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Published in:
SIAM Journal on Mathematical Analysis

Published: 01/01/2021

Document Version:
Final Published version, also known as Publisher's PDF, Publisher's Final version or Version of Record

Publication record in CityU Scholars:
Go to record

Published version (DOI):
10.1137/20M1368811

Publication details:
DUAN, R., LIU, S., & YANG, T. (2021). GLOBAL CLASSICAL SOLUTIONS FOR THE VLASOV-NORDSTRÖM-FOKKER-PLANCK SYSTEM. *SIAM Journal on Mathematical Analysis*, *53*(5), 6164-6190. https://doi.org/10.1137/20M1368811

SIAM J. MATH. ANAL.
Vol. 53, No. 5, pp. 6164–6190

GLOBAL CLASSICAL SOLUTIONS FOR THE VLASOV–NORDSTRÖM–FOKKER–PLANCK SYSTEM*

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Abstract. In this paper we consider the Vlasov–Nordström–Fokker–Planck system in the whole space. The kinetic model is a relativistic generalization of the classical Vlasov–Poisson–Fokker–Planck system in the gravitational case and describes the ensemble motion of collision particles interacting by means of a self-consistent scalar gravitational field satisfying a nonlinear wave equation. We construct the global-in-time classical solutions to the corresponding Cauchy problem with small perturbation of equilibrium states, and we further obtain the polynomial rate of convergence of solutions to the Maxwell–Jüttner distribution function with a constant scalar gravitational field. The proof is based on the dissipative structure analysis of the linearized system together with the nonlinear energy method. In particular, we make use of the Klein–Gordon dissipation feature induced by the coupling and also overcome difficulties due to degenerate dissipation of momentum derivatives.

Key words. Vlasov–Nordström–Fokker–Planck, Klein–Gordon, global existence, large time behaviors

AMS subject classifications. 35Q84, 35Q20

DOI. 10.1137/20M1368811

1. Introduction. The Vlasov–Nordström–Fokker–Planck system was first derived by Alcántara Félix and Calogero [1] in 2011 as a mean-field model coupling the Fokker–Planck dynamics to a relativistic scalar theory of gravity (the Nordström theory). Although it is desirable to obtain more physical models in the framework of general relativity, the Vlasov–Nordström–Fokker–Planck system is of its own interest. Indeed, the authors of [1] also obtained via variation method the existence of steady states for all possible prescribed values of the mass, which is a better result than for the corresponding nonrelativistic model, the Vlasov–Poisson–Fokker–Planck system, for which existence of steady states is known only for small mass. On the other hand, when the particle collisions are not considered, the Vlasov–Nordström–Fokker–Planck system reduces to the Vlasov–Nordström system that can be regarded as a toy model for the full general relativistic Vlasov–Einstein system. Readers may refer to a nice review paper [3].

There have been many mathematical studies on the Vlasov–Nordström system. In 2004, Calogero and Rein [6] studied the global weak solutions for the Vlasov–Nordström system. The first global classical solutions for the Cauchy problem with general initial data was constructed by Calogero [5]. Another interesting result for the

*Received by the editors September 22, 2020; accepted for publication June 11, 2021; published electronically October 28, 2021.
https://doi.org/10.1137/20M1368811
Funding: The work of the first author was partially supported by the NSFC/RGC Joint Research Scheme grant N_CUHK409/19 from RGC in Hong Kong. The work of the second author was supported by the National Natural Science Foundation of China grants 11731008 and 11971201. The work of the third author was supported by the General Research Fund of Hong Kong, CityU grant 11304419, and the National Natural Science Foundation of China grant 11771169.
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Vlasov–Nordström system was given by Bauer et al. in [4], where a radiation formula, similar to the dipole formula in electrodynamics, is rigorously derived. Recently, Fajman, Joudioux, and Smulevici [9] and Wang [15] independently constructed the global classical solutions to the Vlasov–Nordström system for small initial data with the aid of the vector field method.

In the presence of collisions, Alcántara Félix, Calogero, and Pankavich [2] investigated the spatially homogeneous Vlasov–Nordström–Fokker–Planck system, where the global strong solution and long time asymptotic behavior are established on the basis of theory of stochastic differential equations and diffusion processes. However, to our knowledge, the spatially inhomogeneous case has not been attempted yet in the literature.

In the paper, we consider the following Vlasov–Nordström–Fokker–Planck system (or VNFP system for short):

$$\partial_t F + \nabla_p \sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x F - \nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p F = e^{2\phi}\nabla\cdot(\Lambda_\phi\nabla_p F + pF), \tag{1.1}$$

$$\Box\phi = -e^{2\phi}\int_{\mathbb{R}^3}\frac{F(t,x,p)}{\sqrt{e^{2\phi}+|p|^2}}\,\mathrm{d}p + S. \tag{1.2}$$

Here the unknowns are $F = F(t,x,p) \geq 0$ denoting the distribution function of particles with momentum $p\in\mathbb{R}^3$ at time $t>0$ and position $x\in\mathbb{R}^3$, and $\phi=\phi(t,x)$ standing for the scalar gravitational field. Moreover, $\Box := \partial_t^2 - \Delta_x$ is the three-dimensional wave operator, and

$$\Lambda_\phi = [(\Lambda_\phi)_{ij}] = \frac{e^{2\phi}I + p\otimes p}{\sqrt{e^{2\phi}+|p|^2}} \tag{1.3}$$

is the relativistic diffusion matrix induced by the Laplace–Beltrami operator (cf. [1]), and I is the identity matrix. Slightly different from the model originally derived in [1], an additional source term $S=S(t,x)$ is included in the wave equation (1.2) taking into account possible repulsive effect by the background. Throughout the paper, we assume that $S\equiv 1$ is a strictly positive constant. Under this assumption, we are interested in constructing the global-in-time classical solutions to the Cauchy problem of the VNFP system (1.1) and (1.2) supplemented with initial data

$$F(0,x,p)=F_0(x,p),\quad \phi(0,x)=\phi_0(x),\quad \partial_t\phi(0,x)=\phi_1(x). \tag{1.4}$$

It is straightforward to see that after normalization, the VNFP system (1.1) and (1.2) admits a trivial steady state, that is, $\phi\equiv 0$ and F being the global relativistic Maxwellian (the Maxwell–Jüttner distribution) given by

$$J(p) = C_J e^{-p_0},\ p_0 := \sqrt{1+|p|^2},$$

where C_J is a positive constant chosen so that

$$\int_{\mathbb{R}^3} J(p)\,\frac{\mathrm{d}p}{p_0} = 1. \tag{1.5}$$

For any solution $[F(t,x,p),\phi(t,x)]$ to the VNFP system (1.1) and (1.2), let $F = J_\phi + J^{1/2}f$ with

$$J_\phi = C_J e^{-\sqrt{e^{2\phi}+|p|^2}}; \tag{1.6}$$

then $f(t,x,p)$ satisfies

$$\begin{aligned}(1.7)\qquad &\partial_t f+\nabla_p\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x f-\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p f\\ &-\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}J_\phi J^{-1/2}+\frac{1}{2}\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f=-e^{2\phi}L_\phi f,\end{aligned}$$

where

$$L_\phi f:=-J^{-1/2}\nabla\cdot\left[\Lambda_\phi\nabla_p(J^{1/2}f)+pJ^{1/2}f\right]. \tag{1.8}$$

Here, by (1.3) and (1.6) we have used the identity $\Lambda_\phi\nabla_p J_\phi+pJ_\phi=0$ for any function $\phi=\phi(t,x)$. The coupled wave equation (1.2) is transformed into

$$\Box\phi=-e^{2\phi}\int_{\mathbb{R}^3}\frac{J_\phi}{\sqrt{e^{2\phi}+|p|^2}}\,\mathrm{d}p-e^{2\phi}\int_{\mathbb{R}^3}\frac{J^{1/2}f}{\sqrt{e^{2\phi}+|p|^2}}\,\mathrm{d}p+1. \tag{1.9}$$

Correspondingly, the supplemented initial data (1.4) are rewritten as

$$f(0,x,p)=f_0(x,p):=\frac{F_0(x,p)-J_{\phi_0}}{J^{1/2}},\ \ \phi(0,x)=\phi_0(x),\ \ \partial_t\phi(0,x)=\phi_1(x). \tag{1.10}$$

To the end, we expect to construct the global-in-time classical solutions to the reformulated Cauchy problem (1.7), (1.9), and (1.10) when the initial data are suitably small. For this purpose, we introduce some notations as follows. First of all, to capture the dissipative property of the ϕ-dependent linear operator L_ϕ in (1.8) for ϕ near zero, we compare it to the case when $\phi=0$, namely,

$$Lf:=L_0 f=-J^{-1/2}\nabla\cdot\left[\Lambda_0\nabla_p(J^{1/2}f)+pJ^{1/2}f\right] \tag{1.11}$$

with

$$\Lambda_0=[(\Lambda_0)_{ij}]=\frac{I+p\otimes p}{p_0}. \tag{1.12}$$

The basic properties of the linear operator L in (1.11) are summarized as follows:

(a) It holds by direct computations that

$$Lf=-\nabla_p\cdot(\Lambda_0\nabla_p f)+\frac{|p|^2}{4p_0}f-\frac{3}{2}f. \tag{1.13}$$

(b) L is self-adjoint and nonnegative on L^2_p with the null space $\ker L=\mathrm{span}\{J^{1/2}\}$ and

$$\langle Lf,f\rangle=\int_{\mathbb{R}^3}J\nabla_p\left(\frac{\{\mathbf{I}-\mathbf{P}_0\}f}{\sqrt{J}}\right)\Lambda_0\nabla_p\left(\frac{\{\mathbf{I}-\mathbf{P}_0\}f}{\sqrt{J}}\right)\mathrm{d}p\geq 0.$$

Here $\mathbf{P}_0:L^2_p\to\ker L$ is the projection of the form

$$\mathbf{P}_0 f=a(t,x)J^{1/2},\quad a(t,x):=\frac{\langle f,J^{1/2}\rangle}{\langle J,1\rangle}J^{1/2}$$

for any $f=f(t,x,p)$, and $\langle\cdot,\cdot\rangle$ denotes the inner product on L^2_p. One may write $f=\mathbf{P}_0f+\{\mathbf{I}-\mathbf{P}_0\}f$, where $\mathbf{I}$ is the identity operator, $\mathbf{P}_0f$ is called the macroscopic component, and $\{\mathbf{I}-\mathbf{P}_0\}f$ the microscopic component.

(c) Moreover, there is $\delta > 0$ such that

$$\langle Lf, f\rangle \geq \delta |\{\mathbf{I}-\mathbf{P}_0\}f|_D^2, \tag{1.14}$$

where the dissipation norm $|\cdot|_D$ in the p-variable is defined by

$$|g|_D^2 = \int_{\mathbb{R}^3} \left\{ \sum_{i,j=1}^3 (\Lambda_0)_{ij}\partial_{p_i} g \partial_{p_j} g + p_0 g^2 \right\} \mathrm{d}p \tag{1.15}$$

for any $g = g(p)$. cf. [11]. Note by (1.12) that

$$\sum_{i,j=1}^3 (\Lambda_0)_{ij}\partial_{p_i} g \partial_{p_j} g = \frac{1}{p_0}\left|\frac{p}{|p|} \times \nabla_p g\right|^2 + p_0 \left|\frac{p}{|p|}\cdot \nabla_p g\right|^2 \geq \frac{1}{p_0}|\nabla_p g|^2, \tag{1.16}$$

where $\times$ denotes the cross product. Thus, (1.15) has a lower bound as

$$|g|_D^2 \geq \int_{\mathbb{R}^3} \left\{ \frac{1}{p_0}|\nabla_p g|^2 + p_0 g^2 \right\} \mathrm{d}p. \tag{1.17}$$

To state the main results, we introduce some notations of norms for convenience of presentation. First, similar to (1.15), we define the dissipation norm $\|\cdot\|_D$ in both p- and x-variables:

$$\|f\|_D^2 = \int_{\mathbb{R}^3} |f|_D^2 \,\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left\{ \sum_{i,j=1}^3 (\Lambda_0)_{ij}\partial_{p_i} f \partial_{p_j} f + p_0 f^2 \right\} \mathrm{d}p\,\mathrm{d}x. \tag{1.18}$$

Letting $w(p) = p_0$, we also define the following weighted L^2-norm and D-norm, respectively:

$$\|w^l f\|^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} w^{2l}|f|^2 \,\mathrm{d}p\,\mathrm{d}x, \quad \|w^l f\|_D^2 = \int_{\mathbb{R}^3} |w^l f|_D^2 \,\mathrm{d}x,$$

where

$$|w^l f|_D^2 = \int_{\mathbb{R}^3} w^{2l} \left\{ \sum_{i,j=1}^3 (\Lambda_0)_{ij}\partial_{p_i} f \partial_{p_j} f + p_0 f^2 \right\} \mathrm{d}p.$$

Furthermore, we define the following temporal energy functional and higher-order energy functional:

$$\begin{aligned}\mathcal{E}_{N,l}(t) \sim &\sum_{|\alpha|\leq N} \left\{ \|\partial^\alpha f\|^2 + \|\partial_t \partial^\alpha \phi\|^2 + \|\nabla_x \partial^\alpha \phi\|^2 + \|\partial^\alpha \phi\|^2 \right\} \\ &+ \sum_{|\alpha|+|\beta|\leq N, |\alpha|\leq N-1} \left\| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}_0\} f\right\|^2,\end{aligned} \tag{1.19}$$

$$\begin{aligned}\mathcal{E}^h_{N,l}(t) \sim &\sum_{1\leq|\alpha|\leq N} \left\{ \|\partial^\alpha a\|^2 + \|\partial_t \partial^\alpha \phi\|^2 + \|\nabla_x \partial^\alpha \phi\|^2 + \|\partial^\alpha \phi\|^2 \right\} \\ &+ \sum_{|\alpha|\leq N} \|\partial^\alpha \{\mathbf{I}-\mathbf{P}_0\} f\|^2 + \sum_{|\alpha|+|\beta|\leq N, |\alpha|\leq N-1} \left\| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}_0\} f\right\|^2,\end{aligned} \tag{1.20}$$

and the dissipation rate functional corresponding to (1.19):

$$
\begin{aligned}
\mathcal{D}_{N,l}(t) = &\sum_{|\alpha|\leq N} \|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2 + \sum_{|\alpha|\leq N-2} \|\partial^\alpha \nabla_x \partial_t \phi\|^2 + \sum_{|\alpha|\leq N-1} \|\nabla_x \partial^\alpha [a,\phi]\|^2 \\
&+ \sum_{|\alpha|+|\beta|\leq N, |\alpha|\leq N-1} \|w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}_0\}f\|_D^2, \qquad (1.21)
\end{aligned}
$$

where N and l are integers, and $\partial^\alpha = \partial_x^\alpha$, $\partial_\beta = \partial_p^\beta$, and $\partial^\alpha_\beta = \partial_x^\alpha \partial_p^\beta$ for multi-indices α and β.

The first result of the paper is concerned with the global-in-time existence of solutions to the Cauchy problem on the VNFP system.

Theorem 1.1 (global existence). *Let $N \geq 4$ and $l \geq 0$. There are constants $\epsilon_0 > 0$ and $C_0 > 0$ such that if initial data $[f_0, \phi_0, \phi_1]$ satisfy that $F_0(x,p) = J_{\phi_0}(p) + J^{1/2}(p) f_0(x,p) \geq 0$ and $\mathcal{E}_{N,l}(0) < \epsilon_0$ for a temporal energy functional $\mathcal{E}_{N,l}(\cdot)$ to be suitably defined in the proof, then the Cauchy problem* (1.7), (1.9), *and* (1.10) *admits a unique global classical solution $\big[f(t,x,p), \phi(t,x)\big]$ satisfying $F(t,x,p) = J_\phi(p) + J^{1/2}(p) f(t,x,p) \geq 0$ and*

$$
\mathcal{E}_{N,l}(t) + \int_0^t \mathcal{D}_{N,l}(s)\, \mathrm{d}s \leq C_0 \mathcal{E}_{N,l}(0) \qquad (1.22)
$$

for all time $t \geq 0$.

Furthermore, under certain additional conditions on initial data, the global-in-time solutions obtained above decay to zero in time with polynomial rates.

Theorem 1.2 (time decay rates). *Assume that all the conditions listed in Theorem* 1.1 *are valid. Define $l \vee 1 = \max\{l, 1\}$ with $l \geq 0$. Let*

$$
Y_{N,l} = \sqrt{\mathcal{E}_{N,l\vee 1}(0)} + \|[\phi_0, \nabla_x \phi_0, \phi_1]\|_{H^N \cap L^1} + \|f_0\|_{L^2_p L^1_x} \qquad (1.23)
$$

be small enough; then when $N \geq 8$,

$$
\mathcal{E}_{N-2,l}(t) \lesssim (1+t)^{-3/2} Y_{N,l}^2 \qquad (1.24)
$$

holds true for any $t \geq 0$, and when $N \geq 11$,

$$
\mathcal{E}^h_{N-7,l}(t) \lesssim (1+t)^{-5/2} Y_{N,l}^2 \qquad (1.25)
$$

holds true for any $t \geq 0$.

Compared with other Vlasov-type systems that couple with Maxwell or Poisson equations to the kinetic equations such as the Vlasov–Maxwell/Poisson–Fokker–Planck system [11, 13, 16], the VNFP system under consideration demonstrates two significantly different features. One feature is that either the convection term or the diffusion term in the Vlasov–Fokker–Planck equation (1.1) depends on the unknown scalar gravitational field $\phi(t,x)$, and the other is that the gravitational field is governed by the wave equation. The former makes much trouble for treating the classical solutions since the the derivatives will go to the gravitational field when acting higher-order space differentiation, and the latter creates difficulty deducing the dissipation of the gravitational field because the energy is generally conserved for the wave equation. To overcome the first difficulty, we set the perturbation around the local relativistic Maxwellian $J_\phi = C_J e^{-\sqrt{e^{2\phi}+|p|^2}}$ which satisfies the identity

$$
\nabla_p \sqrt{e^{2\phi}+|p|^2} \cdot \nabla_x J_\phi - \nabla_x \sqrt{e^{2\phi}+|p|^2} \cdot \nabla_p J_\phi = 0
$$

so as to reduce the convection term and kill the extra linear term involving the gravitational field. To cope with the second difficulty, we use the higher-order moment method to deduce the independent first-order equations for the macroscopic component and gravitational field in terms of the nonfluid component through the linearized VNFP system, respectively. Moreover, we expand the Lorentzian metric coefficient e^{ϕ} up to the second order to obtain the Klein–Gordon-type equation, which gives rise to the dissipation of the higher-order spatial derivatives of the gravitational field.

The rest of the paper is arranged as follows. In section 2, we carry out the analysis of the linearized relativistic VNFP system so as to obtain the dissipation structure of both the fluid component and the gravitational field. The proof of Theorem 1.1 and Theorem 1.2 is presented in sections 3 and 4, respectively. Finally, the local-in-time existence of the Cauchy problem (1.7), (1.9), and (1.10) is sketched in section 5.

Notations. Throughout the paper, C denotes some generic positive (generally large) constant and λ denotes some generic positive (generally small) constant, where both C and λ may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leqslant CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. We use $|\cdot|_2$ to denote the L^2-norm in $\mathbb{R}^3_p$ and $\|\cdot\|$ to denote the L^2-norm in $\mathbb{R}^3_p \times \mathbb{R}^3_x$ or $\mathbb{R}^3_x$. We also use $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ to denote the inner product over $L^2_{x,p}$ and L^2_p, respectively. We also use $\|\cdot\|_{H^N}$ to stand for the standard Sobolev norm in $\mathbb{R}^3$ with respect to the x-variable.

2. Linear analysis. This section is devoted to obtaining the dissipative estimates of the macroscopic degenerate part $a(t,x)$ and the gravitational field $\phi(t,x)$ through the linearized equations of (1.7) and (1.9). Specifically we intend to capture the dissipation of $[a,\phi](t,x)$ in terms of its higher-order derivatives; see Lemma 2.2 later on.

First of all, we introduce four positive constants for the later use:

$$\int_{\mathbb{R}^3} J\,\mathrm{d}p = \mathcal{C}_1, \quad \int_{\mathbb{R}^3} \frac{|p|^2}{p_0} J\,\mathrm{d}p = \mathcal{C}_2, \quad \int_{\mathbb{R}^3} |p|^2 J\,\mathrm{d}p = \mathcal{C}_3, \quad \int_{\mathbb{R}^3} \frac{p_0+1}{p_0^3} J\,\mathrm{d}p = \mathcal{C}_4.$$

We need to consider the energy dissipation structure of the following linear inhomogeneous equations corresponding to (1.7) and (1.9):

$$\partial_t f + \frac{p}{p_0}\cdot\nabla_x f - \frac{\partial_t\phi}{p_0} J^{1/2} + Lf = h \tag{2.1}$$

and

$$(\Box + 2 - \mathcal{C}_4)\phi = -\int_{\mathbb{R}^3} \frac{J^{1/2} f}{p_0}\,\mathrm{d}p + g, \tag{2.2}$$

where $h = h(t,x,p)$ and $g = g(t,x)$ are given source terms.

Remark 2.1. Notice $\mathcal{C}_4 < 2$, so $\Box + 2 - \mathcal{C}_4$ in the left-hand side of (2.2) is a Klein–Gordon operator. Indeed, it is direct to check that $2 - \frac{p_0+1}{p_0^2} \geq 0$, where the equality holds true if and only if $p = 0$. Therefore, it holds by (1.5) that

$$2 - \mathcal{C}_4 = \int_{\mathbb{R}^3} \left(2 - \frac{p_0+1}{p_0^2}\right) \frac{J}{p_0}\,\mathrm{d}p > 0.$$

Remark 2.2. Taking the inner product of (2.1) with

$$\psi_0 = J^{1/2}, \quad \psi_1 = \left(|p|^2 - \frac{\mathcal{C}_3}{\mathcal{C}_1}\right) J^{1/2}, \quad \psi_2 = pJ^{1/2}$$

with respect to p, respectively, one has

$$\mathcal{C}_1\partial_t a - \partial_t\phi + \nabla_x \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}f, \psi_0\right\rangle = \langle h, \psi_0\rangle, \tag{2.3}$$

$$\left(\mathcal{C}_2 - \frac{\mathcal{C}_3}{\mathcal{C}_1}\right)\partial_t\phi + \langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}f, \psi_1\rangle + \nabla_x \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}f, \psi_1\right\rangle = \langle h - Lf, \psi_1\rangle, \tag{2.4}$$

and

$$\langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}f, \psi_2\rangle + \frac{\mathcal{C}_2}{3}\nabla_x a + \nabla_x \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}f, \psi_2\right\rangle = \langle h - Lf, \psi_2\rangle. \tag{2.5}$$

THEOREM 2.1. *Let $[f(t,x,p), \phi(t,x)]$ be the solution to the linear homogeneous Cauchy problem* (2.1) *and* (2.2) *for $h = g = 0$ supplemented with initial data $[f_0(x,p), \phi_0(x), \phi_1(x)]$. Then, denoting $U(t) := [f, \phi, \nabla_x\phi, \partial_t\phi]$, for $m \geq 0$ and $j \geq 0$ it holds that*

$$\begin{aligned} \|\nabla_x^m U(t)\| \lesssim{}& (1+t)^{-\frac{3}{4}-\frac{m}{2}}\left(\|f_0\|_{L^2_pL^1_x} + \|[\phi_0, \phi_1, \nabla_x\phi_0]\|_{L^1_x}\right) \\ &+ (1+t)^{-\frac{j}{2}}\|\nabla_x^{m+j}[f_0, \phi_0, \phi_1, \nabla_x\phi_0]\| \end{aligned} \tag{2.6}$$

for any $t \geq 0$.

Proof. In what follows, $\hat{g}(\xi)$ for any $g = g(x)$ denotes the Fourier transform with respect to the x-variable. $\Re z$ denotes the real part of a complex number z. For two complex vectors $z_1, z_2 \in \mathbb{C}^3$, $(z_1|z_2) = z_1 \cdot \overline{z_2}$ denotes the dot product on the complex field $\mathbb{C}$, where $\overline{z_2}$ is the complex conjugate of z_2. Setting $h = g = 0$, the Fourier transforms of (2.1), (2.2), (2.3), (2.4), and (2.5) in the x-variable, respectively, give that

$$\partial_t\hat{f} + \frac{i\xi\cdot p}{p_0}\hat{f} - \frac{\partial_t\hat{\phi}}{p_0}J^{1/2} + L\hat{f} = 0, \tag{2.7}$$

$$\left(\partial_t^2 + |\xi|^2 + 2 - \mathcal{C}_4\right)\hat{\phi} = -\int_{\mathbb{R}^3}\frac{J^{1/2}\hat{f}}{p_0}\,\mathrm{d}p, \tag{2.8}$$

$$\mathcal{C}_1\partial_t\hat{a} - \partial_t\hat{\phi} + i\xi \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_0\right\rangle = 0, \tag{2.9}$$

$$\left(\mathcal{C}_2 - \frac{\mathcal{C}_3}{\mathcal{C}_1}\right)\partial_t\hat{\phi} + \left\langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_1\right\rangle + i\xi \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_1\right\rangle = -\langle L\hat{f}, \psi_1\rangle, \tag{2.10}$$

and

$$\left\langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_2\right\rangle + \frac{\mathcal{C}_2}{3}i\xi\hat{a} + i\xi \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_2\right\rangle = -\langle L\hat{f}, \psi_2\rangle. \tag{2.11}$$

We now deduce the Fourier energy estimates. First of all, the basic energy estimate on (2.7) and (2.8) together yields

$$\partial_t\left[|\hat{f}|^2_{L^2_p} + |\partial_t\hat{\phi}|^2 + |\xi|^2|\hat{\phi}|^2 + (2-\mathcal{C}_4)|\hat{\phi}|^2\right] + \lambda|\{\mathbf{I}-\mathbf{P}_0\}\hat{f}|^2_D \leq 0, \tag{2.12}$$

where we recall $\mathcal{C}_4 < 2$ by Remark 2.1 and the coercivity inequality (1.14) has been used. Next, (2.11) together with (2.9) gives rise to

$$\partial_t \Re\left(\left\langle \{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_2\right\rangle | i\xi\hat{a}\right) + \lambda|\xi|^2|\hat{a}|^2 \le \eta|\xi|^2|\partial_t\hat{\phi}|^2 + C_\eta(1+|\xi|^2)|\{\mathbf{I}-\mathbf{P}_0\}\hat{f}|_D^2, \tag{2.13}$$

where $\eta > 0$ is arbitrary to be chosen later. Similarly, from (2.10) and (2.8), one has

$$\begin{aligned}\partial_t \Re\left(\xi\left\langle \{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_1\right\rangle | \xi\partial_t\hat{\phi}\right) + \lambda|\xi|^2|\partial_t\hat{\phi}|^2 &\le \eta|\xi|^2(|\xi|^2+1)|\hat{\phi}|^2 + \eta|\xi|^2|\hat{a}|^2 \\ &\quad + C_\eta|\xi|^2(1+|\xi|^2)|\{\mathbf{I}-\mathbf{P}_0\}\hat{f}|_D^2.\end{aligned} \tag{2.14}$$

To further obtain the dissipation of $|\xi|^2|\hat{\phi}|^2$, we get from the Klein–Gordon-type equation (2.8) that

$$\partial_t \Re\left(\xi\partial_t\hat{\phi}|\xi\hat{\phi}\right) + \lambda|\xi|^2(|\xi|^2+1)|\hat{\phi}|^2 \le C|\xi|^2|\hat{a}|^2 + C|\xi|^2|\{\mathbf{I}-\mathbf{P}_0\}\hat{f}|_D^2 + C|\xi|^2|\partial_t\hat{\phi}|^2, \tag{2.15}$$

where we again have used the fact that $2-\mathcal{C}_4 > 0$. Letting $0 < \kappa_3 \ll \kappa_1, \kappa_2 \ll 1$ and $\eta > 0$ be suitably small, taking the linear combination by

$$(2.12) + \frac{1}{(1+|\xi|^2)^2}\left(\kappa_1 \times (2.13) + \kappa_2 \times (2.14) + \kappa_3 \times (2.15)\right),$$

and adjusting the coefficients, one has

$$\partial_t E(f,\phi)(t,\xi) + \frac{\lambda|\xi|^2}{(1+|\xi|^2)^2}\left|[\hat{a}, \partial_t\hat{\phi}]\right|^2 + \frac{\lambda|\xi|^2}{1+|\xi|^2}|\hat{\phi}|^2 + \lambda|\{\mathbf{I}-\mathbf{P}_0\}\hat{f}|_D^2 \le 0, \tag{2.16}$$

where

$$\begin{aligned}E(f,\phi)(t,\xi) = \left[|\hat{f}|^2_{L^2_p} + |\partial_t\hat{\phi}|^2 + |\xi|^2|\hat{\phi}|^2 + (2-\mathcal{C}_4)|\hat{\phi}|^2\right] + \kappa_1\frac{\Re\left(\left\langle \{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_2\right\rangle | i\xi\hat{a}\right)}{(1+|\xi|^2)^2} \\ + \kappa_2\frac{\Re\left(\xi\left\langle \{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_1\right\rangle | \xi\partial_t\hat{\phi}\right)}{(1+|\xi|^2)^2} + \kappa_3\frac{\Re\left(\xi\partial_t\hat{\phi}|\xi\hat{\phi}\right)}{(1+|\xi|^2)^2}.\end{aligned}$$

It is straightforward to see that

$$E(f,\phi)(t,\xi) \sim |\hat{f}|^2_{L^2_p} + |\partial_t\hat{\phi}|^2 + |\xi|^2|\hat{\phi}|^2 + |\hat{\phi}|^2.$$

Since it holds by (1.17) that $|\{\mathbf{I}-\mathbf{P}_0\}\hat{f}|_D \gtrsim |\{\mathbf{I}-\mathbf{P}_0\}\hat{f}|_{L^2_p}$, one then gets from (2.16) that

$$\partial_t E(f,\phi)(t,\xi) + \lambda\frac{|\xi|^2}{(1+|\xi|^2)^2}E(f,\phi)(t,\xi) \le 0,$$

which further implies (2.6); cf. [7, 8, 12]. This finishes the proof of Theorem 2.1. ◻

As a by-product of the proof of Theorem 2.1, we can also obtain the following dissipation estimates in the physical space for the macroscopic component $a(t,x)$ and the scalar gravitational field $\phi(t,x)$.

LEMMA 2.2. *Let h and g satisfy $\langle \partial^\alpha h, \psi_i\rangle \in L^2((0,\infty)\times\mathbb{R}^3)$ $(i=0,1,2)$ and $\partial^\alpha g \in L^2((0,\infty)\times\mathbb{R}^3)$ with $|\alpha|\le N-1$, respectively. For a smooth solution $[f(t,x,p),\phi(t,x)]$ of (2.1) and (2.2), there exists an interaction functional $\mathcal{E}^{\rm int}$ with*

$$|\mathcal{E}^{\rm int}| \le C\sum_{|\alpha|\le N}\left\{\|\partial^\alpha f\|^2+\|\partial_t\partial^\alpha\phi\|^2+\|\nabla_x\partial^\alpha\phi\|^2+\|\partial^\alpha\phi\|^2\right\} \tag{2.17}$$

for a generic constant $C>0$ such that it holds for any $t\ge 0$ that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^{\rm int}+\lambda\sum_{|\alpha|\le N-1}\|\partial^\alpha\nabla_x[a,\phi]\|^2+\lambda\sum_{|\alpha|\le N-2}\|\partial^\alpha\nabla_x\partial_t\phi\|^2\\ &\quad\lesssim \sum_{|\alpha|\le N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2+\sum_{i=0}^2\sum_{|\alpha|\le N-1}\|\langle\partial^\alpha h,\psi_i\rangle\|^2+\sum_{|\alpha|\le N-2}\|\partial^\alpha\nabla_x g\|^2.\end{aligned} \tag{2.18}$$

Proof. Performing similar calculations as for obtaining (2.13), (2.14), and (2.15), one has

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le N-1}\left(\partial^\alpha\langle\{\mathbf{I}-\mathbf{P}_0\}f,\psi_2\rangle,\nabla_x\partial^\alpha a\right)+\lambda\sum_{|\alpha|\le N-1}\|\nabla_x\partial^\alpha a\|^2\\ &\quad\le \eta\sum_{|\alpha|\le N-2}\|\nabla_x\partial^\alpha\partial_t\phi\|^2+C_\eta\sum_{|\alpha|\le N-1}\|\partial^\alpha[\{\mathbf{I}-\mathbf{P}_0\}f,\nabla_x\{\mathbf{I}-\mathbf{P}_0\}f]\|^2\\ &\qquad+C_\eta\sum_{|\alpha|\le N-1}\|\langle\partial^\alpha h-L\partial^\alpha f,\psi_2\rangle\|^2+C_\eta\sum_{|\alpha|\le N-1}\|\langle\partial^\alpha h,\psi_0\rangle\|^2,\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le N-2}\left(\nabla_x\partial^\alpha\langle\{\mathbf{I}-\mathbf{P}_0\}f,\psi_1\rangle,\nabla_x\partial^\alpha\partial_t\phi\right)+\lambda\sum_{|\alpha|\le N-2}\|\nabla_x\partial^\alpha\partial_t\phi\|^2\\ &\quad\le \eta\sum_{|\alpha|\le N-2}\|\nabla_x[\nabla_x\partial^\alpha\phi,\partial^\alpha\phi,\partial^\alpha a]\|^2\\ &\qquad+C_\eta\sum_{|\alpha|\le N-2}\|\partial^\alpha[\nabla_x\{\mathbf{I}-\mathbf{P}_0\}f,\nabla_x^2\{\mathbf{I}-\mathbf{P}_0\}f]\|^2\\ &\qquad+C_\eta\sum_{|\alpha|\le N-2}\|\langle\partial^\alpha\nabla_x h-L\nabla_x\partial^\alpha f,\psi_1\rangle\|^2+C_\eta\sum_{|\alpha|\le N-2}\|\partial^\alpha\nabla_x g\|^2,\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\le N-2}\left(\nabla_x\partial^\alpha\partial_t\phi,\nabla_x\partial^\alpha\phi\right)+\lambda\sum_{|\alpha|\le N-2}\|\nabla_x[\nabla_x\partial^\alpha\phi,\partial^\alpha\phi]\|^2\\ &\quad\le C\sum_{|\alpha|\le N-2}\|\nabla_x\partial^\alpha a\|^2+C\sum_{|\alpha|\le N-2}\|\partial^\alpha\nabla_x\{\mathbf{I}-\mathbf{P}_0\}f\|^2\\ &\qquad+C\sum_{|\alpha|\le N-2}\|\partial^\alpha\nabla_x g\|^2+C\sum_{|\alpha|\le N-2}\|\nabla_x\partial^\alpha\partial_t\phi\|^2.\end{aligned} \tag{2.21}$$

Now, choosing $0<\kappa_4\ll 1$ and defining

$$\begin{aligned}\mathcal{E}^{\rm int}&=\sum_{|\alpha|\le N-1}\left(\partial^\alpha\langle\{\mathbf{I}-\mathbf{P}_0\}f,\psi_2\rangle,\nabla_x\partial^\alpha a\right)\\ &\quad+\sum_{|\alpha|\le N-2}\left(\nabla_x\partial^\alpha\langle\{\mathbf{I}-\mathbf{P}_0\}f,\psi_1\rangle,\nabla_x\partial^\alpha\partial_t\phi\right)+\kappa_4\sum_{|\alpha|\le N-2}\left(\nabla_x\partial^\alpha\partial_t\phi,\nabla_x\partial^\alpha\phi\right),\end{aligned} \tag{2.22}$$

the desired estimate (2.18) follows from the summation of (2.19), (2.20), and $\kappa_4 \times$ (2.21). It is also direct to check that (2.17) holds true. This ends the proof of Lemma 2.2. ∎

3. Global existence. In this section, we shall complete the proof of Theorem 1.1 for the global existence of solutions to the Cauchy problem (1.7), (1.9), and (1.10). Actually, the global existence follows from the a priori energy estimates and the usual continuity argument based on the local existence. Here, we first focus on the desired energy estimate (1.22) under the a priori assumption that

$$\sup_{0\le t\le T} \mathcal{E}_{N,l}(t) \le \epsilon^2 \tag{3.1}$$

for a suitably small constant $\epsilon > 0$, where $T > 0$ is arbitrary and the energy functional $\mathcal{E}_{N,l}(\cdot)$ is understood as its equivalent form on the right-hand side of (1.19), while the local existence will be sketched in section 5 later. We begin the proof with the following proposition.

Proposition 3.1. *Let* $[f(t,x,p), \phi(t,x)]$ *be a classical solution over* $0 \le t \le T$ *for the Cauchy problem* (1.7), (1.9), *and* (1.10) *satisfying the assumption* (3.1); *then there exists the temporal energy functional* $\mathcal{E}_{N,l}(\cdot)$ *in terms of* (1.19) *such that the estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{N,l}(t) + \lambda \mathcal{D}_{N,l}(t) \le 0 \tag{3.2}$$

holds true for all $0 \le t \le T$, *where* $N \ge 4$, $l \ge 0$, *and* $\mathcal{D}_{N,l}(\cdot)$ *is defined in* (1.21).

Proof. For convenience of presentation, we divide the proof into four steps. First of all, for later use we rewrite (1.7) and (1.9), respectively, as

(3.3)
$$\begin{aligned}
&\partial_t f + \nabla_p \sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x f - \nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p f - \frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi - J)J^{-1/2}\\
&\quad - \frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}J^{1/2} + \frac{1}{2}\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f + e^{2\phi}Lf = e^{2\phi}(Lf - L_\phi f)
\end{aligned}$$

and

(3.4)
$$\Box\phi = -e^{2\phi}\int_{\mathbb{R}^3}\left(\frac{J_\phi}{\sqrt{e^{2\phi}+|p|^2}} - \frac{J}{\sqrt{1+|p|^2}}\right)\mathrm{d}p - e^{2\phi}\int_{\mathbb{R}^3}\frac{J^{1/2}f}{\sqrt{e^{2\phi}+|p|^2}}\,\mathrm{d}p + 1 - e^{2\phi}.$$

Note that the following fact will be frequently used:

$$\sup_{0\le t\le T}\|\phi(t)\|_{L^\infty} \le C\epsilon, \tag{3.5}$$

which holds true due to the a priori assumption (3.1).

Step 1. *Zero order energy estimate.* Taking the inner product of (3.3) and f with respect to (x,p) over $\mathbb{R}^3\times\mathbb{R}^3$ and applying the coercivity estimate (1.14), we get

(3.6)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 - \left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}J^{1/2}, f\right) - \left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi - J)J^{-1/2}, f\right) + \frac{1}{2}\left(\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f, f\right) + \delta\|\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2 \le e^{2\phi}(Lf - L_\phi f, f),$$

where the identity

$$\left(\nabla_p\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x f - \nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p f, f\right) = 0$$

and the fact that $e^{2\phi} \ge e^{-2C\epsilon}$ have been used. In what follows we estimate each inner product term in (3.6).

First, the second term in the left-hand side of (3.6) may give rise to the energy of the scalar gravitational field ϕ by making use of the Klein–Gordon-type equation (3.4). To see this, by Taylor expansion it holds that

$$\begin{cases} e^{2\phi} = 1 + 2e^{2\theta_1\phi}\phi,\ \theta_1\in(0,1);\ e^{2\phi} = 1 + 2\phi + 2e^{2\theta_2\phi}\phi^2,\ \theta_2\in(0,1), \\ \dfrac{J_\phi}{\sqrt{e^{2\phi}+|p|^2}} - \dfrac{J}{\sqrt{1+|p|^2}} = -\dfrac{1+p_0}{p_0^3}J\phi + \dfrac{1}{2}\partial_{zz}\left(\dfrac{J_z}{\sqrt{e^{2z}+|p|^2}}\right)\Big|_{z=\theta_3\phi}\phi^2, \\ \theta_3\in(0,1). \end{cases} \tag{3.7}$$

Applying (3.7) and (3.4), it follows that

$$\begin{aligned} &\left(-\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}J^{1/2}, f\right) \\ &= \left(\Box\phi + e^{2\phi}\int_{\mathbb{R}^3}\left(\frac{J_\phi}{\sqrt{e^{2\phi}+|p|^2}} - \frac{J}{\sqrt{1+|p|^2}}\right)\mathrm{d}p - (1-e^{2\phi}), \partial_t\phi\right) \\ &= ((\Box + 2 - \mathcal{C}_4)\phi, \partial_t\phi) + (\mathcal{H}_1(\phi,p)\phi^2, \partial_t\phi) + (\mathcal{H}_2(\phi,p)\phi^3, \partial_t\phi). \end{aligned} \tag{3.8}$$

Here we have denoted

$$\mathcal{H}_1 = 2e^{2\theta_2\phi} - 2e^{2\theta_1\phi}\frac{1+p_0}{p_0^3}J + \frac{1}{2}\partial_{zz}\left(\frac{J_z}{\sqrt{e^{2z}+|p|^2}}\right)\Big|_{z=\theta_3\phi},$$
$$\mathcal{H}_2 = e^{2\theta_1\phi}\partial_{zz}\left(\frac{J_z}{\sqrt{e^{2z}+|p|^2}}\right)\Big|_{z=\theta_3\phi},$$

which are both bounded functions. With the above estimate in hand, we may rewrite the second term in the right-hand side of (3.8) as

$$(\mathcal{H}_1(\phi,p)\phi^2, \partial_t\phi) = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{H}_1(\phi,p), \phi^3) - \frac{1}{3}(\partial_t\mathcal{H}_1(\phi,p), \phi^3). \tag{3.9}$$

Next, both the second term in the right-hand side of the above identity and the third term in the right-hand side of (3.8) can be bounded by

$$\le C\|\partial_t\phi\|_{L^6}\|\phi\|_{L^6}\|\phi\|_{L^3}^2 \le C\epsilon\|\nabla_x\partial_t\phi\|^2 + C\epsilon\|\nabla_x\phi\|^2, \tag{3.10}$$

according to Hölder's inequality and Sobolev's inequality $\|u\|_{L^6(\mathbb{R}^3)} \le C\|\nabla_x u\|_{L^2(\mathbb{R}^3)}$. Consequently, (3.8) has a lower bound by

(3.11)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\partial_t\phi\|^2+\|\nabla_x\phi\|^2+(2-\mathcal{C}_4)\|\phi\|^2+\frac{2}{3}(\mathcal{H}_1(\phi,p),\phi^3)\right\}-C\epsilon(\|\nabla_x\partial_t\phi\|^2+\|\nabla_x\phi\|^2).$$

Next, we turn to estimate the third term in the left-hand side of (3.6). Here, the key point is that following the same spirit as for treating (3.9), one may convert the cubic term into the quadratic one with the aid of (2.3). To do this, we first write the third term in the left-hand side of (3.6) as

(3.12)
$$\begin{aligned}&\left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi-J)J^{-1/2},f\right)\\ &\quad=\left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi-J),a\right)+\left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi-J)J^{-1/2},\{\mathbf{I}-\mathbf{P}_0\}f\right).\end{aligned}$$

We only estimate the first term in the right-hand side of the above identity (3.12), since the second one can be easily handled. In light of Taylor expansion, one has

$$\left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi-J),a\right)=\left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}\left(\frac{-J}{p_0}\phi+\frac{1}{2}\partial_{zz}J_z\Big|_{z=\theta_4\phi}\phi^2\right),a\right),\ \theta_4\in(0,1).$$

The term corresponding to the second-order expansion is already quadratic, and thus it can be controlled in the same way as (3.10). Corresponding to the first-order expansion, we have

$$\begin{aligned}&-\left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0}\phi,a\right)\\ &\quad=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{2\phi}\phi^2}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0},a\right)+\left(\partial_t\left(\frac{\frac{1}{2}e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\right)\frac{J}{p_0}\phi^2,a\right)\\ &\qquad+\left(\frac{\frac{1}{2}e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0}\phi^2,\partial_t a\right)\\ &\quad=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{2\phi}\phi^2}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0},a\right)+\frac{1}{2}\left(\partial_t\left(\frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\right)\frac{J}{p_0}\phi^2,a\right)\\ &\qquad+\frac{1}{2\mathcal{C}_1}\left(\frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0}\phi^2,\partial_t\phi-\left\langle\frac{p}{p_0}\cdot\nabla_x\{\mathbf{I}-\mathbf{P}_0\}f,\psi_0\right\rangle+\langle h,\psi_0\rangle\right),\end{aligned}$$

where the last identity follows from (2.3) and h is given by

(3.13)
$$\begin{aligned}h=&-\nabla_p\left(\sqrt{e^{2\phi}+|p|^2}-\sqrt{1+|p|^2}\right)\cdot\nabla_x f+\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p f+\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}J_\phi J^{-1/2}\\ &-\frac{\partial_t\phi}{\sqrt{1+|p|^2}}J^{1/2}-\frac{1}{2}\frac{\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot p}{\sqrt{e^{2\phi}+|p|^2}}f+(Lf-e^{2\phi}L_\phi f).\end{aligned}$$

Therefore, by applying Sobolev's inequality, the a priori assumption (3.1), and Cauchy–Schwarz's inequality with $\epsilon>0$, we have

$$\left|-\left(\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0}\phi, a\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{2\phi}\phi^2}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0}, a\right)-\frac{1}{6\mathcal{C}_1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0}\phi^2,\phi\right)\right|$$
$$\lesssim \epsilon\|[\nabla_x\partial_t\phi,\nabla_x\phi]\|^2+\epsilon\|\nabla_x f\|^2+\epsilon\|\langle h,\psi_0\rangle\|^2.$$

The fourth term in the left-hand side of (3.6) can be bounded by

$$\le \epsilon\|\nabla_x\phi\|^2 + C\epsilon\|\nabla_x f\|^2 \tag{3.14}$$

by directly applying Sobolev's inequality, the a priori assumption (3.1), and Cauchy–Schwarz's inequality with $\epsilon>0$.

Finally we deal with the right-hand term of (3.6). We write that

$$\begin{aligned} e^{2\phi}(Lf-L_\phi f,f) &= e^{2\phi}\left((\Lambda_\phi-\Lambda_0)\nabla_p(J^{1/2}f),\nabla_p\left(\frac{\{\mathbf{I}-\mathbf{P}_0\}f}{J^{1/2}}\right)\right)\\ &= e^{2\phi}\left((\Lambda_\phi-\Lambda_0)\nabla_p(J^{1/2})f,\nabla_p(J^{-1/2})\{\mathbf{I}-\mathbf{P}_0\}f\right)\\ &\quad + e^{2\phi}\left((\Lambda_\phi-\Lambda_0)\nabla_p(J^{1/2})f,J^{-1/2}\nabla_p\{\mathbf{I}-\mathbf{P}_0\}f\right)\\ &\quad + e^{2\phi}\left((\Lambda_\phi-\Lambda_0)\nabla_p f,\nabla_p\{\mathbf{I}-\mathbf{P}_0\}f\right)\\ &\quad + e^{2\phi}\left((\Lambda_\phi-\Lambda_0)J^{1/2}\nabla_p f,\nabla_p(J^{-1/2})\{\mathbf{I}-\mathbf{P}_0\}f\right). \end{aligned} \tag{3.15}$$

Note by $\Lambda_\phi-\Lambda_0=\partial_z\Lambda_z|_{z=\theta_5\phi}\phi$ for $\theta_5\in(0,1)$ that

$$(\Lambda_\phi-\Lambda_0)_{ij}=\frac{2e^{2\theta_5\phi}(e^{2\theta_5\phi}+|p|^2)\delta_{ij}-e^{2\theta_5\phi}(e^{2\theta_5\phi}\delta_{ij}+p_ip_j)}{(e^{2\theta_5\phi}+|p|^2)^{3/2}}\phi, \tag{3.16}$$

which hence implies that $|\Lambda_\phi-\Lambda_0|\lesssim|\phi|/p_0$. Therefore, applying this estimate to (3.15), it follows that

$$|e^{2\phi}(Lf-L_\phi f,f)|\lesssim\epsilon\left\|p_0^{-1/2}\nabla_p\{\mathbf{I}-\mathbf{P}_0\}f\right\|^2+\epsilon\|\{\mathbf{I}-\mathbf{P}_0\}f\|^2+\epsilon\|\nabla_x f\|^2, \tag{3.17}$$

according to Sobolev's inequality and the a priori assumption (3.1).

Putting the estimates (3.6), (3.11), (3.14), and (3.17) together, we finally arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^0+\lambda\|\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2\lesssim\epsilon\mathcal{D}_{N,l}(t)+\epsilon\|\langle h,\psi_0\rangle\|^2 \tag{3.18}$$

with

$$\begin{aligned}\mathcal{E}^0 &= \|f\|^2+\|\partial_t\phi\|^2+\|\nabla_x\phi\|^2+(2-\mathcal{C}_4)\|\phi\|^2+\frac{2}{3}(\mathcal{H}_1(\phi,p),\phi^3)\\ &\quad+\frac{1}{2}\left(\frac{e^{2\phi}\phi^2}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0},a\right)-\frac{1}{6\mathcal{C}_1}\left(\frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\frac{J}{p_0}\phi^2,\phi\right),\end{aligned} \tag{3.19}$$

where h is given in (3.13). This completes the first step on the basic energy estimate.

Step 2. Energy estimates with pure space derivatives. Let $0<|\alpha|\le N$; we act ∂^α to (3.3), take the inner product of the resultant identity with $w^{2l}\partial^\alpha f$ over $\mathbb{R}^3\times\mathbb{R}^3$, and apply the coercivity estimate (1.14) to obtain

(3.20)

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha f\|^2 \underbrace{-\left(\frac{e^{2\phi}\partial_t\partial^\alpha\phi}{\sqrt{e^{2\phi}+|p|^2}}J^{1/2},\partial^\alpha f\right)}_{I_1} \\
&\underbrace{-\sum_{\substack{\alpha_1>0\\ \alpha_1+\alpha_2=\alpha}} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\left[\frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\right]\partial_t\partial^{\alpha_2}\phi J^{1/2},\partial^\alpha f\right)}_{I_2} \\
&\underbrace{-\sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\left[\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}\right]\partial^{\alpha_2}(J_\phi-J)J^{-1/2},\partial^\alpha f\right)}_{I_3} \\
&\underbrace{+\sum_{\alpha_1+\alpha_2=\alpha,\alpha_1>0} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\nabla_p\sqrt{e^{2\phi}+|p|^2}\cdot\partial^{\alpha_2}\nabla_x f-\partial^{\alpha_1}\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\partial^{\alpha_2}\nabla_p f,\partial^\alpha f\right)}_{I_4} \\
&\underbrace{+\frac{1}{2}\sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\partial^{\alpha_2}\left(\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f\right),\partial^\alpha f\right)}_{I_5}+\delta\|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2 \\
&\le \underbrace{\sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\partial^{\alpha_2}(Lf-L_\phi f),\partial^\alpha f\right)}_{I_6}\underbrace{-\sum_{\substack{\alpha_1>0\\ \alpha_1+\alpha_2=\alpha}} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})L\partial^{\alpha_2}f,\partial^\alpha f\right)}_{I_7}.
\end{aligned}
$$

We now compute $I_k (1 \le k \le 7)$ term by term. To treat I_1, using an argument similar to that used to obtain (3.8), one has

(3.21)

$$
\begin{aligned}
I_1 &= \left(\Box\partial^\alpha\phi+\partial^\alpha\left[e^{2\phi}\int_{\mathbb{R}^3}\left(\frac{J_\phi}{\sqrt{e^{2\phi}+|p|^2}}-\frac{J}{\sqrt{1+|p|^2}}\right)\mathrm{d}p-(1-e^{2\phi})\right],\partial_t\partial^\alpha\phi\right) \\
&= ((\Box+2-\mathcal{C}_4)\partial^\alpha\phi,\partial_t\partial^\alpha\phi)+(\partial^\alpha(\mathcal{H}_1(\phi,p)\phi^2),\partial_t\partial^\alpha\phi)+(\partial^\alpha(\mathcal{H}_2(\phi,p)\phi^3),\partial_t\partial^\alpha\phi).
\end{aligned}
$$

By Sobolev's inequality $\|u\|_{L^\infty} \le C\|\nabla_x u\|_{H^1}$, both of the last two terms in the right-hand side of the above identity can be controlled by

$$\lesssim \|\partial_t\partial^\alpha\phi\|\mathcal{D}_{N,l}(t) \lesssim \epsilon\mathcal{D}_{N,l}(t).$$

It then follows from (3.21) that

$$I_1+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\partial_t\partial^\alpha\phi\|^2+\|\nabla_x\partial^\alpha\phi\|^2+(2-\mathcal{C}_4)\|\partial^\alpha\phi\|^2\right\} \lesssim \epsilon\mathcal{D}_{N,l}(t).$$

For I_2 and I_3, applying again Sobolev's inequality and Hölder's inequality, we have

$$
\begin{aligned}
|I_2| \le C &\sum_{|\alpha_2|\le|\alpha|/2,\alpha_1>0}\|\partial^\alpha f\|\|\partial_t\partial^{\alpha_2}\phi\|_{L^6}\left\|\partial^{\alpha_1}\left[e^{2\phi}(e^{2\phi}+|p|^2)^{-1/2}\right]\right\|_{L^3} \\
&+C\sum_{|\alpha_1|\le|\alpha|/2,\alpha_1>0}\|\partial^\alpha f\|\|\partial_t\partial^{\alpha_2}\phi\|\left\|\partial^{\alpha_1}\left[e^{2\phi}(e^{2\phi}+|p|^2)^{-1/2}\right]\right\|_{L^\infty}
\end{aligned}
$$

$$\leq C \sum_{|\alpha_2|\leq|\alpha|/2,\alpha_1>0} \|\partial^\alpha f\|\|\partial_t\partial^{\alpha_2}\phi\|_{H^1} \left\|\partial^{\alpha_1}\left[e^{2\phi}(e^{2\phi}+|p|^2)^{-1/2}\right]\right\|_{H^1}$$
$$+ C \sum_{|\alpha_1|\leq|\alpha|/2,\alpha_1>0} \|\partial^\alpha f\|\|\partial_t\partial^{\alpha_2}\phi\| \left\|\partial^{\alpha_1}\left[e^{2\phi}(e^{2\phi}+|p|^2)^{-1/2}\right]\right\|_{H^2} \leq C\epsilon D_N(t)$$

and

$$|I_3| \leq C \sum_{|\alpha_1|\leq|\alpha|/2} \|\partial^\alpha f\| \left\|\partial^{\alpha_1}\left[e^{2\phi}\partial_t\phi(e^{2\phi}+|p|^2)^{-1/2}\right]\right\|_{L^3} \left\|\partial^{\alpha_2}(J_\phi - J)J^{-1/2}\right\|_{L^6}$$
$$+ C \sum_{|\alpha_2|\leq|\alpha|/2} \|\partial^\alpha f\| \left\|\partial^{\alpha_1}\left[e^{2\phi}\partial_t\phi(e^{2\phi}+|p|^2)^{-1/2}\right]\right\| \left\|\partial^{\alpha_2}(J_\phi - J)J^{-1/2}\right\|_{L^\infty}$$
$$\leq C \sum_{|\alpha_1|\leq|\alpha|/2} \|\partial^\alpha f\|\|\partial_t\partial^{\alpha_2}\phi\|_{H^1} \left\|\partial^{\alpha_1}\left[e^{2\phi}(e^{2\phi}+|p|^2)^{-1/2}\right]\right\|_{H^1}$$
$$+ C \sum_{|\alpha_2|\leq|\alpha|/2} \|\partial^\alpha f\|\|\partial_t\partial^{\alpha_2}\phi\| \left\|\nabla_x\partial^{\alpha_1}\left[e^{2\phi}(e^{2\phi}+|p|^2)^{-1/2}\right]\right\|_{H^1} \leq C\epsilon\mathcal{D}_{N,l}(t).$$

In completely same way as above, it holds that $|I_4| + |I_5| \leq C\epsilon\mathcal{D}_{N,l}(t)$.

The estimate of I_6 and I_7 is more complicated, since the diffusive matrix Λ_0 contained in L has a simple eigenvalue $\lambda_1(p) = p_0$ and a double eigenvalue $\lambda_2(p) = 1/p_0$ which makes the D-norm defined in (1.18) degenerate in the direction of the extra moment derivatives. For I_6, as for treating (3.15), we have that

$$\begin{aligned} I_6 = &\sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\nabla_p(J^{1/2})\partial^{\alpha_2}[(\Lambda_\phi-\Lambda_0)f], \nabla_p(J^{-1/2})\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right)\\ &+ \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\nabla_p(J^{1/2})\partial^{\alpha_2}[(\Lambda_\phi-\Lambda_0)f], J^{-1/2}\nabla_p\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right)\\ &+ \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\partial^{\alpha_2}[(\Lambda_\phi-\Lambda_0)\nabla_p f], \nabla_p\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right)\\ &+ \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})J^{1/2}\partial^{\alpha_2}[(\Lambda_\phi-\Lambda_0)\nabla_p f], \nabla_p(J^{-1/2})\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right). \end{aligned} \tag{3.22}$$

Note that thanks to (3.16),

$$\|\sqrt{p_0}\partial^{\alpha_2}[(\Lambda_\phi-\Lambda_0)f]\| \lesssim \epsilon \sum_{1\leq|\alpha|\leq N}\left\{\|\partial^\alpha\phi\| + \left\|p_0^{-1/2}\partial^\alpha f\right\|\right\},$$

where the crucial point is that $\Lambda_\phi - \Lambda$ decays in p with rate $1/p_0$. Using an estimate like that above, it follows from (3.22) that $|I_6| \leq C\epsilon\mathcal{D}_{N,l}(t)$.

It now remains to estimate I_7. Note that one can also write (3.22) for I_7 with $\Lambda_\phi - \Lambda_0$ replaced simply by Λ_0 and the summation restricted over $\alpha_1 > 0$. In light of the Cauchy–Schwarz-type inequality with respect to the inner product induced by Λ_0, it follows that

$$|\langle\Lambda_0\nabla_p g_1, \nabla_p g_2\rangle| \leq (\langle\Lambda_0\nabla_p g_1, \nabla_p g_1\rangle)^{1/2}(\langle\Lambda_0\nabla_p g_2, \nabla_p g_2\rangle)^{1/2} \leq |g_1|_D|g_2|_D$$

for any g_1 and g_2. Moreover, one can always use the energy to control $\partial^{\alpha_1}e^{2\phi}$, since the full energy functional $\mathcal{E}_{N,l}$ contains the $(N+1)$th order derivatives of the scalar gravitational field ϕ. To see this, corresponding to the first term on the right of (3.22), for I_7 we have

$$
\begin{aligned}
&\sum_{\alpha_1+\alpha_2=\alpha,\alpha_1>0} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\Lambda_0\nabla_p\partial^{\alpha_2}f,\nabla_p\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right)\\
&\quad=\sum_{\alpha_1+\alpha_2=\alpha,\alpha_1>0} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\Lambda_0\nabla_p\partial^{\alpha_2}\mathbf{P}_0f,\nabla_p\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right)\\
&\qquad+\sum_{\alpha_1+\alpha_2=\alpha,\alpha_1>0} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\Lambda_0\nabla_p\partial^{\alpha_2}\{\mathbf{I}-\mathbf{P}_0\}f,\nabla_p\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right)\\
&\quad\lesssim\sum_{\substack{1\le|\alpha_1|\le|\alpha|/2\\ \alpha_1+\alpha_2=\alpha}}\left\|p_0^{-1/2}\nabla_p\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right\|\left\|\partial^{\alpha_1}e^{2\phi}\right\|_{L^\infty}\|\partial^{\alpha_2}a\|\\
&\qquad+\sum_{\substack{|\alpha_1|>|\alpha|/2>0\\ \alpha_1+\alpha_2=\alpha}}\left\|p_0^{-1/2}\nabla_p\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\right\|\left\|\partial^{\alpha_1}e^{2\phi}\right\|_{L^3}\|\partial^{\alpha_2}a\|_{L^6}\\
&\qquad+\sum_{\substack{|\alpha_1|>|\alpha|/2>0\\ \alpha_1+\alpha_2=\alpha}}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D\left\|\partial^{\alpha_1}e^{2\phi}\right\|_{L^6}\||\partial^{\alpha_2}\{\mathbf{I}-\mathbf{P}_0\}f|_D\|_{L^3}\\
&\qquad+\sum_{\substack{1\le|\alpha_1|\le|\alpha|/2\\ \alpha_1+\alpha_2=\alpha}}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D\left\|\partial^{\alpha_1}e^{2\phi}\right\|_{L^\infty}\|\partial^{\alpha_2}\{\mathbf{I}-\mathbf{P}_0\}f\|_D\\
&\quad\lesssim\epsilon\mathcal{D}_{N,l}(t),
\end{aligned}
\tag{3.23}
$$

where Sobolev's inequalities $\|u\|_{L^6(\mathbb{R}^3)}\le C\|\nabla u\|$ and $\|u\|_{L^3}\le C\|u\|_{H^1}$ have been used. The other corresponding terms can be handled in identical fashion. Therefore we dispose of the estimate $I_7\lesssim\epsilon\mathcal{D}_{N,l}(t)$.

Now, we plug the above estimates on I_k $(1\le k\le 7)$ altogether into (3.20), take the summation in α over $1\le|\alpha|\le N$, and then combine the resultant estimate with (3.18) in the previous step to obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le N}\left\{\|\partial^\alpha f\|+\|\partial_t\partial^\alpha\phi\|^2+\|\nabla_x\partial^\alpha\phi\|^2+(2-\mathcal{C}_4)\|\partial^\alpha\phi\|^2\right\}+\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^0\\
&\quad+\lambda\sum_{|\alpha|\le N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2\lesssim\epsilon\mathcal{D}_{N,l}(t).
\end{aligned}
\tag{3.24}
$$

Recall (3.3) and (3.4). To further proceed including the dissipation of $a(t,x)$ and $\phi(t,x)$, we may apply the result in Lemma 2.2 to (2.1) and (2.2) with the inhomogeneous source terms h given in (3.13) and g given by

$$
\begin{aligned}
g&=\int_{\mathbb{R}^3}\frac{J^{1/2}f}{\sqrt{1+|p|^2}}\,\mathrm{d}p-e^{2\phi}\int_{\mathbb{R}^3}\frac{J^{1/2}f}{\sqrt{e^{2\phi}+|p|^2}}\,\mathrm{d}p-2e^{2\theta_2\phi}\phi^2\\
&\quad-\frac{1}{2}e^{2\phi}\phi^2\partial_{zz}\left(\frac{J_z}{\sqrt{e^{2z}+|p|^2}}\right)\Bigg|_{z=\theta_3\phi}.
\end{aligned}
\tag{3.25}
$$

Therefore, in terms of (2.18), to obtain the dissipation for $a(t,x)$ and $\phi(t,x)$, it suffices to bound the last two terms on the right-hand side of (2.18). Indeed, by using the same argument as for computing I_2 and I_3 in (3.20), it follows that

$$
\sum_{i=0}^{2}\sum_{|\alpha|\le N-1}\|\langle\partial^\alpha h,\psi_i\rangle\|^2+\sum_{|\alpha|\le N-2}\|\partial^\alpha\nabla_x g\|^2\lesssim\epsilon\mathcal{D}_{N,l}(t).
$$

Applying it to (2.18) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^{\mathrm{int}} + \lambda \sum_{|\alpha|\le N-1} \|\partial^\alpha \nabla_x [a,\phi]\|^2 + \lambda \sum_{|\alpha|\le N-2} \|\partial^\alpha \nabla_x \partial_t \phi\|^2 \lesssim \sum_{|\alpha|\le N} \|\partial^\alpha \{\mathbf{I}-\mathbf{P}_0\} f\|_D^2 + \varepsilon_0 \mathcal{D}_{N,l}(t). \tag{3.26}$$

Let $\kappa_5 > 0$ be suitably small; then taking (3.24) + $\kappa_5 \times$ (3.26) gives

(3.27)
$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le N}\left\{\|\partial^\alpha f\| + \|\partial_t\partial^\alpha\phi\|^2 + \|\nabla_x\partial^\alpha\phi\|^2 + (2-\mathcal{C}_4)\|\partial^\alpha\phi\|^2\right\} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^0 + \kappa_5\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^{\mathrm{int}}\\ &\quad + \lambda\sum_{|\alpha|\le N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2 + \lambda\sum_{|\alpha|\le N-2}\|\partial^\alpha\nabla_x\partial_t\phi\|^2\\ &\quad + \lambda\sum_{|\alpha|\le N-1}\|\nabla_x[a,\phi]\|^2 \lesssim \epsilon\mathcal{D}_{N,l}(t),\end{aligned}$$

where $\mathcal{E}^0$ and $\mathcal{E}^{\mathrm{int}}$ are given in (3.19) and (2.22), respectively. Note that $\mathcal{E}^{\mathrm{int}}$ satisfies (2.17). This ends the second step on the energy estimates with the space derivatives up to order N.

Step 3. *Energy estimates with mixed derivatives and weight.* To obtain the energy estimates for the mixed derivatives, particularly for the pure momentum derivatives, one should pay much attention to the term involving $\partial_t\phi$. Indeed, the term

$$\left(\partial^{\alpha_1}_{\beta_1}\left[\frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\right]\partial^{\alpha_2}_{\beta_2}\left(\partial_t\phi J^{1/2}\right), \partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}_0\}f\right)$$

is out of control when $\alpha = 0$ and hence $\alpha_1 = \alpha_2 = 0$, since $\partial^{\alpha_2}\partial_t\phi$ is dissipative only for $\alpha_2 > 0$ but not for $\alpha_2 = 0$. To overcome this difficulty, we introduce the higher-order projection $\mathbf{P}_1$ as

$$\mathbf{P}_1 f = b(t,x)(1/p_0 - C_0)J^{1/2}, \quad b(t,x) := \frac{\langle f, (1/p_0 - C_0)J^{1/2}\rangle}{\langle (1/p_0 - C_0)^2, J\rangle},$$

where C_0 is chosen such that $C_0\int_{\mathbb{R}^3} J\,\mathrm{d}p = 1$. Furthermore, we denote $\mathbf{P} = \mathbf{P}_0 \oplus \mathbf{P}_1$, and accordingly, we use the notation $\{\mathbf{I}-\mathbf{P}\}f = f - \mathbf{P}f$. The main idea is that the corresponding linear term involving $\partial_t\phi$ vanishes after $\mathbf{P}_1$ acts on the equation.

To proceed, we first rewrite (3.3) as

(3.28)
$$\begin{aligned}&\partial_t f + \nabla_p\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x f - \nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p f - \frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi - J)J^{-1/2}\\ &\quad - \frac{\partial_t\phi}{p_0}J^{1/2} + \left(\frac{1}{p_0} - \frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\right)\partial_t\phi J^{1/2}\\ &\quad + \frac{1}{2}\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f + e^{2\phi}Lf = e^{2\phi}Lf - e^{2\phi}L_\phi f.\end{aligned}$$

Then, we act $\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}$ with $|\alpha|+|\beta|\le N$ and $|\beta|\ge 1$ to (3.28) and take the inner product of the resultant identity with $w^{2l}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2+\left(\partial_\beta^\alpha(e^{2\phi}Lf),w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\quad=\left(\partial_\beta^\alpha\mathbf{P}(e^{2\phi}Lf),w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad-\left(\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}\left\{\nabla_p\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x f\right\},w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad+\left(\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}\left\{\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p f\right\},w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad+\left(\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}\left\{\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi-J)J^{-1/2}\right\},w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad-\left(\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}\left\{\left(\frac{1}{p_0}-\frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\right)\partial_t\phi J^{1/2}\right\},w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad-\frac{1}{2}\left(\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}\left\{\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f\right\},w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad+\left(\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}\left[e^{2\phi}(L-L_\phi)f\right],w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)=\sum_{n=1}^{7}K_n,
\end{aligned}
\tag{3.29}
$$

where K_n $(1\le n\le 7)$ denote the corresponding inner product terms on the right-hand side. We further estimate (3.29) as follows.

First of all, we rewrite the second term on the left-hand side of (3.29) as

$$
\begin{aligned}
&\left(\partial_\beta^\alpha(e^{2\phi}Lf),w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\quad=\left(e^{2\phi}L\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f,w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)+\left(e^{2\phi}L\partial_\beta^\alpha\mathbf{P}_1 f,w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad+\sum_{|\beta_1|\ge1}\left(e^{2\phi}(\partial_{\beta_1}L)\partial_{\beta-\beta_1}^\alpha\{\mathbf{I}-\mathbf{P}_0\}f,w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad+\mathbf{1}_{\{\alpha>0\}}\sum_{|\alpha_1|\ge1}\left((\partial^{\alpha_1}e^{2\phi})(\partial_\beta(L^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}_0\}f),w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right),
\end{aligned}
\tag{3.30}
$$

where $\mathbf{1}_{\{\alpha>0\}}$ is the indicator function meaning that $\mathbf{1}_{\{\alpha>0\}}=1$ for $\alpha>0$ and $\mathbf{1}_{\{\alpha>0\}}=0$ for $\alpha=0$. For the first term on the right, applying the identity (1.13) to L and further using integration by parts in p, it holds that

$$
\begin{aligned}
&\left(e^{2\phi}L\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f,w^{2l}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\quad=\left(e^{2\phi}\Lambda\nabla_p\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f,w^{2l}\nabla_p\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad-\frac{1}{2}\left(e^{2\phi}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f,\nabla_p\cdot\left(\Lambda\nabla_p w^{2l}\right)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad+\int_{\mathbb{R}^3\times\mathbb{R}^3}e^{2\phi}\left(\frac{|p|^2}{4p_0}-\frac{3}{2}\right)w^{2l}|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f|^2\,\mathrm{d}x\,\mathrm{d}p.
\end{aligned}
$$

Here, we recall (3.5). Using (1.16), the first term on the right has a lower bound as

$$
e^{-2C\epsilon}\int_{\mathbb{R}^3\times\mathbb{R}^3}w^{2l}\frac{1}{p_0}|\partial_\beta^\alpha\nabla_p\{\mathbf{I}-\mathbf{P}\}f|^2\,\mathrm{d}x\,\mathrm{d}p.
$$

For the second term, due to $\nabla_p\cdot\left(\Lambda\nabla_p w^{2l}\right)\sim w^{2l}\frac{1}{p_0}$, it can be bounded by

$$
C\int_{\mathbb{R}^3\times\mathbb{R}^3}w^{2l}\frac{1}{p_0}|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f|^2\,\mathrm{d}x\,\mathrm{d}p.
$$

To treat the third term on the right, noticing $\frac{|p|^2}{4p_0} - \frac{3}{2} \geq \frac{1}{m}p_0 - C_m\frac{1}{p_0}$ for $m > 0$ large enough, it is then bounded below by

$$\frac{e^{-2C\epsilon}}{m}\int_{\mathbb{R}^3\times\mathbb{R}^3} w^{2l}p_0|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|^2\,\mathrm{d}x\,\mathrm{d}p - C_m e^{2C\epsilon}\int_{\mathbb{R}^3\times\mathbb{R}^3} w^{2l}\frac{1}{p_0}|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|^2\,\mathrm{d}x\,\mathrm{d}p.$$

Hence, the first term on the right-hand side of (3.30) can be estimated as

$$\begin{aligned}
&\left(e^{2\phi}L\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f, w^{2l}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\quad\geq \lambda\int_{\mathbb{R}^3\times\mathbb{R}^3} w^{2l}\left(\frac{1}{p_0}|\partial^\alpha_\beta\nabla_p\{\mathbf{I}-\mathbf{P}\}f|^2 + p_0|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|^2\right)\mathrm{d}x\,\mathrm{d}p\\
&\qquad - C\sum_{|\beta'|=|\beta|-1}\int_{\mathbb{R}^3\times\mathbb{R}^3} w^{2l}\frac{1}{p_0}|\partial^\alpha_{\beta'}\nabla_p\{\mathbf{I}-\mathbf{P}\}f|^2\,\mathrm{d}x\,\mathrm{d}p
\end{aligned}$$

for a suitable constant $\lambda > 0$, where we recall $|\beta| \geq 1$. Next, by Cauchy–Schwarz's inequality with $\eta > 0$ and the a priori assumption (3.1), the second term on the right-hand side of (3.30) is bounded by

$$\eta\left\|w^l\sqrt{p_0}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2 + C_\eta\|\partial^\alpha b\|^2,$$

and the third term can be dominated by

$$\eta\left\|w^l p_0^{-1/2}\nabla_p\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2 + C_\eta\sum_{|\beta'|\leq|\beta|-1}\left\|w^l\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_D^2 + C_\eta\|\partial^\alpha b\|^2.$$

Performing similar calculations as that of (3.23), one can see that the fourth term on the right-hand side of (3.30) is controlled by

$$\begin{aligned}
&\epsilon\left\|w^l\nabla_p\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_D^2 + \epsilon\sum_{|\alpha'|+|\beta'|\leq N}\left\|w^l\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_D^2 + \epsilon\|\partial^\alpha b\|^2\\
&\quad + \epsilon\sum_{|\alpha|\leq N-1}\|\nabla_x\partial^\alpha\phi\|^2.
\end{aligned}$$

Putting the above estimates altogether into (3.30), we arrive at

(3.31)
$$\begin{aligned}
&\left(e^{2\phi}\partial^\alpha_\beta L\{\mathbf{I}-\mathbf{P}\}f, w^{2l}\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\quad\geq \lambda\left\|w^l\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|_D^2 - \epsilon\sum_{|\alpha'|+|\beta'|\leq N}\left\|w^l\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_D^2\\
&\qquad - C\sum_{|\beta'|\leq|\beta|-1}\left\|w^l\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_D^2 - \epsilon\sum_{|\alpha|\leq N-1}\|\nabla_x\partial^\alpha\phi\|^2 - C\sum_{|\alpha|\leq N}\|\partial^\alpha b\|^2.
\end{aligned}$$

We now turn to estimate K_n $(1 \leq n \leq 7)$ on the right-hand side of (3.29). For K_1, Sobolev's inequality and Cauchy–Schwarz's inequality yield

$$\begin{aligned}
|K_1| &\lesssim (\epsilon+\eta)\|w^l\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + \epsilon\sum_{|\alpha|\leq N-1}\|\partial^\alpha\nabla_x\phi\|^2\\
&\quad + C_\eta\sum_{|\alpha|\leq N-1}\|w^l\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2.
\end{aligned}$$

For the delicate term K_2, we have by Sobolev's inequality, Cauchy–Schwarz's inequality, and the a priori assumption (3.1) that

$$
\begin{aligned}
|K_2| \le & \left| \left(\partial^\alpha_\beta \left\{ \nabla_p \left(\sqrt{e^{2\phi} + |p|^2} \right) \cdot \nabla_x (\{\mathbf{I}-\mathbf{P}\} + \mathbf{P}) f \right\}, w^{2l} \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f \right) \right| \\
& + \left| \left(\partial^\alpha_\beta \mathbf{P} \left\{ \nabla_p \left(\sqrt{e^{2\phi} + |p|^2} \right) \cdot \nabla_x f \right\}, w^{2l} \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f \right) \right| \\
& \le (\eta+\epsilon) \| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f\|_D^2 + C_\eta \sum_{|\alpha'|+|\beta'|\le N, \beta'<\beta} \| w^l \partial^{\alpha'}_{\beta'} \{\mathbf{I}-\mathbf{P}\} f\|_D^2 \\
& + C_\eta \sum_{|\alpha'|\le N-1} \|\partial^{\alpha'} \nabla_x f\|^2,
\end{aligned}
$$

where the key point is that the extra x-derivatives of f are transferred to the scalar gravitational field ϕ by integration by parts and the assumption that $|\beta| \ge 1$ guarantees that the total derivatives acting on f in the second line above would be no more than N. Note that all the other right-hand terms in (3.29) are cubic. Thus by Sobolev's inequality and the a priori assumption (3.1), one has the following estimates:

$$
|K_3|,\ |K_6| \lesssim \epsilon \| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f\|^2 + \epsilon \sum_{|\alpha|\le N-1} \|\partial^\alpha \nabla_x \phi\|^2 + \epsilon \sum_{|\alpha|\ge 1, |\alpha|+|\beta|\le N} \| w^l \partial^\alpha_\beta f\|^2,
$$

$$
|K_4|,\ |K_5| \lesssim \epsilon \| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f\|^2 + \epsilon \sum_{|\alpha|\le N-1} \|\partial^\alpha \nabla_x \phi\|^2 + \epsilon \sum_{1\le|\alpha|\le N-1} \|\partial^\alpha \partial_t \phi\|^2.
$$

For the last term K_7, performing the similar computations as in (3.15), it follows that

$$
|K_7| \lesssim \epsilon \| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f\|_D^2 + \epsilon \sum_{|\alpha|\le N-1} \|\partial^\alpha \nabla_x \phi\|^2 + \epsilon \sum_{|\alpha|\ge 1, |\alpha|+|\beta|\le N} \| w^l \partial^\alpha_\beta f\|_D^2.
$$

Applying all the above estimates on K_n $(1 \le n \le 7)$ together with (3.31) to (3.29), taking the summation in β over $1 \le |\beta| \le N$, and then combining the resultant estimate with (3.18) and (3.27), we obtain that

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} \sum_{1\le|\alpha|\le N} \left\{ \|\partial^\alpha f\|^2 + \|\partial_t \partial^\alpha \phi\|^2 + \|\nabla_x \partial^\alpha \phi\|^2 + (2-\mathcal{C}_4)\|\partial^\alpha \phi\|^2 \right\} \\
& \quad + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{|\alpha|+|\beta|\le N, |\beta|\ge 1} \left\| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f \right\|^2 + \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}^0 + \kappa_5 \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}^{\mathrm{int}} \\
& \quad + \lambda \sum_{|\alpha|+|\beta|\le N, |\beta|\ge 1} \| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}\} f\|_D^2 + \lambda \sum_{|\alpha|\le N} \|\partial^\alpha \{\mathbf{I}-\mathbf{P}_0\} f\|_D^2 \\
& \quad + \lambda \sum_{|\alpha|\le N-2} \|\partial^\alpha \nabla_x \partial_t \phi\|^2 + \lambda \sum_{|\alpha|\le N-1} \|\nabla_x \partial^\alpha [a, \phi]\|^2 \\
& \le C \sum_{|\alpha|\le N-1} \| w^l \partial^\alpha \{\mathbf{I}-\mathbf{P}\} f\|_D^2,
\end{aligned} \tag{3.32}
$$

where we have used the identity that $\|\{\mathbf{I}-\mathbf{P}_0\} f\|^2 = \|\{\mathbf{I}-\mathbf{P}\} f\|^2 + C_b \|b\|^2$ for a constant $C_b > 0$, $\mathcal{E}^0$ and $\mathcal{E}^{\mathrm{int}}$ are given in (3.19) and (2.22), respectively, and $\mathcal{E}^{\mathrm{int}}$ satisfies (2.17). This ends the third step on the energy estimates with mixed derivatives and weight.

Step 4. *Energy estimates with pure space derivatives and weight.* We now turn to control the right-hand term of (3.32). To do this, we divide the computations in two cases, either $\alpha = 0$ or $|\alpha| \geq 1$. For $\alpha = 0$, similar to (3.29), one has with $l \geq 0$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\{\mathbf{I}-\mathbf{P}\}f\|^2 + \left(e^{2\phi}Lf, w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\quad = \left(\mathbf{P}(e^{2\phi}Lf), w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right) - \left(\{\mathbf{I}-\mathbf{P}\}\left\{\nabla_p\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x f\right\}, w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad + \left(\{\mathbf{I}-\mathbf{P}\}\left\{\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p f\right\}, w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad + \left(\{\mathbf{I}-\mathbf{P}\}\left\{\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}(J_\phi - J)J^{-1/2}\right\}, w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad - \left(\{\mathbf{I}-\mathbf{P}\}\left\{\left(\frac{1}{p_0} - \frac{e^{2\phi}}{\sqrt{e^{2\phi}+|p|^2}}\right)\partial_t\phi J^{1/2}\right\}, w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad - \frac{1}{2}\left(\{\mathbf{I}-\mathbf{P}\}\left\{\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f\right\}, w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\qquad + \left(\{\mathbf{I}-\mathbf{P}\}\left[e^{2\phi}(L-L_\phi)f\right], w^{2l}\{\mathbf{I}-\mathbf{P}\}f\right),
\end{aligned}$$

which further implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^l\{\mathbf{I}-\mathbf{P}\}f\|^2 + \lambda\|w^l\{\mathbf{I}-\mathbf{P}\}f\|_D^2 \lesssim \|\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2 + \|\nabla_x f\|^2 + \epsilon\mathcal{D}_{N,l}. \tag{3.33}$$

In the case when $1 \leq |\alpha| \leq N-1$, similar to (3.20), we have for $l \geq 0$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|w^l\partial^\alpha f\right\|^2 - \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\left[\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}\right]\partial^{\alpha_2}J_\phi J^{-1/2}, w^{2l}\partial^\alpha f\right)\\
&\quad + \sum_{\substack{\alpha_1>0\\ \alpha_1+\alpha_2=\alpha}} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\nabla_p\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_x\partial^{\alpha_2}f, w^{2l}\partial^\alpha f\right)\\
&\quad - \sum_{\substack{\alpha_1>0\\ \alpha_1+\alpha_2=\alpha}} \left(\partial^{\alpha_1}\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\nabla_p\partial^{\alpha_2}f, w^{2l}\partial^\alpha f\right)\\
&\quad + \left(\nabla_x\sqrt{e^{2\phi}+|p|^2}\partial^\alpha f, (\nabla_p w^{2l})\partial^\alpha f\right)\\
&\quad + \frac{1}{2}\sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}\nabla_x\sqrt{e^{2\phi}+|p|^2}\cdot\partial^{\alpha_2}\left(\frac{p}{\sqrt{e^{2\phi}+|p|^2}}f\right), w^{2l}\partial^\alpha f\right)\\
&\quad + \left(e^{2\phi}L\partial^\alpha f, w^{2l}\partial^\alpha f\right)\\
&= \sum_{\alpha_1+\alpha_2=\alpha} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})\partial^{\alpha_2}(Lf - L_\phi f), w^{2l}\partial^\alpha f\right)\\
&\quad - \sum_{\substack{\alpha_1>0\\ \alpha_1+\alpha_2=\alpha}} C_\alpha^{\alpha_1}\left((\partial^{\alpha_1}e^{2\phi})L\partial^{\alpha_2}f, w^{2l}\partial^\alpha f\right).
\end{aligned} \tag{3.34}$$

From (1.13) and (3.5), it follows that

$$\begin{aligned}
&\left(e^{2\phi}L\partial^\alpha f, w^{2l}\partial^\alpha f\right)\\
&\quad\geq \delta_C \left\|p_0^{-1/2}w^l\nabla_p\partial^\alpha f\right\|^2 + \delta_C\left\|p_0^{1/2}w^l\partial^\alpha f\right\|^2\\
&\qquad - C\left|\left(\nabla_p\cdot\left(\Lambda\nabla_p w^{2l}\right),(\partial^\alpha f)^2\right)\right| - C\left\|w^l\partial^\alpha f\right\|^2\\
&\quad\geq \delta_C \left\|p_0^{-1/2}w^l\nabla_p\partial^\alpha f\right\|^2 + \frac{\delta_C}{2}\left\|p_0^{1/2}w^l\partial^\alpha f\right\|^2 - C_m\int_{\mathbb{R}^3\times\mathbb{R}^3}|\partial^\alpha f|^2\mathbf{1}_{\{|p|\leq m\}}\,\mathrm{d}x\,\mathrm{d}p,
\end{aligned}$$

where $\delta_C>0$ is a constant. The other terms in (3.34) can be controlled in the same way as the corresponding terms in (3.20). We only give the following final estimate with details of the proof omitted for brevity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leq|\alpha|\leq N-1}\left\|w^l\partial^\alpha f\right\|^2 + \lambda\sum_{1\leq|\alpha|\leq N-1}\left\|w^l\partial^\alpha f\right\|_D^2 \lesssim \varepsilon\mathcal{D}_{N,l} + \sum_{1\leq|\alpha|\leq N-1}\|\partial^\alpha[f,\partial_t\phi]\|^2. \tag{3.35}$$

This ends the last step on energy estimates with pure space derivatives and weight.

To obtain the desired estimate (3.2) with a suitably defined energy functional $\mathcal{E}_{N,l}(\cdot)$, we need to combine all the estimates obtained in the previous steps. Indeed, by choosing $C_1\gg C_2\gg 1$ and taking the linear combination

$$(3.27)\times C_1 + (3.33)\times C_2 + (3.35)\times C_2 + (3.32),$$

we then derive (3.2) with

$$\begin{aligned}
\mathcal{E}_{N,l}(t) = C_1&\left\{\sum_{1\leq|\alpha|\leq N}\left\{\|\partial^\alpha f\|^2 + \|\partial_t\partial^\alpha\phi\|^2 + \|\nabla_x\partial^\alpha\phi\|^2 + \|\partial^\alpha\phi\|^2\right\} + \mathcal{E}^0 + \kappa_5\mathcal{E}^{\mathrm{int}}\right\}\\
&+ C_2\left\{\left\|w^l\{\mathbf{I}-\mathbf{P}\}f\right\|^2 + \sum_{1\leq|\alpha|\leq N-1}\left\|w^l\partial^\alpha f\right\|^2\right\}\\
&+ \sum_{\substack{|\alpha|+|\beta|\leq N\\ |\beta|\geq 1}}\left\|w^l\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2.
\end{aligned}$$

It is direct to verify that

$$\begin{aligned}
\mathcal{E}_{N,l}(t)\sim &\sum_{|\alpha|\leq N}\left\{\|\partial^\alpha f\|^2 + \|\partial_t\partial^\alpha\phi\|^2 + \|\nabla_x\partial^\alpha\phi\|^2 + \|\partial^\alpha\phi\|^2\right\}\\
&+ \sum_{\substack{|\alpha|+|\beta|\leq N\\ |\alpha|\leq N-1}}\left\|w^l\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}_0\}f\right\|^2.
\end{aligned}$$

The proof of Proposition 3.1 is complete. ∎

COROLLARY 3.2. *Let* $[f(t,x,p),\phi(t,x)]$ *be a classical solution over* $0\leq t\leq T$ *for the Cauchy problem* (1.7), (1.9), *and* (1.10) *satisfying the assumption* (3.1)*; then there exist the temporal high-order energy functionals* $\mathcal{E}^h_{N,l}(\cdot)$ *in terms of* (1.20) *such that the estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^h_{N,l}(t) + \lambda\mathcal{D}_{N,l}(t)\leq 0 \tag{3.36}$$

holds true for all $0\leq t\leq T$, *where* $N\geq 4$, $l\geq 0$, *and* $\mathcal{D}_{N,l}(\cdot)$ *is defined in* (1.21).

Proof. The desired estimate (3.36) follows similarly for obtaining (3.2). Thus details are omitted for brevity. ∎

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. As mentioned before, the global existence follows from the a priori energy estimates obtained in Proposition 3.1 together with the local existence with the help of the usual continuity argument. For brevity, we omit the details of the proof. Therefore, (1.22) is a direct consequence of (3.2) by taking the time integration. Furthermore, it is also direct to verify the uniqueness and positivity of the obtained solution, since the solution constructed here enjoys high-order regularity in the x- and p-variables. This establishes the validity of Theorem 1.1. ∎

4. Time decay rates. In this section, we further deduce the time decay rates of the constructed global solution in order to show Theorem 1.2. Here, the proof is based on Theorem 1.1 and Theorem 2.1 as well as the following Sobolev inequalities (cf. [14]).

LEMMA 4.1. *For $u, v \in L^\infty \cap H^m(\mathbb{R}^n)$ with the nonnegative integer m, it holds that*

$$\|\nabla_x^m(uv)\| \lesssim \|u\|_{L_x^\infty}\|\nabla_x^m v\| + \|v\|_{L_x^\infty}\|\nabla_x^m u\|,$$

$$\|\nabla_x^m(uv) - u\nabla_x^m v\| \lesssim \|\nabla_x u\|_{L_x^\infty}\|\nabla_x^{m-1} v\| + \|v\|_{L_x^\infty}\|\nabla_x^m u\|.$$

For $u \in H^2(\mathbb{R}^3)$, it also holds that

$$\|u\|_{L_x^\infty} \lesssim \|\nabla_x u\|^{\frac{1}{2}}\|\nabla_x^2 u\|^{\frac{1}{2}}.$$

Proof of Theorem 1.2. Let $N \geq 8$ and $l \geq 0$. Recall (1.23) for $Y_{N,l}$ small enough. We denote

$$X(t) = \sup_{0\leq s\leq t}(1+s)^{3/2}\mathcal{E}_{N-2,l\vee 1}(s).$$

To show (1.24), it suffices to prove $X(t) \lesssim Y_{N,l}^2$ for any $t \geq 0$. We start from the following lemma.

LEMMA 4.2. *Let $N \geq 8$; then it holds that*

$$\sup_{0\leq s\leq t}\{(1+s)^{\frac{5}{2}}\sum_{1\leq m\leq N-7}\|\nabla_x^m[f,\phi,\nabla_x\phi,\partial_t\phi]\|^2 + (1+s)^{\frac{3}{2}}\|[f,\phi,\nabla_x\phi,\partial_t\phi]\|^2\} \lesssim Y_{N,l}^2 + X^2(t). \tag{4.1}$$

Proof. From Theorem 2.1, one has

$$\begin{aligned}\|[f,\phi,\nabla_x\phi,\partial_t\phi]\| &\lesssim (1+t)^{-\frac{3}{4}}\left(\|f_0\|_{Z_1} + \|[\phi_0,\nabla_x\phi_0,\phi_1]\|_{L_x^1}\right)\\ &\quad + (1+t)^{-\frac{3}{4}}\left(\|\nabla_x^{3/2}f_0\| + \|\nabla_x^{3/2}[\phi_0,\nabla_x\phi_0,\phi_1]\|\right)\\ &\quad + \int_0^t(1+t-s)^{-\frac{3}{4}}\left(\|h\|_{Z_1} + \|g\|_{L^1} + \|\nabla_x^{3/2}[h,g]\|\right)\mathrm{d}s,\end{aligned}\tag{4.2}$$

and for $m \geq 1$

$$\begin{aligned}\|\nabla_x^m[f,\phi,\nabla_x\phi,\partial_t\phi]\| &\lesssim (1+t)^{-\frac{3}{4}-\frac{m}{2}}\left(\|f_0\|_{Z_1} + \|[\phi_0,\nabla_x\phi_0,\phi_1]\|_{L_x^1}\right)\\ &\quad + (1+t)^{-5/4}\left(\|\nabla_x^{5/2+m}f_0\| + \|\nabla_x^{5/2+m}[\phi_0,\nabla_x\phi_0,\phi_1]\|\right)\\ &\quad + \int_0^t(1+t-s)^{-5/4}\left(\|h\|_{Z_1} + \|g\|_{L^1} + \|\nabla_x^{5/2+m}[h,g]\|\right)\mathrm{d}s,\end{aligned}\tag{4.3}$$

where h and g are given by (3.13) and (3.25), respectively, and we have denoted $Z_1 = L^2_p L^1_x$. For the sake of convenience, we rewrite h and g as the high-order terms:

$$\begin{aligned} h = {}& \mathcal{H}_1(\phi, p) \cdot \nabla_x f\phi + \mathcal{H}_2(\phi, p)\nabla_p f \cdot \nabla_x \phi + \mathcal{H}_3(\phi, p)\phi\partial_t\phi + \mathcal{H}_4(\phi, p)\nabla_x\phi f \\ & + \mathcal{H}_5(\phi, p)p_0\nabla^2_p f\phi + \mathcal{H}_6(\phi, p)\nabla_p f\phi + \mathcal{H}_7(\phi, p)p_0 f\phi \end{aligned} \tag{4.4}$$

and

$$g = \mathcal{G}_1(\phi, f)\phi + \mathcal{G}_2(\phi, p)\phi^2, \tag{4.5}$$

where

$$\begin{aligned} &\mathcal{H}_1(\phi, p) \cdot \nabla_x f\phi = -\nabla_p(\sqrt{e^{2\phi} + |p|^2} - \sqrt{1 + |p|^2}) \cdot \nabla_x f, \\ &\mathcal{H}_2(\phi, p)\nabla_p f \cdot \nabla_x\phi = \nabla_x\sqrt{e^{2\phi} + |p|^2} \cdot \nabla_p f \\ &\mathcal{H}_3(\phi, p)\phi\partial_t\phi = \frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi} + |p|^2}} J_\phi J^{-1/2} - \frac{\partial_t\phi}{\sqrt{1 + |p|^2}} J^{1/2}, \\ &\mathcal{H}_4(\phi, p)\nabla_x\phi f = -\frac{1}{2}\frac{\nabla_x\sqrt{e^{2\phi} + |p|^2} \cdot p}{\sqrt{e^{2\phi} + |p|^2}} f, \\ &\mathcal{H}_5(\phi, p)p_0\nabla^2_p f\phi + \mathcal{H}_6(\phi, p)\nabla_p f\phi + \mathcal{H}_7(\phi, p)p_0 f\phi = (Lf - e^{2\phi}L_\phi f), \end{aligned}$$

and

$$\begin{aligned} &\mathcal{G}_1(\phi, f)\phi = \int_{\mathbb{R}^3} \frac{J^{1/2} f}{\sqrt{1 + |p|^2}} \,\mathrm{d}p - e^{2\phi} \int_{\mathbb{R}^3} \frac{J^{1/2} f}{\sqrt{e^{2\phi} + |p|^2}} \,\mathrm{d}p, \\ &\mathcal{G}_2(\phi, p)\phi^2 = -2e^{2\theta_2\phi}\phi^2 - \frac{e^{2\phi}}{2}\phi^2\partial_{zz}\left(\frac{J_z}{\sqrt{e^{2z} + |p|^2}}\right)\Bigg|_{z=\theta_3\phi}. \end{aligned}$$

With (4.4) and (4.5) at hand, by Cauchy–Schwarz's inequality, (3.5), and Lemma 4.1, we obtain that

$$\begin{aligned} \|[h, g]\|_{Z_1} \lesssim {}& \|\phi\|\|\nabla_x f\| + \|\nabla_x\phi\|\|\nabla_p f\| + \|\partial_t\phi\|\|\phi\| + \|\nabla_x\phi\|\|f\| \\ & + \|\phi\| \left\{\left\|p_0\nabla^2_p\{\mathbf{I} - \mathbf{P}_0\}f\right\| + \|\nabla_p\{\mathbf{I} - \mathbf{P}_0\}f\| \right.\\ & \left. + \|p_0\{\mathbf{I} - \mathbf{P}_0\}f\| + \|a\|\right\} + \|\phi\|^2 \\ \lesssim {}& \mathcal{E}_{N-2,l\vee 1}(t) \end{aligned} \tag{4.6}$$

and

$$\sum_{m\le N-7} \left\|\nabla^{m+5/2}_x [h, g]\right\| \lesssim \mathcal{E}_{N-2,l\vee 1}(t). \tag{4.7}$$

Plugging (4.6) and (4.7) into (4.2) and (4.3) and using the fact that

$$\mathcal{E}_{N-2,l\vee 1}(s) \le (1 + s)^{-\frac{3}{2}} X(t), \quad 0 \le s \le t,$$

one further has

$$\sup_{0\le s\le t} \{(1 + s)^{\frac{3}{2}} \|[f, \phi, \nabla_x\phi, \partial_t\phi]\|^2\} \lesssim Y^2_{N,l} + X^2(t)$$

and

$$\sup_{0\le s\le t}\{(1+s)^{\frac{5}{2}}\sum_{1\le m\le N-7}\|\nabla_x^m[f,\phi,\nabla_x\phi,\partial_t\phi]\|^2\}\lesssim Y_{N,l}^2+X^2(t).$$

These two estimates together give the desired estimate (4.1). Thus the proof of Lemma 4.2 is complete. ∎

We now use the method of iteration to proceed with the proof. First of all, we have by Proposition 3.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{N-1,l\vee 1}(t)+\lambda\mathcal{D}_{N-1,l\vee 1}(t)\le 0.$$

Further multiplying the above estimate with $(1+t)^{\frac{1}{2}+\sigma}$ with $\sigma>0$ fixed small enough and taking the time integration, one has

$$(1+t)^{\frac{1}{2}+\sigma}\mathcal{E}_{N-1,l\vee 1}(t)+\int_0^t(1+s)^{\frac{1}{2}+\sigma}\mathcal{D}_{N-1,l\vee 1}(s)\,\mathrm{d}s \lesssim Y_{N,l}^2+\int_0^t(1+s)^{-\frac{1}{2}+\sigma}\mathcal{E}_{N-1,l\vee 1}(s)\,\mathrm{d}s. \tag{4.8}$$

Here, the second term on the right is bounded by $Y_{N,l}^2+X^2(t)$ in terms of (1.22) and (4.1) together with smallness of $\sigma>0$ by noticing

$$\mathcal{E}_{N-1,l\vee 1}(t)\lesssim\mathcal{D}_{N,l\vee 1}(t)+\|[a,\phi,\partial_t\phi]\|^2.$$

Hence, we get from (4.8) that

$$\sup_{0\le s\le t}\left\{(1+s)^{\frac{1}{2}+\sigma}\mathcal{E}_{N-1,l\vee 1}(s)\right\}+\int_0^t(1+s)^{\frac{1}{2}+\sigma}\mathcal{D}_{N-1,l\vee 1}(s)\,\mathrm{d}s\lesssim Y_{N,l}^2+X^2(t). \tag{4.9}$$

Similarly, multiplying the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{N-2,l\vee 1}(t)+\lambda\mathcal{D}_{N-2,l\vee 1}(t)\le 0$$

by $(1+t)^{\frac{3}{2}+\sigma}$ and taking the time integration gives rise to

$$(1+t)^{\frac{3}{2}+\sigma}\mathcal{E}_{N-2,l\vee 1}(t)+\int_0^t(1+s)^{\frac{3}{2}+\sigma}\mathcal{D}_{N-2,l\vee 1}(s)\,\mathrm{d}s \lesssim Y_{N,l}^2+\int_0^t(1+s)^{\frac{1}{2}+\sigma}\mathcal{E}_{N-2,l\vee 1}(s)\,\mathrm{d}s. \tag{4.10}$$

Again, by noticing $\mathcal{E}_{N-2,l\vee 1}(t)\lesssim\mathcal{D}_{N-1,l\vee 1}(t)+\|[a,\phi,\partial_t\phi]\|^2$, the second term on the right-hand side of (4.10) is dominated by $C(1+t)^\sigma(Y_{N,l}^2+X^2(t))$, in terms of (4.9) and (4.1). As a consequence, it follows from (4.10) that

$$\sup_{0\le s\le t}\left\{(1+s)^{\frac{3}{2}}\mathcal{E}_{N-2,l\vee 1}(s)\right\}\lesssim Y_{N,l}^2+X^2(t),$$

which implies that $X(t)\lesssim Y_{N,l}^2$ for any $t\ge 0$ by the continuity argument due to the smallness assumption of $Y_{N,l}$. Therefore (1.24) is valid.

It now remains to prove the desired estimate (1.25) for the time decay of $\mathcal{E}^h_{N-7,l}(t)$ with $N \geq 11$. Recall (1.20) and (1.21). From Proposition 3.1, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^h_{N-7,l\vee 1}(t) + \lambda\mathcal{E}^h_{N-7,l\vee 1}(t) \lesssim \|\nabla_x[a,\phi,\partial_t\phi]\|^2 + \sum_{|\alpha|=N-7}\|\partial^\alpha[\partial_t\phi,\nabla_x\phi]\|^2$$

$$\lesssim \sum_{1\leq|\alpha|\leq N-7}\|\partial^\alpha[f,\phi,\nabla_x\phi,\partial_t\phi]\|^2. \tag{4.11}$$

Here, in terms of (1.24) and (4.1), the right-hand term can be bounded as

$$\sum_{1\leq|\alpha|\leq N-7}\|\partial^\alpha[f,\phi,\nabla_x\phi,\partial_t\phi]\|^2 \lesssim (1+t)^{-5/2}Y^2_{N,l}.$$

Applying the above estimate, (1.25) then follows from (4.11) by Gronwall's inequality. The proof of Theorem 1.2 is thus complete. ∎

5. Local existence. For completeness, this section is devoted to sketching the proof of the local-in-time existence for the Cauchy problem (1.1) and (1.2) with initial data (1.4). We start with constructing a sequence of approximation solutions $\{[F^n(t,x,p),\phi^n(t,x)]\}_{n\geq 0}$ by iteration through solving the Cauchy problems

$$\begin{cases}\partial_t F^{n+1} + \nabla_p\sqrt{e^{2\phi^n}+|p|^2}\cdot\nabla_x F^{n+1} - \nabla_x\sqrt{e^{2\phi^n}+|p|^2}\cdot\nabla_p F^{n+1} \\ \qquad = e^{2\phi^n}\nabla_p\cdot(\Lambda_{\phi^n}\nabla_p F^{n+1} + pF^{n+1}), \\ \Box\phi^{n+1} = -e^{2\phi^n}\displaystyle\int_{\mathbb{R}^3}\frac{F^n(t,x,p)}{\sqrt{e^{2\phi^n}+|p|^2}}\,\mathrm{d}p + 1, \quad n=0,1,\dots,\end{cases} \tag{5.1}$$

with initial data

$$F^{n+1}(0,x,p) = F_0(x,p), \quad \phi^{n+1}(0,x) = \phi_0(x), \quad \partial_t\phi^{n+1}(0,x) = \phi_1(x), \tag{5.2}$$

where we have set $F^0(t,x,p) \equiv F_0(x,p)$ and $\phi^0(t,x) \equiv \phi_0(x)$ for $n=0$. Note that to solve (5.1) and (5.2), letting $F^{n+1} = J_{\phi^n} + J^{1/2}f^{n+1}$, it is equivalent to consider

$$\begin{aligned}&\partial_t f^{n+1} + \nabla_p\sqrt{e^{2\phi^n}+|p|^2}\cdot\nabla_x f^{n+1} - \nabla_x\sqrt{e^{2\phi^n}+|p|^2}\cdot\nabla_p f^{n+1} \\ &\quad - \frac{e^{2\phi^n}\partial_t\phi^n}{\sqrt{e^{2\phi^n}+|p|^2}}J_{\phi^n}J^{-1/2} + \frac{1}{2}\nabla_x\sqrt{e^{2\phi^n}+|p|^2}\cdot\frac{p}{\sqrt{1+|p|^2}}f^{n+1} \\ &\quad + e^{2\phi^n}L_{\phi^n}f^{n+1} = 0, \\ &\quad \Box\phi^{n+1} = -e^{2\phi^n}\int_{\mathbb{R}^3}\frac{J_{\phi^n}}{\sqrt{e^{2\phi^n}+|p|^2}}\,\mathrm{d}p - e^{2\phi^n}\int_{\mathbb{R}^3}\frac{J^{1/2}f^n}{\sqrt{e^{2\phi^n}+|p|^2}}\,\mathrm{d}p + 1\end{aligned}$$

with

$$f^{n+1}(0,x,p) = f_0(x,p),\ \phi^{n+1}(0,x) = \phi_0(x),\ \partial_t\phi^{n+1}(0,x) = \phi_1(x).$$

LEMMA 5.1. *For any $n \geq 1$, the approximation solution $[F^n(t,x,p),\phi^n(t,x)]$ is well-defined with $F^n(t,x,p) \geq 0$. In particular, there exist constants $\epsilon_1 > 0$, $T_1 > 0$ and $C_1 > 0$ such that if $\mathcal{E}_{N,l}(0) \leq \epsilon_1$, then it holds that*

$$\sup_{0\leq t\leq T_1}\mathcal{E}_{N,l}(f^n(t),\phi^n(t)) \leq C_1\mathcal{E}_{N,l}(0) \tag{5.3}$$

for $n = 1, 2, \dots$.

Proof. First of all, for given $[F^n(t,x,p), \phi^n(t,x)]$ with $F^n(t,x,p) \geq 0$, there exists the unique solution $[F^{n+1}(t,x,p), \phi^{n+1}(t,x)]$ to the Cauchy problem (5.1) and (5.2) on the linear inhomogeneous system. Moreover, the first equation of (5.1) implies that $F^{n+1}(t,x,p) \geq 0$ holds true, since $F^0(x,p)$ is nonnegative. The desired estimate (5.3) can be proved via induction on n. For $n = 0$ it is obvious to hold true. Assume that (5.3) is valid for $n \geq 0$. Then, by repeating similar estimates as in Steps 1–4 in section 3, one can further show that (5.3) is also true for $n+1$; such tedious computations are omitted for brevity. This ends the proof of Lemma 5.1. ◻

Once Lemma 5.1 is obtained, the local-in-time solution can be constructed by using a similar argument as that of [10, Theorem 4, page 607]. We then conclude the complete proof of Theorem 1.1.

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