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GRADIENT ESTIMATES FOR ELECTRIC FIELDS WITH MULTISCALE INCLUSIONS IN THE QUASI-STATIC REGIME*

YOUJUN DENG†, XIAOPING FANG‡, AND HONGYU LIU§

Abstract. In this paper, we are concerned with the gradient estimate of the electric field due to two nearly touching dielectric inclusions, which is a central topic in the theory of composite materials. We derive accurate quantitative characterizations of the gradient fields in the transverse electromagnetic case within the quasi-static regime, which clearly indicate the optimal blowup rate or nonblowup of the gradient fields in different scenarios. There are mainly two novelties of our study. First, the sizes of the two material inclusions may be of different scales. Second, we consider our study in the quasi-static regime, whereas most of the existing studies are concerned with the static case.

Key words. composite optical materials, nearly touching inclusions, gradient estimates, blowup, quasi-static, multiscale

AMS subject classifications. 35J25, 35C20, 78A40

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1. Introduction.

Stress concentration is a peculiar phenomenon that widely occurs in continuum mechanics. It is a central topic in the theory of composite materials, where the concentration occurs due to nearly touching material inclusions that are the building blocks of the composite material. The degree of concentration is characterized by the blowup rate of the gradient of the underlying field. There are extensive studies in the literature on the gradient estimates of the underlying fields due to two nearly touching inclusions. We refer the reader to [23, 24] for related results in general elliptic systems, [2, 7, 8] for elastostatics, and [3, 4, 5, 26, 6, 19, 20, 25] for electrostatics for optical materials. The gradient estimates depend on the background field as well as the asymptotic parameter ϵ, which signifies the distance between the closely spaced material inclusions. Generically, the optimal blowup rate of the gradient field is of order $1/\sqrt{\epsilon}$ in two dimensions, whereas it is $(\epsilon|\ln\epsilon|)^{-1}$ in three dimensions. In establishing those results, it is usually assumed that the inclusions are of regular size, i.e., the size is of order $\mathcal{O}(1)$ compared to the asymptotic distance

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parameter $\epsilon \ll 1$. In fact, it is shown in [3, 18] that if the size of the two objects are of the same order as the distance between them, the gradient stays bounded. To the best of our knowledge, there are few studies on the case that the sizes of the inclusions are of different scales. Moreover, very few results are concerned with the gradient estimates for waves in the frequency regime. There is a major difficulty for the latter case, i.e., the maximum principle fails for the wave system (cf. [9]).

In this paper, we study the gradient estimate for the electromagnetic field in the transverse model in $\mathbb{R}^2$ due to nearly touching dielectric inclusions. We consider our study in the quasi-static regime, namely, the size of the inclusion is smaller than the operating wavelength. Nevertheless, we allow the sizes of the inclusions to be of the same scale or different scales. That is, one inclusion may be of regular size, while the size of the other one can be very large (actually, can be related to the asymptotic parameter ϵ). Geometrically, this means that the curvatures of the nearly touching faces of the two inclusions may be in sharply different scales, say, e.g., one is very high while the other is very low (nearly flat); see also [10, 11] for some motivating geometric discussion in different physical setups. In such a general scenario, we derive an accurate gradient estimate of the electric field, which is contained in (2.7) in Theorem 2.1. There are two parts in the asymptotic estimate: the first one accounts for the static effect, whereas the second one accounts for the frequency effect. The static part recovers the known results in the literature if both inclusions are of regular size. It also covers the more general scenario that the two inclusions are of sharply different scales. It is more interesting to note that the frequency part can induce new blowup phenomena. In fact, even if the static part vanishes, there might still be the blowup phenomenon in certain generic scenarios due to the frequency part. In deriving the new gradient estimate, we develop techniques that combine layer-potential operators with asymptotic analysis and singular decomposition of the wave field.

The rest of the paper is organized as follows. In section 2, we present the mathematical setup of our study as well as state the main results of the paper. In section 3, we use a layer potential technique to derive the integral representation of the solution as well as the associated asymptotic expansions. The estimates of the nonsingular and singular parts of the gradient fields are established in sections 4 and 5, respectively.

2. Mathematical setup and statement of the main results.

In this section, we present the mathematical formulation of the transverse electromagnetic scattering with multiscale dielectric inclusions. Then we state the main results in this paper, whose proofs shall be postponed to the subsequent sections.

2.1. Mathematical setup.

Let B_1 and B_2 be two disks in $\mathbb{R}^2$. Let $\mathbf{z}_j \in \mathbb{R}^2$ and $r_j \in \mathbb{R}_+$ be the center and radius of B_j, $j = 1, 2$, respectively. Define $\epsilon := \text{dist}(B_1, B_2)$ and suppose $\epsilon \ll 1$. Here, B_1 and B_2 represent the two dielectric inclusions and they are closely spaced, characterized by the asymptotic distance parameter $\epsilon \in \mathbb{R}_+$. By rigid motions if necessary, we can assume without loss of generality that

$$\mathbf{z}_1 = \left(-r_1 - \frac{\epsilon}{2}, 0\right) \quad \text{and} \quad \mathbf{z}_2 = \left(r_2 + \frac{\epsilon}{2}, 0\right). \tag{2.1}$$

In what follows, we set

$$r_1 = r_{1,\alpha_1}\epsilon^{\alpha_1} \quad \text{and} \quad r_2 = r_{2,\alpha_2}\epsilon^{\alpha_2}, \quad \alpha_j \in \mathbb{R},\ j = 1, 2, \tag{2.2}$$

where r_{1,α_1} and r_{2,α_2} are positive constants that are independent of ϵ. It is pointed out that if one takes $\alpha_1 = \alpha_2 = 0$, then both B_1 and B_2 are of regular size. It is emphasized that α_j can be negative or positive, respectively corresponding to the low- and high-curvature cases. Define

$$\alpha_+ = \max(\alpha_1, \alpha_2) \quad \text{and} \quad \alpha_- = \min(\alpha_1, \alpha_2). \tag{2.3}$$

As mentioned earlier, B_1 and B_2 signify two dielectric inclusions embedded in a uniformly homogeneous medium. The medium parameters are characterized by the electric permittivity ε and magnetic permeability μ. By normalization, we assume that $\varepsilon = \mu = 1$ in $\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}$. Let $\varepsilon = \varepsilon_1$ and $\mu = 1$ in $B_1 \cup B_2$, where $\varepsilon_1 \in \mathbb{R}_+$. We consider the transverse magnetic scattering, which is described by the following system (cf. [12, 15, 21]):

$$\begin{cases} \Delta u^* + \omega^2 u^* = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \\ \nabla \cdot (\frac{1}{\varepsilon_1} \nabla u^*) + \omega^2 u^* = 0 \quad \text{in} \quad B_1 \cup B_2, \\ u^*|_+ = u^*|_-, \quad \frac{\partial u^*}{\partial \nu}\Big|_+ = \frac{1}{\varepsilon_1}\frac{\partial u^*}{\partial \nu}\Big|_- \quad \text{on} \quad \partial B_1 \cup \partial B_2, \\ (u^* - u^i)(\mathbf{x}) \quad \text{satisfies the Sommerfeld radiation condition,} \end{cases} \tag{2.4}$$

where $\omega \in \mathbb{R}_+$ signifies the angular frequency of the wave propagation, and u^i and u^*, respectively, denote the incident and total wave fields. u^i is an entire solution to $\Delta u^i + \omega^2 u^i = 0$ in $\mathbb{R}^2$, and one special case is that it is a plane wave of the form $u^i = e^{i\omega \mathbf{x}\cdot\mathbf{d}}$, where $\mathbf{d} \in \mathbb{S}^2$ signifies the impinging direction. By the Sommerfeld radiation condition, we mean that the scattered wave $u^s(\mathbf{x}) = (u^* - u^i)(\mathbf{x})$ satisfies

$$\lim_{|\mathbf{x}|\to+\infty} |\mathbf{x}|^{1/2} \left(\frac{\partial u^s(\mathbf{x})}{\partial |\mathbf{x}|} - i\omega u^s(\mathbf{x}) \right) = 0. \tag{2.5}$$

Throughout the rest of the paper, we shall consider $\omega \ll 1$ and $\varepsilon_1 = \mathcal{O}(\omega)$.

2.2. Main gradient estimate and discussion. We present our main result in this paper as follows.

THEOREM 2.1. *Suppose* $\omega \cdot \epsilon^{\alpha_-} \ll 1$, *and*

$$u^i = u_0^i + \sum_{j=1}^{\infty} \omega^j u_j^i, \tag{2.6}$$

where the functions u_j^i, $j = 0, 1, 2, \ldots,$ *are independent of* ω. *Let* $u^*(\mathbf{x})$ *be defined in* (2.4). *Then for any bounded set* Ω *containing* $\overline{B}_1$ *and* $\overline{B}_2$, *it holds that*

$$\begin{aligned} \|\nabla u^*\|_{L^\infty(\Omega\setminus\overline{B_1\cup B_2})} \sim & \frac{C_0}{r_-}\epsilon^{\min(\alpha_+,1)/2-1/2}\Big(\partial_{\mathbf{x}_1} u^i(\mathbf{0}) + \frac{1}{\pi}\omega^2 |\ln\omega| \int_{B_1\cup B_2} \partial_{\mathbf{x}_1} u^i \\ & + \mathcal{O}(\omega^2)\Big) + \mathcal{O}(1), \end{aligned} \tag{2.7}$$

where r_- *is defined by*

$$r_- = \begin{cases} \frac{\alpha_2-\alpha_-}{\alpha_2-\alpha_1} r_{1,\alpha_1} + \frac{\alpha_1-\alpha_-}{\alpha_1-\alpha_2} r_{2,\alpha_2}, & \alpha_1 \neq \alpha_2, \\ r_{1,\alpha_1} + r_{2,\alpha_2}, & \alpha_1 = \alpha_2, \end{cases} \tag{2.8}$$

and $C_0 > 0$ *is the coefficient of the leading order term of* τ *defined in* (2.18) *in what follows.*

Remark 2.1. It is worth mentioning that if $\alpha_1 = \alpha_2 = 0$, then from (2.18), one has

$$C_0 = \sqrt{2r_{1,0}r_{2,0}(r_{1,0}+r_{2,0})},$$

and there holds the estimate

$$\|\nabla u^*\|_{L^\infty(\Omega\setminus\overline{B_1\cup B_2})} \sim \sqrt{\frac{2r_{1,0}r_{2,0}}{(r_{1,0}+r_{2,0})}}\epsilon^{-1/2}\Big(\partial_{\mathbf{x}_1}u^i(\mathbf{0}) + \frac{1}{\pi}\omega^2|\ln\omega|\int_{B_1\cup B_2}\partial_{\mathbf{x}_1}u^i + \mathcal{O}(\omega^2)\Big) + \mathcal{O}(1),$$

which recovers the blowup estimate for the static case (see [3, 5]).

Remark 2.2. It can be seen that if one inclusion is of high curvature, i.e., $\alpha_+ > 0$ and $\partial_{\mathbf{x}_1}u_0^i(\mathbf{0}) \neq 0$, then the blowup rate is $\epsilon^{\min(\alpha_+,1)/2-1/2}$, which is less than $\epsilon^{-1/2}$. No blow up occurs in the case that $\alpha_+ \geq 1$.

Remark 2.3. We emphasize that the estimate (2.7) also holds for the low curvature case, i.e., $\alpha_+ < 0$. In such a case, the blowup rate is $\epsilon^{\alpha_+/2-1/2}$, which is bigger than $\epsilon^{-1/2}$, if $\partial_{\mathbf{x}_1}u_0^i(\mathbf{0}) \neq 0$. Moreover, even if $\partial_{\mathbf{x}_1}u_0^i(\mathbf{0}) = 0$, one can still have the blowup if $\partial_{\mathbf{x}_1}u_1^i(\mathbf{0}) \neq 0$ and

$$-\log_\epsilon\omega < \alpha_+ < 1 - 2\log_\epsilon\omega,$$

or

$$\int_{B_1\cup B_2}\partial_{\mathbf{x}_1}u_0^i \neq 0,$$

where α_+satisfies

$$-\log_\epsilon\omega < \alpha_+ < 1 - 2\log_\epsilon(\omega^2|\ln\omega|).$$

2.3. Key decompositions. In this subsection, we present the main auxiliary results that we shall derive in order to prove the main result in Theorem 2.1, whose proofs are deferred to subsequent sections. To estimate the gradient field of the solution to (2.4), we shall decompose the system into several parts. We first introduce the following system:

$$\begin{cases} \Delta u + \omega^2 u = 0 & \text{in} \quad \mathbb{R}^2\setminus\overline{B_1\cup B_2},\\ u = \lambda_1 + \mathcal{O}(\omega^2) & \text{on} \quad \partial B_1,\\ u = \lambda_2 + \mathcal{O}(\omega^2) & \text{on} \quad \partial B_2,\\ (u-u^i)(\mathbf{x}) & \text{satisfies the Sommerfeld radiation condition,}\end{cases} \tag{2.9}$$

where the constants λ_j, $j = 1, 2$, are determined by

$$\int_{\partial B_j}\partial_\nu u|_+ = \mathcal{O}(\omega^2), \quad j = 1, 2, \tag{2.10}$$

and they are unique up to $\mathcal{O}(\omega^2)$.

We have the following result.

LEMMA 2.1. *Let u^* and u be the solution to systems* (2.4) *and* (2.9), *respectively. Then it holds that*

$$\nabla u^* = \nabla u + C\omega + \mathcal{O}(\omega^2) \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \tag{2.11}$$

where C is a generic constant that does not depend on ω and ϵ.

In what follows, we shall decompose the solution to (2.9) into two parts as follows:

$$u(\mathbf{x}) = a q_\omega(\mathbf{x}) + b(\mathbf{x}), \tag{2.12}$$

where $q_\omega(\mathbf{x})$ is the solution to

$$\begin{cases} \Delta q_\omega + \omega^2 q_\omega = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \\ q_\omega(\mathbf{x}) \quad \text{satisfies the Sommerfeld radiation condition.} \end{cases} \tag{2.13}$$

The concrete form of q_ω will be shown in the next section. Let q_0 be the singular function defined by

$$q_0(\mathbf{x}) := \frac{1}{2\pi}(\ln|\mathbf{x} - \mathbf{p}_1| - \ln|\mathbf{x} - \mathbf{p}_2|), \tag{2.14}$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ denote the fixed points of the reflection R_1R_2 and R_2R_1, respectively. Here, the reflection R_j with respect to ∂B_j, centering at $\mathbf{z}_j$ and of radius r_j, are defined by

$$R_j(\mathbf{x}) := \frac{r_j^2(\mathbf{x} - \mathbf{z}_j)}{|\mathbf{x} - \mathbf{z}_j|^2} + \mathbf{z}_j, \quad j = 1, 2. \tag{2.15}$$

If $\alpha_1 = \alpha_2 = 0$, then it is proved in [26] that $\mathbf{p}_j$, $j = 1, 2$, admits the following asymptotic expansion:

$$\mathbf{p}_1 = \left(-\sqrt{2}\sqrt{\frac{r_{1,0}r_{2,0}}{r_{1,0} + r_{2,0}}}\sqrt{\epsilon} + \mathcal{O}(\epsilon), 0\right)^T, \quad \mathbf{p}_2 = \left(\sqrt{2}\sqrt{\frac{r_{1,0}r_{2,0}}{r_{1,0} + r_{2,0}}}\sqrt{\epsilon} + \mathcal{O}(\epsilon), 0\right)^T. \tag{2.16}$$

In this paper, we shall consider the case that $\alpha_1, \alpha_2 \neq 0$, and we derive the explicit forms of $\mathbf{p}_1$ and $\mathbf{p}_2$ as follows:

$$\begin{aligned} \mathbf{p}_1 &= \left(-\frac{(r_1 - r_2)\epsilon/2 + \sqrt{\epsilon}\tau(r_1, r_2, \epsilon)}{r_1 + r_2 + \epsilon}, 0\right)^T, \\ \mathbf{p}_2 &= \left(\frac{(r_2 - r_1)\epsilon/2 + \sqrt{\epsilon}\tau(r_1, r_2, \epsilon)}{r_1 + r_2 + \epsilon}, 0\right)^T, \end{aligned} \tag{2.17}$$

where

$$\tau(r_1, r_2, \epsilon) = \sqrt{2r_1r_2(r_1 + r_2) + (r_1^2 + 3r_1r_2 + r_2^2)\epsilon + (r_1 + r_2)\epsilon^2 + \epsilon^3/4}. \tag{2.18}$$

Formula (2.17) can be verified by straightforward computations. It can be seen that q_0 is the solution to the following equation (see [25]):

$$\begin{cases} \Delta q_0 = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \\ q_0 = C_j \quad \text{on} \quad \partial B_j, \\ \displaystyle\int_{\partial B_j} \partial_\nu q_0|_+ = (-1)^j, \quad j = 1, 2, \\ q_0(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}) \quad \text{as} \quad |\mathbf{x}| \to \infty, \end{cases} \tag{2.19}$$

where C_j, $j=1,2$, are

$$C_j = (-1)^{j-1}\frac{1}{2\pi}\ln\frac{-(2r_j+\epsilon)\sqrt{\epsilon}+2\tau}{(2r_j+\epsilon)\sqrt{\epsilon}+2\tau}, \quad j=1,2. \tag{2.20}$$

In what follows, we define $b(\mathbf{x})$ in (2.12) by

$$b(\mathbf{x}) := u(\mathbf{x}) - \frac{\lambda_1-\lambda_2}{C_1-C_2}q_\omega(\mathbf{x}), \tag{2.21}$$

where u, λ_j and C_j, $j=1,2$, are defined in (2.9) and (2.19). We shall prove the following critical result.

LEMMA 2.2. *Suppose $\omega\cdot\epsilon^{\alpha-}\ll 1$. Let $b(\mathbf{x})$ be defined in* (2.21). *Then for any bounded set Ω containing $\overline{B}_1$ and $\overline{B}_2$, there is a constant C which is independent of ϵ and ω such that*

$$\|\nabla b\|_{L^\infty(\Omega\setminus\overline{B_1\cup B_2})} \le C(1+\mathcal{O}(\omega^2)). \tag{2.22}$$

3. Quantitative approximations of the solution.

3.1. Layer potentials. Before the estimation of the gradient field, we introduce some necessary notation and results on the layer potential operators (cf. [15, 16, 14, 17]), which shall be used in our subsequent analysis. Let $\Gamma_\omega(\mathbf{x})$ be the fundamental solution to PDE operator $\Delta+\omega^2$ in $\mathbb{R}^2$, given by

$$\Gamma_\omega(\mathbf{x}) = -\frac{i}{4}H_0^{(1)}(\omega|\mathbf{x}|), \tag{3.1}$$

where $H_0^{(1)}(\omega|\mathbf{x}|)$ is the Hankel function of the first kind and zeroth order. We mention that if $\omega=0$, then $\Gamma_0(\mathbf{x})=\frac{1}{2\pi}\ln|\mathbf{x}|$. For any bounded $C^{2,\alpha}$ domain $B\subset\mathbb{R}^2$, $\alpha>0$, we denote by $\mathcal{S}_B^\omega : L^2(\partial B)\to H^1(\mathbb{R}^2\setminus\partial B)$ the single-layer potential operator given by

$$\mathcal{S}_B^\omega[\phi](\mathbf{x}) := \int_{\partial B}\Gamma_\omega(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\,ds_{\mathbf{y}}, \tag{3.2}$$

and $(\mathcal{K}_B^\omega)^* : L^2(\partial B)\to L^2(\partial B)$ the Neumann–Poincaré operator

$$(\mathcal{K}_B^\omega)^*[\phi](\mathbf{x}) := \text{p.v.}\quad \int_{\partial B}\frac{\partial\Gamma_\omega(\mathbf{x}-\mathbf{y})}{\partial\nu_{\mathbf{x}}}\phi(\mathbf{y})\,ds_{\mathbf{y}}, \tag{3.3}$$

where p.v. stands for the Cauchy principle value. In (3.3) and also in what follows, unless otherwise specified, ν signifies the exterior unit normal vector to the boundary of the concerned domain. We also introduce the double-layer potential $\mathcal{D}_B^\omega : L^2(\partial D)\to H^1(\mathbb{R}^2\setminus\partial B)$ given by

$$\mathcal{D}_B^\omega[\phi](\mathbf{x}) := \int_{\partial B}\frac{\partial\Gamma_\omega(\mathbf{x}-\mathbf{y})}{\partial\nu_{\mathbf{y}}}\phi(\mathbf{y})\,ds_{\mathbf{y}}. \tag{3.4}$$

It is known that the single-layer potential operator $\mathcal{S}_B^\omega$ is continuous across ∂B and satisfies the trace formula

$$\frac{\partial}{\partial\nu}\mathcal{S}_B^\omega[\phi]\Big|_\pm = \left(\pm\frac{1}{2}I+(\mathcal{K}_B^k)^*\right)[\phi] \quad \text{on} \quad \partial B, \tag{3.5}$$

where $\frac{\partial}{\partial \nu}$ stands for the normal derivative and the subscripts $\pm$ indicate the limits from outside and inside of a given inclusion B, respectively. The double-layer potential operator $\mathcal{D}_B^\omega$ satisfies the following trace formula across ∂B:

$$\mathcal{D}_B^\omega[\phi]\Big|_\pm = \left(\mp\frac{1}{2}I + \mathcal{K}_B^k\right)[\phi] \quad \text{on} \quad \partial B. \tag{3.6}$$

When $\omega = 0$ the operators $\mathcal{S}_B^0$ and $\mathcal{D}_B^0$ stand for the single-layer potential operator and double-layer potential with kernel function Γ_0.

3.2. Asymptotic estimates. Recall that the Bessel function $J_0(\omega|\mathbf{x})$ and the Neumann function $Y_0(\omega|\mathbf{x})$ admit the following integral formula (see, e.g., [1]):

$$\begin{aligned} J_0(\omega|\mathbf{x}|) &= \frac{1}{\pi}\int_0^\pi \cos(\omega|\mathbf{x}|\cos\theta)d\theta, \\ Y_0(\omega|\mathbf{x}|) &= \frac{4}{\pi^2}\int_0^{\pi/2} \cos(\omega|\mathbf{x}|\cos\theta)(\gamma + \ln(2\omega|\mathbf{x}|\sin^2\theta))d\theta, \end{aligned} \tag{3.7}$$

where $\gamma = 0.5772\ldots$ is the Euler–Mascheroni constant. The Hankel function appearing in (3.1) can be represented by

$$H_0^{(1)}(\omega|\mathbf{x}|) = -\frac{i}{4}J_0(\omega|\mathbf{x}|) + \frac{1}{4}Y_0(\omega|\mathbf{x}|). \tag{3.8}$$

Note that for ω sufficiently small, one has the asymptotic result

$$\Gamma_\omega(\mathbf{x}) = a_\omega + \Gamma_0(\mathbf{x}) + A_\omega(\mathbf{x}), \tag{3.9}$$

where a_ω is a constant defined by

$$a_\omega := -\frac{i}{4} + \frac{\gamma}{2\pi} + \frac{1}{2\pi}\ln\frac{\omega}{2} = -\frac{i}{4} + \frac{\gamma}{2\pi} - \frac{1}{2\pi}\ln 2 + \frac{1}{2\pi}\ln\omega,$$

and the function $A_\omega(\mathbf{x})$ is defined by

$$\begin{aligned} A_\omega(\mathbf{x}) := {} & \frac{i}{4\pi}|\mathbf{x}|\int_0^\pi \sin(\eta|\mathbf{x}|\cos\theta)\cos\theta d\theta \\ & - \frac{1}{\pi^2}|\mathbf{x}|\int_0^{\pi/2} \sin(\eta|\mathbf{x}|\cos\theta)\cos\theta(\gamma + \ln(2\omega|\mathbf{x}|\sin^2\theta))d\theta, \end{aligned} \tag{3.10}$$

where $\eta \in (0, \omega)$ is some fixed positive number. It is worth mentioning that A_ω is a smooth function in $\mathbb{R}^2$ for any $\omega \in \mathbb{R}_+$. In addition, one has

$$A_\omega(\mathbf{x}) = -\frac{1}{4\pi}|\mathbf{x}|^2\omega^2\ln\omega + \mathcal{O}(\omega^2). \tag{3.11}$$

We define the boundary integral operator $\mathcal{A}_B^\omega$ by

$$\mathcal{A}_B^\omega[\phi](\mathbf{x}) := \int_{\partial B} A_\omega(\mathbf{x} - \mathbf{y})\phi(\mathbf{y})ds_\mathbf{y}. \tag{3.12}$$

In what follows, we let q_ω be the following singular function:

$$q_\omega := \Gamma_\omega(\mathbf{x} - \mathbf{p}_1) - \Gamma_\omega(\mathbf{x} - \mathbf{p}_2) = q_0 + A_\omega(\mathbf{x} - \mathbf{p}_1) - A_\omega(\mathbf{x} - \mathbf{p}_2). \tag{3.13}$$

3.3. First approximation. We next consider the solution to (2.4). By applying the layer potential techniques, one can represent the solution to (2.4) by

$$u^* = \begin{cases} u^i + \mathcal{S}_{B_c}^{\omega}[\varphi_1^*] & \text{in} \quad \mathbb{R}^2 \setminus \overline{B_c}, \\ \mathcal{S}_{B_c}^{k_c}[\varphi_2^*] & \text{in} \quad B_c, \end{cases} \tag{3.14}$$

where $B_c := B_1 \cup B_2$ and $k_c = \omega\sqrt{\varepsilon_1}$. By using the transmission conditions across ∂B_c, there holds

$$\mathbf{A}_{B_c}^{\omega}[\boldsymbol{\varphi}^*] = \boldsymbol{U} \quad \text{on} \quad \partial B_c, \tag{3.15}$$

where the operator $\mathbf{A}_{B_c}^{\omega} : H^{-1/2}(\partial B_c) \times H^{-1/2}(\partial B_c) \to H^{1/2}(\partial B_c) \times H^{-1/2}(\partial B_c)$ is defined by

$$\mathbf{A}_{B_c}^{\omega} := \begin{pmatrix} -\mathcal{S}_{B_c}^{\omega} & \mathcal{S}_{B_c}^{k_c} \\ -\big(\frac{I}{2} + (\mathcal{K}_{B_c}^{\omega})^*\big) & \frac{1}{\varepsilon_1}\big(-\frac{I}{2} + (\mathcal{K}_{B_c}^{k_c})^*\big) \end{pmatrix} \tag{3.16}$$

and

$$\boldsymbol{\varphi}^* = \begin{pmatrix} \varphi_1^* \\ \varphi_2^* \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} u^i \\ \frac{\partial u^i}{\partial \nu} \end{pmatrix}. \tag{3.17}$$

For later use, we define the operator $\mathbb{S}$ by

$$\mathbb{S} := \begin{pmatrix} \mathcal{S}_{B_1}^0|_{\partial B_1} & \mathcal{S}_{B_2}^0|_{\partial B_1} \\ \mathcal{S}_{B_1}^0|_{\partial B_2} & \mathcal{S}_{B_2}^0|_{\partial B_2} \end{pmatrix} \tag{3.18}$$

and the operator $\mathbb{K}^*$ by

$$\mathbb{K}^* := \begin{pmatrix} (\mathcal{K}_{B_1}^0)^* & \partial_{\nu_1}\mathcal{S}_{B_2}^0 \\ \partial_{\nu_2}\mathcal{S}_{B_1}^0 & (\mathcal{K}_{B_2}^0)^* \end{pmatrix}, \tag{3.19}$$

where ν_1 and ν_2 are the unit normal directions to ∂B_1 and ∂B_2, respectively. It can be verified that $\mathbb{S} = \mathcal{S}_{B_c}^0$ and $\mathbb{K}^* = (\mathcal{K}_{B_c}^0)^*$. Similar to Calderón-type identities (cf. [16, 13, 22]), we have the identity

$$\mathbb{S}\mathbb{K}^* = \mathbb{K}\mathbb{S}, \tag{3.20}$$

where $\mathbb{K}$ is the adjoint operator of $\mathbb{K}^*$ given by

$$\mathbb{K} := \begin{pmatrix} \mathcal{K}_{B_1}^0 & \mathcal{D}_{B_2}^0|_{\partial B_1} \\ \mathcal{D}_{B_1}^0|_{\partial B_2} & \mathcal{K}_{B_2}^0 \end{pmatrix}.$$

For completeness and convenient reference to the reader, we shall present the proof to the identity (3.20) in the appendix A.

Proof of Lemma 2.1. By using the asymptotic estimates in the previous section, one can derive the following asymptotic expansions for the layer potentials:

$$\begin{aligned} \mathcal{S}_{B_c}^{\omega}[\varphi] &= a_\omega \int_{\partial B_c} \varphi + \mathcal{S}_{B_c}^0[\varphi] + \mathcal{A}_{B_c}^{\omega}[\varphi], \\ (\mathcal{K}_{B_c}^{\omega})^*[\varphi] &= (\mathcal{K}_{B_c}^0)^*[\varphi] + \partial_\nu \mathcal{A}_{B_c}^{\omega}[\varphi]. \end{aligned} \tag{3.21}$$

By using (3.15) and the definition of a_ω, one has

$$\ln\omega \int_{\partial B_c} \varphi_1^* = \ln(\omega\sqrt{\varepsilon_1}) \int_{\partial B_c} \varphi_2^* + \mathcal{O}(\omega \ln\omega). \tag{3.22}$$

We declare that there holds the decomposition $u^* = u + u'$ in $\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}$, where u' is the solution to

$$\begin{cases} \Delta u' + \omega^2 u' = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \\ u' = \mathcal{O}(\omega) \quad \text{on} \quad \partial B_1 \cup \partial B_2, \\ u'(\mathbf{x}) \quad \text{satisfies the Sommerfeld radiation condition,} \end{cases} \tag{3.23}$$

together with the relation

$$\int_{\partial B_j} \partial_\nu u' = \mathcal{O}(\omega^2), \quad j = 1, 2. \tag{3.24}$$

In fact, by noting that $u^i = u_0^i + u_1^i \omega + \mathcal{O}(\omega^2)$ (see (2.6)) and

$$a_\omega = a_0 + \frac{1}{2\pi} \ln \omega, \quad a_0 := -\frac{i}{4} + \frac{\gamma}{2\pi} - \frac{1}{2\pi} \ln 2,$$

the first equation in (3.15) can be rewritten as

$$a_0 \int_{\partial B_1 \cup \partial B_2} \boldsymbol{\varphi}^* + \mathbb{S}[\boldsymbol{\varphi}^*] + \mathbb{A}^\omega[\boldsymbol{\varphi}^*] = u_0^i + \mathcal{C} \omega \ln \omega + u_1^i \omega + \mathcal{O}(\omega^2) \quad \text{on} \quad \partial B_1 \cup \partial B_2,$$

where $\mathcal{C}$ is some constant. The operator $\mathbb{A}^\omega$ is given by

$$\mathbb{A}^\omega := \begin{pmatrix} \mathcal{A}_{B_1}^\omega|_{\partial B_1} & \mathcal{A}_{B_2}^\omega|_{\partial B_1} \\ \mathcal{A}_{B_1}^\omega|_{\partial B_2} & \mathcal{A}_{B_2}^\omega|_{\partial B_2} \end{pmatrix}. \tag{3.25}$$

By (3.11), one can show that $\mathbb{A}^\omega = \mathcal{O}(\omega^2 \ln \omega)$. Thus we can assume that φ_1^* and φ_2^* admit the following asymptotic expansions:

$$\varphi_1^* = \varphi_{1,0}^* + \omega \ln \omega \varphi_{1,1}^* + \omega \varphi_{1,2}^* + \omega^2 \ln \omega \varphi_{1,3}^* + \mathcal{O}(\omega^2),$$
$$\varphi_2^* = \varphi_{2,0}^* + \omega \ln \omega \varphi_{2,1}^* + \omega \varphi_{2,2}^* + \omega^2 \ln \omega \varphi_{2,3}^* + \mathcal{O}(\omega^2).$$

It then follows from (3.15) and the asymptotic expansion (3.21) that

$$\left(-\frac{I}{2} + \mathbb{K}^*\right)[\varphi_{2,0}^* + \omega \ln \omega \varphi_{2,1}^*] = 0, \quad \int_{\partial B_j} \varphi_1^* = \mathcal{O}(\omega^2). \tag{3.26}$$

Thus one has

$$\mathbb{S}[\varphi_{2,0}^*] = \lambda_{1,1} \chi(\partial B_1) + \lambda_{2,1} \chi(\partial B_2), \quad \mathbb{S}[\varphi_{2,1}^*] = \lambda_{1,2} \chi(\partial B_1) + \lambda_{2,2} \chi(\partial B_2), \tag{3.27}$$

where $\lambda_{j,l}$, $j, l = 1, 2$, are constants. It follows by straightforward computations that

$$\begin{aligned} \mathcal{S}_{B_c}^0[\varphi_{2,2}^*] &= -2\frac{\varepsilon_1}{\omega} \left(\mathcal{S}_{B_c}^0[\varphi_{2,0}^*] + \mathcal{S}_{B_c}^0[\partial_\nu u_0^i] - \left(-\frac{I}{2} + \mathcal{K}_{B_c}^0\right)[u_0^i] \right), \\ \mathcal{S}_{B_c}^0[\varphi_{2,3}^*] &= -2\frac{\varepsilon_1}{\omega} \mathcal{S}_{B_c}^0[\varphi_{2,1}^*]. \end{aligned} \tag{3.28}$$

One thus has

$$\begin{aligned} \mathcal{S}_{B_c}^{k_c}[\varphi_2^*] &= \mathcal{S}_{B_c}^0[\varphi_{2,0}^* + \omega \ln \omega \varphi_{2,1}^* + \omega^2 \ln \omega \varphi_{2,3}^*] + \omega \mathcal{S}_{B_c}^0[\varphi_{2,2}^*] + \mathcal{O}(\omega^2) \\ &= \begin{cases} \lambda_1 - 2\varepsilon_1 \mathcal{S}_{B_c}^0[\partial_\nu u_0^i] - \varepsilon_1 u_0^i + \mathcal{O}(\omega^2) & \text{on} \quad \partial B_1, \\ \lambda_2 - 2\varepsilon_1 \mathcal{S}_{B_c}^0[\partial_\nu u_0^i] - \varepsilon_1 u_0^i + \mathcal{O}(\omega^2) & \text{on} \quad \partial B_2. \end{cases} \end{aligned} \tag{3.29}$$

One can thus set

$$u = u^i + \mathcal{S}_{B_c}^{\omega}[\varphi_1^*] - 2\varepsilon_1 \mathcal{S}_{B_c}^0[\partial_\nu u_0^i] - \varepsilon_1 u_0^i + \mathcal{O}(\omega^2),$$

and the higher order term is arranged such that $\Delta u + \omega^2 u = 0$ holds in $\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}$. Now it is readily verified that $u' = u^* - u$ satisfies (3.23). More precisely, one has

$$u' = -2\varepsilon_1 \mathcal{S}_{B_c}^0[\partial_\nu u_0^i] - \varepsilon_1 u_0^i + \mathcal{O}(\omega^2) \quad \text{on} \quad \partial B_1 \cup \partial B_2.$$

Suppose $u' = \varepsilon_1 u_1' + \mathcal{O}(\omega^2)$, where u_1' is the solution to

$$\begin{cases} \Delta u_1' = 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \\ u_1' = -2\mathcal{S}_{B_c}^0[\partial_\nu u_0^i] - u_0^i & \text{on} \quad \partial B_1 \cup \partial B_2, \\ u_1'(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}). \end{cases} \tag{3.30}$$

We mention that $\nabla u_1'$ is uniformly bounded with respect to the distance ϵ. In fact, the solution to (3.30) can be represented by

$$u_1' = \mathcal{S}_{B_c}[\varphi'](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2 \setminus \overline{B_1 \cup B_2},$$

where the φ' satisfy

$$\int_{\partial B_c} \varphi' = 0$$

and

$$\left(-\frac{I}{2} + \mathbb{K}^*\right)[\varphi'] = -2\mathbb{K}^*[\partial_\nu u_0^i] \quad \text{on} \quad \partial B_c. \tag{3.31}$$

One can show that

$$u_1'(\zeta_1) - u_1'(-\zeta_1) = \epsilon(2\partial_\nu u_0^i(\zeta_1) - \partial_\nu u_0^i(-\zeta_1)) + \mathcal{O}(\epsilon^2),$$

where $\zeta_1 = (\frac{\epsilon}{2}, 0)$. One thus has

$$\begin{aligned} \nabla u_1'(\zeta_1) &= \nu(\zeta_1) \cdot \nabla u_1'(\zeta_1)\nu(\zeta_1) + \partial_T u_1'(\zeta_1) T(\zeta_1) \\ &= \partial_{\mathbf{x}_1} u_1'(\zeta_1)(-1, 0) + \partial_T u_1'(\zeta_1) T(\zeta_1) \\ &= (-1, 0) \lim_{\epsilon \to 0} \frac{u_1'(-\zeta_1) - u_1'(\zeta_1)}{\epsilon} - (0, 1)\partial_T(2\mathcal{S}_{B_c}^0[\partial_\nu u_0^i] + u_0^i)(\zeta_1) \\ &= -2\nabla u_0^i(\zeta_1) + \nabla u_0^i(-\zeta_1) + \mathcal{O}(\epsilon), \end{aligned} \tag{3.32}$$

where ∂_T stands for the tangential derivative and T is the unit tangential vector. Hence u_1' is uniformly bounded and the proof is complete. ∎

3.4. Further approximation. In order to prove the main result, we need to estimate the key quantities on the right-hand side of (2.12), where

$$a = \frac{\lambda_1 - \lambda_2}{C_1 - C_2}.$$

By using (3.11), one has

$$
\begin{aligned}
\lambda_2-\lambda_1 &= \int_{\partial B_2} u\partial_\nu q_0 + \int_{\partial B_1} u\partial_\nu q_0 + \mathcal{O}(\omega^2)\\
&= \int_{\partial B_2}(u-u^i)\partial_\nu q_\omega + \int_{\partial B_1}(u-u^i)\partial_\nu q_\omega + \int_{\partial B_1\cup\partial B_2} u^i\partial_\nu q_0\\
&\quad + \frac{1}{2\pi}\omega^2\ln\omega\int_{B_1\cup B_2}\nabla u^i\cdot(\mathbf{p}_1-\mathbf{p}_2)+\mathcal{O}(\omega^2)\\
&= \int_{\partial B_1\cup\partial B_2}\partial_\nu(u-u^i)q_\omega + \int_{\partial B_1\cup\partial B_2} u^i\partial_\nu q_0 + \mathcal{O}(\omega^2)\\
&= \frac{1}{\pi}\omega^2\ln\omega\int_{B_1\cup B_2}\nabla u^i\cdot(\mathbf{p}_1-\mathbf{p}_2) + \int_{\partial B_1\cup\partial B_2} u^i\partial_\nu q_0 + \mathcal{O}(\omega^2)\\
&= u^i(\mathbf{p}_1)-u^i(\mathbf{p}_2)+\frac{1}{\pi}\omega^2\ln\omega\int_{B_1\cup B_2}\nabla u^i\cdot(\mathbf{p}_1-\mathbf{p}_2)+\mathcal{O}(\omega^2),
\end{aligned}
\tag{3.33}
$$

where we have used the results

$$
\begin{aligned}
&\int_{\partial B_1\cup\partial B_2}(u-u^i)\partial_\nu(q_\omega-q_0)\\
&\quad= \frac{1}{4\pi}\omega^2\ln\omega\int_{\partial B_1\cup\partial B_2}(u-u^i)\partial_\nu(|\mathbf{x}-\mathbf{p}_2|^2-|\mathbf{x}-\mathbf{p}_1|^2)\\
&\quad= \frac{1}{2\pi}\omega^2\ln\omega\int_{\partial B_1\cup\partial B_2}(u-u^i)\nu\cdot(\mathbf{p}_1-\mathbf{p}_2)\\
&\quad= \frac{1}{2\pi}\omega^2\ln\omega\int_{B_1\cup B_2}\nabla u^i\cdot(\mathbf{p}_1-\mathbf{p}_2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\partial B_1\cup\partial B_2}\partial_\nu(u-u^i)(q_\omega-q_0)\\
&\quad= \frac{1}{2\pi}\omega^2\ln\omega\left(r_1^2\int_{\partial B_1}\partial_\nu(u-u^i)\frac{\mathbf{x}-\mathbf{z}_1}{|\mathbf{x}-\mathbf{z}_1|^2}+r_2^2\int_{\partial B_2}\partial_\nu(u-u^i)\frac{\mathbf{x}-\mathbf{z}_2}{|\mathbf{x}-\mathbf{z}_2|^2}\right)\\
&\qquad\cdot(\mathbf{p}_1-\mathbf{p}_2)+\mathcal{O}(\omega^2) = \frac{1}{2\pi}\omega^2\ln\omega\int_{B_1\cup B_2}\nabla u^i\cdot(\mathbf{p}_1-\mathbf{p}_2)+\mathcal{O}(\omega^2).
\end{aligned}
$$

Moreover, one has

$$
\begin{aligned}
\nabla q_\omega &= \nabla q_0 + \omega^2\ln\omega\frac{1}{2\pi}(\mathbf{p}_1-\mathbf{p}_2)+\mathcal{O}(\omega^2)\\
&= \frac{1}{2\pi}\left(\frac{\mathbf{x}-\mathbf{p}_1}{|\mathbf{x}-\mathbf{p}_1|^2}-\frac{\mathbf{x}-\mathbf{p}_2}{|\mathbf{x}-\mathbf{p}_2|^2}\right)+\omega^2\ln\omega\frac{1}{2\pi}(\mathbf{p}_1-\mathbf{p}_2)+\mathcal{O}(\omega^2).
\end{aligned}
\tag{3.34}
$$

4. Estimate of $b(\mathbf{x})$.

By definition of (2.21), one finds that $b(\mathbf{x})$ is the solution to

$$
\begin{cases}
\Delta b+\omega^2 b = 0 & \text{in } \mathbb{R}^2\setminus\overline{B_1\cup B_2},\\
b = \left(\lambda_2 C_1-\lambda_1 C_2\right)/(C_1-C_2) & \text{on } \partial B_1\cup\partial B_2,\\
(b-u^i)(\mathbf{x}) & \text{satisfies the Sommerfeld radiation condition.}
\end{cases}
\tag{4.1}
$$

By using layer potential techniques, one can represent b in (4.1) by

$$
b(\mathbf{x}) = u^i(\mathbf{x}) + \mathcal{S}^\omega_{B_1}[\varphi_1](\mathbf{x}) + \mathcal{S}^\omega_{B_2}[\varphi_2](\mathbf{x}),
\tag{4.2}
$$

where $\varphi_1 \in L^2(\partial B_1)$ and $\varphi_2 \in L^2(\partial B_2)$ satisfy

$$u^i(\mathbf{x}) + \mathcal{S}^\omega_{B_1}[\varphi_1](\mathbf{x}) + \mathcal{S}^\omega_{B_2}[\varphi_2](\mathbf{x}) = \tilde{C}_1, \quad \mathbf{x} \in \partial B_1 \cup \partial B_2, \tag{4.3}$$

with $\tilde{C}_1 := \big(\lambda_2 C_1 - \lambda_1 C_2\big)/(C_1 - C_2)$.

Note that it is proved in [18] that $\nabla b(\mathbf{x})$ is uniformly bounded if $\omega = 0$. We need some further analysis on the solution b. First, by using (4.3) and the expansion (3.8) one has

$$a_\omega \int_{\partial B_1 \cup \partial B_2} \boldsymbol{\varphi} + \mathbb{S}[\boldsymbol{\varphi}] + \mathbb{A}^\omega[\boldsymbol{\varphi}] = \tilde{C}_1 - u^i \quad \text{on} \quad \partial B_1 \cup \partial B_2 \tag{4.4}$$

for ω sufficiently small. Here $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ and the operator $\mathbb{S}$ is given by (3.18). By using the definition of a_ω there holds

$$\int_{\partial B_1 \cup \partial B_2} \boldsymbol{\varphi} = \mathcal{O}(\omega). \tag{4.5}$$

Suppose $\varphi_j = \varphi_{j,0} + \mathcal{O}(\omega)$, $j = 1, 2$. Direct asymptotic analysis shows that

$$b(\mathbf{x}) = u^i(\mathbf{x}) + b_0(\mathbf{x}) + \mathcal{O}(\omega^2), \tag{4.6}$$

where $b_0 = \mathcal{S}^0_{B_1}[\varphi_{1,0}](\mathbf{x}) + \mathcal{S}^0_{B_2}[\varphi_{2,0}](\mathbf{x})$ is the harmonic function which satisfies

$$\begin{cases} \Delta b_0 = 0 & \text{in} \quad \mathbb{R}^2 \setminus \overline{B_1 \cup B_2}, \\ b_0 = \tilde{C}_1 - u^i & \text{on} \quad \partial B_1 \cup \partial B_2, \\ b_0(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1}). \end{cases} \tag{4.7}$$

Proof of Lemma 2.2. By the asymptotic result in (4.6), it is sufficient to prove that ∇b_0, where b_0 is the solution to (4.7), is uniformly bounded in $\mathbb{R}^2 \setminus (B_1 \cup B_2)$. Since ∇b_0 is harmonic in $\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}$, and $\nabla b_0 = \mathcal{O}(|\mathbf{x}|^{-2})$, the function $|\nabla b_0|_{l^\infty}$ in $\mathbb{R}^2 \setminus (B_1 \cup B_2)$ attains its maximum on the boundary $\partial B_1 \cup \partial B_2$. Note that b_0 is smooth on $\mathbb{R}^2 \setminus \overline{B_1 \cup B_2}$. It is enough to show that ∇b_0 is uniformly bounded with respect to ϵ on the two points ζ_1 and $-\zeta_1$, where $\zeta_1 = (\frac{\epsilon}{2}, 0)$. Since $b_0(\zeta_1) = \tilde{C}_1 - u^i(\zeta_1)$, one has

$$\begin{aligned} \nabla b_0(\zeta_1) &= \nu(\zeta_1) \cdot \nabla b_0(\zeta_1) \nu(\zeta_1) + \partial_T b_0(\zeta_1) T(\zeta_1) \\ &= \partial_{\mathbf{x}_1} b_0(\zeta_1)(-1, 0) + \partial_T u^i(\zeta_1) T(\zeta_1) \\ &= (-1, 0) \lim_{\epsilon \to 0} \frac{b_0(-\zeta_1) - b_0(\zeta_1)}{\epsilon} + (0, 1)\partial_T u^i(\zeta_1) \\ &= \nabla u^i(\zeta_1), \end{aligned} \tag{4.8}$$

where ∂_T stands for the tangential derivative and T is the unit tangential vector. Since $\nabla u^i(\zeta_1)$ is uniformly bounded, one thus has verified that $\nabla b_0(\zeta_1)$ is uniformly bounded with respect to ϵ. Similarly, one can show that $\nabla b_0(-\zeta_1)$ is also uniformly bounded. ∎

Remark 4.1. We mention that the bound on ∇b_0 can be shown by following an argument similar to that in [5] and [18]. Here, we provide a different proof.

5. Estimate of $q_\omega(\mathbf{x})$. In this section, we shall estimate the singular function $q_\omega(\mathbf{x})$. The asymptotic result (3.34) shows that one only needs to estimate ∇q_0. We mention that if r_1 and r_2 are constants which do not depend on ϵ, the estimate of ∇q_0 is well settled in [18]. We shall consider the case that r_1 and r_2 depend on ϵ. Note that

$$|\mathbf{x}-\mathbf{p}_1| \geq \frac{-(2r_2+\epsilon)\epsilon+\sqrt{\epsilon}\tau}{r_1+r_2+\epsilon} \quad \text{and} \quad |\mathbf{x}-\mathbf{p}_2| \geq \frac{-(2r_1+\epsilon)\epsilon+\sqrt{\epsilon}\tau}{r_1+r_2+\epsilon}$$

hold for $\mathbf{x}\in\mathbb{R}^2\setminus\overline{B_1\cup B_2}$. It follows that

$$\begin{aligned} &\left\|\frac{\mathbf{x}-\mathbf{p}_1}{|\mathbf{x}-\mathbf{p}_1|^2}-\frac{\mathbf{x}-\mathbf{p}_2}{|\mathbf{x}-\mathbf{p}_2|^2}\right\|_{L^\infty(\mathbb{R}^2\setminus\overline{B_1\cup B_2})} \\ &\quad\leq \left\|\frac{\mathbf{x}-\mathbf{p}_1}{|\mathbf{x}-\mathbf{p}_1|^2}\right\|_{L^\infty(\mathbb{R}^2\setminus\overline{B_1\cup B_2})}+\left\|\frac{\mathbf{x}-\mathbf{p}_2}{|\mathbf{x}-\mathbf{p}_2|^2}\right\|_{L^\infty(\mathbb{R}^2\setminus\overline{B_1\cup B_2})} \\ &\quad\leq \Big(\frac{1}{-(2r_1+\epsilon)\epsilon/2+\sqrt{\epsilon}\tau}+\frac{1}{-(2r_2+\epsilon)\epsilon/2+\sqrt{\epsilon}\tau}\Big)(r_1+r_2+\epsilon). \end{aligned} \tag{5.1}$$

On the other hand, setting $\mathbf{x}=(\frac{\epsilon}{2},0)^T$, one has

$$\begin{aligned} &\left|\frac{\mathbf{x}-\mathbf{p}_1}{|\mathbf{x}-\mathbf{p}_1|^2}-\frac{\mathbf{x}-\mathbf{p}_2}{|\mathbf{x}-\mathbf{p}_2|^2}\right|_{l^\infty} \\ &\quad= \Big(\frac{1}{-(2r_1+\epsilon)\epsilon/2+\sqrt{\epsilon}\tau}+\frac{1}{(2r_1+\epsilon)\epsilon/2+\sqrt{\epsilon}\tau}\Big)(r_1+r_2+\epsilon). \end{aligned}$$

Similarly, setting $\mathbf{x}=(-\frac{\epsilon}{2},0)^T$, one has

$$\begin{aligned} &\left|\frac{\mathbf{x}-\mathbf{p}_1}{|\mathbf{x}-\mathbf{p}_1|^2}-\frac{\mathbf{x}-\mathbf{p}_2}{|\mathbf{x}-\mathbf{p}_2|^2}\right|_{l^\infty} \\ &\quad= \Big(\frac{1}{-(2r_2+\epsilon)\epsilon/2+\sqrt{\epsilon}\tau}+\frac{1}{(2r_2+\epsilon)\epsilon/2+\sqrt{\epsilon}\tau}\Big)(r_1+r_2+\epsilon). \end{aligned}$$

Thus there holds

$$\frac{r_1+r_2+\epsilon}{-(\max(r_1,r_2)+\epsilon/2)\epsilon+\sqrt{\epsilon}\tau} \leq \|\nabla q_0\|_{L^\infty(\mathbb{R}^2\setminus\overline{B_1\cup B_2})} \leq 2\frac{r_1+r_2+\epsilon}{-(\max(r_1,r_2)+\epsilon/2)\epsilon+\sqrt{\epsilon}\tau}. \tag{5.2}$$

Proof of Theorem 2.1. It is sufficient to show the estimation for ∇u in $\Omega\setminus\overline{B_1\cup B_2}$. By using (2.19) one has

$$C_1-C_2=\frac{1}{2\pi}\left(\ln\left(1-\frac{2(r_1+\epsilon/2)\sqrt{\epsilon}}{(r_1+\epsilon/2)\sqrt{\epsilon}+\tau}\right)+\ln\left(1-\frac{2(r_2+\epsilon/2)\sqrt{\epsilon}}{(r_2+\epsilon/2)\sqrt{\epsilon}+\tau}\right)\right). \tag{5.3}$$

First, if $\alpha_- < 1$, then one has

$$\tau = C_0\epsilon^{\alpha_-+\min(\alpha_+,1)/2}(1+o(1)), \tag{5.4}$$

where $C_0>0$ does not depend on ϵ and is a generic constant which may vary for different choice of α_1 and α_2. It follows that

$$C_1-C_2=\begin{cases} -2\frac{r_-}{C_0}\epsilon^{1/2-\alpha_+/2}(1+o(1)), & \alpha_+<1, \\ \ln\left(1-\frac{2r_-}{r_-+C_0}\right), & \alpha_+\geq 1, \end{cases}$$

where α_- is defined as in (2.8). By (5.2), one has

$$\|\nabla q_0\|_{L^\infty(\mathbb{R}^2\setminus\overline{B_1\cup B_2})} \sim \begin{cases} \frac{r_-}{C_0}\epsilon^{-\frac{1}{2}-\frac{\alpha_+}{2}}, & \alpha_+ < 1, \\ \frac{r_-}{C_0}\epsilon^{-1}, & \alpha_+ \geq 1. \end{cases}$$

Finally by (3.33), one can derive that

$$\begin{aligned} \lambda_1 - \lambda_2 &= \nabla u^i(\mathbf{p}_2)\cdot(\mathbf{p}_2-\mathbf{p}_1) + \frac{1}{\pi}\omega^2\ln\omega\int_{B_1\cup B_2}\nabla u^i\cdot(\mathbf{p}_2-\mathbf{p}_1) + \mathcal{O}(\omega^2+|\mathbf{p}_1-\mathbf{p}_2|^2) \\ &= 2\frac{1}{r_-}\epsilon^{1/2+\min(\alpha_+,1)/2}(1+o(1))\left(\partial_{\mathbf{x}_1}u^i(\mathbf{0}) + \frac{1}{\pi}\omega^2\ln\omega\int_{B_1\cup B_2}\partial_{\mathbf{x}_1}u^i + \mathcal{O}(\omega^2)\right), \end{aligned}$$

which together with Lemma 2.2 further yields that

(5.5)

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\sim \left|\frac{\lambda_1-\lambda_2}{C_1-C_2}\right|\|\nabla q_\omega\|_{L^\infty} + \mathcal{O}(\omega^2) \\ &\sim \frac{C_0}{r_-}\epsilon^{\min(\alpha_+,1)/2-1/2}\left(\partial_{\mathbf{x}_1}u^i(\mathbf{0}) + \frac{1}{\pi}\omega^2|\ln\omega|\int_{B_1\cup B_2}\partial_{\mathbf{x}_1}u^i + \mathcal{O}(\omega^2)\right) + \mathcal{O}(1). \end{aligned}$$

In the above estimation, L^∞ stands for $L^\infty(\Omega\setminus\overline{B_1\cup B_2})$. Similarly, if $\alpha_- \geq 1$, then one can derive that

$$\tau = C_0\epsilon^{3/2}(1+o(1)).$$

The estimates $C_1 - C_2 = \mathcal{O}(1)$, $\|\nabla q_0\|_{L^\infty(\mathbb{R}^2\setminus\overline{B_1\cup B_2})} = O(\epsilon^{-1})$, and $\mathbf{p}_2 - \mathbf{p}_1 = O(\epsilon)$ then follow, and thus $\|\nabla u\|_{L^\infty(\Omega\setminus\overline{B_1\cup B_2})}$ is uniformly bounded. The proof is complete. ∎

Appendix A. Calderón-type identity. In this appendix, we prove the Calderón-type identity (3.20). By straightforward computations one can show that

$$\mathbb{SK}^* = \begin{pmatrix} \mathcal{S}^0_{B_1}(\mathcal{K}^0_{B_1})^*|_{\partial B_1} + \mathcal{S}^0_{B_2}\partial_{\nu_2}\mathcal{S}^0_{B_1}|_{\partial B_1} & \mathcal{S}^0_{B_1}\partial_{\nu_1}\mathcal{S}^0_{B_2}|_{\partial B_1} + \mathcal{S}^0_{B_2}(\mathcal{K}^0_{B_2})^*|_{\partial B_1} \\ \mathcal{S}^0_{B_1}(\mathcal{K}^0_{B_1})^*|_{\partial B_2} + \mathcal{S}^0_{B_2}\partial_{\nu_2}\mathcal{S}^0_{B_1}|_{\partial B_2} & \mathcal{S}^0_{B_1}\partial_{\nu_1}\mathcal{S}^0_{B_2}|_{\partial B_2} + \mathcal{S}^0_{B_2}(\mathcal{K}^0_{B_2})^*|_{\partial B_2} \end{pmatrix}$$

and

$$\mathbb{SK}^* = \begin{pmatrix} \mathcal{K}^0_{B_1}\mathcal{S}^0_{B_1} + \mathcal{D}^0_{B_2}\mathcal{S}^0_{B_1}|_{\partial B_1} & \mathcal{K}^0_{B_1}\mathcal{S}^0_{B_2} + \mathcal{D}^0_{B_2}\mathcal{S}^0_{B_2}|_{\partial B_1} \\ \mathcal{D}^0_{B_1}\mathcal{S}^0_{B_1}|_{\partial B_2} + \mathcal{K}^0_{B_2}\mathcal{S}^0_{B_1} & \mathcal{D}^0_{B_1}\mathcal{S}^0_{B_2}|_{\partial B_2} + \mathcal{K}^0_{B_2}\mathcal{S}^0_{B_2} \end{pmatrix}.$$

Note that there holds the Calderón identity

$$\mathcal{S}^0_{B_1}(\mathcal{K}^0_{B_1})^*|_{\partial B_1} = \mathcal{K}^0_{B_1}\mathcal{S}^0_{B_1}, \quad \mathcal{S}^0_{B_2}(\mathcal{K}^0_{B_2})^*|_{\partial B_2} = \mathcal{K}^0_{B_2}\mathcal{S}^0_{B_2}.$$

We first show the identity

$$\mathcal{S}^0_{B_2}\partial_{\nu_2}\mathcal{S}^0_{B_1}|_{\partial B_1} = \mathcal{D}^0_{B_2}\mathcal{S}^0_{B_1}|_{\partial B_1}.$$

In fact, letting $\varphi \in L^2(\partial B_1)$ and through integration by parts, there holds

$$\begin{aligned}
\mathcal{S}^0_{B_2}\partial_{\nu_2}\mathcal{S}^0_{B_1}[\varphi](\mathbf{x}) &= \int_{\partial B_2}\Gamma_0(\mathbf{x}-\mathbf{y})\int_{\partial B_1}\frac{\partial\Gamma_0(\mathbf{y}-\mathbf{z})}{\partial_{\nu_\mathbf{y}}}\varphi(\mathbf{z})ds_\mathbf{z}ds_\mathbf{y}\\
&= \int_{\partial B_1}\int_{\partial B_2}\Gamma_0(\mathbf{x}-\mathbf{y})\frac{\partial\Gamma_0(\mathbf{y}-\mathbf{z})}{\partial_{\nu_\mathbf{y}}}ds_\mathbf{y}\varphi(\mathbf{z})ds_\mathbf{z}\\
&= \int_{\partial B_1}\int_{\partial B_2}\frac{\partial\Gamma_0(\mathbf{x}-\mathbf{y})}{\partial_{\nu_\mathbf{y}}}\Gamma_0(\mathbf{y}-\mathbf{z})ds_\mathbf{y}\varphi(\mathbf{z})ds_\mathbf{z}\\
&= \int_{\partial B_2}\frac{\partial\Gamma_0(\mathbf{x}-\mathbf{y})}{\partial_{\nu_\mathbf{y}}}\int_{\partial B_1}\Gamma_0(\mathbf{y}-\mathbf{z})\varphi(\mathbf{z})ds_\mathbf{z}ds_\mathbf{y} = \mathcal{D}^0_{B_2}\mathcal{S}^0_{B_1}[\varphi](\mathbf{x})
\end{aligned}$$

for any $\mathbf{x}\in\partial B_1$. Next, by integration by parts again, one has

$$\begin{aligned}
&\mathcal{S}^0_{B_1}\partial_{\nu_1}\mathcal{S}^0_{B_2}[\varphi](\mathbf{x}) + \mathcal{S}^0_{B_2}(\mathcal{K}^0_{B_2})^*[\varphi](\mathbf{x})\\
&= \int_{\partial B_1}\Gamma_0(\mathbf{x}-\mathbf{y})\int_{\partial B_2}\frac{\partial\Gamma_0(\mathbf{y}-\mathbf{z})}{\partial_{\nu_\mathbf{y}}}\varphi(\mathbf{z})ds_\mathbf{z}ds_\mathbf{y}\\
&\quad + \int_{\partial B_2}\Gamma_0(\mathbf{x}-\mathbf{y})\int_{\partial B_2}\frac{\partial\Gamma_0(\mathbf{y}-\mathbf{z})}{\partial_{\nu_\mathbf{y}}}\varphi(\mathbf{z})ds_\mathbf{z}\Big|_-ds_\mathbf{y} + \frac{1}{2}\int_{\partial B_2}\Gamma_0(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})\Big|_-ds_\mathbf{y}\\
&= \int_{\partial B_2}\int_{\partial B_1}\frac{\partial\Gamma_0(\mathbf{x}-\mathbf{y})}{\partial_{\nu_\mathbf{y}}}\Big|_-\Gamma_0(\mathbf{y}-\mathbf{z})ds_\mathbf{y}\varphi(\mathbf{z})ds_\mathbf{z} - \frac{1}{2}\int_{\partial B_2}\Gamma_0(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})\Big|_-ds_\mathbf{y}\\
&\quad + \int_{\partial B_2}\int_{\partial B_2}\frac{\partial\Gamma_0(\mathbf{x}-\mathbf{y})}{\partial_{\nu_\mathbf{y}}}\Gamma_0(\mathbf{y}-\mathbf{z})ds_\mathbf{y}\varphi(\mathbf{z})ds_\mathbf{z}\\
&= \mathcal{K}^0_{B_1}\mathcal{S}^0_{B_2}[\varphi](\mathbf{x}) + \mathcal{D}^0_{B_2}\mathcal{S}^0_{B_2}[\varphi](\mathbf{x})
\end{aligned}$$

for any $\mathbf{x}\in\partial B_1$. Similarly, one can show that

$$\mathcal{S}^0_{B_1}\partial_{\nu_1}\mathcal{S}^0_{B_2}|_{\partial B_2} = \mathcal{D}^0_{B_1}\mathcal{S}^0_{B_2}|_{\partial B_2}$$

and

$$\mathcal{S}^0_{B_1}(\mathcal{K}^0_{B_1})^*|_{\partial B_2} + \mathcal{S}^0_{B_2}\partial_{\nu_2}\mathcal{S}^0_{B_1}|_{\partial B_2} = \mathcal{D}^0_{B_1}\mathcal{S}^0_{B_1}|_{\partial B_2} + \mathcal{K}^0_{B_2}\mathcal{S}^0_{B_1}.$$

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