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Event-Triggered H_∞ Output Consensus of Heterogeneous Linear Multi-Agent Systems ☆

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Abstract

This paper addresses the leader-following H_∞ output consensus problem of heterogeneous linear multi-agent systems (MASs) under directed graphs. Both state feedback and output feedback based event-triggered control (ETC) protocols are developed. First, to avoid continuous communication between neighboring agents, the open-loop estimation method is used in the proposed ETC protocols. Second, a unified time- and event-triggering mechanism with an explicit minimum inter-event time (MIET) is further developed based on the open-loop estimation method so that Zeno behavior can be excluded. It is shown that under the proposed ETC protocols, the controlled MAS can achieve H_∞ output consensus. Finally, two numerical examples are provided to verify effectiveness of the proposed ETC protocols.

Keywords:
Event-triggered control protocol, H_∞ output consensus, Heterogeneous, Multi-agent systems, Leader-following

1. Introduction

Cooperative control of multi-agent systems (MASs) has attracted considerable attention in the control community these years [1–4]. Plenty of significant results have been obtained in various topics such as consensus, formation, and cooperative output regulation [5–8], to name just a few. One of the most striking differences for cooperative control of MASs, in comparison to control of single systems, is that communication mechanism among agents should be designed adequately to achieve control objectives.

It is well known that continuous communication for MASs, which would incur excessive usage of communication resources and energy, is neither efficient nor realistic for real-world

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applications due to limited communication bandwidth and on-board power supply of each agent. To reduce communication burden, event-triggered control (ETC) techniques, which were first proposed for some single systems [9, 10], have been widely investigated for cooperative control of MASs in the last decade [11–21]. In earlier works [11, 12], in which consensus of MASs with single-integrator dynamics was studied, the information transmission between neighboring agents would not be executed until a predefined event-triggering condition was satisfied. Thus, communication burden was reduced. Then, event-triggered cooperative control methods were extended to MASs with homogeneous linear dynamics [13, 14], and heterogeneous linear dynamics [15–17] successively. Recently, fully distributed ETC protocols were further developed [17–19], in which no global knowledge of the communication network was needed. Moreover, communication topologies were extended from fixed to time-varying graphs [20, 21]. For other significant results, one can refer to [22] and references therein.

It should be pointed out that MASs in real-world applications are often subject to disturbances and/or noises caused by environment or measurement. In the presence of these unstructured disturbances, various cooperative control objectives have been introduced. For example, consensus with bounded errors was investigated for MASs in [23–25]. Compared with consensus with bounded errors, the well-known H_∞ performance technique is a more useful tool to investigate the disturbance attenuation property of the controlled MASs [26, 27]. Though the event-triggered cooperative control of MASs without disturbance has been intensively studied [11–21], few results on event-triggered H_∞ consensus for continuous-time MASs have been reported. That is the motivation underpinning the current research. In fact, the event-triggered H_∞ consensus problem is challenging in the following two aspects. **1)** The ETC protocols are challenging to design and analyze for H_∞ consensus of MASs with heterogeneous dynamics. The H_∞ consensus problem for MASs is often formulated as a set of matrix inequalities, whose solvability is the key to the control protocol design. In contrast to [26, 27], in which continuous communication was needed, and [28], in which only homogeneous linear MASs were studied, the matrix inequalities for event-triggered H_∞ consensus of heterogeneous MASs are more challenging to obtain, and their solvability is more difficult to be ensured. **2)** It is challenging to exclude Zeno behavior in the presence of external disturbances or measurement noises. Zeno behavior means an infinite number of events in a finite time interval. Note that to ensure the feasibility of ETC protocols, it is necessary to exclude Zeno behavior, and two approaches have been developed for this purpose: excluding Zeno behavior by the argument of contradiction [17, 18], and providing a positive minimum inter-event time (MIET) [12, 13, 16]. Compared with the first approach, a positive MIET is more desirable for physical implementations. As revealed in [29], many existing event-triggering mechanisms (ETMs) cannot ensure a nonzero MIET for continuous-time systems in this scenario. As a result, most existing works on event-triggered H_∞ cooperative control were restricted to discrete-time MASs or based on periodic sampled-data communication mechanisms so that Zeno behavior was avoided [30–33].

Motivated by the above observations, the event-triggered leader-following H_∞ output consensus problem of heterogeneous linear MASs via intermittent communication is addressed in this paper. The main contributions of this work in comparison with those existing relevant works are summarized as follows:

1) By adopting the open-loop estimation method as in [24], a unified time- and event-triggering mechanism with an explicit MIET in [16] is further developed in this work. The proposed ETC protocol thus has two advantages. First, in contrast to [24, 25], a positive MIET can be ensured in the presence of external disturbances. Second, contrary to [32, 33], both the controller and event-triggering mechanism of the ETC protocol depend only on event-triggered communication at the triggering instants of each agent.

2) Unlike [23–25], in which only consensus with bounded errors was obtained for MASs subject to external disturbances, a novel Lyapunov functional is developed so that the H_∞ consensus analysis can be accomplished for the MAS under the proposed event-triggering mechanism. Specifically, the new Lyapunov functional is designed to be piecewise continuous so that the system states for $t<0$ are not needed in the H_∞ consensus analysis, which is developed from the analytical approaches for time-delay systems in [34]. Moreover, the disturbances considered in this work are root-mean-squared (RMS) bounded rather than norm bounded in [23–25].

3) Contrary to [28, 35, 36], where only homogeneous MASs were considered, sufficient conditions to ensure a desired H_∞ consensus performance cannot be directly derived from the dynamics of each agent in heterogeneous MASs, since the states of the agents may have different dimensions. To overcome this challenge, a hierarchical event-triggered control method is applied to obtain the sufficient conditions. In contrast with [33], the feasibility of the obtained conditions for the H_∞ consensus problem with some positive disturbance attenuation indexes can be theoretically guaranteed. Specifically, the H_∞ consensus problem under the proposed ETC protocol for heterogeneous MASs is formulated as a set of matrix inequalities, whose solvability is theoretically ensured under some loose assumptions.

The rest of this paper is organized as follows. In Section 2, preliminaries including some useful technical lemmas, and problem formulation are presented. In Section 3, two ETC protocols are proposed to solve the leader-following H_∞ output consensus problem of heterogeneous linear MASs. Two numerical examples are provided to verify effectiveness of the proposed ETC protocols in Section 4, which is followed by some conclusions in Section 5.

2. Preliminaries and Problem Formulation

In this section, notations, preliminaries on graph theory, some useful lemmas and problem formulation are presented.

2.1. Notations

$\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent the sets of real vectors with dimension n and real matrices with dimension $m \times n$, respectively. $\mathbb{Z}_{>0}$ represents the set of positive integers. 1_n is the column vector of dimension n whose entries all equal 1. $\|\cdot\|$ denotes the Euclidean norm for vectors or 2-norm for matrices. For a matrix $M \in \mathbb{R}^{n\times n}$, denote its transpose as M^{T}. For $M_i \in \mathbb{R}^{m\times n}$, $i = 1, \dots, N$, $\mathrm{diag}\{M_1, \dots, M_N\} \in \mathbb{R}^{mN\times nN}$ represents a block diagonal matrix whose diagonal elements are $M_i, i = 1, ..., N$. Similarly, $\mathrm{diag}\{k_1, \dots, k_N\} \in \mathbb{R}^{N\times N}$ is a diagonal matrix whose diagonal elements are $k_i \in \mathbb{R}$, $i = 1, \dots, N$. Denote the symmetric terms in a matrix by $\star$. For vectors $x_i \in \mathbb{R}^{n_i}$, $i = 1, \dots, N$, define $\mathrm{col}(x_1, \dots, x_N) = (x_1^{\mathrm{T}}, \dots, x_N^{\mathrm{T}})^{\mathrm{T}}$. The symbol $\otimes$ represents the Kronecker product. For a function $f : [0, +\infty) \to \mathbb{R}$, $D^+f(t)$ denotes its upper Dini derivative, where $D^+f(t) = \limsup_{h\to 0^+} \frac{f(t+h)-f(t)}{h}$.

2.2. Graph Theory

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is adopted to denote a directed graph, where the set of nodes $\mathcal{V} = \{1, \dots, N\}$ represents N agents in the system, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges among agents. The edges are described by the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. For $i, j \in \mathcal{V}$, $(j, i) \in \mathcal{E}$ implies agent i can receive information from agent j. Hence, agent j is called an in-neighbor of agent i. In this case, we denote $a_{ij} = 1$, otherwise $a_{ij} = 0$. In this paper, we assume that $a_{ii} = 0$ for all the agents. Denote the set of agent i's in-neighbors by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The Laplacian matrix of the digraph $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ij} = \sum_{k=1}^{N} a_{ik}$ for $i = j$. There is a path from agent j to agent i if the graph contains a sequence of edges in the form of $\{(j, k_1), (k_1, k_2), \dots, (k_m, i)\}$. A directed graph contains a spanning tree if there is a node such that there exists a path from it to every other node, and this node is called the root of the spanning tree.

For a leader-following MAS, the leader agent is denoted as node 0 by convention. Consequently, the graph is denoted by $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, where $\bar{\mathcal{V}} = \mathcal{V} \cup \{0\}$ and $\bar{\mathcal{E}} \subseteq \bar{\mathcal{V}} \times \bar{\mathcal{V}}$. If a follower i can access the leader directly, then $a_{i0} = 1$, $i \in \mathcal{V}$, otherwise $a_{i0} = 0$. The leader receives no information from any follower, that is, $a_{0i} = 0$, $\forall i \in \mathcal{V}$. Furthermore, define $\Lambda = \mathrm{diag}\{a_{10}, \dots, a_{N0}\}$ and $\mathcal{H} = \mathcal{L} + \Lambda$.

2.3. Some Technical Lemmas

Lemma 2.1. *(Young's Inequality, [37]) Assume that D, F and X are matrices with appropriate dimensions and X is positive definite. For any vectors $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, the following inequality holds,*

$$2x^{\mathrm{T}}DFy \le x^{\mathrm{T}}DXD^{\mathrm{T}}x + y^{\mathrm{T}}F^{\mathrm{T}}X^{-1}Fy. \tag{1}$$

Lemma 2.2. *(Jensen's Inequality, [38]) For any constant symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$, scalar $h > 0$, vector function $f : [0, h] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then*

$$h\int_0^h f^{\mathrm{T}}(\tau)Xf(\tau)\mathrm{d}\tau \geq \left[\int_0^h f(\tau)\,\mathrm{d}\,\tau\right]^{\mathrm{T}} X \left[\int_0^h f(\tau)\,\mathrm{d}\tau\right]. \tag{2}$$

Lemma 2.3. *([39]) If $\bar{\mathcal{G}}$ contains a directed spanning tree with node 0 as the root, then there exists a positive definite diagonal matrix $R = \mathrm{diag}\{r_1, \ldots, r_N\}$, such that $R\mathcal{H} + \mathcal{H}^{\mathrm{T}}R > 0$. One such matrix R can be obtained by $(r_1, \ldots, r_N)^{\mathrm{T}} = (\mathcal{H}^{\mathrm{T}})^{-1}1_N$. Denote the smallest eigenvalue of $R\mathcal{H} + \mathcal{H}^{\mathrm{T}}R$ by $\lambda_1 > 0$.*

2.4. Problem Formulation

Consider a leader-following heterogeneous linear MAS with N followers described as follows:

$$\begin{aligned}\dot{x}_i(t) =& A_i x_i(t) + B_i u_i(t) + B_{wi} w_i(t),\\ y_i(t) =& C_i x_i(t),\end{aligned} \tag{3}$$

and a leader described as follows:

$$\begin{aligned}\dot{v}(t) =& Sv(t) + B_{w0} w_0(t),\\ y_0(t) =& C_0 v(t),\end{aligned} \tag{4}$$

where $i = 1, \ldots, N$, $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, $w_i(t) \in \mathbb{R}^{p_i}$, $y_i(t) \in \mathbb{R}^l$ represent the state, input, disturbance and output of follower i respectively. $v(t) \in \mathbb{R}^q$, $w_0(t) \in \mathbb{R}^{p_0}$ and $y_0(t) \in \mathbb{R}^l$ are the state, disturbance and output of the leader respectively. The matrices A_i, B_i, B_{wi}, C_i and S have compatible dimensions. The output consensus error of each follower, $z_i(t) \in \mathbb{R}^l$, is defined as follows:

$$z_i(t) = y_i(t) - y_0(t), \quad i = 1, \ldots, N. \tag{5}$$

Denote $z(t) = \mathrm{col}\big(z_1(t), \ldots, z_N(t)\big)$, and $w(t) = \mathrm{col}\big(w_0(t), w_1(t), \ldots, w_N(t)\big)$. Now we are ready to present the definition of event-triggered H_∞ output consensus of the heterogeneous linear MAS composed of (3) and (4).

Definition 2.1. *(Event-triggered leader-following H_∞ output consensus problem) Given an H_∞ performance index $\gamma > 0$, design a distributed ETC protocol such that the following two properties hold for the heterogeneous linear MAS composed of (3) and (4).*

1) If $w(t) \equiv 0$, the controlled MAS achieves leader-following output consensus asymptotically for any initial conditions, that is,

$$\lim_{t\to+\infty} z_i(t) = 0, \quad i = 1, \ldots, N. \tag{6}$$

2) If $w(t) \neq 0$, the following inequality holds for γ with zero-initial condition $\|v(0)\| = \|x_i(0)\| = \|z(0)\| = 0$,

$$\frac{1}{\mathfrak{T}}\int_0^{\mathfrak{T}} \|z(t)\|^2 - \gamma^2\|w(t)\|^2 \,\mathrm{d}\,t < 0, \forall \mathfrak{T} > 0. \tag{7}$$

To solve the H_∞ output consensus problem via distributed ETC protocols, the following assumptions are needed.

Assumption 1. *The communication topology contains a spanning tree with the leader agent as its root node.*

Assumption 2. *The following linear matrix equations,*

$$\begin{aligned} \Pi_i S &= A_i \Pi_i + B_i \Gamma_i, \\ C_i \Pi_i &= C_0, \end{aligned} \tag{8}$$

have solution pairs, (Π_i, Γ_i), $i = 1, \ldots, N$.

Assumption 3. *(A_i, B_i), $i = 1, \ldots, N$, are all stabilizable.*

Assumption 4. *The disturbance signal $w(t)$ satisfies*

$$\lim_{\mathfrak{T}\to+\infty} \left[\frac{1}{\mathfrak{T}}\int_0^{\mathfrak{T}} \|w(t)\|^2 \,\mathrm{d}\,t\right]^{\frac{1}{2}} < +\infty. \tag{9}$$

Remark 1. *Assumptions 1-3 are commonly used in cooperative control of heterogeneous linear MASs [15–18]. Equations in (8) are regulator equations, the solvability of which is necessary for solving the output consensus problem of heterogeneous linear MASs [6]. A sufficient condition to ensure Assumption 2 is given as follows [40]:*

$$\operatorname{rank}\begin{pmatrix} A_i - \rho I & B_i \\ C_i & 0 \end{pmatrix} = n_i + l, \quad \forall \rho \in \sigma(S), i = 1, \ldots, N,$$

where $\sigma(S)$ represents the spectrum of S. Assumption 4 means that the external disturbance signal $w(t)$ is RMS bounded, which implies the disturbance is not only limited to L_2 or norm bounded signals [36].

3. Main Results

In this section, the main results are presented. Two distributed ETC protocols based on state feedback and output feedback respectively are proposed to tackle the leader-following H_∞ output consensus problem of heterogeneous MASs.

3.1. Event-triggered H_∞ Output Consensus Based on State Feedback

In this subsection, we assume that each agent can measure its own state. The distributed ETC protocol based on state feedback is designed as follows:

$$u_i(t) = K_{i1}x_i(t) + K_{i2}\eta_i(t), \tag{10a}$$

$$\dot{\eta}_i(t) = S\eta_i(t) - \mu \sum_{j\in\mathcal{N}_i} a_{ij}[\hat{\eta}_i(t) - \hat{\eta}_j(t)], \tag{10b}$$

where the positive constant μ, controller gain matrices $K_{i1} \in \mathbb{R}^{m_i\times n_i}$ and $K_{i2} \in \mathbb{R}^{m_i\times q}$ will be designed later. As in [15–17], $\eta_i(t) \in \mathbb{R}^q$ in (10b) is the state of agent i's distributed observer which is used to estimate the state of the leader. The initial value of each η_i is selected as $\eta_i(0) = 0$. $\hat{\eta}_i(t) \in \mathbb{R}^q$ and $\hat{\eta}_j(t) \in \mathbb{R}^q$ represent the open-loop estimates of the observer states of agent i and its in-neighboring agent j respectively which are defined as follows:

$$\hat{\eta}_i(t) = \mathrm{e}^{S(t-t_k^i)}\,\eta_i(t_k^i), \quad t \in [t_k^i, t_{k+1}^i), \tag{11a}$$

$$\hat{\eta}_j(t) = \mathrm{e}^{S(t-t_{k'}^j)}\,\eta_j(t_{k'}^j), \quad t_{k'}^j = \max_{k\in\mathbb{Z}_{>0}}\{t_k^j | t_k^j \le t\}, \tag{11b}$$

where $t_{k'}^j$ represents the latest triggering time instant of agent j. In particular, the open-loop estimate of the leader agent is defined as follows:

$$\hat{\eta}_0(t) = \mathrm{e}^{S(t-t_{k'}^0)}\,v(t_{k'}^0), \quad t_{k'}^0 = \max_{k\in\mathbb{Z}_{>0}}\{t_k^0 | t_k^0 \le t\}. \tag{12}$$

The triggering time sequence of each agent i, $\{t_k^i | k = 1, 2, \ldots\}$ is determined by the following unified time- and event-triggering mechanism:

$$t_{k+1}^i = \max\left\{\inf\left\{t > t_k^i \middle| \|e_i(t)\| \ge \rho_i \left\|\sum_{j=0}^N a_{ij}[\hat{\eta}_i(t) - \hat{\eta}_j(t)]\right\|\right\}, t_k^i + h_i\right\}, \tag{13}$$

where $i = 0, 1, \ldots, N$, $e_i(t)$ is the estimation error of the distributed observer state of agent i, which is defined as follows:

$$e_i(t) = \hat{\eta}_i(t) - \eta_i(t), t \in [t_k^i, t_{k+1}^i), i = 0, 1, \ldots, N, \tag{14}$$

with $\eta_0(t) = v(t)$.

As in [16, 33], the proposed triggering mechanism (13) contains two parts. The first part is an event-triggering condition, and the second one is a time-triggering condition. Note that agent i will not be triggered at time instant t, if $t - t_k^i < h_i$, $t_k^i = \max_{l\in\mathbb{Z}_{>0}}\{t_l^i | t_l^i \le t\}$, which implies that the MIET of agent i is exactly h_i. Therefore, Zeno behavior is strictly excluded for all the agents. In addition, it can be seen from the controller (10) and the triggering mechanism (13) that each agent i only uses its neighboring agents' information $\eta_j(t^j_{k'})$ and its own information. Thus, the proposed ETC protocol is distributed, and only event-based communication is required.

Note that both the controller (10) and the triggering mechanism (13) are designed based on the open-loop estimation method used in [24], which is different from the combinational measurement technique in [16] and the zero-order holder in [33]. Hence, the ETC protocol composed of (10) and (13) can be applied to MASs subject to unknown disturbances in contrast with [16], and the MAS under the ETC protocol of this work converges faster than [33]. It should be pointed out that $e_i(t)$ is reset to 0 after t^i_k immediately, i.e., $e_i(t^{i+}_k) = 0$, $i = 0, \ldots, N$. When implementing the above ETC protocol, all the agents are designed to be triggered at $t = 0$, that is $t^i_1 = 0$, $\forall i = 0, 1, \ldots, N$.

Next, the main result of this subsection will be presented.

Theorem 1. *Under Assumptions 1-4, the leader-following H_∞ output consensus problem of the heterogeneous linear MAS (3) and (4) with the disturbance attenuation index γ can be solved by the ETC protocol consisting of (10) and (13) with controller gain matrices K_{i1}, $K_{i2} = \Gamma_i - K_{i1}\Pi_i$, and positive parameters μ, ρ_i, h_i, if there exist positive definite matrices P_1, P_2, positive parameters δ_i, σ_i, $i = 1, \ldots, 3$ and c such that the following inequalities hold,*

$$\Sigma_1 < 0, \Sigma_2 < 0, \tag{15}$$

and

$$\Psi_i < 0, \ \ i = 1, \ldots, 5, \tag{16}$$

where

$$\Sigma_1 = \begin{bmatrix} \bar{A}^{\mathrm{T}} P_1 + P_1 \bar{A} + C^{\mathrm{T}} C & \star & \star \\ \bar{B}_w^{\mathrm{T}} P_1 & -(\gamma^2 - \delta_1) I & \star \\ \bar{B}_1^{\mathrm{T}} P_1 & 0 & -\delta_2 I \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} \Xi & \star & \star \\ -\mu Q^{\mathrm{T}} \otimes P_2 & -\sigma_1 I & \star \\ -R \otimes P_2 & 0 & -\sigma_2 I \end{bmatrix},$$

$$\Xi = R \otimes (P_2 S + S^{\mathrm{T}} P_2) - \mu(R\mathcal{H} + \mathcal{H}^{\mathrm{T}} R) \otimes P_2 + \delta_3 I,$$

$$\Psi_1 = 3\mu^2 c \bar{h}^2 \|\mathcal{H}\|^2 + \delta_2 - \delta_3,$$

$$\Psi_2 = \frac{\sigma_3 - 1}{\sigma_3} \Psi_1 + \bar{\rho}^2 \|\mathcal{H}\|^2 [3c\bar{h}^2 \Upsilon + \sigma_1 - (\sigma_3 - 1)(N+1)\Psi_1],$$

$$\Psi_3 = 3c\bar{h}^2 \Upsilon + \sigma_1 - (\sigma_3 - 1)(N + 1)\Psi_1 - c,$$

$$\Psi_4{=}2ch_0{}^2\|S\|^2{+}6\mu^2c\sum_{i=1}^{N}a_{i0}h_i{}^2{+}\sigma_1{-}(\sigma_3-1)(N{+}1)\Psi_1{-}c,$$
$$\Psi_5{=}(\sigma_2N+2ch_0{}^2)\|B_{w0}\|^2-\delta_1,$$

$P_1 = \mathrm{diag}\{P_{11},\ldots,P_{1N}\}$, $P_{1i}\in\mathbb{R}^{n_i\times n_i}$, $\bar{A} = A+BK_1$, $A = \mathrm{diag}\{A_1,\ldots,A_N\}$, $B = \mathrm{diag}\{B_1,\ldots,B_N\}$, $K_1=\mathrm{diag}\{K_{11},\ldots,K_{N1}\}$, $C=\mathrm{diag}\{C_1,\ldots,C_N\}$, $Q{=}R\mathcal{H}M$, $M{=}(-1_N,\ I_N)$, $P_2{\in}\mathbb{R}^{q\times q}$, $\Pi=\mathrm{diag}\{\Pi_1,\ldots,\Pi_N\}$, $\bar{B}_w=\big(-\Pi(1_N\otimes B_{w0}),\ \mathrm{diag}\{B_{w1},\ldots,B_{wN}\}\big)$, $\bar{B}_1=\mathrm{diag}\{B_1K_{12},\ldots,B_NK_{N2}\}$, $\Upsilon=\|S\|^2+2\mu^2\|\mathcal{H}\|^2$, $\bar{\rho}=\max\limits_{1\le i\le N}\{\rho_i\}$, and $\bar{h}=\max\limits_{1\le i\le N}\{h_i\}$. For brevity, all the identity matrices above have compatible dimensions.

Proof. Let $\xi_i(t)=x_i(t)-\Pi_iv(t)$, and $\varepsilon_i(t)=\eta_i(t)-v(t)$. Then the output consensus error vector $z(t)$ can be expressed as follows:

$$z(t)=C\xi(t), \tag{17}$$

where $\xi(t){=}\mathrm{col}\big(\xi_1(t),\ldots,\xi_N(t)\big)$. Define $\hat{\varepsilon}_i(t)=\hat{\eta}_i(t)-\hat{\eta}_0(t)$, then we obtain

$$\hat{\varepsilon}(t)=\varepsilon(t)+(M\otimes I_q)e(t), \tag{18}$$

where $e(t){=}\mathrm{col}\big(e_0(t),e_1(t),\ldots,e_N(t)\big)$, $\varepsilon(t){=}\mathrm{col}\big(\varepsilon_1(t),\ldots,\varepsilon_N(t)\big)$, and $\hat{\varepsilon}(t){=}\mathrm{col}\big(\hat{\varepsilon}_1(t),\ldots,\hat{\varepsilon}_N(t)\big)$. According to (3), (4), (8), (10), (11) and (14), one has

$$\dot{\xi}(t)=\bar{A}\xi(t)+\bar{B}_1\varepsilon(t)+\bar{B}_ww(t), \tag{19a}$$
$$\dot{\varepsilon}(t)=(I_N\otimes S-\mu\mathcal{H}\otimes I_q)\varepsilon(t)-\mu(\mathcal{H}M\otimes I_q)e(t)-1_N\otimes B_{w_0}w_0(t). \tag{19b}$$

In addition, one has the upper Dini derivative of $e_i(t)$ as follows:

$$\begin{aligned}&D^+e_0(t){=}Se_0(t){-}B_{w0}w_0(t),\\&D^+e_i(t){=}Se_i(t){+}\mu(\mathcal{H}_i{\otimes}I_q)\varepsilon(t){+}\mu(\mathcal{H}_iM{\otimes}I_q)e(t),\end{aligned} \tag{20}$$

where $i{=}1,\ldots,N$, and $\mathcal{H}_i$ represents the ith row of matrix $\mathcal{H}$.

Now we are ready to analyze the H_∞ performance of the closed-loop system. Consider the following Lyapunov functional candidate,

$$\begin{aligned}V(t)=&V_1(t)+V_2(t)+V_3(t),\\V_1(t)=&\xi^{\mathrm{T}}(t)P_1\xi(t),\ V_2(t)=\varepsilon^{\mathrm{T}}(t)(R\otimes P_2)\varepsilon(t),\ V_3(t)=c\sum_{i=0}^{N}V_{3i}(t),\end{aligned} \tag{21}$$

where

$$V_{3i}(t){=}\begin{cases}h_i\Big\{\int_{-t}^{0}\int_{t+\tau}^{t}[D^+e_i(v)]^{\mathrm{T}}[D^+e_i(v)]\,\mathrm{d}\,v\,\mathrm{d}\,\tau+(h_i-t)\int_0^t[D^+e_i(v)]^{\mathrm{T}}[D^+e_i(v)]\,\mathrm{d}\,v\Big\},\ 0{\le}t{\le}h_i,\\ h_i\int_{-h_i}^{0}\int_{t+\tau}^{t}[D^+e_i(v)]^{\mathrm{T}}[D^+e_i(v)]\,\mathrm{d}\,v\,\mathrm{d}\,\tau,\quad h_i<t.\end{cases}$$

One can verify that $V_{3i}(t) \geq 0$, $i = 0, \ldots, N$ are continuous over $[0, +\infty)$. So is $V(t)$.

Substituting (19a) into the derivative of $V_1(t)$ yields

$$\dot{V}_1(t) = \xi^{\mathrm{T}}(t)(P_1\bar{A} + \bar{A}^{\mathrm{T}}P_1)\xi(t) + 2\xi^{\mathrm{T}}(t)P_1\bar{B}_1\varepsilon(t) + 2\xi^{\mathrm{T}}(t)P_1\bar{B}_w w(t). \tag{22}$$

Considering (17), one has

$$\dot{V}_1(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + z^{\mathrm{T}}(t)z(t) = \chi_1^{\mathrm{T}}(t)\Sigma_1\chi_1(t) + \delta_2\varepsilon^{\mathrm{T}}(t)\varepsilon(t) - \delta_1 w^{\mathrm{T}}(t)w(t), \tag{23}$$

where $\chi_1(t) = \mathrm{col}\,\big(\xi(t), w(t), \varepsilon(t)\big)$. According to (19b) and (21), we have

$$\begin{aligned}\dot{V}_2(t) =& \varepsilon^{\mathrm{T}}(t)[R \otimes (P_2 S + S^{\mathrm{T}}P_2) - \mu(R\mathcal{H} + \mathcal{H}^{\mathrm{T}}R) \otimes P_2]\varepsilon(t) \\ &- 2\mu\varepsilon^{\mathrm{T}}(t)(Q \otimes P_2)e(t) - 2\varepsilon^{\mathrm{T}}(t)(R \otimes P_2)[1_N \otimes B_{w_0}w_0(t)],\end{aligned} \tag{24}$$

where $Q = R\mathcal{H}M$. Let $\chi_2(t) = \mathrm{col}\,\big(\varepsilon(t), e(t), 1_N \otimes B_{w_0}w_0(t)\big)$, it then follows that

$$\dot{V}_2(t) = \chi_2^{\mathrm{T}}(t)\Sigma_2\chi_2(t) + \sigma_1 e^{\mathrm{T}}(t)e(t) + \sigma_2 N w_0^{\mathrm{T}}(t)B_{w0}^{\mathrm{T}}B_{w0}w_0(t) - \delta_3\varepsilon^{\mathrm{T}}(t)\varepsilon(t), \tag{25}$$

where $(1_N \otimes B_{w_0}w_0)^{\mathrm{T}}(1_N \otimes B_{w_0}w_0) = N w_0^{\mathrm{T}} B_{w_0}^{\mathrm{T}} B_{w_0} w_0$ is adopted in the equality. Then, one has

$$\begin{aligned}\dot{V}_3(t) =& c\sum_{i=0}^{N} h_i^2 [D^+e_i(t)]^{\mathrm{T}}[D^+e_i(t)] - c\sum_{\{i\in\bar{\mathcal{V}}|h_i<t\}} h_i \int_{t-h_i}^{t} [D^+e_i(\tau)]^{\mathrm{T}}[D^+e_i(\tau)]\,\mathrm{d}\,\tau \\ &- c\sum_{\{i\in\bar{\mathcal{V}}|0\leq t\leq h_i\}} h_i \int_0^t [D^+e_i(\tau)]^{\mathrm{T}}[D^+e_i(\tau)]\,\mathrm{d}\,\tau.\end{aligned} \tag{26}$$

Applying (20) and Lemma 2.1 to the first item of (26) yields

$$\begin{aligned}\dot{V}_3(t) \leq& 3c\bar{e}^{\mathrm{T}}(t)(\Lambda_h^2 \otimes S^{\mathrm{T}}S + 2\mu^2\mathcal{H}^{\mathrm{T}}\Lambda_h^2\mathcal{H} \otimes I_q)\bar{e}(t) \\ &+ 3\mu^2 c\varepsilon^{T}(t)\left(\mathcal{H}^{\mathrm{T}}\Lambda_h^2\mathcal{H} \otimes I_q\right)\varepsilon(t) + e_0^{\mathrm{T}}(t)\left[6\mu^2 c\sum_{i=1}^{N} a_{i0}h_i^2 I_q + 2ch_0^2 S^{\mathrm{T}}S\right]e_0(t) \\ &+ 2ch_0^2 w_0^{\mathrm{T}}(t)B_{w0}^{\mathrm{T}}B_{w0}w_0(t) - c\sum_{\{i\in\bar{\mathcal{V}}|h_i<t\}} h_i \int_{t-h_i}^{t} [D^+e_i(\tau)]^{\mathrm{T}}[D^+e_i(\tau)]\,\mathrm{d}\,\tau \\ &- c\sum_{\{i\in\bar{\mathcal{V}}|0\leq t\leq h_i\}} h_i \int_0^t [D^+e_i(\tau)]^{\mathrm{T}}[D^+e_i(\tau)]\,\mathrm{d}\,\tau,\end{aligned} \tag{27}$$

where $\bar{e}(t) = \mathrm{col}\,\big(e_1(t), \ldots, e_N(t)\big)$, and $\Lambda_h = \mathrm{diag}\{h_1, \ldots, h_N\}$. For ease of presentation, denote the set of triggering time instants of all the agents as $\Omega = \{t_m : m \in \mathbb{Z}_{>0}\}$ with $t_m < t_{m+1}$. Note that at least one agent is triggered at every $t_m \in \Omega$.

Next, we will consider $\dot{V}(t)$ for $t \in (t_m, t_{m+1})$, i.e., $t \notin \Omega$. One obtains that $V(t)$ is continuous and differentiable in each subinterval (t_m, t_{m+1}). According to the unified time- and event-triggering mechanism (13), the followers can be divided into the following two sets:

$$\begin{cases} S_1(t) = \left\{ i \in \mathcal{V} \middle| \|e_i(t)\| < \rho_i \left\| \sum_{j=0}^{N} a_{ij}[\hat{\eta}_i(t) - \hat{\eta}_j(t)] \right\| \right\}, \\ S_2(t) = \left\{ i \in \mathcal{V} \middle| \|e_i(t)\| \geq \rho_i \left\| \sum_{j=0}^{N} a_{ij}[\hat{\eta}_i(t) - \hat{\eta}_j(t)] \right\| \right\}. \end{cases} \tag{28}$$

It then follows from (28) that $S_1(t) \bigcup S_2(t) = \mathcal{V}$. Accordingly, the following two cases will be considered.

Case I: $i \in S_1(t)$. In this case, the condition $\|e_i(t)\| < \rho_i \|(\mathcal{H}_i \otimes I_q)\hat{\varepsilon}(t)\|$ holds, which yields

$$\begin{aligned} \sum_{i \in S_1(t)} e_i^{\mathrm{T}}(t) e_i(t) < & \sum_{i \in S_1(t)} \rho_i^2 \hat{\varepsilon}^{\mathrm{T}}(t)(\mathcal{H}_i^{\mathrm{T}} \mathcal{H}_i \otimes I_q)\hat{\varepsilon}(t) \\ & < \hat{\varepsilon}^{\mathrm{T}}(t)(\mathcal{H}^{\mathrm{T}} \Lambda_\rho^2 \mathcal{H} \otimes I_q)\hat{\varepsilon}(t) < \bar{\rho}^2 \|\mathcal{H}\|^2 \hat{\varepsilon}^{\mathrm{T}}(t)\hat{\varepsilon}(t). \end{aligned} \tag{29}$$

Case II: $i \in S_2(t)$. First, assume that $t - t_k^i \geq h_i$ holds, where t_k^i represents the latest triggering time instant of agent i before t. According to ETM (13), one then has $t_{k+1}^i = t$, i.e., $t \in \Omega$, which contradicts to the hypothesis. One thus has $t - t_k^i < h_i$ for $i \in S_2(t)$. To deal with the last two items of (27), $S_2(t)$ is further divided into two parts, i.e., $S_2(t) = S_2^1(t) \bigcup S_2^2(t)$, where $S_2^1(t) = \{i \in S_2(t) | h_i < t\}$ and $S_2^2(t) = \{i \in S_2(t) | 0 \leq t \leq h_i\}$. It then follows that

$$\begin{aligned} & c \sum_{\{i \in \mathcal{V} | h_i < t\}} h_i \int_{t-h_i}^{t} [D^+ e_i(\tau)]^{\mathrm{T}} [D^+ e_i(\tau)] \,\mathrm{d}\,\tau \\ \geq & c \sum_{i \in S_2^1(t)} h_i \int_{t-h_i}^{t} [D^+ e_i(\tau)]^{\mathrm{T}} [D^+ e_i(\tau)] \,\mathrm{d}\,\tau \geq c \sum_{i \in S_2^1(t)} h_i \int_{t_k^{i+}}^{t} [D^+ e_i(\tau)]^{\mathrm{T}} [D^+ e_i(\tau)] \,\mathrm{d}\,\tau \\ \geq & c \sum_{i \in S_2^1(t)} \left[\int_{t_k^{i+}}^{t} D^+ e_i(\tau) d\tau \right]^{\mathrm{T}} \left[\int_{t_k^{i+}}^{t} D^+ e_i(\tau) \,\mathrm{d}\,\tau \right] = c \sum_{i \in S_2^1(t)} e_i^{\mathrm{T}}(t) e_i(t), \end{aligned} \tag{30}$$

where Lemma 2.2 is used to get the third inequality. $e(t_k^{i+}) = 0$ is used to obtain the last equality. For followers $i \in S_2^2(t)$, we have

$$\begin{aligned} c \sum_{\{i \in \mathcal{V} | 0 \leq t \leq h_i\}} h_i \int_{0}^{t} [D^+ e_i(\tau)]^{\mathrm{T}} [D^+ e_i(\tau)] \,\mathrm{d}\,\tau \geq & c \sum_{i \in S_2^2(t)} h_i \int_{0}^{t} [D^+ e_i(\tau)]^{\mathrm{T}} [D^+ e_i(\tau)] \,\mathrm{d}\,\tau \\ \geq & c \sum_{i \in S_2^2(t)} \left[\int_{0}^{t} D^+ e_i(\tau) d\tau \right]^{\mathrm{T}} \left[\int_{0}^{t} D^+ e_i(\tau) \,\mathrm{d} \right] \tau = c \sum_{i \in S_2^2(t)} e_i^{\mathrm{T}}(t) e_i(t). \end{aligned} \tag{31}$$

For the leader agent 0, the condition $t - t_k^0 < h_0$, $t_k^0 = \max\limits_{l\in\mathbb{Z}_{>0}}\{t_l^0 | t_l^0 \le t\}$ always holds, since $a_{0i} = 0$, $\forall i \in \mathcal{V}$. Similar to (30) and (31), one derives

$$\begin{aligned} &ch_0\int_0^t [D^+e_0(\tau)]^{\mathrm{T}}[D^+e_0(\tau)]\mathrm{d}\tau \ge ce_0^{\mathrm{T}}(t)e_0(t), 0\le t\le h_0,\\ &ch_0\int_{t-h_0}^t [D^+e_0(\tau)]^{\mathrm{T}}[D^+e_0(\tau)]\mathrm{d}\tau \ge ce_0^{\mathrm{T}}(t)e_0(t), h_0 < t. \end{aligned} \tag{32}$$

By substituting (30)-(32) into (27), one has

$$\begin{aligned} \dot{V}_3(t) \le & 3c\bar{h}^2\Upsilon \sum_{i\in S_1(t)} e_i^{\mathrm{T}}(t)e_i(t) + (3c\bar{h}^2\Upsilon - c)\sum_{i\in S_2(t)} e_i^{\mathrm{T}}(t)e_i(t)\\ & + 3\mu^2 c\bar{h}^2\varepsilon^{\mathrm{T}}(t)\left(\mathcal{H}^{\mathrm{T}}\mathcal{H}\otimes I_q\right)\varepsilon(t) + 2ch_0^2 w_0^{\mathrm{T}}(t)B_{w0}^{\mathrm{T}}B_{w0}w_0(t)\\ & + e_0^{\mathrm{T}}(t)\left[6\mu^2 c\sum_{i=1}^N a_{i0}h_i^2 I_q + 2ch_0^2 S^{\mathrm{T}}S - cI_q\right]e_0(t), \end{aligned} \tag{33}$$

where $\Upsilon = \|S\|^2 + 2\mu^2\|\mathcal{H}\|^2$. According to (23), (25) and (33), we obtain

$$\begin{aligned} &\dot{V}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + z^{\mathrm{T}}(t)z(t)\\ &\le \chi_1^{\mathrm{T}}(t)\Sigma_1\chi_1(t) + \chi_2^{\mathrm{T}}(t)\Sigma_2\chi_2(t) + \Psi_1\varepsilon^{\mathrm{T}}(t)\varepsilon(t)\\ &\quad + (3c\bar{h}^2\Upsilon + \sigma_1)\sum_{i\in S_1(t)} e_i^{\mathrm{T}}(t)e_i(t) + (3c\bar{h}^2\Upsilon + \sigma_1 - c)\sum_{i\in S_2(t)} e_i^{\mathrm{T}}(t)e_i(t)\\ &\quad + e_0^{\mathrm{T}}(t)\left[2ch_0^2 S^{\mathrm{T}}S + \left(6\mu^2 c\sum_{i=1}^N a_{i0}h_i^2 + \sigma_1 - c\right)I_q\right]e_0(t)\\ &\quad + (\sigma_2 N + 2ch_0^2)w_0^{\mathrm{T}}(t)B_{w0}^{\mathrm{T}}B_{w0}w_0(t) - \delta_1 w^{\mathrm{T}}(t)w(t). \end{aligned} \tag{34}$$

It follows from (18) and Lemma 2.1 that

$$\varepsilon^{\mathrm{T}}(t)\varepsilon(t) \ge \frac{\sigma_3 - 1}{\sigma_3}\hat{\varepsilon}^{\mathrm{T}}(t)\hat{\varepsilon}(t) - (\sigma_3 - 1)(N+1)e^{\mathrm{T}}(t)e(t), \tag{35}$$

where $\sigma_3 > 1$. Substituting (29) and (35) into (34), one derives

$$\begin{aligned} &\dot{V}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + z^{\mathrm{T}}(t)z(t)\\ &\le \chi_1^{\mathrm{T}}(t)\Sigma_1\chi_1(t) + \chi_2^{\mathrm{T}}(t)\Sigma_2\chi_2(t) + \Psi_2\hat{\varepsilon}^{\mathrm{T}}(t)\hat{\varepsilon}(t)\\ &\quad + \Psi_3\sum_{i\in S_2(t)} e_i^{\mathrm{T}}(t)e_i(t) + \Psi_4 e_0^{\mathrm{T}}(t)e_0(t) + \Psi_5 w_0^{\mathrm{T}}(t)w_0(t). \end{aligned} \tag{36}$$

Considering that $V(t)$ is continuous on $[0, +\infty)$, one then has

$$\begin{aligned} &V(t_m^+) - V(t_m) = 0, \\ &V(t_m) - V(t_m^-) = 0, \qquad \forall t_m \in \Omega. \end{aligned} \tag{37}$$

If $w(t) \equiv 0$, substituting (15), (16) into (36), we obtain that $\dot{V}(t)$, $t \in (t_m, t_{m+1})$ satisfies

$$\dot{V}(t) \leq -\|z(t)\|^2. \tag{38}$$

It follows from (37) and (38) that the MAS achieves output consensus asymptotically for any initial conditions.

If $w(t) \neq 0$, it follows from (36) that

$$\dot{V}(t) < \gamma^2\|w(t)\|^2 - \|z(t)\|^2. \tag{39}$$

Furthermore, one can verify that $V(0) = 0$ with the zero-initial condition. It then follows from (37) and (39) that

$$\begin{aligned} &\frac{1}{\mathfrak{T}}\int_0^{\mathfrak{T}} \|z(t)\|^2 - \gamma^2\|w(t)\|^2 \,\mathrm{d}\, t < -\frac{1}{\mathfrak{T}}\int_0^{\mathfrak{T}} \dot{V}(t)\,\mathrm{d}\, t \\ &= -\frac{1}{\mathfrak{T}}\left[\int_{t_1}^{t_2} \dot{V}(t)\,\mathrm{d}\, t + \cdots + \int_{t_m}^{\mathfrak{T}} \dot{V}(t)\,\mathrm{d}\, t\right] \\ &= -\frac{1}{\mathfrak{T}}\left[V(\mathfrak{T}) - V(t_m^+) + V(t_m^-) - \cdots + V(t_2^-) - V(0)\right] \\ &= -\frac{1}{\mathfrak{T}}V(\mathfrak{T}) \leq 0, \qquad \forall \mathfrak{T} > 0. \end{aligned} \tag{40}$$

The proof is thus completed. ■

The design of ETC protocol composed of (10) and (13) is mainly motivated by [16, 33]. Note that only the states of the distributed observers are transmitted between neighboring agents in implementation. Moreover, owing to the open-loop estimation method (11), no continuous or periodic communication is required between neighboring followers for the distributed controller and the ETM, which is the main advantage of this work over [33]. $V_3(t)$ in (21) is partly inspired by [34], in which a class of time-delay systems were investigated. However, since the states of the MAS for $t < 0$ are not concerned in the leader-following H_∞ output consensus problem, which is different from the analysis of time-delay systems, $V_{3i}(t)$ is designed to be of two parts to avoid its dependence on the states for $t < 0$.

A sufficient condition to solve the leader-following H_∞ output consensus problem is provided by inequalities in (15) and (16) in Theorem 1. Thus, in what follows, the solvability of these inequalities will be analyzed.

According to Assumption 3, there exist K_{i1} such that $A_i+B_iK_{i1}$, $i=1,\dots,N$ are Hurwitz. By Theorem 4.6 in [41], for a positive definite symmetric matrix $Y=C^{\mathrm{T}}C+\nu I$, $\nu>0$, there exists a unique positive definite P_1 such that

$$\bar{A}^{\mathrm{T}}P_1+P_1\bar{A}+Y=0. \tag{41}$$

Choosing $\tilde{\gamma}^2>\frac{\nu}{2\|P_1\bar{B}_w\bar{B}_w^{\mathrm{T}}P_1\|}$ and $\delta_2>\frac{\nu}{2\|P_1\bar{B}_1\bar{B}_1^{\mathrm{T}}P_1\|}$ leads to

$$\frac{1}{\tilde{\gamma}^2}P_1\bar{B}_w\bar{B}_w^{\mathrm{T}}P_1+\frac{1}{\delta_2}P_1\bar{B}_1\bar{B}_1^{\mathrm{T}}P_1<\nu I, \tag{42}$$

where $\tilde{\gamma}^2\triangleq\gamma^2-\delta_1$. It then follows from (41) and (42) that

$$\bar{A}^{\mathrm{T}}P_1+P_1\bar{A}+C^{\mathrm{T}}C+\frac{1}{\tilde{\gamma}^2}P_1\bar{B}_w\bar{B}_w^{\mathrm{T}}P_1+\frac{1}{\delta_2}P_1\bar{B}_1\bar{B}_1^{\mathrm{T}}P_1<0, \tag{43}$$

which implies $\Sigma_1<0$ in (15) by Schur complement.

Via Schur complement again, $\Sigma_2<0$ holds if and only if

$$\Xi+\left(\frac{\mu^2}{\sigma_1}QQ^{\mathrm{T}}+\frac{1}{\sigma_2}R^2\right)\otimes P_2^2<0. \tag{44}$$

According to Lemma 2.3, we obtain

$$\begin{aligned}\Xi&=R\otimes(P_2S+S^{\mathrm{T}}P_2)-\mu(R\mathcal{H}+\mathcal{H}^{\mathrm{T}}R)\otimes P_2+\delta_3I_{Nq}\\&\le R\otimes(P_2S+S^{\mathrm{T}}P_2)-\mu\lambda_1I_N\otimes P_2+\delta_3I_{Nq}.\end{aligned}$$

Thus, with choices of $\sigma_1=\mu^2$, $\sigma_2\in\left(0,\frac{\delta_1}{N\|B_{w0}\|^2}\right)$, $\delta_3\in(\delta_2,+\infty)$, and $P_2>0$, (44) can be ensured by selecting

$$\mu>\frac{\bar{r}\|P_2S+S^{\mathrm{T}}P_2\|+\delta_3+(\|Q\|^2+\bar{r}^2/\sigma_2)\|P_2\|^2}{\lambda_1\lambda_{\min}(P_2)}, \tag{45}$$

where $\bar{r}=\max\limits_{i=1,\dots,N}\{r_i\}$, and $\lambda_{\min}(P_2)$ represents the smallest eigenvalue of P_2. In this case, $\Sigma_2<0$ is thus ensured with selected parameters.

Furthermore, one can verify that $\Psi_i<0$, $i=1,\dots,5$ hold by selecting other parameters c, h_i and ρ_i as follows:

$$\begin{aligned}c>\max\big\{&3\Delta_1\Upsilon+\sigma_1-(\sigma_3-1)(N+1)\Delta_2,\\&2\Delta_0\|S\|^2+6\mu^2N\Delta_1+\sigma_1-(\sigma_3-1)(N+1)\Delta_2\big\},\end{aligned} \tag{46a}$$

$$h_0<\sqrt{\frac{\Delta_0}{c}},\quad h_i<\sqrt{\frac{\Delta_1}{c}},\quad \rho_i<\sqrt{\frac{(1-\sigma_3)\Psi_1}{c\sigma_3\|\mathcal{H}\|^2}}, \tag{46b}$$

where $\Delta_0 < \frac{\delta_1}{2\|B_{w0}\|^2} - \frac{\sigma_2 N}{2}$, $\Delta_1 < \frac{\delta_3-\delta_2}{3\mu^2\|\mathcal{H}\|^2}$, $\Delta_2 = \delta_2 - \delta_3$, and $\sigma_3 > 1$. It is then concluded that inequalities (15) and (16) hold with those selected parameters for $\gamma = \sqrt{\tilde{\gamma}^2 + \delta_1}$.

It can thus be seen that there exists a disturbance attenuation index γ for MASs satisfying Assumptions 1-3 such that the H_∞ output consensus problem can be solved by the proposed ETC protocol in this work in contrast with [33]. This advantage is further illustrated by Example I in Section 4. In addition, for given gain matrices K_{i1} and parameter μ, (15) becomes two linear matrix inequalities (LMIs).

Remark 2. *Note that the parameters h_i and ρ_i in (13) affect the triggering and communication frequency of the MAS directly. Larger values of h_i and ρ_i yield less triggering and communication times. Besides, larger value of μ increases the convergence rate of the MAS under the control law (10). However, (46) shows that the upper bounds of h_i and ρ_i decrease as μ increases. Thus, there exists a trade-off in the design of parameters h_i, ρ_i and μ.*

3.2. Event-triggered H_∞ Output Consensus Based on Output Feedback

In some applications, the states of the agents may not be available. Hence, we design an ETC protocol base on output feedback in this subsection.

First, the following assumption is needed.

Assumption 5. *(C_0, S) and (C_i, A_i), $\forall i = 1, \ldots, N$, are detectable.*

For the leader agent, a Luenberger observer is designed as follows:

$$\dot{\hat{v}}(t) = S\hat{v}(t) + L_0[C_0\hat{v}(t) - y_0(t)], \tag{47}$$

where $\hat{v}(t)$ is the observer state, and L_0 is the observer gain matrix. The leader is assumed to broadcast its observer state $\hat{v}(t_k^0)$ to its neighboring followers in this subsection. The distributed ETC protocol in this case is designed as follows:

$$\begin{aligned} u_i(t) =& K_{i1}\hat{x}_i(t) + K_{i2}\eta_i(t), \\ \dot{\hat{x}}_i(t) =& A_i\hat{x}_i(t) + B_i u_i(t) + L_i[C_i\hat{x}_i(t) - y_i(t)], \\ \dot{\eta}_i(t) =& S\eta_i(t) - \mu \sum_{j\in\mathcal{N}_i} a_{ij}[\hat{\eta}_i(t) - \hat{\eta}_j(t)], \end{aligned} \tag{48}$$

where $\hat{x}_i(t)$ represents the state of agent i's Luenberger observer, L_i is the observer gain matrix. The initial value of $\hat{x}_i(t)$ is set to $\hat{x}_i(0) = 0$. Other matrices and variables are the same as those in (11). The ETM is also defined as in (13) with $e_0(t)$ being defined as follows:

$$e_0(t) = \hat{\eta}_0(t) - \hat{v}(t), t \in [t_k^0, t_{k+1}^0), \tag{49}$$

where $\hat{\eta}_0(t) = \mathrm{e}^{S(t-t_k^0)}\,\hat{v}(t_k^0)$.

Next, the main result of this subsection will be presented.

Theorem 2. *Under Assumptions 1-5, the leader-following H_∞ output consensus problem of the heterogeneous linear MAS (3) and (4) with the disturbance attenuation index γ can be solved by the ETC protocol consisting of (48) and (13) with observer gain matrices L_i, controller gain matrices K_{i1}, $K_{i2} = \Gamma_i - K_{i1}\Pi_i$, and positive parameters μ, ρ_i, h_i, if there exist positive definite matrices P_1, P_2, P_3, P_{30}, positive parameters δ_i, $i = 1, \dots, 6$, σ_i, $i = 1, \dots, 4$ and c such that the following inequalities hold,*

$$\Sigma_i < 0, \ \ i = 1, \dots, 4, \tag{50}$$

and

$$\Psi_i < 0, \ \ i = 1, \dots, 6, \tag{51}$$

where

$$\begin{aligned}
\Sigma_1 =& \begin{bmatrix} \bar{A}^{\mathrm{T}} P_1 + P_1 \bar{A} + C^{\mathrm{T}} C & \star & \star & \star & \star \\ \bar{B}_w^{\mathrm{T}} P_1 & -(\gamma^2 - \delta_1) I & \star & \star & \star \\ \bar{B}_1^{\mathrm{T}} P_1 & 0 & -\delta_2 I & \star & \star \\ \bar{B}_1^{\mathrm{T}} P_1 & 0 & 0 & -\delta_4 I & \star \\ \bar{B}_2^{\mathrm{T}} P_1 & 0 & 0 & 0 & -\delta_5 I \end{bmatrix}, \\
\Sigma_3 =& \begin{bmatrix} P_3 \bar{A}_1 + \bar{A}_1^{\mathrm{T}} P_3 + \delta_5 I & \star \\ P_3 & -\sigma_4 I \end{bmatrix}, \\
\Sigma_4 =& \begin{bmatrix} P_{30} \bar{A}_0 + \bar{A}_0^{\mathrm{T}} P_{30} + \delta_6 I & \star \\ P_{30} & -\sigma_4 I \end{bmatrix}, \\
\Psi_5 =& \ (\sigma_2 N + 2 c h_0^2) \|L_0 C_0\|^2 - \delta_6 + \delta_4 N, \\
\Psi_6 =& \ \sigma_4 \|B_w\|^2 - \delta_1,
\end{aligned}$$

$\bar{B}_2 = \mathrm{diag}\{B_1 K_{11}, \dots, B_N K_{N1}\}$, $P_3 = \mathrm{diag}\{P_{31}, \dots, P_{3N}\}$, $P_{3i} \in \mathbb{R}^{n_i \times n_i}$, $\bar{A}_1 = \mathrm{diag}\{A_1 + L_1 C_1, \dots, A_N + L_N C_N\}$, $P_{30} \in \mathbb{R}^{q \times q}$, $\bar{A}_0 = S + L_0 C_0$, *and* $B_w = \mathrm{diag}\{B_{w0}, B_{w1}, \dots, B_{wN}\}$. *Other matrices and variables are the same as those given in Theorem 1, and all the identity matrices above have compatible dimensions.*

Proof. Let $\xi_i(t) = x_i(t) - \Pi_i v(t)$, $\varepsilon_i(t) = \eta_i(t) - \eta_0(t)$, $\phi_i(t) = \hat{x}_i(t) - x_i(t)$, and $\phi_0(t) = \eta_0(t) - v(t)$. According to (3), (4), (8), (11), (14) and (47)-(49), one has

$$\begin{aligned}
\dot{\xi}(t) =& \bar{A}\xi(t) + \bar{B}_1[\varepsilon(t) + 1_N \otimes \phi_0(t)] + \bar{B}_2 \phi(t) + \bar{B}_w w(t), \\
\dot{\varepsilon}(t) =& (I_N \otimes S - \mu \mathcal{H} \otimes I_q)\varepsilon(t) - \mu(\mathcal{H} M \otimes I_q) e(t) - 1_N \otimes L_0 C_0 \phi_0(t), \\
\dot{\phi}(t) =& \bar{A}_1 \phi(t) - \tilde{B}_w \bar{w}(t), \\
\dot{\phi}_0(t) =& \bar{A}_0 \phi_0(t) - B_{w0} w_0(t),
\end{aligned} \tag{52}$$

where $\tilde{B}_w = \mathrm{diag}\{B_{w1}, \ldots, B_{wN}\}$, $\phi(t) = \mathrm{col}\big(\phi_1(t), \ldots, \phi_N(t)\big)$, and $\bar{w}(t) = \mathrm{col}\,\big(w_1(t), \ldots, w_N(t)\big)$. Furthermore, one obtains the upper Dini derivative of $e_i(t)$ as follows:

$$\begin{aligned} D^+e_0(t) &= Se_0(t) - L_0C_0\phi_0(t), \\ D^+e_i(t) &= Se_i(t) + \mu(\mathcal{H}_i \otimes I_q)\varepsilon(t) + \mu(\mathcal{H}_iM \otimes I_q)e(t), \end{aligned} \tag{53}$$

where $i = 1, \ldots, N$.

Then, consider the following Lyapunov functional candidate,

$$\begin{aligned} V(t) =& V_1(t) + V_2(t) + V_3(t) + V_4(t), \\ V_1(t) =& \xi^{\mathrm{T}}(t)P_1\xi(t), \quad V_2(t) = \varepsilon^{\mathrm{T}}(t)(R \otimes P_2)\varepsilon(t), \\ V_3(t) =& \phi^{\mathrm{T}}(t)P_3\phi(t) + \phi_0^{\mathrm{T}}(t)P_{30}\phi_0(t), \quad V_4(t) = c\sum_{i=0}^{N} V_{4i}(t), \end{aligned} \tag{54}$$

where

$$V_{4i}(t) = \begin{cases} h_i \left\{ \int_{-t}^{0}\int_{t+\tau}^{t} [D^+e_i(v)]^{\mathrm{T}}[D^+e_i(v)]\,\mathrm{d}\,v\,\mathrm{d}\,\tau + h_i - t) \int_0^t [D^+e_i(v)]^{\mathrm{T}}[D^+e_i(v)]\,\mathrm{d}\,v \right\}, 0 \le t \le h_i, \\ h_i \int_{-h_i}^{0}\int_{t+\tau}^{t} [D^+e_i(v)]^{\mathrm{T}}[D^+e_i(v)]\,\mathrm{d}\,v\,\mathrm{d}\,\tau, \quad h_i < t. \end{cases}$$

Similar to Theorem 1, we have

$$\begin{aligned} &\dot{V}_1(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + z^{\mathrm{T}}(t)z(t) \\ &= \chi_1^{\mathrm{T}}(t)\Sigma_1\chi_1(t) + \delta_2\varepsilon^{\mathrm{T}}(t)\varepsilon(t) - \delta_1 w^{\mathrm{T}}(t)w(t) + \delta_4 N\phi_0^{\mathrm{T}}(t)\phi_0(t) + \delta_5\phi^{\mathrm{T}}(t)\phi(t), \end{aligned} \tag{55}$$

where $\chi_1(t) = \mathrm{col}\,\big(\xi(t), w(t), \varepsilon(t), 1_N \otimes \phi_0(t), \phi(t)\big)$. For $V_2(t)$, we have

$$\begin{aligned} \dot{V}_2(t) =& \varepsilon^{\mathrm{T}}(t)[R \otimes (P_2S + S^{\mathrm{T}}P_2) - \mu(R\mathcal{H} + \mathcal{H}^{\mathrm{T}}R) \otimes P_2]\varepsilon(t) \\ & - 2\mu\varepsilon^{\mathrm{T}}(t)(Q \otimes P_2)e(t) - 2\varepsilon^{\mathrm{T}}(t)(R \otimes P_2)[1_N \otimes L_0C_0\phi_0(t)] \\ =& \chi_2^{\mathrm{T}}(t)\Sigma_2\chi_2(t) + \sigma_1 e^{\mathrm{T}}(t)e(t) + \sigma_2 N\phi_0^T(t)C_0^TL_0^TL_0C_0\phi_0(t) - \delta_3\varepsilon^{\mathrm{T}}(t)\varepsilon(t), \end{aligned} \tag{56}$$

where $\chi_2(t) = \mathrm{col}\,\big(\varepsilon(t), e(t), 1_N \otimes L_0C_0\phi_0(t)\big)$. According to (52) and (54), one has

$$\begin{aligned} \dot{V}_3(t) =& \phi^{\mathrm{T}}(t)(P_3\bar{A}_1 + \bar{A}_1^{\mathrm{T}}P_3)\phi(t) + \phi_0^{\mathrm{T}}(t)(P_{30}\bar{A}_0 + \bar{A}_0^{\mathrm{T}}P_{30})\phi_0(t) \\ & - 2\phi^{\mathrm{T}}(t)P_3\tilde{B}_w\bar{w}(t) - 2\phi_0^{\mathrm{T}}P_{30}B_{w0}w_0(t). \end{aligned}$$

By Lemma 2.1, $\Sigma_3 < 0$ and $\Sigma_4 < 0$, one further has

$$\begin{aligned} \dot{V}_3(t) \le& \phi^{\mathrm{T}}(t)\left(P_3\bar{A}_1 + \bar{A}_1^{\mathrm{T}}P_3 + \frac{1}{\sigma_4}P_3^2\right)\phi(t) \\ & + \phi_0^{\mathrm{T}}(t)\left(P_{30}\bar{A}_0 + \bar{A}_0^{\mathrm{T}}P_{30} + \frac{1}{\sigma_4}P_{30}^2\right)\phi_0(t) + \sigma_4 w^{\mathrm{T}}(t)B_w^{\mathrm{T}}B_w w(t) \\ \le& -\delta_5\phi^{\mathrm{T}}(t)\phi(t) - \delta_6\phi_0^{\mathrm{T}}(t)\phi_0(t) + \sigma_4 w^{\mathrm{T}}(t)B_w^{\mathrm{T}}B_w w(t). \end{aligned} \tag{57}$$

Similar to (26)-(33) in Theorem 1, one derives $\dot{V}_4(t)$ as follows:

$$\begin{aligned}\dot{V}_4(t) \leq & 3c\bar{h}^2\Upsilon \sum_{i\in S_1(t)} e_i^{\mathrm{T}}(t)e_i(t) + (3c\bar{h}^2\Upsilon - c)\sum_{i\in S_2(t)} e_i^{\mathrm{T}}(t)e_i(t) + 3\mu^2 c\bar{h}^2\varepsilon^T(t)\left(\mathcal{H}^{\mathrm{T}}\mathcal{H}\otimes I_q\right)\varepsilon(t)\\ & + e_0^{\mathrm{T}}(t)\left(6\mu^2 c\sum_{i=1}^{N} a_{i0}h_i^2 I_q + 2ch_0^2 S^{\mathrm{T}}S - cI_q\right)e_0(t)\\ & + 2ch_0^2\phi_0^{\mathrm{T}}(t)C_0^{\mathrm{T}}L_0^{\mathrm{T}}L_0C_0\phi_0(t),\ t\in(t_m, t_{m+1}). \end{aligned} \tag{58}$$

Summing up (55)-(58) yields

$$\begin{aligned}&\dot{V}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + z^{\mathrm{T}}(t)z(t)\\ &\quad \leq \chi_1^{\mathrm{T}}(t)\Sigma_1\chi_1(t) + \chi_2^{\mathrm{T}}(t)\Sigma_2\chi_2(t) + \Psi_1\varepsilon^{\mathrm{T}}(t)\varepsilon(t)\\ &\quad\quad + (3c\bar{h}^2\Upsilon + \sigma_1)\sum_{i\in S_1(t)} e_i^{\mathrm{T}}(t)e_i(t) + (3c\bar{h}^2\Upsilon + \sigma_1 - c)\sum_{i\in S_2(t)} e_i^{\mathrm{T}}(t)e_i(t)\\ &\quad\quad + e_0^{\mathrm{T}}(t)\left[2ch_0^2 S^{\mathrm{T}}S + \left(6\mu^2 c\sum_{i=1}^{N} a_{i0}h_i^2 + \sigma_1 - c\right)I_q\right]e_0(t) + \Psi_5\phi_0^{\mathrm{T}}(t)\phi_0(t) + \Psi_6 w^{\mathrm{T}}(t)w(t). \end{aligned} \tag{59}$$

Recalling (29) and (35), we have

$$\begin{aligned}&\dot{V}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + z^{\mathrm{T}}(t)z(t)\\ &\quad \leq \chi_1^{\mathrm{T}}(t)\Sigma_1\chi_1(t) + \chi_2^{\mathrm{T}}(t)\Sigma_2\chi_2(t) + \Psi_2\hat{\varepsilon}^{\mathrm{T}}(t)\hat{\varepsilon}(t) + \Psi_3\sum_{i\in S_2(t)} e_i^{\mathrm{T}}(t)e_i(t)\\ &\quad\quad + \Psi_4 e_0^{\mathrm{T}}(t)e_0(t) + \Psi_5\phi_0^{\mathrm{T}}(t)\phi_0(t) + \Psi_6 w^{\mathrm{T}}(t)w(t). \end{aligned} \tag{60}$$

Following similar analysis procedures in Theorem 1, the proof can be completed. ■

Similarly, the solvability of (50) and (51) for some γ will be analyzed next. For Hurwitz matrix $\bar{A}$, there exists unique P_1 such that (41) holds. Thus, $\Sigma_1 < 0$ can be ensured by choosing the parameters as follows:

$$\tilde{\gamma}^2 > \frac{\nu}{4\|P_1\bar{B}_w\bar{B}_w^{\mathrm{T}}P_1\|}, \delta_2 > \frac{\nu}{4\|P_1\bar{B}_1\bar{B}_1^{\mathrm{T}}P_1\|}, \delta_4 > \frac{\nu}{4\|P_1\bar{B}_1\bar{B}_1^{\mathrm{T}}P_1\|}, \delta_5 > \frac{\nu}{4\|P_1\bar{B}_2\bar{B}_2^{\mathrm{T}}P_1\|},$$

where $\tilde{\gamma}^2 \triangleq \gamma^2 - \delta_1$. Moreover, one can verify that $\Sigma_2 < 0$ holds with choices of $\sigma_1 = \mu^2$, $\sigma_2 > 0$, $\delta_3 \in (\delta_2, +\infty)$, and μ satisfying (45).

Under Assumption 5, we can design observer gain matrices L_i, $i = 0, \ldots, N$ such that $\bar{A}_0$ and $\bar{A}_1$ are Hurwitz. Hence, there exist unique P_3 and P_{30} such that

$$\begin{aligned} P_3\bar{A}_1 + \bar{A}_1^{\mathrm{T}}P_3 + \delta_5 I + \nu_1 I = 0,\\ P_{30}\bar{A}_0 + \bar{A}_0^{\mathrm{T}}P_{30} + \delta_6 I + \nu_1 I = 0, \end{aligned}$$

with $\nu_1 > 0$, and $\delta_6 > \delta_4 N + \sigma_2 N\|L_0 C_0\|^2$. Thus, $\Sigma_3 < 0$ and $\Sigma_4 < 0$ hold by choosing $\sigma_4 > \max\left\{\frac{\nu_1}{\|P_{30}\|^2}, \frac{\nu_1}{\|P_3\|^2}\right\}$ according to Schur complement.

Moreover, $\Psi_i < 0$, $i = 1, \ldots, 6$ hold by selecting c, h_i, ρ_i satisfying (46) with $\Delta_0 < \frac{\delta_6 - \delta_4 N}{2\|L_0 C_0\|^2} - \frac{\sigma_2 N}{2}$, $\Delta_1 < \frac{\delta_3 - \delta_2}{3\mu^2\|\mathcal{H}\|^2}$, $\Delta_2 = \delta_2 - \delta_3$, $\sigma_3 > 1$ and $\delta_1 > \sigma_4\|B_w\|^2$.

It is then concluded that inequalities (50) and (51) hold with those selected parameters for $\gamma = \sqrt{\tilde{\gamma}^2 + \delta_1}$.

4. Numerical Examples

In this section, two numerical examples are provided to verify effectiveness of the proposed ETC protocols. To show the advantages of this work, some comparisons are also provided.

Example I: Consider a group of robots described as follows [23]:

$$J_i \ddot{q}_i + D_i \dot{q}_i + M_i g l_i \sin(q_i) = u_i + w_i, \quad i = 0, 1, \ldots, 4, \tag{61}$$

where q_i, u_i and w_i represent the angle of the rigid link of the robots, the control input and disturbances, respectively. J_i, D_i, M_i and l_i are the rotation inertia of the servo motor, the damping coefficient, the mass of the link and the length between the joint and the mass center, respectively. In this example, the parameters of the MAS described by (61) are selected as follows:

$$\begin{aligned} &J_0 = 2, && M_0 g l_0 = 2,\ D_i = 0.002, \\ &J_1 = J_2 = 1, && M_1 g l_1 = M_2 g l_2 = 0.5, \\ &J_3 = J_4 = 0.833, && M_3 g l_3 = M_4 g l_4 = 0.25. \end{aligned}$$

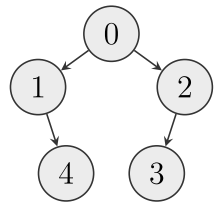
Neglecting the damping effects and linearizing the MAS around zero as in [23], we get the dynamics of the MAS in the form of (3) and (4) by denoting $x_i = (q_i, \dot{q}_i)^{\mathrm{T}}$ and $v = (q_0, \dot{q}_0)^{\mathrm{T}}$ with

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$
$$A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ -0.3 & 0 \end{bmatrix}, B_3 = B_4 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},$$

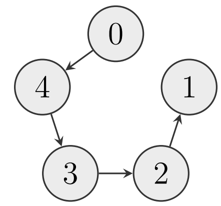
and $B_{wi} = B_i$, $C_i = [1\ \ 0]$. The solution pairs (Π_i, Γ_i) for (8) in Assumption 2 can be found as follows:

$$\Pi_i = I_2,\ \Gamma_1 = \Gamma_2 = \begin{bmatrix} -0.5 & 0 \end{bmatrix},\ \Gamma_3 = \Gamma_4 = \begin{bmatrix} -0.583 & 0 \end{bmatrix}.$$

The directed communication topology is shown in Figure 1 (a).



(a) Example I.



(b) Example II.

Figure 1: The communication graphs of the examples.

Consider the case of the state-feedback ETC protocol with $\gamma = 4$. According to Theorem 1, the feedback gain matrices K_{i1}, $i = 1, 2, 3, 4$ are selected as follows:

$$K_{11} = K_{21} = \begin{bmatrix} -4.5 & -4 \end{bmatrix}, K_{31} = K_{41} = \begin{bmatrix} -3.92 & -3.33 \end{bmatrix}.$$

By solving (15) and (16), other parameters in the ETC protocol are given as follows:

$$\mu = 2, h_i = 0.01, \ \rho_i = 0.077, \ i = 0, 1, \ldots, 4.$$

Simulations are carried out in Matlab/Simulink software package. For the case that there is no disturbance, i.e., $w(t) \equiv 0$, the initial conditions of all the agents are arbitrarily chosen as follows:

$$\begin{aligned} &v(0) = \begin{pmatrix} 0 & -0.04 \end{pmatrix}^{\mathrm{T}}, x_1(0) = \begin{pmatrix} -0.4 & -0.02 \end{pmatrix}^{\mathrm{T}}, x_2(0) = \begin{pmatrix} 0.3 & 0.14 \end{pmatrix}^{\mathrm{T}}, \\ &x_3(0) = \begin{pmatrix} -0.3 & -0.20 \end{pmatrix}^{\mathrm{T}}, x_4(0) = \begin{pmatrix} 0.2 & 0.16 \end{pmatrix}^{\mathrm{T}}. \end{aligned} \tag{62}$$

The time responses of $z_i(t)$, $i = 1, \ldots, 4$ are given in Figure 2, which show that all the agents achieve output consensus asymptotically. The triggering times of agents 1-4 within the simulation time period are 237, 237, 254 and 254, respectively.

For the case $w(t) \neq 0$ with zero initial condition, we select the disturbance signals satisfying Assumption 4 as follows:

$$\begin{aligned} &w_0(t) = 0.17\cos(10t + 1.2), w_1(t) = -0.12\cos(4t + 2), w_2(t) = 0.15\cos(4t + 1.3), \\ &w_3(t) = 0.13\cos(10t + 0.5), w_4(t) = 0.10\cos(6t - 3). \end{aligned} \tag{63}$$

The time responses of $z_i(t)$, $i = 1, \ldots, 4$ and $\sqrt{\frac{\int_0^t z^{\mathrm{T}}(\tau) z(\tau)\,\mathrm{d}\,\tau}{\int_0^t w^{\mathrm{T}}(\tau) w(\tau)\,\mathrm{d}\,\tau}}$ are shown in Figure 3 and Figure 4, respectively. It can be seen that the predefined H_∞ performance, i.e., $\gamma = 4$, is achieved. In this case, the triggering times of agents 1-4 within the selected time period are 230, 230, 244 and 244, respectively. Thus, it can be concluded that the H_∞ output consensus problem is solved by the state-feedback ETC protocol.

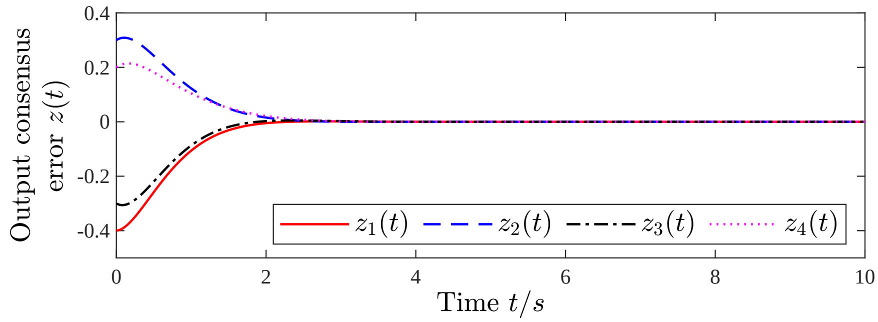


Figure 2: Time responses of $z_i(t)$ via distributed state-feedback ETC protocol with initial condition (62) and $w(t) \equiv 0$ in Example I.

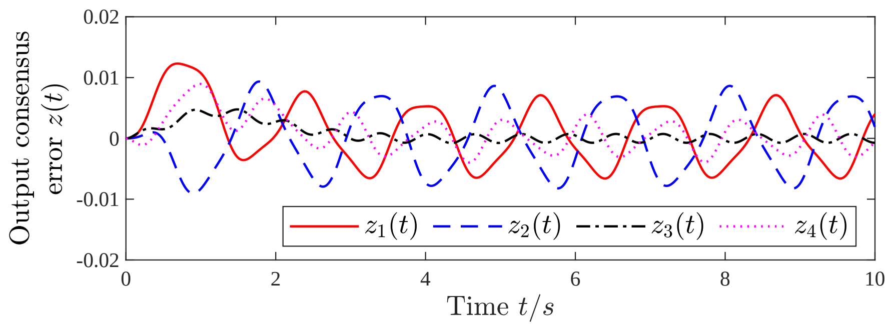


Figure 3: Time responses of $z_i(t)$ via distributed state-feedback ETC protocol with zero initial condition and disturbances (63) in Example I.

Then consider the case of the output-feedback ETC protocol with $\gamma = 13.7$. According to Theorem 2, the observer gain matrices L_i, $i = 0, \dots, 4$ in (47) and (48) are designed as follows:

$$L_0 = \begin{bmatrix} -1.2 & -0.7 \end{bmatrix}^{\mathrm{T}},\ L_1 = L_2 = \begin{bmatrix} -1.5 & -1.5 \end{bmatrix}^{\mathrm{T}},\ L_3 = L_4 = \begin{bmatrix} -1.5 & -1.7 \end{bmatrix}^{\mathrm{T}}.$$

The controller gain matrices K_{i1}, $i = 1, \dots, 4$ are selected as follows:

$$K_{11} = K_{21} = \begin{bmatrix} -12 & -5 \end{bmatrix}, K_{31} = K_{41} = \begin{bmatrix} -10.2 & -4.2 \end{bmatrix}.$$

By solving (50) and (51), other parameters are given as follows:

$$\mu = 3.5,\ h_i = 0.007,\ \rho_i = 0.073,\ i = 0, 1, \dots, 4.$$

For the case that $w(t) \equiv 0$, the initial states for the MAS are also selected as in (62). In this case, we get the time responses of $z_i(t)$, $i = 1, \dots, 4$ as given in Figure 5, which show that the MAS achieves output consensus asymptotically. The triggering times of all the followers within the selected time period are 461, 461, 460 and 460, respectively. For the case $w(t) \neq 0$ with

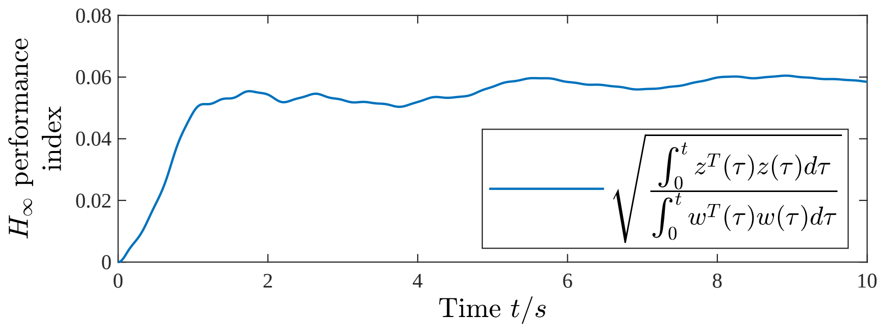


Figure 4: H_∞ performance index via distributed state-feedback ETC protocol with zero initial condition and disturbances (63) in Example I.

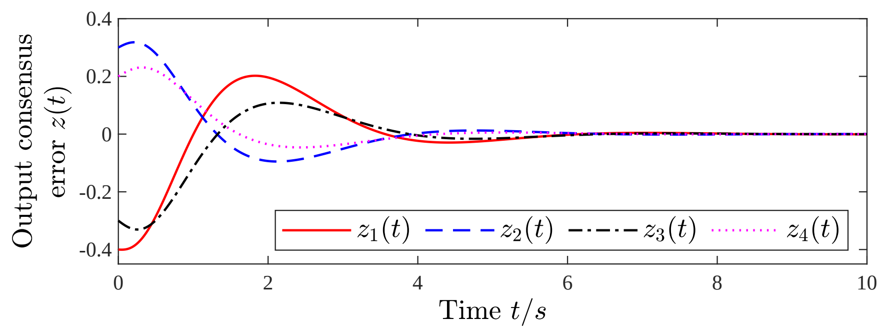


Figure 5: Time responses of $z_i(t)$ via distributed output-feedback ETC protocol with initial condition (62) and $w(t) \equiv 0$ in Example I.

zero initial condition, the disturbance signals are also chosen as in (63). The time responses of $z_i(t)$, $i = 1, \ldots, 4$ and $\sqrt{\frac{\int_0^t z^{\mathrm{T}}(\tau)z(\tau)\,\mathrm{d}\,\tau}{\int_0^t w^{\mathrm{T}}(\tau)w(\tau)\,\mathrm{d}\,\tau}}$ are shown in Figure 6 and Figure 7, respectively. It can be seen that the predefined H_∞ performance, i.e., $\gamma = 13.7$, is achieved. The triggering times of agents 1-4 within the simulation time period are 453, 453, 462 and 462, respectively. Thus, it can be concluded that the H_∞ output consensus problem is solved by the output-feedback ETC protocol.

It is noted that the H_∞ output consensus problem of this example cannot be solved via the ETC protocols designed in [33] for any $\gamma > 0$, which illustrates the advantages of our ETC protocols.

Example II: For comparison purpose, we apply the proposed output-feedback ETC pro-

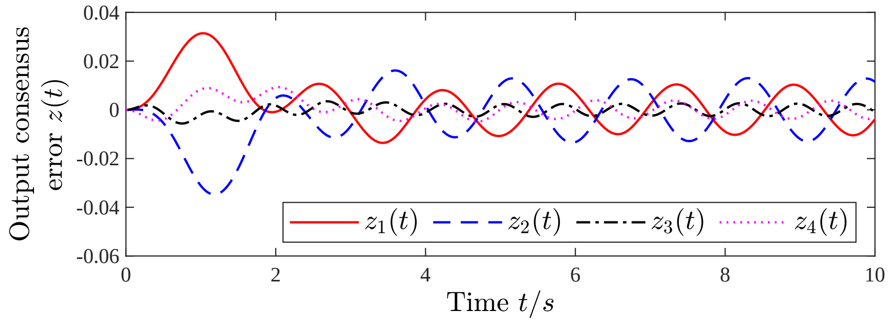


Figure 6: Time responses of $z_i(t)$ via distributed output-feedback ETC protocol with zero initial condition and disturbances (63) in Example I.

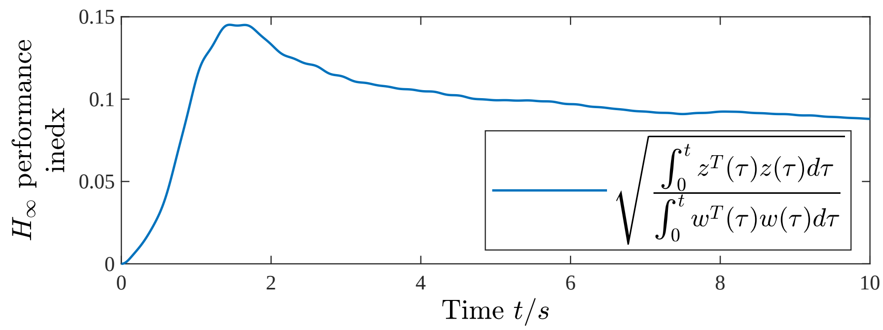


Figure 7: H_∞ performance index via distributed output-feedback ETC protocol with zero initial condition and disturbances (63) in Example I.

tocol to the example in [33]. The MAS is described by (3) and (4) with

$$A_1 = A_2 = A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, B_{w4} = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix},$$

$$B_{w0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_{w1} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, B_{w2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, B_{w3} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},$$

$$C_0 = C_1 = C_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, C_3 = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}, C_4 = \begin{bmatrix} 0.1 & -0.1 & 0.1 \end{bmatrix}.$$

The solution pair to (8) can be found as follows:

$$\Pi_1 = \Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \Pi_4 = \begin{bmatrix} 0.46 & 0.46 \\ -0.38 & -0.38 \\ 0.15 & 0.15 \end{bmatrix},$$
$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \Gamma_4 = \begin{bmatrix} -0.23 & -0.23 \end{bmatrix}.$$

The directed communication graph is shown in Figure 1 (b).

As in [33], set $\gamma = 2$. One can verify that the H_∞ output consensus problem can be solved by the ETC protocol consisting of (13) and (48) with

$$K_{11} = K_{21} = \begin{bmatrix} -1 & -4 \end{bmatrix}, K_{31} = \begin{bmatrix} 4 & -1 \end{bmatrix}, K_{41} = \begin{bmatrix} -5 & 4 & -1 \end{bmatrix},$$
$$L_1 = L_2 = \begin{bmatrix} -35 & -5 \end{bmatrix}^{\mathrm{T}}, L_3 = \begin{bmatrix} -5 & 35 \end{bmatrix}^{\mathrm{T}}, L_4 = \begin{bmatrix} -10 & 20 & -30 \end{bmatrix}^{\mathrm{T}},$$
$$L_0 = \begin{bmatrix} -10 & -10 \end{bmatrix}^{\mathrm{T}}, \mu = 17, h_i = 0.001, \rho_i = 0.02, i = 0, \ldots, 4.$$

For the case $w(t) \neq 0$ with zero initial condition, the disturbance signals are given to be the same as those in [33]:

$$w_1(t) = 0.5\,\mathrm{e}^{-t}, w_2(t) = -1.5\,\mathrm{e}^{-0.5t}, w_3(t) = 2\,\mathrm{e}^{-1.5t}, w_4(t) = -2.5\,\mathrm{e}^{-2.5t}.$$

The time response of $\sqrt{\frac{\int_0^t z^{\mathrm{T}}(\tau) z(\tau)\,\mathrm{d}\,\tau}{\int_0^t w^{\mathrm{T}}(\tau) w(\tau)\,\mathrm{d}\,\tau}}$ is shown in Figure 8. It can be seen that the predefined H_∞ performance, i.e., $\gamma = 2$, is achieved.

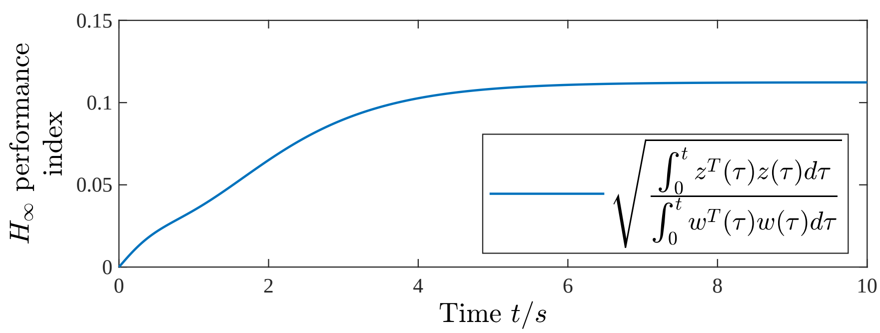


Figure 8: H_∞ performance index in Example II.

Then for the case $w(t) \equiv 0$, the initial conditions of all the agents are chosen such that the MAS has the same initial output consensus error as in [33]. The time responses of $z_i(t)$, $i = 1, \ldots, 4$ are given in Figure 9, which show that all the agents achieve output consensus asymptotically. It can also be observed from Figure 9 that it takes 6.62s for the output consensus error to converge within ±0.02. In contrast, it can be seen from Figure 4 in [33] that it takes about 16.41s with the ETC protocol proposed therein, which shows that the MAS under the ETC protocol proposed in this work achieves output consensus faster than that in [33]. Furthermore, the average communication periods of each agent via the output-feedback ETC protocols in this work and [33] are listed in Table 1.

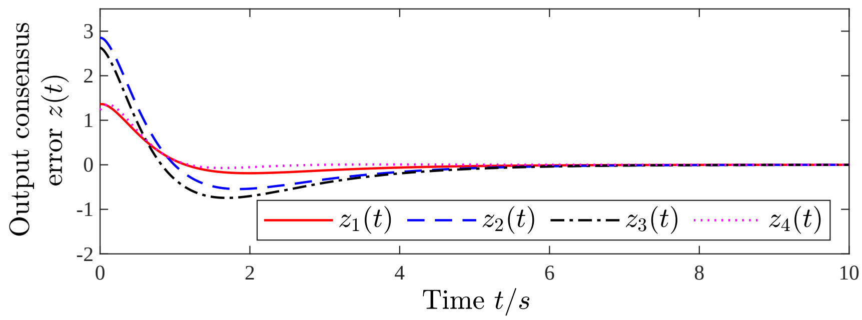


Figure 9: Time responses of $z_i(t)$ via distributed output-feedback ETC protocol in Example II.

Table 1: Comparison of Average Communication Periods of Each Agent via The ETC Protocols in This Work and [33].

Agents		1	2	3	4
Average Communication Period[1]	This work	0.00125	0.00124	0.00126	0.001
	[33]	0.001			

[1] Average communication period in this table represents the average period an agent receives information from its in-neighbors.

It can be observed from Table 1 that the ETC protocol proposed in this work can reduce the communication burden of the MAS compared with that in [33].

5. Conclusion

In this work, the event-triggered H_∞ output consensus problem of heterogeneous linear MASs has been studied. Two ETC protocols based on state feedback and output feedback have been proposed respectively. Owing to the open-loop estimation method, no continuous communication or measurement between neighboring agents has been required in the proposed protocols. By adopting the unified time- and event-triggering mechanism, Zeno behavior has been excluded for all the agents in the presence of unknown disturbances. Moreover, an explicit MIET has been ensured for each agent. A possible future work is to design a fully distributed

ETC protocol for the H_∞ consensus problem that is independent of any global information of the MAS such as the leader's system matrix S and the Laplacian matrix of the communication topology as in [42, 43].

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