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Toward a finite-time energy-saving robust control method for active suspension systems: exploiting beneficial state-coupling, disturbance, and nonlinearities

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Abstract—A novel control method of addressing coupling and disturbance influences for finite-time energy-saving robust control of active suspension systems is investigated. By elaborately constructing coupling and disturbance effect indicators, the pros and cons of coupling and disturbance influences on active suspension systems are discussed, and then a finite-time coupling and disturbance effects-triggered control method is designed via a second-order sliding mode control technique. Importantly, the good/bad coupling effects are assessed through a well-designed nonlinear function. By means of determining if the sign of disturbances conforms to the expected motion or not, the addition of beneficial disturbance effects or removal of detrimental disturbance effects is implemented. Noticeably, by employing a bioinspired nonlinear reference model, beneficial nonlinear stiffness and damping effects are thus utilized, leading to the possibility of energy-saving performance. As a result, the proposed control method exhibits a unique feature, i.e., fully employing potential contribution from the coupling and disturbance effects, and presents a totally new coupling and disturbance effects-triggered control framework, leading to obvious performance improvement. Benchmark experimental conclusions are devoted to distinguishing the advantages and effectiveness of the designed tracking method.

Index Terms—Active suspension systems, coupling effect, disturbance effect, bioinspired reference model, finite-time convergence

I. INTRODUCTION

Suspension systems have the capacity to enhance vehicle riding comfortability as well as driving security, and thus they have become the focus of the attention of researchers and vehicle manufacturers [1-3]. Compared to those passive suspension systems (PSSs), active suspension systems (ASSs) possess absolutely great potential to significantly improve ride comfort as well as vehicle maneuverability [4-5]. However, there are still several critical issues, for instance, nonlinear coupling, inevitable disturbance, slowness of convergence rate, and high control energy consumption, making isolation control of ASS quite challenging [6-7].

To deal with the coupling issue for the ASS, decoupling control [8-9], dynamic inversion control [10-11], and robust control [12-22] are proposed. Even accurately, the decoupling control methods, which are developed to guarantee that each control only affects a single output, are also called static decoupling. Nevertheless, for the frequently changing active suspension model, it is hard to obtain real-time accurate decoupling behavior. For the dynamic inversion control methods, negative system dynamics are deleted and selected desired ones instead. However, these dynamic inversion methods require precise model information, which is difficult or even impossible to be obtained for the ASS under heavy uncertainties. Additionally, robust control methods, including sliding mode control [12-14], adaptive control [15-16], model predictive control [17], H-infinity control [18-19], and observer-based control [20-22] are applied to the ASS by regarding the nonlinear coupling terms as uncertainties/disturbances. However, nonlinear coupling and inevitable disturbance are considered by most existing robust control methods to be completely detrimental elements, and thus their effects on the ASS are suppressed or eliminated directly. Actually, bearing some similarity to the magnitude and direction characteristics of the vector, coupling’s/disturbance’s effect on the ASS is two-sided. Unfortunately, the latter, which can reflect the coupling’s/disturbance’s effect on the systems, often gets overlooked. As the saying goes, every coin has two sides, there are two different sorts of opinions about the coupling/disturbance influences. To make the best of beneficial coupling/disturbance effects, a specific coupling/disturbance
effect indicator should be constructed to judge them [7, 23-25]. In addition, to better apply the constructed coupling/disturbance indicator to the controller development, the indicator must be able to connect with the system stability/tracking performance.

Furthermore, toward low power demand requirements, researchers employ damping as well as nonlinear stiffness of great benefits, generated by the bioinspired nonlinear dynamics, in the vehicle suspension control [26-30]. More precisely, Pan et al. [26] construct a bioinspired reference model, which is inspired by the biological systems’ limb motion dynamics, in tracking control systems. Although the model only takes one layer and does not consider the damping effect, the outcome still provides an advisable means for the energy-saving control problem. In [27-30], the even more common bioinspired nonlinear dynamics with two layers together with nonlinear damping consequences are constructed which demonstrate the vital significance of damping and nonlinear stiffness in the development of control methods to obtain better performance (not only for isolation). Therefore, more careful consideration is required in the design of robust controllers.

Unlike the asymptotical stability of most existing control methods, one of the main goals is to obtain the finite-time convergence of the ASS, and hence a better dynamic performance could also be guaranteed. As known, second-order SMC methods could reach finite-time convergence as well as improve the robustness of the system under heavy uncertainties and inevitable disturbances [31-33]. Consequently, a finite-time energy-saving robust control method is carried out by using the state coupling, disturbance, and nonlinearities of great benefits, as well as the second-order SMC technique. To sum up, we could come to the main contributions of the paper in the following:

1. A coupling effect recognition and coupling effect-employment control framework is introduced, solving the problem of strong coupling behavior by retaining the good coupling effect and deleting the bad coupling effect in the Lyapunov stability sense.
2. A disturbance effect-employment control method is established, comprising the novel disturbance observer, disturbance effect indicator, as well as control manipulation, and it can include the estimated disturbance information into the control method development on the basis of assessment of the disturbance effect. By utilizing beneficial disturbance effects, the tracking control performance improvement can be obtained.
3. By conducting advantageous nonlinear stiffness as well as damping of a constructed bioinspired nonlinear dynamics, beneficial inherent nonlinearities of the ASS are not eliminated, and instead retained in the system, leading to obviously less energy cost.
4. By elaborately constructing a specific nonlinear coefficient function, the designed control method can achieve finite-time convergence. Additionally, the robustness in terms of parametric uncertainties as well as external disturbances is improved for the ASS via the employment of the second-order SMC technique.

The remaining parts are illustrated as follows. The mathematical model is depicted in Section 2. Main conclusions including the bioinspired reference model, novel disturbance observer, coupling/disturbance effect indicators, as well as finite-time disturbance and coupling effects-employment control method, and relevant stability analysis are provided in Section 3. A series of experimental conclusions are described in Section 4. In Section 5, we make the conclusions.

Consider an ASS depicted in Fig. 1 with the following dynamics [12-22, 26-30]:

$$s_m\ddot{v}_s = -U_s(v_s, v_u) - U_d(\dot{v}_s, \dot{v}_u) + d + F(t)$$

(1)

$$u_m\ddot{v}_u = U_s(v_s, v_u) + U_d(\dot{v}_s, \dot{v}_u) - U_s(v_s, v_u) - U_d(\dot{v}_s, \dot{v}_u) + d_2 - F(t)$$

(2)

with $s_m$ as well as $u_m$ denoting the sprung and unsprung masses separately, $v_s, v_u$ as well as $v$, representing the sprung mass, unsprung mass, together with the road input displacements, respectively, $d$ as well as $d_2$ referring to the disturbances, $F(t)$ referring to the control input, $U_s(v_s, v_u)$ and $U_d(\dot{v}_s, \dot{v}_u)$ denoting the spring force and damper force, respectively, $U_s(v_s, v_u)$ and $U_d(\dot{v}_s, \dot{v}_u)$ representing the tire elastic and damping forces separately, which are described as follows:

$$U_s(v_s, v_u) = \mu_s(v_s - v_u)$$

(3)

$$U_d(\dot{v}_s, \dot{v}_u) = \mu_d(\dot{v}_s - \dot{v}_u)$$

(4)

$$U_s(v_s, v_u) = \mu_s(v_s - v_u)$$

(5)

$$U_d(\dot{v}_s, \dot{v}_u) = \mu_d(\dot{v}_s - \dot{v}_u)$$

(6)

where $\mu$ and $\mu_s$ refer to the stiffness as well as the damping coefficient, $\mu$ as well as $\mu_d$ represent the tire stiffness and damping coefficients separately, which have the following forms:

$$\mu = \bar{\mu}_s(1 + R_s)$$

(7)

$$\mu_d = \bar{\mu}_d(1 + R_d)$$

(8)

$$\mu = \bar{\mu}_s(1 + R_s)$$

(9)

Fig. 1. An ASS system
\[
\mu_c = \bar{\mu}_s (1 + R_s) \tag{10}
\]
with \(\mu, \bar{\mu}_s, \bar{\mu}\), as well as \(\bar{\mu}_s\), standing for nominal values of \(\mu, \bar{\mu}_s, \mu\), as well as \(\bar{\mu}\), respectively, \(R_s, \bar{R}_s, \bar{R}_s\), and \(\bar{R}_s\) referring to perturbation ranges coefficients.

From (1) and (2), we can propose the dynamic model of the suspension space as follows:
\[
\begin{align*}
\frac{s_n}{s_n + u_n} \ddot{v}_v &= -[U_1(v_v, v_v) + U_2(\dot{v}_v, \dot{v}_v)] \\
&\quad + \frac{s_n}{s_n + u_n} [U_1(v_v, v_v) + U_2(\dot{v}_v, \dot{v}_v)] \\
&\quad + \frac{u_n}{s_n + u_n} d_1 - \frac{s_n}{s_n + u_n} d_2 + F(t) \\
&\quad - \bar{\mu}_v v_v - \bar{\mu}_d \dot{v}_v + \frac{s_n}{s_n + u_n} (\bar{\mu}_v v_v + \bar{\mu}_d \dot{v}_v) + D + F(t)
\end{align*}
\]
where \(v_v = v_v - v_m\) referring to the tire travel, \(v_v = v_v - v_m\) standing for the suspension space, \(\bar{s}_n\) and \(\bar{u}_n\) representing nominal values of the sprung and unsprung masses, respectively, \(D\) illustrating the uncertain/unknown disturbance, which is described as follows:
\[
D = -\bar{\mu}_v R_v v_v - \bar{\mu}_d R_d v_v + \left(\frac{s_n}{s_n + u_n} - \frac{s_n}{s_n + u_n}\right) (\bar{\mu}_v R_v v_v + \bar{\mu}_d R_d v_v)
\tag{12}
\]
Substituting \(v_v = v_v - v_m\) into (2), (2) can be rewritten into
\[
u_v = \mu_v v_v + \mu_d \dot{v}_v - \mu_v v_v - \mu_d \dot{v}_v + d_2 - F(t)
\tag{13}
\]
Subsequently, the suspension space’s tracking error \(e\) is introduced in the following:
\[
e = v_v - v_m
\tag{14}
\]
where \(v_m\) refers to the desired trajectory of \(v_v\), which will be constructed later.

From (11) and (14), it is obtained that
\[
\bar{m} \ddot{e} = -\bar{\mu}_e \ddot{e} - \bar{\mu}_d \ddot{e} + \frac{s_n}{s_n + u_n} (\bar{\mu}_v v_v + \bar{\mu}_d \dot{v}_v) - \bar{\mu}_v v_m - \bar{\mu}_d \dot{v}_m - \bar{\mu}_v v_m + \bar{\mu}_d \dot{v}_m + \theta + F(t)
\tag{15}
\]
with \(\bar{m} = \frac{s_n}{s_n + u_n}\) referring to an auxiliary constant, and \(\theta\) standing for the lumped disturbance, which is represented by
\[
\theta = \left(\frac{s_n}{s_n + u_n} - \frac{s_n}{s_n + u_n}\right) \ddot{v}_m + \left(\frac{s_n}{s_n + u_n} - \frac{s_n}{s_n + u_n}\right) \dot{e} + D
\tag{16}
\]
Then, to facilitate the following control method design, we construct the sliding surface as
\[
d = \lambda_e \dot{e}
\tag{17}
\]
with \(\lambda = \lambda_e \bar{\mu}_v \bar{\mu}_d\) standing for a positive control gain, \(\lambda_e\) referring to an auxiliary constant.

Substituting (17) into (15), we have
\[
\bar{m} \ddot{d} = -\lambda_e \ddot{d} + \bar{\mu}_v \mu_v + \bar{\mu}_d \mu_d - \bar{\mu}_v \mu_v - \bar{\mu}_d \mu_d + \ddot{d} - F(t)
\tag{18}
\]
From (13), (14), and (17), it is easy to conclude that
\[
u_v = \lambda_e \mu_v + \mu_v \mu_v + \mu_v \mu_v + \mu_v \mu_v + \mu_v \mu_v + \mu_v \mu_v + \ddot{d} - F(t)
\tag{19}
\]
with \(\ddot{d} = \mu_v - \mu_v\) \(v_v + (\mu_v - \mu_v) \dot{v} + d_2\) referring to the unknown disturbance.

It is noted that, \(\frac{s_n}{s_n + u_n} (\bar{\mu}_v v_v + \bar{\mu}_d \dot{v}_v)\) in (18) refers to the state-coupling effect resulting from the tire travel acceleration system (19).

For a better assessment of the control performance, we employ two indexes including root mean square values of the body acceleration and actuator power consumption, which are described as [34]:
\[
\text{RMS}(\dot{v}_v) = \sqrt{\frac{1}{T} \int_0^T \dot{v}_v^2 dt} \tag{20}
\]
\[
\text{RMS}(Q) = \sqrt{\frac{1}{T} \int_0^T Q^2 dt} \tag{21}
\]
with \(T\) referring to the experimental time, \(Q\) being the function as follows:
\[
Q = \begin{cases} F(t) \dot{v}_v, & \text{If } F(t) \dot{v}_v > 0 \\ 0, & \text{else} \end{cases}
\tag{22}
\]

Assumption 1: As for the ASS, the time-varying lumped disturbance \(g\) as well as its first derivative in terms of time are bounded [24, 35], in the sense that
\[
|\dot{g}| \leq a_g, \quad |\ddot{g}| \leq a_g
\tag{23}
\]
where \(a_g\) and \(a_d\) stand for the top limitations of \(g\) as well as \(\dot{g}\), respectively.

Assumption 2: The displacement between the unsprung mass and the road surface as well as its first derivative are bounded and continuous functions, which satisfy
\[
\frac{1}{2} \frac{s_n}{s_n + u_n} (\bar{\mu}_v v_v + \bar{\mu}_d \dot{v}_v) \leq \Omega, \quad \text{with a positive constant } \Omega.
\]

III. MAIN RESULTS

In order to produce positive nonlinear properties for the ASS, an X-shaped bioinspired reference model is introduced; a nonlinear disturbance observer is deliberately conformed to exert an estimation of the lumped disturbance accurately; two indicators for state-coupling and disturbance effects are employed to evaluate the merit and demerit of the state-coupling and disturbance effects; and finally, the finite-time coupling and disturbance effects-triggered energy-saving control method design process and corresponding stability as well as finite-time convergence analyzes are illustrated.

A. Bioinspired Reference Model

Fig. 2 shows a multilayer X-shaped bioinspired structure, which is composed of springs, rotating joints, as well as connecting rods. As discussed in previous results [26-30], the X-shaped structure is capable of suppressing vibration in a beneficial nonlinear way, including its nonlinear stiffness and damping properties. In Fig. 2(b), the chart of bioinspired nonlinear dynamics is provided, where \(M\) refers to the isolated object mass, \(l_i\) and \(l_i\) stand for lengths of connecting rods, \(\dot{\vartheta}_i\) and \(\theta_i\) represent initial angles. Additionally, Fig. 2(c) illustrates the deformation analysis, with \(x_i\) as well as \(x_2\) referring to horizontal motions of relation rods, \(\phi_i\) as well as \(\phi_i\) representing rotational motions of connection rods, \(n = 2\)
denoting layer number, and \( v_{rd} \) denoting relative displacement between the isolated object and the base, which is calculated as [26-30]:

\[
MV_{vd} + b_1 + \frac{s}{n} v_{rd} + \mu_1 v_{rd} + \mu_2 n h b_1 v_{rd} = -MV_{vd}
\]  

(24)

with \( s \), standing for the spring stiffness in the vertical direction, \( n_1 = 3n + 1 \) referring to a great many joints, \( \mu \) as well as \( \mu_2 \) representing the air resistance coefficient and the friction-related coefficient, respectively, \( b_1 \) as well as \( b_1 \) denoting auxiliary functions, which are represented by

\[
b_1 = \frac{s_E}{2n} \left[ l_1 \cos \theta_1 - \sqrt{l_1^2 - l_2^2} + l_2 \cos \theta_2 - \sqrt{l_2^2 - l_2^2} \right] + \frac{1}{\sqrt{l_1^2 - l_2^2} + \sqrt{l_2^2 - l_2^2}}
\]

\[
b_2 = \left[ \frac{l_1}{2n\sqrt{l_1^2 - l_2^2}} + \frac{l_2}{2n\sqrt{l_2^2 - l_2^2}} \right]
\]

(25)

wherein \( s_h \) refers to the spring stiffness in the horizontal direction, and \( E \) stands for an auxiliary function, which is expressed as

\[
E = l_1 \sin \theta_1 + \frac{v_{rd}}{2n} l_2 \sin \theta_2 + \frac{v_{rd}}{2n}
\]

(26)

Fig.2. (a) An ostrich and its legs. (b) Chart of the bioinspired nonlinear dynamics. (c) Deformation analysis (layer number \( n=2 \))

Subsequently, to guarantee that \( v_{rd} \), \( v_{rd} \), and \( v_{rd} \) are always remained within allowable scopes, we revise (24) as follows:

\[
\begin{align*}
\dot{v}_{rd} &= \lambda_1, \\
MV_{vd} + b_1 + \frac{s}{n} v_{rd} + \mu_1 v_{rd} + \mu_2 n h b_1 v_{rd} &= -MV_{vd}, \\
\dot{v}_{rd} &= \delta_1, & v_{rd} &< \lambda_1 \\
MV_{vd} + b_1 + \frac{s}{n} v_{rd} + \mu_1 v_{rd} + \mu_2 n h b_1 v_{rd} &= -MV_{vd}, \\
\dot{v}_{rd} &= \delta_1, & v_{rd} &< \delta_1 \\
MV_{vd} + b_1 + \frac{s}{n} v_{rd} + \mu_1 v_{rd} + \mu_2 n h b_1 v_{rd} &= -MV_{vd}, \\
\dot{v}_{rd} &= \beta_1, & \lambda_1 &< v_{rd} \\
MV_{vd} + b_1 + \frac{s}{n} v_{rd} + \mu_1 v_{rd} + \mu_2 n h b_1 v_{rd} &= -MV_{vd}, \\
\dot{v}_{rd} &= \beta_2, & v_{rd} &< \beta_2
\end{align*}
\]

(27)

(28)

(29)

with \( \lambda_1 \) and \( \lambda_2 \) referring to the top and bottom limitations of \( v_{rd} \), respectively, \( \delta_1 \) as well as \( \delta_2 \) representing the top and bottom limitations of \( v_{rd} \), respectively, \( \beta_1 \) as well as \( \beta_2 \) denoting the top and bottom limitations of \( v_{rd} \), respectively.

**B. Novel Disturbance Observer Development**

Existing disturbance observers can be roughly divided into linear and nonlinear ones. For nonlinear systems, the disturbance estimation performance of the linear disturbance observer methods may be unsatisfactory due to the linearization of the original nonlinear systems [36, 37]. The fundamental idea of nonlinear disturbance observers is to estimate the unknown disturbances with the state feedback before the control action is executed. As a result, the disturbance effects can be suppressed which enhances the robustness of the systems [38, 39]. Unlike existing nonlinear disturbance observers, the adaptive control method is introduced into the designed nonlinear observer to avoid using accurate model parameters, guaranteeing high estimation precision in the presence of uncertain system parameters.

To clearly illustrate the following disturbance observer design, we have the definition concerning the observation error \( \hat{\theta} \) as

\[
\hat{\theta} = \theta - \theta
\]

(30)

where \( \theta \) refers to the observation of \( \theta \). From (18), we construct the novel disturbance observer as

\[
\hat{\theta} = c_1 + c_2
\]

(31)

\[
c_1 = -Lc_1 - L \left[ c_2 + F(t) - \lambda_d d + \frac{s}{s_n + b_n} (\bar{v}_d - \bar{v}_d) \right] - \bar{v}_d - \bar{v}_d - m \lambda d
\]

(32)

\[
c_2 = Lmd
\]

(33)

where \( c_1 \) and \( c_2 \) stand for two auxiliary functions, \( L \) refers to the positive observer gain.

**Theorem 1:** Under the deliberately constructed observer (30)-(33), the observed lumped disturbance \( \hat{\theta} \) and the observation error \( \theta \) are always kept within the permitted ranges as follows:

\[
|\hat{\theta}| \leq b_\theta, \quad |\hat{\theta}| \leq b_\theta
\]

(34)

where \( b_\theta = |\hat{\theta}(0)| + a_\theta \) and \( b_\theta = |\hat{\theta}(0)| + a_\theta \) refer to the upper bounds of \( \hat{\theta} \) and \( \theta \) respectively. \( \hat{\theta}(0) \) and \( \theta(0) \) stand for the initial values of \( \hat{\theta} \) and \( \theta \) respectively.

**Proof:** Differentiating (31), and inserting (31)-(33) into the resulting equation, one has

\[
\hat{\theta} = c_1 + c_2
\]

(35)

\[
\hat{\theta} = -Lc_1 - L \left[ c_2 + F(t) - \lambda_d d + \frac{s}{s_n + b_n} (\bar{v}_d - \bar{v}_d) \right] + Lmd
\]

(36)

Solving (35), it is obtained that

\[
\hat{\theta} \leq \hat{\theta}(0) e^{\lambda_d t} + a_\theta
\]

(37)

From (36), it is directly derived that
\[
\hat{\theta} = -L\hat{\theta} + \hat{\theta}
\]

By solving the differential equation of (37), one has
\[
\hat{\theta} \leq \hat{\theta}(0)e^{-\alpha t} + \frac{\alpha}{L}
\]
\[
\rightarrow |\hat{\theta}| < |\hat{\theta}(0)| + \frac{\alpha}{L}
\]
\[
\leq h_t
\]

(38)

From (35) and (38), it is easy to conclude that
\[
|\hat{\theta}| < Lh_t
\]

(39)

To sum up, theorem 1 is proved.

**Remark 1:** It should be pointed out that the proposed nonlinear disturbance observer does not require the exact dynamic model of the ASS system. In addition, it also unifies the existing linear and nonlinear disturbance observers in a general framework.

**C. Influence Indicators of State-Coupling and Disturbance**

Until recently, the coupling problem has not been dealt with to a large extent, and the coupling analysis requires further study for the ASS. On account of its highly nonlinear features, the state-coupling term is usually considered as a known part by assuming that the states are measurable, thereby directly being canceled out. State couplings could exert crucial/essential influences on stability and entire control performance of a controlled system subject to disturbance and measurement noise etc. [31-33]. Most existing control ways for the ASS in the literature do not involve a meaningful state-coupling effect analysis, which could potentially limit further performance development of resulting controllers. As a result, it is significant to assess the state-coupling influence on the ASS quantitatively, which will be carried out in this research.

It is important that the amplitude and directionality of disturbances potentially present at least two significant influences on an ASS. The disturbance amplitude is closely connected with the stability of the system and can be well tackled in the controller design with a boundedness analysis. The disturbance directionality, which is hardly/seldom studied, could be related to how the tracking performance is influenced by disturbances. For instance, those avoidable time-varying disturbances might be used to provide a direction for the control effort. If the sign of the time-varying disturbances conforms to the expected motion, the time-varying disturbances may have the ability to foster the control performance. Consequently, the relationship between the disturbance effect and the tracking performance is worth researching deeply.

To make full use of beneficial state couplings and disturbances, the definitions of two indicators for state-coupling and disturbance effects are proposed and defined as follows.

**Definition 1:** As for (18), the state-coupling effect indicator is constructed as follows:

\[
J_1 = \text{sgn} [d(\bar{\mu}_v + \bar{\mu}_d)]
\]

(40)

Based on the introduced state-coupling effect indicator, the state-coupling effect on the system (18) is defined in the following:

\[
J_1 < 0, \text{ State-coupling effect is beneficial}
\]
\[
J_1 > 0, \text{ State-coupling effect is detrimental}
\]
\[
J_1 = 0, \text{ State-coupling effect is nil}
\]

**Definition 2:** Consider the disturbance system (18), the disturbance effect indicator is represented by

\[
J_2 = \text{sgn} [d\hat{\theta}]
\]

(42)

and then the disturbance effect on the system (18) is constructed as

\[
J_2 < 0, \text{ Disturbance effect is beneficial}
\]
\[
J_2 > 0, \text{ Disturbance effect is detrimental}
\]
\[
J_2 = 0, \text{ Disturbance effect is nil}
\]

**D. Finite-Time Disturbance and Coupling Effects-Triggered Energy Saving Control Method and Stability Analysis**

**Theorem 2:** As for (18), the finite-time disturbance and coupling effects-triggered energy saving control method is constructed as follows:

\[
u(t) = -k_1 |d| \text{sgn}(d) - k_2 \int_0^t \text{sgn}(d) dt - \hat{\theta}J_2 + \hat{\theta}_d d
\]

(44)

\[
- \frac{\bar{\mu}_v}{\bar{s}_n + \bar{\mu}_v}(\bar{\mu}_v\bar{v}_n + \bar{\mu}_d\bar{v}_d) + \bar{\mu}_v\bar{v}_n + \bar{\mu}_d v_n + \bar{\mu}_d v_m + \bar{\mu}_v v_m + \bar{\mu}_d v_m
\]

where \(k_1 \) and \(k_2 \) stand for positive control gains, \(\bar{s}_n \) and \(\bar{\mu}_v \) refer to the nominal values of the sprung and unsprung masses, respectively, which are positive in practical application, and the function \(g \) is defined as

\[
g = \begin{cases} 
1 & J_2 \geq 0 \\
1-|d| & J_2 < 0 
\end{cases}
\]

(45)

\[
J_2 \text{ denotes the following logic function:}
\]

\[
H(J_2) = \begin{cases} 
1 & J_2 \geq 0 \\
0 & J_2 < 0 
\end{cases}
\]

(46)

then the sliding surface reaches 0 in finite time, provided that the control gains are chosen to satisfy

\[
k_1 > 0, k_2 > \bar{s} + \left(2\frac{\bar{\mu}_v}{k_1} + \bar{\Omega}\right)^2
\]

(47)

such that the tracking error reaches 0 in a finite time.

To better understand the process of the controller design, we give the overall control system in Fig. 3.

**Proof:** Substituting (44) into (18), one has

\[
\bar{m}\ddot{d} = -k_1 |d| \text{sgn}(d) + \eta + (1-g)\bar{s}_n + \bar{\mu}_v(\bar{\mu}_v\bar{v}_n + \bar{\mu}_d\bar{v}_d)
\]

(48)

\[
\dot{\eta} = -k_2 \text{sgn}(d) + \hat{\theta} - \hat{\theta}H(J_2)
\]

If \(J_2 \geq 0\), i.e., \(d(\bar{\mu}_v + \bar{\mu}_d) \geq 0\), \((1-g)\bar{s}_n + \bar{\mu}_v(\bar{\mu}_v\bar{v}_n + \bar{\mu}_d\bar{v}_d) = 0\),

(48) is calculated as

\[
\bar{m}\ddot{d} = -k_1 |d| \text{sgn}(d) + \eta
\]

(49)

\[
\dot{\eta} = k_2 \text{sgn}(d) + \hat{\theta} - \hat{\theta}H(J_2)
\]

The new variable is defined as

\[
\alpha = [\sigma, \dot{\sigma}]^T
\]

(50)

\[
= [d^T \text{sgn}(d) \ \eta]^T
\]

(51)

The Lyapunov function is designed as

\[
V_\alpha(t) = \frac{\bar{m}}{2}\sigma^T P\sigma
\]
where
\[
P = \begin{bmatrix} k_i^2 + 4k_i & -k_i \\ -k_i & 2 \end{bmatrix}
\]

Differentiating (51), we have
\[
V_i(t) = \dot{\varphi}^T P \varphi
\]
\[
= \varphi^T \left[ \frac{1}{2|d|^2} \right] \begin{bmatrix} -k_i \alpha_i + \eta \\ -k_i \text{sgn}(d) + \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i) \end{bmatrix}
\]
\[
= \begin{bmatrix} \alpha_i & \alpha_i \end{bmatrix} \begin{bmatrix} k_i^2 + 4k_i & -k_i \\ -k_i & 2 \end{bmatrix} \begin{bmatrix} \alpha_i \end{bmatrix} - k_i \text{sgn}(d) + \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i)
\]
\[
\leq -k_i \varphi^T W \varphi + \left( \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i) \right) \varphi^T \varphi
\]

where
\[
\varphi^T = [-k_i \ 2]
\]
\[
W = \begin{bmatrix} k_i^2 & -k_i - 2k_i^2 \\ -k_i & 1 \end{bmatrix}
\]

Note that \( |\hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i)| \leq \Lambda_o \), where \( \Lambda_o = L \rho_p + L \alpha_p + \alpha_s \) and we can write that \( \hat{\vartheta} = \hat{\alpha} \text{sgn}(d) \), where \( \hat{\alpha} \) refers to the bounded function so that \( 0 < \hat{\alpha} \leq \Lambda_o \). Consequently, it is derived that
\[
V_i(t) = -\frac{k_i}{2|d|^2} \begin{bmatrix} \varphi^T W \varphi + 2 \hat{\alpha} \alpha_i [-k_i \ 2] \alpha_i \end{bmatrix}
\]
\[
= -\frac{k_i}{2|d|^2} \varphi^T W \varphi
\]

where
\[
W_i = \begin{bmatrix} k_i^2 + 2k_i + 2\hat{\alpha} & -k_i - 2k_i^2 \\ -k_i & 2\hat{\alpha} \end{bmatrix} \begin{bmatrix} \alpha_i \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}
\]

To guarantee the controlled ASS stability, it only needs to satisfy that \( W_i \) is positive definite, i.e.,
\[
k_i^2 + 2k_i + 2\hat{\alpha} - \left(k_i + \frac{2\hat{\alpha}}{k_i} \right)^2 > 0
\]

It is easily obtained that (58) holds if the following conditions are satisfied:
\[
k_i > 0, \ k_i > \hat{\alpha} + \frac{2\hat{\alpha}^2}{k_i^2}
\]

From (51), it is easy to obtain that
\[
\lambda_{\text{max}}(P)||\alpha||^2 \leq V_i \leq \lambda_{\text{min}}(P)||\alpha||^2
\]

Therefore, one has
\[
V_i(t) \leq -m \varphi^T \varphi
\]

where \( m = \frac{\lambda_{\text{min}}(W_i)}{\lambda_{\text{max}}(P)} \). Then, the state vector \( \varphi \) converges to 0 in finite time.

If \( J_i < 0 \), \( d(\hat{\mu}_i v + \hat{\mu}_i \dot{v}) < 0 \), in this case
\[
(1-g) \frac{\hat{y}}{\hat{y} + \hat{u}_n}(\hat{\mu}_i v + \hat{\mu}_i \dot{v}) = \frac{\hat{y}}{\hat{y} + \hat{u}_n} ||\dot{d}(\hat{\mu}_i v + \hat{\mu}_i \dot{v})||
\]

Accordingly revised as
\[
\dot{\hat{y}} = -k_i \text{sgn}(d) + \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i)
\]

The same Lyapunov candidate function is selected as
\[
V_i(t) = \frac{\hat{\alpha}}{2} \varphi^T P \varphi
\]

Taking the time derivative of (63), one has
\[
V_i(t) = \frac{\hat{\alpha}}{2} \varphi^T P \varphi
\]
\[
= \varphi^T \left[ \frac{1}{2|d|^2} \right] \begin{bmatrix} -k_i \alpha_i + \eta \\ -k_i \text{sgn}(d) + \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i) \end{bmatrix}
\]
\[
= \begin{bmatrix} \alpha_i & \alpha_i \end{bmatrix} \begin{bmatrix} k_i^2 + 4k_i \alpha_i & -k_i \alpha_i \\ -k_i \alpha_i & 2 \end{bmatrix} \begin{bmatrix} \alpha_i \end{bmatrix} - k_i \text{sgn}(d) + \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i)
\]
\[
\leq -\frac{k_i}{2|d|^2} \varphi^T W \varphi + \left( \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i) \right) \varphi^T \varphi
\]

where
\[
\varphi^T = [-k_i \ 2]
\]

\[
W = \begin{bmatrix} k_i^2 & -k_i - 2k_i^2 \\ -k_i & 1 \end{bmatrix}
\]

It is noted that \( \frac{\hat{y}}{\hat{y} + \hat{u}_n}(\hat{\mu}_i v + \hat{\mu}_i \dot{v}) \leq \Omega \), and thus we can obtain that
\[
\frac{\hat{y}}{\hat{y} + \hat{u}_n}(\hat{\mu}_i v + \hat{\mu}_i \dot{v}) \leq \Omega
\]

As a consequence, it is derived that
\[
V_i(t) \leq -\frac{k_i}{2|d|^2} \varphi^T W \varphi + \left( \hat{\vartheta} - \hat{\vartheta} \hat{H}(J_i) \right) \varphi^T \varphi
\]
\[
\leq -\frac{k_i}{2|d|^2} \varphi^T W \varphi
\]

where
\[
W_i = \begin{bmatrix} k_i^2 + 2k_i + 2\hat{\alpha} & -k_i - 2k_i^2 \\ -k_i & 2\hat{\alpha} \end{bmatrix} \begin{bmatrix} \alpha_i \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}
\]

Similarly, it is proved that if the following conditions are satisfied:
and the matrix \( W_1 \) is positive definite, and then it is derived that

\[
V_i(t) \leq -m_i V_{i1}^2(t)
\]

where \( m_i = \frac{\lambda_{\text{max}}(W_i) \lambda_{\text{max}}^2(P)}{\lambda_{\text{max}}(P)} \).

From (61) and (69), it is concluded that

\[
V_i(t) \leq -m V_{i2}^2(t)
\]

where \( m = \min(m_1, m_2) \).

After calculating (70), it is not difficult to obtain that

\[
\sqrt{V_i(t)} - \sqrt{V_i(0)} \leq -\frac{1}{2} mt
\]

It is assumed that the state vector \( \mathbf{v} \) will converge to 0 at the finite time \( t_f \), namely, \( V_i(t_f) = 0 \). Then, from (71), we can conclude that

\[
t_f \leq \frac{2V_{i1}^2(0)}{m}
\]

This completes the proof.

**Remark 2:** To avoid the chattering phenomenon associated with the proposed energy-saving robust control method, hyperbolic tangent functions are employed to replace sign functions. Then, (44) is revised as

\[
u(t) = -k_1|d|^\frac{1}{2} \tanh(d) - k_2 \int_0^t \tanh(d) d\tau - \delta H(J_2) + \lambda_p \dot{e}
\]

\[-g \frac{F_s}{s_n} (\hat{\mu}_v + \dot{\bar{\mu}}_v) + \hat{\mu}_v \dot{v}_m + \bar{\mu}_v v_m + \bar{\mu}_e \dot{e} - \bar{\mu} \lambda e
\]

---

**IV. EXPERIMENTAL RESULTS AND ANALYSIS**

To further evaluate the practical control performance concerning the designed control method, several hardware experimental conclusions are carried out on a quarter-car ASS platform, which can refer to [28-30]. The nominal systematic parameters are illustrated as follows:

- \( s_n = 2.45 \text{ kg}, u_n = 1 \text{ kg}, \mu_s = 900 \text{ N/m,}\)
- \( \mu_s = 8 \text{ Ns/m}, \mu_i = 1250 \text{ N/m}, \mu_e = 5 \text{ Ns/m}\)

And the actual parameters show fluctuate closely around their nominal values by 10%.

The next two road profiles are considered to verify the control performance of the proposed control method regarding ride comfort, energy conservation, suspension stroke, and tire deflection under uncertain dynamics.

1) Sinusoidal road profile

The sinusoidal road profile is set as

\[ v_i = \lambda \sin(6\pi t) \]

with \( l = 0.2 \text{ cm} \) standing for the vibration amplitude.

2) Random road profile

The random road profile is illustrated in Fig. 4.

To have a better comparison, we consider the following three cases.

Case 1: the PSS;
Case 2: Extended state observer Tracking (ESOT) method;
Case 3: Proposed control method (44).

---

**Fig. 3.** Block diagram for the overall control system

**Fig. 4.** Random road profile
The control gains are tuned as follows:

\[ k_1 = 100, \quad k_2 = 20, \quad \lambda_p = 20 \]

Additionally, the bioinspired reference model-related parameters are chosen as:

\[
M = 22 \text{ kg}, \quad \mu_i = 5 \text{ Ns/m}, \quad \mu_o = 0.15 \text{ Ns/m}, \quad \theta_0 = \frac{\theta}{6} \text{ rad},
\]

\[
l_1 = 0.1 \text{ m}, \quad l_2 = 0.2 \text{ m}, \quad s_v = 500 \text{ N/m}, \quad s_c = 350 \text{ N/m}
\]
these three control methods with respect to sinusoidal and random road profiles separately. Table I gives the RMS of the vehicle body acceleration concerning two diverse road profiles. It could be obviously noticed in Figs. 5 and 8 that, compared to the PSS and the ESOT method, the magnitude of the vehicle body acceleration is drastically downgraded, indicating that the designed control method could achieve vibration isolation to a large degree. Consequently, the ride comfort of the proposed control method is greatly fostered. To promote the performance comparison of the PSS, the ESOT method, as well as the proposed tracking controller, the spectrum of the normalized acceleration signal could be determined as: the decibel value $20\log_{10}(\tilde{\nu}(y/y_x))$, with $\tilde{\nu}(\cdot)$ standing for the Fourier Transformation. The reference acceleration signal for normalization (noted by $y_x$) is equal to 1 m/s$^2$, i.e., -20dB is correspondent to 0.1 m/s$^2$. Conclusions for sinusoidal and random road profiles are illustrated in Figs. 5(c) and 8(c) separately. As could be clearly noticed, the vibration level of the sinusoidal road profile under the proposed controller descends by 26.97dB at 3Hz, which is better than that under the ESOT method (18.56dB). In accordance with the experimental effects in Fig. 8(c), we could demonstrate that the vibration level of random road under the proposed controller declines during the whole frequency period, superior to that under the ESOT method. From Table I, it is depicted that, in comparison with the PSS, the ESOT method together with the proposed controller could reduce the body acceleration of the vehicle by respectively 85.29%, 91.56% regarding the sinusoidal road profile, and 45.74%, 61.34% regarding random road profiles.

In addition, Figs. 6 and 9 depict the control force as well as its frequency component under sinusoidal and random road profiles. Also, Table II shows the RMS of the energy consumption regarding two road profiles. It can be seen from Fig. 6 and 9 that the control input of the proposed control method degrades dramatically compared to that of the ESOT method. What’s more, one conclusion can be obtained that high-frequency components in the ESOT method are bigger than the proposed tracking control method. As is acknowledged, the increase of the actuator bandwidth requirement will improve the cost of the controlled ASS as well as the risk of input saturation [26-30]. Actuator bandwidth always has top and bottom limitations in practical applications. Therefore, compared with the ESOT method, the proposed tracking controller supplies higher energy efficiency and lower actuator bandwidth characteristics. Moreover, the experimental results in Fig. 6(c) illustrate that in the case of sinusoidal road profiles, the control force amplitude of the ESOT method is 3.05N at 3Hz, while that of the proposed control method is only 2N. From Fig. 9(c), as is noticed in the case of random road profiles, the control force of the proposed control method is smaller than that of ESOT in almost all frequency bands. Comparative experimental results show that better vibration suppression performance can be obtained with even a lower control force. As is confirmed in Table II, compared with the ESOT method, we know that the RMS of the energy consumption descends by 60.11% in terms of sinusoidal road profile, and 66.67% in terms of random road profile.

![Fig. 7. Suspension space and tire deflection for sinusoidal road profile](image)

<table>
<thead>
<tr>
<th>Control method</th>
<th>Sinusoidal road surfaces</th>
<th>Random road surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>ESOT method</td>
<td>0.0722</td>
<td>0.0033</td>
</tr>
<tr>
<td>Proposed control method</td>
<td>0.0288 (%60.11%)</td>
<td>0.0011 (%66.67%)</td>
</tr>
</tbody>
</table>

Table II. RMS of the energy consumption regarding different road surfaces
As depicted in [28-30], the maximum suspension space is 3.8 cm in the quarter-car ASS platform. Consequently, from Figs. 7 and 10, it is clearly shown that the suspension space as well as the tire deflection of the PSS, the ESOT method, and the proposed control method are all kept within reasonable limits. The aforementioned results clearly demonstrate that the proposed controller could keep the drivers and occupants safe on the road.
Fig. 10. Suspension space and tire deflection for random road profile

V. CONCLUSIONS

This paper investigates the coupling, disturbance, energy-saving, as well as finite-time convergence issues of robust control for the ASS. The control performance of the ASS can be improved by recognizing coupling effects and injecting good coupling effects into the designed controller; the tracking performance could be enhanced by recognizing disturbances effects and taking full advantage of helpful disturbance effects; the energy-saving performance can be significantly enhanced by using advantageous bioinspired nonlinear stiffness as well as damping properties. Experimental results illustrate that the designed algorithm possesses better vibration isolation performance as well as better energy-saving performance than other existing control methods, which validates the superior performance of the designed algorithm. In our future work, we will apply the designed energy-saving robust control method to real vehicle suspensions to further validate its superior vibration suppression performance and energy conservation. In addition, the practical finite-time convergence performance will be tested.

REFERENCES

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