-7)!]^\sigma}{\tilde{\rho}^{m-7}} |\bar{a}(s)|_{\rho,\sigma}^2 ds \\ &\leq CC_* \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \int_0^t \frac{|\bar{a}(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds, \end{aligned}$$

where in the last inequality we have used (2.12) and $\sigma = 3/2$. Finally, using (2.18) and the condition (2.8) we have by recalling $\xi = (f\partial_x + h\partial_z)f$ that

$$\begin{aligned} &\left\| \langle z \rangle^\ell (\partial_z \xi) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z} \right\|_{L^2} \\ &\leq \|\langle z \rangle^{\ell+1} \partial_z \xi\|_{L_x^\infty(L_z^2)} \left\| \langle z \rangle^{-1} \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z} \right\|_{L_x^2(L_z^\infty)} \leq CC_*^2 \frac{[(m-7)!]^\sigma}{\rho^{m-7}} |\bar{a}|_{\rho,\sigma}. \end{aligned}$$

This with (2.19) yields

$$- \int_0^t \left(\langle z \rangle^\ell (\partial_z \xi) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z}, \langle z \rangle^\ell \psi_m \right)_{L^2} ds \leq CC_*^3 \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \int_0^t |\bar{a}|_{\rho,\sigma}^2 ds.$$

Putting the above inequalities into (2.16) gives

$$(2.21) \quad \|\langle z \rangle^\ell \psi_m(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_z (\langle z \rangle^\ell \psi_m)\|_{L^2}^2 dt \leq \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \|(u_0, f_0)\|_{2\rho_0,\sigma}^2 \\ + CC_*^3 \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \left(\int_0^t (|\bar{a}(s)|_{\rho,\sigma}^2 + |\bar{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds \right).$$

Note that

$$\begin{aligned} \|\langle z \rangle^\ell \partial_x^m u\|_{L^2}^2 &\leq 2 \|\langle z \rangle^\ell \psi_m\|_{L^2}^2 + 2 \left\| \langle z \rangle^\ell (\partial_z u) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z} \right\|_{L^2}^2 \\ &\leq 2 \|\langle z \rangle^\ell \psi_m\|_{L^2}^2 + CC_*^2 \|\partial_x^{m-1} \mathcal{U}\|_{L^2}^2 \end{aligned}$$

due to the definition (2.15) of ψ_m . Hence, the two above estimates together with Proposition 2.7 give

$$\begin{aligned} \|\langle z \rangle^\ell \partial_x^m u(t)\|_{L^2}^2 &\leq 2 \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \|(u_0, f_0)\|_{2\rho_0,\sigma}^2 \\ &\quad + CC_*^3 \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \left(\int_0^t (|\bar{a}(s)|_{\rho,\sigma}^2 + |\bar{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^t \|\langle z \rangle^\ell \partial_z \partial_x^m u\|_{L^2}^2 ds &\leq \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m u)\|_{L^2}^2 ds + C \int_0^t \|\langle z \rangle^\ell \partial_x^m u\|_{L^2}^2 ds \\ &\leq 2 \int_0^t \|\partial_z (\langle z \rangle^\ell \psi_m)\|_{L^2}^2 ds + CC_*^2 \int_0^t \left(\|\partial_x^{m-1} \mathcal{U}\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_x^m u\|_{L^2}^2 \right) ds \\ &\leq 2 \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \|(u_0, f_0)\|_{2\rho, \sigma}^2 \\ &\quad + CC_*^3 \frac{[(m-7)!]^{2\sigma}}{\rho^{2(m-7)}} \left(\int_0^t (|\bar{a}(s)|_{\rho, \sigma}^2 + |\bar{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\bar{\rho}, \sigma}^2}{\bar{\rho} - \rho} ds \right), \end{aligned}$$

where in the last inequality we have used (2.21) and the estimates (2.17)–(2.18). Then we obtain the estimate on $\partial_x^m u$ when $m \geq 7$, and the estimate for $m \leq 6$ is straightforward. It remains to estimate $\partial_x^m f$. For this, consider

$$\varphi_m = \partial_x^m f - (\partial_z f) \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z},$$

which satisfies $\partial_z \varphi_m|_{z=0} = 0$ and solves

$$(2.22) \quad \begin{aligned} (\partial_t + u\partial_x + w\partial_z - \mu\partial_z^2)\varphi_m &= (\mu - \nu)(\partial_z f)\partial_z \partial_x^{m-1} \mathcal{U} + \partial_x^m \eta + \tilde{F}_m - \tilde{L}_m \\ &\quad - [\partial_z \eta - (\partial_z u)\partial_x f + (\partial_z f)\partial_x u] \int_0^z \partial_x^{m-1} \mathcal{U} d\tilde{z}, \end{aligned}$$

where

$$\tilde{F}_m = - \sum_{j=1}^m \binom{m}{j} (\partial_x^j u) \partial_x^{m-j+1} f - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j w) \partial_x^{m-j} \partial_z f,$$

and

$$\tilde{L}_m = -(\partial_z f) \sum_{j=1}^{m-1} \binom{m-1}{j} \left[(\partial_x^j u) \int_0^z \partial_x^{m-j} \mathcal{U} d\tilde{z} + (\partial_x^j w) \partial_x^{m-1-j} \mathcal{U} \right] - 2\mu(\partial_z^2 f) \partial_x^{m-1} \mathcal{U}.$$

Observe

$$\begin{aligned} &\left((\mu - \nu) \langle z \rangle^\ell (\partial_z f) \partial_z \partial_x^{m-1} \mathcal{U}, \langle z \rangle^\ell \varphi_m \right)_{L^2} \\ &\quad \leq \frac{1}{2} \|\partial_z (\langle z \rangle^\ell \varphi_m)\|_{L^2}^2 + CC_*^2 \left(\|\partial_x^{m-1} \mathcal{U}\|_{L^2}^2 + \|\langle z \rangle^\ell \varphi_m\|_{L^2}^2 \right), \end{aligned}$$

and the other terms on the right side of (2.22) can be treated similarly as for $\partial_x^m u$. Then the estimate (2.21) also holds with ψ_m replaced by φ_m . The proof of the proposition is completed. \square

2.4. Tangential derivatives of ξ and η . We now turn to estimate the tangential derivatives of ξ and η which are defined in (2.3), that is, $\xi = f\partial_x f + h\partial_z f$ and $\eta = f\partial_x u + h\partial_z u$.

PROPOSITION 2.9. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned} & \sup_{m \geq 6} \frac{\rho^{2(m-6)}}{[(m-6)!]^{2\sigma}} m^2 (\|\partial_x^m \xi(t)\|_{L^2}^2 + \|\partial_x^m \eta(t)\|_{L^2}^2) + \sup_{m \leq 5} (\|\partial_x^m \xi(t)\|_{L^2}^2 + \|\partial_x^m \eta(t)\|_{L^2}^2) \\ & \leq C (\|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 + \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^4) \\ & \quad + e^{CC_*^2} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where $C_* \geq 1$ is the constant given in (2.8).

The proof relies on a newly observed cancellation property of ξ and η . Precisely, we use the equations in (2.2) for u, f , and h to derive the equations for η and ξ :

$$\begin{aligned} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\eta &= (f\partial_x + h\partial_z)\xi + 2\nu[(\partial_x f)\partial_z^2 u - (\partial_z f)\partial_x \partial_z u] \\ & \quad + (\mu - \nu)[(\partial_x u)\partial_z^2 f - (\partial_z u)\partial_x \partial_z f] \end{aligned}$$

and

$$(\partial_t + u\partial_x + w\partial_z - \mu\partial_z^2)\xi = (f\partial_x + h\partial_z)\eta + 2\mu[(\partial_x f)\partial_z^2 f - (\partial_z f)\partial_x \partial_z f],$$

where the loss of tangential derivative term $\partial_x w$ is canceled. Now we apply $\langle z \rangle^\ell \partial_x^m$ to the above equations for ξ and η to get

$$(2.23) \quad \begin{cases} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2) \langle z \rangle^\ell \partial_x^m \eta = (f\partial_x + h\partial_z) \langle z \rangle^\ell \partial_x^m \xi + P_m, \\ (\partial_t + u\partial_x + w\partial_z - \mu\partial_z^2) \langle z \rangle^\ell \partial_x^m \xi = (f\partial_x + h\partial_z) \langle z \rangle^\ell \partial_x^m \eta + Q_m, \end{cases}$$

where

$$\begin{aligned} P_m &= \langle z \rangle^\ell \sum_{j=1}^m \binom{m}{j} [(\partial_x^j f)\partial_x^{m-j+1} \xi + (\partial_x^j h)\partial_x^{m-j} \partial_z \xi] \\ & \quad - \langle z \rangle^\ell \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u)\partial_x^{m-j+1} \eta + (\partial_x^j w)\partial_x^{m-j} \partial_z \eta] \\ & \quad + 2\nu \langle z \rangle^\ell \partial_x^m [(\partial_x f)\partial_z^2 u - (\partial_z f)\partial_x \partial_z u] \\ & \quad + (\mu - \nu) \langle z \rangle^\ell \partial_x^m [(\partial_x u)\partial_z^2 f - (\partial_z u)\partial_x \partial_z f] \\ & \quad + w(\partial_z \langle z \rangle^\ell) \partial_x^m \eta - 2\nu(\partial_z \langle z \rangle^\ell) \partial_z \partial_x^m \eta - \nu(\partial_z^2 \langle z \rangle^\ell) \partial_x^m \eta - h(\partial_z \langle z \rangle^\ell) \partial_x^m \xi, \end{aligned}$$

and

$$\begin{aligned} Q_m &= \langle z \rangle^\ell \sum_{j=1}^m \binom{m}{j} [(\partial_x^j f)\partial_x^{m-j+1} \eta + (\partial_x^j h)\partial_x^{m-j} \partial_z \eta] \\ & \quad - \langle z \rangle^\ell \sum_{j=1}^m \binom{m}{j} [(\partial_x^j u)\partial_x^{m-j+1} \xi + (\partial_x^j w)\partial_x^{m-j} \partial_z \xi] \\ & \quad + 2\mu \langle z \rangle^\ell \partial_x^m [(\partial_x f)\partial_z^2 f - (\partial_z f)\partial_x \partial_z f] + w(\partial_z \langle z \rangle^\ell) \partial_x^m \xi \\ & \quad - 2\mu(\partial_z \langle z \rangle^\ell) \partial_z \partial_x^m \xi - \mu(\partial_z^2 \langle z \rangle^\ell) \partial_x^m \xi - h(\partial_z \langle z \rangle^\ell) \partial_x^m \eta. \end{aligned}$$

Now we take the inner product with $m^2 \langle z \rangle^\ell \partial_x^m \eta$ for the first equation in (2.23) and with $m^2 \langle z \rangle^\ell \partial_x^m \xi$ for the second one, and then take summation. Since $\partial_z \xi|_{z=0} =$

$\eta|_{z=0} = 0$ and the first terms on the right side of (2.23) are canceled by symmetry as well as a divergence-free condition, we have

$$\begin{aligned}
 & \frac{m^2}{2} \left(\| \langle z \rangle^\ell \partial_x^m \eta(t) \|_{L^2}^2 + \| \langle z \rangle^\ell \partial_x^m \xi(t) \|_{L^2}^2 \right) + \nu m^2 \int_0^t \| \partial_z (\langle z \rangle^\ell \partial_x^m \eta) \|_{L^2}^2 ds \\
 & + \mu m^2 \int_0^t \| \partial_z (\langle z \rangle^\ell \partial_x^m \xi) \|_{L^2}^2 ds \\
 (2.24) \quad & = \frac{m^2}{2} \left(\| \langle z \rangle^\ell \partial_x^m \eta(0) \|_{L^2}^2 + \| \langle z \rangle^\ell \partial_x^m \xi(0) \|_{L^2}^2 \right) + m^2 \int_0^t (P_m, \langle z \rangle^\ell \partial_x^m \eta)_{L^2} ds \\
 & + m^2 \int_0^t (Q_m, \langle z \rangle^\ell \partial_x^m \xi)_{L^2} ds.
 \end{aligned}$$

The following lemmas are about the estimation on the terms in the above equality.

LEMMA 2.10. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned}
 & m^2 \int_0^t \left(\langle z \rangle^\ell \sum_{1 \leq j \leq m} \binom{m}{j} (\partial_x^j f) \partial_x^{m-j+1} \xi, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\
 & \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t |\bar{a}(s)|_{\rho, \sigma}^3 ds + C_*^2 \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right).
 \end{aligned}$$

Proof. First, note that

$$\begin{aligned}
 & m \sum_{j=1}^m \binom{m}{j} \| \langle z \rangle^\ell (\partial_x^j f) \partial_x^{m-j+1} \xi \|_{L^2} \\
 (2.25) \quad & \leq m \sum_{j=1}^{[m/2]} \binom{m}{j} \| \partial_x^j f \|_{L^\infty} \| \langle z \rangle^\ell \partial_x^{m-j+1} \xi \|_{L^2} \\
 & + m \sum_{j=[m/2]+1}^m \binom{m}{j} \| \partial_x^j f \|_{L_x^2(L_x^\infty)} \| \langle z \rangle^\ell \partial_x^{m-j+1} \xi \|_{L_x^\infty(L_x^2)},
 \end{aligned}$$

where $[p]$ denotes the largest integer less than or equal to p . By the Sobolev embedding inequalities

$$\begin{cases} \|F\|_{L^\infty(\mathbb{R}_x)} \leq \sqrt{2} \left(\|F\|_{L_x^2} + \|\partial_x F\|_{L_x^2} \right), \\ \|F\|_{L^\infty} \leq 2 \left(\|F\|_{L^2} + \|\partial_x F\|_{L^2} + \|\partial_z F\|_{L^2} + \|\partial_x \partial_z F\|_{L^2} \right) \end{cases}$$

and the estimates (2.17)–(2.20) as well as (2.8), we have

$$\begin{aligned}
 & m \sum_{j=1}^{[m/2]} \binom{m}{j} \| \partial_x^j f \|_{L^\infty} \| \langle z \rangle^\ell \partial_x^{m-j+1} \xi \|_{L^2} \\
 & \leq Cm \sum_{j=5}^{[m/2]} \frac{m!}{j!(m-j)!} \frac{[(j-5)!]^\sigma}{\rho^{j-5}} \frac{1}{m-j} \frac{[(m-j-5)!]^\sigma}{\rho^{m-j-5}} |\bar{a}|_{\rho, \sigma}^2 \\
 (2.26) \quad & + CC_* m \sum_{1 \leq j \leq 4} \frac{m!}{j!(m-j)!} \frac{1}{m-j} \frac{[(m-j-5)!]^\sigma}{\tilde{\rho}^{m-j-5}} |\bar{a}|_{\tilde{\rho}, \sigma}.
 \end{aligned}$$

Direct calculation shows

$$m \sum_{1 \leq j \leq 4} \frac{m!}{j!(m-j)!} \frac{1}{m-j} \frac{[(m-j-5)!]^\sigma}{\tilde{\rho}^{m-j-5}} |\bar{a}|_{\tilde{\rho}, \sigma} \leq Cm \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}} |\bar{a}|_{\tilde{\rho}, \sigma}.$$

Moreover, by using $m/(m-j) \leq C$ for $j \leq [m/2]$, we have

$$\begin{aligned} & m \sum_{j=5}^{[m/2]} \frac{m!}{j!(m-j)!} \frac{[(j-5)!]^\sigma}{\rho^{j-5}} \frac{1}{m-j} \frac{[(m-j-5)!]^\sigma}{\rho^{m-j-5}} |\bar{a}|_{\rho, \sigma}^2 \\ & \leq C \frac{|\bar{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \sum_{j=5}^{[m/2]} \frac{m! [(j-5)!]^{\sigma-1} [(m-j-5)!]^{\sigma-1}}{j^5 (m-j)^5} \\ & \leq C \frac{|\bar{a}|_{\rho, \sigma}^2}{\rho^{m-6}} \sum_{j=5}^{[m/2]} \frac{(m-6)! m^6 [(m-6)!]^{\sigma-1}}{j^5 m^5 m^{4(\sigma-1)}} \leq C \frac{[(m-6)!]^\sigma}{\rho^{m-6}} |\bar{a}|_{\rho, \sigma}^2, \end{aligned}$$

where in the last inequality we have used $\sigma = 3/2$. Combining the above inequalities with (2.26) gives

$$\begin{aligned} & m \sum_{j=1}^{[m/2]} \binom{m}{j} \|\partial_x^j f\|_{L^\infty} \|\langle z \rangle^\ell \partial_x^{m-j+1} \xi\|_{L^2} \\ & \leq C \frac{[(m-6)!]^\sigma}{\rho^{m-6}} |\bar{a}|_{\rho, \sigma}^2 + CC_* m \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}} |\bar{a}|_{\tilde{\rho}, \sigma}. \end{aligned}$$

Similarly, recalling $\xi = f \partial_x f + h \partial_z f$, we have

$$\begin{aligned} & m \sum_{j=[m/2]+1}^m \binom{m}{j} \|\partial_x^j f\|_{L_x^2(L_x^\infty)} \|\langle z \rangle^\ell \partial_x^{m-j+1} \xi\|_{L_x^\infty(L_x^2)} \\ & \leq Cm \sum_{j=[m/2]+1}^{m-4} \frac{m!}{j!(m-j)!} \frac{[(j-6)!]^\sigma}{\rho^{j-6}} \frac{1}{m-j} \frac{[(m-j-4)!]^\sigma}{\rho^{m-j-4}} |\bar{a}|_{\rho, \sigma}^2 \\ & \quad + CC_*^2 m \sum_{j=m-3}^m \frac{m!}{j!(m-j)!} \frac{[(j-6)!]^\sigma}{\tilde{\rho}^{j-6}} |\bar{a}|_{\rho, \sigma} \\ & \leq C \frac{[(m-6)!]^\sigma}{\rho^{m-6}} |\bar{a}|_{\rho, \sigma}^2 + CC_*^2 m \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}} |\bar{a}|_{\tilde{\rho}, \sigma}. \end{aligned}$$

Putting these inequalities into (2.25) gives

$$\begin{aligned} (2.27) \quad & m \sum_{j=1}^m \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j f) \partial_x^{m-j+1} \xi\|_{L^2} \\ & \leq C \frac{[(m-6)!]^\sigma}{\rho^{m-6}} |\bar{a}|_{\rho, \sigma}^2 + CC_*^2 m \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}} |\bar{a}|_{\tilde{\rho}, \sigma}. \end{aligned}$$

This with (2.20) gives

$$\begin{aligned} & m^2 \int_0^t \left(\langle z \rangle^\ell \sum_{1 \leq j \leq m} \binom{m}{j} (\partial_x^j f) \partial_x^{m-j+1} \xi, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \leq \int_0^t m \sum_{j=1}^m \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j f) \partial_x^{m-j+1} \xi\|_{L^2} \times \left(m \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} \right) ds \\ & \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t |\bar{a}(s)|_{\rho,\sigma}^3 ds + CC_*^2 \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t m \frac{\rho^{2(m-6)}}{\tilde{\rho}^{2(m-6)}} |\bar{a}(s)|_{\tilde{\rho},\sigma}^2 ds \\ & \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t |\bar{a}(s)|_{\rho,\sigma}^3 ds + C_*^2 \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where in the last inequality we have used (2.12). The proof of the lemma is completed. \square

LEMMA 2.11. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned} & m^2 \int_0^t \left(\langle z \rangle^\ell \sum_{1 \leq j \leq m} \binom{m}{j} (\partial_x^j h) \partial_x^{m-j} \partial_z \xi, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \leq \frac{\nu}{6} m^2 \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \left(|\bar{a}(s)|_{\rho,\sigma}^3 + |\bar{a}(s)|_{\rho,\sigma}^4 \right) ds \\ & \quad + CC_*^2 \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

Proof. It follows from integration by parts that

$$(2.28) \quad m^2 \int_0^t \left(\langle z \rangle^\ell \sum_{1 \leq j \leq m} \binom{m}{j} (\partial_x^j h) \partial_x^{m-j} \partial_z \xi, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \leq \sum_{k=1}^4 J_k$$

with

$$\begin{aligned} J_1 &= m \int_0^t \sum_{j=m-1}^m \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j h) \partial_x^{m-j} \partial_z \xi\|_{L^2} \times \left(m \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} \right) ds, \\ J_2 &= m \int_0^t \sum_{j=1}^{m-2} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j h) \partial_x^{m-j} \xi\|_{L^2} \times \left(m \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2} \right) ds, \\ J_3 &= m \int_0^t \sum_{j=1}^{m-2} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^{j+1} f) \partial_x^{m-j} \xi\|_{L^2} \times \left(m \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} \right) ds, \\ J_4 &= 2m \int_0^t \sum_{j=1}^{m-2} \binom{m}{j} \|\partial_z \langle z \rangle^\ell (\partial_x^j h) \partial_x^{m-j} \xi\|_{L^2} \times \left(m \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} \right) ds. \end{aligned}$$

Note that $\|\partial_x^j w\|_{L_x^\infty} \leq C \|\langle z \rangle^\ell \partial_x^{j+1} u\|_{L_x^2}$ for $\ell > 1/2$, and a similar estimate holds for $\partial_x^j h$. Then it follows from (2.17) and (2.8) that, for any $0 < r \leq \rho_0$,

$$\|\partial_x^j w\|_{L_x^2(L_x^\infty)} + \|\partial_x^j h\|_{L_x^2(L_x^\infty)} \leq \begin{cases} C \frac{[(j-6)!]^\sigma}{r^{(j-6)}} |\bar{a}|_{r,\sigma} & \text{if } j \geq 6, \\ C |\bar{a}|_{r,\sigma} & \text{if } j \leq 5. \end{cases}$$

Thus as for the proof of (2.27), when $\sigma = 3/2$, we have

$$\begin{aligned} & m \sum_{j=1}^{m-2} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j h) \partial_x^{m-j} \xi\|_{L^2} + m \sum_{j=1}^{m-2} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^{j+1} f) \partial_x^{m-j} \xi\|_{L^2} \\ & \leq C \frac{[(m-6)!]^\sigma}{\rho^{m-6}} |\bar{a}|_{\rho, \sigma}^2. \end{aligned}$$

Thus, by (2.20), we have

$$J_2 + J_3 + J_4 \leq \frac{\nu}{6} m^2 \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t (|\bar{a}(s)|_{\rho, \sigma}^3 + |\bar{a}(s)|_{\rho, \sigma}^4) ds.$$

Finally, by (2.8) and (2.20), direct calculation gives

$$J_1 \leq CC_*^2 \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds.$$

Combining the above estimates with (2.28) completes the proof of the lemma. \square

LEMMA 2.12. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned} & -m^2 \int_0^t \left(\langle z \rangle^\ell \sum_{1 \leq j \leq m} \binom{m}{j} [(\partial_x^j u) \partial_x^{m-j+1} \eta + (\partial_x^j w) \partial_x^{m-j} \partial_z \eta], \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \leq \frac{\nu}{6} m^2 \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t (|\bar{a}(s)|_{\rho, \sigma}^3 + |\bar{a}(s)|_{\rho, \sigma}^4) ds \\ & \quad + CC_*^2 \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

We omit the proof of this lemma because it is almost the same as those for Lemmas 2.10 and 2.11.

LEMMA 2.13. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned} & 2\nu m^2 \int_0^t \left(\langle z \rangle^\ell \partial_x^m [(\partial_x f) \partial_z^2 u - (\partial_z f) \partial_x \partial_z u], \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \quad + (\mu - \nu) m^2 \int_0^t \left(\langle z \rangle^\ell \partial_x^m [(\partial_x u) \partial_z^2 f - (\partial_z u) \partial_x \partial_z f], \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \leq \frac{\nu}{6} m^2 \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 \\ & \quad + e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\bar{a}(s)|_{\rho, \sigma}^2 + |\bar{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof. By the definition (2.5) of λ and δ , we can derive

$$\begin{aligned} (\partial_x f) \partial_z^2 u - (\partial_z f) \partial_x \partial_z u &= \left(\delta + (\partial_z f) \int_0^z \mathcal{U} d\tilde{z} \right) \partial_z^2 u - (\partial_z f) \partial_z \left(\lambda + (\partial_z u) \int_0^z \mathcal{U} d\tilde{z} \right) \\ &= \delta \partial_z^2 u - (\partial_z f) \partial_z \lambda - (\partial_z f) (\partial_z u) \mathcal{U}. \end{aligned}$$

Thus

$$(2.29) \quad 2\nu m^2 \int_0^t \left(\langle z \rangle^\ell \partial_x^m [(\partial_x f) \partial_z^2 u - (\partial_z f) \partial_x \partial_z u], \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \leq \sum_{j=1}^4 K_j$$

with

$$\begin{cases} K_1 = 2\nu m^2 \int_0^t \sum_{0 \leq j \leq 4} \binom{m}{j} \left(\langle z \rangle^\ell (\partial_x^j \delta) \partial_x^{m-j} \partial_z^2 u, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds, \\ K_2 = 2\nu m^2 \int_0^t \sum_{j=5}^m \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j \delta) \partial_x^{m-j} \partial_z^2 u\|_{L^2} \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} ds, \\ K_3 = -2\nu m^2 \int_0^t \left(\langle z \rangle^\ell \partial_x^m [(\partial_z f) \partial_z \lambda], \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds, \\ K_4 = 2\nu m^2 \int_0^t \|\langle z \rangle^\ell \partial_x^m [(\partial_z f) (\partial_z u) \mathcal{U}]\|_{L^2} \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} ds. \end{cases}$$

To estimate $K_j, 1 \leq j \leq 4$, we need the following estimates from [21, Lemma 5.2]:

$$(2.30) \quad \forall t \in [0, T], \quad \sum_{k \leq 9} \left\| \langle z \rangle^{-\ell} \int_0^z \partial_x^k \mathcal{U}(t) dz \right\|_{L^2} + \sum_{\substack{k+j \leq 8 \\ 0 \leq j \leq 2}} \|\partial_x^k \partial_z^j \mathcal{U}(t)\|_{L^2} \leq e^{CC_*^2},$$

and

$$(2.31) \quad \forall t \in [0, T], \quad \sum_{\substack{k+j \leq 8 \\ 0 \leq j \leq 2}} \|\partial_x^k \partial_z^j \lambda(t)\|_{L^2} \leq e^{CC_*^2},$$

where $C_* \geq 1$ is the constant in (2.8). Then by (2.30), (2.11), (2.17), and (2.20) as well as (2.8), following the proof for Lemma 2.10, we obtain

$$(2.32) \quad K_2 + K_4 \leq e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\bar{a}(s)|_{\rho,\sigma}^3 + |\bar{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right).$$

As for K_3 , we first write it as

$$\begin{aligned} K_3 &\leq 2\nu m^2 \int_0^t \sum_{0 \leq j \leq m-5} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j \partial_z f) \partial_x^{m-j} \lambda\|_{L^2} \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2} ds \\ &\quad + 2\nu m^2 \int_0^t \sum_{0 \leq j \leq m-5} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j \partial_z^2 f) \partial_x^{m-j} \lambda\|_{L^2} \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} ds \\ &\quad + 2\nu m^2 \int_0^t \sum_{0 \leq j \leq m-5} \binom{m}{j} \|(\partial_z \langle z \rangle^\ell) (\partial_x^j \partial_z f) \partial_x^{m-j} \lambda\|_{L^2} \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} ds \\ &\quad + 2\nu m^2 \int_0^t \sum_{m-4 \leq j \leq m} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j \partial_z f) \partial_x^{m-j} \partial_z \lambda\|_{L^2} \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} ds. \end{aligned}$$

Then by (2.31), it holds that

$$(2.33) \quad \begin{aligned} K_3 &\leq \frac{\nu}{24} m^2 \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2}^2 ds \\ &\quad + e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\bar{a}(s)|_{\rho,\sigma}^3 + |\bar{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

It remains to estimate K_1 . Again, note that

$$\begin{aligned} K_1 &\leq 2\nu m^2 \int_0^t \sum_{0 \leq j \leq 4} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j \delta) \partial_x^{m-j} \partial_z u\|_{L^2} \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2} ds \\ &\quad + 2\nu m^2 \int_0^t \sum_{0 \leq j \leq 4} \binom{m}{j} \|\langle z \rangle^\ell (\partial_x^j \partial_z \delta) \partial_x^{m-j} \partial_z u\|_{L^2} \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} ds \\ &\quad + 2\nu m^2 \int_0^t \sum_{0 \leq j \leq 4} \binom{m}{j} \|(\partial_z \langle z \rangle^\ell) (\partial_x^j \delta) \partial_x^{m-j} \partial_z u\|_{L^2} \|\langle z \rangle^\ell \partial_x^m \eta\|_{L^2} ds. \end{aligned}$$

Observe that the estimate (2.31) holds when λ is replaced by δ . Thus direct calculation shows that

$$\begin{aligned} K_1 &\leq \frac{\nu}{24} m^2 \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2}^2 ds \\ &\quad + e^{CC_*^2} m^2 \sum_{0 \leq j \leq 4} \left(\frac{m!}{j!(m-j)!} \right)^2 \int_0^t \|\langle z \rangle^\ell \partial_x^{m-j} \partial_z u\|_{L^2}^2 ds \\ &\quad + e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

As for the second term on the right side, we have

$$\|\langle z \rangle^\ell \partial_x^{m-j} \partial_z u\|_{L^2}^2 \leq \|\langle z \rangle^\ell \partial_x^{m-j-1} \partial_z u\|_{L^2} \|\langle z \rangle^\ell \partial_x^{m-j+1} \partial_z u\|_{L^2}.$$

Hence

$$\begin{aligned} &e^{CC_*^2} m^2 \sum_{0 \leq j \leq 4} \left(\frac{m!}{j!(m-j)!} \right)^2 \int_0^t \|\langle z \rangle^\ell (\partial_x^j \delta) \partial_x^{m-j} \partial_z u\|_{L^2}^2 ds \\ &\leq e^{CC_*^2} m^4 \sum_{0 \leq j \leq 4} \left(\frac{m!}{j!(m-j)!} \right)^2 \int_0^t \|\langle z \rangle^\ell \partial_x^{m-j-1} \partial_z u\|_{L^2}^2 ds \\ &\quad + \sum_{0 \leq j \leq 4} \left(\frac{m!}{j!(m-j)!} \right)^2 \int_0^t \|\langle z \rangle^\ell \partial_x^{m-j+1} \partial_z u\|_{L^2}^2 ds \\ &\leq e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 \\ &\quad + CC_*^3 \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where we have used Proposition 2.8 to control the second term in the first inequality. Thus, we can conclude that

$$\begin{aligned} K_1 &\leq \frac{\nu}{24} m^2 \int_0^t \|\partial_z (\langle z \rangle^\ell \partial_x^m \eta)\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \|(u_0, f_0)\|_{2\rho_0, \sigma}^2 \\ &\quad + e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Putting the above inequality and the estimates (2.32)–(2.33) into (2.29) gives the upper bound for the first term on the left side in Lemma 2.13. The second term can be estimated similarly and we omit the details. Then the proof of the lemma is completed. \square

LEMMA 2.14. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned} & m^2 \int_0^t \left(w(\partial_z \langle z \rangle^\ell) \partial_x^m \eta - 2\nu(\partial_z \langle z \rangle^\ell) \partial_z \partial_x^m \eta, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \quad + m^2 \int_0^t \left(-\nu(\partial_z^2 \langle z \rangle^\ell) \partial_x^m \eta - h(\partial_z \langle z \rangle^\ell) \partial_x^m \xi, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^3 \right) ds. \end{aligned}$$

Since the proof of this lemma follows from direct calculation and (2.20), we omit it for brevity. And now we are ready to prove Proposition 2.9.

Proof of Proposition 2.9. In view of the representation of P_m given in (2.23), we combine the estimates in Lemmas 2.10–2.14 to conclude

$$\begin{aligned} & m^2 \int_0^t \left(P_m, \langle z \rangle^\ell \partial_x^m \eta \right)_{L^2} ds \\ & \leq \frac{\nu}{2} m^2 \int_0^t \left\| \partial_z (\langle z \rangle^\ell \partial_x^m \eta) \right\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 \\ & \quad + e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

A similar upper bound holds for

$$m^2 \int_0^t \left(Q_m, \langle z \rangle^\ell \partial_x^m \xi \right)_{L^2} ds.$$

Then by (2.24), we have

$$\begin{aligned} & m^2 \left(\|\langle z \rangle^\ell \partial_x^m \eta(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_x^m \xi(t)\|_{L^2}^2 \right) \\ & \leq m^2 \left(\|\langle z \rangle^\ell \partial_x^m \eta(0)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_x^m \xi(0)\|_{L^2}^2 \right) + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 \\ & \quad + e^{CC_*^2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Moreover, following the argument for proving Lemma 2.10 we can derive

$$\begin{aligned} & m \|\langle z \rangle^\ell \partial_x^m \xi(0)\|_{L^2} \\ & \leq m \sum_{j \leq m} \binom{m}{j} \left(\|\langle z \rangle^\ell (\partial_x^j f_0) \partial_x^{m-j+1} f_0\|_{L^2} + \|\langle z \rangle^\ell (\partial_x^j h(0)) \partial_x^{m-j} \partial_z f_0\|_{L^2} \right) \\ & \leq Cm \frac{[(m-6)!]^\sigma}{(2\rho_0)^{m-6}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 \leq C_1 \frac{[(m-6)!]^\sigma}{\rho^{m-6}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2, \end{aligned}$$

where in the last inequality we have used the fact that $\rho \leq \rho_0$ as well as $h|_{t=0} = -\int_0^z \partial_x f_0 d\tilde{z}$. A similar upper bound holds for $m \|\langle z \rangle^\ell \partial_x^m \eta(0)\|_{L^2}$. Thus combining the above inequalities yields the desired estimate in Proposition 2.9 for $m \geq 6$. The estimate for $m \leq 5$ is straightforward so that the proof of the proposition is completed. \square

2.5. Tangential derivatives of λ and δ . The estimate on the tangential derivatives of λ and δ defined in (2.5) is given in the following proposition.

PROPOSITION 2.15. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned} & \sup_{m \geq 6} \frac{\rho^{2(m-6)}}{[(m-6)!]^{2\sigma}} m \|\partial_x^m \lambda\|_{L^2}^2 + \sup_{m \leq 5} \|\partial_x^m \lambda\|_{L^2}^2 \\ & \leq C \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 + C \int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^3 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds + e^{CC_*} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

The above estimate also holds when λ is replaced by δ .

Proof. We apply ∂_x to the equation for u in (2.2) and multiply (2.4) by $\partial_z u$, and then the subtraction of these two equations gives the equation for λ :

$$(\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\lambda = \partial_x \xi - (\partial_x u)\partial_x u - (\partial_z \xi) \int_0^z \mathcal{U} d\tilde{z} + 2\nu(\partial_z^2 u)\mathcal{U}.$$

Thus

$$\begin{aligned} (\partial_t + u\partial_x + w\partial_z - \nu\partial_z^2)\partial_x^m \lambda &= \partial_x^{m+1} \xi - \partial_x^m \left[(\partial_x u)\partial_x u + (\partial_z \xi) \int_0^z \mathcal{U} d\tilde{z} - 2\nu(\partial_z^2 u)\mathcal{U} \right] \\ &\quad - \sum_{j=1}^m \binom{m}{j} \left[(\partial_x^j u)\partial_x^{m-j+1} \lambda + (\partial_x^j w)\partial_x^{m-j} \partial_z \lambda \right]. \end{aligned}$$

Taking an inner product with $m\partial_x^m \lambda$ and observing $\lambda|_{z=0} = 0$, we obtain

$$\begin{aligned} & \frac{m}{2} \|\partial_x^m \lambda(t)\|_{L^2}^2 + m\nu \int_0^t \|\partial_z \partial_x^m \lambda(s)\|_{L^2}^2 ds \\ & \leq \frac{m}{2} \|\partial_x^m \lambda(0)\|_{L^2}^2 + m \int_0^t (\partial_x^{m+1} \xi, \partial_x^m \lambda)_{L^2} ds \\ (2.34) \quad & - m \int_0^t \left(\partial_x^m \left[(\partial_x u)\partial_x u + (\partial_z \xi) \int_0^z \mathcal{U} d\tilde{z} - 2\nu(\partial_z^2 u)\mathcal{U} \right], \partial_x^m \lambda \right)_{L^2} ds \\ & - m \int_0^t \left(\sum_{j=1}^m \binom{m}{j} [(\partial_x^j u)\partial_x^{m-j+1} \lambda + (\partial_x^j w)\partial_x^{m-j} \partial_z \lambda], \partial_x^m \lambda \right)_{L^2} ds. \end{aligned}$$

By (2.11) and (2.20), when $\sigma = 3/2$ we have

$$\begin{aligned} m \int_0^t (\partial_x^{m+1} \xi, \partial_x^m \lambda)_{L^2} ds &\leq \int_0^t m^{-1/2} \frac{[(m-5)!]^\sigma}{\tilde{\rho}^{m-5}} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-6}} |\vec{a}(s)|_{\tilde{\rho}, \sigma}^2 ds \\ &\leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{m^{\sigma-\frac{1}{2}} \rho^{2(m-6)}}{\tilde{\rho}} |\vec{a}(s)|_{\tilde{\rho}, \sigma}^2 ds \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds, \end{aligned}$$

where in the last inequality we have used (2.12). Moreover, similar to the proofs of Lemmas 2.10 and 2.11, we obtain

$$\begin{aligned}
 & - m \int_0^t \left(\partial_x^m \left[(\partial_x u) \partial_x u + (\partial_z \xi) \int_0^z \mathcal{U} dz - 2\nu (\partial_z^2 u) \mathcal{U} \right], \partial_x^m \lambda \right)_{L^2} ds \\
 & - m \int_0^t \left(\sum_{j=1}^m \binom{m}{j} [(\partial_x^j u) \partial_x^{m-j+1} \lambda + (\partial_x^j w) \partial_x^{m-j} \partial_z \lambda], \partial_x^m \lambda \right)_{L^2} ds \\
 & \leq \frac{\nu}{2} m \int_0^t \|\partial_z \partial_x^m \lambda\|_{L^2}^2 ds + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^3 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds \\
 & + e^{CC_*} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds.
 \end{aligned}$$

Combining the above inequalities with (2.34) gives

$$\begin{aligned}
 m \|\partial_x^m \lambda\|_{L^2}^2 & \leq m \|\partial_x^m \lambda(0)\|_{L^2}^2 + C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \left(|\vec{a}(s)|_{\rho, \sigma}^3 + |\vec{a}(s)|_{\rho, \sigma}^4 \right) ds \\
 & + e^{CC_*} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds.
 \end{aligned}$$

Note that $\lambda|_{t=0} = \partial_x u_0$. Hence

$$m \|\partial_x^m \lambda(0)\|_{L^2}^2 \leq m \frac{[(m-6)!]^{2\sigma}}{(2\rho_0)^{2(m-6)}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-6)}} \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2$$

because $\rho \leq \rho_0$. Thus we obtain the desired estimate on λ for $m \geq 6$. Again, the estimate for $m \leq 5$ is straightforward. Note that the upper bound for δ can be derived similarly because there is no nonzero boundary terms in the integration by parts due to the fact that $\partial_z \delta|_{z=0} = 0$. The proof of the proposition is completed. \square

2.6. Time derivatives. The estimate involving t -derivatives is stated as follows. Note it is only in this estimate that we need the normal derivatives of the initial data u_0 and f_0 up to the eighth order.

PROPOSITION 2.16. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned}
 & \sup_{\substack{1 \leq i \leq 4 \\ m+i \geq 7}} \frac{\rho^{2(m+i-7)}}{[(m+i-7)!]^{2\sigma}} \left(\|\langle z \rangle^\ell \partial_t^i \partial_x^m u(t)\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_t^i \partial_x^m f(t)\|_{L^2}^2 \right) \\
 & + \sup_{\substack{1 \leq i \leq 4 \\ m+i \geq 7}} \frac{\rho^{2(m+i-7)}}{[(m+i-7)!]^{2\sigma}} \int_0^t \left(\|\langle z \rangle^\ell \partial_z \partial_t^i \partial_x^m u\|_{L^2}^2 + \|\langle z \rangle^\ell \partial_z \partial_t^i \partial_x^m f\|_{L^2}^2 \right) ds \\
 & + \sup_{\substack{1 \leq i \leq 4 \\ m+i \geq 6}} \frac{\rho^{m+i-6}}{[(m+i-6)!]^\sigma} \|\partial_t^i \partial_x^m \mathcal{U}(t)\|_{L^2} \\
 & + \sup_{\substack{1 \leq i \leq 4 \\ m+i \geq 6}} \frac{\rho^{m+i-6}}{[(m+i-6)!]^\sigma} \left(m^{1/2} \|\partial_t^i \partial_x^m \lambda(t)\|_{L^2} + m^{1/2} \|\partial_t^i \partial_x^m \delta(t)\|_{L^2} \right) \\
 & + \sup_{\substack{1 \leq i \leq 4 \\ m+i \geq 6}} \frac{\rho^{m+i-6}}{[(m+i-6)!]^\sigma} \left(m \|\langle z \rangle^\ell \partial_t^i \partial_x^m \xi(t)\|_{L^2} + m \|\langle z \rangle^\ell \partial_t^i \partial_x^m \eta(t)\|_{L^2} \right) \\
 & + \sup_{\substack{1 \leq i \leq 4 \\ m+i \leq 6}} \left(\|\partial_t^i \partial_x^m u(t)\|_{L^2} + \|\partial_t^i \partial_x^m f(t)\|_{L^2} \right) + \sup_{\substack{1 \leq i \leq 4 \\ m+i \leq 5}} \|\partial_t^i \partial_x^m \mathcal{U}(t)\|_{L^2}
 \end{aligned}$$

$$\begin{aligned}
 & + \sup_{\substack{1 \leq i \leq 4 \\ m+i \leq 5}} (\|\partial_t^i \partial_x^m \lambda\|_{L^2} + \|\partial_t^i \partial_x^m \delta\|_{L^2} + \|\langle z \rangle^\ell \partial_t^i \partial_x^m \xi\|_{L^2} + \|\langle z \rangle^\ell \partial_t^i \partial_x^m \eta\|_{L^2}) \\
 & \leq C (\|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 + \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^4) \\
 & \quad + e^{CC_*^2} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right).
 \end{aligned}$$

Proof. The proof is similar to those in the previous subsections 2.2–2.5, with the tangential derivatives ∂_x^m replaced by $\partial_t^i \partial_x^m$. The main difference arises from the initial data. Note that

$$\partial_t u|_{t=0} = \nu \partial_z^2 u_0 - u_0 \partial_x u_0 - w_0 \partial_z u_0 + f_0 \partial_x f_0 + h_0 \partial_z f_0$$

with $w_0 = -\int_0^z \partial_x u_0 d\tilde{z}$ and $h_0 = -\int_0^z \partial_x f_0 d\tilde{z}$. Similar expressions hold for $\partial_t^i u|_{t=0}$, $2 \leq i \leq 4$, in terms of u_0 and f_0 . Thus we can control the terms $\partial_t^i u|_{t=0}$, $1 \leq i \leq 4$, by the initial data u_0 and f_0 if the normal derivatives of u_0 and f_0 are up to the eighth order rather than the fourth. Other than the difference in the order of differentiation on the initial data, there is no essential difference from the argument in the previous subsections on tangential derivatives. Hence, we omit the details of the proof. \square

2.7. Normal derivatives of u and f . It remains to estimate the normal derivatives of u and f in the a priori estimate and it is given in the following proposition.

PROPOSITION 2.17. *Under Assumption 2.5 we have, for any $t \in [0, T]$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0 \leq 1$,*

$$\begin{aligned}
 & \sup_{\substack{1 \leq i+j \leq 4 \\ m+i+j \geq 7}} \frac{\rho^{2(m+i+j-7)}}{[(m+i+j-7)!]^{2\sigma}} \|\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u(t)\|_{L^2}^2 \\
 & + \sup_{\substack{1 \leq i+j \leq 4 \\ m+i+j \geq 7}} \frac{\rho^{2(m+i+j-7)}}{[(m+i+j-7)!]^{2\sigma}} \int_0^t \|\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^{j+1} u(s)\|_{L^2}^2 ds \\
 & + \sup_{\substack{1 \leq i+j \leq 4 \\ m+i+j \leq 6}} \|\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u(t)\|_{L^2}^2 \\
 & \leq C (\|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^2 + \|(u_0, f_0)\|_{2\rho_0, \sigma, 8}^4) \\
 & \quad + e^{CC_*^2} \left(\int_0^t (|\vec{a}(s)|_{\rho, \sigma}^2 + |\vec{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right),
 \end{aligned}$$

where $C_* \geq 1$ is the constant given in (2.8). The above estimate also holds when $\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u$ is replaced by $\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f$.

Proof. We apply induction on j , the order of normal derivatives. The validity for $j = 0$ follows from Proposition 2.16. Now for given $j \geq 1$ and for any i and m , suppose the estimate

(2.35)

$$\begin{aligned}
 & \|\langle z \rangle^{\ell+k} \partial_t^i \partial_x^m \partial_z^k u(t)\|_{L^2}^2 + \|\langle z \rangle^{\ell+k} \partial_t^i \partial_x^m \partial_z^k f(t)\|_{L^2}^2 \\
 & + \int_0^t \left(\|\langle z \rangle^{\ell+k} \partial_t^i \partial_x^m \partial_z^{k+1} u(s)\|_{L^2}^2 + \|\langle z \rangle^{\ell+k} \partial_t^i \partial_x^m \partial_z^{k+1} f(s)\|_{L^2}^2 \right) ds
 \end{aligned}$$

$$\begin{aligned} &\leq C \frac{[(m+i+k-7)!]^{2\sigma}}{\rho^{2(m+i+k-7)}} (\| (u_0, f_0) \|_{2\rho_0, \sigma, 8}^2 + \| (u_0, f_0) \|_{2\rho_0, \sigma, 8}^4) \\ &\quad + e^{CC_*^2} \frac{[(m+i+k-7)!]^{2\sigma}}{\rho^{2(m+i+k-7)}} \left(\int_0^t (|\bar{a}(s)|_{\rho, \sigma}^2 + |\bar{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right) \end{aligned}$$

holds for any $k \leq j - 1$ with $i + k \leq 4$ and $m + i + k \geq 7$; we will show the above estimate holds for $k = j$. To do so, applying $\langle z \rangle^{\ell+j} \partial_t^i \partial_z^j$, $i + j \leq 4$, to (2.13), and observing

$$\partial_x^m \xi = (f \partial_x + h \partial_z) \partial_x^m f + \underbrace{\sum_{1 \leq k \leq m} \binom{m}{k} [(\partial_x^k f) \partial_x^{m-k+1} f + (\partial_x^k h) \partial_x^{m-k} \partial_z f]}_{\stackrel{\text{def}}{=} H_m},$$

we obtain, with H_m defined above,

$$\begin{aligned} &(\partial_t + u \partial_x + w \partial_z - \nu \partial_z^2) \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u \\ &= (f \partial_x + h \partial_z) \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j \partial_x^m f + \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j H_m \\ &\quad + [\langle z \rangle^{\ell+j} \partial_t^i \partial_z^j, f \partial_x + h \partial_z] \partial_x^m f - \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j [(\partial_x^m w) \partial_z u] \\ &\quad + \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j F_m + [u \partial_x + w \partial_z - \nu \partial_z^2, \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j] \partial_x^m u, \end{aligned}$$

where F_m is defined in (2.13) and $[T_1, T_2] = T_1 T_2 - T_2 T_1$ stands for the commutator of two operators T_1, T_2 . Following the argument used in the proof of Lemma 2.10 (see also the proof of [21, Lemma 4.3]), we have

$$\begin{aligned} &(- \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j [(\partial_x^m w) \partial_z u] + \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j F_m, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u)_{L^2} \\ &\quad + ([u \partial_x + w \partial_z - \nu \partial_z^2, \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j] \partial_x^m u, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u)_{L^2} \\ &\leq \frac{\nu}{4} \| \partial_z (\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u) \|_{L^2}^2 + C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} (|\bar{a}|_{\rho, \sigma}^2 + |\bar{a}|_{\rho, \sigma}^4) \end{aligned}$$

and

$$\begin{aligned} &(\langle z \rangle^{\ell+j} \partial_t^i \partial_z^j H_m, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u)_{L^2} \\ &\quad + ([f \partial_x + h \partial_z, \langle z \rangle^{\ell+j} \partial_t^i \partial_z^j] \partial_x^m f, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u)_{L^2} \\ &\leq \frac{\nu}{4} \| \partial_z (\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u) \|_{L^2}^2 + C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} (|\bar{a}|_{\rho, \sigma}^2 + |\bar{a}|_{\rho, \sigma}^4). \end{aligned}$$

Hence, combining the above inequalities gives

$$\begin{aligned} (2.36) \quad &\frac{1}{2} \frac{d}{dt} \| \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u \|_{L^2}^2 + \frac{\nu}{2} \| \partial_z (\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u) \|_{L^2}^2 \\ &\leq \left| \nu \int_{\mathbb{R}} (\partial_t^i \partial_x^m \partial_z^j u) (\partial_t^i \partial_x^m \partial_z^{j+1} u) |_{z=0} dx \right| + C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} (|\bar{a}|_{\rho, \sigma}^2 + |\bar{a}|_{\rho, \sigma}^4) \\ &\quad + ((f \partial_x + h \partial_z) \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u)_{L^2}. \end{aligned}$$

For the boundary term, since $\| \omega \|_{L^\infty(\mathbb{R}_+)}^2 \leq 2 \| \partial_z \omega \|_{L^2(\mathbb{R}_+)} \| \omega \|_{L^2(\mathbb{R}_+)}$ if $\omega \rightarrow 0$ as $z \rightarrow +\infty$ and

$$\begin{aligned} \nu \partial_t^i \partial_x^m \partial_z^{j+1} u|_{z=0} &= \partial_t^{i+1} \partial_x^m \partial_z^{j-1} u|_{z=0} + \partial_t^i \partial_x^m \partial_z^{j-1} (u \partial_x u + w \partial_z u - f \partial_x f - h \partial_z f)|_{z=0} \\ &= \partial_t^{i+1} \partial_x^m \partial_z^{j-1} u|_{z=0} + \partial_t^i \partial_x^m \partial_z^{j-1} (u \partial_x u - f \partial_x f)|_{z=0} \\ &\quad + \sum_{k=1}^{j-1} \binom{j-1}{k} \partial_t^i \partial_x^m \left((\partial_z^k w) \partial_z^{j-k} u - (\partial_z^k h) \partial_z^{j-k} f \right) \Big|_{z=0}, \end{aligned}$$

we have

$$\begin{aligned} \nu^2 \frac{\rho}{m^\sigma} \|(\partial_t^i \partial_x^m \partial_z^{j+1} u)|_{z=0}\|_{L_x^2}^2 &\leq C \frac{\rho}{m^\sigma} \|\partial_t^{i+1} \partial_x^m \partial_z^{j-1} u\|_{L^2} \|\partial_t^{i+1} \partial_x^m \partial_z^j u\|_{L^2} \\ &\quad + C \frac{\rho}{m^\sigma} \|\partial_t^i \partial_x^m \partial_z^{j-1} (u \partial_x u - f \partial_x f)\|_{L^2} \|\partial_t^i \partial_x^m \partial_z^j (u \partial_x u - f \partial_x f)\|_{L^2} \\ &\quad + C \sum_{k=1}^{j-1} \frac{\rho}{m^\sigma} \|\partial_t^i \partial_x^m \left((\partial_z^k w) \partial_z^{j-k} u - (\partial_z^k h) \partial_z^{j-k} f \right)\|_{L^2} \\ &\quad \times \|\partial_z \partial_t^i \partial_x^m \left((\partial_z^k w) \partial_z^{j-k} u - (\partial_z^k h) \partial_z^{j-k} f \right)\|_{L^2} \\ &\leq \|\partial_t^{i+1} \partial_x^m \partial_z^j u\|_{L^2}^2 + C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} \left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4 \right), \end{aligned}$$

where we have used again the argument similar to the proof of Lemma 2.10, and the estimate for $p + q \leq 4$,

$$\|\langle z \rangle^{\ell+q} \partial_t^p \partial_x^m \partial_z^q u\|_{L^2} + \|\langle z \rangle^{\ell+q} \partial_t^p \partial_x^m \partial_z^q f\|_{L^2} \leq \frac{[(m+p+q-7)!]^\sigma}{\rho^{(m+p+q-7)}} |\vec{a}|_{\rho,\sigma}.$$

Moreover,

$$\begin{aligned} \frac{m^\sigma}{\rho} \|(\partial_t^i \partial_x^m \partial_z^j u)|_{z=0}\|_{L_x^2}^2 &\leq C \frac{m^\sigma}{\rho} \|\partial_t^i \partial_x^m \partial_z^j u\|_{L^2} \|\partial_t^i \partial_x^m \partial_z^{j+1} u\|_{L^2} \\ &\leq \frac{\nu}{4} \|\partial_t^i \partial_x^m \partial_z^{j+1} u\|_{L^2}^2 + C \frac{m^{2\sigma}}{\rho^2} \|\partial_t^i \partial_x^m \partial_z^j u\|_{L^2}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \left| \nu \int_{\mathbb{R}} (\partial_t^i \partial_x^m \partial_z^j u) (\partial_t^i \partial_x^m \partial_z^{j+1} u) \Big|_{z=0} dx \right| &\leq \frac{\nu}{4} \|\partial_t^i \partial_x^m \partial_z^{j+1} u\|_{L^2}^2 + \|\partial_t^{i+1} \partial_x^m \partial_z^j u\|_{L^2}^2 \\ &\quad + C \frac{m^{2\sigma}}{\rho^2} \|\partial_t^i \partial_x^m \partial_z^j u\|_{L^2}^2 + C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} \left(|\vec{a}|_{\rho,\sigma}^2 + |\vec{a}|_{\rho,\sigma}^4 \right), \end{aligned}$$

which together with (2.36) implies

$$\begin{aligned} &\|\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u(t)\|_{L^2}^2 + \frac{\nu}{2} \int_0^t \|\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^{j+1} u(s)\|_{L^2}^2 ds \\ &\leq \|\langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u(0)\|_{L^2}^2 + C \frac{m^{2\sigma}}{\rho^2} \int_0^t \|\partial_t^i \partial_x^m \partial_z^j u(s)\|_{L^2}^2 ds + \int_0^t \|\partial_t^{i+1} \partial_x^m \partial_z^j u\|_{L^2}^2 ds \\ &\quad + C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} \int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds \\ &\quad + \left((f \partial_x + h \partial_z) \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u \right)_{L^2} \\ &\leq C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} (\|(u_0, f_0)\|_{2\rho_0,\sigma,8}^2 + \|(u_0, f_0)\|_{2\rho_0,\sigma,8}^4) \\ &\quad + e^{CC_*^2} \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} \left(\int_0^t (|\vec{a}(s)|_{\rho,\sigma}^2 + |\vec{a}(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|\vec{a}(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right) \\ &\quad + \left((f \partial_x + h \partial_z) \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u \right)_{L^2}, \end{aligned}$$

where in the last inequality we have used the induction assumption (2.35). Similarly,

$$\begin{aligned} & \| \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f(t) \|_{L^2}^2 + \frac{\mu}{2} \int_0^t \| \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^{j+1} f(s) \|_{L^2}^2 ds \\ & \leq C \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} (\| (u_0, f_0) \|_{2\rho_0, \sigma, 8}^2 + \| (u_0, f_0) \|_{2\rho_0, \sigma, 8}^4) \\ & \quad + e^{CC_*^2} \frac{[(m+i+j-7)!]^{2\sigma}}{\rho^{2(m+i+j-7)}} \left(\int_0^t (|\bar{a}(s)|_{\rho, \sigma}^2 + |\bar{a}(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|\bar{a}(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right) \\ & \quad + \left((f\partial_x + h\partial_z) \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f \right)_{L^2}. \end{aligned}$$

Taking the summation of the above two estimates and noticing

$$\begin{aligned} & \left((f\partial_x + h\partial_z) \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u \right)_{L^2} \\ & \quad + \left((f\partial_x + h\partial_z) \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j u, \langle z \rangle^{\ell+j} \partial_t^i \partial_x^m \partial_z^j f \right)_{L^2} = 0, \end{aligned}$$

we show (2.35) holds for $k = j$. Thus the proof of the proposition is completed. \square

2.8. 2D MHD boundary layer. By combining the estimates in Propositions 2.7–2.9 and 2.15–2.17, we obtain the a priori estimate (2.9) in Theorem 2.6 that enables us to prove the well-posedness of the MHD boundary layer system (2.2). Precisely, for given initial datum $(u_0, f_0) \in X_{2\rho_0, \sigma, 8}$, as in [24, section 7], we first construct the approximate solutions $(u_\varepsilon, f_\varepsilon) \in L^\infty([0, T_\varepsilon]; X_{3\rho_0/2, \sigma, 8})$ to the regularized MHD boundary layer system

$$(2.37) \quad \begin{cases} (\partial_t + u_\varepsilon \partial_x + w_\varepsilon \partial_z - \varepsilon \partial_x^2 - \nu \partial_z^2) u_\varepsilon = \xi_\varepsilon, \\ (\partial_t + u_\varepsilon \partial_x + w_\varepsilon \partial_z - \varepsilon \partial_x^2 - \mu \partial_z^2) f_\varepsilon = \eta_\varepsilon, \\ (\partial_t + u_\varepsilon \partial_x + w_\varepsilon \partial_z - \varepsilon \partial_x^2 - \mu \partial_z^2) h_\varepsilon = f_\varepsilon \partial_x w_\varepsilon - h_\varepsilon \partial_x u_\varepsilon, \\ \partial_x u_\varepsilon + \partial_z w_\varepsilon = \partial_x f_\varepsilon + \partial_z h_\varepsilon = 0, \\ (u_\varepsilon, w_\varepsilon)|_{z=0} = (\partial_z f_\varepsilon, h_\varepsilon)|_{z=0} = (0, 0), \quad (u_\varepsilon, f_\varepsilon)|_{z \rightarrow +\infty} = (0, 0), \\ (u_\varepsilon, f_\varepsilon)|_{t=0} = (u_0, f_0), \end{cases}$$

with $\xi_\varepsilon = (f_\varepsilon \partial_x + h_\varepsilon \partial_z) f_\varepsilon$ and $\eta_\varepsilon = (f_\varepsilon \partial_x + h_\varepsilon \partial_z) u_\varepsilon$. Then we derive a uniform estimate on the approximate solutions $(u_\varepsilon, f_\varepsilon)$ so that we can take the $\varepsilon \rightarrow 0$ to have existence of a solution in a time interval independent of ε . For this, we define \bar{a}_ε in the same way as \bar{a} given in Definition 2.2, with the functions replaced accordingly by those derived from (2.37). Then the a priori estimate (2.9) in Theorem 2.6 holds with \bar{a} replaced by \bar{a}_ε . Hence, we can derive, repeating the argument in [21, section 6], the uniform upper bound with respect to ε of the approximate solutions $(u_\varepsilon, f_\varepsilon)$ in $L^\infty([0, T]; X_{\rho, \sigma, 4})$ for some $0 < \rho < \rho_0$ and some T independent of ε . By taking $\varepsilon \rightarrow 0$, we conclude, by a compactness argument that the limit (u, f) solves (2.2). The uniqueness of the solution follows from a similar argument used in [24, subsection 8.2]. Therefore, we complete the proof of Theorem 2.1 for $\sigma = 3/2$. We remark that as in [23, section 8], it is straightforward to modify the proof for $1 < \sigma < 3/2$.

3. 3D MHD boundary layer. Now we consider the 3D MHD boundary layer and use (u, v, w) and (f, g, h) to denote velocity and magnetic fields, respectively, and denote by (x, y, z) the spatial variables in \mathbb{R}_+^3 . Then the MHD boundary layer system (1.3) in 3D space is

$$(3.1) \quad \begin{cases} (\partial_t + u\partial_x + v\partial_y + w\partial_z - \nu\partial_z^2)u - (f\partial_x + g\partial_y + h\partial_z)f = 0, \\ (\partial_t + u\partial_x + v\partial_y + w\partial_z - \nu\partial_z^2)v - (f\partial_x + g\partial_y + h\partial_z)g = 0, \\ (\partial_t + u\partial_x + v\partial_y + w\partial_z - \mu\partial_z^2)f - (f\partial_x + g\partial_y + h\partial_z)u = 0, \\ (\partial_t + u\partial_x + v\partial_y + w\partial_z - \mu\partial_z^2)g - (f\partial_x + g\partial_y + h\partial_z)v = 0, \\ (\partial_t + u\partial_x + v\partial_y + w\partial_z - \mu\partial_z^2)h = f\partial_x w + g\partial_y w - h\partial_x u - h\partial_y v, \end{cases}$$

with the divergence-free and initial-boundary conditions

$$\begin{cases} \partial_x u + \partial_y v + \partial_z w = \partial_x f + \partial_y g + \partial_z h = 0, \\ (u, v, w)|_{z=0} = (\partial_z f, \partial_z g, h)|_{z=0} = (0, 0, 0) \quad (u, v, f, g)|_{z \rightarrow +\infty} = (0, 0, 0, 0), \\ (u, v)|_{t=0} = (u_0, v_0), \quad (f, g)|_{t=0} = (f_0, g_0). \end{cases}$$

The proof of well-posedness of this system in Gevrey function space with index $3/2$ is similar to the proof for the 2D case with a slight modification. Precisely, instead of the scalar auxiliary functions \mathcal{U} , λ , and δ in (2.4) and (2.5) we introduce here the vector-valued functions $\mathbf{U} = (\mathcal{U}_1, \mathcal{U}_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \tilde{\lambda}_1, \tilde{\lambda}_2)$, and $\boldsymbol{\delta} = (\delta_1, \delta_2, \tilde{\delta}_1, \tilde{\delta}_2)$, where

$$\begin{cases} (\partial_t + u\partial_x + v\partial_y + w\partial_z - \nu\partial_z^2) \int_0^z \mathcal{U}_1(t, x, y, \tilde{z}) d\tilde{z} = -\partial_x w(t, x, y, z), \\ (\partial_t + u\partial_x + v\partial_y + w\partial_z - \nu\partial_z^2) \int_0^z \mathcal{U}_2(t, x, y, \tilde{z}) d\tilde{z} = -\partial_y w(t, x, y, z), \\ \mathcal{U}_j|_{t=0} = 0, \quad \partial_z \mathcal{U}_j|_{z=0} = \mathcal{U}_j|_{z \rightarrow +\infty} = 0, \quad j = 1, 2, \end{cases}$$

and

$$\begin{cases} \lambda_1 = \partial_x u - (\partial_z u) \int_0^z \mathcal{U}_1 d\tilde{z}, & \lambda_2 = \partial_y u - (\partial_z u) \int_0^z \mathcal{U}_2 d\tilde{z}, \\ \tilde{\lambda}_1 = \partial_x v - (\partial_z v) \int_0^z \mathcal{U}_1 d\tilde{z}, & \tilde{\lambda}_2 = \partial_y v - (\partial_z v) \int_0^z \mathcal{U}_2 d\tilde{z}, \\ \delta_1 = \partial_x f - (\partial_z f) \int_0^z \mathcal{U}_1 d\tilde{z}, & \delta_2 = \partial_y f - (\partial_z f) \int_0^z \mathcal{U}_2 d\tilde{z}, \\ \tilde{\delta}_1 = \partial_x g - (\partial_z g) \int_0^z \mathcal{U}_1 d\tilde{z}, & \tilde{\delta}_2 = \partial_y g - (\partial_z g) \int_0^z \mathcal{U}_2 d\tilde{z}. \end{cases}$$

Moreover, corresponding to (2.3), set $\boldsymbol{\xi} = (\xi_1, \xi_2)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2)$ by

$$\begin{cases} \xi_1 = (f\partial_x + g\partial_y + h\partial_z)f, & \xi_2 = (f\partial_x + g\partial_y + h\partial_z)g, \\ \eta_1 = (f\partial_x + g\partial_y + h\partial_z)u, & \eta_2 = (f\partial_x + g\partial_y + h\partial_z)v. \end{cases}$$

We remark that as in the 2D case, here we can also apply the cancellation mechanism so that the highest order term $\partial_x w$ does not appear in the evolution equations solved by $\xi_j, \eta_j, j = 1, 2$. With the above functions, we set accordingly $\vec{a} = (u, v, f, g, \mathbf{U}, \boldsymbol{\lambda}, \boldsymbol{\delta}, \boldsymbol{\xi}, \boldsymbol{\eta})$ and define $|\vec{a}|_{\rho, \sigma}$ in the same way as in Definition 2.2 with the tangential derivative ∂_x^m replaced by $\partial_x^{\alpha_1} \partial_y^{\alpha_2}$. Then the a priori estimate stated in Theorem 2.6 also holds for the new function \vec{a} . For example, we can repeat the argument used in subsection 2.2 to get the desired estimate on the L^2 norm of $\partial_x^m \mathbf{U}$ and $\partial_y^m \mathbf{U}$, and the estimate for $\partial_x^{\alpha_1} \partial_y^{\alpha_2} \mathbf{U}$ will follow from the inequality

$$\|\partial_x^{\alpha_1} \partial_y^{\alpha_2} \mathbf{U}\|_{L^2(\mathbb{R}_+^3)} \leq C \left(\|\partial_x^{\alpha_1 + \alpha_2} \mathbf{U}\|_{L^2(\mathbb{R}_+^3)} + \|\partial_y^{\alpha_1 + \alpha_2} \mathbf{U}\|_{L^2(\mathbb{R}_+^3)} \right).$$

A similar argument holds for the estimates on the other functions λ, δ, \dots . Again from the a priori estimate we can derive the existence and uniqueness of a solution to the 3D MHD boundary layer system (3.1) with corresponding initial and boundary conditions. Since there is no extra difficulty in the proof for the 3D case, we omit the details for brevity.

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