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# High-Order Approximation of Heteroclinic Bifurcations in Truncated 2D-Normal Forms for the Generic Cases of Hopf-Zero and Nonresonant Double Hopf Singularities*

B. W. Qin†, K. W. Chung‡, A. Algaba§, and A. J. Rodríguez-Luis¶

**Abstract.** Based on the nonlinear time transformation method, in this paper we propose a recursive algorithm for arbitrary order approximation of heteroclinic orbits. This approach works fine for a wide class of systems that are perturbations of non-Hamiltonian integrable planar vector fields. Specifically, our method can provide an approximation up to any desired order, both for the locus where the heteroclinic bifurcation occurs in the parameter space and for the orbit in the phase space. We also give proofs that guarantee the existence and uniqueness of the solution. As illustrations, we apply it to the generic Hopf-zero and nonresonant double Hopf singularities which give rise to an intricate bifurcation scenario in the cases where a heteroclinic cycle appears in the corresponding unfolding of normal forms. The obtained high-order approximations are expressed in terms of symbolic nonzero normal form coefficients which improve the existing first-order approximations in the literature. They are also in excellent agreement with numerical continuations. Note that the derived formulas can be practically important in many control engineering applications in which certain constants of the problem may not be known in advance.

**Key words.** asymptotic expansion, heteroclinic orbit, integrable system, Hopf-zero singularity, double Hopf singularity

**AMS subject classifications.** 34E05, 34E10, 37C29, 37M20, 41A60

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**1. Introduction.** When an equilibrium in a family of autonomous systems has a zero eigenvalue and a pair of purely imaginary eigenvalues ($\lambda = 0, \pm\omega i$), generically, a Hopf-zero bifurcation occurs. In the literature, this bifurcation is also called fold-Hopf, saddle-node-Hopf, and Gavrilov–Guckenheimer. This codimension-two bifurcation appears, in the corresponding parameter plane, in the tangential intersection point between a curve of Hopf bifurcations and a curve of saddle-node bifurcations (see, for instance, [44, 55, 68]). When the system is $\mathbb{Z}_2$-symmetric, the eigenvalues $0, \pm\omega i$ lead to a Hopf-pitchfork bifurcation.

The importance of this bifurcation as an organizing center (for systems of a dimension greater than or equal to three) justifies the large number of works in which it is considered. For instance, some aspects of the computation of the normal form of the Hopf-zero bifurcation are treated in [5, 34, 25, 38, 40, 39, 58, 21], the Hopf-saddle-node bifurcation and some of its degeneracies are studied in [44, 53], and the Hopf-pitchfork bifurcation and some of its degeneracies are analyzed in [7, 6, 8, 12, 33]. It is well known that the more complex situation appears when a heteroclinic cycle exists in the corresponding unfolding of a planar normal form. In this case, for example, resonances and Arnold's tongues related to Hopf-zero and Hopf-pitchfork bifurcations are present (see, for instance, [14, 15, 9]), and heteroclinic bifurcations, which carry out Shilnikov homoclinic orbits and homoclinic tangle [20, 36], lead to a local birth of chaos [20, 37, 35, 13, 16]. Moreover, the presence of global bifurcations is analyzed in [20, 37, 23, 56, 17, 10], and the importance of the Hopf-zero bifurcation in the analysis of some systems can be found in [13, 11, 57, 43].

On the other hand, from the beginnings of the theory of dynamical systems, homoclinic and heteroclinic orbits have been identified as important organizing centers for the behavior of systems (see, for instance, [44, 55, 68, 48]). Since it is not usual to obtain an exact homoclinic or heteroclinic orbit (see, for instance, [10]), great efforts have been made in the development of analytical methods that provide the maximum possible information on them. The procedure most used to ensure their existence is the Melnikov function, which, in addition, provides a first-order approximation (see, for example, [44, 55, 68, 59, 2]). Many other perturbative methods have been developed, namely Poincaré–Lindstedt, regular perturbations, multiple scales, and nonlinear time transformation (NTT) (see, for instance, [26, 27, 18, 19, 28, 46, 47, 22, 64]).

In this context, our paper analyzes the existence, uniqueness, and computation, up to any order of approximation, of heteroclinic orbits in planar systems that are perturbations of a non-Hamiltonian integrable planar vector field. Using the NTT method for the first time in this type of system, we are able to approximate these global connections both in the parameter space and in the phase space. Until now, the scope of application of the NTT method was essentially reserved for differential equations of second order coming from Mechanics, that is, for systems already in normal form of Liénard type or even for systems that are perturbations of a Hamiltonian system. In these cases, the NTT method has shown itself to be a more efficient alternative for the calculation of global connections than classical methods (Melnikov, multiple scales, Lindstedt–Poincaré, for example) [1, 4, 62, 60].

As a first application of the obtained results, heteroclinic connections are studied in the cubic parametric normal form system

$$\begin{aligned} \dot{r} &= \mu_1 r + arz, \\ \dot{z} &= \mu_2 - r^2 - z^2 + sz^3 \quad (a > 0,\ s \neq 0). \end{aligned} \tag{1.1}$$

This two-parametric system is related to the unfolding of the generic Hopf-zero singularity in the case III[1] [44, sect. 7.4]. Recall that a Hopf-zero singularity on its "center manifold" is of three-dimensional state space, and its normal form also has the same dimension. However, the angular component within a cylindrical coordinate system can be ignored, and planar system (1.1) is then obtained (for more details see [44, sect. 7.4]).

In a second application we consider a two-parametric unfolding of a normal form system related to the nonresonant double Hopf singularity (also valid for the Hopf-pitchfork singularity) in the case VIa [44, sect. 7.5]

$$\begin{aligned} \dot{r}_1 &= r_1(\mu_1 + r_1^2 + br_2^2), \\ \dot{r}_2 &= r_2(\mu_2 + cr_1^2 - r_2^2) + kr_2^5, \end{aligned} \tag{1.2}$$

with $c < 0 < b$, $A = -1 - bc > 0$, and $k < 0$. The double Hopf singularity (also called Hopf-Hopf) appears when there are two purely imaginary pairs of eigenvalues at an equilibrium (see, for instance, [44, 55, 49, 69, 67, 52, 65]). Its normal form is in a four-dimensional state space, but the phase or azimuthal components can be ignored.

As is well known, the presence of heteroclinic connections gives rise to the most intricate bifurcation scenario in a neighborhood of both singularities [44, 55, 68].

Our method will provide for both systems, for all values of the symbolic constants ($a$ and $s$ in system (1.1) and $b$, $c$, and $k$ in system (1.2)), the approximation of the heteroclinic bifurcations, up to any desired order, both in the $(\mu_1, \mu_2)$-parameter plane and in the phase plane. This result clearly improves the first-order approximations found in the literature [51, 44, 55, 29].

This high-order approximation is sufficient for many bifurcation analysis and bifurcation control applications. Due to the singular nature of many control systems and their modeling imperfections, it is fundamentally important to derive high-order approximations of the transition sets in any meaningful bifurcation control engineering applications. This paper raises this ideal goal and successfully accomplishes that. Certain constants of the problem may not be known in advance for many control engineering applications. Yet, a feedback bifurcation controller design is required for such singular control systems; see, for instance, [24, 41, 42, 45, 50, 54]. The derived formulas here are presented in terms of several nonzero symbolic constant coefficients of the normal form systems. This is practically important in many such control engineering applications.

On the other hand, note that a good approximation of the heteroclinic curve in the parameter plane of the unfolding of normal form delimits an area where chaos and homoclinic tangle can appear in the three-dimensional system. In addition, a good approximation of

[1]We replace the coefficient $f$ with $s$ to avoid the repetition of the symbol. The symbol $f$ will be used to indicate a function in the following section.

the heteroclinic orbit in the phase plane allows one to calculate a good approximation of the heteroclinic surface in the corresponding three-dimensional system (see, for instance, [66] where the authors compute the surfaces of heteroclinic orbits in the Michelson system).

The paper is organized as follows. In section 2 we develop a recursive algorithm for heteroclinic orbits in planar systems that are perturbations of a non-Hamiltonian integrable planar vector field, giving proofs to guarantee the existence and uniqueness of the solution. The application of the procedure to the generic Hopf-zero normal form appears in section 3. The case of the nonresonant double Hopf normal form is considered in section 4. Finally, some conclusions are included.

## 2. The nonlinear time transformation method.

In this section, we introduce the NTT method for finding the analytical approximation of heteroclinic orbits in perturbed integrable planar vector fields. An algorithm will be demonstrated to obtain the asymptotic expansions up to any desired order.

We consider the vector fields in $\mathbb{R}^2$ of the form

$$\begin{aligned} \dot{x} &= f(x,y) + \varepsilon\left[h_1(x,y) + \mu h_2(x,y)\right], \\ \dot{y} &= g(x,y) + \varepsilon\left[h_3(x,y) + \mu h_4(x,y)\right], \end{aligned} \tag{2.1}$$

where functions $g(x,y)$, $f(x,y)$, and $h_i(x,y)$, $i = 1,2,3,4$, are smooth, and $|\varepsilon| \ll 1$ and $\mu$ are perturbation and control parameters, respectively. Additionally, we assume that there exists a nonzero integrating factor $u(x,y)$ such that

$$(uf)_x + (ug)_y = 0, \tag{2.2}$$

where the subscripts denote the partial derivatives with respect to the corresponding variables. Then the unperturbed system of (2.1) is integrable with the first integral $I(x,y)$ satisfying

$$\frac{\partial I}{\partial y} = -uf \quad \text{and} \quad \frac{\partial I}{\partial x} = ug. \tag{2.3}$$

Note that the unperturbed system is Hamiltonian if $u(x,y) \equiv C \in \mathbb{R} \setminus \{0\}$. Moreover, we also assume that the unperturbed system (when $\varepsilon = 0$) possesses a heteroclinic orbit $\Gamma_0$ connecting two hyperbolic saddles.

Following the notation of [22, 1, 4], we introduce the NTT $\varphi(t,\varepsilon)$ of the form

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Phi(\varphi,\varepsilon), \quad \Phi(\varphi + 2\pi, \varepsilon) = \Phi(\varphi,\varepsilon). \tag{2.4}$$

The function $\Phi(\varphi,\varepsilon)$ depending on the value of $\varepsilon$ shows the relation between the physical time variable $t$ and the new one $\varphi$. It will be regarded as one unknown when solving the problem. Through the procedure of the NTT method, its power series solution will be explicitly solved. Moreover, we let $\Phi$ be periodic in $\varphi$ so that we can convert a heteroclinic orbit which evolves over an infinite time interval into one with a periodic interval. The period is chosen as $2\pi$ so that trigonometric functions can be used. If $\varphi(t,\varepsilon) \to \varphi_\pm$ as $t \to \pm\infty$ for the heteroclinic orbit, then we should impose the following condition:

$$\Phi(\varphi_\pm, \varepsilon) = 0 \quad \text{for all} \quad \varepsilon, \tag{2.5}$$

since the orbit has “infinite period”.

Then, we obtain system (2.1) in the $\varphi$ domain as

$$x'\Phi = f(x,y) + \varepsilon[h_1(x,y) + \mu h_2(x,y)],$$
$$y'\Phi = g(x,y) + \varepsilon[h_3(x,y) + \mu h_4(x,y)], \tag{2.6}$$

where $'$ denotes the derivative with respect to $\varphi$.

This system has three unknowns, $x(\varphi,\varepsilon)$, $y(\varphi,\varepsilon)$, and $\Phi(\varphi,\varepsilon)$, in contrast with system (2.1), which only has two, $x(t)$ and $y(t)$. However, we can reduce that number by choosing a smooth function for $x(\varphi,\varepsilon)$ such that it describes the connecting orbit. Then, we can solve $y(\varphi,\varepsilon)$ and $\Phi(\varphi,\varepsilon)$ from (2.6) with the perturbation method, and the value of the parameter $\mu$ for the global bifurcation can also be calculated.

Note that the bifurcation value of $\mu$ provided by the NTT method does not depend on the function $x(\varphi,\varepsilon)$ chosen. Usually, we use trigonometric functions for $x(\varphi,\varepsilon)$ due to the $2\pi$-periodicity. Obviously, the expression of $x(\varphi,\varepsilon)$ depends on the shape of the heteroclinic orbit, and it could be much more complicated than that in an autonomous oscillator. For the heteroclinic orbit $(x(t),y(t))$ in an autonomous oscillator, it always lies in the upper-half (resp., lower-half) plane, and $x(t)$ is strictly increasing (resp., decreasing) since we have $\dot{x} = y$. However, for a general planar vector field, it is possible to possess a more complex orbit which has several turns (i.e., $\dot{x}(t) = 0$ for some $t$). To approximate such orbits, a complicated expression for $x(\varphi,\varepsilon)$ is needed, and, consequently, the computations and discussions become tedious and cumbersome. Therefore, in this paper, we only consider the simplest case—that the heteroclinic orbit does not have turns. More precisely, we give the following definition.

Definition 2.1. *A heteroclinic orbit $(x(t),y(t))$ of a planar vector field is simple if it connects two hyperbolic saddles and $x(t)$ is a strictly monotonic function.*

We remark that, generically, the heteroclinic orbits associated with Hopf-zero and double Hopf singularities are not necessarily simple. Nevertheless, by using a nonlinear coordinate transformation, the system involving such singularities can be converted into corresponding truncated normal forms. Furthermore, as we will see later (in sections 3 and 4), the heteroclinic orbits in both truncated normal forms (1.1) and (1.2) are simple under certain coordinate rotation. Now, for a simple heteroclinic orbit, we can always use the following expression to find the approximation:

$$x(\varphi,\varepsilon) = p(\varepsilon)\cos\varphi + q(\varepsilon), \quad p(\varepsilon) < 0, \tag{2.7}$$

where $p(\varepsilon)$ and $q(\varepsilon)$ are symbolic constants to be determined. Recall that the heteroclinic orbit that we study evolves over a periodic time interval with period $2\pi$. Now, with (2.7) provided, the heteroclinic orbit approaches the left equilibrium as $\varphi \to 0$ and $\varphi \to 2\pi$, and approaches the right one as $\varphi \to \pi$. As $\varphi$ increases from 0 to $\pi$, (2.7) describes the heteroclinic orbit once from the left equilibrium to the right one. As $\varphi$ keeps increasing from $\pi$ to $2\pi$, it describes the orbit again in a reverse direction. We note that it is enough to consider system (2.6) for $\varphi \in [0,\pi]$ due to the symmetry (i.e., $x(\varphi,\varepsilon) = x(2\pi - \varphi,\varepsilon)$).

Next, we seek a perturbation solution for the heteroclinic orbit. Assume that the analytical result can be expressed by power series of $\varepsilon$ as

$$x(\varphi,\varepsilon) = p(\varepsilon)\cos\varphi + q(\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i x_i(\varphi) = \sum_{i=0}^{\infty}\varepsilon^i(p_i\cos\varphi + q_i),$$

$$y(\varphi,\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i y_i(\varphi), \quad \Phi(\varphi,\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i\Phi_i(\varphi), \quad \mu(\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i\mu_i, \tag{2.8}$$

where $p_i$, $q_i$, and $\mu_i$ are reals, and $y_i(\varphi)$ and $\Phi_i(\varphi)$ are $2\pi$-periodic functions in $\varphi$.

Substituting (2.8) into (2.6), expanding the functions $g(x(\varphi,\varepsilon))$, $f(x(\varphi,\varepsilon), y(\varphi,\varepsilon))$, and $h(x(\varphi,\varepsilon), y(\varphi,\varepsilon))$ in Taylor series around $\varepsilon = 0$, and equating the coefficients of equal powers of $\varepsilon$, we obtain the following equations for $\mathcal{O}(\varepsilon^0)$:

$$x_0'\Phi_0 = f_0, \tag{2.9a}$$

$$y_0'\Phi_0 = g_0. \tag{2.9b}$$

For $\mathcal{O}(\varepsilon^i)(i \in \mathbb{Z}^+)$ we obtain

(2.10a)

$$\sum_{k=0}^{i} x_k'\Phi_{i-k} = \frac{1}{i!}\frac{\mathrm{d}^i f(x,y)}{\mathrm{d}\varepsilon^i}\bigg|_{\varepsilon=0} + \frac{1}{(i-1)!}\frac{\mathrm{d}^{(i-1)}h_1(x,y)}{\mathrm{d}\varepsilon^{(i-1)}}\bigg|_{\varepsilon=0} + \sum_{k=0}^{i-1}\mu_{i-k-1}\frac{1}{k!}\frac{\mathrm{d}^k h_2(x,y)}{\mathrm{d}\varepsilon^k}\bigg|_{\varepsilon=0},$$

(2.10b)

$$\sum_{k=0}^{i} y_k'\Phi_{i-k} = \frac{1}{i!}\frac{\mathrm{d}^i g(x,y)}{\mathrm{d}\varepsilon^i}\bigg|_{\varepsilon=0} + \frac{1}{(i-1)!}\frac{\mathrm{d}^{(i-1)}h_3(x,y)}{\mathrm{d}\varepsilon^{(i-1)}}\bigg|_{\varepsilon=0} + \sum_{k=0}^{i-1}\mu_{i-k-1}\frac{1}{k!}\frac{\mathrm{d}^k h_4(x,y)}{\mathrm{d}\varepsilon^k}\bigg|_{\varepsilon=0},$$

where $f_0 = f(x_0(\varphi), y_0(\varphi))$ and $g_0 = g(x_0(\varphi), y_0(\varphi))$.

### 2.1. The zero-order solution.

First, we are going to find the zero-order solution $p_0$, $q_0$, $y_0$, and $\Phi_0$ from (2.9a)–(2.9b). Assume that the unperturbed simple heteroclinic orbit connects two saddles $E_1 = (\alpha_1, \beta_1)$ and $E_2 = (\alpha_2, \beta_2)$ $(\alpha_1 < \alpha_2)$. It is worth pointing out that the two equilibria remain invariant under the NTT, that is, the positions of them are identical in both systems (2.1) and (2.6). Then, we have

$$p_0 + q_0 = \alpha_1 \quad \text{and} \quad q_0 - p_0 = \alpha_2, \tag{2.11}$$

from which $p_0$ and $q_0$ can be determined as

$$p_0 = \frac{1}{2}(\alpha_1 - \alpha_2), \quad q_0 = \frac{1}{2}(\alpha_1 + \alpha_2). \tag{2.12}$$

As we have mentioned previously, the unperturbed system of (2.1) is integrable with a first integral $I(x,y)$. Then, every orbit in the unperturbed system is given by the level curve $I(x,y) = C$, $C \in \mathbb{R}$. Since the heteroclinic orbit connects the equilibrium $E_1$, the orbit $(x_0, y_0)$ satisfies

$$I(x_0, y_0) = I(\alpha_1, \beta_1), \tag{2.13}$$

from which $y_0$ can be solved. We remark that $I(x_0, y_0) = I(\alpha_2, \beta_2)$ gives the same function as (2.13) since the heteroclinic orbit also connects the equilibrium $E_2$. Moreover, we can derive the following property for $y_0$:

$$y_0(0) = \beta_1 \quad \text{and} \quad y_0(\pi) = \beta_2. \tag{2.14}$$

Then, from (2.9a), $\Phi_0$ can be obtained as

$$\Phi_0 = \frac{f(x_0, y_0)}{x_0'}. \tag{2.15}$$

For the denominator, we have $x_0'(\varphi) = 0$ but $x_0''(\varphi) \neq 0$ at $\varphi \in \{0, \pi\}$. We also have $f(x_0(0), y_0(0)) = f(x_0(\pi), y_0(\pi)) = 0$ for the numerator. Now, we want to show that $\Phi_0$ vanishes at $\varphi \in \{0, \pi\}$ so that the "infinite period" of the orbit can be guaranteed. Taking the derivative of (2.13) with respect to $\varphi$, we get

$$\frac{\partial I(x_0, y_0)}{\partial x_0} x_0' + \frac{\partial I(x_0, y_0)}{\partial y} y_0' = u(x_0, y_0) g_0 x_0' - u(x_0, y_0) f_0 y_0' = 0 \quad \text{for all} \quad \varphi \in [0, \pi]. \tag{2.16}$$

Thus, we have

$$g_0 x_0' = f_0 y_0' \quad \text{for all} \quad \varphi \in [0, \pi]. \tag{2.17}$$

The computation of derivative of (2.17) at $\varphi = 0$ leads to

$$\frac{\partial f_0}{\partial y_0}(x_0(0), y_0(0))[y_0'(0)]^2 = 0. \tag{2.18}$$

Consequently, we have

$$\Phi_0(0) = \lim_{\varphi \to 0} \Phi_0 = \lim_{\varphi \to 0} \frac{f(x_0, y_0)}{x_0'} = \lim_{\varphi \to 0} \frac{1}{x_0''} \left( \frac{\partial f_0}{\partial x_0} x_0' + \frac{\partial f_0}{\partial y_0} y_0' \right) = 0. \tag{2.19}$$

Thus, $\Phi_0$ is continuous and vanishes at $\varphi = 0$, which implies that it is smooth at the same value. We can also get that $\Phi_0$ vanishes and is smooth at $\varphi = \pi$ in an analogous way. Consequently, $\Phi_0$ is smooth in the interval $[0, \pi]$.

**2.2. The $i$th-order ($i \in \mathbb{Z}^+$) solution.** Once the zero-order solution is obtained, the $i$th-order solution can be found with an iterative algorithm. In order to find an appropriate solution for the heteroclinic orbit, we look for a solution satisfying

$$\Phi_i(0) = \Phi_i(\pi) = 0 \quad \text{for} \quad i \in \mathbb{Z}^+, \tag{2.20}$$

which immediately implies that $\Phi(0, \varepsilon) = \Phi(\pi, \varepsilon) = 0$ for all $\varepsilon$, and, therefore, it can be guaranteed that the solution obtained by the NTT method approaches the two equilibria as $\varphi \to 0$ and $\varphi \to \pi$. Assume that, for the equations in $\mathcal{O}(\varepsilon^i)$, $p_j$, $q_j$, $\mu_{j-1}$, $y_j$, and $\Phi_j$ ($j = 1, 2, \ldots, i-1$) are inductively known constants and functions, and they are obtained iteratively. Then, we are going to solve $p_i$, $q_i$, $\mu_{i-1}$, $y_i$, and $\Phi_i$ from (2.10a)–(2.10b). We want to rewrite the two equations such that the left-hand sides only contain the terms with unknowns.

To this end, we expand the terms on the right-hand side of (2.10b) by the multivariate Faà di Bruno formula [30]. For $f(x,y)$, we have

$$\left.\frac{\mathrm{d}^i f(x,y)}{\mathrm{d}\varepsilon^i}\right|_{\varepsilon=0} = \sum_{1\le\alpha_1+\alpha_2\le i} \frac{\partial^{(\alpha_1+\alpha_2)} f(x_0,y_0)}{\partial x_0^{\alpha_1}\partial y_0^{\alpha_2}} \sum_{s=1}^{i} \sum_{p_{s,i}(\alpha_1,\alpha_2)} i! \prod_{j=1}^{s} \frac{x_{\beta_j}^{m_j} y_{\beta_j}^{n_j}}{m_j! n_j!}, \tag{2.21}$$

where

$$p_{s,i}(\alpha_1,\alpha_2) = \{(m_1, m_2, \dots, m_s, n_1, n_2, \dots, n_s, \beta_1, \beta_2, \dots, \beta_s) : m_j, n_j, \beta_j \in \mathbb{N}\cup\{0\}, m_j^2 + n_j^2 \ne 0,$$
$$0 < \beta_1 < \cdots < \beta_s,\ \sum_{j=1}^{s} m_j = \alpha_1,\ \sum_{j=1}^{s} n_j = \alpha_2, \text{ and } \sum_{j=1}^{s}(m_j+n_j)\beta_j = i\}.$$

Analogously, the other terms can also be expanded using the above formula. Then, we can find the terms with unknowns in the expansions, and (2.10a)–(2.10b) can be rewritten as

$$x_0'\Phi_i + x_i'\Phi_0 - \frac{\partial f_0}{\partial x_0}x_i - \frac{\partial f_0}{\partial y_0}y_i - \mu_{i-1}h_2(x_0,y_0) = R_{i,1}(\varphi), \tag{2.22a}$$

$$y_0'\Phi_i + y_i'\Phi_0 - \frac{\partial g_0}{\partial x_0}x_i - \frac{\partial g_0}{\partial y_0}y_i - \mu_{i-1}h_4(x_0,y_0) = R_{i,2}(\varphi), \tag{2.22b}$$

where $R_{i,1}$ and $R_{i,2}$ represent the remaining terms in the corresponding equation.

Eliminating $\Phi_i$ between (2.22a) and (2.22b) leads to

$$\begin{aligned} -x_0'\Phi_0 y_i' + &\left(x_0'\frac{\partial g_0}{\partial y_0} - y_0'\frac{\partial f_0}{\partial y_0}\right) y_i + y_0'\Phi_0 x_i' + \left(x_0'\frac{\partial g_0}{\partial x_0} - y_0'\frac{\partial f_0}{\partial x_0}\right) x_i \\ &+ \mu_{i-1}[x_0'h_4(x_0,y_0) - y_0'h_2(x_0,y_0)] = R_{i,1}x_0' - R_{i,2}y_0'. \end{aligned} \tag{2.23}$$

Before solving (2.23), we first define the following function:

$$G(w,v)(\varphi) = u(x_0(\varphi), y_0(\varphi))\left[w(\varphi)y_0'(\varphi) - v(\varphi)x_0'(\varphi)\right]. \tag{2.24}$$

Multiplying both sides of (2.23) with the integrating factor $u_0 = u(x_0,y_0)$ and considering the zero-order equation, given by (2.9a)–(2.9b), we obtain

$$\begin{aligned} -u_0 f_0 y_i' - G\left(\frac{\partial f_0}{\partial y_0}, \frac{\partial g_0}{\partial y_0}\right) y_i + u_0 g_0 x_i' - G\left(\frac{\partial f_0}{\partial x_0}, \frac{\partial g_0}{\partial x_0}\right) x_i \\ - \mu_{i-1} G(h_2(x_0,y_0), h_4(x_0,y_0)) = G(R_{i,1}, R_{i,2}). \end{aligned} \tag{2.25}$$

From (2.9a)–(2.9b), we have

$$f_0 y_0' = g_0 x_0'. \tag{2.26}$$

Then, differentiating $u_0 g_0$ with respect to $\varphi$ and considering (2.2) lead to

$$\begin{aligned} (u_0g_0)' &= \frac{\partial g_0}{\partial x_0}u_0x_0' + \frac{\partial u_0}{\partial x_0}g_0x_0' + \frac{\partial u_0}{\partial y_0}g_0y_0' + \frac{\partial g_0}{\partial y_0}u_0y_0' \\ &= \frac{\partial g_0}{\partial x_0}u_0x_0' + \left(\frac{\partial u_0}{\partial x_0}f_0 + \frac{\partial u_0}{\partial y_0}g_0 + \frac{\partial g_0}{\partial y_0}u_0\right)y_0' = u_0\left(\frac{\partial g_0}{\partial x_0}x_0' - \frac{\partial f_0}{\partial x_0}y_0'\right) = -G\left(\frac{\partial f_0}{\partial x_0}, \frac{\partial g_0}{\partial x_0}\right). \end{aligned} \tag{2.27}$$

Analogously, we can also derive

$$(u_0 f_0)' = G\left(\frac{\partial f_0}{\partial y_0}, \frac{\partial g_0}{\partial y_0}\right). \tag{2.28}$$

Substituting (2.27) and (2.28) into (2.25), we obtain

$$\begin{aligned} &-u_0 f_0 y_i' - (u_0 f_0)' y_i + u_0 g_0 x_i' + (u_0 g_0)' x_i - \mu_{i-1} G(h_2(x_0, y_0), h_4(x_0, y_0)) \\ &\quad = (-u_0 f_0 y_i + u_0 g_0 x_i)' - \mu_{i-1} G(h_2(x_0, y_0), h_4(x_0, y_0)) = G(R_{i,1}, R_{i,2}). \end{aligned} \tag{2.29}$$

Integrating (2.29) with respect to $\varphi$ leads to

$$-u_0 f_0 y_i + u_0 g_0 x_i - \mu_{i-1} \int_0^\varphi G(h_2(x_0, y_0), h_4(x_0, y_0))(\tau)\mathrm{d}\tau = \int_0^\varphi G(R_{i,1}, R_{i,2})(\tau)\mathrm{d}\tau. \tag{2.30}$$

To solve $p_i$, $q_i$, and $\mu_{i-1}$, we consider the conditions at two hyperbolic saddles. The heteroclinic orbit approaches the equilibria as $\varphi \to 0$ and $\varphi \to \pi$. We can obtain four algebraic equations with unknowns $p_i$, $q_i$, $y_i(0)$, $y_i(\pi)$, and $\mu_{i-1}$ by letting $\varphi = 0$ and $\varphi = \pi$ in (2.22a) and (2.22b). Moreover, if we let $\varphi = \pi$ in (2.30), we can get another algebraic equation with the same unknowns. Then, we have the following system of linear equations:

$$\mathbf{A}(p_i, q_i, y_i(0), y_i(\pi), \mu_{i-1})^\top = \mathbf{B}_i, \tag{2.31}$$

with

(2.32)

$$\mathbf{A} = \begin{pmatrix} -\left.\frac{\partial f_0}{\partial x_0}\right|_{\varphi=0} & -\left.\frac{\partial f_0}{\partial x_0}\right|_{\varphi=0} & -\left.\frac{\partial f_0}{\partial y_0}\right|_{\varphi=0} & 0 & -h_2(x_0(0), y_0(0)) \\ \left.\frac{\partial f_0}{\partial x_0}\right|_{\varphi=\pi} & -\left.\frac{\partial f_0}{\partial x_0}\right|_{\varphi=\pi} & 0 & -\left.\frac{\partial f_0}{\partial y_0}\right|_{\varphi=\pi} & -h_2(x_0(\pi), y_0(\pi)) \\ -\left.\frac{\partial g_0}{\partial x_0}\right|_{\varphi=0} & -\left.\frac{\partial g_0}{\partial x_0}\right|_{\varphi=0} & -\left.\frac{\partial g_0}{\partial y_0}\right|_{\varphi=0} & 0 & -h_4(x_0(0), y_0(0)) \\ \left.\frac{\partial g_0}{\partial x_0}\right|_{\varphi=\pi} & -\left.\frac{\partial g_0}{\partial x_0}\right|_{\varphi=\pi} & 0 & -\left.\frac{\partial g_0}{\partial y_0}\right|_{\varphi=\pi} & -h_4(x_0(\pi), y_0(\pi)) \\ 0 & 0 & 0 & 0 & -\int_0^\pi G(h_2(x_0, y_0), h_4(x_0, y_0))\mathrm{d}\varphi \end{pmatrix}$$

and

$$\mathbf{B}_i = \begin{pmatrix} R_{i,1}(0) \\ R_{i,1}(\pi) \\ R_{i,2}(0) \\ R_{i,2}(\pi) \\ \int_0^\pi G(R_{i,1}, R_{i,2})\mathrm{d}\varphi \end{pmatrix}. \tag{2.33}$$

Proposition 2.2. *System (2.31) has a unique solution if and only if*

$$\int_{\Gamma_0} u(x, y)h_2(x, y)dy - u(x, y)h_4(x, y)\mathrm{d}x \neq 0,$$

*where $u(x, y)$ is the integrating factor satisfying (2.2) and $\Gamma_0$ stands for the heteroclinic orbit.*

*Proof.* System (2.31) has a unique solution if and only if

$$\det(\mathbf{A}) = -2J(0)J(\pi)\int_0^\pi G(h_2(x_0,y_0),h_4(x_0,y_0))\mathrm{d}\varphi \neq 0, \tag{2.34}$$

where

$$J(\varphi) = \frac{\partial f_0}{\partial x_0}\frac{\partial g_0}{\partial y_0} - \frac{\partial f_0}{\partial y_0}\frac{\partial g_0}{\partial x_0}.$$

Since the zero-order solution describes the unperturbed heteroclinic orbit as $\varphi$ increases from 0 to $\pi$, we have

$$\int_0^\pi G(h_2(x_0,y_0),h_4(x_0,y_0))\mathrm{d}\varphi = \int_{\Gamma_0} u(x,y)h_2(x,y)dy - u(x,y)h_4(x,y)\mathrm{d}x.$$

Hence, this proposition is equivalent to $J(0)J(\pi) \neq 0$.

Recall that the zero-order solution approaches the two equilibria as $\varphi \to 0$ and $\varphi \to \pi$. Thus, $J(0)$ and $J(\pi)$ are the determinants of the linearization matrix at $E_1$ and $E_2$, respectively. Since the two equilibria are hyperbolic saddles, we have $J(0) < 0$ and $J(\pi) < 0$. This completes the proof. ∎

Once the unique solution of (2.31) is determined, $y_i$ and $\Phi_i$ can be solved from (2.30) and (2.22a), respectively.

Since the coefficient of $\Phi_i$ in (2.22a) vanishes at $\varphi \in \{0,\pi\}$, it is necessary to show that $\Phi_i$ is smooth at the same values. To this end, we introduce the following proposition.

Proposition 2.3. *$\Phi_i(\varphi)$ obtained from (2.22a) is smooth in the interval $[0,\pi]$.*

*Proof.* From (2.22a), $\Phi_i(\varphi)$ can be expressed as

$$\Phi_i(\varphi) = \frac{R_{i,1}(\varphi) - x_i'\Phi_0 + \dfrac{\partial f_0}{\partial x_0}x_i + \dfrac{\partial f_0}{\partial y_0}y_i + \mu_{i-1}h_2(x_0,y_0)}{x_0'(\varphi)} :\equiv \frac{F(\varphi)}{x_0'(\varphi)}. \tag{2.35}$$

Recall that $x_0'(\varphi) = 0$ and $x_0''(\varphi) \neq 0$ for $\varphi \in \{0,\pi\}$. In addition, from the first two equations of system (2.31), $F(\varphi)$ also vanishes at $\varphi \in \{0,\pi\}$. This implies that $\Phi_i$ is smooth at these values. ∎

The uniqueness of $y_i$ and $\Phi_i$ can be verified by the following corollary.

Corollary 2.4. *$y_i$ and $\Phi_i$ obtained from (2.30) and (2.22a) are unique.*

*Proof.* Assume that $(y_i,\Phi_i)$ and $(y_i^*,\Phi_i^*)$ are two sets of solutions to (2.30) and (2.22a). Let $v = y_i - y_i^*$, $\Psi = \Phi_i - \Phi_i^*$. By Proposition 2.2, $x_i$ and $\mu_{i-1}$ are unique. Then, $v$ and $\Psi$ satisfy

$$x_0'\Psi - \frac{\partial f_0}{\partial y_0}v = 0 \quad \text{and} \quad -u_0 f_0 v = 0. \tag{2.36}$$

As the determinant of the coefficient matrix of this linear system is

$$-x_0'(\varphi)u_0(\varphi)f_0(\varphi) \neq 0 \;\text{ for all } \varphi \in (0,\pi),$$

it follows that $v(\varphi) = \Psi(\varphi) = 0$ for all $\varphi \in (0, \pi)$. As both functions are continuous in $[0, \pi]$, they must satisfy

$$v(0) = v(\pi) = \Psi(0) = \Psi(\pi) = 0,$$

and, therefore, $v(\varphi) \equiv 0$, $\Psi(\varphi) \equiv 0$ in $[0, \pi]$. This completes the proof. ■

In general, we have not been able to prove that $\Phi_i(0) = \Phi_i(\pi) = 0$ for all $i \geq 0$. However, the following two propositions give sufficient conditions to guarantee that the solution we are calculating corresponds to a heteroclinic orbit. They will be useful in the two cases studied in sections 3 and 4.

Proposition 2.5. *If $y_0'(0) \neq 0$ and $y_0'(\pi) \neq 0$, then $\Phi_i(0) = \Phi_i(\pi) = 0$.*

*Proof.* From the third equation of system (2.31) together with (2.22b), one is able to derive $y_0'(0)\Phi_i(0) = 0$. By continuity, we have $\Phi_i(0) = 0$ if $y_0'(0) \neq 0$. Considering the fourth equation of system (2.31) together with (2.22b), we can also deduce that $\Phi_i(\pi) = 0$ if $y_0'(\pi) \neq 0$ in an analogous fashion. ■

Proposition 2.6. *If $det(\mathbf{M}(0)) \neq 0$ and $det(\mathbf{M}(\pi)) \neq 0$, then $\Phi_i(0) = \Phi_i(\pi) = 0$, where*

$$\mathbf{M}(\varphi) = \begin{pmatrix} x_0''(\varphi) & -\left.\dfrac{\partial f_0}{\partial y_0}\right|_{\varphi} \\ y_0''(\varphi) & \Phi_0'(\varphi) - \left.\dfrac{\partial g_0}{\partial y_0}\right|_{\varphi} \end{pmatrix}. \tag{2.37}$$

*Proof.* The computation of the derivatives of (2.22a) and (2.22b) at $\varphi = 0$ leads to

$$\begin{pmatrix} x_0''(0) & -\left.\dfrac{\partial f_0}{\partial y_0}\right|_{\varphi=0} \\ y_0''(0) & \Phi_0'(0) - \left.\dfrac{\partial g_0}{\partial y_0}\right|_{\varphi=0} \end{pmatrix} \begin{pmatrix} \Phi_i(0) \\ y_i'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.38}$$

Therefore, if $det(\mathbf{M}(0)) \neq 0$, then $\Phi_i(0) = y_i'(0) = 0$. Analogously, we can obtain that $\Phi_i(\pi) = y_i'(\pi) = 0$. ■

Propositions 2.5 and 2.6 give sufficient conditions to ensure that $\Phi_i(0) = \Phi_i(\pi) = 0$ for all $i$, or equivalently

$$\Phi(0, \varepsilon) = \Phi(\pi, \varepsilon) = 0 \quad \text{for all} \quad \varepsilon. \tag{2.39}$$

These conditions guarantee that the solution provided by the NTT method corresponds to a heteroclinic orbit because it approaches the two equilibria as $\varphi \to 0$ and $\varphi \to \pi$.

For any perturbed integrable planar system which admits a simple heteroclinic orbit, one is able to find the corresponding asymptotic expansions with the above procedure. For convenience, we summarize it in Algorithm 2.1.

## 3. Applications to generic Hopf-zero singularity.

In this section, we first apply the method presented in section 2 to the cubic parametric normal form system (1.1) related to the unfolding of the generic Hopf-zero singularity (see, for instance, [44, 55, 68, 29]). Our

**Algorithm 2.1** Finding asymptotic expansions for the heteroclinic bifurcation.

**Input:** Planar perturbed integrable system which admits a simple heteroclinic orbit; desired order $N$

**Output:** Asymptotic expansions for the heteroclinic bifurcation up to desired order

1: Define the NTT (2.4) and transform the equation into $\varphi$ domain
2: Write the phase variables $x(\varphi, \varepsilon)$, $y(\varphi, \varepsilon)$, nonlinear function $\Phi(\varphi, \varepsilon)$, and the control parameter $\mu(\varepsilon)$ into power series of $\varepsilon$ as (2.8)
3: Let $\varepsilon = 0$ in (2.6) and do the items 4–8 below
4: Find two hyperbolic equilibria $E_1$ and $E_2$
5: Solve $p_0$ and $q_0$ from (2.12)
6: Compute the first integral and the unperturbed heteroclinic orbit $\Gamma_0$
7: Solve $y_0$ from $\Gamma_0$
8: Solve $\Phi_0$ from one of the unperturbed equation
9: Find matrix $\mathbf{A}$ and verify that the determinant is nonzero
10: Substitute the power series obtained in item 2 into the perturbed system (2.6)
11: **for** each integer $i \in [1, N]$ **do**
12: Use the required information from the previous items and find the coefficients of $\varepsilon^i$ in the two equations (2.22a)–(2.22b) and compute $R_{i,1}(\varphi)$, $R_{i,2}(\varphi)$
13: Evaluate vector $\mathbf{B}_i$
14: Solve $p_i$, $q_i$, and $\mu_{i-1}$ from (2.31)
15: Find $y_i$ from (2.30)
16: Solve $\Phi_i$ from (2.22a)
17: **end for**
18: **return** The expansions of $x(\varphi, \varepsilon)$, $y(\varphi, \varepsilon)$, $\Phi(\varphi, \varepsilon)$, and $\mu(\varepsilon)$ up to order $N$

goal is to find the asymptotic expansions of the heteroclinic bifurcation curve $\mu_1 = \mu_1^*(\mu_2)$ in the parameter space $\mu_1$-$\mu_2$ for both $|\mu_1|$ and $|\mu_2| \ll 1$. At the same time, we will obtain approximations for the heteroclinic orbits in the phase space. The situation we consider corresponds to the case III discussed in [44, sect. 7.4]. Note that only a first-order approximation has been obtained by the Melnikov method [51, 44]. However, as we are going to see, our procedure allows us to obtain the approximation up to any desired order for all values of the nonzero normal form coefficients $a$ and $s$.

**3.1. Bifurcation analysis.** Before applying the NTT method to approximate the heteroclinic orbit, we give some information on the equilibria and on the bifurcation set close to the origin in the parameter space $\mu_1$-$\mu_2$.

Note that the system is symmetric with respect to the $z$-axis, and it is also invariant under the transformation

$$(r(t), z(t), a, \mu_1, \mu_2, s) \to (r(-t), -z(-t), a, -\mu_1, \mu_2, -s). \tag{3.1}$$

Thus, it is sufficient to consider the system on the half plane $r \geq 0$ with $s > 0$.

A partial bifurcation set is depicted in Figure 1. For $s < 0$, it can be found in [44, Figure 7.4.10], where, in the same section, the authors carefully study the bifurcations of equilibria when $s = 0$. Here, we consider system (1.1) when $s$ is nonzero.

System (1.1) possesses up to five equilibria. Three of them are located on the $z$-axis, $E_a = (0, z_a)$, $E_b = (0, z_b)$, and $E_c = (0, z_c)$, where $z_a \leq z_b \leq z_c$, and are determined by the algebraic equation

$$\mu_2 - z^2 + sz^3 = 0. \tag{3.2}$$

The other two equilibria are given by

$$E_\pm = \left(\pm r^*, -\frac{\mu_1}{a}\right) = \left(\pm\frac{\sqrt{a(a^3\mu_2 - s\mu_1^3 - a\mu_1^2)}}{a^2}, -\frac{\mu_1}{a}\right). \tag{3.3}$$

Then, we can deduce that the algebraic equation (3.2) has three real solutions if $\mu_2 \in \big(0, 4/(27s^2)\big)$. Otherwise, it has one, and only one, real solution. Consequently, there exist two saddle-node bifurcations **SN**. One occurs when $\mu_2 = 0$ (see Figure 1), and it involves the two equilibria $E_a$ and $E_b$. The other one happens for $\mu_2 = 4/(27s^2)$ and involves $E_b$ and $E_c$, but it is out of the scope of our study because it is far from the origin. In addition, we can also deduce that $z_a \leq 0 \leq z_b \leq 2/(3s) \leq z_c$.

The eigenvalues of the linearization matrix at the three equilibria are $3sz^2 - 2z$ and $az + \mu_1$, $z \in \{z_a, z_b, z_c\}$. Therefore, for $\mu_1 < 0$, the equilibrium $E_a$ is always a saddle while $E_c$ is always an unstable node if they exist. Remark that the equilibrium $E_c$ is not shown in Figure 1 because it is not in the neighborhood of the origin. For the other equilibrium $E_b$, the stability changes when it exhibits a pitchfork bifurcation **P** (see Figure 1) together with the two equilibria $E_\pm$, and this bifurcation is given by the transition variety

$$a^3\mu_2 - \mu_1^2(a + s\mu_1) = 0. \tag{3.4}$$

$E_b$ is a stable node for the parameters between the curves **P** and **SN**, and it becomes a saddle when parameters pass through the curve **P**.

For the other two equilibria $E_\pm$, the characteristic polynomial of their linearization matrix is given by

$$p(\lambda) = \lambda^2 - \frac{\mu_1(2a + 3s\mu_1)}{a^2}\lambda + \frac{2(a^3\mu_2 - s\mu_1^3 - a\mu_1^2)}{a^2}. \tag{3.5}$$

It can be easily derived from (3.5) that the equilibria $E_\pm$ undergo a subcritical Hopf bifurcation **H** when $\mu_1 = 0$ and $\mu_2 > 0$. We remark that there exists another Hopf bifurcation if $\mu_1 = -2a/(3s)$ and $\mu_2 > 4/(27s^2)$. This bifurcation occurs far from the origin; therefore, it is not shown in the partial bifurcation set. As parameters cross the Hopf bifurcation **H**, an unstable periodic orbit arises, and it collides with the two saddles $E_a$ and $E_b$ as $\mu_1$ decreases, giving rise to a heteroclinic cycle formed by two orbits. One of them is structurally stable as it lies in the invariant subspace $r = 0$. The important orbit is the second one, which is located in the right-half plane.

### 3.2. Asymptotic expansions.

Our goal is to find an asymptotic expansion for the curve **Het** where the heteroclinic cycle exists (see Figure 1) with the NTT method. Recall that the situation we consider corresponds to the case III discussed in [44, sect. 7.4]. Obviously,

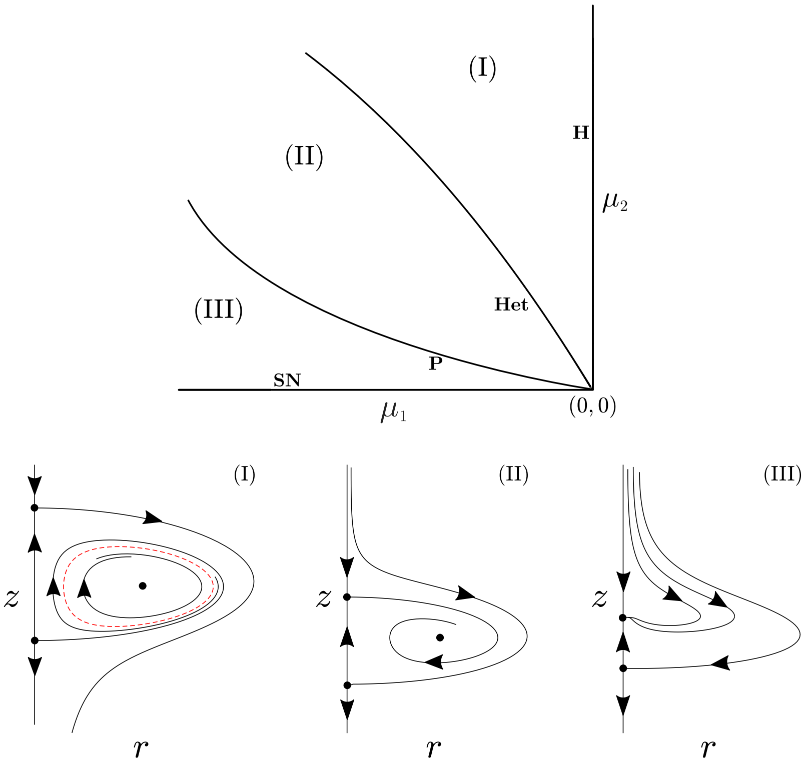


**Figure 1.** *Partial qualitative bifurcation set for system* (1.1) *with* $a > 0$ *and* $s > 0$.

according to Definition 2.1, the heteroclinic orbit is not simple. In order to apply the method introduced in section 2, we first rotate the original coordinates counterclockwise about the origin by $\pi/2$ with the linear transformation

$$\tilde{r} = -z, \quad \tilde{z} = r, \tag{3.6}$$

and, in this way, the heteroclinic orbit becomes simple. Thus, system (1.1) reads as

$$\begin{aligned} \dot{\tilde{r}} &= -\mu_2 + \tilde{r}^2 + \tilde{z}^2 + s\tilde{r}^3, \\ \dot{\tilde{z}} &= -a\tilde{r}\tilde{z} + \mu_1\tilde{z}, \end{aligned} \tag{3.7}$$

with $a > 0$.

To study the heteroclinic orbit, we use the following blow-up transformation [44, eq. (7.4.28)]:

$$\tilde{r} = \varepsilon x, \ \tilde{z} = \varepsilon y, \ \mu_1 = \varepsilon^2\nu_1, \ \mu_2 = \varepsilon^2\nu_2, \ t = \varepsilon^{-1}\tau. \tag{3.8}$$

Without loss of generality, we let $\nu_2 = 1$. Then, system (3.7) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -1 + x^2 + y^2 + \varepsilon s x^3,$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = -axy + \varepsilon \nu_1 y. \tag{3.9}$$

The unperturbed system is integrable with the integrating factor $u(x,y) = y^{\left(\frac{2}{a}-1\right)}$, and a first integral is given by

$$I(x,y) = \frac{a}{2}\, y^{\frac{2}{a}} \left(1 - x^2 - \frac{1}{1+a} y^2\right). \tag{3.10}$$

Note that a special case occurs when $a = 2$, namely the unperturbed system becomes Hamiltonian (the only case considered in [44, sect. 7.4]).

Applying the NTT (2.4) to time $\tau$ in (3.9) leads to

$$x'\Phi = -1 + x^2 + y^2 + \varepsilon s x^3, \tag{3.11a}$$

$$y'\Phi = -axy + \varepsilon \nu_1 y. \tag{3.11b}$$

System (3.11) is in the form of (2.6) with

$$\mu = \nu_1, \quad f(x,y) = -1 + x^2 + y^2, \quad g(x,y) = -axy,$$
$$h_1(x,y) = sx^3, \quad h_2(x,y) = h_3(x,y) = 0, \quad h_4(x,y) = y.$$

We are going to seek a perturbation solution in the form of

$$x(\varphi,\varepsilon) = p(\varepsilon)\cos\varphi + q(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i x_i(\varphi) = \sum_{i=0}^{\infty} \varepsilon^i (p_i \cos\varphi + q_i),$$

$$y(\varphi,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i y_i(\varphi), \quad \Phi(\varphi,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \Phi_i(\varphi), \quad \nu_1(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \nu_{1,i}. \tag{3.12}$$

The heteroclinic orbit in the unperturbed system connects the hyperbolic saddles $(\pm 1, 0)$. Therefore, from (2.13) and (3.10), the orbit $\Gamma_0$ is given by

$$1 - x_0^2 - \frac{1}{1+a} y_0^2 = 0, \quad -1 \le x_0 \le 1. \tag{3.13}$$

Moreover, $p_0$ and $q_0$ can be determined from (2.12) as

$$p_0 = -1, \quad q_0 = 0. \tag{3.14}$$

Then, from (3.13) and (2.15), we are able to find $y_0$ and $\Phi_0$ as

$$y_0 = \sqrt{a+1}\,\sin\varphi \quad \text{and} \quad \Phi_0 = a\sin\varphi. \tag{3.15}$$

Next, Algorithm 2.1 can be applied to look for the $i$th-order solution ($i \in \mathbb{Z}^+$) iteratively. It can be easily verified that

$$\int_{\Gamma_0} u(x,y)h_2(x,y)dy - u(x,y)h_4(x,y)\mathrm{d}x = -\sqrt{a+1}\int_0^\pi (\sin\varphi)^{\frac{2}{a}+1}\mathrm{d}\varphi < 0. \tag{3.16}$$

Thus, by Proposition 2.2, one can obtain a unique solution in each order. Moreover, in this particular case, $y_0'(0) = \sqrt{a+1} \neq 0$ and $y_0'(\pi) = -\sqrt{a+1} \neq 0$. Therefore, from Proposition 2.5 we have that $\Phi(0,\varepsilon) = \Phi(\pi,\varepsilon) = 0$ for all $\varepsilon$, which guarantees that the solution provided by the NTT method corresponds to a heteroclinic orbit.

However, the calculations are very cumbersome due to the presence of fractional power if we find $y_i$ from (2.30) by taking the integration. For instance, in the first order, the hypergeometric functions appear in the solution, and it is not easy to simplify those functions even using an algebraic symbolic computational software such as Maple. To overcome this difficulty, we take advantage of the uniqueness of the solution. We note that the solutions of $y_i$ can be expressed in the following form:

$$y_i = \sum_{k=1}^{i+1} \gamma_{i,k}\sin(k\varphi), \quad \gamma_{i,k} \in \mathbb{R}. \tag{3.17}$$

Then, the reals $\gamma_{i,k}$ can be found by equating the constant and the coefficient of harmonic terms to zero. The detailed procedure is given in Algorithm 3.1, and some Maple codes are commented on correspondingly.

**Algorithm 3.1** Computing heteroclinic bifurcation for generic Hopf-zero singularity.

**Input:** Equations (3.11a)–(3.11b); series (3.12); the zero-order solution; desired order $N$
**Output:** Solutions $x_i$, $y_i$, $\Phi_i$, and $\nu_{1,i-1}$ for $i = 1, 2, \ldots, N$

1: Substituting the series (3.12) into (3.11a) and (3.11b)
2: **for** each integer $i \in [1, N]$ **do**
3: Define $EQ_{1,i}$ :=coefficient of $\varepsilon^i$ in (3.11a)
4: Define $EQ_{2,i}$ :=coefficient of $\varepsilon^i$ in (3.11b)
//Use code *coeff(·, ·)* to collect the coefficients.
5: Solve $\Phi_i$ from $EQ_{1,i}$
//Use code *solve(·, ·)* to solve the equation.
6: Define $y_i := \sum_{k=1}^{i+1} \gamma_{i,k}\sin(k\varphi)$
7: Equating the constant and coefficients of the harmonic terms in $EQ_{2,i}$ to find the values of $\gamma_{i,k}$, $p_i$, $q_i$, and $\nu_{1,i-1}$
//Use code *coeff(·, ·)* to collect the coefficients and *solve(·, ·)* to solve the algebraic equations.
8: **end for**
9: **return** Expressions of $x_i$, $y_i$, and $\Phi_i$, and values of $\nu_{1,i-1}$ for $i = 1, 2, \ldots, N$

By Corollary 2.4, the solution obtained in such a way is equivalent to that solved from integration, and it is very efficient to obtain the high-order solution. For instance, up to the fourth order, the solutions of $x$, $y$, and $\Phi$ are given by

$$\begin{aligned} x(\varphi,\varepsilon) &= -\cos\varphi - \frac{1}{2}\varepsilon s - \frac{5}{8}(\varepsilon s)^2\cos\varphi - (\varepsilon s)^3 - \frac{231}{128}(\varepsilon s)^4\cos\varphi, \\ y(\varphi,\varepsilon) &= \sqrt{a+1}\sin\varphi - \frac{\sqrt{a+1}}{2(3a+2)}\varepsilon s\sin(2\varphi) + (\varepsilon s)^2[\gamma_{2,1}\sin\varphi + \gamma_{2,3}\sin(3\varphi)] \\ &\quad + (\varepsilon s)^3[\gamma_{3,2}\sin(2\varphi) + \gamma_{3,4}\sin(4\varphi)], \\ \Phi(\varphi,\varepsilon) &= a\sin\varphi + \frac{a}{2(3a+2)}\varepsilon s\sin(2\varphi) + (\varepsilon s)^2[\eta_{2,1}\sin\varphi + \eta_{2,3}\sin(3\varphi)] \\ &\quad + (\varepsilon s)^3[\eta_{3,2}\sin(2\varphi) + \eta_{3,4}\sin(4\varphi)], \end{aligned} \tag{3.18}$$

where the coefficients $\gamma_{i,j}$ and $\eta_{i,j}$ are as given in Appendix A. The solution of $\nu_1(\varepsilon)$ up to the seventh order is given by

$$\nu_1(\varepsilon) = -\frac{3a^2 s}{2(3a+2)} + \nu_{1,2}\varepsilon^2 + \nu_{1,4}\varepsilon^4 + \nu_{1,6}\varepsilon^6, \tag{3.19}$$

where the values of $\nu_{1,2}$, $\nu_{1,4}$, and $\nu_{1,6}$ can be found in Appendix A. According to the transformation (3.8), the heteroclinic transition variety in the parameter space is obtained as

$$\mu_1 = -\frac{3a^2 s}{2(3a+2)}\mu_2 + \nu_{1,2}\,\mu_2^2 + \nu_{1,4}\,\mu_2^3 + \nu_{1,6}\,\mu_2^4 + \mathcal{O}(\mu_2^5). \tag{3.20}$$

After several changes of variables, it is possible to check that the first term of the above expression coincides with the first-order approximation given in [51, eq. (3.18)] and [29, sect. 4.6]. To the best of our knowledge, this is the first time that the high-order terms appear in the literature.

We have mentioned that the unperturbed system is Hamiltonian when $a = 2$, and, in this

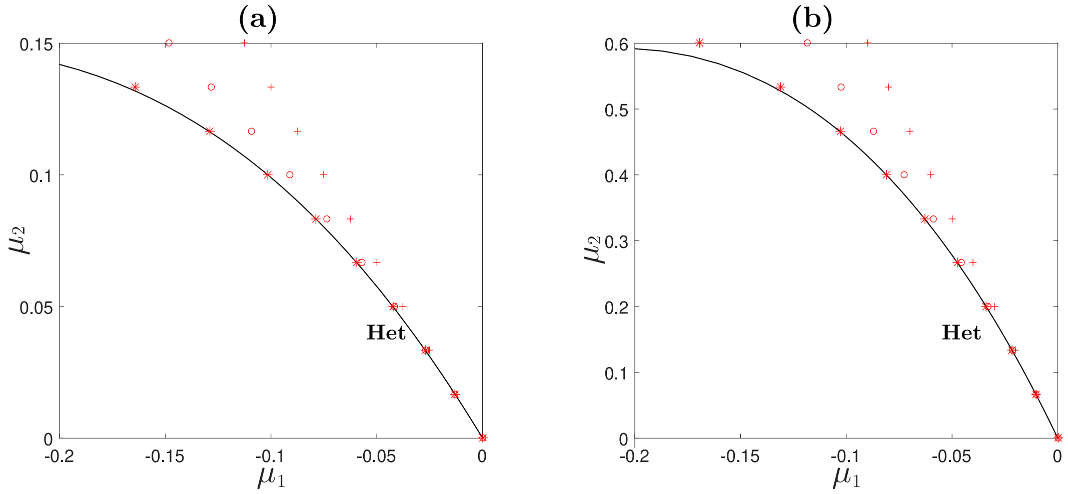


**Figure 2.** *Comparisons between numerical continuations (solid) and analytical approximations (+: 1st-order; o: 3rd-order; *: 20th-order) for the heteroclinic bifurcation of* (1.1) *with* (a) $a = 2$, $s = 1$; (b) $a = 1$, $s = 0.5$.

case, the approximation of the heteroclinic bifurcation is given by

$$\begin{aligned}\mu_1 = & -\frac{3}{4}s\mu_2 - \frac{16323}{10240}s^3\mu_2^2 - \frac{2077080483}{367001600}s^5\mu_2^3 - \frac{160516256539767}{6576668672000}s^7\mu_2^4 \\ & - \frac{945906025841649195O3}{8102455803904000000}s^9\mu_2^5 - \frac{39575152137288106444731487481}{66441693263527149568000000}s^{11}\mu_2^6 \\ & - \frac{4055742743672597591294101830976851 71}{12769561911703809929773056000000000}s^{13}\mu_2^7 \\ & - \frac{233480478521932867001788799116176270542419402 49}{133508895775619317908448108924108800000000000}s^{15}\mu_2^8 \\ & - \frac{65420687252303433828150157542609470100697321420145251834 1}{66303738944809827416237825927684550977126400000000000}s^{17}\mu_2^9 \\ & - \frac{7474169428770955683782838800120278257895008036935382806269341461 39}{13171214614642328266567841336780852475234778538311680000000000}s^{19}\mu_2^{10} \\ & + \mathcal{O}\left(\mu_2^{11}\right).\end{aligned} \tag{3.21}$$

Note that the first term is equivalent to that obtained with the Melnikov function (see [44, eq. (7.4.40)]). Thus, with the NTT method, we have greatly improved the approximation for $a = 2$. Note that, for $a = 1$, the corresponding unperturbed system when $s = 0$ appears in [66]. The authors obtain the equivalent equations [66, eqs. (3.11) and (3.13)] when applying the method of multiple scales to the Michelson system.

To illustrate the validity of the approximations obtained by the NTT method, the analytical results are compared in Figures 2 and 3 with numerical continuations carried out with

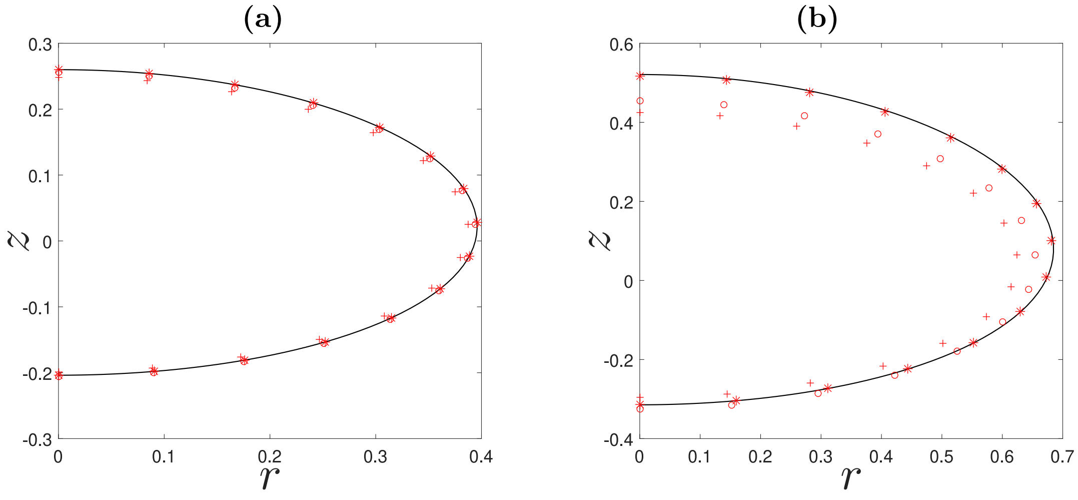


**Figure 3.** *Comparisons between numerical continuations (solid) and analytical approximations (+: 1st-order; o: 3rd-order; *: 20th-order) for the heteroclinic orbit of* (1.1) *with* $a = 2$, $s = 1$ *for* (a) $\mu_2 = 0.05$; (b) $\mu_2 = 0.13$.

AUTO [32] and MatCont [31]. They are in good agreement, and the high-order solutions greatly improve the accuracy for parameter values far from the origin (see Figure 2). Note that, as can be seen in Figure 3, the method also provides an excellent approximation to the heteroclinic orbit in the phase plane.

It is worth pointing out that the power series solutions (3.18) of phase variables $x(\varphi, \varepsilon)$ and $y(\varphi, \varepsilon)$ are expressed in terms of the new time variable $\varphi$ which reparametrized the heteroclinic orbits in the phase plane. Usually, when starting a numerical continuation of such orbits, expressions of phase variables in original time $t$ are helpful. In order to obtain the evolution of phase variables over $t$, one needs to consider the relation between the two time variables. Recall that the NTT rescales the time $t$ in an infinite interval $(-\infty, +\infty)$ to time $\varphi$ in a finite interval $[0, \pi]$. Assume that we have $t = 0$ when $\varphi = \pi/2$. Then, according to transformation (2.4) and power series (3.12), the physical time $t$ can be expressed in terms of $\varphi$ as

$$t(\varphi, \varepsilon) = \int_{\frac{\pi}{2}}^{\varphi} \frac{1}{\Phi(\theta, \varepsilon)} \mathrm{d}\theta = \sum_{i=0}^{\infty} \varepsilon^i \int_{\frac{\pi}{2}}^{\varphi} T_i(\theta) \mathrm{d}\theta = \sum_{i=0}^{\infty} \varepsilon^i t_i(\varphi), \tag{3.22}$$

where $T_i(\theta)$ can be expressed in terms of $\Phi_j$ with $j = 0, 1, \ldots, i$. For instance, $T_0 = 1/\Phi_0$ and $T_1 = -\Phi_1/\Phi_0^2$. The terms $t_i(\varphi)$ with $i = 0, 1$ can then be solved as

$$t_0(\varphi) = \frac{1}{a} \ln\left(\tan\frac{\varphi}{2}\right), \qquad t_1(\varphi) = -\frac{s}{a(3a+2)} \ln(\sin\varphi). \tag{3.23}$$

In such a way, once the power series of $\Phi$ is found up to $i$th order, the physical time $t$ can be approximated by a power series given as (3.22) with the same order. Such an approximation provides a mapping from $\varphi \in [0, \pi]$ to $t \in (-\infty, +\infty)$. Although it is not an easy task to

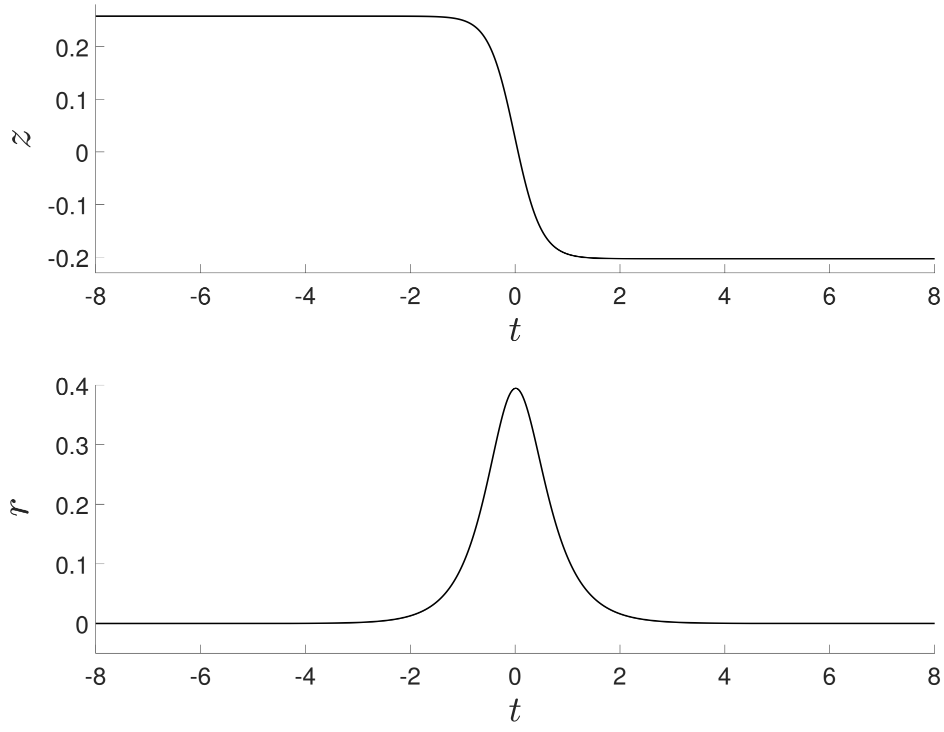


**Figure 4.** *The approximations of $z(t)$ (upper) and $r(t)$ (lower) evolving over physical time $t$ for the heteroclinic orbit of (1.1) with $a = 2$, $s = 1$, and $\mu_2 = 0.05$, which corresponds to the orbit shown by empty circles in Figure 3 (a). We use power series (3.18) and (3.22) up to $\mathcal{O}(\varepsilon^3)$ to draw the time series. A fixed $\varphi$ in $[0, \pi]$ determines a unique set of values $(z, r, t)$.*

obtain an explicit expression of phase variable in terms of $t$, one can still find a mapping from $t$ to the phase variable using (3.22); see Figure 4.

We now end this section by summarizing the result in the following theorem.

Theorem 3.1. *Consider the two-parameter unfolding of the generic Hopf-zero singularity in cylindrical polar coordinates without azimuthal component as (1.1), with $a > 0$ and $s \neq 0$. The asymptotic expansion for the heteroclinic bifurcation in the parameter space $\mu_1$-$\mu_2$ follows (3.20). Specifically, when $a = 2$, it is given by (3.21).*

**4. Application to nonresonant double Hopf singularity.** In this section, we consider system (1.2), a two parametric unfolding of a normal form system related to the nonresonant double Hopf singularity (also valid for the Hopf-pitchfork singularity) (see, for instance, [44, 55, 68]). Note that it is symmetric with respect to the axis $r_1 = 0$ and $r_2 = 0$. There are several distinct cases depending on the values of the nonzero normal form coefficients. Here, we assume that $c < 0 < b$, $A = -1 - bc > 0$, and $k < 0$, which corresponds to the case VIa in [44, sect. 7.5]. We focus our attention on the parameter space $\mu_1$-$\mu_2$ with $\mu_1 < 0$ and $\mu_2 > 0$ in which the system admits closed orbits.

**4.1. Bifurcation analysis.** Again, we first provide some information on the partial bifurcation set. It is depicted in Figure 5.

Besides $(r_1, r_2) = (0, 0)$, the system possesses up to three equilibria in the positive quadrant with $\mu_1 < 0$ and $\mu_2 > 0$ close to the origin, namely

$$E_1 = \left(\sqrt{-\mu_1}, 0\right), \quad E_2 = \left(0, \sqrt{\frac{1-\sqrt{1-4k\mu_2}}{2k}}\right), \quad E_3 = (r_1^*, r_2^*) = (\sqrt{-\mu_1 - b(r^*)^2}, r^*),$$

where

$$r^* = \sqrt{\frac{\sqrt{A^2 + 4k(c\mu_1 - \mu_2)} - A}{2k}}. \tag{4.1}$$

Then, we deduce that there exist two pitchfork bifurcations. One occurs when $\mu_2 = c\mu_1$, and it involves the equilibria $(r_1^*, \pm r_2^*)$ and $E_1$. The corresponding curve is depicted as $\mathbf{P}_1$ in Figure 5. The other one corresponds to curve $\mathbf{P}_2$, and it happens when $\mu_2 = -(1/b)\mu_1 - (k/b^2)\mu_1^2$ and the equilibria $(\pm r_1^*, r_2^*)$ and $E_2$ are involved. In addition, the equilibrium $(r_1^*, r_2^*)$ undergoes a Hopf bifurcation $\mathbf{H}$ along the curve given by

$$\mu_2 = \frac{1}{8k}(2bc - b + 1)\left[b + 1 - \sqrt{(b+1)^2 + 8k\mu_1}\right] + \frac{2c-1}{2}\mu_1. \tag{4.2}$$

A stable periodic orbit arises from the Hopf bifurcation. It collides with the three equilibria $(0, 0)$, $E_1$, and $E_2$ as the amplitude of it increases, and, consequently, a heteroclinic cycle formed by three orbits appears. Two of them are structurally stable as they lie in the invariant subspaces $r_1 = 0$ and $r_2 = 0$, whereas the third one is located in the positive quadrant connecting the equilibria $E_1$ and $E_2$.

**4.2. Asymptotic expansions.** Now, we are going to investigate the third heteroclinic orbit using the NTT method. In order to simplify the calculation, we introduce the following transformation:

$$R_1 = r_1^2, \quad R_2 = r_2^2, \quad \text{and} \quad \tau = 2t. \tag{4.3}$$

Then, we obtain the new system as

$$\begin{aligned} \frac{\mathrm{d}R_1}{\mathrm{d}\tau} &= R_1(\mu_1 + R_1 + bR_2), \\ \frac{\mathrm{d}R_2}{\mathrm{d}\tau} &= R_2(\mu_2 + cR_1 - R_2) + kR_2^3, \end{aligned} \tag{4.4}$$

with the restrictions $R_1 \geq 0$ and $R_2 \geq 0$.

To study the heteroclinic orbit in system (4.4), we use the following blow-up transformation [44, eq. (7.5.17)]:

$$R_1 = \varepsilon x, \;\; R_2 = \varepsilon y, \;\; \mu_1 = \varepsilon \nu_1, \;\; \mu_2 = \varepsilon \nu_1 \left(\frac{c-1}{1+b}\right) + \varepsilon^2 \nu_2, \;\; \tau = \varepsilon^{-1}\xi. \tag{4.5}$$

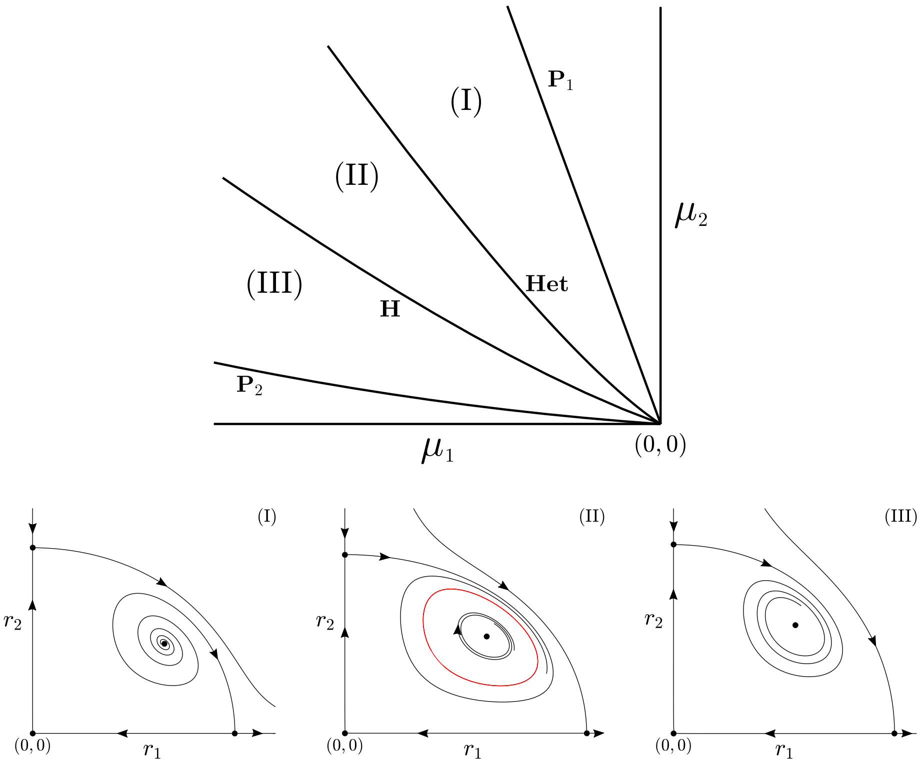


**Figure 5.** *Partial qualitative bifurcation set for system* (1.2) *with* $c < 0 < b$, $A = -1 - bc > 0$, *and* $k < 0$.

Without loss of generality, we let $\nu_1 = -1$. Then, system (4.4) becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\xi} &= x\,(-1 + x + by), \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} &= y\left(\frac{1-c}{1+b} + cx - y\right) + \varepsilon y(\nu_2 + ky^2). \end{aligned} \tag{4.6}$$

The unperturbed system is integrable with the integrating factor $u(x, y) = x^{m-1}y^{n-1}$, where

$$m = \frac{1-c}{A}, \quad n = \frac{1+b}{A}, \quad A = -1 - bc, \tag{4.7}$$

and the first integral is given by

$$I(x, y) = -\frac{1}{n+1}x^m y^n \left(-1 + x + \frac{1+b}{1-c}y\right). \tag{4.8}$$

Applying the NTT (2.4) to time $\xi$ in (4.6) leads to

$$\begin{aligned} x'\Phi &= x\,(-1 + x + by), \\ y'\Phi &= y\left(\frac{1-c}{1+b} + cx - y\right) + \varepsilon y(\nu_2 + ky^2). \end{aligned} \tag{4.9}$$

This system is in the form of (2.6) with

$$\mu = \nu_2, \quad f(x,y) = x(-1+x+by), \quad g(x,y) = y\left(\frac{1-c}{1+b} + cx - y\right),$$
$$h_1(x,y) = h_2(x,y) = 0, \quad h_3(x,y) = ky^3, \quad h_4(x,y) = y.$$

Once more, we are going to seek a perturbation solution in the form of

$$x(\varphi,\varepsilon) = p(\varepsilon)\cos\varphi + q(\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i x_i(\varphi) = \sum_{i=0}^{\infty}\varepsilon^i(p_i\cos\varphi + q_i),$$
$$y(\varphi,\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i y_i(\varphi), \;\; \Phi(\varphi,\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i\Phi_i(\varphi), \quad \nu_2(\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i\nu_{2,i}. \tag{4.10}$$

The heteroclinic orbit in the unperturbed system connects the hyperbolic saddles $(1,0)$ and $(0, \frac{1-c}{1+b})$. Therefore, from (2.13) and (4.8), the orbit $\Gamma_0$ is given by the function

$$-1 + x_0 + \frac{1+b}{1-c}y_0 = 0, \quad 0 \le x_0 \le 1. \tag{4.11}$$

To find the zero-order solution, $p_0$ and $q_0$ can be determined from (2.12) as

$$p_0 = -\frac{1}{2} \quad \text{and} \quad q_0 = \frac{1}{2}. \tag{4.12}$$

Then, from (4.11) and (2.15), we are able to find $y_0$ and $\Phi_0$ as

$$y_0 = \frac{1-c}{2(1+b)}(1+\cos\varphi) \quad \text{and} \quad \Phi_0 = \frac{A}{1+b}\sin\varphi. \tag{4.13}$$

Again, we apply Algorithm 2.1 to find the $i$th-order solution ($i \in \mathbb{Z}^+$) iteratively. It can be easily verified that

$$\int_{\Gamma_0} u(x,y)h_2(x,y)dy - u(x,y)h_4(x,y)\mathrm{d}x = \frac{(-1)^n m^n}{n^{n-1}(n+bm)(m+1)}B(m,n+1) \neq 0, \tag{4.14}$$

where $B(\cdot,\cdot)$ is the Beta function. Thus, by Proposition 2.2, one can obtain a unique solution in each order. Moreover, in this particular case, $det(\mathbf{M}(0)) = -c/2 \neq 0$ and $det(\mathbf{M}(\pi)) = b(1-c)/(2(1+b)) \neq 0$. Therefore, from Proposition 2.6 we have that $\Phi(0,\varepsilon) = \Phi(\pi,\varepsilon) = 0$ for all $\varepsilon$, which guarantees that the solution provided by the NTT method corresponds to a heteroclinic orbit.

We note that the solutions of $y_i$ can be expressed in the following form:

$$y_i = \sum_{j=0}^{i+1}\zeta_{i,j}(b,c,k)\cos(j\varphi), \quad \zeta_{i,j}(b,c,k) \in \mathbb{R}. \tag{4.15}$$

Moreover, we also note that the abscissas of the two hyperbolic saddles are independent of the perturbation parameter $\varepsilon$. Therefore, we have $p_i = q_i = 0$, that is, $x_i = 0$ for $i \in \mathbb{Z}^+$. Consequently, the solution of $x$ is

$$x = \frac{1}{2}(1 - \cos\varphi). \tag{4.16}$$

Then, substituting (4.15) into (2.29), and equating the constant and the coefficient of harmonic terms to zero, one is able to determine the values of $\zeta_{i,j}$. The procedure is analogous to that provided in Algorithm 3.1. We observe that $\zeta_{i,j}$ $(i \in \mathbb{Z}^+)$ can be expressed as

$$\zeta_{i,j}(b,c,k) = \Psi_i(b,c,k)\tilde{\zeta}_{i,j}(b,c), \tag{4.17}$$

with

$$\Psi_i(b,c,k) = \left[\frac{k^i(1-c)^{(i+1)}}{(2+2b)^{(1+2i)}} B^{-\lfloor\frac{i-1}{2}\rfloor} \prod_{m=0}^{i-1}(B - mA)^{m-i}\right], \quad i \in \mathbb{Z}^+,$$

where $B = 2bc - b + c$. For $i = 1, 2, 3$, the coefficients $\tilde{\zeta}_{i,j}$ are given in Appendix B. Moreover, $\Phi_i$ and $\nu_{2,i}$ can be expressed as

$$\Phi_i = b\Psi_i \sum_{j=1}^{i+1} \sigma_{i,j} \sin(j\varphi), \ \ i \in \mathbb{Z}^+, \tag{4.18a}$$

$$\nu_{2,i} = b(b - 2bc - 1)2^{(3+2i)}\Psi_{i+1}\tilde{\nu}_{2,i}, \ \ i \in \mathbb{N}, \tag{4.18b}$$

where $\sigma_{i,j}$ $(i = 1, 2, 3)$ and $\tilde{\nu}_{2,i}$ $(i = 0, 1, 2, 3)$ are given in Appendices C and D, respectively. It follows from the transformation (4.5) that the relation between the original system parameters $\mu_1$ and $\mu_2$ can be obtained as

$$\begin{aligned} \mu_2 &= \left(\frac{c-1}{1+b}\right)\mu_1 + \sum_{i=0}^{\infty}(-1)^i \nu_{2,i}\,\mu_1^{i+2} \\ &= \left(\frac{c-1}{1+b}\right)\mu_1 - \frac{bk(c-1)^2(1-b+2bc)}{(1+b)^3(c-b+2bc)}\mu_1^2 - \nu_{2,1}\,\mu_1^3 + \nu_{2,2}\,\mu_1^4 - \nu_{2,3}\,\mu_1^5 + \mathcal{O}(\mu_1^6), \end{aligned} \tag{4.19}$$

where $\nu_{2,i}$ $(i = 1, 2, 3)$ are given explicitly in Appendix E. As far as we know, this is the first time that the approximation to the locus where this heteroclinic orbit exists is computed for all values of nonzero normal form coefficients $b$, $c$, and $k$. This result clearly improves the first-order approximation given in [29, sect. 4.7] and [55, sect. 8.6].

The original system (1.2) is considered in [44, sect. 7.5] for $b = -c = 3$. In this case, the asymptotic expansion of the heteroclinic bifurcation obtained with the NTT method is given by

$$\begin{aligned}\mu_2 = & -\mu_1 - \frac{5}{8}k\mu_1^2 + \frac{115}{256}k^2\mu_1^3 - \frac{335}{768}k^3\mu_1^4 + \frac{861995}{1769472}k^4\mu_1^5 - \frac{200400265}{339738624}k^5\mu_1^6 \\ & + \frac{687578001545}{913217421312}k^6\mu_1^7 - \frac{204031800502045}{204560702373888}k^7\mu_1^8 + \frac{280075516618888925}{206197187992879104}k^8\mu_1^9 \\ & - \frac{523668255735616144595}{277129020662429515776}k^9\mu_1^{10} + \frac{912907469140339974016565}{341422953456113163436032}k^{10}\mu_1^{11} \\ & - \frac{4356893935253684269216961987 5}{11357093123764148268536168448}k^{11}\mu_1^{12} \\ & + \frac{7291881763265281455048104373827 2025}{13096454649854875104159019718148096}k^{12}\mu_1^{13} \\ & - \frac{6161618383104334850001889001347216877 3885}{7551101436194724468055590953010956599296}k^{13}\mu_1^{14} \\ & + \frac{210002056539436902482345752702614723502432932575}{17415135446701637819574481637292981248782761984}k^{14}\mu_1^{15} \\ & - \frac{1802270961814228619418485745860608230708480019295 53925}{1004114913531744352745897632450383795649816977368 6784}k^{15}\mu_1^{16} \\ & + \mathcal{O}(\mu_1^{17}).\end{aligned} \tag{4.20}$$

Again, the first two terms are the same as that obtained with the Melnikov function (see [44, eq. (7.5.26)]). With the present method, we have greatly improved the approximation for $b = -c = 3$.

To illustrate the validity of the approximation obtained by the NTT method, the analytical results are compared in Figure 6 with numerical continuations obtained by AUTO and MatCont. They are in good agreement, and the high-order solutions greatly improve the accuracy for parameter values far from the origin. We also remark that one is able to obtain the evolution of phase variables over physical time $t$ by taking (3.22) into account.

We end this section by summarizing the result in a theorem.

Theorem 4.1. *Consider the two-parameter unfolding of the nonresonant double Hopf singularity in cylindrical polar coordinates without azimuthal component as* (1.2)*, with* $c < 0 < b$*,* $-1 - bc > 0$*, and* $k < 0$*. The asymptotic expansion for the heteroclinic bifurcation in the parameter space* $\mu_1$*-*$\mu_2$ *follows* (4.19)*. Specifically, when* $b = -c = 3$*, it is given by* (4.20)*.*

## 5. Conclusions.

In this paper we have developed an iterative procedure, based on the NTT method, to study heteroclinic orbits in perturbed non-Hamiltonian integrable systems. This approach allows one to obtain the asymptotic expansions up to any desired order, not only in the parameter space but also in the phase space. We have also given sufficient conditions to verify the existence and uniqueness of the solution.

The usefulness of the proposed method has been illustrated by applying it to two codimension-two singularities. In the case of the generic Hopf-zero bifurcation (see (1.1)), previous studies

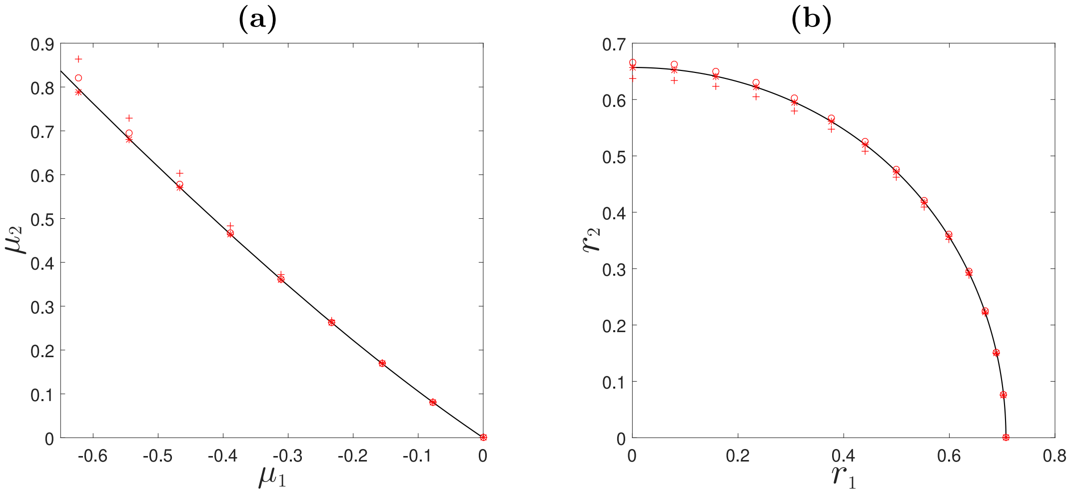


**Figure 6.** *Comparisons between numerical continuations (solid) and analytical approximations (+: 1st-order; o: 3rd-order; *: 10th-order) for the heteroclinic bifurcation of* (1.2) *with* $b = -c = 3$, $k = -1$ *in* (a) *parameter space;* (b) *phase space* ($\mu_1 = -0.5$).

only obtained a first-order approximation for the heteroclinic curve. However, we are able to find high-order asymptotic expansions for all values of the normal form coefficient $a > 0$. Moreover, the corresponding results have a very good agreement with the numerical results, both in the parameter space and in the phase space.

For the nonresonant double Hopf and Hopf-pitchfork bifurcations (see (1.2)), the results found in the literature only provided the first-order approximation for the locus where the global connection exists. Our method also gives high-order approximations that are valid for all values of the normal form coefficients $b$, $c$, and $k$. Their agreement with the numerical continuation is noteworthy for the curve of heteroclinic bifurcation in the parameter space and for the associated orbit in the phase space.

As was stated in the introduction, the derived formulas presented in terms of several nonzero symbolic constant coefficients of the normal form are practically important in many bifurcation control engineering applications. Indeed, one may argue that there are many numerical softwares that can very accurately locate the heteroclinic orbits, and there may not be any need for an analytic approximation. The substantial contribution here is that numerical softwares are unable to symbolically locate the heteroclinic orbits associated with systems with unknown symbolic constants. Therefore, our results here facilitate the accomplishment of this. The proposed procedure provides a powerful approach for highly accurate symbolic bifurcation control of singular systems of these types. This approach, of course, needs to be complemented with a normal form computation equipped with bifurcation parameters and symbolic constant coefficients in terms of the original parametric system.

On the other hand, the better approximations we have obtained for both the heteroclinic curve and the heteroclinic orbit will help in detection of complex behavior in the corresponding three-dimensional system (see, for instance, [66], where the surfaces of heteroclinic orbits are computed in the Michelson system). Knowing more precisely in normal form the locus of the heteroclinic bifurcation can be useful in numerically locating the wedge (limited by curves that are exponentially tangent at the singularity) where the homoclinic tangle occurs [20, 37]. Having a higher-order approximation allows further separation in the plane of parameters from the point where the Hopf-zero bifurcation occurs and, in this way, it may be easier to find the entanglement region as the wedge widens [36, Figure 3]. Also, by considering the relation between the two time variables, i.e., (3.22), we are able to obtain the approximation of phase variables evolving over physical time $t$. Such an approximation may be helpful for starting a numerical continuation.

Finally, with this work we show a new usefulness of the NTT method, which, until now, had been very efficient in the study of global connections in systems that appear as perturbations of a Hamiltonian system [1, 4, 62, 60] and also in the analysis of the canard explosion in singularly perturbed systems [3, 61, 63].

## Appendix A. Coefficients $\gamma_{i,j}$ and $\eta_{i,j}$ in (3.18) and $\nu_{1,i}$ in (3.19).

$$\gamma_{2,1} = \frac{90a^4 + 147a^3 + 56a^2 - 12a - 5}{8\sqrt{a+1}(2a+1)(3a+2)^2},$$

$$\gamma_{2,3} = \frac{(a-1)\sqrt{a+1}}{8(2a+1)(3a+2)^2},$$

$$\gamma_{3,2} = -\frac{3(450a^6 + 1845a^5 + 3213a^4 + 3010a^3 + 1579a^2 + 428a + 44)}{8\sqrt{a+1}(2a+1)(5a+2)(3a+2)^4},$$

$$\gamma_{3,4} = \frac{\sqrt{a+1}(11a^2 + 15a - 2)}{16(2a+1)(5a+2)(3a+2)^3},$$

$$\eta_{2,1} = \frac{a(90a^3 + 165a^2 + 82a + 2)}{8(2a+1)(3a+2)^2},$$

$$\eta_{2,3} = \frac{3a(a+1)}{4(2a+1)(3a+2)^2},$$

$$\eta_{3,2} = \frac{a(1350a^5 + 4995a^4 + 6900a^3 + 4360a^2 + 1140a + 56)}{8(2a+1)(5a+2)(3a+2)^4},$$

$$\eta_{3,4} = \frac{3a(a+1)(a+3)}{4(2a+1)(5a+2)(3a+2)^3},$$

$$\nu_{1,2} = -\frac{9a^3s^3(360a^4 + 1134a^3 + 1341a^2 + 712a + 144)}{4(2a+1)(5a+2)(3a+2)^4},$$

$$\begin{aligned}\nu_{1,4} = &-\frac{27a^4s^5}{4(3a+1)(7a+2)(2a+1)^2(5a+2)^2(3a+2)^7}(2381400a^{10} + 15966720a^9 \\ &+ 48311154a^8 + 86899986a^7 + 102941937a^6 + 83928167a^5 + 47686000a^4 \\ &+ 18633320a^3 + 4786792a^2 + 728784a + 49824),\end{aligned}$$

$$\nu_{1,6} = -\frac{81a^5s^7}{8(4a+1)(9a+2)(3a+1)^2(7a+2)^2(2a+1)^3(5a+2)^3(3a+2)^{10}}$$
$$\times (1388832480000a^{18} + 14874836760000a^{17} + 75200785819200a^{16}$$
$$+ 238467531749040a^{15} + 531566725617396a^{14} + 884557588857408a^{13}$$
$$+ 1138808032909785a^{12} + 1159776473334021a^{11} + 947067282357321a^{10}$$
$$+ 624626832462493a^9 + 333419845878098a^8 + 143640452620234a^7$$
$$+ 49544795072020a^6 + 13487374822616a^5 + 2831768549904a^4$$
$$+ 442070023232a^3 + 48274310784a^2 + 3287402496a + 104965632).$$

## Appendix B. Expressions of $\tilde{\zeta}_{i,j}$, $i = 1, 2, 3$ of (4.17).

$$\tilde{\zeta}_{1,0} = 11bc - 7b + 3c + 1,$$

$$\tilde{\zeta}_{1,1} = 12bc - 8b + 4c,$$

$$\tilde{\zeta}_{1,2} = (c-1)(1+b),$$

$$\tilde{\zeta}_{2,0} = 832b^3c^3 - 1350b^3c^2 + 756b^2c^3 + 692b^3c - 462b^2c^2 + 200bc^3 - 110b^3 - 232b^2c + 150bc^2$$
$$+ 20c^3 + 130b^2 - 140bc + 30c^2 - 18b + 16c - 2,$$

$$\tilde{\zeta}_{2,1} = 924b^3c^3 - 1527b^3c^2 + 918b^2c^3 + 794b^3c - 715b^2c^2 + 280bc^3 - 127b^3 - 140b^2c + 79bc^2$$
$$+ 30c^3 + 129b^2 - 166bc + 35c^2 - b - 1,$$

$$\tilde{\zeta}_{2,2} = 2(c-1)(1+b)(48b^2c^2 - 45b^2c + 38bc^2 + 9b^2 - 6bc + 6c^2 - 8b + 7c - 1),$$

$$\tilde{\zeta}_{2,3} = (c-1)^2(1+b)^2(4bc - b + 2c + 1),$$

$$\tilde{\zeta}_{3,0} = 646412b^7c^7 - 2169150b^7c^6 + 1497128b^6c^7 + 3041806b^7c^5 - 3492721b^6c^6 + 1426055b^5c^7$$
$$- 2309133b^7c^4 + 2553153b^6c^5 - 1710113b^5c^6 + 725127b^4c^7 + 1024358b^7c^3 + 6263b^6c^4$$
$$- 787931b^5c^5 + 35988b^4c^6 + 214966b^3c^7 - 265530b^7c^2 - 945284b^6c^3 + 1862200b^5c^4$$
$$- 1256028b^4c^5 + 293796b^3c^6 + 38178b^2c^7 + 37264b^7c + 511615b^6c^2 - 710753b^5c^3$$
$$+ 448468b^4c^4 - 296196b^3c^5 + 99571b^2c^6 + 3895bc^7 - 2187b^7 - 112453b^6c - 73459b^5c^2$$
$$+ 364897b^4c^3 - 256217b^3c^4 + 30757b^2c^5 + 14075bc^6 + 175c^7 + 9179b^6 + 87237b^5c$$
$$- 193426b^4c^2 + 165304b^3c^3 - 82717b^2c^4 + 16721bc^5 + 810c^6 - 12596b^5 + 3092b^4c$$
$$+ 30524b^3c^2 - 23822b^2c^3 + 1198bc^4 + 1398c^5 + 6282b^4 - 17210b^3c + 16943b^2c^2 - 7613bc^3$$
$$+ 1234c^4 - 567b^3 + 1831b^2c - 1615bc^2 + 305c^3 - 101b^2 + 229bc - 92c^2 - 10b + 10c,$$

$$\tilde{\zeta}_{3,1} = 722976b^7c^7 - 2446912b^7c^6 + 1744384b^6c^7 + 3456528b^7c^5 - 4230144b^6c^6 + 1749032b^5c^7$$
$$- 2639772b^7c^4 + 3375500b^6c^5 - 2420800b^5c^6 + 946488b^4c^7 + 1176528b^7c^3 - 393692b^6c^4$$
$$- 396636b^5c^5 - 246552b^4c^6 + 300752b^3c^7 - 306032b^7c^2 - 895000b^6c^3 + 2013720b^5c^4$$
$$- 1400080b^4c^5 + 277648b^3c^6 + 56944b^2c^7 + 43056b^7c + 538208b^6c^2 - 930560b^5c^3$$

$$
\begin{aligned}
&+ 759168b^4c^4 - 464232b^3c^5 + 119968b^2c^6 + 6056bc^7 - 2532b^7 - 122564b^6c - 1992b^5c^2\\
&+ 269352b^4c^3 - 177212b^3c^4 - 11700b^2c^5 + 19504bc^6 + 280c^7 + 10188b^6 + 80740b^5c\\
&- 214272b^4c^2 + 209584b^3c^3 - 100876b^2c^4 + 14900bc^5 + 1208c^6 - 12864b^5 + 15072b^4c\\
&+ 4208b^3c^2 - 1032b^2c^3 - 6592bc^4 + 1720c^5 + 5224b^4 - 16392b^3c + 17424b^2c^2 - 7104bc^3\\
&+ 744c^4 + 44b^3 - 36b^2c + 8bc^2 - 72c^3 - 52b^2 + 116bc - 48c^2 - 8b + 8c,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\zeta}_{3,2} = {}& 4(c-1)(1+b)(20316b^6c^6 - 53706b^6c^5 + 46268b^5c^6 + 57240b^6c^4 - 80527b^5c^5 + 42423b^4c^6\\
&- 31479b^6c^3 + 35994b^5c^4 - 31583b^4c^5 + 19940b^3c^6 + 9435b^6c^2 + 10822b^5c^3 - 27613b^4c^4\\
&+ 7410b^3c^5 + 5050b^2c^6 - 1467b^6c - 13416b^5c^2 + 28919b^4c^3 - 25080b^3c^4 + 8240b^2c^5 + 656bc^6\\
&+ 93b^6 + 3809b^5c - 4272b^4c^2 + 1908b^3c^3 - 3498b^2c^4 + 1981bc^5 + 35c^6 - 358b^5 - 1804b^4c\\
&+ 5564b^3c^2 - 4761b^2c^3 + 1022bc^4 + 153c^5 + 410b^4 - 982b^3c + 1083b^2c^2 - 794bc^3 + 207c^4\\
&- 120b^3 + 389b^2c - 324bc^2 + 57c^3 - 23b^2 + 53bc - 22c^2 - 2b + 2c),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\zeta}_{3,3} = {}& 4(c-1)^2(1+b)^2(2bc - b + c)(604b^4c^4 - 850b^4c^3 + 954b^3c^4 + 417b^4c^2 - 402b^3c^3 + 538b^2c^4\\
&- 88b^4c - 375b^3c^2 + 380b^2c^3 + 126bc^4 + 7b^4 + 210b^3c - 345b^2c^2 + 258bc^3 + 10c^4 - 27b^3\\
&- 64b^2c + 51bc^2 + 38c^3 + 31b^2 - 66bc + 36c^2 - 9b + 8c - 2),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\zeta}_{3,4} = {}& (c-1)^3(1+b)^3(3bc - b + c + 1)(2bc - b + c)(22b^2c^2 - 8b^2c + 21bc^2 + b^2 + 12bc + 5c^2\\
&- 3b + 8c + 2).
\end{aligned}
$$

## Appendix C. Expressions of $\sigma_{i,j}$, $i = 1, 2, 3$ of (4.18a).

$$\sigma_{1,1} = 10bc - 6b + 2c + 2,$$

$$\sigma_{1,2} = (c-1)(1+b),$$

$$
\begin{aligned}
\sigma_{2,1} = {}& 740b^3c^3 - 1173b^3c^2 + 594b^2c^3 + 590b^3c - 209b^2c^2 + 120bc^3 - 93b^3 - 324b^2c + 221bc^2\\
&+ 10c^3 + 131b^2 - 114bc + 25c^2 - 35b + 32c - 3,
\end{aligned}
$$

$$\sigma_{2,2} = 8(c-1)(1+b)(11b^2c^2 - 10b^2c + 8bc^2 + 2b^2 + c^2 - 2b + 2c),$$

$$\sigma_{2,3} = (c-1)^2(1+b)^2(4bc - b + 2c + 1),$$

$$
\begin{aligned}
\sigma_{3,1} = {}& 569848b^7c^7 - 1891388b^7c^6 + 1249872b^6c^7 + 2627084b^7c^5 - 2755298b^6c^6 + 1103078b^5c^7\\
&- 1978494b^7c^4 + 1730806b^6c^5 - 999426b^5c^6 + 503766b^4c^7 + 872188b^7c^3 + 406218b^6c^4\\
&- 1179226b^5c^5 + 318528b^4c^6 + 129180b^3c^7 - 225028b^7c^2 - 995568b^6c^3 + 1710680b^5c^4\\
&- 1111976b^4c^5 + 309944b^3c^6 + 19412b^2c^7 + 31472b^7c + 485022b^6c^2 - 490946b^5c^3\\
&+ 137768b^4c^4 - 128160b^3c^5 + 79174b^2c^6 + 1734bc^7 - 1842b^7 - 102342b^6c - 144926b^5c^2\\
&+ 460442b^4c^3 - 335222b^3c^4 + 73214b^2c^5 + 8646bc^6 + 70c^7 + 8170b^6 + 93734b^5c\\
&- 172580b^4c^2 + 121024b^3c^3 - 64558b^2c^4 + 18542bc^5 + 412c^6 - 12328b^5 - 8888b^4c\\
&+ 56840b^3c^2 - 46612b^2c^3 + 8988bc^4 + 1076c^5 + 7340b^4 - 18028b^3c + 16462b^2c^2 - 8122bc^3
\end{aligned}
$$

$$+ 1724c^4 - 1178b^3 + 3698b^2c - 3238bc^2 + 682c^3 - 150b^2 + 342bc - 136c^2 - 12b + 12c,$$

$$\sigma_{3,2} = 2(c-1)(1+b)(35932b^6c^6 - 93786b^6c^5 + 78156b^5c^6 + 99044b^6c^4 - 127915b^5c^5 + 67439b^4c^6 - 54157b^6c^3 + 46668b^5c^4 - 35575b^4c^5 + 29196b^3c^6 + 16185b^6c^2 + 27696b^5c^3 - 59845b^4c^4 + 22742b^3c^5 + 6610b^2c^6 - 2513b^6c - 25714b^5c^2 + 47927b^4c^3 - 40000b^3c^4 + 15596b^2c^5 + 744bc^6 + 159b^6 + 6931b^5c - 3612b^4c^2 - 4112b^3c^3 - 1242b^2c^4 + 3125bc^5 + 35c^6 - 638b^5 - 4176b^4c + 10952b^3c^2 - 9575b^2c^3 + 2900bc^4 + 197c^5 + 802b^4 - 1174b^3c + 641b^2c^2 - 768bc^3 + 411c^4 - 324b^3 + 931b^2c - 822bc^2 + 221c^3 - b^2 + 3bc + 2c^2 + 2b - 2c),$$

$$\sigma_{3,3} = 2(c-1)^2(1+b)^2(2bc - b + c)(1142b^4c^4 - 1588b^4c^3 + 1757b^3c^4 + 777b^4c^2 - 636b^3c^3 + 955b^2c^4 - 164b^4c - 750b^3c^2 + 800b^2c^3 + 211bc^4 + 13b^4 + 400b^3c - 588b^2c^2 + 492bc^3 + 15c^4 - 51b^3 - 148b^2c + 150bc^2 + 68c^3 + 61b^2 - 112bc + 75c^2 - 21b + 24c - 2),$$

$$\sigma_{3,4} = (c-1)^3(1+b)^3(3bc - b + c + 1)(2bc - b + c)(22b^2c^2 - 8b^2c + 21bc^2 + b^2 + 12bc + 5c^2 - 3b + 8c + 2).$$

## Appendix D. Expressions of $\tilde{\nu}_{2,i}$, $i = 0, 1, 2, 3$ of (4.18b).

$$\tilde{\nu}_{2,0} = 1,$$

$$\tilde{\nu}_{2,1} = 2(9b^2c^2 - 9b^2c + 5bc^2 + 2b^2 + bc - 2b + 2c),$$

$$\tilde{\nu}_{2,2} = 3432b^6c^6 - 9406b^6c^5 + 6612b^5c^6 + 10425b^6c^4 - 10852b^5c^5 + 4744b^4c^6 - 5980b^6c^3 + 3168b^5c^4 - 1012b^4c^5 + 1500b^3c^6 + 1873b^6c^2 + 3768b^5c^3 - 7165b^4c^4 + 3020b^3c^5 + 176b^2c^6 - 304b^6c - 3100b^5c^2 + 4962b^4c^3 - 3456b^3c^4 + 1298b^2c^5 + 20b^6 + 844b^5c - 201b^4c^2 - 992b^3c^3 + 363b^2c^4 + 152bc^5 - 80b^5 - 528b^4c + 1292b^3c^2 - 1072b^2c^3 + 344bc^4 + 100b^4 - 124b^3c + 23b^2c^2 - 40bc^3 + 33c^4 - 40b^3 + 112b^2c - 96bc^2 + 26c^3 + c^2,$$

$$\tilde{\nu}_{2,3} = 2(932880b^{10}c^{10} - 3895636b^{10}c^9 + 2821320b^9c^{10} + 7151032b^{10}c^8 - 8095844b^9c^9 + 3626472b^8c^{10} - 7600969b^{1,0}c^7 + 7562642b^9c^8 - 4973140b^8c^9 + 2566772b^7c^{10} + 5183200b^{10}c^6 + 307629b^9c^7 - 4538800b^8c^8 + 1075296b^7c^9 + 1079784b^6c^{10} - 2370880b^10c^5 - 6313888b^9c^6 + 12443915b^8c^7 - 9220764b^7c^8 + 2915320b^6c^9 + 269816b^5c^{10} + 737286b^{10}c^4 + 5872358b^9c^5 - 7734119b^8c^6 + 4797016b^7c^7 - 3450196b^6c^8 + 1600684b^5c^9 + 37056b^4c^{10} - 154047b^{10}c^3 - 2814462b^9c^4 - 158240b^8c^5 + 5186900b^7c^6 - 4368990b^6c^7 + 778400b^5c^8 + 425964b^4c^9 + 2156b^3c^{10} + 20714b^{10}c^2 + 807169b^9c^3 + 2455692b^8c^4 - 5786704b^7c^5 + 5305036b^6c^6 - 3171642b^5c^7 + 916728b^4c^8 + 56632b^3c^9 - 1620b^{10}c - 139700b^9c^2 - 1326743b^8c^3 + 1427596b^7c^4 + 231212b^6c^5 - 451132b^5c^6 - 294758b^4c^7 + 263580b^3c^8 + 2980b^2c^9 + 56b^10 + 13496b^9c + 336507b^8c^2 + 458136b^7c^3 - 1898872b^6c^4 + 2084688b^5c^5 - 1181254b^4c^6 + 253608b^3c^7 + 32244b^2c^8 - 560b^9 - 43064b^8c - 322760b^7c^2 + 607742b^6c^3 - 367960b^5c^4 + 218620b^4c^5 - 210708b^3c^6 + 79879b^2c^7 + 1390bc^8 + 2240b^8 + 65024b^7c + 40976b^6c^2 - 344410b^5c^3$$

$$+ 429280b^4c^4 - 231864b^3c^5 + 36484b^2c^6 + 8045bc^7 - 4592b^7 - 43748b^6c + 107980b^5c^2$$
$$- 98402b^4c^3 + 68740b^3c^4 - 43804b^2c^5 + 12028bc^6 + 219c^7 + 5096b^6 + 4520b^5c - 38338b^4c^2$$
$$+ 50552b^3c^3 - 26038b^2c^4 + 3442bc^5 + 733c^6 - 2912b^5 + 7792b^4c - 8376b^3c^2 + 5953b^2c^3$$
$$- 3162bc^4 + 756c^5 + 672b^4 - 2400b^3c + 3022b^2c^2 - 1559bc^3 + 284c^4 - 24bc^2 + 25c^3 - c^2).$$

## Appendix E. Expressions of $\nu_{2,i}$, $i = 1, 2, 3$ of (4.19).

$$\nu_{2,1} = \frac{2bk^2(c-1)^3(1-b+2bc)}{(1+b)^5(c-b+2bc)^2(1+c-b+3bc)} \left[2c - (2-c-5c^2)b + (1-3c)(2-3c)b^2\right],$$

$$\nu_{2,2} = \frac{bk^3(c-1)^4(1-b+2bc)}{(1+b)^7(c-b+2bc)^4(1+c-b+3bc)^2(2+c-b+4bc)}$$
$$\times \big[ - c^2(1 + 26c + 33c^2) + 8c^2(12 + 5c - 43c^2 - 19c^3)b$$
$$- c(112 + 23c - 1072c^2 + 363c^3 + 1298c^4 + 176c^5)b^2$$
$$+ (40 + 124c - 1292c^2 + 992c^3 + 3456c^4 - 3020c^5 - 1500c^6)b^3$$
$$- (100 - 528c - 201c^2 + 4962c^3 - 7165c^4 - 1012c^5 + 4744c^6)b^4$$
$$+ (80 - 844c + 3100c^2 - 3768c^3 - 3168c^4 + 10852c^5 - 6612c^6)b^5$$
$$- (1 - 2c)(1 - 3c)(1 - 4c)(20 - 124c + 237c^2 - 143c^3)b^6\big],$$

$$\nu_{2,3} = \frac{2bk^4(c-1)^5(1-b+2bc)}{(1+b)^9(c-b+2bc)^5(1+c-b+3bc)^3(2+c-b+4bc)^2(3+c-b+5bc)}$$
$$\times \big[ - c^2(1 - 25c - 284c^2 - 756c^3 - 733c^4 - 219c^5)$$
$$- c^2(24 + 1559c + 3162c^2 - 3442c^3 - 12028c^4 - 8045c^5 - 1390c^6)b$$
$$+ c^2(3022 + 5953c - 26038c^2 - 43804c^3 + 34684c^4 + 79879c^5 + 32244c^6 + 2980c^7)b^2$$
$$- c(2400 + 8376c - 50552c^2 - 68740c^3 + 231864c^4 + 210708c^5 - 253608c^6$$
$$- 263580c^7 - 56632c^8 - 2156c^9)b^3$$
$$+ (672 + 7792c - 38338c^2 - 98402c^3 + 429280c^4 + 218620c^5 - 1181254c^6$$
$$- 294758c^7 + 916728c^8 + 425964c^9 + 37056c^{10})b^4$$
$$- (2912 - 4520c - 107980c^2 + 344410c^3 + 367960c^4 - 2084688c^5 + 451132c^6$$
$$+ 3171642c^7 - 778400c^8 - 1600684c^9 - 269816c^{10})b^5$$
$$+ (5096 - 43748c + 40976c^2 + 607742c^3 - 1898872c^4 + 231212c^5 + 5305036c^6$$
$$- 4368990c^7 - 3450196c^8 + 2915320c^9 + 1079784c^{10})b^6$$
$$- (4592 - 65024c + 322760c^2 - 458136c^3 - 1427596c^4 + 5786704c^5 - 5186900c^6$$
$$- 4797016c^7 + 9220764c^8 - 1075296c^9 - 2566772c^{10})b^7$$
$$+ (2240 - 43064c + 336507c^2 - 1326743c^3 + 2455692c^4 - 158240c^5 - 7734119c^6$$
$$+ 12443915c^7 - 4538800c^8 - 4973140c^9 + 3626472c^{10})b^8$$

$$
\begin{aligned}
&- (560 - 13496c + 139700c^2 - 807169c^3 + 2814462c^4 - 5872358c^5 + 6313888c^6 \\
&- 307629c^7 - 7562642c^8 + 8095844c^9 - 2821320c^{10})b^9 \\
&+ (1-2c)(1-3c)(1-4c)(1-5c)(56 - 836c + 5034c^2 - 15591c^3 + 26134c^4 \\
&- 22487c^5 + 7774c^6)b^{10}\big].
\end{aligned}
$$

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