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Published in:
IEEE Transactions on Network Science and Engineering

Published: 01/11/2022

Document Version:
Post-print, also known as Accepted Author Manuscript, Peer-reviewed or Author Final version

Publication record in CityU Scholars:
Go to record

Published version (DOI):
10.1109/TNSE.2022.3196805

Publication details:

Citing this paper
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Synchronization Control of Complex Dynamical Networks: Invariant Pinning Impulsive Controller with Asynchronous Actuation  

Yifan Sun, Lulu Li and Daniel W. C. Ho, Fellow, IEEE

Abstract—This article revisits the pinning impulsive control issue for complex dynamical networks (CDNs) using a new invariant pinning impulsive strategy. The word invariant means that the set of pinned nodes is fixed and does not change throughout the control process. First, it is revealed that each pinning strategy can naturally induce a hierarchical network structure. Based on that, a new multi-layer method is proposed to realize the invariant pinning. Next, under some mild assumptions, distinct Zeno-free controllers are designed using the event-triggered fashion in each layer of the network. Different from most existing works, the design of events here allows asynchronous actuation of the impulsive controllers. Then, the proposed strategy is proved to be also effective for impulsive sequences subject to average impulsive interval (AII). Finally, numerical examples are provided to show the validity of the main results, even for large scale networks or chaotic systems.

Index Terms—Synchronization, pinning impulsive control, asynchronous impulses, event-triggered control.

I. INTRODUCTION

Cooperative behavior of complex dynamical networks (CDNs) has attracted much attention from the scientific community in recent years due to its vast applications in engineering, biological systems, social science and so on [1]–[3]. To regulate a system to achieve expected cooperative behavior, one usually needs to design proper external control inputs, which promotes thriving development of cooperative control issue for CDNs [4]–[7].

As is already known, discontinuous input seems to be more reasonable in some certain real-world problems, such as the population control of insect by releasing its natural enemies [3], [8]. For such systems, many traditional control schemes including classical state/output feedback control [9], [10] or adaptive control [11], [12] cannot be implemented, and impulsive control method seems to be a proper choice because of its discontinuous actuation [13]–[15]. On the other hand, it is well-known that impulsive controllers can stabilize systems using only small impulsive gain [16]. Many impulsive control schemes have been developed in the past few decades, e.g., the occasional proportional feedback control, the predictive Poincaré control, and so forth [17]. For CDNs, however, impulsive control can be difficult to apply if one has to control all nodes as in [18], [19]. It is because exerting impulsive input on all nodes may tremendously increase the cost. As a contrast, pinning strategy makes the network control easier to implement since only a small fraction of nodes needs to be controlled directly. For this reason, more and more researchers are considering the pinning impulsive control strategy. The advantages are that only a small part of nodes (or even a single one) in the network are connected to the virtual leader and generate impulsive signals, and the whole network can be regulated to converge towards the leader [20]–[22].

There are mainly two types of pinning impulsive control strategy: single pinning [21], [24], which exerts impulsive input on only one node in the network; or switching pinning [20], [25], [26], which allows the leader to pin multiple nodes in a network, but the pinned nodes need to be re-selected at each impulse instant based on all nodes’ real-time errors. The latter strategy can be difficult to implement, because it leads to a time-varying set of pinned nodes and might be costly in practice. Furthermore, it demands that the leader has access to all nodes’ error information, which cannot be met in many real cases [24]. As for the former one (which is also called the impulsive pinning control), although the re-selecting problem no longer exists, it always requires that only one node be pinned. According to the results in [20], smaller fraction of pinned nodes may bring higher impulse frequency. More precisely, if one denotes by \( l \) the number of pinned nodes and by \( N \) the network scale, then, the impulsive interval should be very close to 0 when \( l/N \) is sufficiently small. Since \( l \) always equals to 1 for single pinning, the impulsive input should be updated at an extremely fast speed in a large-scale network, and no existing devices can keep up with such action rate. The discussion above inspires the idea of invariant pinning impulsive control of this article. That is, one or more nodes are selected in prior as the pinning set, and the set does not change throughout the control process. Under existing frameworks, however, such an idea significantly increases the difficulty of analyzing the...
network dynamics. Consequently, the following question of interest naturally arises:

**Q1)** How to modify existing frameworks and develop the invariant pinning impulsive synchronization control strategy?

There is another interesting question in impulsive control issue of networks. In most existing results, the nodes equipped with impulsive controllers should actuate synchronously [3], [22], [27]–[29]. This is sometimes difficult to implement, especially for the recently proposed event-triggered impulsive control method [30], [31]. As we all know, to be distributed is a basic requirement for event-based protocols in CDNs, but synchronous triggering is clearly a global requirement. It seems more reasonable that the nodes generate their impulse instants independent with each other. In [3] and [22], impulsive coupling controllers are designed to synchronize heterogeneous coupled CDNs. Invariant pinning is realized, but the impulsive sequences are required to be synchronous both in these two works. In [31], Xu et al. proposed an edge event-triggered communication protocol. Although the information is asynchronously shared on each edge of the network, unified impulses are still necessary. In [32], Tan et al. take a step forward on this problem, and the impulsive controller permitting asynchronous actuation is designed for linear multi-agent systems. However, it is required in [32] that

$$(I_N \otimes A)T P + P(I_N \otimes A) + \alpha(I_N \otimes I_n) \leq 0$$

for some positive scalar $\alpha$ and positive definite matrix $P$, where $A$ is the linear dynamic of the system. This basically means that the system is asymptotically stable even without any control input, and the impulsive controller mainly plays the role of accelerating the convergence. Summarizing the above discussion, one may ask:

**Q2)** Can we design an asynchronously actuated impulsive control protocol, such that more general dynamical networks can be regulated to achieve synchronization?

From the above technical points of view, it seems desirable to solve the questions **Q1**) and **Q2**). However, to the best of our knowledge, little progress has been made on this due to two difficulties under existing frameworks: (1) the mathematical derivation at impulse instants cannot be carried on without reordering the errors of all nodes, and (2) it can be extremely difficult to analyze the Zeno behavior if impulse sequences are not unified. In this article, we will refocus on the pinning impulsive control issue for CDNs and propose a new approach to solve the aforementioned problems. There are four main contributions of this article.

1) This article reveals that any pinning strategy can naturally induce a hierarchical network structure. Based on that, we introduce a multi-layer idea to design distinct control mechanisms for each layer.

2) With the multi-layer controller design, we prove that the network can achieve synchronization by using the invariant pinning strategy, and the choice of pinned nodes only needs to satisfy a basic assumption.

3) Event-based impulsive controller is designed for each pinned node without exhibiting the Zeno behavior. It is no longer required that all nodes should trigger at the same time, which makes the control strategy more flexible.

4) Our framework significantly enhances the practicability of existing event-triggered impulsive strategy both in the sense of triggering and pinning. We also show that the proposed impulsive method is effective for the commonly-used average impulsive interval (AII) type of impulses.

**Notations:** Throughout this article, the derivative of a function $x$ with respect to $t$ is denoted by $\dot{x}(t)$, and the upper-right Dini derivative is denoted by $D^+ x(t)$. For a vector $v \in \mathbb{R}^n$, the Euclidean norm of $v$ is denoted by $|v|$. A diagonal matrix with diagonal entries $a_1, \cdots, a_N$ is denoted by $\text{diag}(a_1, \cdots, a_N)$. The Kronecker product of two matrices $A_1$ and $A_2$ is denoted by $A_1 \otimes A_2$, and the 2-norm of $A_1$ is denoted by $||A_1||$. Please see TABLE I for a summary of other specifically defined notations used in this article.

II. PROBLEM FORMULATION

This article investigates control strategy for synchronizing the following dynamic network composed of $N$ nodes:

$$\dot{x}_i(t) = f(x_i(t)) + v_i(t), \quad i = 1, \cdots, N,$$

where $x_i(\cdot) \in \mathbb{R}^n$ is the state of node $i$, $f(\cdot)$ is the continuous nonlinear dynamic, and $v_i(\cdot)$ is the control input to be designed. Design of $v_i(\cdot)$ relies on the information over the network, as in many previous studies [33]. Moreover, the network is pinned by a leader node 0, whose dynamic is given as follows:

$$\dot{x}_0(t) = f(x_0(t)).$$

<table>
<thead>
<tr>
<th>TABLE I MODEL DEFINITIONS</th>
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<tr>
<td>$x_i(t)$ (or $x_0(t)$)</td>
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Assume that the $N$ nodes form an undirected simple graph $G = (S_N, \mathcal{E})$, where $S_N = \{1, 2, \cdots, N\}$ and $\mathcal{E} \subseteq S_N \times S_N$ are the sets of vertices and edges, respectively. Let $A = (a_{ij})_{N \times N}$ be the adjacency matrix of $G$, where $a_{ij} \geq 0$, and $a_{ij} > 0$ if and only if there is a directed edge from node $j$ to node $i$. Additionally, $L = (l_{ij})_{N \times N}$, the Laplacian of $G$, is assumed to be an undirected simple graph, it always holds that $a_{ij} = a_{ji}$ and $\sum_{j=1}^{N} l_{ij} = 0$. The system dynamic $f(\cdot)$ is assumed to satisfy the following assumption.

**Assumption 1:** The nonlinear function $f(\cdot)$ is Lipschitz continuous with a Lipschitz constant $F$, that is, for any $y_1, y_2 \in \mathbb{R}^n$, it always holds that

$$|f(y_1) - f(y_2)| \leq F|y_1 - y_2|.$$

In this article, we aim at regulating the CDNs (1) with a leader (2) to achieve exponential synchronization defined as follows:

**Definition 1:** (Exponential synchronization) The system (1) (2) is said to achieve exponential synchronization, if it holds for any $i \in S_N$ that

$$|\delta_i(t)| \leq \Omega^* e^{-\zeta(t-t_0)}, \quad \forall t \geq t_0$$

where $\delta_i(t) \triangleq x_i(t) - x_0(t)$, $\Omega^*$ and $\zeta$ are two positive constants.

### III. Multi-layer controller design

This section gives a detailed controller design by adopting an invariant pinning impulsive control strategy. By invariant here, we mean that the set of pinned nodes is fixed. Without loss of generality, let the first $m_p$ nodes in the network be connected to the leader, and denote $S_p = \{1, 2, \cdots, m_p\} \subseteq S_N$. Once the set $S_p$ is determined, it will never change during the control process. The choice of $S_p$ should satisfy the following assumption.

**Assumption 2:** There is at least one path from the leader to each node $i \in S_N$.

**Remark 1:** When investigating the pinning impulsive control method, a basic question always needs to be considered: which node(s) in the network should be pinned to achieve synchronization? According to Assumption 2, it would be enough to have a spanning tree with the leader as its root.

The invariant pinning set $S_p$ is a major difference compared with most existing works [20], [25], [26]. For $i \in S_p$, in particular, it is no longer guaranteed that the synchronization errors $x_i(t) - x_0(t)$ are the largest ones, which makes analysis of system dynamic more difficult at impulse instants. This will be overcome by a new multi-layer idea when designing the controllers, and we shall introduce it before proceeding.

Under conventional leader-follower framework [33], all followers (including pinned and unpinned nodes) are in one layer (Fig. 1(a)) and each node in the layer uses local information from all neighbors to generate the control input. Nonetheless, the idea of this article is different due to that the pinned nodes are restricted only to use those information from leader. More precisely, when any node is selected in set $S_p$ to satisfy Assumption 2, the selected node $i \in S_p$ has to stop using any local information from neighbors except using information from leader (information is still sent from its neighbors to node $i$). In this way, $S_p$ becomes a super-layer with some semi-leaders in it, as can be seen in Fig. 1(b). In the figure, the set of unpinned nodes is denoted as $S_p = S_N \setminus S_p$. It is an under-layer in which nodes work the same way as in Fig. 1(a). Clearly, any choice of the pinning nodes in set $S_p$ naturally induces a hierarchical network structure as described above.

![Fig. 1. The conventional leader-follower framework & the multi-layer idea.](image-url)
A. Controller design for the impulse layer

The nodes in the impulse layer \( S_p \) are regulated by the following impulsive controller

\[
v_i(t) = \xi_i(t) \triangleq (\mu_i - 1) \sum_{k=1}^{+\infty} \delta_i(t) \Delta(t - t_k^i),
\]

where \( \mu_i \) is the impulsive strength of node \( i \), and \( \Delta(\cdot) \) is the Dirac impulse. Moreover, \( t_k^i \) stands for the \( k \)-th actuating instant of \( i \), which can be generated by many methods, such as the periodical method, the \( \mathcal{AII} \) approach, or the event-triggered protocol. This article mainly considers the event-triggered impulsive controller for reducing the actuating cost as much as possible. Specifically, for a node \( i \in S_p \), the actuating instant is generated by the following protocol:

\[
t_{k+1}^i = \inf \left\{ t > t_k^i \mid |\delta_i(t)| \geq \tilde{\delta}_0 e^{-\gamma_L(t-t_0)} \right\},
\]

where \( \tilde{\delta}_0 > 0 \) is a constant satisfying \( \tilde{\delta}_0 \geq |\delta_i(t_0)| \), and \( \gamma_L > 0 \) is a design parameter. Under the action of event-triggered mechanism (ETM) \(^4\), it can be ensured that

\[
|\delta_i(t)| \leq \tilde{\delta}_0 e^{-\gamma_L(t-t_0)}, \quad \forall i \in S_p.
\]

Remark 2: For CDNs, there have been many works considering the impulsive control strategy, most of which require that the impulses of all nodes should happen synchronously \(^3\). As another way of thinking, Chen et al. in \(^27\) consider a single pinning impulsive control strategy. Although the synchronous impulse is overcome, the single pinning strategy may not give full play to the advantages of the impulsive controller. As a contrast, the event-triggered impulsive controller \(^3\)\(^-\)\(^4\) in this article allows multiple nodes in \( S_p \) to generate the impulses independently. This is one of the major contributions of this article since, in many cases, especially for event-based controllers, asynchronous actuation can be more reasonable due to the distributed requirements in CDNs.

Remark 3: Tan et al. have proposed a distributed event-triggered impulsive controller for linear multi-agent systems in \(^32\), where asynchronous triggering is permitted. However, note that the controller there plays a completely different role compared to the one in this article. In fact, the theorem 1 in \(^32\) requires a positive definite matrix \( P \) and a positive \( \alpha \) such that

\[
(I_N \otimes A)^T P + P(I_N \otimes A) + \alpha(I_N \otimes I_n) \leq 0,
\]

which indicates that the system is stable itself, and the impulsive controller is for accelerating the synchronization. As a contrast, the synchronization control strategy in this article is also valid for unstable, or even chaotic systems, which will be shown in example \(^1\) in Section \( \mathcal{VI} \).

B. Controller design for the interaction layer

For the nodes in the interaction layer, the control inputs are generated based on the states of their own and their neighbors. More specifically,

\[
v_i(t) = \tilde{u}_i(t) \triangleq u_i(t_{k_i}^i), \quad \forall i \in S_p,
\]

where \( t_{k_i}^i \) is the actuating instant, and

\[
u_i(t) = c \sum_{j=1}^{N} a_{ij} (x_j(t) - x_i(t))
\]

with \( c > 0 \) representing the coupling strength. We denote \( e_i(t) \triangleq x_i(t) - u_i(t) \) for designing the ETM in \( S_p \) later. Moreover, although \( u_i(t) \) does not act on the impulse layer, we still formally define \( u_i(t) \) as in (7) for \( i \in S_p \) for further discussion.

For \( i \in S_p \), the actuating instants \( t_{k}^i \) are generated by the following event-triggered protocol:

\[
t_{k+1} = \inf \left\{ t > t_k^i \mid |e_i(t)| \geq \chi_i(t) e^{-\gamma_L(t-t_0)} \right\},
\]

where \( \chi_i > 0 \) is a constant satisfying \( \chi_i \geq |e_i(t_0)| \), and \( \gamma_L > 0 \) is a design parameter.

IV. MAIN RESULTS

In this section, our goal is to give a detailed parametric design for dynamic network \(^1\). The network will be shown to achieve synchronization as defined in Definition \(^7\). In order to deal with the asynchronous impulses, we shall first analyze each node in the impulse layer \( S_p \) individually. Then, all nodes in \( S_p \) are considered as a whole by constructing a proper Lyapunov candidate function (LCF). The following lemma is critical for obtaining the convergence of such an LCF.

Lemma 1: Suppose the Laplacian \( \mathcal{L} \) has the following form:

\[
\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{12} & \mathcal{L}_{22} \end{pmatrix}
\]

with \( \mathcal{L}_{11} \in \mathbb{R}^{m_p \times m_p} \). Then, under Assumption \(^2\), \( \mathcal{L}_{22} \) is positive definite.

Proof. It directly follows from Assumption \(^2\) that \( \mathcal{L}_{22} \) is symmetric, and it remains to show that all eigenvalues of \( \mathcal{L}_{22} \) have positive real parts.

By removing all nodes \( i \in S_p \) and associated edges, we obtain a subgraph \( \tilde{G} \), whose Laplacian is denoted by \( \tilde{\mathcal{L}} \). Moreover, construct a virtual leader node \( v_0 \), and define the pinning matrix \( \mathcal{P} = \text{diag}\{p_{m_p+1,0}, \cdots, p_{N,0}\} \) as

\[
p_{i0} = - \sum_{j \in S_p} l_{ij}, \quad i \in S_p.
\]

If \( p_{i0} > 0 \), then node \( i \) is regarded as being pinned by \( v_0 \), or it is not pinned otherwise. It is clear that \( \mathcal{L}_{22} = \tilde{\mathcal{L}} + \mathcal{P} \), and according to \(^34\), it suffices to show that there is a path
from \( t_0 \) to any \( i \in \mathcal{S}_p \). Equivalently, we will prove that for some sequence \( j_1, j_2, \cdots, j_m \in \mathcal{S}_p \), it holds that \( p_{j_1,0} > 0 \), \( l_{j_2,j_1} < 0 \), \( \cdots \), \( l_{j_m,j_{m-1}} < 0 \). In view of Assumption 2 we can always find \( j_0 \in \mathcal{S}_p \) and \( j_1, j_2, \cdots, j_m \in \mathcal{S}_p \) such that \( l_{j_1,j_0} < 0 \), \( \cdots \), \( l_{j_m,j_{m-1}} < 0 \). Since \( \mathcal{S}_p \cap \mathcal{S}_p = \emptyset \), \( p_{j_1,0} \geq -l_{j_1,j_0} > 0 \). The proof is finished.

In light of Lemma 7 there exists a positive constant \( \varrho \) such that
\[
y^T (L_{22} \otimes I_n) y \geq \varrho |y|^2. \tag{9}
\]
for any vector \( y \in \mathbb{R}^{n(N-m_p)} \).

Now we are ready to give the main result of this article.

**Theorem 1:** Under Assumptions 7–2, the dynamical network (1)–(2) under control law (3), (6)–(7) can achieve exponential synchronization without exhibiting the Zeno behavior, if the following conditions are satisfied:

1. The coupling strength \( c > \mathcal{F} / \varrho \), where \( \mathcal{F} \) is given in Assumption 7 and \( \varrho \) satisfies (9);
2. \( \gamma_L \geq \gamma_F \), where \( \gamma_L \) and \( \gamma_F \) are, respectively, given in (4) and (8);
3. The impulsive strength satisfies \(-1 < \mu_i < 1\).

**Proof.** See Appendix VIII-A

**Remark 4:** The difficulties of proving Theorem 7 are mainly twofold:

1. *How to analyze the impact of asynchronous impulses on system dynamics?*
2. *For the interaction layer \( \mathcal{S}_p \), how to exclude the Zeno behavior without having a strictly positive minimum inter-event time (MIET)?*

As mentioned at the beginning of this section, the first problem is solved by using the specially constructed LCF, which makes the analysis easier at impulse instants. Lemma 7 is crucial for the convergence of such an LCF. The difficulty in 2) is overcome by showing that any interval with a constant length contains finite times of triggers. For details, one can refer to the part after (18) in Appendix VIII-A

**Remark 5:** Zhang et al. have proposed an asynchronous impulsive control scheme for multi-agent systems in [35], where the consensus criterion is given in term of linear matrix inequality (LMI). However, the LMI condition there involves the matrix
\[ S(t_k) \equiv \text{diag}\{s_1(t_k), \cdots, s_N(t_k)\}, \]
where \( s_i(t_k) = 0 \) if node \( i \) is free from impulse at \( t_k \), or \( s_i(t_k) = 1 \) otherwise. If the impulses are asynchronous, \( S(t_k) \) can have at most \( 2^N - 1 \) values, and the LMI condition will be difficult to verify when the network scale is large. As a contrast, the conditions in Theorem 7 are easier to verify even if for a large scale network.

**V. Flexibly Designed Impulsive Controller**

In Section III the impulse sequence of each node in \( \mathcal{S}_p \) is generated by the event (4). It should be noted that our strategy is valid for more types of impulsive controllers, for instance, the ones subject to the following commonly-used All restriction.

**Definition 2:** (All) Consider the sequence \( \{t_k\}_{k=1,2,\cdots} \), which corresponds to the impulse instants of node \( i \). The positive constant \( T_a^{(i)} \) is called the All of node \( i \), if there exists a positive integer \( N_0^{(i)} \), such that
\[
\frac{t-t_0}{T_a^{(i)}} - N_0^{(i)} \leq N^{(i)}(t,t_0) \leq \frac{t-t_0}{T_a^{(i)}} + N_0^{(i)},
\]
where \( N^{(i)}(t,t_0) \) represents the number of impulses of node \( i \) on \( [t_0, t) \).

When the impulsive controller is designed by using the All, we have the following result.

**Theorem 2:** Suppose the All of node \( i \in \mathcal{S}_p \) is \( T_a^{(i)} \) as defined in Definition 2. Then, under Assumptions 7–2, the dynamical network (1)–(2) under control law (3), (6)–(7) can achieve exponential synchronization if the following conditions are satisfied:

1. \( c > \mathcal{F} / \varrho \);
2. \( \frac{\ln \mu_i}{T_a^{(i)}} + \mathcal{F} + \gamma_F \leq 0 \), \( \forall i \in \mathcal{S}_p \).

Moreover, the ETM (8) is free from the Zeno behavior.

**Proof.** See Appendix VIII-B

As an important special case of Theorem 7 we have the following results when impulses of all nodes are synchronous.

**Corollary 1:** Suppose that the impulse sequence of all nodes is designed to be unified, that is, it holds that \( t_k = t_k \), \( T_a^{(i)} = T_a \) and \( N_0^{(i)} = N_0 \) in Definition 2 for \( i \in \mathcal{S}_p \). Under Assumptions 7–2, the dynamical network (1)–(2) under control law (3), (6)–(7) can achieve exponential synchronization if the following are satisfied:

1. \( c > \mathcal{F} / \varrho \);
2. \( \frac{\ln \mu_i}{T_a} + \mathcal{F} + \gamma_F \leq 0 \), \( \forall i \in \mathcal{S}_p \).

**Proof.** This corollary directly follows from Theorem 7.

When the impulsive strengths are also the same, Corollary 7 reduces to the following.

**Corollary 2:** With unified impulsive strength \( \mu \) and impulse sequence \( \{t_k\}_{k=1,2,\cdots} \), the dynamical network (1)–(2) satisfying Assumptions 7–2 under control law (3), (6)–(7) can achieve exponential synchronization if

1. \( c > \mathcal{F} / \varrho \);
2. \( \frac{\ln \mu}{T_a} + \mathcal{F} + \gamma_F \leq 0 \).

**Remark 6:** In this work, results involving communication delays might be obtained by resorting to some effective
methods, e.g., the delay compensation approach in [36]. Yet, the difficulty of excluding the Zeno behavior significantly increases after taking delays into account. Further, a new ETM technique may need to develop, and we leave this as an important future topic.

VI. NUMERICAL EXAMPLES

This section provides two numerical examples to illustrate the validity of our main results. In the first example, a network with only $7$ nodes is considered. It is clearly shown that our strategy can be applied to a chaotic system, for which the impulsive controller in [32] may not work. Moreover, the Zeno behavior is excluded. The second example considers a scale-free network with 100 nodes and shows the effectiveness of our results in large scale complex networks.

Example 1: Consider $x_i(t) \in \mathbb{R}^3$. The system dynamic $f(x)$ is given by the following Chua’s circuit [37]
\[
 f(x_i) = Cx_i + [h(x_{i1}), 0, 0]^T,
\]
where
\[
 C = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix},
\]
\[
 h(x) = m_1 x + 0.5(m_0 - m_1) (|x + 1| - |x - 1|)
\]
with $\alpha = 15.6$, $\beta = 28$, $m_0 = -1.143$ and $m_1 = -0.714$. Moreover, the weighted Laplacian is given as
\[
 L = \begin{pmatrix} -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & -6 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & -4 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 & 0 & -4 \end{pmatrix}.
\]
It can be verified that Assumption [7] is satisfied with $\mathcal{F} = 32.1447$. The nodes 1 and 2 are in the impulse layer $\mathcal{S}_p$, and all other nodes are in the interaction layer. Fig. [2] shows the chaotic dynamic of the leader $x_0$ with initial value $[0.7, 0, 0]^T$.

Now we apply appropriate control input to the system according to our main results. In the impulse layer, the impulsive strengths are chosen to be $\mu_1 = 0.35$, $\mu_2 = 0.7$, and the triggering parameter $\gamma_L = 0.05$. In the interaction layer, the coupling strength is designed as $c = 32$, and the triggering parameter $\gamma_F = 0.039$. It can be calculated that $\rho = 1.0376$ in [9], $\alpha = 0.1$, and the conditions in Theorem [7] are all satisfied. Figs. [3] and [4] show the system dynamics under control input, and Fig. [5] shows the inter-event time (IET) of all nodes in the interaction layer. It can be clearly observed that the system achieves synchronization, and the IET has a strictly positive lower bound (in the simulation, the sampling period is 0.001).

Fig. 2. The chaotic dynamic of the leader $x_0$ in example [1].

Fig. 3. Dynamics of the impulse layer in example [1]. Plots (a), (b), and (c) show the 1-st, 2-nd and 3-rd dimensional dynamics, respectively.

Fig. 4. Dynamics of the interaction layer in example [1]. Plots (a), (b) and (c) show the 1-st, 2-nd and 3-rd dimensional dynamics, respectively.
For real-world impulsive controllers, due to limited device capacity, the events should not be too intensive within a short period of time. In example 1 however, although our strategy realize the control objective, the IET of nodes 1 and 2 is small as shown in Fig. 6(a). This is mainly due to the large Lipschitz constant $\mathcal{F} = 32.1447$ of a chaotic system. More precisely, it can be seen in inequality (18) that $t_{k+1}^i - t_k^i$ is inversely proportional to $\mathcal{F}$. In order to have a larger IET, the inequality (18) provides two possible ways: 1) increasing the value of $1 - |\mu_i|$, or 2) decreasing the value of $\gamma_L$. When $\mathcal{F}$ is large as in example 1, the second way has little impact, and thus we need to set the impulsive strengths $\mu_i$ smaller. In example 1 if we set $\mu_1 = \mu_2 = 0.1$, the IET dramatically increases as shown in Fig. 6(b). State trajectories after changing $\mu_1$ and $\mu_2$ are plotted in Figs. 7 to 8. As compared to Figs. 3 and 4, there is little change in the convergence rate, which is not surprising: according to (5) and (12), the convergence rate mainly depends on the parameters $\gamma_L$ and $\gamma_F$. However, they remain unchanged in the simulation of Figs. 7 to 8.

When $\mathcal{F}$ is relatively small, the effect of decreasing $\gamma_L$ will become obvious, which consists with the inequality (18). As a trade-off, the convergence rate will also decrease. The impact of adjusting $\gamma_L$ will be simulated in example 2.

In Figs. 3 and 4 it takes about 40 seconds for the network to achieve synchronization. For faster convergence rate, we now set $\gamma_L = \gamma_F = 0.5$, which is much larger than the setting above. Figs. 9 and 10 show the system dynamic after changing $\gamma_L$ and $\gamma_F$, and it only takes around 10 seconds to achieve the expected synchronization. In fact, the convergence rate can be even faster if we choose larger $\gamma_L$ and $\gamma_F$, but this will lead to more frequent triggers and increase energy consumption.

Example 2: Now we consider a scale-free network composed of 100 1-dimensional nodes, in which the nodes 1 to 3 are pinned by the leader, and all the other 97 nodes are in the interaction layer. In this example, the network topology is randomly generated with $\rho = 0.3138$ in (9), and the initial conditions are randomly chosen from $[-10, 10]$. In
addition, \( f(x) = 0.5 \tanh x \), satisfying Assumption 2 with \( \mathcal{F} = 0.5 \). The impulsive strengths are \( \mu_1 = 0.7 \), \( \mu_2 = 0.5 \) and \( \mu_3 = 0.3 \). By setting \( c = 2 \), \( \gamma_L = 0.15 \) and \( \gamma_F = 0.1 \), all conditions in Theorem 7 are satisfied, and the system achieves synchronization as shown in Fig. 11. Moreover, the asynchronous impulse instants of the nodes 1 to 3 are shown in Fig. 12.

In order to reduce the impulse frequency, we now select a smaller \( \gamma_L = 0.05 \), and \( \gamma_F \) is also set to be 0.05 due to the condition 2) of Theorem 7. Fig. 13 clearly shows the reduction in the number of impulses, but the synchronization rate also decreases as compared to Fig. 11.

VII. CONCLUSION

This article has enriched the results on pinning impulsive synchronization control of CDNs. It has been revealed that the proposed pinning strategy can naturally determine a hierarchical network structure, based on which a multi-layer control mechanism has been introduced. Under some regular assumptions, the invariant pinning impulsive strategy has been developed with the set of pinned nodes fixed throughout the control process. It has been proved that synchronization can be achieved even if the impulsive controllers actuate asynchronously. Both event-based and \( AII \)-based impulsive controllers are investigated, and numerical examples have been given to show the effectiveness in chaotic systems and large-scale networks.

Under the framework of this article, the strictly positive MIET cannot be obtained in the interaction layer \( S_p \). Thus, it may lead to high actuation rate and is a main drawback of the present work. Future topics include proposing a new framework to handle this issue. In addition, it would be interesting and challenging to take communication delays into account in the network model.

VIII. APPENDIX

A. Proof of Theorem 7

The proof contains two parts. The first part mainly proves the exponential synchronization, and the second part shows how we exclude the Zeno behavior. Difficulties of proving
Theorem 1 mainly lie on deriving (12), which indicates the synchronization in interaction layer $S_p$. Different from the traditional framework which constructs the Lyapunov impulses, we separately consider the nodes in $S_p$ utilizing the multi-layer structure. This guarantees the continuity of the LCF even if there are impulses, and makes the analysis at impulse instants easier to carry on.

Part 1): Exponential synchronization

It directly follows from (5) that any $i \in S_p$ achieves synchronization.

For $i \in S_p$, recalling the notation $\delta_i(t) = x_i(t) - x_0(t)$, it can be derived from (1), (2), (6)–(7) that

$$\dot{\delta}_i(t) = f(x_i(t)) - f(x_0(t)) + u_i(t) + e_i(t) = f(x_i(t)) - f(x_0(t)) + \sum_{j \in S_p} a_{ij} \delta_j(t) - e_{i}(t), \quad \forall i \in S_p,$$

where $e_i(t) = \hat{u}_i(t) - u_i(t)$. Construct the LCF as $V_p(t) = (1/2) \sum_{i \in S_p} \delta_i^T(t) \delta_i(t)$. Since all nodes in $S_p$ are free from impulses, $V_p(t)$ is continuous by definition. So we have

$$D^+ V_p(t) = \sum_{i \in S_p} \delta_i^T(t) \left[ f(x_i(t)) - f(x_0(t)) - \sum_{j \in S_p} a_{ij} \delta_j(t) + e_i(t) \right]$$

$$\leq \Omega_1 \|\delta(t)\|^2 - \frac{1}{\sqrt{2}} \|\delta(t)\| - \frac{\Omega_1}{\sqrt{2} \alpha} e^{-\gamma_F (t-t_0)},$$

where $\delta_0 \equiv \max_{i \in S_p} \{\delta_i(0)\}$, and $\chi_0 \equiv \max_{i \in S_p} \{\chi_i(0)\}$. For simplicity, let $\Omega_1 \equiv c \hat{a}(N - m_p) m_p \delta_0 + \frac{\sqrt{N - m_p}}{\chi_0} \chi_0$, and it can be further deduced from conditions 1) and 2) that

$$D^+ V_p(t) \leq - \sqrt{2} \alpha |\delta(t)| \left( \frac{1}{\sqrt{2}} |\delta(t)| - \frac{\Omega_1}{\sqrt{2} \alpha} e^{-\gamma_F (t-t_0)} \right),$$

where $\alpha = c \hat{a} - 2 \Omega > 0$, which exists due to the condition 1). By using (11), we will prove that

$$V_p(t) \leq \Omega_2 e^{-2 \gamma_F (t-t_0)}, \quad \forall t \geq t_0,$$

where $\Omega_2 = 0 \max \{V_p(t_0), (\Omega_1/\sqrt{2} \alpha) \epsilon \}^2$, and $\epsilon > 0$ is an arbitrary small constant. Assume by contradiction that (12) is false. Since (12) is obviously true for $t = t_0$ and $V_p(t)$ is continuous, there must exists a $t \geq t_0$ such that

$$V_p(t) \leq \Omega_2 e^{-2 \gamma_F (t-t_0)} \geq \left( \frac{\Omega_1}{\sqrt{2} \alpha} + \epsilon \right)^2 e^{-2 \gamma_F (i-t_0)} \geq 0,$$

which contradicts against (14). Therefore, (12) is true, which indicates the exponential synchronization for $i \in S_p$.

Part 2): Zeno-free

We first show that the impulse layer is free from the Zeno behavior. For $i \in S_p$, consider an arbitrary triggering instant $t^+_k$, and $t^+_{k+1}$ will be estimated. Construct an auxiliary function $\varphi^T_k(t) \equiv \delta_i(t) e^{\gamma_L (t-t_0)}$, and it can be derived from (1) and (3) that

$$\dot{\delta}_i(t^+_k) = \mu_i \hat{\delta}_i(t^-_k), \quad i \in S_p.$$

By (4), we also have

$$|\varphi^T_k(t^+_k)| = |\delta_0|, \quad i \in S_p$$

and

$$|\varphi^T_k(t^+_k)| = \mu_i |\hat{\delta}_i|, \quad i \in S_p.$$

On the other hand,

$$\frac{d}{dt} |\varphi^T_k(t)| \leq |\dot{\varphi}^T_k(t)| = |\dot{\delta}_i(t) + \gamma_L \delta_i(t) e^{\gamma_L (t-t_0)}|$$

$$\leq (\hat{a} + \gamma_L) |\delta_i(t)| e^{\gamma_L (t-t_0)} \leq (\hat{a} + \gamma_L) |\hat{\delta}_i(t)| e^{\gamma_L (t-t_0)} \leq (\hat{a} + \gamma_L) |\hat{\delta}_i(t)|, \quad i \in S_p,$$
where the second inequality uses Assumption 7 and the last line is because of 5. By combining condition 3), (15)−(17) with the fact that \( \delta_0 > 0 \), we can infer that

\[
t^i_{k+1} - t^i_k \geq 1 - \frac{|\mu_1|}{\mathcal{F} + \gamma_L} \triangleq \tau_i > 0, \quad i \in \mathcal{S}_p. \tag{18}
\]

Now we consider the interaction layer \( \mathcal{S}_\mathcal{P} \). Since there may exist impulses in \( \mathcal{S}_p \), \( e_i(t) \) can be discontinuous in \( t \) even if no event is triggered. For this reason, the positive MIET can no longer be guaranteed as in (13), and we merely show that any finite interval only contains finite triggering instants. Let us first consider the simplest case, in which for two consecutive triggering instants \( t^i_k \) and \( t^i_{k+1} \), there is no impulse instant on \( (t^i_k, t^i_{k+1}) \). In other words, \( e_i(t) \) is continuous on \( (t^i_k, t^i_{k+1}) \), and \( t^i_{k+1} \) is not an impulse instant for any \( \ell \in \mathcal{S}_\mathcal{P} \). We similarly construct \( \phi^F_i(t) \triangleq e_i(t)e^{\mathcal{F}(t-t_0)} \), and it follows from the definition of \( e_i(t) \) and (8) that

\[
|\phi^F_i(t^i_k)| = 0, \quad i \in \mathcal{S}_\mathcal{P} \tag{19}
\]

and

\[
|\phi^F_i(t^i_{k+1})| = \chi_0, \quad i \in \mathcal{S}_\mathcal{P}. \tag{20}
\]

On the other hand, due to the continuity of \( e_i(t) \) on \( (t^i_k, t^i_{k+1}) \), it can be calculated that

\[
\frac{d}{dt}|\phi^F_i(t)| \leq |\phi^F_i(t)|
= |e_i(t)e^{\mathcal{F}(t-t_0)} + \mathcal{F}e_i(t)e^{\mathcal{F}(t-t_0)}|
\leq (|\dot{u}_i(t)| + \mathcal{F}|e_i(t)|)e^{\mathcal{F}(t-t_0)},
\forall t \in (t^i_k, t^i_{k+1}). \tag{21}
\]

Recalling the formal definition of \( u_i(t) \) for \( i \in \mathcal{S}_p \) below 4, we can denote \( u(t) \triangleq (u^T_1(t), \ldots, u^T_p(t))^T \). Since it is assumed to be no impulse, \( u(t) \) is also differentiable on \( (t^i_k, t^i_{k+1}) \). Hence, we further have

\[
\frac{d}{dt}|\phi^F_i(t)| \leq (|\dot{u}(t)| + \mathcal{F}|e_i(t)|)e^{\mathcal{F}(t-t_0)}
\leq (c|\mathcal{L}||\delta(t)| + \mathcal{F}|e_i(t)|)e^{\mathcal{F}(t-t_0)},
\forall t \in (t^i_k, t^i_{k+1}), \tag{21}
\]

where \( \delta(t) \triangleq (\delta_1^T(t), \ldots, \delta_p^T(t))^T \). Since \( \mathcal{F} \leq \gamma_L \) in condition 2), it follows from (5) that

\[
|\delta_i(t)| \leq \bar{\delta}_0 e^{-\gamma_L(t-t_0)}, \quad i \in \mathcal{S}_\mathcal{P}. \tag{22}
\]

Moreover, we conclude from (12) that

\[
|\delta_i(t)| \leq \sqrt{2\Omega_3} \cdot e^{-\gamma_L(t-t_0)}, \quad i \in \mathcal{S}_\mathcal{P}.
\]

By denoting \( \Omega_3 \triangleq \sqrt{N} \cdot \max_{i \in \mathcal{S}_\mathcal{P}} \{ \sqrt{2\Omega_2} \cdot \bar{\delta}_0 \} \), it holds that

\[
|\delta(t)| \leq \Omega_3 e^{-\gamma_L(t-t_0)}. \tag{22}
\]

The inequalities (21), (22) and the ETM (8) indicate that

\[
\frac{d}{dt}|\phi^F_i(t)| \leq c|\mathcal{L}||\Omega_3 + \gamma_{\mathcal{F}}\chi_0|, \quad \forall t \in (t^i_k, t^i_{k+1}). \tag{23}
\]

In view of (19), (20) and (23), it is finally obtained that

\[
t^i_{k+1} - t^i_k \geq \frac{\chi_0}{c|\mathcal{L}||\Omega_3 + \gamma_{\mathcal{F}}\chi_0|} \triangleq \tau_i > 0. \tag{24}
\]

Now we take the impulses into consideration, and it will be proved that there are no more than \( m_p + 1 \) triggering instants on \( [t, t + \tau_i] \) for any \( t \geq t_0 \). Here, \( \tau_i \triangleq \min\{\tau_1, \ldots, \tau_{m_p}, \tau_i\} > 0, \tau_i \) is given in (18), and \( \tau_i \) is given in (24). Apparently, we have the following two facts due to the definition of \( \tau_i \):

1) There are at most \( m_p \) impulse instants on \( [t, t + \tau_i] \);
2) Between any two consecutive impulse instants on \( [t, t + \tau_i] \), there are at most one triggering instants (since \( \tau_i \leq \tau_i \), and there is no impulse between any two consecutive impulse instants).

Hence, there are at most \( m_p + 1 \) triggering instants on \( [t, t + \tau_i] \). This indicates that the Zeno behavior is impossible, and the proof is completed.

\section{B. Proof of Theorem 2}

\textbf{Part 1): Exponential synchronization}

We first consider the impulse layer \( \mathcal{S}_p \), and define \( V_i(t) = |\delta_i(t)|^2 \) for \( i \in \mathcal{S}_p \). By simple calculation, we have

\[
D^+V_i(t) \leq 2\mathcal{F}V_i(t), \quad \forall t \in [t^i_k, t^i_{k+1}),
\]

which further indicates that

\[
V_i(t) \leq V_i(t_k^i) \cdot \exp(2\mathcal{F}(t - t_k^i)), \quad \forall t \in [t_k^i, t_{k+1}^i). \tag{25}
\]

On the other hand, at an impulse instant, we can infer from (3) that

\[
V_i(t_k^i) = \mu_2^2 V_i(t_k^i), \quad k = 1, 2, \ldots \tag{26}
\]

By applying the comparison lemma and mathematical induction to (25) and (26), we easily obtain that

\[
V_i(t) \leq \mu_2^{2k} V_i(t_0) \cdot \exp(2\mathcal{F}(t - t_0)), \quad \forall t \in [t_k, t_{k+1}),
\]

which is equivalent to

\[
|\delta_i(t)| \leq |\mu_2|^k |\delta_i(t_0)| \cdot \exp(\mathcal{F}(t - t_0)), \quad \forall t \in [t_k, t_{k+1}).
\]

Since condition 2') implies that \( |\mu_k| < 1 \), it follows from Definition 2 that

\[
|\delta_i(t)| \leq |\mu_k|^{-N_0^{(i)}} |\delta_i(t_0)| \exp\left(\left|\frac{1}{T_0^{(i)}} + \mathcal{F}\right|(t - t_0)\right).
\]

By condition 2'), we further have

\[
|\delta_i(t)| \leq |\mu_k|^{-N_0^{(i)}} |\delta_i(t_0)| \exp(-\gamma_F(t - t_0)), \tag{27}
\]

which indicates the synchronization of \( \mathcal{S}_p \).

For the interaction layer, since (5) is guaranteed by taking \( \bar{\delta}_0 = |\mu_k|^{-N_0^{(i)}} |\delta_i(t_0)| \), the remaining proof is the same as in Theorem 7 and is omitted due to space limitation.
Part 2): Zeno-free

By the definition of All, there are at most $1 + N_0^i$ impulses on $[t, t + T_0^i)$ for any $t \geq 0$. Therefore, the Zeno behavior cannot occur in the impulse layer.

For any node $i$ in the interaction layer, since the ETM does not change, the inequality (24) is still true for any impulse-free time interval $(t_k^i, t_{k+1}^i)$. Moreover, by denoting $\tilde{T}_a = \min_{j \in S_i} \{T_a^{(j)}\}$ and $N_0 = \max_{j \in S_i} \{N_0^{(j)}\}$, there are no more than $m_p (1 + N_0)$ impulses on $[t, t + \tilde{T}_a)$ for any $t \geq 0$. Hence, there are at most $1 + m_p (1 + N_0)$ events on $[t, t + \min\{T_a^i, \tilde{T}_a\})$, where $\tilde{T}_a$ is given in (24). The proof is completed.

References

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