ON AN INVERSE PROBLEM FOR THE PLATE EQUATION WITH PASSIVE MEASUREMENT

Gao, Yixian; Liu, Hongyu; Liu, Yang

Published in:
SIAM Journal on Applied Mathematics

Published: 01/01/2023

Document Version:
Final Published version, also known as Publisher's PDF, Publisher's Final version or Version of Record

Publication record in CityU Scholars:
Go to record

Published version (DOI):
10.1137/22M1499881

Publication details:
Gao, Y., Liu, H., & Liu, Y. (2023). ON AN INVERSE PROBLEM FOR THE PLATE EQUATION WITH PASSIVE MEASUREMENT. *SIAM Journal on Applied Mathematics*, *83*(3), 1196-1214. Advance online publication. https://doi.org/10.1137/22M1499881

SIAM J. APPL. MATH. © 2023 Society for Industrial and Applied Mathematics
Vol. 83, No. 3, pp. 1196–1214

ON AN INVERSE PROBLEM FOR THE PLATE EQUATION WITH PASSIVE MEASUREMENT*

YIXIAN GAO†, HONGYU LIU‡, AND YANG LIU§

Abstract. This paper focuses on an inverse problem associated with the plate equation which is derived from models in fluid mechanics and elasticity. We establish the unique identifying results in simultaneously determining both the unknown density and the internal sources from the passive boundary measurement. The proof mainly relies on the asymptotic analysis and harmonic analysis on integral transforms.

Key words. plate equation, density, internal sources, passive boundary measurement

MSC codes. 35R30, 31B10, 74K20

DOI. 10.1137/22M1499881

1. Introduction. Let Ω be a compact set in $\mathbb{R}^3$ such that $\mathbb{R}^3\backslash\Omega$ is connected. Consider the following plate equation:

$$\begin{cases}\rho(\boldsymbol{x})\partial_t^2 u(t,\boldsymbol{x})+\Delta^2 u(t,\boldsymbol{x})=0, & (t,\boldsymbol{x})\in\mathbb{R}_+\times\mathbb{R}^3,\\ u(0,\boldsymbol{x})=f(\boldsymbol{x}), \quad \partial_t u(0,\boldsymbol{x})=g(\boldsymbol{x}), & \boldsymbol{x}\in\mathbb{R}^3,\end{cases} \tag{1.1}$$

where the sources $f(\boldsymbol{x}), g(\boldsymbol{x})\in L^\infty(\mathbb{R}^3)$ with $\mathrm{supp}(f)\subset\Omega$ and $\mathrm{supp}(g)\subset\Omega$ and the density function $\rho(\boldsymbol{x})\in L^\infty(\mathbb{R}^3)$ is nonnegative and $\mathrm{supp}(\rho(\boldsymbol{x})-1)\subset\Omega$. Associated with (1.1), we introduce the boundary measurement as follows:

$$\Lambda_{\rho,f,g}(t,\boldsymbol{x})=(u(t,\boldsymbol{x}),\,\Delta u(t,\boldsymbol{x})), \quad (t,\boldsymbol{x})\in\mathbb{R}_+\times\partial\Omega. \tag{1.2}$$

The inverse problem that we are concerned with is to determine both the unknown density $\rho(\boldsymbol{x})$ and the internal sources $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ simultaneously by knowledge of $\Lambda_{\rho,f,g}(t,\boldsymbol{x})$, $(t,\boldsymbol{x})\in\mathbb{R}_+\times\partial\Omega$. That is,

$$\Lambda_{\rho,f,g}(t,\boldsymbol{x})\big|_{(t,\boldsymbol{x})\in\mathbb{R}_+\times\partial\Omega}\longrightarrow\rho,f,g. \tag{1.3}$$

In the physical setup, $\Lambda_{\rho,f,g}(t,\boldsymbol{x})$ is referred to as the *passive measurement* since it is generated by certain unknown sources, namely, f and g. This is in contrast to

*Received by the editors May 31, 2022; accepted for publication (in revised form) January 18, 2023; published electronically June 9, 2023.
https://doi.org/10.1137/22M1499881
Funding: The research of the first author was supported by NSFC grants 11871140 and 12071065 and National Key R&D Program of China 2020YFA0714102. The research of the second author was supported by NSFC/RGC Joint Research Scheme N_CityU101/21, ANR/RGC Joint Research Scheme A-CityU203/19, and the Hong Kong RGC General Research Funds (projects 11311122, 11300821, and 12301420).

†School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, Jilin 130024, People's Republic of China (gaoyx643@nenu.edu.cn).
‡Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, People's Republic of China (hongyu.liuip@gmail.com, hongyliu@cityu.edu.hk).
§School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, People's Republic of China (liuy694@nenu.edu.cn).

the *active measurement*, where one actively sends a priori known probing sources and collects the responses for identification purposes. Here, we would like to point out that the terminology "passive measurement" has different meanings in other fields, such as electronics and information engineering, which should be clear from the context.

The plate equation is a significant portion of engineering construction that arises from the study of mathematical models in fluid mechanics and elasticity. A mechanical background of the boundary value problem is the elastic bending problem of thin plates. Inverse scattering problems have played an important role in diverse scientific areas, such as radar, sonar, geophysical exploration, and medical imaging. Some literature has concentrated on inverse scattering problems for biharmonic operators; see [9, 11, 20]. Compared with second-order differential equations, as acoustic, elastic, and electromagnetic waves, the research of the inverse scattering problem of the biharmonic equation is not as extensive as the results of second-order differential equations. The increase of the order leads to the failure of the methods which work for the second-order equations. A detailed description of the properties of the solution can be found in [19].

Global uniqueness results of recovering potential function or medium parameters associated with biharmonic or polyharmonic operators by active measurements can be found in [1, 11, 17, 22]. To the best of our knowledge, no uniqueness identifiability result is known in the literature in determining unknown medium parameters associated with the biharmonic operator by the passive measurement. Indeed, in our study of (1.3), we aim at determining both the medium parameter ρ and the unknown sources f, g from the associated passive measurement. In recent years, simultaneously determining both an unknown source and its surrounding medium by the associated passive measurement has received considerable interest in the literature due to its practical importance in emerging applications. In [10, 18], the authors proved the uniqueness in determining both an acoustic density and an internal source for the scalar wave equation by the passive measurement, which arises in thermo- and photo-acoustic tomography. In [7], the authors established unique recovery results in simultaneously determining an unknown internal current and its surrounding medium by the passive measurement associated with a Maxwell system, which arises in brain imaging. In [4, 5], similar inverse problems were considered associated with the geodynamical system which arises in geomagnetic anomaly detection. We also refer the reader to [2, 6, 13, 14, 15, 16] for more related studies in different physical and mathematical setups. These results show that it is possible to prove the uniqueness of two or more unknowns simultaneously by the passive measurement. Motivated by [2, 7, 10, 14, 18], we are interested in recovering both the density ρ and the sources f, g by the passive boundary measurement for the biharmonic plate equation (1.1).

In this article, we shall make use of the temporal Fourier transform, converting the time-dependent problem (1.1) into the frequency domain. To that end, we need to require that the plate equation of (1.1) is exponentially decaying in time, which can guarantee the well-posedness of the temporal Fourier transform. In order to appeal for a general study, we shall always assume the exponentially decaying in time for the plate equation. Nevertheless, we would like to emphasize that such a property is satisfied by generic mediums and sources. In what follows, we refer to (f, g, ρ) as admissible if the aforementioned time decaying property is fulfilled. In fact, one can refer to the case of the acoustic wave equation (cf. Theorem 6.1, p. 113, of [12]) and derive general admissibility conditions by following similar arguments. However, this is not the focus of the current study, and our main purpose is to study the related inverse problem. The main methods to obtain the uniqueness results are by performing certain asymptotic analysis in the low-frequency regime. We derive some integral identities involving source functions and density coupling. Combining those integral identities and using harmonic analysis techniques, the uniqueness results are obtained. The uniqueness relies on assuming that the density ρ and internal source functions f, g along an arbitrary direction vector $\boldsymbol{\iota}$ satisfy $\nabla\rho\cdot\boldsymbol{\iota}=\nabla f\cdot\boldsymbol{\iota}=\nabla g\cdot\boldsymbol{\iota}=0$. Additionally, the above assumption can be replaced by setting the size of the corresponding parameters to obtain a more general uniqueness result.

The paper is organized as follows. In section 2, we introduce the temporal Fourier transform, converting the time domain problem (1.1) into the frequency domain, and give some notations. In section 3, we derive the asymptotic expansions of the solutions with respect to the frequency κ and get some integral identities involving the source functions and the density, which are coupled together. Then the uniqueness results are established by a natural admissibility assumption on the parameters. The more general uniqueness results are given in section 4.

2. Problem formulation. In this section, we introduce the temporal Fourier transform to convert the time domain problem (1.1) into the frequency domain and introduce some notations.

Our argument depends on the temporal Fourier transform of the function $u(t,\boldsymbol{x})$ defined by

$$\hat{u}(\omega,\boldsymbol{x}):=\frac{1}{2\pi}\int_0^\infty u(t,\boldsymbol{x})e^{\mathrm{i}wt}\,\mathrm{d}t,\qquad(\omega,\boldsymbol{x})\in\mathbb{R}_+\times\mathbb{R}^3.$$

Applying the temporal Fourier transform to (1.1) and assuming that $\kappa=\omega^{1/2}$ yields that $\hat{u}(\kappa,\boldsymbol{x})$ satisfies

(2.1)
$$\Delta^2\hat{u}(\kappa,\boldsymbol{x})-\kappa^4\rho(\boldsymbol{x})\hat{u}(\kappa,\boldsymbol{x})=-\frac{\mathrm{i}\kappa^2}{2\pi}\rho(\boldsymbol{x})f(\boldsymbol{x})+\frac{1}{2\pi}\rho(\boldsymbol{x})g(\boldsymbol{x}),\quad(\kappa,\boldsymbol{x})\in\mathbb{R}_+\times\mathbb{R}^3,$$

and the boundary measurement (1.2) in the frequency domain

$$\hat{\Lambda}_{\rho,f,g}(\kappa,\boldsymbol{x})=\left(\hat{u}(\kappa,\boldsymbol{x}),\,\Delta\hat{u}(\kappa,\boldsymbol{x})\right),\quad(\kappa,\boldsymbol{x})\in\mathbb{R}_+\times\partial\Omega.\tag{2.2}$$

To ensure the well-posedness of (2.1), we impose an analogue of the Sommerfeld radiation conditions

$$\lim_{r\to\infty}r\left(\partial_r\hat{u}(\kappa,\boldsymbol{x})-\mathrm{i}\kappa\hat{u}(\kappa,\boldsymbol{x})\right)=0,\qquad\lim_{r\to\infty}r\left(\partial_r(\Delta\hat{u}(\kappa,\boldsymbol{x}))-\mathrm{i}\kappa(\Delta\hat{u}(\kappa,\boldsymbol{x}))\right)=0\tag{2.3}$$

uniformly in all directions $\hat{\boldsymbol{x}}=\boldsymbol{x}/|\boldsymbol{x}|$ with $r=|\boldsymbol{x}|$ (cf. [21]).

One of the key technical ingredients to establish the unique recovery results is first by performing certain asymptotic analysis in the low-frequency regime to derive certain integral identities involving the source function and density, which are coupled together.

We introduce some notations. The fundamental solution for biharmonic operator $\Delta^2-\kappa^4$ in $\mathbb{R}^3$ is

$$G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)=\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\quad\text{for }\boldsymbol{x}\neq\boldsymbol{y}.\tag{2.4}$$

Notice that when $\kappa=0$, the fundamental solution to Δ^2 is

$$G_0(|\boldsymbol{x}-\boldsymbol{y}|)=-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{8\pi}\quad\text{for }\boldsymbol{x}\neq\boldsymbol{y},$$

and the fundamental solution for $-\Delta$ is

$$g_0(|\boldsymbol{x}-\boldsymbol{y}|) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \quad \text{for } \boldsymbol{x} \neq \boldsymbol{y}.$$

3. The uniqueness results for density and internal sources. In this section, we will prove the uniqueness for both the unknown density ρ and the internal sources f and g.

3.1. Auxiliary results. Before proving the uniqueness results, we first derive several auxiliary results.

LEMMA 3.1. *Let $\hat{u}(\kappa,\boldsymbol{x}) \in H^2_{loc}(\mathbb{R}^3)$ be the solution of (2.1) and (2.3). Then $\hat{u}(\kappa,\boldsymbol{x})$ is uniquely given by the following integral equation:*

$$\begin{aligned}\hat{u}(\kappa,\boldsymbol{x}) =& \kappa^4 \int_{\mathbb{R}^3} (\rho(\boldsymbol{y})-1)\hat{u}(\kappa,\boldsymbol{y}) G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y} - \frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3} \rho(\boldsymbol{y}) f(\boldsymbol{y}) G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^3} \rho(\boldsymbol{y}) g(\boldsymbol{y}) G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\mathbb{R}^3. \end{aligned} \tag{3.1}$$

In addition, taking the expansion of $e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|} - e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}$ as $\kappa \to +0$, we have

$$\begin{aligned}\hat{u}(\kappa,\boldsymbol{x}) =& \frac{1}{2\pi\kappa}\int_{\mathbb{R}^3} \frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{1}{8\pi}\rho(\boldsymbol{y})g(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|\,\mathrm{d}\boldsymbol{y}\\ &+ \frac{\kappa}{2\pi}\int_{\mathbb{R}^3}\frac{(1-\mathrm{i})}{8\pi}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}\\ &- \frac{\mathrm{i}\kappa}{2\pi}\int_{\mathbb{R}^3}\frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \mathcal{O}(\kappa^2), \quad \boldsymbol{x}\in B_R, \end{aligned} \tag{3.2}$$

where $B_R := B(0,R)$ is a central ball of radius $R\in\mathbb{R}_+$ and satisfies $\Omega\subset B_R$.

Proof. From the regularity theorem in [8], it is easy to verify that the solution $\hat{u}\in H^4_{loc}(\mathbb{R}^3)$. By applying the fundamental solution to (2.1), we obtain a Lippmann–Schwinger integral equation

$$\begin{aligned}\hat{u}(\kappa,\boldsymbol{x}) =& \kappa^4 \int_{\mathbb{R}^3} (\rho(\boldsymbol{y})-1)\hat{u}(\kappa,\boldsymbol{y}) G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\ &- \frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3} \rho(\boldsymbol{y}) f(\boldsymbol{y}) G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^3} \rho(\boldsymbol{y}) g(\boldsymbol{y}) G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}. \end{aligned} \tag{3.3}$$

Since $\rho(x)-1=0$ in $\mathbb{R}^3\backslash\bar{\Omega}$ and $\rho\in L^\infty(\mathbb{R}^3)$, we assume that $\Omega\subset B_R$ and $\mathcal{K}_{\rho,\kappa} : C(\overline{B_R}) \longrightarrow C(\overline{B_R})$ satisfies

$$\mathcal{K}_{\rho,\kappa}(\hat{u}) = \kappa^4\int_{B_R}(\rho(\boldsymbol{y})-1)\hat{u}(\kappa,\boldsymbol{y})G_\kappa(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}.$$

Supposing that $M := \sup_{|\boldsymbol{x}|\le R}|\rho(\boldsymbol{x})-1|$ and $\kappa^2 < \frac{2}{MR^2}$, we have $\|\mathcal{K}_{\rho,\kappa}\|_{L^\infty(B_R)} \le 1$. Therefore, there exsits a Neumann series

$$(I-\mathcal{K}_{\rho,\kappa})^{-1} = I + \mathcal{K}_{\rho,\kappa} + \mathcal{K}^2_{\rho,\kappa} + \cdots.$$

Taking $\kappa\longrightarrow +0$ and replacing $e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|} - e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}$ by the series expansion

$$e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|} - e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|} = \kappa(\mathrm{i}+1)|\boldsymbol{x}-\boldsymbol{y}| - \kappa^2|\boldsymbol{x}-\boldsymbol{y}|^2 + \frac{\kappa^3(1-\mathrm{i})}{3!}|\boldsymbol{x}-\boldsymbol{y}|^3 + \mathcal{O}(\kappa^5),$$

we calculate that

$$(I-\mathcal{K}_{\rho,\kappa})^{-1}=I+\mathcal{O}(\kappa^3). \tag{3.4}$$

Additionally, substituting (2.4) into (3.3) implies that

$$\begin{aligned}\hat{u}(\kappa,\boldsymbol{x})&=(I-\mathcal{K}_{\rho,\kappa})^{-1}\Bigg(-\frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\&\quad+\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\Bigg)\\&=\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{(\mathrm{i}+1)}{8\pi\kappa}-\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|}{8\pi}+\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{\kappa(1-\mathrm{i})}{3!}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{8\pi}\,\mathrm{d}\boldsymbol{y}\\&\quad-\frac{\mathrm{i}\kappa}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{(\mathrm{i}+1)|\boldsymbol{x}-\boldsymbol{y}|}{8\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}+\mathcal{O}(\kappa^2),\quad \boldsymbol{x}\in B_R.\end{aligned}$$

The proof is completed. ∎

Lemma 3.2. *Let* $\hat{u}(\kappa,\boldsymbol{x})\in H^2_{loc}(\mathbb{R}^3)$ *be the solution of* (2.1) *and* (2.3). *Then the integral equation* (3.1) *can be rewritten as*

$$\hat{u}(\kappa,\boldsymbol{x})=\sum_{m=-1}^{3}M_m(\boldsymbol{x})\kappa^m+\mathcal{O}(\kappa^4)\quad \textit{as}\,\kappa\to+0,\quad \boldsymbol{x}\in B_R, \tag{3.5}$$

where

$$\begin{aligned}M_{-1}&:=\frac{1}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},\\M_0&:=\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})G_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y},\\M_1&:=-\frac{\mathrm{i}}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}+\frac{1}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y},\\M_2&:=\left(\frac{\mathrm{i}+1}{8\pi}\right)\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)M_{-1}\,\mathrm{d}\boldsymbol{y}-\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})G_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y},\\M_3&:=\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)M_{-1}\,G_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}+\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)M_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\&\quad-\frac{\mathrm{i}}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}+\frac{1}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^4}{5!}\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Taking the Laplacian for (3.5) *with respect to* $\boldsymbol{x}$ *implies that*

$$\begin{aligned}\Delta_{\boldsymbol{x}}\hat{u}(\kappa,\boldsymbol{x})&=-\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}+\frac{\kappa}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\&\quad+\frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}-\kappa^3\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)M_{-1}\,g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\&\quad-\frac{\mathrm{i}\kappa^3}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\&\quad+\frac{\kappa^3}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}+\mathcal{O}(\kappa^4)\quad \textit{as}\,\kappa\to+0,\quad \boldsymbol{x}\in B_R.\end{aligned} \tag{3.6}$$

Proof. Plugging (3.2) into (3.1) and using the series expansion

$$e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}=\kappa(\mathrm{i}+1)|\boldsymbol{x}-\boldsymbol{y}|-\kappa^2|\boldsymbol{x}-\boldsymbol{y}|^2+\frac{\kappa^3(1-\mathrm{i})}{3!}|\boldsymbol{x}-\boldsymbol{y}|^3 + \frac{\kappa^5(\mathrm{i}+1)}{5!}|\boldsymbol{x}-\boldsymbol{y}|^5+\mathcal{O}(\kappa^6) \quad as\,\kappa\to+0$$

uniformly for $\boldsymbol{x}\in B_R$, we get

(3.7)

$$\begin{aligned}\hat{u}(\kappa,\boldsymbol{x})=\kappa^4\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)\Bigg(&\frac{1}{2\pi\kappa}\int_{\mathbb{R}^3}\frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{z})g(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}-\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{z})g(\boldsymbol{z})\frac{|\boldsymbol{y}-\boldsymbol{z}|}{8\pi}\,\mathrm{d}\boldsymbol{z}\\
&+\frac{\kappa}{2\pi}\int_{\mathbb{R}^3}\frac{(1-\mathrm{i})}{8\pi}\rho(\boldsymbol{z})g(\boldsymbol{z})\frac{|\boldsymbol{y}-\boldsymbol{z}|^2}{3!}\,\mathrm{d}\boldsymbol{z}\\
&-\frac{\mathrm{i}\kappa}{2\pi}\int_{\mathbb{R}^3}\frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{z})f(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}+\mathcal{O}(\kappa^2)\Bigg)\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\
&-\frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\
&+\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\
:=&I_1+I_2+I_3+\mathcal{O}(\kappa^4).\end{aligned}$$

By direct calculation, one has

$$\begin{aligned}I_1=\kappa^4\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)\Bigg(&\frac{1}{2\pi\kappa}\int_{\mathbb{R}^3}\frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{z})g(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}-\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{z})g(\boldsymbol{z})\frac{|\boldsymbol{y}-\boldsymbol{z}|}{8\pi}\,\mathrm{d}\boldsymbol{z}\\
&+\frac{\kappa}{2\pi}\int_{\mathbb{R}^3}\frac{(1-\mathrm{i})}{8\pi}\rho(\boldsymbol{z})g(\boldsymbol{z})\frac{|\boldsymbol{y}-\boldsymbol{z}|^2}{3!}\,\mathrm{d}\boldsymbol{z}\\
&-\frac{\mathrm{i}\kappa}{2\pi}\int_{\mathbb{R}^3}\frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{z})f(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}+\mathcal{O}(\kappa^2)\Bigg)\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\
=\kappa^2\int_{\mathbb{R}^3}&(\rho(\boldsymbol{y})-1)\frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{z})g(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\frac{(\mathrm{i}+1)}{8\pi}\,\mathrm{d}\boldsymbol{y}\\
+\kappa^3\int_{\mathbb{R}^3}&(\rho(\boldsymbol{y})-1)\frac{1}{2\pi}\int_{\mathbb{R}^3}\frac{(\mathrm{i}+1)}{8\pi}\rho(\boldsymbol{z})g(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,G_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\
+\kappa^3\int_{\mathbb{R}^3}&(\rho(\boldsymbol{y})-1)\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{z})g(\boldsymbol{z})G_0(|\boldsymbol{y}-\boldsymbol{z}|)\,\mathrm{d}\boldsymbol{z}\,\frac{(\mathrm{i}+1)}{8\pi}\,\mathrm{d}\boldsymbol{y}+\mathcal{O}(\kappa^4),\qquad \boldsymbol{x}\in B_R,\end{aligned}$$

$$\begin{aligned}I_2=&-\frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\
=&-\frac{\mathrm{i}\kappa}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}-\frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})G_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\
&-\frac{\mathrm{i}\kappa^3}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}+\mathcal{O}(\kappa^4),\qquad \boldsymbol{x}\in B_R,\end{aligned}$$

and

$$
\begin{aligned}
I_3 =& \frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{e^{\mathrm{i}\kappa|\boldsymbol{x}-\boldsymbol{y}|}-e^{-\kappa|\boldsymbol{x}-\boldsymbol{y}|}}{8\pi\kappa^2|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\\
=& \frac{1}{2\pi\kappa}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}+\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})G_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\
&+\frac{\kappa}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}\\
&+\frac{\kappa^3}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^4}{5!}\,\mathrm{d}\boldsymbol{y}+\mathcal{O}(\kappa^4),\qquad \boldsymbol{x}\in B_R.
\end{aligned}
$$

Taking the Laplacian on both sides of equality (3.7) with respect to $\boldsymbol{x}$, we obtain

$$
\begin{aligned}
\Delta_{\boldsymbol{x}}\hat{u}(\kappa,\boldsymbol{x}) =& -\frac{\kappa^3}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)\int_{\mathbb{R}^3}\rho(\boldsymbol{z})g(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\\
&+\frac{\mathrm{i}\kappa^2}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}-\frac{\mathrm{i}\kappa^3}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\
&-\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}+\frac{\kappa}{2\pi}\frac{(1-\mathrm{i})}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\
&+\frac{\kappa^3}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}+\mathcal{O}(\kappa^4),\qquad \boldsymbol{x}\in B_R
\end{aligned}
$$

as $\kappa\to+0$. The proof is completed. ∎

Remark 3.1. Let $n\in\mathbb{N}\cup\{0\}$. Then the solution $\hat{u}(\kappa,\boldsymbol{x})$ can be represented as

$$
\hat{u}(\kappa,\boldsymbol{x})=\sum_{m=-1}^{n+1}M_m(\boldsymbol{x})\kappa^m+M_{n+2}(\boldsymbol{x})\kappa^{n+2}+\mathcal{O}(\kappa^{n+3})
$$

and

$$
\Delta_{\boldsymbol{x}}\hat{u}(\kappa,\boldsymbol{x})=\sum_{m=0}^{n+2}N_m(\boldsymbol{x})\kappa^m+N_{n+3}(\boldsymbol{x})\kappa^{n+3}+\mathcal{O}(\kappa^{n+4})\qquad\text{as }\kappa\to+0,
$$

where

$$
\begin{aligned}
M_{n+2}(\boldsymbol{x}):=&\sum_{m=0}^{n}\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)M_{n-m-1}(\boldsymbol{y})\frac{\mathrm{i}^{m+1}-(-1)^{m+1}}{8\pi}\frac{|\boldsymbol{x}-\boldsymbol{y}|^m}{(m+1)!}\,\mathrm{d}\boldsymbol{y}\\
&-\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{\mathrm{i}^{n+2}-(-1)^{n+2}}{8\pi}\frac{|\boldsymbol{x}-\boldsymbol{y}|^{n+1}}{(n+2)!}\,\mathrm{d}\boldsymbol{y}\\
&+\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{\mathrm{i}^{n+4}-(-1)^{n+4}}{8\pi}\frac{|\boldsymbol{x}-\boldsymbol{y}|^{n+3}}{(n+4)!}\,\mathrm{d}\boldsymbol{y}
\end{aligned}
$$

and

$$
\begin{aligned}
N_{n+3}:=&\sum_{m=1}^{n+1}\int_{\mathbb{R}^3}(\rho(\boldsymbol{y})-1)M_{n-m}(\boldsymbol{y})\frac{\mathrm{i}^{m+1}-(-1)^{m+1}}{8\pi}\frac{|\boldsymbol{x}-\boldsymbol{y}|^{m-2}}{(m-1)!}\,\mathrm{d}\boldsymbol{y}\\
&-\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})f(\boldsymbol{y})\frac{\mathrm{i}^{n+3}-(-1)^{n+3}}{8\pi}\frac{|\boldsymbol{x}-\boldsymbol{y}|^{n}}{(n+1)!}\,\mathrm{d}\boldsymbol{y}\\
&+\frac{1}{2\pi}\int_{\mathbb{R}^3}\rho(\boldsymbol{y})g(\boldsymbol{y})\frac{\mathrm{i}^{n+5}-(-1)^{n+5}}{8\pi}\frac{|\boldsymbol{x}-\boldsymbol{y}|^{n+2}}{(n+3)!}\,\mathrm{d}\boldsymbol{y},\qquad \boldsymbol{x}\in B_R.
\end{aligned}
$$

In addition, $N_m(\boldsymbol{x})\kappa^m, m=0,1,2$, corresponds to the first three terms in (3.6), respectively.

THEOREM 3.1. *Assume that* (ρ_1, f_1, g_1) *and* (ρ_2, f_2, g_2) *are two sets of admissible configurations and supported in* Ω. *If*

$$\Lambda_{\rho_1,f_1,g_1}(t,\boldsymbol{x}) = \Lambda_{\rho_2,f_2,g_2}(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in \mathbb{R}_+ \times \partial\Omega, \tag{3.8}$$

then for any harmonic function $h(\boldsymbol{x})$, *we have*

$$\int_{\mathbb{R}^3} (\rho_1 f_1 - \rho_2 f_2)(\boldsymbol{x}) h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0, \tag{3.9}$$

$$\int_{\mathbb{R}^3} (\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{x}) h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0. \tag{3.10}$$

Furthermore, for any $\boldsymbol{x} \in \partial B_R$, *the following holds:*

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\rho_1(\boldsymbol{y}) - 1) \int_{\mathbb{R}^3} \rho_1(\boldsymbol{z}) g_1(\boldsymbol{z})\, \mathrm{d}\boldsymbol{z}\, g_0(|\boldsymbol{x}-\boldsymbol{y}|)\, \mathrm{d}\boldsymbol{y} \\
&\quad + \int_{\mathbb{R}^3} \rho_1(\boldsymbol{y}) f_1(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} - \int_{\mathbb{R}^3} \rho_1(\boldsymbol{y}) g_1(\boldsymbol{y}) \frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\, \mathrm{d}\boldsymbol{y} \\
&= \int_{\mathbb{R}^3} (\rho_2(\boldsymbol{y}) - 1) \int_{\mathbb{R}^3} \rho_2(\boldsymbol{z}) g_2(\boldsymbol{z})\, \mathrm{d}\boldsymbol{z}\, g_0(|\boldsymbol{x}-\boldsymbol{y}|)\, \mathrm{d}\boldsymbol{y} \\
&\quad + \int_{\mathbb{R}^3} \rho_2(\boldsymbol{y}) f_2(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} - \int_{\mathbb{R}^3} \rho_2(\boldsymbol{y}) g_2(\boldsymbol{y}) \frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\, \mathrm{d}\boldsymbol{y}.
\end{aligned} \tag{3.11}$$

Proof. Using the temporal Fourier transform, let $\hat{u}_1(\kappa, \boldsymbol{x})$ and $\hat{u}_2(\kappa, \boldsymbol{x})$ denote the solution of (2.1), corresponding to (ρ_1, f_1, g_1) and (ρ_2, f_2, g_2), respectively. It follows from (2.2) and (3.8) that

$$(\hat{u}_1(\boldsymbol{x}),\, \Delta\hat{u}_1(\boldsymbol{x})) = (\hat{u}_2(\boldsymbol{x}),\, \Delta\hat{u}_2(\boldsymbol{x})), \quad \boldsymbol{x} \in \partial\Omega.$$

According to the prior information of density and internal sources, it is easy to verify that both $\hat{u}_1$ and $\hat{u}_2$ satisfy the same equation:

$$\Delta^2 u - \kappa^4 u = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\Omega}.$$

Additionally, from the uniqueness of exterior boundary value problem in Theorem A.1, we have

$$(\hat{u}_1(\boldsymbol{x}),\, \Delta\hat{u}_1(\boldsymbol{x})) = (\hat{u}_2(\boldsymbol{x}),\, \Delta\hat{u}_2(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{R}^3 \backslash \Omega.$$

Due to $\partial B_R \subset \mathbb{R}^3 \backslash \overline{\Omega}$, we can obtain

$$(\hat{u}_1(\boldsymbol{x}),\, \Delta\hat{u}_1(\boldsymbol{x})) = (\hat{u}_2(\boldsymbol{x}),\, \Delta\hat{u}_2(\boldsymbol{x})), \quad \boldsymbol{x} \in \partial B_R. \tag{3.12}$$

Combining (3.6) and (3.12), we imply the integral identities

$$\frac{1}{2\pi} \int_{B_R} \rho_1(\boldsymbol{y}) g_1(\boldsymbol{y}) g_0(|\boldsymbol{x}-\boldsymbol{y}|)\, \mathrm{d}\boldsymbol{y} = \frac{1}{2\pi} \int_{B_R} \rho_2(\boldsymbol{y}) g_2(\boldsymbol{y}) g_0(|\boldsymbol{x}-\boldsymbol{y}|)\, \mathrm{d}\boldsymbol{y}, \tag{3.13}$$

$$\frac{\mathrm{i}\kappa^2}{2\pi} \int_{B_R} \rho_1(\boldsymbol{y}) f_1(\boldsymbol{y}) g_0(|\boldsymbol{x}-\boldsymbol{y}|)\, \mathrm{d}\boldsymbol{y} = \frac{\mathrm{i}\kappa^2}{2\pi} \int_{B_R} \rho_2(\boldsymbol{y}) f_2(\boldsymbol{y}) g_0(|\boldsymbol{x}-\boldsymbol{y}|)\, \mathrm{d}\boldsymbol{y}, \tag{3.14}$$

and

$$\frac{\kappa^3}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\Bigg(\int_{B_R}(\rho_1(\boldsymbol{y})-1)\int_{B_R}\rho_1(\boldsymbol{z})g_1(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y} + \int_{B_R}\rho_1(\boldsymbol{y})f_1(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \int_{B_R}\rho_1(\boldsymbol{y})g_1(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}\Bigg)$$
$$= \frac{\kappa^3}{2\pi}\frac{(\mathrm{i}+1)}{8\pi}\Bigg(\int_{B_R}(\rho_2(\boldsymbol{y})-1)\int_{B_R}\rho_2(\boldsymbol{z})g_2(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y} + \int_{B_R}\rho_2(\boldsymbol{y})f_2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \int_{B_R}\rho_2(\boldsymbol{y})g_2(\boldsymbol{y})\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{3!}\,\mathrm{d}\boldsymbol{y}\Bigg)$$

for $\boldsymbol{x}\in\partial B_R$.

Note that the fundamental solution of $-\Delta$ can be written as

$$\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}=\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{|\boldsymbol{y}|^m}{|\boldsymbol{x}|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\quad \text{for } |\boldsymbol{x}|>|\boldsymbol{y}|, \tag{3.15}$$

where $Y_m^n(\cdot)$ denotes the spherical harmonics of order $m\in\mathbb{N}\cup\{0\}$ for $n=-m,\cdots,m$.

Substituting (3.15) into (3.13) and (3.14), we calculate that

$$\int_{B_R}\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|\boldsymbol{x}|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\rho_1(\boldsymbol{y})f_1(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}$$
$$=\int_{B_R}\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|\boldsymbol{x}|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\rho_2(\boldsymbol{y})f_2(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}\quad\text{for } \boldsymbol{x}\in\partial B_R$$

and

$$\int_{B_R}\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|\boldsymbol{x}|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\rho_1(\boldsymbol{y})g_1(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}$$
$$=\int_{B_R}\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|\boldsymbol{x}|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\rho_2(\boldsymbol{y})g_2(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}\quad\text{for } \boldsymbol{x}\in\partial B_R.$$

That is,

$$\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|R|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\int_{B_R}\rho_1(\boldsymbol{y})f_1(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}$$
$$=\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|R|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\int_{B_R}\rho_2(\boldsymbol{y})f_2(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}\quad\text{for } \boldsymbol{x}\in\partial B_R$$

and

$$\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|R|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\int_{B_R}\rho_1(\boldsymbol{y})g_1(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}$$
$$=\sum_{m=0}^{\infty}\sum_{n=-m}^{m}\frac{1}{2m+1}\frac{1}{|R|^{m+1}}Y_m^n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\int_{B_R}\rho_2(\boldsymbol{y})g_2(\boldsymbol{y})|\boldsymbol{y}|^m\overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right)\,\mathrm{d}\boldsymbol{y}\quad\text{for } \boldsymbol{x}\in\partial B_R.$$

Indeed, $\{Y_m^n(\cdot)\}_{m=0,1,2,\ldots;n=-m,\ldots,m}$ is a complete orthonormal basis of $L^2(\mathbb{S}^2)$ (note that $\mathbb{S}^2$ means the unit sphere). We get

$$\int_{B_R} (\rho_1(\boldsymbol{y})f_1(\boldsymbol{y}) - \rho_2(\boldsymbol{y})f_2(\boldsymbol{y}))|\boldsymbol{y}|^m \overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right) \mathrm{d}\boldsymbol{y} = 0$$

and

$$\int_{B_R} (\rho_1(\boldsymbol{y})g_1(\boldsymbol{y}) - \rho_2(\boldsymbol{y})g_2(\boldsymbol{y}))|\boldsymbol{y}|^m \overline{Y}_m^n\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right) \mathrm{d}\boldsymbol{y} = 0,$$

where $m \in \mathbb{N} \cup \{0\}$ for $n = -m, \ldots, m$.

Since $|\boldsymbol{y}|^m Y_m^n(\frac{\boldsymbol{y}}{|\boldsymbol{y}|})$ for $m = 0, 1, 2, \ldots$ and $n = -m, \ldots, m$ can yield all the homogeneous harmonic function $h(\cdot)$, which means

$$\begin{aligned} \int_{B_R} (\rho_1 f_1 - \rho_2 f_2)(\boldsymbol{y}) h(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} &= 0, \quad \boldsymbol{x} \in \partial B_R, \\ \int_{B_R} (\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{y}) h(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} &= 0, \quad \boldsymbol{x} \in \partial B_R. \end{aligned} \tag{3.16}$$

∎

Remark 3.2. Since f_i, g_i are supported in Ω and $\operatorname{supp}(\rho_i - 1) \subset \Omega, i = 1, 2$, we note that the integral domains in (3.16) can be replaced by Ω.

3.2. The uniqueness results.

THEOREM 3.2. *Assume that (ρ_1, f_1, g_1) and (ρ_2, f_2, g_2) are two sets of admissible configurations and supported in Ω, respectively. Furthermore, suppose that*

$$F(\boldsymbol{x}) := (\rho_1 f_1 - \rho_2 f_2)(\boldsymbol{x}), \quad G(\boldsymbol{x}) := (\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

satisfy either of the following conditions:

(i) *$F(\boldsymbol{x}) = h_1(\boldsymbol{x})$ and $G(\boldsymbol{x}) = h_2(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega$, where $h_1(\boldsymbol{x})$ and $h_2(\boldsymbol{x})$ are harmonic functions in $\mathbb{R}^3$;*

(ii) *$\nabla F(\boldsymbol{x}) \cdot \boldsymbol{\iota} = 0$ and $\nabla G(\boldsymbol{x}) \cdot \boldsymbol{\iota} = 0$, where $\boldsymbol{\iota}$ is an arbitrary direction vector in $\mathbb{R}^3$.*

Then

$$F(\boldsymbol{x}) = G(\boldsymbol{x}) = 0 \quad \textit{for a.e. } \boldsymbol{x} \in \Omega.$$

Proof. For the first case, taking $h(\boldsymbol{x}) = h_1(\boldsymbol{x})$ and $h(\boldsymbol{x}) = h_2(\boldsymbol{x})$ into (3.9) and (3.10), respectively, we get

$$\int_\Omega h_1^2(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega h_2^2(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0.$$

It shows that $F(\boldsymbol{x}) = G(\boldsymbol{x}) = 0$.

For the second case, because of the rotation invariance of the biharmonic operator Δ^2, the vector $\boldsymbol{\iota}$ can be rotated appropriately to any coordinate axis. Without loss of generality, we assume that $\boldsymbol{\iota}$ rotates to the x_3-axis. Then we have

$$\partial_{x_3} F(\boldsymbol{x}) = \partial_{x_3} G(\boldsymbol{x}) = 0,$$

which means $F(\boldsymbol{x}), G(\boldsymbol{x})$ only depend on the variables x_1, x_2 for $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Denote

$$h(\boldsymbol{x}) = e^{\mathrm{i}\tilde{\boldsymbol{\xi}} \cdot \boldsymbol{x}}, \quad \boldsymbol{x} \in \mathbb{R}^3, \tag{3.17}$$

to be a harmonic function, where

$$\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}_1 + \mathrm{i}\boldsymbol{\xi}_2, \quad \boldsymbol{\xi}_1 = (\xi_1, \xi_2, 0)^\top \in \mathbb{R}^3, \quad \boldsymbol{\xi}_2 = (0, 0, \xi_3)^\top \in \mathbb{R}^3,$$

satisfying $\xi_1^2 + \xi_2^2 = \xi_3^2$.

Plugging (3.17) into (3.9), we compute

$$\begin{aligned}\int_{\mathbb{R}^3} F(x_1, x_2) e^{\mathrm{i}\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} &= \int_{B_R} F(x_1, x_2) e^{\mathrm{i}\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} F(x_1, x_2) e^{\mathrm{i}\xi_1\cdot x_1 + \mathrm{i}\xi_2\cdot x_2}\,\mathrm{d}x_1 \mathrm{d}x_2 \int_{\{x_3; (x_1,x_2,x_3)\in B_R\}} e^{-\xi_3 x_3} \mathrm{d}x_3 = 0.\end{aligned}$$

It follows from the priori information of ρ_i, f_i and $g_i, i = 1, 2$ that

$$0 = \int_{\mathbb{R}^2} F(x_1, x_2) e^{\mathrm{i}\xi_1\cdot x_1 + \mathrm{i}\xi_2\cdot x_2}\,\mathrm{d}x_1 \mathrm{d}x_2 = (\mathcal{F}F)(\boldsymbol{\xi}_1),$$

which holds for any $(\xi_1, \xi_2) \in \mathbb{R}^2$. Since $(\mathcal{F}F)(\boldsymbol{\xi}_1)$ means the Fourier transform of $F(x_1, x_2)$, it is clearly shown that $F(\boldsymbol{x}) = 0$. We can state $G(\boldsymbol{x}) = 0$ by using the same methods. The proof is completed. ∎

Now we discuss the uniqueness for density and internal sources by using above orthogonality results.

THEOREM 3.3. *Assume that (ρ_1, f_1, g_1) and (ρ_2, f_2, g_2) are two sets of configurations and supported in Ω, which satisfy both conditions:*

(i) *ρ_1 and ρ_2 are positive constants;*

(ii) *$\nabla f_i(\boldsymbol{x}) \cdot \boldsymbol{\iota} = 0$ and $\nabla g_i(\boldsymbol{x}) \cdot \boldsymbol{\iota} = 0, i = 1, 2,$ where $\boldsymbol{\iota}$ is an arbitrary direction vector in $\mathbb{R}^3$.*

If

$$\Lambda_{\rho_1,f_1,g_1}(t, \boldsymbol{x}) = \Lambda_{\rho_2,f_2,g_2}(t, \boldsymbol{x}), \quad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \partial\Omega, \tag{3.18}$$

and supposing that

$$\int_\Omega g_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \neq 0, \quad i = 1, 2, \quad \boldsymbol{x} \in \Omega, \tag{3.19}$$

then

$$\rho_1 = \rho_2, \quad f_1(\boldsymbol{x}) = f_2(\boldsymbol{x}), \quad g_1(\boldsymbol{x}) = g_2(\boldsymbol{x}).$$

Proof. By rotation invariance, without loss of generality, we still assume that $\boldsymbol{\iota}$ rotates to the x_3-axis. Then $\rho_1 f_1(\boldsymbol{x}) - \rho_2 f_2(\boldsymbol{x})$ and $\rho_1 g_1(\boldsymbol{x}) - \rho_2 g_2(\boldsymbol{x})$ only depend on the variables x_1, x_2. By Theorem 3.2 and (3.18), we deduce that

$$\rho_1 f_1(\boldsymbol{x}) = \rho_2 f_2(\boldsymbol{x}), \quad \rho_1 g_1(\boldsymbol{x}) = \rho_2 g_2(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

It can be seen that

$$\int_{B_R} \rho_1 f_1(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{B_R} \rho_2 f_2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{3.20}$$

$$\int_{B_R} \rho_1 g_1(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{B_R} \rho_2 g_2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{3.21}$$

and

$$\int_{B_R} \rho_1 g_1(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^2\,\mathrm{d}\boldsymbol{y} = \int_{B_R} \rho_2 g_2(\boldsymbol{y})|\boldsymbol{x}-\boldsymbol{y}|^2\,\mathrm{d}\boldsymbol{y}. \tag{3.22}$$

Substituting (3.20)–(3.22) into (3.11) implies that

$$(\rho_1-\rho_2)\int_{B_R}\left(\int_{B_R}\rho_1 g_1(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right) g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}=0,\ \boldsymbol{x}\in\partial B_R.$$

From the assumption (3.19), we have

$$\rho_1\int_{B_R} g_1(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\neq 0.$$

It is easy to verify that

$$\int_{B_R}\left(\int_{B_R}\rho_1 g_1(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right) g_0(|\boldsymbol{x}-\boldsymbol{y}|)\,\mathrm{d}\boldsymbol{y}\neq 0 \quad \text{for } \boldsymbol{x}\in\partial B_R.$$

Therefore, we have $\rho_1 = \rho_2$, implying that$f_1(\boldsymbol{x}) = f_2(\boldsymbol{x})$ and $g_1(\boldsymbol{x}) = g_2(\boldsymbol{x})$, respectively. ∎

Next, we prove the uniqueness for density and internal sources in a domain with an anomalous inclusion. Let $\rho_0(\boldsymbol{x})$ be a positive background density which is known in advance and $\varrho_i, i=1,2$ be a positive constant denoting different anomalous inclusion supported in $\Omega_0\subset\Omega$.

COROLLARY 3.1. *Assume that (ρ_1, f_1, g_1) and (ρ_2, f_2, g_2) are two sets of configurations and supported in Ω, which satisfy the following conditions:*
(i) $\rho_i(\boldsymbol{x})=\rho_0(\boldsymbol{x})+\varrho_i\chi_{\Omega_0}$ *with $\varrho_i, i=1,2$ is a constant;*
(ii) $\nabla\rho_0(\boldsymbol{x})\cdot\boldsymbol{\iota}=0$, $\nabla f_i(\boldsymbol{x})\cdot\boldsymbol{\iota}=0$ *and* $\nabla g_i(\boldsymbol{x})\cdot\boldsymbol{\iota}=0, i=1,2$, *where $\boldsymbol{\iota}$ is an arbitrary direction vector in $\mathbb{R}^3$.*
If

$$\Lambda_{\rho_1,f_1,g_1}(t,\boldsymbol{x})=\Lambda_{\rho_2,f_2,g_2}(t,\boldsymbol{x}),\quad (t,\boldsymbol{x})\in\mathbb{R}_+\times\partial\Omega,$$

and supposing that

$$\int_{\Omega_0}\left(\int_{B_R}\rho_1(\boldsymbol{z})g_1(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right)h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\neq 0,\quad \boldsymbol{x}\in\Omega_0, \tag{3.23}$$

for any harmonic function $h(\boldsymbol{x})$,then

$$\varrho_1=\varrho_2,\quad f_1(\boldsymbol{x})=f_2(\boldsymbol{x}),\quad g_1(\boldsymbol{x})=g_2(\boldsymbol{x}).$$

Proof. We use analogue analysis as Theorem 3.3 and assume that ρ_0, f_i, and g_i only depend on the variables x_1, x_2 and write them as $\rho_0(\boldsymbol{x}) = \rho_0(x_1,x_2), f_i(\boldsymbol{x}) = f_i(x_1,x_2), g_i(\boldsymbol{x})=g_i(x_1,x_2)$ for $(x_1,x_2)\in\mathbb{R}^2, i=1,2$. Then we deduce that

$$\rho_1(\boldsymbol{x})f_1(\boldsymbol{x})=\rho_2(\boldsymbol{x})f_2(\boldsymbol{x}),\quad \rho_1(\boldsymbol{x})g_1(\boldsymbol{x})=\rho_2(\boldsymbol{x})g_2(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega.$$

It is easy to see that

$$\int_{B_R}\rho_1(\boldsymbol{y})f_1(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}=\int_{B_R}\rho_2(\boldsymbol{y})f_2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$
$$\int_{B_R}\rho_1(\boldsymbol{y})g_1(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}=\int_{B_R}\rho_2(\boldsymbol{y})g_2(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$

and

$$\int_{B_R} \rho_1(\boldsymbol{y}) g_1(\boldsymbol{y}) |\boldsymbol{x}-\boldsymbol{y}|^2 \, \mathrm{d}\boldsymbol{y} = \int_{B_R} \rho_2(\boldsymbol{y}) g_2(\boldsymbol{y}) |\boldsymbol{x}-\boldsymbol{y}|^2 \, \mathrm{d}\boldsymbol{y} \quad \text{for } \boldsymbol{x} \in \partial B_R.$$

Taking the above identities into (3.11) implies that

$$\begin{aligned}
&\int_{B_R} (\rho_1(\boldsymbol{y}) - 1) \left(\int_{B_R} \rho_1(\boldsymbol{z}) g_1(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z} \right) g_0(|\boldsymbol{x}-\boldsymbol{y}|) \, \mathrm{d}\boldsymbol{y} \\
&\quad = \int_{B_R} (\rho_2(\boldsymbol{y}) - 1) \left(\int_{B_R} \rho_2(\boldsymbol{z}) g_2(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z} \right) g_0(|\boldsymbol{x}-\boldsymbol{y}|) \, \mathrm{d}\boldsymbol{y} \quad \text{for } \boldsymbol{x} \in \partial B_R.
\end{aligned}$$

It follows from the proof of Theorem 3.1 that

$$\int_{B_R} (\rho_1 - \rho_2)(\boldsymbol{y}) \left(\int_{B_R} \rho_1(\boldsymbol{z}) g_1(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z} \right) h(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = 0 \quad \text{for } \boldsymbol{x} \in \partial B_R, \tag{3.24}$$

where $h(\cdot)$ is any harmonic function.

Substituting $\rho_i = \rho_0(\boldsymbol{x}) + \varrho_i \chi_{\Omega_0}$ into (3.24), we have

$$(\varrho_1 - \varrho_2) \int_{\Omega_0} \left(\int_{B_R} \rho_1(\boldsymbol{z}) g_1(\boldsymbol{z}) \, \mathrm{d}\boldsymbol{z} \right) h(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = 0.$$

Because of the condition (3.23), we get

$$\varrho_1 = \varrho_2,$$

which yields $g_1(\boldsymbol{x}) = g_2(\boldsymbol{x})$ and $f_1(\boldsymbol{x}) = f_2(\boldsymbol{x})$. ∎

Remark 3.3. In fact, the condition (3.19) is a special form for (3.23). There are other ways to achieve the nonzero condition for (3.23).

Besides the assumptions of density and internal sources in Theorem 3.3 and Corollary 3.1, we also consider whether there are other circumstances in which a more general uniqueness result can be derived. The following example illustrates a more general result.

Example 3.1. Let (ρ_1, f_1, g_1) and (ρ_2, f_2, g_2) be two sets of configurations and supported in Ω, which satisfy

$$(\rho_2, f_2, g_2) = (\rho_1 + a, f_1 + b, g_1 + c)$$

with $a(\boldsymbol{x}), b(\boldsymbol{x}), c(\boldsymbol{x}) \in L^\infty(\mathbb{R}^3)$ being nonnegative and supported in Ω. Furthermore, suppose that $f_1(\boldsymbol{x}), g_1(\boldsymbol{x}) > 0$. If

$$\Lambda_{\rho_1, f_1, g_1}(t, \boldsymbol{x}) = \Lambda_{\rho_2, f_2, g_2}(t, \boldsymbol{x}), \qquad (t, \boldsymbol{x}) \in \mathbb{R}_+ \times \partial\Omega,$$

then

$$a(\boldsymbol{x}) = b(\boldsymbol{x}) = c(\boldsymbol{x}) = 0.$$

Proof. Assume that at least one of $a(\boldsymbol{x}), b(\boldsymbol{x})$ and $c(\boldsymbol{x})$ is not zero. Without loss of generality, we set $a \neq 0$. Then we have

$$\begin{aligned}
(\rho_1 f_1 - \rho_2 f_2)(\boldsymbol{x}) &= (\rho_1 f_1 - (\rho_1 f_1 + a f_1 + b\rho_1 + ab))(\boldsymbol{x}) = -(a f_1 + b\rho_1 + ab)(\boldsymbol{x}) < 0, \\
(\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{x}) &= (\rho_1 g_1 - (\rho_1 g_1 + a g_1 + c\rho_1 + ac))(\boldsymbol{x}) = -(a g_1 + c\rho_1 + ac)(\boldsymbol{x}) < 0,
\end{aligned}$$

which yields

$$\int_{\Omega}(\rho_1 f_1-\rho_2 f_2)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}<0, \tag{3.25}$$

$$\int_{\Omega}(\rho_1 g_1-\rho_2 g_2)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}<0. \tag{3.26}$$

It follows from (3.9) and (3.10) that

$$\int_{\Omega}(\rho_1 f_1-\rho_2 f_2)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0,$$
$$\int_{\Omega}(\rho_1 g_1-\rho_2 g_2)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0$$

when taking $h(\boldsymbol{x})=1$. Therefore, the inequalities (3.25) and (3.26) are contradiction. The proof is completed. ∎

It can be seen from the above example that a more general uniqueness result can be obtained if additional assumptions of density and sources are considered. The detailed process will be shown in the following section.

4. Extension to general results.

In this section, the previous assumptions of density and internal sources will be replaced by assuming some size relationships of density, internal sources, and their coupling term, which implies more general unique results.

LEMMA 4.1. *Assume that (ρ_1,f_1,g_1) and (ρ_2,f_2,g_2) are two sets of configurations and supported in Ω. If*

$$\Lambda_{\rho_1,f_1,g_1}(t,\boldsymbol{x})=\Lambda_{\rho_2,f_2,g_2}(t,\boldsymbol{x}),\qquad (t,\boldsymbol{x})\in\mathbb{R}_+\times\partial\Omega,$$

and satisfies

$$(\rho_1 g_1)(\boldsymbol{x})\le(\rho_2 g_2)(\boldsymbol{x})\quad \textit{or}\quad (\rho_1 g_1)(\boldsymbol{x})\ge(\rho_2 g_2)(\boldsymbol{x}),\qquad \boldsymbol{x}\in\Omega, \tag{4.1}$$

then

$$\rho_1(\boldsymbol{x})g_1(\boldsymbol{x})=\rho_2(\boldsymbol{x})g_2(\boldsymbol{x}). \tag{4.2}$$

In addition, if

$$\int_{\mathbb{R}^3}(\rho_i g_i)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\neq 0,\qquad i=1,2,$$

then we have

$$\int_{\mathbb{R}^3}(\rho_1-\rho_2)(\boldsymbol{x})h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0, \tag{4.3}$$

where $h(\boldsymbol{x})$ is any harmonic function in $\mathbb{R}^3$.

Proof. Let $h(\boldsymbol{x})=1$. Then it follows from (3.9) and (3.10) that

$$\int_{B_R}(\rho_1 f_1)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}=\int_{B_R}(\rho_2 f_2)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{4.4}$$

$$\int_{B_R}(\rho_1 g_1)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}=\int_{B_R}(\rho_2 g_2)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{4.5}$$

Given by the conditions (4.1), if

$$(\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{x}) > 0 \quad \text{or} \quad (\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{x}) < 0,$$

then we have

$$\int_{B_R} (\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} > 0 \quad \text{or} \quad \int_{B_R} (\rho_1 g_1 - \rho_2 g_2)(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} < 0.$$

This is a contradiction with (4.5). Therefore, we get

$$(\rho_1 g_1)(\boldsymbol{x}) = (\rho_2 g_2)(\boldsymbol{x}),$$

implying that

$$\int_{B_R} (\rho_1 g_1)(\boldsymbol{y}) |\boldsymbol{x} - \boldsymbol{y}|^2\, \mathrm{d}\boldsymbol{y} = \int_{B_R} (\rho_2 g_2)(\boldsymbol{y}) |\boldsymbol{x} - \boldsymbol{y}|^2\, \mathrm{d}\boldsymbol{y}. \tag{4.6}$$

Substituting (4.4)–(4.6) into (3.11), we obtain

$$\int_{B_R} (\rho_1 - \rho_2)(\boldsymbol{y})\, g_0(|\boldsymbol{x} - \boldsymbol{y}|)\, \mathrm{d}\boldsymbol{y} = 0.$$

Repeating the process of (3.9) and (3.10) in Theorem 3.1, we derive an orthogonal relation

$$\int_{\mathbb{R}^3} (\rho_1 - \rho_2)(\boldsymbol{x})\, h(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0$$

for any harmonic function $h(\boldsymbol{x})$ in $\mathbb{R}^3$. ∎

COROLLARY 4.1. *With the same assumptions of Lemma* 4.1*, if* $\rho_i(\boldsymbol{x}), i = 1, 2$*, satisfies either of the conditions*

(i) $(\rho_1 - \rho_2)(\boldsymbol{x})$ *is a harmonic function in* $\mathbb{R}^3$*;*

(ii) $\rho_1(\boldsymbol{x}) \leq \rho_2(\boldsymbol{x})$ *or* $\rho_1(\boldsymbol{x}) \geq \rho_2(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$,

then

$$\rho_1(\boldsymbol{x}) = \rho_2(\boldsymbol{x}) \quad \text{and} \quad g_1(\boldsymbol{x}) = g_2(\boldsymbol{x}).$$

Furthermore, suppose that

$$f_1(\boldsymbol{x}) \leq f_2(\boldsymbol{x}) \quad \text{or} \quad f_1(\boldsymbol{x}) \geq f_2(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{4.7}$$

Then

$$f_1(\boldsymbol{x}) = f_2(\boldsymbol{x}).$$

Proof. For the first case, taking $(\rho_1 - \rho_2)(\boldsymbol{x}) = h(\boldsymbol{x})$ into (4.3) implies that

$$\int_\Omega h^2(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0.$$

Thus, we can obtain $\rho_1(\boldsymbol{x}) = \rho_2(\boldsymbol{x})$.

For the second case, substituting $h(\boldsymbol{x}) = 1$ into (4.3), we get

$$\int_\Omega (\rho_1 - \rho_2)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0.$$

By using the conditions of $\rho_1(\boldsymbol{x})$ and $\rho_2(\boldsymbol{x})$, we deduce that $\rho_1(\boldsymbol{x})=\rho_2(\boldsymbol{x})$. It follows from (4.2) that $g_1(\boldsymbol{x})=g_2(\boldsymbol{x})$.

Let $\rho_1(\boldsymbol{x})=\rho_2(\boldsymbol{x})=\rho(\boldsymbol{x})$ and $h(\boldsymbol{x})=1$. Plugging them into (3.9), we have

$$\int_\Omega \rho(\boldsymbol{y})(f_1-f_2)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}=0.$$

Therefore, given by (4.7), $f_1(\boldsymbol{x})=f_2(\boldsymbol{x})$ is proved. ∎

Appendix A. The well-posedness for the exterior boundary problem. We prove the well-posedness for the exterior boundary value problem

$$\begin{cases}\Delta^2\hat{u}-\kappa^4\hat{u}=0 & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega},\\ \hat{u}=\phi_1, \quad \Delta\hat{u}=\phi_2 & \text{on} \quad \partial\Omega\end{cases} \tag{A.1}$$

with the Sommerfeld radiation conditions

$$\lim_{r\to\infty} r\left(\partial_r\hat{u}(\kappa,\boldsymbol{x})-\mathrm{i}\kappa\hat{u}(\kappa,\boldsymbol{x})\right)=0, \qquad \lim_{r\to\infty} r\left(\partial_r(\Delta\hat{u}(\kappa,\boldsymbol{x}))-\mathrm{i}\kappa(\Delta\hat{u}(\kappa,\boldsymbol{x}))\right)=0. \tag{A.2}$$

Theorem A.1. *There exists a unique solution for* (A.1) *with the Sommerfeld conditions* (A.2).

Proof. Let $\tilde{\hat{u}}=\hat{u}_1-\hat{u}_2$, where $\hat{u}_1$ and $\hat{u}_2$ are solutions of (A.1)–(A.2). Then it satisfies

$$\begin{cases}\Delta^2\tilde{\hat{u}}-\kappa^4\tilde{\hat{u}}=0 & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega},\\ \tilde{\hat{u}}=0, \quad \Delta\tilde{\hat{u}}=0 & \text{on} \quad \partial\Omega\end{cases} \tag{A.3}$$

with the Sommerfeld conditions (A.2).

Denote that $\Omega_r=B_r\backslash\Omega$, where B_r is a sphere of radius r and center at the origin. Let $\Delta\tilde{\hat{u}}=-\kappa^2 w$. Then the first equation in (A.3) turns to

$$\begin{cases}\Delta\tilde{\hat{u}}+\kappa^2 w=0 & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega},\\ \Delta w+\kappa^2\tilde{\hat{u}}=0 & \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega}.\end{cases}$$

Taking the inner product of the first equation with $\tilde{\hat{u}}$ and the second equation with w over Ω_r and summing together, we compute

$$\int_{\Omega_r} \Delta\tilde{\hat{u}}\bar{\tilde{\hat{u}}}+\kappa^2 w\bar{\tilde{\hat{u}}}+\Delta w\bar{w}+\kappa^2\tilde{\hat{u}}\bar{w}\,\mathrm{d}\boldsymbol{x}=0, \tag{A.4}$$

$$\int_{\Omega_r} \tilde{\hat{u}}\Delta\bar{\tilde{\hat{u}}}+\tilde{\hat{u}}\kappa^2\bar{w}+w\Delta\bar{w}+w\kappa^2\bar{\tilde{\hat{u}}}\,\mathrm{d}\boldsymbol{x}=0. \tag{A.5}$$

Subtracting (A.5) with (A.4) and taking integration by parts, we have

$$\begin{aligned}0&=\int_{\Omega_r} \Delta\tilde{\hat{u}}\bar{\tilde{\hat{u}}}+\kappa^2 w\bar{\tilde{\hat{u}}}+\Delta w\bar{w}+\kappa^2\tilde{\hat{u}}\bar{w}\,\mathrm{d}\boldsymbol{x}-\int_{\Omega_r} \tilde{\hat{u}}\Delta\bar{\tilde{\hat{u}}}+\tilde{\hat{u}}\kappa^2\bar{w}+w\Delta\bar{w}+w\kappa^2\bar{\tilde{\hat{u}}}\,\mathrm{d}\boldsymbol{x}\\ &=\int_{\Omega_r} \Delta\tilde{\hat{u}}\bar{\tilde{\hat{u}}}+\Delta w\bar{w}\,\mathrm{d}\boldsymbol{x}-\int_{\Omega_r} \tilde{\hat{u}}\Delta\bar{\tilde{\hat{u}}}+w\Delta\bar{w}\,\mathrm{d}\boldsymbol{x}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial B_r} \partial_\nu \tilde{\hat{u}}\bar{\tilde{\hat{u}}} + \partial_\nu w\bar{w}\,\mathrm{d}S - \int_{\partial\Omega} \partial_\nu \tilde{\hat{u}}\bar{\tilde{\hat{u}}} + \partial_\nu w\bar{w}\,\mathrm{d}S - \int_{\Omega_r} |\nabla\tilde{\hat{u}}|^2 + |\nabla w|^2\,\mathrm{d}\boldsymbol{x} \\
&\quad - \int_{\partial B_r} \tilde{\hat{u}}\partial_\nu\bar{\tilde{\hat{u}}} + w\partial_\nu\bar{w}\,\mathrm{d}S + \int_{\partial\Omega} \tilde{\hat{u}}\partial_\nu\bar{\tilde{\hat{u}}} + w\partial_\nu\bar{w}\,\mathrm{d}S + \int_{\Omega_r} |\nabla\tilde{\hat{u}}|^2 + |\nabla w|^2\,\mathrm{d}\boldsymbol{x} \\
&= 2\mathrm{iIm}\int_{\partial B_r} \partial_\nu \tilde{\hat{u}}\bar{\tilde{\hat{u}}} + \partial_\nu\bar{w}w\,\mathrm{d}S \\
&= 2\mathrm{iIm}\int_{\partial B_r} \mathrm{i}\kappa|\tilde{\hat{u}}|^2 + \mathrm{i}\kappa|w|^2\,\mathrm{d}S + \mathcal{O}\left(\frac{1}{r^2}\right)(2\mathrm{iIm}\int_{\partial B_r} \bar{\tilde{\hat{u}}} + \bar{w}\,\mathrm{d}S) \\
&= 2\mathrm{i}\kappa\int_{\partial B_r} |\tilde{\hat{u}}|^2 + |w|^2\,\mathrm{d}S + \mathcal{O}\left(\frac{1}{r^2}\right)(2\mathrm{iIm}\int_{\partial B_r} \bar{\tilde{\hat{u}}} + \bar{w}\,\mathrm{d}S).
\end{aligned}
$$

Letting $r \to \infty$, we obtain the identity

$$
\lim_{r\to\infty}\int_{\partial B_r} |\tilde{\hat{u}}|^2 + |w|^2\,\mathrm{d}S = 0.
$$

It follows from Rellich's lemma that $\tilde{\hat{u}} = 0$ in $\mathbb{R}^3\backslash\overline{\Omega}$.

Next, we prove the existence of the solution. The proof depends on the boundary integral equation.

Assume that u_H and u_M are the solutions of $\Delta u + \kappa^2 u = 0$ and $\Delta u - \kappa^2 u = 0$, respectively. Then $\hat{u} = u_H + u_M$ is a solution to $\Delta^2\hat{u} - \kappa^4\hat{u} = 0$.

Let $G_H(|\boldsymbol{x}-\boldsymbol{y}|)$ be the Green function of the Helmholtz equation

$$
\Delta u + \kappa^2 u = 0
$$

and $G_M(|\boldsymbol{x}-\boldsymbol{y}|)$ be the Green function of the modified Helmholtz equation

$$
\Delta u - \kappa^2 u = 0.
$$

Given integrable functions φ and ψ, we define the single-layer potential and the double-layer potential as

$$
\begin{aligned}
v_s(\boldsymbol{x}) &:= \int_{\partial\Omega} G_H(|\boldsymbol{x}-\boldsymbol{y}|)\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\partial\Omega} G_M(|\boldsymbol{x}-\boldsymbol{y}|)\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\mathbb{R}^3\backslash\partial\Omega, \\
v_d(\boldsymbol{x}) &:= \int_{\partial\Omega} \frac{\partial G_H(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial\nu(\boldsymbol{y})}\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\partial\Omega} \frac{\partial G_M(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial\nu(\boldsymbol{y})}\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\mathbb{R}^3\backslash\partial\Omega.
\end{aligned}
$$

Taking a solution of a combination of the double- and single-layer potential,

$$
\hat{u}(\boldsymbol{x}) = v_d(\boldsymbol{x}) - \mathrm{i}\gamma v_s(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^3\backslash\partial\Omega, \tag{A.6}
$$

where γ is a nonzero constant. By the jump relations, we see that $\hat{u}$ given by (A.6) in $\mathbb{R}^3\backslash\partial\Omega$ solves the exterior problem (A.1) provided that (φ,ψ) is a solution of the integral equation

$$
\begin{cases}
(-\mathrm{i}\gamma S^H + K^H + \frac{1}{2})\varphi + (-\mathrm{i}\gamma S^M + K^M + \frac{1}{2})\psi = \phi_1, \\
(-\mathrm{i}\gamma S^H + K^H + \frac{1}{2})\varphi - (-\mathrm{i}\gamma S^M + K^M + \frac{1}{2})\psi = -\frac{\phi_2}{\kappa^2},
\end{cases}
$$

where

$$
\begin{aligned}
(S^H\varphi)(\boldsymbol{x}) &:= \int_{\partial\Omega} G_H(|\boldsymbol{x}-\boldsymbol{y}|)\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\partial\Omega,\\
(S^M\psi)(\boldsymbol{x}) &:= \int_{\partial\Omega} G_M(|\boldsymbol{x}-\boldsymbol{y}|)\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\partial\Omega,\\
(K^H\varphi)(\boldsymbol{x}) &:= \int_{\partial\Omega} \frac{\partial G_H(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial\nu(\boldsymbol{y})}\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\partial\Omega,\\
(K^M\psi)(\boldsymbol{x}) &:= \int_{\partial\Omega} \frac{\partial G_M(|\boldsymbol{x}-\boldsymbol{y}|)}{\partial\nu(\boldsymbol{y})}\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\partial\Omega.
\end{aligned}
$$

By direct calculation, we have

$$
\begin{cases}
(-\mathrm{i}\gamma S^H + K^H + \frac{1}{2})\varphi = \frac{1}{2}(\phi_1 - \frac{\phi_2}{\kappa^2}),\\
(-\mathrm{i}\gamma S^M + K^M + \frac{1}{2})\psi = \frac{1}{2}(\phi_1 + \frac{\phi_2}{\kappa^2}).
\end{cases} \tag{A.7}
$$

Therefore, the existence of (φ,ψ) to (A.7) can be established by the Riesz–Fredholm theory with the compactness of S^H, S^M, K^H, and K^M (see [3]). Then the representation (A.6) is a solution for the exterior boundary value problem (A.1). The proof is completed. ∎

Acknowledgment. The authors would like to thank the anonymous referees for many constructive and insightful comments and suggestions, which have led to significant improvements on the presentation and result of the paper.

REFERENCES

[1] Y. M. Assylbekov and Y. Yang, *Determining the first order perturbation of a polyharmonic operator on admissible manifolds*, J. Differential Equations, 262 (2017), pp. 590–614.

[2] X. Cao and H. Liu, *Determining a fractional Helmholtz equation with unknown source and scattering potential*, Commun. Math. Sci., 17 (2019), pp. 1861–1876.

[3] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, Springer-Verlag, Berlin, 2019.

[4] Y. Deng, J. Li, and H. Liu, *On identifying magnetized anomalies using geomagnetic monitoring*, Arch. Ration. Mech. Anal., 231 (2019), pp. 153–187.

[5] Y. Deng, J. Li, and H. Liu, *On identifying magnetized anomalies using geomagnetic monitoring within a magnetohydrodynamic model*, Arch. Ration. Mech. Anal., 235 (2020), pp. 691–721.

[6] Y. Deng, H. Liu, and W. Y. Tsui, *Identifying varying magnetic anomalies using geomagnetic monitoring*, Discrete Contin. Dyn. Syst., 40 (2020), pp. 6411–6440.

[7] Y. Deng, H. Liu, and G. Uhlmann, *On an inverse boundary problem arising in brain imaging*, J. Differential Equations, 267 (2019), pp. 2471–2502.

[8] L. C. Evans, *Partial Differential Equations,* Graduate Studies in Mathematics 19, 2nd ed., American Mathematical Society, Providence, RI, 2010.

[9] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic Boundary Value Problems*, Lecture Notes in Mathematics 1991, Springer-Verlag, Berlin, 2010.

[10] C. Knox and A. Moradifam, *Determining both the source of a wave and its speed in a medium from boundary measurements*, Inverse Problems, 36 (2020), 025002.

[11] K. Krupchyk, M. Lassas, and G. Uhlmann, *Determining a first order perturbation of the biharmonic operator by partial boundary measurements*, J. Funct. Anal., 262 (2012), pp. 1781–1801.

[12] R. Leis, *Initial-boundary value problems in mathematical physics*, in Modern Mathematical Methods in Diffraction Theory and Its Applications in Engineering (Freudenstadt, 1996), Methoden Verfahren der mathematischen Physik 42, Peter Lang, Frankfurt, 1997, pp. 125–144.

[13] J. Li, H. Liu, and S. Ma, *Determining a random Schrödinger equation with unknown source and potential*, SIAM J. Math. Anal., 51 (2019), pp. 3465–3491.

[14] J. Li, H. Liu, and S. Ma, *Determining a random schrödinger operator: Both potential and source are random*, Comm. Math. Phys., 381 (2021), pp. 527–556.

[15] Y.-H. Lin, H. Liu, X. Liu, and S. Zhang, *Determining a Nonlinear Hyperbolic System with Unknown Sources and Nonlinearity*, preprint, arXiv:2107.10219, 2023.

[16] Y.-H. Lin, H. Liu, X. Liu, and S. Zhang, *Simultaneous recoveries for semilinear parabolic systems*, Inverse Problems, 38 (2022), 115006.

[17] B. Liu, *Stability estimates in a partial data inverse boundary value problem for biharmonic operators at high frequencies*, Inverse Probl. Imaging, 14 (2020), pp. 783–796.

[18] H. Liu and G. Uhlmann, *Determining both sound speed and internal source in thermo- and photo-acoustic tomography*, Inverse Problems, 31 (2015), 105005.

[19] S. Mayboroda and V. Maz'ya, *Boundedness of the gradient of a solution and Wiener test of order one for the biharmonic equation*, Invent. Math., 175 (2009), pp. 287–334.

[20] N. V. Movchan, R. C. McPhedran, A. B. Movchan, and C. G. Poulton, *Wave scattering by platonic grating stacks*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 465 (2009), pp. 3383–3400.

[21] T. Tyni and V. Serov, *Scattering problems for perturbations of the multidimensional biharmonic operator*, Inverse Probl. Imaging, 12 (2018), pp. 205–227.

[22] Y. Yang, *Determining the first order perturbation of a bi-harmonic operator on bounded and unbounded domains from partial data*, J. Differential Equations, 257 (2014), pp. 3607–3639.