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RESEARCH ARTICLE

# Distributed Optimal Coordination of Uncertain Nonlinear Multi-Agent Systems over Unbalanced Directed Networks via Output Feedback

Jin Zhang[1,2] | Lu Liu[2] | Xinghu Wang[1] | Haibo Ji[1]

[1]Department of Automation, University of Science and Technology of China, Hefei, China
[2]Department of Biomedical Engineering, City University of Hong Kong, Kowloon, Hong Kong

**Correspondence**
Lu Liu, Department of Biomedical Engineering, City University of Hong Kong, Kowloon, Hong Kong.
Email: luliu45@cityu.edu.hk

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**Abstract**

In this article, a novel observer-based output feedback control approach is proposed to address the distributed optimal coordination problem of uncertain nonlinear multi-agent systems in the normal form over unbalanced directed graphs. The main challenges of the concerned problem lie in unbalanced directed graphs and nonlinearities of multi-agent systems with their agent states not available for feedback control. Based on a two-layer controller structure, a distributed optimal coordinator is first designed to convert the considered problem into a reference-tracking problem. Then a decentralized output feedback controller is developed to stabilize the resulting augmented system. A high-gain observer is exploited in controller design to estimate the agent states in the presence of uncertainties and disturbances so that the proposed controller relies only on agent outputs. The semi-global convergence of the agent outputs toward the optimal solution that minimizes the sum of all local cost functions is proved under standard assumptions. A key feature of the obtained results is that the nonlinear agents under consideration are only required to be locally Lipschitz and possess globally asymptotically stable and locally exponentially stable zero dynamics.

**KEYWORDS:**
distributed optimal coordination, nonlinear systems, unbalanced directed networks, output feedback, high-gain observer

## 1 | INTRODUCTION

In the past decade, distributed optimization has made remarkable advancement and is widely applied in a broad range of fields, such as machine learning, power systems and so on [1, 2]. Distributed optimization aims at achieving an optimal consensus, which minimizes the sum of all local cost functions attached to individual agents, in a distributed manner. A considerable volume of works on solving distributed optimization problems have been reported, in both discrete-time [3–5] and continuous-time settings [6, 7]. Many engineering tasks can be formulated as a distributed optimal coordination (DOC) problem of multi-agent systems with general agent dynamics, such as economic dispatch in power systems [8] and source seeking in multi-robot systems [9]. So far, much effort has been devoted to the DOC problem of multi-agent systems with double integrators agent dynamics [10] and high-order linear agent dynamics [11, 12].

More recently, DOC problems of more general multi-agent systems with nonlinear agent dynamics have been investigated [13–17]. The DOC problem of multi-agent systems with nonlinear agent dynamics in output feedback form over undirected networks is addressed in [13]. Later, the authors in [14] develop an adaptive controller to tackle the difficulty brought by unknown nonlinear agent dynamics though still over undirected networks. It is worth noting that the controller developed in [14] is based on a two-layer framework, which is comprised of an optimal coordinator generating the optimal trajectory and a decentralized output feedback controller driving each agent to track its individual optimal coordinator. Then, in our preliminary work [15], the two-layer framework is extended to solve the DOC problem of disturbed second-order nonlinear systems but over unbalanced directed networks. In [16, 17], the DOC problem of more general nonlinear multi-agent systems in the normal form over undirected or balanced directed networks are addressed, where the inverse dynamics of agents are unfortunately required to be input-to-state stable (ISS). Moreover, the controllers developed in [16, 17] are based on agent state feedback, which may not be always available for many practical multi-agent systems.

Due to limited bandwidth or other constraints, the information exchange between agents may be unidirectional. Therefore, the study of unbalanced directed graphs has both theoretical and practical significance. By virtue of the topology balancing technique in [18], the authors in [19] propose a distributed control strategy to address unbalanced directed networks. Nevertheless, this control strategy cannot be adopted when the left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix is not known *a priori*. To remove such a restriction, a novel distributed algorithm with its gradient term being divided by an auxiliary variable is proposed in [20] to address the DOC problem for unbalanced directed networks. Nevertheless, one common limitation of the above-mentioned works is that only single integrator agent dynamics are considered.

Motivated by the above observation, this article investigates the DOC problem of uncertain nonlinear multi-agent systems with nonlinear agent dynamics in the normal form over *unbalanced* directed networks via *output* feedback. To address main challenges brought by unbalanced directed networks and uncertain nonlinear agent dynamics, the concerned DOC problem is first converted to a reference-tracking problem by developing a distributed optimal coordinator, and the resulting augmented system is then stabilized by an observer-based output feedback controller. A high-gain observer is developed to estimate agent states in the presence of uncertainties and disturbances, and semi-global convergence of the closed-loop system is then established. The main contributions of this work are summarized as follows.

1) In contrast to linear agent dynamics or nonlinear agent dynamics in output feedback form considered in [12–15, 19–21], the nonlinear agent dynamics in the normal form are more general and include the above-mentioned agent dynamics as special cases [22]. Furthermore, in virtue of a high-gain observer, only agent outputs are needed for controller design in this article, which greatly enhances its applicability in practice.

2) Unlike the works [6, 7, 13, 14, 16, 17] that study undirected or balanced directed graphs, this work focuses on unbalanced directed networks, which are more general and also more challenging. Specifically, the proposed distributed controller does not rely on the left eigenvector, which is some global information about network connectivity. It is shown that, under the proposed controller, the imbalance resulting from unbalanced directed networks is tackled, and the agent outputs are driven to the optimal solution.

3) Compared with the existing works [16, 17], where the inverse dynamics of agents are assumed to be ISS, this work only requires the zero dynamics of agents to be globally asymptotically stable and locally exponentially stable. It is shown by semi-global stability analysis that the convergence can be achieved via a linear controller instead of a nonlinear one even under this less stringent assumption. It is noted that a linear controller is advantageous in both theoretical design and practical implementation, and thus has significant engineering implications.

The rest of this article is organized as follows. Some preliminaries and the problem formulation are given in Section 2. The main results of this article and two simulation examples are provided in Section 3 and Section 4, respectively. The conclusion and future works are stated in Section 5.

*Notations*: Let $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{N\times N}$ be the sets of real numbers, $n$-order real vectors and $N$-dimensional real square matrices, respectively. $I_n$ refers to the $n$-dimensional identity matrix. Let $\mathbf{0}_n$ and $\mathbf{1}_n$, or simply $\mathbf{0}$ and $\mathbf{1}$, represent the $n$-dimensional column vector in which all entries are equal to 0 and 1, respectively. $\|\cdot\|$ denotes the Euclidean norm of vectors or induced 2-norm of matrices. $x^{\mathrm{T}}$ and $A^{\mathrm{T}}$ refer to the transpose of vector $x$ and matrix $A$, respectively. col $\left(x_1, x_2, \dots, x_n\right)$ represents a column vector with $x_1, x_2, \dots, x_n$ being its elements. diag $\left(x_1, x_2, \dots, x_n\right)$ represents a diagonal matrix with $x_1, x_2, \dots, x_n$ being its diagonal elements. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f$ denotes its gradient. $\bar{Q}^n_R \triangleq \{x = \mathrm{col}(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \leq R, i = 1, \dots, n\}$ and $\bar{\Omega}_c(V(x)) \triangleq \{x \in \mathbb{R}^n : V(x) \leq c\}$ are compact sets, where $V : \mathbb{R}^n \to \mathbb{R}$ is a positive definite function.

## 2 | PRELIMINARIES AND PROBLEM FORMULATION

In this section, we present some preliminaries on graph theory and convex analysis, and then formulate the problem under consideration.

### 2.1 | Graph Theory

A directed graph, a digraph in short, can be described by a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, which is composed of a set $\mathcal{V} = \{1, \ldots, N\}$ of nodes, a collection $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ of ordered pairs of nodes, called edges, and an adjacency matrix $\mathcal{A}$. For $i, j \in \mathcal{V}$, the ordered pair $(j, i) \in \mathcal{E}$ denotes an edge from $j$ to $i$, in this case, node $j$ is called an in-neighbor of node $i$. $\mathcal{N}_i$ denotes the set containing all the in-neighbors of agent $i$. A directed path is an ordered sequence of nodes in which any pair of consecutive nodes is a directed edge. A self-loop is an edge from a node to itself. A digraph is said to be strongly connected if, for any node, there exists a directed path from any other node to itself. The adjacency matrix $\mathcal{A} = \left[a_{ij}\right] \in \mathbb{R}^{N\times N}$ is defined as $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, otherwise $a_{ij} = 0$. Besides, $a_{ii} = 0$ for all $i$ since there is no self-loop. Moreover, the associated Laplacian matrix $\mathcal{L} = \left[l_{ij}\right] \in \mathbb{R}^{N\times N}$ is defined as $l_{ii} = \sum_{j=1}^{N} a_{ij}$, and $l_{ij} = -a_{ij}$ for $i \neq j$. A digraph $\mathcal{G}$ is weight balanced if and only if $\mathbf{1}_N^{\mathrm{T}}\mathcal{L} = \mathbf{0}_N^{\mathrm{T}}$. One may refer to [23] for more details on graph theory.

**Lemma 1.** [18, 23] Let $\mathcal{L}$ be the Laplacian matrix associated with a strongly connected directed graph $\mathcal{G}$. Then the following statements hold.

i) There exists a positive left eigenvector $\varrho = \left(\varrho_1, \varrho_2, \ldots, \varrho_N\right)^{\mathrm{T}}$ associated with the zero eigenvalue of the Laplacian matrix such that $\varrho^{\mathrm{T}}\mathcal{L} = \mathbf{0}_N^{\mathrm{T}}$ and $\sum_{i=1}^{N} \varrho_i = 1$.

ii) Let $R = \mathrm{diag}\left(\varrho_1, \varrho_2, \ldots, \varrho_N\right)$ and $\bar{\mathcal{L}} = \left(R\mathcal{L} + \mathcal{L}^{\mathrm{T}}R\right)/2$. Then $\bar{\mathcal{L}}$ is positive semidefinite, and its eigenvalues can be ordered as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N$.

### 2.2 | Convex Analysis

A differentiable function $c : \mathbb{R}^n \to \mathbb{R}$ is said to be $\varpi$-strongly convex on $\mathbb{R}^n$ if there exists a constant $\varpi > 0$ such that $(x - y)^{\mathrm{T}}(\nabla c(x) - \nabla c(y)) \geq \varpi\|x - y\|^2$ for all $x, y \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *Lipschitz continuous*, or simply *Lipschitz*, if there exists a constant $L > 0$ such that the following *Lipschitz condition* is satisfied,

$$\|f(x) - f(y)\| \leq L\|x - y\|. \tag{1}$$

If condition (1) is satisfied for every $x, y \in \mathbb{R}^n$, then $f$ is said to be *globally Lipschitz*. If condition (1) is satisfied for every $x, y \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a compact set, then $f$ is said to be *locally Lipschitz* in $\Omega$. One may refer to [24] for more details.

### 2.3 | Problem Formulation

Consider a networked system of $N$ dynamical agents over an unbalanced directed graph. The dynamics of each agent, labeled by $1, 2, \ldots, N$, are in the normal form described as follows,

$$\begin{aligned}
\dot{z}_i &= f_{i0}\left(z_i, x_{i1}, v, w\right), \\
\dot{x}_{ik} &= x_{i(k+1)}, \quad k = 1, 2, \ldots n_i - 1, \\
\dot{x}_{in_i} &= f_{i1}\left(z_i, x_i, v, w\right) + b_i(w)u_i, \\
y_i &= x_{i1}, \quad i = 1, 2, \ldots, N,
\end{aligned} \tag{2}$$

where $\mathrm{col}(z_i, x_i)$ is the state with $z_i \in \mathbb{R}^{n_{z_i}}$ and $x_i = \mathrm{col}(x_{i1}, x_{i2}, \ldots, x_{in_i}) \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ are the control input and measurement output, respectively. $w \in \mathbb{R}^{n_w}$ represents the parametric uncertainty. $v \in \mathbb{R}^{n_v}$ represents the disturbance, which is an exogenous signal generated by an autonomous exosystem,

$$\dot{v} = Sv, \tag{3}$$

where $S \in \mathbb{R}^{n_v\times n_v}$. For $i = 1, 2, \ldots, N$, it is assumed that $f_{i0}$, $f_{i1}$ and $b_i$ are sufficiently smooth functions that satisfy $f_{i0}(0, 0, 0, w) = 0$, $f_{i1}(0, 0, 0, w) = 0$, and $b_i(w) > 0$ for all $w \in \mathbb{R}^{n_w}$.

*Remark 1.* The DOC problem of multi-agent systems with nonlinear agent dynamics in the normal form over undirected or balanced directed networks has been investigated in [16, 17]. However, the following significant differences should be noted. Firstly, this article considers the same DOC problem of uncertain nonlinear multi-agent systems with nonlinear agent dynamics in the same normal form but over unbalanced directed networks. Secondly, the inverse dynamics of agents are assumed to be ISS in [16, 17], while this work only requires the zero dynamics of agents to be globally asymptotically stable as well as locally exponentially stable. Thirdly, it is assumed that the nonlinear functions in [16] are globally Lipschitz, while it is not required in this article. Fourthly, external disturbances generated by the exosystem (3) are considered in this work, while they are not considered in [16, 17]. It is noted that the exosystem (3) is able to produce several typical external signals ( e.g., sinusoidal, step and ramp type signals).

It is assumed that each agent $i$ is assigned with an individual local cost function $c_i(\cdot) : \mathbb{R} \to \mathbb{R}$, i.e., the local cost function $c_i(\cdot)$ is available only for agent $i$. The global cost function is defined as $c(s) = \sum_{i=1}^{N} c_i(s)$, where $s \in \mathbb{R}$ represents its decision variable. To cooperatively seek the optimal solution $s^\star \in \mathbb{R}$ of the global cost function, for each agent, only avaliable information from its in-neighbors and itself can be exploited for the controller design in a distributed manner. Specifically, the controller is expected to take the following form,

$$u_i = \kappa_{i1}\big(\nabla c_i, y_i, v_i\big), \quad \dot{v}_i = \kappa_{i2}\big(\nabla c_i, y_i, v_j, j \in \bar{\mathcal{N}}_i\big), \tag{4}$$

where $\kappa_{i1}$ and $\kappa_{i2}$ are functions vanishing at the origin, $\bar{\mathcal{N}}_i = \mathcal{N}_i \cup \{i\}$ is a set containing the in-neighbors of agent $i$ and itself, $v_i \in \mathbb{R}^{n_{v_i}}$ is a state of the dynamic controller with its dimention $n_{v_i}$ to be specified later. Let $x_c = \mathrm{col}(z_1, x_1, v_1, \ldots, z_N, x_N, v_N)$ and $n_c = \sum_{i=1}^{N}(n_{z_i} + n_i + n_{v_i})$. Then the following problem can be formulated.

**Problem 1.** Consider the multi-agent system (2) and the exosystem (3) under the directed graph $\mathcal{G}$ with local cost functions $c_i(\cdot), i = 1, 2, \ldots, N$. Given any constant $R > 0$ and any nonempty compact set $\mathbb{V}_0 \times \mathbb{W} \subseteq \mathbb{R}^{n_v \times n_w}$ containing the origin, design a distributed dynamic output feedback controller of the form (4) such that, for any initial $x_c(0) \in \bar{Q}_R^{n_c}$ and $\mathrm{col}(v(0), w) \in \mathbb{V}_0 \times \mathbb{W}$, the trajectories of the closed-loop system consisting of (2) and (4) exist and are bounded for all $t \geq 0$, moreover, all the outputs $y_i, i = 1, 2, \ldots, N$ converge to the optimal solution $s^\star$ that minimizes $c(s) = \sum_{i=1}^{n} c_i(s)$ as time goes to infinity.

The following assumptions are needed for solving Problem 1.

**Assumption 1.** For $i = 1, 2, \ldots, N$, the local cost function $c_i$ is continuously differentiable and $\varpi_i$-strongly convex, and $\nabla c_i$ is globally Lipschitz on $\mathbb{R}$ with constant $l_i$.

**Assumption 2.** The directed graph $\mathcal{G}$ is strongly connected.

*Remark 2.* Under Assumption 1, the existence and uniqueness of the optimal solution $s^\star \in \mathbb{R}$ can be guaranteed. Under Assumption 2, one is able to obtain a weight-balanced digraph by virtue of the topology balancing technique as in [18, 23]. Assumptions 1 and 2 are commonly used in solving the distributed optimization problem over directed graphs, see, for example [7, 20].

**Assumption 3.** For $i = 1, 2, \ldots, N$, there exists a sufficiently smooth function $z_i^\star(s, v, w)$ with $z_i^\star(0, 0, w) = 0$ such that, for any $\mathrm{col}(v, w) \in \mathbb{R}^{n_v \times n_w}$ and $s \in \mathbb{R}$,

$$\frac{\partial z_i^\star(s, v, w)}{\partial v} S v = f_{i0}(z_i^\star(s, v, w), s, v, w). \tag{5}$$

**Assumption 4.** The exosystem is neutrally stable, i.e., all the eigenvalues of $S$ are semi-simple with zero real parts.

*Remark 3.* Assumption 3 is commonly used in solving the cooperative output regulation problem of multi-agent systems with nonlinear agent dynamics in the normal form [25, 26]. Under Assumption 4, given any compact set $\mathbb{V}_0$, it can be shown that for any $v(0) \in \mathbb{V}_0$, the trajectory $v(t)$ of the exosystem (3) remains in some compact set $\mathbb{V}$ for all $t \geq 0$.

## 3 | MAIN RESULTS

In this section, based on a two-layer controller structure, a distributed output feedback controller is developed to solve Problem 1. The architecture of the two-layer strategy is depicted in Fig. 1. In the upper layer, the concerned DOC problem is first converted

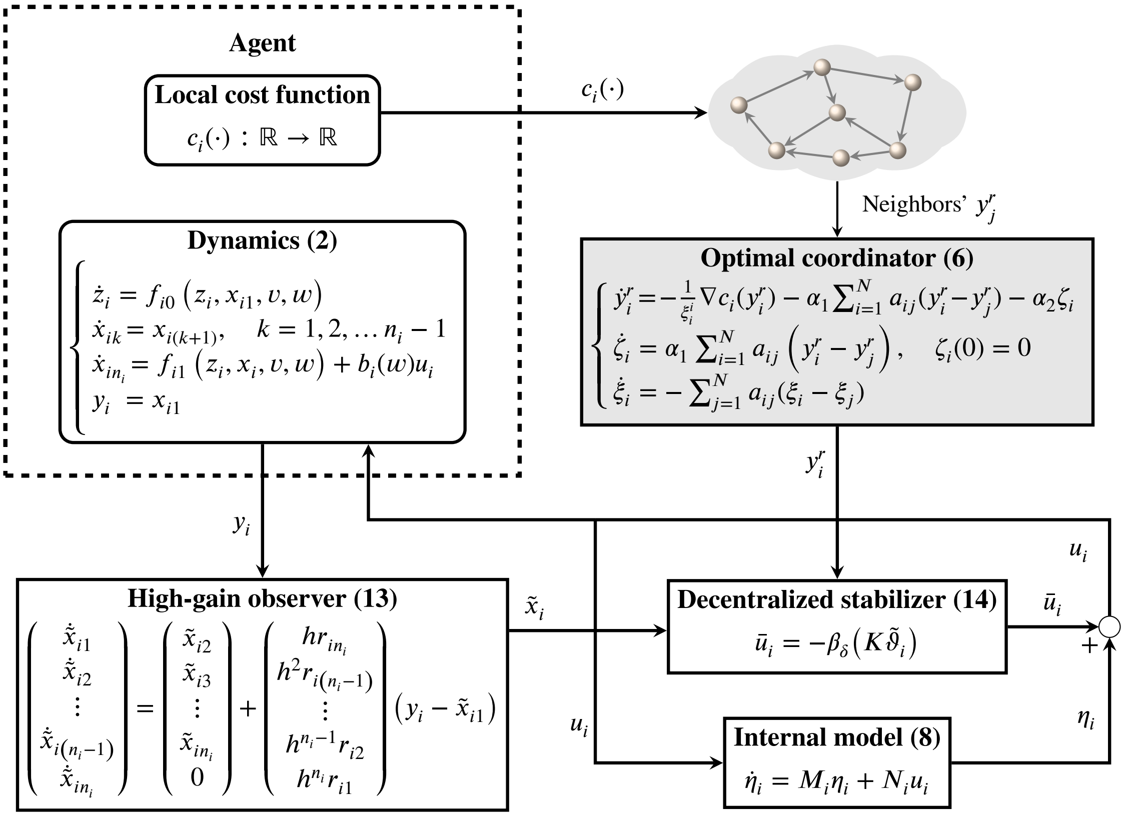


**FIGURE 1** Design diagram for distributed optimal coordination.

to a reference-tracking problem by designing a distributed optimal coordinator for each agent, which cooperates with others to generate a local reference signal that eventually converges to the optimal solution. Then in the lower layer, by virtue of an internal model for handling the external disturbance and a high-gain observer for estimating the agent states, a decentralized output feedback stabilizer is proposed to address the augmented system composed of the resulting error system and the internal model.

## 3.1 | Distributed Optimal Coordinator Design

In this subsection, inspired by our preliminary work [15], a distributed optimal coordinator is designed to generate the optimal solution for the concerned multi-agent system over the unbalanced directed network. Then the proposed optimal coordinator is embedded into the feedback loop to convert the DOC problem to a reference-tracking problem, which will be addressed in the next subsection. Specifically, the following optimal coordinator is designed for each agent $i$,

$$\begin{cases} \dot y_i^r = -\dfrac{1}{\xi_i^i}\nabla c_i(y_i^r) - \alpha_1 \displaystyle\sum_{i=1}^N a_{ij}(y_i^r - y_j^r) - \alpha_2\zeta_i, \\ \dot\zeta_i = \alpha_1 \displaystyle\sum_{i=1}^N a_{ij}\left(y_i^r - y_j^r\right), \quad \zeta_i(0) = 0, \\ \dot\xi_i = -\displaystyle\sum_{j=1}^N a_{ij}(\xi_i - \xi_j), \quad i = 1, 2, \ldots, N, \end{cases} \tag{6}$$

where $\alpha_1$ and $\alpha_2$ are two positive constants, $y_i^r \in \mathbb{R}$ represents the generated reference signal for agent $i$, $\zeta_i \in \mathbb{R}$ and $\xi_i \in \mathbb{R}^N$ are auxiliary variables, with $\xi_i^k$ being the $k$-th component of $\xi_i$ and initial value $\xi_i(0)$ satisfying $\xi_i^i(0) = 1$, otherwise $\xi_i^k(0) = 0$ for all $k \neq i$. With these choices, it is shown in [15] that $\xi_i^i(t) > 0$ for all $t \geq 0$, which means that the algorithm (6) is well defined.

It should be emphasized that, under Assumption 2, it can be proved by using Lemma 2.1 in [18] that $\lim_{t\to\infty} \xi_i^i = \varrho_i$, where $\varrho_i$ represents the $i$-th component of the left eigenvector $\varrho$ corresponding to the zero eigenvalue of the Laplacian matrix [15]. Note that the optimal coordinator of each agent only requires information from its neighbors and itself, and thus is distributed.

*Remark 4.* Based on the topology balancing technique developed in [18, 23], a distributed optimization algorithm similar to the optimal coordinator (6), but with its gradient term being divided by the $i$-th component of the left eigenvector $\varrho$ corresponding to the zero eigenvalue of the Laplacian matrix, is proposed in [19]. However, it is worth noting that the left eigenvector is a kind of global information, and thus may not be known *a priori*. Instead, the gradient term $\nabla c_i$ in (6) is divided by $\xi_i^i$, which is generated by the autonomous system $\dot{\xi}_i = -\sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j)$ and provides an alternative so that the explicit dependence on the left eigenvector can be removed.

The following lemma shows that the distributed optimal coordinator (6) is capable of generating the optimal solution $s^\star$. Its proof can be found in Theorem 1 in our preliminary result [15].

**Lemma 2.** Under Assumptions 1 and 2, there exist sufficiently large positive constants $\alpha_1$ and $\alpha_2$ such that the generated reference signals $y_i^r, i = 1, 2, \dots, N$ of the distributed optimal coordinator (6) are bounded for all $t \geq 0$, and exponentially converge to the optimal solution $s^\star$ that minimizes the global cost function.

Compared with second-order nonlinear systems in [15], from which the distributed optimal coordinator (6) is adopted, we consider more general nonlinear systems in the normal form in this article. Moreover, it is assumed that agent states are unavailable for feedback due to uncertainties and disturbances in the higher-order systems. Therefore, it remains challenging to force all the agents to achieve optimal output consensus in this more practical case.

In what follows, by taking $y_i^r$ as a reference to be tracked by agent $i$, we convert Problem 1 into a reference-tracking problem. To this end, define $x_i^\star = \mathrm{col}(y_i^r, \mathbf{0}_{n_i-1})$. Let $\bar{z}_i = z_i - z_i^\star\left(s^\star, v, w\right)$ and $\bar{x}_i = x_i - x_i^\star$. Then, the dynamics of $\bar{z}_i$ and $\bar{x}_i$ can be described as follows,

$$\begin{cases} \dot{\bar{z}}_i = \bar{f}_{i0}\left(\bar{z}_i, \bar{x}_{i1}, y_i^r, v, w\right) + \hat{f}_{i0}\left(y_i^r, s^\star, v, w\right), \\ \dot{\bar{x}}_{i1} = \bar{x}_{i2} - \dot{y}_i^r, \\ \dot{\bar{x}}_{ik} = \bar{x}_{i(k+1)}, \quad k = 2, 3, \dots n_i - 1, \\ \dot{\bar{x}}_{in_i} = \bar{f}_{i1}\left(\bar{z}_i, \bar{x}_i, y_i^r, v, w\right) + \bar{f}_{i2}\left(y_i^r, s^\star, v, w\right) + b_i(w)\left(u_i - u_i^\star\left(s^\star, v, w\right)\right), \end{cases} \tag{7}$$

where

$$\begin{aligned} \bar{f}_{i0}\left(\bar{z}_i, \bar{x}_{i1}, y_i^r, v, w\right) =& f_{i0}\left(\bar{z}_i + z_i^\star, \bar{x}_{i1} + y_i^r, v, w\right) - f_{i0}\left(z_i^\star, y_i^r, v, w\right), \\ \hat{f}_{i0}\left(y_i^r, s^\star, v, w\right) =& f_{i0}\left(z_i^\star, y_i^r, v, w\right) - f_{i0}\left(z_i^\star, s^\star, v, w\right), \\ \bar{f}_{i1}\left(\bar{z}_i, \bar{x}_i, y_i^r, v, w\right) =& f_{i1}\left(\bar{z}_i + z_i^\star, \bar{x}_i + x_i^\star, v, w\right) - f_{i1}\left(z_i^\star, x_i^\star, v, w\right), \\ \bar{f}_{i2}\left(y_i^r, s^\star, v, w\right) =& f_{i1}\left(z_i^\star, y_i^r, \mathbf{0}, v, w\right) - f_{i1}\left(z_i^\star, s^\star, \mathbf{0}, v, w\right), \\ u_i^\star\left(s^\star, v, w\right) =& -f_{i1}\left(z_i^\star, s^\star, \mathbf{0}, v, w\right)/b_i(w). \end{aligned}$$

By the smoothness of $f_{ik}, k = 0, 1$, it is known that $\bar{f}_{ik}, k = 0, 1$ are sufficiently smooth functions satisfyinig $\bar{f}_{ik}\left(\mathbf{0}, \mathbf{0}, y_i^r, v, w\right) = 0$ for all $y_i^r \in \mathbb{R}$ and $\mathrm{col}(v, w) \in \mathbb{V} \times \mathbb{W}$. Note that $u_i^\star\left(s^\star, v, w\right)$ is well-defined for all $w \in \mathbb{W}$ since it is assumed that $b_i(w) > 0$. Nevertheless,it should be pointed out that the feed-forward term $u_i^\star\left(s^\star, v, w\right)$ in (7) is unavailable for feedback due to the presence of uncertain parameter $w$. To address this issue, an additional standard assumption [27] is needed.

**Assumption 5.** For $i = 1, 2, \dots, N$, $u_i^\star\left(s^\star, v, w\right)$ is a polynomial in $v$ with coefficients depending on $s^\star$ and $w$.

Now, we are ready to design an internal model to generate the feed-forward term $u_i^\star\left(s^\star, v, w\right)$. Under Assumptions 4 and 5, there exist integers $s_i, i = 1, 2, \dots, N$ such that for all $w \in \mathbb{W}$, one has $\frac{\mathrm{d}^{s_i} u_i^\star(s^\star, v, w)}{\mathrm{d}t^{s_i}} = \ell_{i1} u_i^\star\left(s^\star, v, w\right) + \ell_{i2} \frac{\mathrm{d}u_i^\star(s^\star, v, w)}{\mathrm{d}t} + \cdots + \ell_{is_i} \frac{\mathrm{d}^{(s_i-1)} u_i^\star(s^\star, v, w)}{\mathrm{d}t^{(s_i-1)}}$, where $\ell_{i1}, \ell_{i2}, \dots \ell_{is_i}$, $i = 1, 2, \dots, N$ are scalars such that the roots of the polynomials $P_i(\sigma) = \sigma^{s_i} - \ell_{i1} - \ell_{i2}\sigma - \cdots - \ell_{is_i}\sigma^{s_i-1}$ are distinct with zero real parts. Define

$$\Phi_i = \left[\begin{array}{c|c} \mathbf{0}_{(s_i-1)\times 1} & I_{s_i-1} \\ \hline \ell_{i1} & \ell_{i2}, \dots, \ell_{is_i} \end{array}\right], \quad \Gamma_i = \begin{bmatrix} 1 \\ \mathbf{0}_{(s_i-1)\times 1} \end{bmatrix}^T.$$

Let $\tau_i(s^\star, v, w) = \mathrm{col}\left(u_i^\star, \mathrm{d}u_i^\star/\mathrm{d}t, \dots, \mathrm{d}^{(s_i-1)}u_i^\star/\mathrm{d}t^{(s_i-1)}\right)$. One thus has $\dot{\tau}_i = \Phi_i\tau_i, u_i^\star = \Gamma_i\tau_i$. Let $\left(M_i, N_i\right), i = 1, 2, \dots, N$ be any controllable pairs, where $M_i \in \mathbb{R}^{s_i\times s_i}$ is a Hurwitz matrix, and $N_i \in \mathbb{R}^{s_i\times 1}$ is a column vector. We proposed the internal model for agent $i$ as follows,

$$\dot{\eta}_i = M_i\eta_i + N_iu_i, \quad i = 1, 2, \dots, N. \tag{8}$$

Since the spectra of $\Phi_i$ and $M_i$ are disjoint, the Sylvester equation $T_i\Phi_i - M_iT_i = N_i\Gamma_i$ is satisfied with a nonsingular matrix $T_i$. Let $\bar{\eta}_i = \eta_i - T_i\tau_i\left(s^\star, v, w\right)$ and $\bar{u}_i = u_i - \Gamma_iT_i^{-1}\eta_i$. One then has $\dot{\bar{\eta}}_i = \left(M_i + N_i\Gamma_iT_i^{-1}\right)\bar{\eta}_i + N_i\bar{u}_i$. Thus, the following augmented error system can be obtained,

$$\begin{cases} \dot{\bar{z}}_i = \bar{f}_{i0}\left(\bar{z}_i, \bar{y}_i, y_i^r, v, w\right) + \hat{f}_{i0}\left(y_i^r, s^\star, v, w\right), \\ \dot{\bar{x}}_i = A_i\bar{x}_i + B_i\Big(\bar{f}_{i1}\left(\bar{z}_i, \bar{x}_i, y_i^r, v, w\right) + \bar{f}_{i2}\left(y_i^r, s^\star, v, w\right) + b_i(w)\Gamma_iT_i^{-1}\bar{\eta}_i + b_i(w)\bar{u}_i\Big) - E_i\dot{y}_i^r, \\ \dot{\bar{\eta}}_i = \left(M_i + N_i\Gamma_iT_i^{-1}\right)\bar{\eta}_i + N_i\bar{u}_i, \\ \bar{y}_i = \bar{x}_{i1}, \end{cases} \tag{9}$$

where

$$A_i = \left[\begin{array}{c|c} \mathbf{0}_{n_i-1} & I_{n_i-1} \\ \hline 0 & \mathbf{0}^{\mathrm{T}}_{n_i-1} \end{array}\right], B_i = \begin{bmatrix} \mathbf{0}_{n_i-1} \\ 1 \end{bmatrix}, E_i = \begin{bmatrix} 1 \\ \mathbf{0}_{n_i-1} \end{bmatrix}.$$

In what follows, to stabilize the augmented error system (9) semi-globally, a distributed dynamic output feedback controller of the following form will be developed,

$$\bar{u}_i = \beta_\delta\big(\bar{\kappa}_{i1}(\bar{x}_i, \bar{v}_i)\big), \quad \dot{\bar{v}}_i = \bar{\kappa}_{i2}\big(y_i, \bar{v}_j, j \in \bar{\mathcal{N}}_i\big), \tag{10}$$

with $\beta_\delta(\cdot)$ being a saturation function defined as follows,

$$\beta_\delta(r) = \begin{cases} r & \text{if } |r| < \delta, \\ \mathrm{sgn}(r)\delta & \text{if } |r| \geq \delta, \end{cases}$$

where $\delta > 0$ is a constant to be designed, $\bar{\kappa}_{i1}$ and $\bar{\kappa}_{i2}$ are sufficiently smooth functions vanishing at the origin, $\bar{\mathcal{N}}_i = \mathcal{N}_i \cup \{i\}$ is defined to be the same as that in (4), and $\bar{v}_i \in \mathbb{R}^{n_{\bar{v}_i}}$ is the state of the dynamic controller with its dimention $n_{\bar{v}_i}$ to be specificed later. Let $\bar{x}_c = \mathrm{col}(\bar{z}_1, \bar{x}_1, \bar{\eta}_1, \bar{v}_1, y_1^r, \zeta_1, \dots, \bar{z}_N, \bar{x}_N, \bar{\eta}_N, \bar{v}_N, y_N^r, \zeta_N)$ and $n_c = \sum_{i=1}^N(n_{z_i} + n_i + s_i + n_{\bar{v}_i} + 2)$. The reference-tracking problem is defined as follows.

**Problem 2.** Consider the augmented error system (9) and the optimal coordinator (6). Under Assumptions 1-4, given any constant $\bar{R} > 0$ and any nonempty compact set $\mathbb{V} \times \mathbb{W} \subseteq \mathbb{R}^{n_v+n_w}$ containing the origin, design a distributed dynamic output feedback controller of the form (10) such that, for any $\bar{x}_c(0) \in \bar{Q}_{\bar{R}}^{n_c}$ and $\mathrm{col}(v, w) \in \mathbb{V} \times \mathbb{W}$, the trajectories of the closed-loop system composed of (6), (9) and (10) starting from $\bar{x}_c(0)$ are bounded for all $t \geq 0$, and the output errors $\bar{y}_i, i = 1, 2, \dots, N$ tend to zero as time goes to infinity.

The following lemma shows that Problem 1 is solved as long as Problem 2 is solved.

**Lemma 3.** Under Assumptions 1-4, if the reference-tracking Problem 2 is solved by a dynamic output feedback controller of the form (10), then the Problem 1 can be solved by a distributed dynamic controller composed of (6), (8) and (10).

*Proof.* As stated in Remark 3, given any compact set $\mathbb{V}_0 \subseteq \mathbb{R}^{n_v}$, there exists a compact set $\mathbb{V} \subseteq \mathbb{R}^{n_v}$ such that $v(t) \in \mathbb{V}$ for all $t \geq 0$. By comparing the definitions of $x_c$ and $\bar{x}_c$, given any $R > 0$ and $x_c(0) \in \bar{Q}_R^{n_c}$, there exists $\bar{R} > 0$ such that $\bar{x}_c(0) \in \bar{Q}_{\bar{R}}^{n_c}$. Note that Problem 2 is solved means that the trajectory $\bar{x}_c(t)$ of the closed-loop system starting from $\bar{x}_c(0) \in \bar{Q}_{\bar{R}}^{n_c}$ is bounded for all $t \geq 0$, and the output errors $\bar{y}_i, i = 1, 2, \dots, N$ converge to zero. Thus, on the one hand, by similar arguments as in [25], for any $x_c(0) \in \bar{Q}_R^{n_c}$ and $\mathrm{col}(v, w) \in \mathbb{V} \times \mathbb{W}$, $x_c(t)$ can be shown to be bounded for all $t \geq 0$. On the other hand, one obtains that $\lim_{t\to\infty} y_i = y_i^r$. It follows from Lemma 2 that $y_i^r$ is bounded for all $t \geq 0$ and $\lim_{t\to\infty} y_i^r = s^\star$. Therefore, by the triangle inequality $|y_i - s^\star| \leq |y_i - y_i^r| + |y_i^r - s^\star|$, one can conclude that $\lim_{t\to\infty} y_i = s^\star$. In summary, Problem 1 is solved as long as Problem 2 is solved. □

## 3.2 Reference-Tracking Controller Design

In this subsection, we focus on developing an output feedback controller to solve the reference-tracking Problem 2 for the augmented error system (9). At first, we transfer (9) to a new system of relative degree one. To this end, define

$$\begin{aligned}\hat{x}_{ik} =& \frac{\bar{x}_{ik}}{g^{k-1}}, \quad k = 1, \dots, n_i - 1,\\ \vartheta_i =& \bar{x}_{in_i} + g\gamma_{i(n_i-1)}\bar{x}_{i(n_i-1)} + \cdots + g^{n_i-1}\gamma_{i1}\bar{x}_{i1},\\ \tilde{\eta}_i =& \bar{\eta}_i - b_i^{-1}(w)N_i\vartheta_i,\end{aligned}$$

where $g > 0$ is a constant to be determined later, and the coefficients $\gamma_{ik}, k = 1, 2, \dots, n_i - 1$ are chosen such that the polynomials $\lambda^{n_i-1} + \gamma_{i(n_i-1)}\lambda^{n_i-2} + \cdots + \gamma_{i2}\lambda + \gamma_{i1}, i = 1, 2, \dots, N$ are all Hurwitz. Then, one can obtain that

$$\begin{cases}\dot{\hat{x}}_{i1} = g\hat{x}_{i2} - \dot{y}_i^r,\\ \dot{\hat{x}}_{ik} = g\hat{x}_{i(k+1)}, \quad k = 2, 3, \dots, n_i - 2,\\ \dot{\hat{x}}_{i(n_i-1)} = \dfrac{\vartheta_i}{g^{n_i-2}} - g\gamma_{i(n_i-1)}\hat{x}_{i(n_i-1)} - \cdots - g\gamma_{i1}\hat{x}_{i1},\\ \dot{\vartheta}_i = \hat{f}_{i1}\left(\bar{z}_i, \hat{x}_{ia}, \vartheta_i, y_i^r, g, v, w\right) + b_i(w)\Gamma_i T_i^{-1}\tilde{\eta}_i + \Gamma_i T_i^{-1} N_i \vartheta_i + b_i(w)\bar{u}_i + \varepsilon_i\left(y_i^r, \dot{y}_i^r, s^\star, v, w\right),\end{cases}$$

where

$$\begin{aligned}\hat{f}_{i1}\left(\bar{z}_i, \hat{x}_{ia}, \vartheta_i, y_i^r, g, v, w\right) =& \bar{f}_{i1}\Big(\bar{z}_i, \hat{x}_{i1}, g\hat{x}_{i2}, \dots, g^{n_i-2}\hat{x}_{i(n_i-1)}, \vartheta_i - g^{n_i-1}\left(\gamma_{i(n_i-1)}\hat{x}_{i(n_i-1)} + \cdots + \gamma_{i1}\hat{x}_{i1}\right), y_i^r, v, w\Big)\\ &+ g\gamma_{i(n_i-1)}\Big(\vartheta_i - g^{n_i-1}\left(\gamma_{i(n_i-1)}\hat{x}_{i(n_i-1)} + \cdots + \gamma_{i1}\hat{x}_{i1}\right)\Big) + g^{n_i}\left(\gamma_{i(n_i-2)}\hat{x}_{i(n_i-1)} + \cdots + \gamma_{i2}\hat{x}_{i3} + \gamma_{i1}\hat{x}_{i2}\right),\\ \varepsilon_i\left(y_i^r, \dot{y}_i^r, s^\star, v, w\right) =& \bar{f}_{i2}\left(y_i^r, s^\star, v, w\right) - g^{(n_i-1)}\gamma_{i1}\dot{y}_i^r.\end{aligned}$$

Let $\hat{x}_{ia} = \text{col}\left(\hat{x}_{i1}, \hat{x}_{i2}, \dots \hat{x}_{i(n_i-1)}\right)$. Then the augmented error system (9) can be rewritten as follows,

$$\begin{cases}\dot{\bar{z}}_i = \tilde{f}_{i0}\left(\bar{z}_i, \hat{x}_{ia}, y_i^r, v, w\right) + \hat{f}_{i0}\left(y_i^r, s^\star, v, w\right),\\ \dot{\hat{x}}_{ia} = gA_{ci}\hat{x}_{ia} + B_{ci}(g)\vartheta_i - E_i\dot{y}_i^r,\\ \dot{\tilde{\eta}}_i = M_i\tilde{\eta}_i + \tilde{f}_{i1}\left(\bar{z}_i, \hat{x}_{ia}, \vartheta_i, y_i^r, g, v, w\right) - b_i^{-1}(w)N_i\varepsilon_i,\\ \dot{\vartheta}_i = \tilde{f}_{i2}\left(\bar{z}_i, \hat{x}_{ia}, \tilde{\eta}_i, \vartheta_i, y_i^r, g, v, w\right) + b_i(w)\bar{u}_i + \varepsilon_i,\end{cases} \tag{11}$$

where

$$\begin{aligned}A_{ci} =& \left[\begin{array}{c|c}\mathbf{0}_{n_i-2} & I_{n_i-2}\\ \hline -\gamma_{i1} & -\gamma_{i2}, \dots, -\gamma_{i(n_i-1)}\end{array}\right], \quad B_{ci}(g) = \begin{bmatrix}\mathbf{0}_{n_i-2}\\ 1/g^{n_i-2}\end{bmatrix}, \quad E_i = \begin{bmatrix}1\\ \mathbf{0}_{n_i-1}\end{bmatrix},\\ \tilde{f}_{i0}\left(\bar{z}_i, \hat{x}_{ia}, y_i^r, v, w\right) =& \bar{f}_{i0}\left(\bar{z}_i, \bar{x}_{i1}, y_i^r, v, w\right),\\ \tilde{f}_{i1}\left(\bar{z}_i, \hat{x}_{ia}, \vartheta_i, y_i^r, g, v, w\right) =& b_i^{-1}(w)M_iN_i\vartheta_i - b_i^{-1}(w)N_i\hat{f}_{i1}\left(\bar{z}_i, \hat{x}_{ia}, \vartheta_i, y_i^r, g, v, w\right),\\ \tilde{f}_{i2}\left(\bar{z}_i, \hat{x}_{ia}, \tilde{\eta}_i, \vartheta_i, y_i^r, g, v, w\right) =& \hat{f}_{i1}\left(\bar{z}_i, \hat{x}_{ia}, \vartheta_i, y_i^r, g, v, w\right) + b_i(w)\Gamma_iT_i^{-1}\tilde{\eta}_i + \Gamma_iT_i^{-1}N_i\vartheta_i.\end{aligned}$$

Now we are ready to design the decentralized output feedback controller $\bar{u}_i$ for the augmented system (11) under the following assumption.

**Assumption 6.** For each $i = 1, 2, \dots, N$, there exists a continuously differentiable function $V_{i0} : \mathbb{R}^{n_{z_i}} \to \mathbb{R}$ such that, for all $y_i^r \in \mathbb{R}$ and $\text{col}(v, w) \in \mathbb{V} \times \mathbb{W}$, along the trajectories of the zero dynamics $\dot{\bar{z}}_i = \bar{f}_{i0}(\bar{z}_i, 0, y_i^r, v, w)$, the following inequality is satisfied,

$$\frac{\partial V_{i0}}{\partial \bar{z}_i}\bar{f}_{i0}(\bar{z}_i, 0, y_i^r, v, w) \le -a_{i0}\left\|\bar{z}_i\right\|^2, \tag{12}$$

where $a_{i0} > 0$ is a known constant.

*Remark 5.* According to Theorem 4.10 in [22], the inequality (12) implies that the zero dynamics of agents are globally asymptotically stable as well as locally exponentially stable, which is less stringent than the assumption that the inverse dynamics of agents are ISS in [16, 17]. In particular, the locally exponential stability is needed to drive all trajectories to the origin instead of its neighborhood [28]. By choosing sufficiently large control gains, it is shown by semi-global stability analysis that $\mathrm{col}(\bar{z}_i, \bar{x}_{i1})$ can still be guaranteed to remain in a compact set under such a relaxed assumption.

### 3.2.1 | High-Gain Observer Design

Due to the uncertainties in agent dynamics (2), it is more challenging to estimate the agent states since no perfect knowledge of the nonlinear functions $f_{i1}, i = 1, 2, \ldots, N$ can be utilized for designing an observer. Fortunately, the high gain observer is robust to a certain level of uncertainties and can still work in this case. Specifically, a distributed high-gain observer is proposed for each agent as follows,

$$\begin{pmatrix} \dot{\tilde{x}}_{i1} \\ \dot{\tilde{x}}_{i2} \\ \vdots \\ \dot{\tilde{x}}_{i(n_i-1)} \\ \dot{\tilde{x}}_{in_i} \end{pmatrix} = \begin{pmatrix} \tilde{x}_{i2} \\ \tilde{x}_{i3} \\ \vdots \\ \tilde{x}_{in_i} \\ 0 \end{pmatrix} + \begin{pmatrix} hr_{in_i} \\ h^2 r_{i(n_i-1)} \\ \vdots \\ h^{n_i-1} r_{i2} \\ h^{n_i} r_{i1} \end{pmatrix} \left(y_i - \tilde{x}_{i1}\right), \tag{13}$$

where $\tilde{x}_{ik}, k = 1, 2, \ldots, n_i$ are the estimations of the agent states, $r_{ik}, k = 1, 2, \ldots, n_i$ are cofficients such that the polynomials $\lambda^{n_i} + r_{in_i}\lambda^{n_i-1} + \cdots + r_{i2}\lambda + r_{i1}, i = 1, 2, \ldots, N$ are all Hurwitz, and $h > 0$ is a sufficiently large constant to be specified later.

### 3.2.2 | Decentralized Output Feedback Stabilizer Design

With the proposed distributed high-gain observer (13), the decentralized output feedback stabilizer for the augmented system (11) is developed as follows,

$$\bar{u}_i = -\beta_\delta\left(K\tilde{\vartheta}_i\right), \tag{14}$$

where $K$ and $\delta$ are positive constants to be designed later, $\tilde{\vartheta}_i = \tilde{x}_{in_i} + g\gamma_{i(n_i-1)}\tilde{x}_{i(n_i-1)} + \cdots + g^{n_i-2}\gamma_{i2}\tilde{x}_{i2} + g^{n_i-1}\gamma_{i1}\left(\tilde{x}_{i1} - y_i^r\right)$, with $\tilde{x}_{ik}, k = 1, 2, \ldots, n_i$ being generated by the high-gain observer (13) and $y_i^r$ being generated by the optimal coordinator (6).

*Remark 6.* The high-gain observer is exploited owing to its robustness in estimating the agent states in the presence of uncertainties. However, it may also exhibit an impulsive-like behavior known as the peaking phenomenon, which may even interact with nonlinearities leading to a finite escape time. Hence, the saturation function $\beta_\delta(\cdot)$ is utilized in (14) to avoid the peaking phenomenon and to ensure the control input in a compact set of interest.

The solvability of Problem 2 is summarized as follows.

**Theorem 1.** Under Assumption 6, given any arbitrarily large constant $\bar{R} > 0$, there exist sufficiently large positive constants $g$, $K$, $\delta$ and $h$ depending on $\bar{R}$ such that the equilibrium point of the closed-loop system consisting of (6), (11), (13) and (14) is uniformly locally asymptotically stable with its region of attraction containing $\bar{Q}_{\bar{R}}^{n_c}$, where $n_c = \sum_{i=1}^{N}\left(n_{z_i} + 2n_i + s_i + 2\right)$. In other words, the semi-global reference-tracking Problem 2 is solvable.

Before giving the proof of Theorem 1, some coordinate transformations are introduced to redescribe the resulting closed-loop system. Let $\tilde{y}_i^r = y_i^r - \bar{y}_i^r$ and $\tilde{\zeta}_i = \zeta_i - \bar{\zeta}_i$, where $\mathrm{col}(y_i^r, \zeta_i)$ is generated by (6) with $\mathrm{col}(\bar{y}_i^r, \bar{\zeta}_i)$ being its corresponding equilibrium point. Define $X_i = \mathrm{col}\left(\bar{z}_i, \hat{x}_{ia}, \tilde{\eta}_i, \vartheta_i, \tilde{y}_i^r, \tilde{\zeta}_i\right)$. Let $e_{ik} = h^{n_i-k}\left(x_{ik} - \tilde{x}_{ik}\right), k = 1, 2, \ldots, n_i$ and $e_i = \mathrm{col}\left(e_{i1}, e_{i2}, \ldots, e_{in_i}\right)$. It follows that $\tilde{x}_i = x_i - H_i^{-1}e_i$, where $H_i = \mathrm{diag}\left(h^{n_i-1}, h^{n_i-2}, \ldots, h, 1\right)$. One then has $\tilde{\vartheta}_i = \vartheta_i - E_{i2}(h)e_i = E_{i1}X_i - E_{i2}(h)e_i$, where

$$E_{i1} = \left[\mathbf{0}_{1\times(n_{z_i}+s_i+n_i-1)}, 1, \mathbf{0}_{1\times 2}\right],$$
$$E_{i2}(h) = \left[\frac{g^{n_i-1}\gamma_{i1}}{h^{n_i-1}}, \frac{g^{n_i-2}\gamma_{i2}}{h^{n_i-2}}, \ldots, \frac{g\gamma_{i(n_i-1)}}{h}, 1\right].$$

It can be obtained from (13) and (14) that

$$\bar{u}_i = -\beta_\delta\left(KE_{i1}X_i - KE_{i2}(h)e_i\right), \tag{15}$$
$$\dot{e}_i = hE_{i3}e_i + E_{i4}\dot{X}_i, \tag{16}$$

where

$$E_{i3} = \begin{bmatrix} -r_{in_i} & 1 & 0 & \cdots & 0 \\ -r_{i(n_i-1)} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_{i2} & 0 & 0 & \cdots & 1 \\ -r_{i1} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{i4} = \begin{bmatrix} \mathbf{0}_{(n_i-1)\times n_{z_i}} & \mathbf{0}_{(n_i-1)\times(n_i-1)} & \mathbf{0}_{(n_i-1)\times s_i} & \mathbf{0}_{(n_i-1)\times 1} & \mathbf{0}_{(n_i-1)\times 2} \\ \mathbf{0}_{1\times n_{z_i}} & -g^{n_i-1}\left[\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{i(n_i-1)}\right] & \mathbf{0}_{1\times s_i} & 1 & \mathbf{0}_{1\times 2} \end{bmatrix}.$$

Thus, the closed-loop system composed of (6), (11), (13) and (14) can be described by the closed-loop system composed of (6), (11), (15) and (16). Let $e = \text{col}\left(e_1, e_2, \ldots, e_N\right)$ and $X = \text{col}\left(X_1, X_2, \ldots, X_N\right)$. Then, we can obtain the following two crucial lemmas, whose proofs are given in the Appendix.

**Lemma 4.** Under Assumption 6, given any arbitrarily large $\bar{R} > 0$, there exist positive constants $g$, $K$ and $c$ that depend on $\bar{R}$, and a continuously differentiable positive definite function $W_X(\cdot)$, such that the following equation is satisfied,

$$\bar{Q}_{\bar{R}}^{\bar{n}_c} \subseteq \bar{\Omega}_c\left(W_X(X)\right), \tag{17}$$

where $\bar{n}_c = \sum_{i=1}^{N}\left(n_{z_i} + n_i + s_i + 2\right)$. Moreover, given any $\epsilon > 0$, for all $X \in \bar{\Omega}_{c+\epsilon}\left(W_X(X)\right)$ and all $\text{col}(v, w) \in \mathbb{V} \times \mathbb{W}$, the derivative of $W_X(\cdot)$ along the trajectories of the closed-loop system composed of (6), (11) and the following decentralized state feedback controller,

$$\bar{u}_i = -K\vartheta_i, \tag{18}$$

satisfies

$$\dot{W}_X(X)\Big|_{(6)+(11)+(18)} \leq -a\|X\|^2, \tag{19}$$

where $a > 0$ is a constant to be determined later.

**Lemma 5.** There exists a continuously differentiable positive definite function $W_e(e)$ such that, for all $\text{col}(v, w) \in \mathbb{V} \times \mathbb{W}$, the derivative of $W_e(e)$ along the trajectories of (16) satisfies

$$\dot{W}_e(e)\Big|_{(16)} \leq -\frac{h}{2}\|e\|^2 + \frac{\varrho}{h}\|\dot{X}\|^2, \tag{20}$$

where $\varrho > 0$ is a constant independent of $h$.

Based on Lemmas 4 and 5, the proof of Theorem 1 is presented as follows.

**Proof of Theorem 1:** For any arbitrarily large $\bar{R} > 0$, $g$ and $K$ are chosen to be the same as those in Lemma 4. For any arbitrarily small $\epsilon > 0$, the bound $\delta$ of the saturation function is chosen as

$$\delta = K\left(\max_{\substack{X\in\bar{\Omega}_{c+\epsilon}(W_X(X)) \\ i=1,\ldots,N}}\left\{E_{i1}X_i\right\} + \frac{\epsilon}{2}\max_{i=1,\ldots,N}\left\{\left\|E_{i2}(1)\right\|\right\}\right). \tag{21}$$

Now, we are ready to prove that the trajectory of $\text{col}(X(t), e(t))$ starting from any $\text{col}(X(0), e(0)) \in \bar{Q}_{\bar{R}}^{n_c}$ will asymptotically converge to the origin by choosing sufficiently large $h$. To this end, consider the positive definite Lyapunov function $W(X, e) = W_X(X) + W_e(e)$, where $W_X(\cdot)$ and $W_e(\cdot)$ are defined to be the same as those in Lemmas 4 and 5, respectively. The rest of the proof can be accomplished in the following two steps.

**Step 1.** Show that the trajectory of $\text{col}(X(t), e(t))$ enters the compact set $\bar{\Omega}_{c+\epsilon}(W(X, e))$ in a finite time $T$. By substituting the saturated output feedback controller (15) into (11), one obtains that

$$\dot{\vartheta}_i = \tilde{f}_{i2} - b_i(w)K\vartheta_i + \varepsilon_i + b_i(w)(\bar{u}_i + K\vartheta_i). \tag{22}$$

Note that $4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g) \geq 1$, where $\varsigma_{i6}$, $\varsigma_{i7}$ and $\varsigma_{i8}(g)$ are chosen as the same as those in the proof of Lemma 4. Then the derivative of $V_{i3}(\vartheta_i)/(4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g))$ along the trajectories of (22) can be described as follows,

$$\frac{\dot{V}_{i3}(\vartheta_i)}{4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)} \leq \frac{\vartheta_i(\tilde{f}_{i2} - b_i(w)K\vartheta_i + \varepsilon_i)}{4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)} + \left|b_i(w)\vartheta_i(\bar{u}_i + K\vartheta_i)\right|.$$

Reconsider $W_X(X)$ in Lemma 4. Its derivative along the trajectories of the closed-loop system composed of (6), (11) and (15) can be described as follows,

$$\dot{W}_X(X)|_{(6)+(11)+(15)} \leq \dot{W}_X(X)|_{(6)+(11)+(18)} + \Big|\sum_{i=1}^{N} b_i(w)\vartheta_i(\bar{u}_i + K\vartheta_i)\Big|. \tag{23}$$

On the one hand, it can be obtained from (15) that $\bar{u}_i$ is upper bounded by $\delta$, which is independent of $h$. Consequently, for all $X \in \bar{\Omega}_{c+\epsilon}\left(W_X(X)\right)$ and all $\mathrm{col}(v, w) \in \mathbb{V}\times\mathbb{W}$, one has $\left|\sum_{i=1}^{N} b_i(w)\vartheta_i(\bar{u}_i + K\vartheta_i)\right| \leq \delta_1$, where $\delta_1 > 0$ is a constant independent of $h$. By using (19), one further deduces that

$$\dot{W}_X(X)|_{(6)+(11)+(15)} \leq -a\|X\|^2 + \delta_1 \leq \delta_1. \tag{24}$$

Meanwhile, it follows from (17) that $W_X(X(0)) \in \bar{\Omega}_c\left(W_X(X)\right) \subseteq \bar{\Omega}_{c+\epsilon}\left(W_X(X)\right)$. Thus, by using (24) and letting $T = \frac{\epsilon}{2\delta_1}$, for all $t \in [0, T]$, we have

$$W_X(X(t)) \leq W_X(X(0)) + \delta_1 t \leq c + \frac{\epsilon}{2}. \tag{25}$$

In other words, it is shown in (25) that $\|X(t)\|$ is upper bounded for all $t \in [0, T]$, with its bound independent of $h$.

On the other hand, it can be shown from the closed-loop system composed of (6), (11) and (15) that $\varrho\|\dot{X}\|^2 \leq \delta_2$, where $\delta_2 > 0$ is a constant independent of $h$. Thus, by referring to (20), one has $\dot{W}_e(e)\Big|_{(16)} \leq -\frac{h}{2\underline{\lambda}}W_e + \frac{\delta_2}{h}$, where $\underline{\lambda} \triangleq \min_{i=1,2,\ldots,N}\{\lambda_{\min}\left(P_{i3}\right), 1\}$.

By the comparison lemma [22], it follows that $W_e(e(t)) \leq \exp\{-\frac{h}{2\underline{\lambda}}t\}W_e(e(0)) + \frac{2\underline{\lambda}\delta_2}{h^2}$, for all $t \in [0, T]$. Let $h_1 = \sqrt{\frac{8\underline{\lambda}\delta_2}{\epsilon}}$. Then, for all $t \in [0, T]$, choosing $h \geq h_1$ leads to

$$W_e(e(t)) \leq \exp\{-\frac{h}{2\underline{\lambda}}t\}W_e(e(0)) + \frac{\epsilon}{4}. \tag{26}$$

By recalling $e_{ij} = h^{n_i - j}\left(x_{ij} - \tilde{x}_{ij}\right)$, it can be proved that $W_e(e(0))$ is bounded by a polynomial of $h$. One then has $\lim_{h\to+\infty} \exp\{-\frac{h}{2\underline{\lambda}}T\}\ W_e(e(0)) = 0$. Thus, there exists a positive constant $h_2$ such that, for all $h > h_2$, $\exp\{-\frac{h}{2\underline{\lambda}}T\}W_e(e(0)) < \frac{\epsilon}{4}$. By using (25) and (26), one can obtain that $W(X(T), e(T)) < c + \epsilon$. In other words, the trajectory of $\mathrm{col}(X(t), e(t))$ enters the compact set $\bar{\Omega}_{c+\epsilon}(W(X, e))$ within the time $T$.

**Step 2.** Show that the trajectory of $\mathrm{col}(X(t), e(t))$ remains in the compact set $\bar{\Omega}_{c+\epsilon}\left(W_X(X)\right) \times \bar{\Omega}_{\epsilon/2}\left(W_e(e)\right)$, and asymptotically converges to the origin as time goes to infinity. It can be observered from (26) that $\exp\{-\frac{h}{2\underline{\lambda}}t\}W_e(e(0))$ is decreasing as time goes to infinity. Thus, by choosing any $h \geq \max\left\{h_1, h_2\right\}$, we have $W_e(e(t)) < \epsilon/2$ for all $t \geq T$, that is, $e(t)$ remains in the compact set $\bar{\Omega}_{\epsilon/2}\left(W_e(e)\right)$ for all $t \geq T$. Note that $\|E_{i2}(h)\| \leq \|E_{i2}(1)\|$ for all $h \geq 1$. It then follows from (21) that $|\bar{u}_i| \leq \delta$, which implies that the saturation function will not trigger. As a consequence, the decentralized output feedback controller and the function in the resulting closed-loop system are smooth. Thus, it follows from Lemma 7.8 in [27] that, for all $h \geq 1$, all $\mathrm{col}(v, w) \in \mathbb{V}\times\mathbb{W}$ and all $\mathrm{col}(X, e) \in \bar{\Omega}_{c+\epsilon}\left(W_X(X)\right) \times \bar{\Omega}_{\epsilon/2}\left(W_e(e)\right)$, the following inequalities are satisfied,

$$\Big|\sum_{i=1}^{N} b_i(w)\vartheta_i(\bar{u}_i + K\vartheta_i)\Big| \leq \frac{\delta_3}{\varsigma}\|X\|^2 + \varsigma\delta_4\|e\|^2, \tag{27}$$

$$\varrho\|\dot{X}\|^2 \leq \delta_5\|X\|^2 + \delta_6\|e\|^2, \tag{28}$$

where $\varsigma$ and $\delta_l, l = 3, 4, 5, 6$ are some positive constants independent of $h$. Then by substituting (27) and (28) into (23) and (20) respectively, for all $\mathrm{col}(v, w) \in \mathbb{V}\times\mathbb{W}$ and all $\mathrm{col}(X, e) \in \bar{\Omega}_{c+\epsilon}\left(W_X(X)\right) \times \bar{\Omega}_{\epsilon/2}\left(W_e(e)\right)$, the derivative of $W(X, e)$ along the trajectories of the closed-loop system consisting of (6), (11), (15) and (16) can be described as follows.

$$\dot{W}(X, e) \leq -\left(a - \frac{\delta_3}{\varsigma} - \frac{\delta_5}{h}\right)\|X\|^2 - \left(\frac{h}{2} - \varsigma\delta_4 - \frac{\delta_6}{h}\right)\|e\|^2.$$

Choose $\varsigma > \frac{\delta_3}{a}$ and $h_3 > \max\{\frac{\varsigma\delta_5}{a\varsigma - \delta_3}, 2(\varsigma\delta_4 + \delta_6), 1\}$ successively. Then, for all $h \geq \max_{i=1,2,3}\{h_i, 1\}$, one can obtain that

$$\dot{W}(X, e)|_{(6)+(11)+(15)+(16)} \leq -\underline{a}\|\ \mathrm{col}(X, e)\|^2, \tag{29}$$

where $\underline{a} = \min\{a - \frac{\delta_3}{\varsigma} - \frac{\delta_5}{h},\ \frac{h}{2} - \varsigma\delta_4 - \frac{\delta_6}{h}\}$.

Note that $\bar{\Omega}_{c+\epsilon}(W(X, e)) \subseteq \bar{\Omega}_{c+\epsilon}\left(W_X(X)\right) \times \bar{\Omega}_{\epsilon/2}\left(W_e(e)\right)$. It can be concluded that, once the trajectory of $\mathrm{col}(X(t), e(t))$ enters $\bar{\Omega}_{c+\epsilon}\left(W(X, e)\right)$, it remains in the compact set $\bar{\Omega}_{c+\epsilon}\left(W_X(X)\right) \times \bar{\Omega}_{\epsilon/2}\left(W_e(e)\right)$. In addition, it is shown in (29) that the trajectory of $\mathrm{col}(X(t), e(t))$ converges to the origin as time goes to infinity. Therefore, it is proved that the equilibrium point of the closed-loop system is semi-globally asymptotically stable, with its region of attraction containing $\bar{Q}^{n_c}_{\bar{R}}$. □

Based on Lemma 3 and Theorem 1, the main result of this work is summaried as follows.

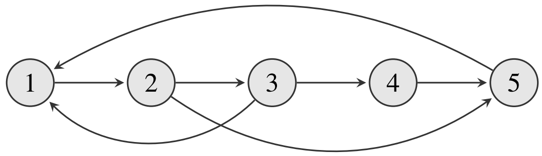


**FIGURE 2** An unbalanced directed network.

**Theorem 2.** Under Assumptions 1–6, given any $R > 0$ and any compact set $\mathbb{V}_0 \times \mathbb{W} \subseteq \mathbb{R}^{n_v+n_w}$, there exist sufficiently large $g$, $K$, $\delta$ and $h$ that depend on $R$ such that, for any $x_c(0) \in \bar{Q}_R^{n_c}$ and any $\mathrm{col}(v(0), w) \in \mathbb{V}_0 \times \mathbb{W}$, the distributed optimal coordination Problem 1 is solved by the following distributed output feedback controller,

$$u_i = -\beta_\delta\left(K\tilde{\vartheta}_i\right) + \Gamma_i T_i^{-1}\eta_i, \tag{30}$$

$$\dot{\eta}_i = M_i\eta_i + N_i u_i, \quad i = 1, 2, \ldots, N, \tag{31}$$

where $\tilde{\vartheta}_i = \tilde{x}_{in_i} + g\gamma_{i(n_i-1)}\tilde{x}_{i(n_i-1)} + \cdots + g^{n_i-2}\gamma_{i2}\tilde{x}_{i2} + g^{n_i-1}\gamma_{i1}\left(\tilde{x}_{i1} - y_i^r\right)$, with $\tilde{x}_{ik}, k = 1, 2, \ldots, n_i$ being generated by the distributed high-gain observer (13) and $y_i^r$ being generated by the distributed optimal coordinator (6).

*Remark 7.* Due to the utilization of the saturated output feedback controller (30), the selection of the initial agent conditions is restricted in a positively invariant compact subset of the attraction region. Hence, only a semi-global stability of the resulting closed-loop system can be achieved by the proposed controller. It is worth noting that the controller (30) is designed by saturating the linear state feedback controller (18). Compared with the nonlinear controller developed in [17], a linear controller has many advantages in both theoretical design and practical implementation.

## 4 | ILLUSTRATIVE EXAMPLES

In this section, the effectiveness of the proposed distributed output feedback controller is verified by two illustrative examples.

### 4.1 | Example 1

We start with a simplified but practical example of optimal rendezvous problems. Consider a team of five networked agents with the agent dynamics being described by single-link manipulators with flexible joints [17, 29] as follows,

$$\begin{aligned} J_{i1}\ddot{q}_{i1} + M_i g_0 L_i \sin q_{i1} + k_i\left(q_{i1} - q_{i2}\right) &= 0, \\ J_{i2}\ddot{q}_{i2} - k_i\left(q_{i1} - q_{i2}\right) &= \tau_i, \end{aligned} \tag{32}$$

where $q_{i1}, q_{i2}$ represent the angular positions, $J_{i1}, J_{i2}$ denote moments of inertia, $M_i$ is the mass, $g_0$ is gravitational constant, $L_i$ represents the length, $k_i$ is the spring constant, and $\tau_i$ is the torque input. It is assumed that $J_{i1} = \left(1 + w_{i1}\right)\bar{J}_{i1}$, $J_{i2} = \left(1 + w_{i2}\right)\bar{J}_{i2}$, $M_i = \left(1 + w_{i3}\right)\bar{M}_i$ and $L_i = \left(1 + w_{i4}\right)\bar{L}_i$, where $\bar{J}_{i1}$, $\bar{J}_{i2}$, $\bar{M}_i$ and $\bar{L}_i$ are the nominal values, while $w_{ik}, k = 1, 2, 3, 4$ represent the uncertain parameters. The information transmission among single-link manipulators is described by the unbalanced directed network $\mathcal{G}$ in Fig. 2. It can be verified that the directed network is strongly connected, and Assumption 2 is thus satisfied.

Let $x_i = \mathrm{col}\left(q_{i1}, \dot{q}_{i1}, q_{i1}^{(2)}, q_{i1}^{(3)}\right)$. Then, the system (32) can be rewritten as the form (2) with $y_i = q_{i1}$, $n_i = 4$, $f_{i1}\left(x_i, w\right) = -x_{i3}\left(\frac{M_i L_i}{J_{i1}}\cos\left(x_{i1}\right) + \frac{k_i}{J_{i1}} + \frac{k_i}{J_{i2}}\right) + \frac{M_i g L_i}{J_{i1}}\left(x_{i2}^2 - \frac{k_i}{J_{i2}}\right)\sin\left(x_{i1}\right)$, and $b_i(w) = \frac{k_i}{J_{i1}J_{i2}}$. Define the local cost functions as $c_i\left(y_i\right) = \frac{1}{2}\left\|y_i - q_{i1}(0)\right\|^2$, for $i = 1, \ldots, 5$. Then, the optimal rendezvous problem is to steer the five manipulators to rendezvous at a common position that minimizes the total angular distance from their starting positions to the final position. It can be verified that the optimal solution is $s^\star = \frac{1}{4}\sum_{i=1}^4 q_{i1}(0)$.

In the simulation, let $\bar{J}_{i1} = 1$, $\bar{J}_{i2} = 1$, $\bar{M}_i = 1$, $\bar{L}_i = 1$, and $k_i = 1$ for simplicity. The uncertain parameters are randomly chosen such that $w_{ij} > 0, j = 1, 2, 3, 4$. The initial values of $q_{i1}(0)$ are chosen as $q_{i1}(0) = i$ for $i = 1, 2, 3, 4$. Let the control parameters of the distributed output feedback controller in Theorem 2 be $K = 10$, $\delta = 4 \times 10^4$, $g = 4$, $h = 10$, $\gamma_{i1} = 6$, $\gamma_{i2} = 11$, and $\gamma_{i3} = 6$. The simulation results are shown in Fig. 3. It can be observed that $y_i$'s converge to the optimal solution $s^\star = 3$

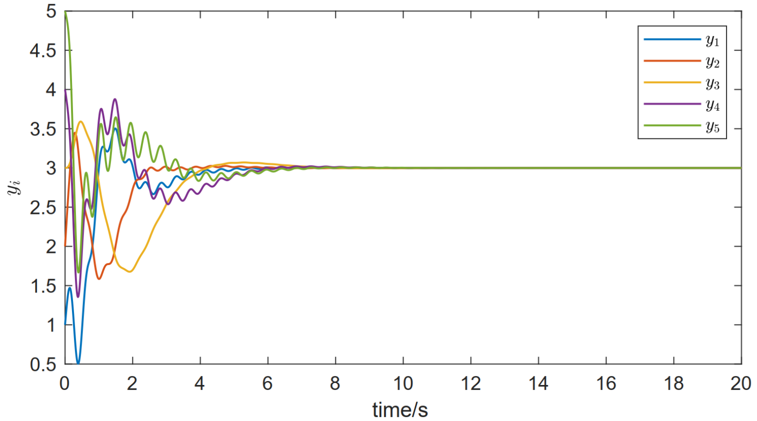


**FIGURE 3** Trajectories of the agent outputs $y_i, i = 1, \ldots, 5$.

as time tends to infinity. Therefore, it is shown that our proposed distributed output feedback controller solves the concerned optimal rendezvous problem.

## 4.2 | Example 2

Next, we further provide another example with heterogeneous agent dynamics and more complex local cost functions. Specifically, reconsider the group of five agents over the unbalanced directed graph $\mathcal{G}$ in Fig. 2, but with their dynamics being described by the following uncertain nonlinear systems [26],

$$\begin{aligned}
\dot{z}_{i1} &= p_{i1} z_{i1} + x_{i1} + A_{w1} \sin(\theta t), \\
\dot{x}_{i1} &= x_{i2} \\
\dot{x}_{i2} &= p_{i2} z_{i1} x_{i1} x_{i2} + p_{i3} x_{i2} + A_{w2} \cos(\theta t) + b_i(w) u_i, \\
y_i &= x_{i1}, \quad i = 1, 2,
\end{aligned}$$

and

$$\begin{aligned}
\dot{x}_{i1} &= x_{i2}, \\
\dot{x}_{i2} &= x_{i3}, \\
\dot{x}_{i3} &= p_{i1} x_{i3} + p_{i2} x_{i2} + p_{i3} x_{i1}^3 + A_{w3} \sin(\theta t) + b_i(w) u_i, \\
y_i &= x_{i1}, \quad i = 3, 4, 5,
\end{aligned}$$

where $A_{w1} \sin(\theta t)$, $A_{w2} \cos(\theta t)$ and $A_{w3} \sin(\theta t)$ are external disturbances, with $A_{wk} = \mu_{ik}(w) A,\ k = 1, 2, 3$ being the uncertain amplitudes and $\theta$ being the angular frequency. For $i = 1, \ldots, 5$, it is assumed that $\mu_{ik}(w) = 1 + 0.1k,\ k = 1, 2, 3$. Note that $\sin(\theta t) = v_1$ and $\cos(\theta t) = v_2$, where $v = \mathrm{col}(v_1, v_2)$ is generated by the exosystem (3) with $S = [0,\ \theta; -\theta,\ 0]$. Thus, Assumption 4 is satisfied. In addition, for $i = 1, \ldots, 5$, it is assumed that $p_i = (p_{i1}, p_{i2}, p_{i3})$ satisfies $p_i = \bar{p}_i + w_i$, where $\bar{p}_i = (\bar{p}_{i1}, \bar{p}_{i2}, \bar{p}_{i3})$ denotes the nominal value of $p_i$, $w_i = (w_{i1}, w_{i2}, w_{i3}) \in \mathbb{W}$ denotes the uncertainty. For $i = 1, \ldots, 5$, let $b_i(w) = 1 + w_{i1}$. Moreover, it can be verified that $z_i^\star(s, v, w)$ in Assumption 3 can be calculated as $z_i^\star(s, v, w) = -\frac{p_{i1} A_{w1}}{p_{i1}^2 + \theta^2} v_1 - \frac{\theta A_{w1}}{p_{i1}^2 + \theta^2} v_2 + \frac{p_{i1} a_{i2}}{\theta} s,\ i = 1, 2$. It is worth pointing out that the nonlinear functions in the agent dynamics are not globally Lipschitz.

In this example, we consider more general local cost functions as follows,

$$\begin{aligned}
&c_1 = 0.25 e^{-0.2s} + 0.5 e^{0.5s}, \quad c_2 = 0.5(s-2)^2 + e^{0.1s}, \quad c_3 = 0.2 s \ln(1+s^2) + s^2, \\
&c_4 = 0.4 \frac{s}{\sqrt{1+s^2}} + 0.5 s^2, \quad c_5 = 0.6 s^2 (\ln(s^2 + 0.5) + 1) + 0.3 s^2 / \sqrt{s^2 + 5}.
\end{aligned}$$

It can be verified that all local cost functions are strongly convex, and the global solution takes an approximate value $s^\star = 0.266$. Therefore, Assumptions 1 and 2 are satisfied. It can be calculated that $u_i^\star\left(s^\star, v, w\right) = -b_i(w)^{-1} A_{w2} v_2$ for $i = 1, 2$, and $u_i^\star\left(s^\star, v, w\right) = -b_i(w)^{-1}\left(p_{i3} s^{\star 3} + A_{w3} v_1\right)$ for $i = 3, 4, 5$. Thus, Assumption 5 is also satisfied. Specifically, one can deduce

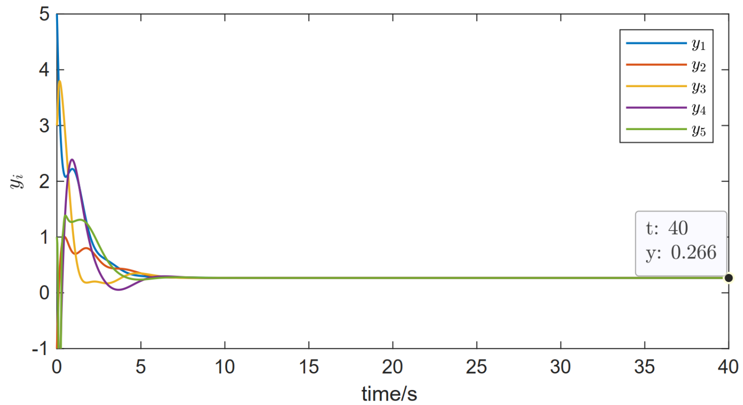


**FIGURE 4** Trajectories of the agent outputs $y_i, i = 1, \dots, 5$.

that $\frac{\mathrm{d}^2 u_i^\star(s^\star, v, w)}{\mathrm{d}t^2} = -\theta^2 u_i^\star(s^\star, v, w)$ for $i = 1, 2$, and $\frac{\mathrm{d}^3 u_i^\star(s^\star, v, w)}{\mathrm{d}t^3} = -\theta^2 \frac{\mathrm{d}u_i^\star(s^\star, v, w)}{\mathrm{d}t}$ for $i = 3, 4, 5$. Thus, by Theorem 1, the optimal coordination problem for the concerned nonlinear multi-system is solvable.

In the simulation, choose $A = 10$ and $\theta = 0.8$. Let $\bar{p}_{i1} = \bar{p}_{i2} = \bar{p}_{i3} = i$, and $w_i = (w_{i1}, w_{i2}, w_{i3})$ be randomly generated such that $p_{i1}, i = 1, \dots, 5$ are negative, and $b_i(w), i = 1, \dots, 5$ are nonzero. Let $M_i = [0, 1; -2, -3]$, $N_i = [0; 1]$ for $i = 1, 2$, and $M_i = [-3, -7, -5;\ 1, 0, 0;\ 0, 1, 0]$, $N_i = [1;\ 0;\ 0]$ for $i = 3, 4, 5$. The other initial conditions are randomly chosen. Then apply the distributed output feedback controller (30) with $K = 4 \times 10^4$ and $h = 100$. The simulation result is presented in Fig. 4. It can be observed from Fig. 4 that all the agent outputs $y_i, i = 1, \dots, 5$ eventually converge to the optimal solution $s^\star = 0.266$.

## 5 | CONCLUSION

In this article, the distributed optimal coordination problem of uncertain nonlinear multi-agent systems in the normal form over unbalanced directed networks has been solved by a novel distributed output feedback controller. Based on a two-layer controller structure, the concerned problem is first converted to a reference-tracking problem by developing a distributed optimal coordinator, and the obtained augmented system is then stabilized by a high-gain observer based output feedback controller. It has been proven that all the agent outputs converge to the optimal solution of the sum of local cost functions semi-globally. The effectiveness of the control scheme has been illustrated by a simulation example. Applying the proposed distributed controller to practical systems, such as multi-robot systems and power systems, will be considered in our future research.

□

## APPENDIX

## A PROOF OF LEMMA 4

Note that $A_{ci}$ and $M_i$ are Hurwitz. Then, there exist positive definite matrices $P_{i1}$ and $P_{i2}$ such that $A_{ci}^T P_{i1} + P_{i1} A_{ci} \leq -I_{n_i-1}$, and $M_i^T P_{i2} + P_{i2} M_i \leq -I_{s_i}$. It is shown in Lemma 2 that $(y_i^r, \zeta_i)$ generated by (6) converges to its equilibrium $(\bar{y}_i^r, \bar{\zeta}_i)$ exponentially. Thus, by applying Theorem 4.14 in [22], there exists a positive definite Lyapunov function $\tilde{V}_i(\tilde{y}_i^r, \tilde{\zeta}_i)$ such that $\dot{\tilde{V}}_i|_{(6)} \leq -\varsigma_{i0}\left( \|\tilde{y}_i^r\|^2 + \|\tilde{\zeta}_i\|^2 \right)$ for a constant $\varsigma_{i0} > 0$, where $\tilde{y}_i^r = y_i^r - \bar{y}_i^r$ and $\tilde{\zeta}_i = \zeta_i - \bar{\zeta}_i$.

Consider the following positive definite functions, $V_{i1}\left(\hat{x}_{ia}\right) = \hat{x}_{ia}^T P_{i1} \hat{x}_{ia}$, $V_{i2}\left(\tilde{\eta}_i\right) = \tilde{\eta}_i^T P_{i2} \tilde{\eta}_i$, $V_{i3}\left(\vartheta_i\right) = \frac{1}{2}\vartheta_i^2$. Then for any $\bar{R} > 0$, there exists $\bar{c}_i > 0$ such that $\bar{Q}_{\bar{R}}^{n_{z_i}+n_i+s_i+2} \subseteq \bar{\Omega}_{\bar{c}_i}\left(V_{i0}\left(\bar{z}_i\right)\right) \times \bar{\Omega}_{\bar{c}_i}\left(V_{i1}\left(\hat{x}_{ia}\right)\right) \times \bar{\Omega}_{\bar{c}_i}\left(V_{i2}\left(\tilde{\eta}_i\right)\right) \times \bar{\Omega}_{\bar{c}_i}\left(V_{i3}\left(\vartheta_i\right)\right) \times \bar{\Omega}_{\bar{c}_i}\left(\tilde{V}_i(\tilde{y}_i^r, \tilde{\zeta}_i)\right)$. Let $c = (4 + \mu) \sum_{i=1}^{\mathbb{N}} \bar{c}_i$, where $\mu$ is a positive constant to be determined. Note that $\tilde{f}_{i0}\left(\bar{z}_i, \hat{x}_{ia}, y_i^r, v, w\right) - \tilde{f}_{i0}\left(\bar{z}_i, 0, y_i^r, v, w\right)$ is sufficiently smooth and vanishes at $\hat{x}_{ia} = 0$, and $\partial V_{i0}\left(\bar{z}_i\right)/\partial \bar{z}_i$ is continuously differentiable with $\partial V_{i0}(0)/\partial \bar{z}_i = 0$. Then, by Lemma 2 in [25], for all $\bar{z}_i \in \bar{\Omega}_{c+\epsilon}\left(V_{i0}\left(\bar{z}_i\right)\right)$, $\hat{x}_{ia} \in \bar{\Omega}_{c+\epsilon}\left(V_{i1}\left(\hat{x}_{ia}\right)\right)$, $y_i^r \in \bar{\Omega}_{c+\epsilon}\left(\tilde{V}_i(\tilde{y}_i^r, \tilde{\zeta}_i)\right)$, and all $\mathrm{col}(v, w) \in \mathbb{V} \times \mathbb{W}$, the following inequalities are satisfied,

$$\left\|\partial V_{i0}(\bar z_i)/\partial \bar z_i\right\| \leq \varsigma_{i1}\|\bar z_i\|, \tag{A1}$$

$$\left\|\tilde f_{i0}\left(\bar z_i, \hat x_{ia}, y_i^r, v, w\right) - \tilde f_{i0}\left(\bar z_i, 0, y_i^r, v, w\right)\right\| \leq \varsigma_{i2}\left\|\hat x_{ia}\right\|, \tag{A2}$$

$$\left\|\hat f_{i0}\left(y_i^r, s^\star, v, w\right)\right\| \leq \varsigma_{i3}\left\|\tilde y_i^r\right\|, \tag{A3}$$

for some constants $\varsigma_{il} \geq 1, l = 1,2,3$. Therefore, the derivatives of $V_{ik}(\cdot), k = 0,1,2,3$ along the trajectories of (11) can be described as follows,

$$\begin{aligned}
\dot V_{i0}(\bar z_i) &\leq -\tfrac{a_{i0}}{2}\|\bar z_i\|^2 + \tfrac{\varsigma_{i1}^2\varsigma_{i2}^2}{a_{i0}}\|\hat x_{ia}\|^2 + \tfrac{\varsigma_{i1}^2\varsigma_{i3}^2}{a_{i0}}\|\tilde y_i^r\|^2,\\
\dot V_{i1}(\hat x_{ia}) &\leq -(g-2)\|\hat x_{ia}\|^2 + \|P_{i1}\|^2\vartheta_i^2 + \|P_{i1}\|^2\|\dot y_i^r\|^2,\\
\dot V_{i2}(\tilde\eta_i) &\leq -\tfrac{1}{2}\|\tilde\eta_i\|^2 + 4\|P_{i2}\tilde f_{i1}\|^2 + 4\|P_{i2}b_i^{-1}(w)N_i\|^2\|\varepsilon_i\|^2,\\
\dot V_{i3}(\vartheta_i) &\leq -(b_i(w)K-1)\vartheta_i^2 + \tfrac{1}{2}\|\tilde f_{i2}\|^2 + \tfrac{1}{2}\|\varepsilon_i\|^2.
\end{aligned}$$

By recalling the definitions of $\tilde f_{i1}(\cdot)$ and $\tilde f_{i2}(\cdot)$, it follows from Lemma 7.8 in [27] that, for all $\mathrm{col}(v,w) \in \mathbb{V}\times\mathbb{W}$,

$$\begin{aligned}
4\|P_{i2}\tilde f_{i1}(\cdot)\|^2 \leq& \varsigma_{i4}(g)\big(\rho_{i1}\left(\bar z_i\right)\left\|\bar z_i\right\|^2 + \rho_{i2}\left(\hat x_{ia}\right)\left\|\hat x_{ia}\right\|^2 + \rho_{i3}\left(\vartheta_i\right)\vartheta_i^2\big),\\
\frac{1}{2}\|\tilde f_{i2}(\cdot)\|^2 \leq& \varsigma_{i5}(g)\big(\rho_{i4}\left(\bar z_i\right)\left\|\bar z_i\right\|^2 + \rho_{i5}\left(\hat x_{ia}\right)\left\|\hat x_{ia}\right\|^2 + \rho_{i6}\left(\vartheta_i\right)\vartheta_i^2\big) + \varsigma_{i6}\left\|\tilde\eta_i\right\|^2,
\end{aligned}$$

for some smooth functions $\varsigma_{il}(\cdot) \geq 1, l = 4,5$, $\rho_{ik}(\cdot) \geq 1, k = 1,2,\dots,6$, and a positive constant $\varsigma_{i6} \geq 1$. Thus, for all $\bar z_i \in \bar\Omega_{c+\epsilon}\left(V_{i0}\left(\bar z_i\right)\right)$, $\hat x_{ia} \in \bar\Omega_{c+\epsilon}\left(V_{i1}\left(\hat x_{ia}\right)\right)$, and all $\mathrm{col}(v,w) \in \mathbb{V}\times\mathbb{W}$, one has

$$\max\left\{\rho_{i1}\left(\bar z_i\right) + \tfrac{\rho_{i4}\left(\bar z_i\right)}{4\varsigma_{i6}},\ \rho_{i2}\left(\hat x_{ia}\right) + \tfrac{\rho_{i5}\left(\hat x_{ia}\right)}{4\varsigma_{i6}}\right\} \leq \tfrac{a_{i0}\varsigma_{i7}}{4}, \tag{A4}$$

where $\varsigma_{i7} \geq 1$ is a constant. Furthermore, by letting $\varsigma_{i8}(g) = \max\{\varsigma_{i4}(g), \varsigma_{i5}(g)\}$, we have

$$\left\|P_{i1}\right\|^2 + \tfrac{\rho_{i3}(\vartheta_i)}{\varsigma_{i7}} + \tfrac{\rho_{i6}(\vartheta_i)}{4\varsigma_{i6}\varsigma_{i7}} \leq \varsigma_{i9}, \tag{A5}$$

for all $\vartheta_i \in \bar\Omega_{4(c+\epsilon)\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)}\left(V_{i3}(\vartheta_i)\right)$ and all $\mathrm{col}(v,w) \in \mathbb{V}\times\mathbb{W}$, where $\varsigma_{i9} \geq 1$ is a constant.

Consider the Lyapunov function candidate $W_X(X) = \sum_{i=1}^N W_i\left(X_i\right)$, where $W_i\left(X_i\right) = V_{i0}(\bar z_i) + V_{i1}(\hat x_{ia}) + V_{i2}(\tilde\eta_i)/(\varsigma_{i7}\varsigma_{i8}(g)) + V_{i3}(\vartheta_i)/(4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)) + \mu\tilde V_i(\tilde y_i^r, \tilde\zeta_i)$. Since $\varsigma_{i7}\varsigma_{i8}(g) \geq 1$ and $4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g) \geq 1$, it follows from $X_i \in \bar\Omega_{\bar c_i}\left(V_{i0}(\bar z_i)\right) \times \bar\Omega_{\bar c_i}\left(V_{i1}(\hat x_{ia})\right) \times \bar\Omega_{\bar c_i}\left(V_{i2}(\tilde\eta_i)\right) \times \bar\Omega_{\bar c_i}\left(V_{i3}(\vartheta_i)\right) \times \bar\Omega_{\bar c_i}\left(\tilde V_i(\tilde y_i^r, \tilde\zeta_i)\right)$ that $X \in \bar\Omega_c\left(W_X(X)\right)$, i.e., equation (17) is satisfied.

In addition, it is noted that $X \in \bar\Omega_{c+\epsilon}\left(W_X(X)\right)$ implies $\bar z_i \in \bar\Omega_{c+\epsilon}\left(V_{i0}(\bar z_i)\right)$, $\hat x_{ia} \in \bar\Omega_{c+\epsilon}\left(V_{i1}(\hat x_{ia})\right)$, $\tilde y_i^r \in \bar\Omega_{c+\epsilon}\left(\tilde V_i(\tilde y_i^r, \tilde\zeta_i)\right)$ and $\vartheta_i \in \bar\Omega_{4(c+\epsilon)\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)}\left(V_{i3}(\vartheta_i)\right)$. Therefore, inequalities $(A1)$–$(A5)$ are still satisfied for all $X \in \bar\Omega_{c+\epsilon}\left(W_X(X)\right)$. Moreover, it can be obtained that

$$\left(\tfrac{4\|b_i^{-1}(w)P_{i2}N_i\|^2}{\varsigma_{i7}\varsigma_{i8}(g)} + \tfrac{1}{8\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)}\right)\|\varepsilon_i\|^2 + \tfrac{\varsigma_{i1}^2\varsigma_{i2}^2}{a_{i0}}\|\tilde y_i^r\|^2 + \|P_{i1}\|^2\|\dot y_i^r\|^2 \leq \varsigma_{i10}\|\tilde y_i^r\|^2 + \varsigma_{i11}\|\dot y_i^r\|^2,$$

for some constants $\varsigma_{il} \geq 1, l = 10,11$. Thus, for all $X \in \bar\Omega_{c+\epsilon}\left(W_X(X)\right)$ and all $\mathrm{col}(v,w) \in \mathbb{V}\times\mathbb{W}$, the derivative of $W_i(\cdot)$ along the trajectories of the closed-loop system composed of (6), (11) and (18) can be described as follows,

$$\begin{aligned}
\dot W_i\left(\bar z_i, \hat x_{ia}, \tilde\eta_i, \vartheta_i, \tilde y_i^r, \tilde\zeta_i\right)\Big|_{(6)+(11)+(18)} \leq& -\tfrac{a_{i0}}{4}\left\|\bar z_i\right\|^2 - \left(g - 2 - \tfrac{\varsigma_{i1}^2\varsigma_{i2}^2}{a_{i0}} - \tfrac{a_{i0}}{4}\right)\|\hat x_{ia}\|^2\\
&- \tfrac{1}{4\varsigma_{i7}\varsigma_{i8}(g)}\left\|\tilde\eta_i\right\|^2 - \left(\tfrac{b_i(w)K-1}{4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)} - \varsigma_{i9}\right)\vartheta_i^2\\
&+ \varsigma_{i10}\|\tilde y_i^r\|^2 + \varsigma_{i11}\|\dot y_i^r\|^2 - \mu\varsigma_{i0}\left(\left\|\tilde y_i^r\right\|^2 + \|\tilde\zeta_i\|^2\right).
\end{aligned}$$

Under Assumption 1, it can be proved that $\dot y_i^r$ is Lipschitz in $\mathrm{col}(\tilde y_i^r, \tilde z_i)$ by using similar arguments as those in [17]. One thus has $\varsigma_{i11}\|\dot y_i^r\|^2 \leq \mu_1\left(\left\|\tilde y_i^r\right\|^2 + \|\tilde\zeta_i\|^2\right)$, where $\mu_1 > 0$ is a constant. Then, by successively choosing $g > \max_{i=1,2,\dots,N}\left\{2 + \tfrac{\varsigma_{i1}^2\varsigma_{i2}^2}{a_{i0}} + \tfrac{a_{i0}}{4}\right\}$, $K > \max_{i=1,2,\dots,N}\left\{\tfrac{1}{\min_{w\in\mathbb{W}}\{b_i(w)\}}(1 + 4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)\varsigma_{i9})\right\}$, $\mu > \max_{i=1,2,\dots,N}\left\{\tfrac{\varsigma_{i10}+\mu_1}{\varsigma_{i0}}\right\}$, and $a = \min_{i=1,2,\dots,N}\left\{\tfrac{a_{i0}}{4},\ g - 2 - \tfrac{\varsigma_{i1}^2\varsigma_{i2}^2}{a_{i0}} - \tfrac{a_{i0}}{4},\ \tfrac{1}{4\varsigma_{i7}\varsigma_{i8}(g)},\ \tfrac{b_i(w)K-1}{4\varsigma_{i6}\varsigma_{i7}\varsigma_{i8}(g)} - \varsigma_{i9},\ \mu\varsigma_{i0} - \mu_1 - \varsigma_{i10}\right\}$, one can obtain that $\dot W_i\left(X_i\right)\Big|_{(6)+(11)+(18)} \leq -a\left\|X_i\right\|^2$. This indicates that inequality (19) is satisfied, and the proof is thus completed. □

# B PROOF OF LEMMA 5

Under appropriate choices of $r_{ik}, k = 1, 2, \ldots, n_i$, matrix $E_{i3}$ defined in (16) is Hurwitz. Thus, there exists a positive definite matrix $P_{i3}$ such that $E_{i3}^T P_{i3} + P_{i3} E_{i3} \leq -I_{n_i}$. Consider the positive definite function $W_e(e) = \sum_{i=1}^{N} e_i^{\mathrm{T}} P_{i3} e_i$. Then, for all $\mathrm{col}(v, w) \in \mathbb{V} \times \mathbb{W}$, its derivative along the trajectories of (16) can be deduced as follows,

$$\dot{W}_e(e)\Big|_{(16)} \leq -\frac{h}{2} \sum_{i=1}^{N} \|e_i\|^2 + \frac{2}{h} \sum_{i=1}^{N} \|P_{i3} E_{i4}\|^2 \|\dot{X}_i\|^2.$$

Let $\varrho = \max_{i=1,2,\ldots,N} \left\{2\|P_{i3} E_{i4}\|^2\right\}$. Then the inequality (20) is satisfied, and the proof is thus completed. □

# CONFLICT OF INTEREST

The authors declared that they have no conflict of interest to this work.

# DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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