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Published in:

IEEE Transactions on Cybernetics

Published: 01/02/2024

Document Version:

Post-print, also known as Accepted Author Manuscript, Peer-reviewed or Author Final version

Publication record in CityU Scholars:

[Go to record](#)

Published version (DOI):

[10.1109/TCYB.2022.3191938](https://doi.org/10.1109/TCYB.2022.3191938)

Publication details:

Zhang, J., Liu, L., Ji, H., & Wang, X. (2024). Optimal Output Consensus of Heterogeneous Linear Multiagent Systems Over Weight-Unbalanced Directed Networks. *IEEE Transactions on Cybernetics*, 54(2), 1167-1177. <https://doi.org/10.1109/TCYB.2022.3191938>

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Optimal Output Consensus of Heterogeneous Linear Multi-Agent Systems over Weight-Unbalanced Directed Networks

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Abstract—This paper investigates the distributed optimal output consensus problem of heterogeneous linear multi-agent systems over weight-unbalanced directed networks. A novel distributed continuous-time state feedback controller is proposed to steer the outputs of all the agents to converge to the optimal solution of the global cost function. Under the standard condition that the unbalanced digraph is strongly connected and the local cost functions are strongly convex with global Lipschitz gradients, the exponential convergence of the closed-loop multi-agent system is established. Then, the proposed state feedback control law is extended to an observer-based output feedback setting. Two examples are finally provided to illustrate the effectiveness of the proposed control schemes.

Index Terms—Optimal output consensus, linear systems, weight-unbalanced, directed networks.

I. INTRODUCTION

OVER the past decade, the distributed optimization problem (DOP) has attracted increasing attention due to its broad applications in sensor networks, distributed parameter estimation, power systems and machine learning; see, for example, [1], [2]. In the DOP, each agent is assigned with an individual local cost function. The objective is to minimize the sum of all local cost functions in a distributed manner by using only neighboring information and local computation. Seminal works on this topic can be traced back to [3], [4], and for recent progress, one may refer to the relevant reviews [5], [6] and references therein.

Most of the existing works focus on discrete-time dynamics, to name a few, see [7]–[12]. However, since many practical systems operate in the continuous-time setting, such as unmanned vehicles and robots, a few efforts have been made recently for the case of continuous-time dynamics [13]–[15]. Based on the first-order gradients, the authors in [13] propose a distributed proportional-integral (PI) control scheme for multi-agent systems over undirected graphs. Then the approach is extended to deal with weighted-balanced directed networks in [14]. To eliminate the requirement for additional information communication by PI feedback, the modified Lagrangian based (MLB) algorithm is then developed in [15] at the expense of special initialization.

To handle the DOP on weight-unbalanced directed graphs, some consensus based protocols are commonly used, such as

the push-pull based protocols [9], [10] and the push-sum based protocols [8], [16]. However, the above-mentioned protocols usually involve certain global information, including in-degree [16] and out-degree [8], [10], which might not be available in general directed networks. A distributed continuous-time control strategy is designed in [17] to deal with the weight-unbalanced directed graphs, but it cannot achieve the optimal solution when the left eigenvector is not available in advance. To handle the imbalance, a distributed discrete-time algorithm is proposed in [11] with the gradient being divided by an additional variable, which is designed to exponentially converges to the left eigenvector corresponding to the eigenvalue one of the row-stochastic matrix. More recently, the discrete-time algorithm in [11] is extended to a continuous-time version in [18], and the explicit dependency on the left eigenvector is removed in comparison with the algorithm in [17].

It is worth mentioning that the conventional DOP in the aforementioned works can be regarded as a distributed optimal output consensus (OOC) problem with single integrator dynamics. However, there are quite a few engineering tasks in practice that could be reformulated as the OOC problem for more general agent dynamics, such as the economic dispatch in power systems [19], rigid body attitude formation control [20] and source seeking in multi-robot systems [21]. Recently, many efforts are dedicated to solving the OOC for double integrators [22], high-order linear systems [23], [24] and nonlinear systems [25], [26], over undirected graphs. It can be noted that the control design in such scenarios is much more challenging. Typically, the control design for the OOC problem can be classified into two types. The first type is a control scheme based on the two-layer structure consisting of an optimal signal generator and a reference-tracking controller [24], [26]. However, this type of control design requires the agent dynamics to be minimum-phase and have well-defined vector relative degrees [24], and may fail in the scenario when some of the optimal signal generators result in an augmented system for which the reference-tracking problem cannot be solved. Different from the first type, the second one concentrates on developing integrated control laws, which avoids the requirement on the optimal signal as a reference [27], [28]. The authors in [27] propose two adaptive control laws to address the OOC problem for homogeneous linear multi-agent systems. However, the controllers can only be applied when the gradients satisfy a specific structure, and, certain global information is needed to verify their applicability. In a recent work [28], the OOC problem of heterogeneous linear multi-agent systems is reformulated as a special output regulation problem, which is then solved by designing a new controller based on the solutions of well-designed linear matrix equations. Unfortunately, the aforementioned controllers can

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only be applied to undirected graphs.

In our preliminary work [29], the OOC problem for homogeneous linear multi-agent systems over weight-unbalanced directed graphs is solved by designing a distributed state feedback controller. The design of the state feedback controller requires a prior knowledge of the left eigenvector corresponding to the eigenvalue zero of the Laplacian matrix \mathcal{L} . In this paper, we consider the OOC problem over weight-unbalanced directed graphs for heterogeneous linear multi-agent systems. A novel distributed state feedback controller is firstly proposed by introducing an additional variable to avoid the explicit dependence on the left eigenvector. To tackle heterogeneous linear agent dynamics, we take advantage of the well-designed matrix equations from [28], which serves as a modification of regulator equations in [30]. Then the proposed state feedback controller is further extended to an observer-based output feedback one. The main contributions of this paper are summarized as follows:

1) This work investigates the OOC problem for general linear multi-agent systems on weight-unbalanced digraphs. Compared with most existing works, the scenarios considered in this paper are much more general and thus more applicable in practice. On one hand, in contrary to integrator-type agent dynamics discussed in [14], [18], we consider more general heterogeneous linear systems. On the other hand, unlike the works of solving the OOC problem for linear multi-agent systems on undirected networks [24], [27], [28], we focus on more general and thus more challenging weight-unbalanced directed networks. To address the challenges arising from the asymmetry of the Laplacian matrices corresponding to directed graphs, we utilize some useful results from Kronecker matrix algebra and the direct sum operation of vectors instead of the commonly used orthogonal transformation.

2) When the state information is not measurable, which is often the case in practice, we extend the newly developed state feedback controller to an observer-based output feedback one so that the considered OOC problem can still be solved. Therefore, compared with the existing results in [23], [27], our work further expands the scope of distributed optimization in practical applications.

The rest of this paper is organized as follows. Preliminaries and problem formulation are presented in Section II and III, respectively. Design of control laws and analysis of the resulting closed-loop systems are provided in Section IV followed by illustrative examples in Section V. Finally, the conclusion is stated in Section VI.

Notations: Let \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote the sets of real vectors of dimension n and real matrices of dimension $n \times m$, respectively. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote the column vector of dimension n with all entries equal to one and zero, respectively. Let I_n denote the identity matrix of dimension n . Let \otimes denote the Kronecker product of matrices. For a matrix $A \in \mathbb{R}^{n \times n}$, A^T and $\text{tr}(A)$ represent its transpose and trace, respectively. $\|\cdot\|$ represents the Euclidean norm of vectors or the induced 2-norm of matrices. $\text{col}(x_1, x_2, \dots, x_n)$ represents a column vector with x_1, x_2, \dots, x_n being its elements. $\text{diag}(x_1, x_2, \dots, x_n)$ represents a diagonal matrix with x_1, x_2, \dots, x_n being its diagonal elements; $\text{diag}(B_1, B_2, \dots, B_N)$ represents a block

diagonal matrix with matrices $B_i \in \mathbb{R}^{n_i \times p_i}$, $i \in 1, 2, \dots, N$ being its diagonal block elements. For a differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, ∇f is its gradient.

II. PRELIMINARIES

In this section, we present some preliminaries on graph theory, convex analysis, Kronecker matrix algebra, and perturbed system theory.

A. Graph Theory

A directed graph (in short, a digraph) is adopted to depict the agent information flow. A weighted directed graph of order N is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is a set with N vertices called nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of ordered pairs of nodes called edges, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is an associated weighted adjacency matrix. For $i, j \in \mathcal{V}$, the ordered pair $(j, i) \in \mathcal{E}$ denotes an edge from j to i , that is, the i th agent can receive information from the j th agent, but not vice versa. In this case, j is called an in-neighbor of i , and i is called an out-neighbor of j . A directed path is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is a directed edge. A digraph is strongly connected if there exists a directed path in each direction between each pair of nodes. The associated adjacency matrix \mathcal{A} is defined as $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, otherwise $a_{ij} = 0$, and $a_{ii} = 0$ for all $i \in \mathcal{V}$ since it is assumed that there are no self-loops in a digraph. Furthermore, the Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ associated with the digraph \mathcal{G} is defined as $l_{ii} = \sum_{j=1}^N a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. A digraph \mathcal{G} is called weight balanced iff $\mathbf{1}_N^T \mathcal{L} = \mathbf{0}_N^T$. For a more detailed introduction of graph theory, please refer to [31].

Lemma 1. [32], [33] *Assume that the unbalanced directed graph \mathcal{G} is strongly connected. Let \mathcal{L} be the associated Laplacian matrix. Then*

- *there exists a positive left eigenvector $r = (r_1, r_2, \dots, r_N)^T$ associated with the eigenvalue zero such that $r^T \mathcal{L} = \mathbf{0}_N^T$ and $\sum_{i=1}^N r_i = 1$;*
- *$\bar{\mathcal{L}} = (R\mathcal{L} + \mathcal{L}^T R) / 2$ is positive semidefinite, where $R = \text{diag}(r_1, r_2, \dots, r_N)$, and the eigenvalues of $\bar{\mathcal{L}}$ can be ordered as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_N$; and*
- *$\exp(-\mathcal{L}t)$ is a nonnegative matrix with positive diagonal entries for all $t > 0$, and $\lim_{t \rightarrow \infty} \exp(-\mathcal{L}t) = \mathbf{1}_N r^T$.*

B. Convex Analysis

In this subsection, the definitions of strong convexity and Lipschitz continuity are given, see [34] for more details.

A continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strongly convex on \mathbb{R}^n if there exists a positive constant m such that $(x - y)^T (\nabla f(x) - \nabla f(y)) \geq m \|x - y\|^2$ for all $x, y \in \mathbb{R}^n$. A function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is globally Lipschitz on \mathbb{R}^n if there exists a positive constant M such that $\|g(x) - g(y)\| \leq M \|x - y\|$ for all $x, y \in \mathbb{R}^n$.

C. Kronecker Matrix Algebra

For $E \in \mathbb{R}^{n \times m}$, Let $\text{col}_i(E) \in \mathbb{R}^n$ be the i th column of matrix E , and denote

$$\text{vec}(E) \triangleq \begin{pmatrix} \text{col}_1(E) \\ \vdots \\ \text{col}_m(E) \end{pmatrix} \in \mathbb{R}^{nm}$$

as the column vector of dimension nm obtained by stacking $\text{col}_i(E)$. The following two crucial lemmas are given in [35].

Lemma 2. Let $E \in \mathbb{R}^{n \times m}$ and $F \in \mathbb{R}^{m \times n}$. Then, $\text{tr}(EF) = \text{tr}(FE) = (\text{vec}(E^T))^T \text{vec}(F)$. Additionally, for real column vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, $\text{tr}(ba^T) = a^T b$.

Lemma 3. Let $E \in \mathbb{R}^{n \times m}$, $F \in \mathbb{R}^{m \times l}$, and $G \in \mathbb{R}^{l \times k}$. Then, $\text{vec}(EFG) = (G^T \otimes E) \text{vec}(F)$.

D. Perturbed System Theory

Last but not least, the theory of perturbed systems which facilitates subsequent analysis is addressed.

Lemma 4. Consider the perturbed system

$$\dot{x} = g(t, x) + \kappa(t, x). \quad (1)$$

Let $x = 0$ be an exponentially stable equilibrium point of the nominal system $\dot{x} = g(t, x)$, where g is continuously differentiable and the Jacobian matrix $[\partial g / \partial x]$ is bounded on \mathbb{R}^n . Suppose the perturbation term $\kappa(t, x)$ satisfies $\kappa(t, 0) = 0$ and $\|\kappa(t, x)\| \leq \gamma(t)\|x\|$, where $\lim_{t \rightarrow \infty} \gamma(t) = 0$. Then, the origin is an exponentially stable equilibrium point of the perturbed system (1).

Proof. It can be proved by a simple combination of Corollary 9.1 and Lemma 9.5 in [36]. \square

III. PROBLEM FORMULATION

Consider a multi-agent system of N heterogeneous agents described by the following dynamics,

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i, \\ y_i &= C_i x_i, \quad i = 1, 2, \dots, N, \end{aligned} \quad (2)$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{p_i}$ and $y_i \in \mathbb{R}^q$ are the state, control input and output of the i th agent, respectively. $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times p_i}$, and $C_i \in \mathbb{R}^{q \times n_i}$ are constant matrices.

A local cost function $f_i(y) : \mathbb{R}^q \rightarrow \mathbb{R}$ is assigned to each agent i for $i = 1, 2, \dots, N$, which is only available to agent i . Define the global cost function as $f(y) = \sum_{i=1}^N f_i(y)$. The objective of this work is to design distributed controllers such that the outputs of all the agents are steered to the optimal solution y^* of the following optimization problem,

$$\min_{y \in \mathbb{R}^q} f(y). \quad (3)$$

Remark 1. Unlike the output consensus problem of heterogeneous linear multi-agent systems [37], [38], we consider the more general and difficult optimal output consensus problem in this paper. The differences and also challenges in this scenario are to steer the outputs of all the agents not only to achieve consensus but also to reach the optimal solution of the global

cost function. In this case, the design of distributed controllers is more challenging.

To achieve the objective, we need some standard assumptions.

Assumption 1. For $i = 1, 2, \dots, N$, the local cost function f_i is continuously differentiable and strongly convex with constant m_i , and ∇f_i is globally Lipschitz on \mathbb{R}^q with constant M_i .

Remark 2. The strong convexities of the local cost functions in Assumption 1 guarantee that the optimal solution $y^* \in \mathbb{R}^q$ is existing and unique. Assumption 1 is standard and thus commonly used in many existing works, see [15], [18].

Assumption 2. The communication directed graph \mathcal{G} is strongly connected.

Define $\tilde{f}(Y) = \sum_{i=1}^N f_i(y_i)$ with $Y = \text{col}(y_1, y_2, \dots, y_N) \in \mathbb{R}^{qN}$. Then similar to the previous works [17], [28], under Assumption 2, we can reformulate problem (3) as

$$\begin{aligned} \min_{Y \in \mathbb{R}^{qN}} \tilde{f}(Y), \quad \tilde{f}(Y) &= \sum_{i=1}^N f_i(y_i), \\ \text{subject to} \quad (\mathcal{L} \otimes I_q) Y &= \mathbf{0}. \end{aligned} \quad (4)$$

To handle linear agent dynamics, we need an additional assumption.

Assumption 3. For $i = 1, 2, \dots, N$, (A_i, B_i) is stabilizable and

$$\text{rank} \begin{bmatrix} C_i B_i & \mathbf{0}_{q \times p_i} \\ -A_i B_i & B_i \end{bmatrix} = n_i + q.$$

Lemma 5. [28] Suppose that Assumption 3 holds. Then for $i = 1, 2, \dots, N$, the linear matrix equations

$$C_i \Psi_i - I_q = \mathbf{0}_{q \times q}, \quad (5a)$$

$$B_i \Phi_i - A_i \Psi_i = \mathbf{0}_{n_i \times q}, \quad (5b)$$

$$B_i \Upsilon_i - \Psi_i = \mathbf{0}_{n_i \times q} \quad (5c)$$

have solution triplets $(\Upsilon_i, \Phi_i, \Psi_i)$.

Remark 3. Although a two-layer control scheme is proposed in [24] to handle the OOC problem for linear systems having well-defined vector relative degrees, it would fail to achieve the objective if the agent dynamics are non-minimum phase. This would thus limit the scope of the algorithm's application given that many practical systems are non-minimum phase. On the contrary, our approach in this work does not suffer such a restriction.

IV. MAIN RESULTS

In this section, a distributed state feedback controller is firstly developed to deal with the OOC problem over weight-unbalanced digraphs. Then the proposed distributed state feedback controller is extended to a distributed observer-based output feedback controller when the state information is not measurable.

A. Distributed state feedback controller

In this subsection, we design the distributed state feedback controller. Based on the relative outputs and individual agent state, the controller for each agent is designed as follows,

$$u_i = -K_i x_i + \Upsilon_i \omega_i - (\Phi_i - K_i \Psi_i) \rho_i, \quad (6a)$$

$$\dot{\rho}_i = \omega_i := -\frac{\nabla f_i(y_i)}{z_i^i} - \gamma_1 \sum_{j=1}^N a_{ij} (y_i - y_j) - \gamma_2 v_i, \quad (6b)$$

$$\dot{v}_i = \gamma_1 \sum_{j=1}^N a_{ij} (y_i - y_j), \quad (6c)$$

$$\dot{z}_i = -\sum_{j=1}^N a_{ij} (z_i - z_j), \quad (6d)$$

where $z_i \in \mathbb{R}^N$, $i = 1, 2, \dots, N$, with z_i^i being its i th component and with its initial value $z_i^i(0)$ satisfying $z_i^i(0) = 1$, $z_i^j(0) = 0$ for all $j \neq i$. $\rho_i \in \mathbb{R}^q$ and $v_i \in \mathbb{R}^q$ are auxiliary variables with initial value $v_i(0) = 0$, $\omega_i \in \mathbb{R}^q$ is an intermediate state. γ_1 and γ_2 are positive constants, $K_i \in \mathbb{R}^{p_i \times n_i}$ is a feedback gain matrix such that $A_i - B_i K_i$ is Hurwitz, and $(\Upsilon_i, \Phi_i, \Psi_i)$ are the solution triplets of linear matrix equations (5). It is worth noting that control law (6a) is composed of the relative outputs and their first-order and second-order integrals. Equations (6b) and (6c) can be seen as a modification to the well-known MLB algorithm in [15] if setting $\rho_i = y_i$ and $z_i^i = 1$.

Let $x = \text{col}(x_1, x_2, \dots, x_N)$, $\rho = \text{col}(\rho_1, \rho_2, \dots, \rho_N)$, $v = \text{col}(v_1, v_2, \dots, v_N)$, $z = \text{col}(z_1, z_2, \dots, z_N)$, $Z_N = \text{diag}(z_1^1, z_2^2, \dots, z_N^N)$, $A = \text{diag}(A_1, A_2, \dots, A_N)$, $B = \text{diag}(B_1, B_2, \dots, B_N)$, $C = \text{diag}(C_1, C_2, \dots, C_N)$, $K = \text{diag}(K_1, K_2, \dots, K_N)$, $\Upsilon = \text{diag}(\Upsilon_1, \Upsilon_2, \dots, \Upsilon_N)$, $\Phi = \text{diag}(\Phi_1, \Phi_2, \dots, \Phi_N)$, $\Psi = \text{diag}(\Psi_1, \Psi_2, \dots, \Psi_N)$, and $\nabla f(Y) = \text{col}(\nabla f_1(y_1), \nabla f_2(y_2), \dots, \nabla f_N(y_N))$. Then, by substituting the control law (6) into the dynamics (2), the closed-loop system can be written in the following compact form,

$$\dot{x} = (A - BK)x + B\Upsilon\dot{\rho} - (B\Phi - BK\Psi)\rho, \quad (7a)$$

$$\dot{\rho} = -(Z_N^{-1} \otimes I_q) \nabla \tilde{f}(Y) - \gamma_1 (\mathcal{L} \otimes I_q) Y - \gamma_2 v, \quad (7b)$$

$$\dot{v} = \gamma_1 (\mathcal{L} \otimes I_q) Y, \quad (7c)$$

$$\dot{z} = -(\mathcal{L} \otimes I_N) z. \quad (7d)$$

Remark 4. Under Assumption 2, one can obtain from Lemma 1 that $z_i^i(t) > 0$ for all $t \geq 0$, which indicates that Z_N^{-1} is well defined [18]. The term $(Z_N^{-1} \otimes I_q) \nabla \tilde{f}(Y)$ in (7b) is used to tackle the imbalance caused by employing only asymmetric Laplacian matrix \mathcal{L} , where Z_N^{-1} serves as the role of R^{-1} in the algorithm in [17]. Thus, the purpose of introducing (7d) is to estimate the left eigenvector r associated with the eigenvalue zero of \mathcal{L} to reduce the restrictive requirement on global information.

To proceed, we first show that the matrix Z_N^{-1} will exponentially tend to R^{-1} . By using Lemma 1, one can obtain that $\lim_{t \rightarrow \infty} z(t) = \lim_{t \rightarrow \infty} \exp(-(\mathcal{L} \otimes I_N)t) z(0) = (\mathbf{1}_N r^T \otimes I_N) z(0) = \mathbf{1}_N \otimes r$. This implies that $\lim_{t \rightarrow \infty} Z_N^{-1} = R^{-1}$ exponentially.

Hereinafter, to cope with the difficulties generated by asymmetric information transmission, we utilize the theories of perturbed systems and input-to-state stability. Define a new variable $\xi = \text{col}(x, \rho, v)$. Then, (7a)-(7c) can be rewritten as follows,

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{\rho} \\ \dot{v} \end{pmatrix}}_{\xi} = \underbrace{\begin{pmatrix} (A - BK)x + B\Upsilon\dot{\rho} - (B\Phi - BK\Psi)\rho \\ -(R^{-1} \otimes I_q) \nabla \tilde{f}(Y) - \gamma_1 (\mathcal{L} \otimes I_q) Y - \gamma_2 v \\ \gamma_1 (\mathcal{L} \otimes I_q) Y \end{pmatrix}}_{g(\xi)} + \underbrace{\begin{pmatrix} \mathbf{0}_{Nq} \\ ((R^{-1} - Z_N^{-1}) \otimes I_q) (\nabla \tilde{f}(Y) - \nabla \tilde{f}(\bar{Y})) \\ \mathbf{0}_{Nq} \end{pmatrix}}_{\kappa(t, \xi)} + \underbrace{\begin{pmatrix} \mathbf{0}_{Nq} \\ ((R^{-1} - Z_N^{-1}) \otimes I_q) \nabla \tilde{f}(\bar{Y}) \\ \mathbf{0}_{Nq} \end{pmatrix}}_{\omega(t)}, \quad (8)$$

where $\bar{Y} = C\bar{x}$, with \bar{x} being the component of the equilibrium point $\bar{\xi} = \text{col}(\bar{x}, \bar{\rho}, \bar{v})$ of the following system,

$$\dot{\xi} = g(\xi). \quad (9)$$

In what follows, our primary goal is to show that $\bar{Y} = \text{col}(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_N)$ is the optimal solution of problem (4) and the equilibrium point of the closed-loop system (7) is exponentially stable.

Lemma 6. Consider system (9) and suppose that Assumptions 1-3 hold. Then \bar{Y} is the optimal solution of problem (4).

Proof. In light of (5b) and (9), the point $\bar{\xi} = \text{col}(\bar{x}, \bar{\rho}, \bar{v})$ satisfies

$$\mathbf{0} = (A - BK)(\bar{x} - \Psi\bar{\rho}), \quad (10a)$$

$$\mathbf{0} = -(R^{-1} \otimes I_q) \nabla \tilde{f}(\bar{Y}) - \gamma_2 \bar{v}, \quad (10b)$$

$$\mathbf{0} = \gamma_1 (\mathcal{L} \otimes I_q) \bar{Y}. \quad (10c)$$

It follows from (10c) that the vector \bar{Y} belongs to the nullspace of $\mathcal{L} \otimes I_q$. Thus, we have $\bar{Y} = \mathbf{1}_N \otimes \tau$ for some constant vector $\tau \in \mathbb{R}^q$. Left multiplying $\dot{v} = \gamma_1 (\mathcal{L} \otimes I_q) Y$ by $r^T \otimes I_q$ leads to $(r^T \otimes I_q) \dot{v} = \mathbf{0}$. Then, with the assumption that $v(0) = \mathbf{0}$, one obtains $(r^T \otimes I_q) \bar{v} = \mathbf{0}$. Next, left multiplying (10b) by $r^T \otimes I_q$ results in $(\mathbf{1}_N^T \otimes I_q) \nabla \tilde{f}(\bar{Y}) = \mathbf{0}$, which is equivalent to

$$\sum_{i=1}^N \nabla f_i(\bar{y}_i) = \mathbf{0}. \quad (11)$$

Replacing \bar{y}_i by τ in (11) results in $\sum_{i=1}^N \nabla f_i(\tau) = \mathbf{0}$. Note that the optimality condition is $\sum_{i=1}^N \nabla f_i(y^*) = \mathbf{0}$. One thus has $\tau = y^*$. Therefore, it can be concluded that $\bar{Y} = \mathbf{1}_N \otimes y^*$ is the optimal solution of problem (4). \square

Remark 5. In essence, we only need to meet the requirement of $(r^T \otimes I_q) v(0) = \mathbf{0}$ for the initial value of $v(0)$. However, since v is an internal variable, we can directly set $v(0) = \mathbf{0}$ for simplicity.

The first main result of this paper is presented below.

Theorem 1. Consider system (2) and suppose Assumptions 1-3 hold. Then there exist positive constants γ_1 and γ_2 such that the output Y exponentially converges to the optimal value $\bar{Y} = \mathbf{1}_N \otimes y^*$ of problem (4) under the state feedback controller (6), with y^* being the optimal solution to problem (3).

Remark 6. The main feature of this paper is that both general high-order dynamics and weight-unbalanced directed graphs can be handled via our proposed controllers. On one hand, different from existing works that require the agent dynamics to be integrator-type [14], [18] or high-order but minimum phase systems [22]–[24], the general dynamics considered in this work are allowed to be non-minimum phase. On the other hand, in contrary to most existing works where undirected graphs [7], [13] or balanced directed graphs [14], [15] are considered, the proposed controller (6) is able to tackle weight-unbalanced directed graphs without a prior knowledge of the left eigenvector corresponding to the eigenvalue zero of the Laplacian matrix \mathcal{L} . It is worth noting that the left eigenvector is a kind of global information, which is required in [17], but may be unavailable in practical applications.

Remark 7. Compared with asymptotic convergence, exponential convergence has many advantages from the perspective of stability analysis and control synthesis. Contrary to existing works [14], [18], [24], [28] where only asymptotic convergence is obtained, the much more desirable exponential convergence can be established in this paper.

Proof. The proof can be accomplished by the following three steps.

Step 1: The exponential stability of system (9) is established. Introduce the variables substitution $\tilde{x} = x - \bar{x}$, $\tilde{\rho} = \rho - \bar{\rho}$ and $\tilde{v} = v - \bar{v}$ such that the new equilibrium point is transferred to the origin. Note that (5c) and $Y = Cx$ hold. Thus, the dynamics of \tilde{x} , $\tilde{\rho}$ and \tilde{v} satisfy

$$\dot{\tilde{x}} = A_c (\tilde{x} - \Psi \tilde{\rho}) + \Psi \dot{\tilde{\rho}}, \quad (12a)$$

$$\dot{\tilde{\rho}} = - (R^{-1} \otimes I_q) h - \gamma_1 (\mathcal{L} \otimes I_q) C \tilde{x} - \gamma_2 \tilde{v}, \quad (12b)$$

$$\dot{\tilde{v}} = \gamma_1 (\mathcal{L} \otimes I_q) C \tilde{x}, \quad (12c)$$

where $h = \nabla \tilde{f}(C(\bar{x} + \tilde{x})) - \nabla \tilde{f}(C\bar{x})$, and $A_c = A - BK$ is Hurwitz.

Consider the following positive definite function,

$$V_1 = \frac{1}{2} (C \tilde{x})^T (R \otimes I_q) (C \tilde{x}) + \frac{1}{2} (C \tilde{x} + \tilde{v})^T (R \otimes I_q) (C \tilde{x} + \tilde{v}).$$

By using (5a), the derivative of V_1 along (12) is given by

$$\begin{aligned} \dot{V}_1 = & -2 (C \tilde{x})^T h - \gamma_1 (C \tilde{x})^T (R \mathcal{L} \otimes I_q) (C \tilde{x}) \\ & + 2 (C \tilde{x})^T (R \otimes I_q) C A_c (\tilde{x} - \Psi \tilde{\rho}) - \tilde{v}^T h \\ & - 2\gamma_2 (C \tilde{x})^T (R \otimes I_q) \tilde{v} - \gamma_2 \tilde{v}^T (R \otimes I_q) \tilde{v} \\ & + \tilde{v}^T (R \otimes I_q) C A_c (\tilde{x} - \Psi \tilde{\rho}). \end{aligned} \quad (13)$$

Using Lemma 1, we have $(C \tilde{x})^T (R \mathcal{L} \otimes I_q) (C \tilde{x}) = (C \tilde{x})^T (\bar{\mathcal{L}} \otimes I_q) (C \tilde{x})$. By referring to the definition of R in

Lemma 1, one has $\tilde{v}^T (R \otimes I_q) \tilde{v} \geq r_{\min} \|\tilde{v}\|^2$, where $r_{\min} = \min \{r_1, r_2, \dots, r_N\}$. Let $M = \max \{M_1, M_2, \dots, M_N\}$ and $m = \min \{m_1, m_2, \dots, m_N\}$ respectively. One then has $\|h\| \leq M \|C \tilde{x}\|$ and $(C \tilde{x})^T h \geq m \|C \tilde{x}\|^2$ by Assumption 1. Note that the inequality $a^T b \leq \frac{1}{\delta} \|a\|^2 + \frac{\delta}{4} \|b\|^2$ is always tenable for any $\delta > 0$. Thus, the following inequality is satisfied,

$$-\tilde{v}^T h \leq \frac{1}{\delta} \|h\|^2 + \frac{\delta}{4} \|\tilde{v}\|^2 \leq \frac{M^2}{\delta} \|C \tilde{x}\|^2 + \frac{\delta}{4} \|\tilde{v}\|^2. \quad (14)$$

With the obtained facts, (13) can be rewritten as follows,

$$\begin{aligned} \dot{V}_1 \leq & - \left(2m - \frac{M^2}{\delta}\right) \|C \tilde{x}\|^2 - \left(\gamma_2 r_{\min} - \frac{\delta}{4}\right) \|\tilde{v}\|^2 \\ & - \gamma_1 (C \tilde{x})^T (\bar{\mathcal{L}} \otimes I_q) (C \tilde{x}) - 2\gamma_2 (C \tilde{x})^T (R \otimes I_q) \tilde{v} \\ & + 2 (C \tilde{x})^T (R \otimes I_q) C A_c (\tilde{x} - \Psi \tilde{\rho}) \\ & + \tilde{v}^T (R \otimes I_q) C A_c (\tilde{x} - \Psi \tilde{\rho}). \end{aligned} \quad (15)$$

Let $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_N$ denote the ordered eigenvalues of the Laplacian matrix $\bar{\mathcal{L}}$. There exist orthonormal vectors $\mathbf{1}_N$ and $\eta_i, i = 2, 3, \dots, N$, such that $\bar{\mathcal{L}} \mathbf{1}_N = \mathbf{0}$ and $\bar{\mathcal{L}} \eta_i = \lambda_i \eta_i$. Define $\Lambda = (C_1 \tilde{x}_1, \dots, C_N \tilde{x}_N) \in \mathbb{R}^{q \times N}$ and construct $D = (d_1, d_2, \dots, d_N) \in \mathbb{R}^{q \times N}$ with vectors $d_1 = \Lambda \mathbf{1}_N \in \mathbb{R}^q$, $d_i = \Lambda \eta_i \in \mathbb{R}^q, i = 2, 3, \dots, N$. Then, it can be verified that $D = \Lambda Q$, where $Q = (\mathbf{1}_N, \eta_2, \dots, \eta_N) \in \mathbb{R}^{N \times N}$. Noticing that Q is an orthogonal matrix, we have $\Lambda = D Q^T$. By using Lemma 3, one then has $C \tilde{x} = \text{vec}(\Lambda) = \text{vec}(D Q^T) = (I_N \otimes D) \text{vec}(Q^T)$. Thus, by applying Lemma 2 and Lemma 3, it follows that

$$\begin{aligned} & (C \tilde{x})^T (\bar{\mathcal{L}} \otimes I_q) (C \tilde{x}) \\ & = (\text{vec}(Q^T))^T (\bar{\mathcal{L}} \otimes D^T D) \text{vec}(Q^T) \\ & = (\text{vec}(Q^T))^T \text{vec}(D^T D Q^T \bar{\mathcal{L}}) \\ & = \text{tr}(Q D^T D Q^T \bar{\mathcal{L}}) = \text{tr}(D Q^T \bar{\mathcal{L}} Q D^T) \\ & = \sum_{i=2}^N \eta_i^T \bar{\mathcal{L}} \eta_i d_i^T d_i = \sum_{i=2}^N \lambda_i \eta_i^T \eta_i d_i^T d_i \\ & \geq \sum_{i=2}^N \lambda_2 \eta_i^T \eta_i d_i^T d_i = \lambda_2 \sum_{i=2}^N d_i^T d_i = \lambda_2 \|s\|^2, \end{aligned} \quad (16)$$

where $s = \|\text{col}(d_2, d_3, \dots, d_N)\|$. Similarly, define $\Pi = (\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_N) \in \mathbb{R}^{q \times N}$. One then has

$$\begin{aligned} & (C \tilde{x})^T (R \otimes I_q) \tilde{v} \\ & = (\text{vec}(Q^T))^T (R \otimes D^T) \text{vec}(\Pi) \\ & = (\text{vec}(Q^T))^T \text{vec}(D^T \Pi R) \\ & = \text{tr}(Q D^T \Pi R) = \text{tr}(\Pi R Q D^T) \\ & = \text{tr}(\Pi R \mathbf{1}_N d_1^T) + \text{tr}(\Pi R H), \end{aligned} \quad (17)$$

where $H = \sum_{i=2}^N \eta_i d_i^T$. Observing that $(r^T \otimes I_q) v = \mathbf{0}$, one thus concludes that $\Pi R \mathbf{1}_N = \sum_{i=1}^N r_i \tilde{v}_i = \mathbf{0}$ by simple computation. Therefore, (17) can be rewritten as follows,

$$\begin{aligned} (C \tilde{x})^T (R \otimes I_q) \tilde{v} & = \text{tr}(H \Pi R) = (\text{vec}(H^T))^T \text{vec}(\Pi R) \\ & = (\text{vec}(H^T))^T (R \otimes I_q) \tilde{v}. \end{aligned} \quad (18)$$

Note that $\|\text{vec}(H^T)\|^2 = (\text{vec}(H^T))^T \text{vec}(H^T) = \text{tr}(HH^T) = \|s\|^2$ holds. Then, by using equation (18) together with $r_i < 1$, $i = 1, 2, \dots, N$, we have the following inequality,

$$\begin{aligned} -2\gamma_2 (C\tilde{x})^T (R \otimes I_q) \tilde{v} &\leq \frac{\gamma_2^2}{\delta} \|\text{vec}(H^T)\|^2 + \delta \|\tilde{v}\|^2 \\ &= \frac{\gamma_2^2}{\delta} \|s\|^2 + \delta \|\tilde{v}\|^2. \end{aligned} \quad (19)$$

Meanwhile, we have

$$\begin{aligned} 2(C\tilde{x})^T (R \otimes I_q) C A_c (\tilde{x} - \Psi\tilde{\rho}) &\leq \frac{\|C\|^2}{\delta} \|C\tilde{x}\|^2 + \delta \|A_c\|^2 \|\tilde{x} - \Psi\tilde{\rho}\|^2, \end{aligned} \quad (20)$$

$$\begin{aligned} \tilde{v}^T (R \otimes I_q) C A_c (\tilde{x} - \Psi\tilde{\rho}) &\leq \frac{\|C\|^2}{4\delta} \|\tilde{v}\|^2 + \delta \|A_c\|^2 \|\tilde{x} - \Psi\tilde{\rho}\|^2. \end{aligned} \quad (21)$$

Substituting (16), (19)–(21) into (15) leads to

$$\begin{aligned} \dot{V}_1 &\leq -\left(2m - \frac{M^2 + \|C\|^2}{\delta}\right) \|C\tilde{x}\|^2 \\ &\quad - \left(\gamma_2 r_{\min} - \frac{5\delta}{4} - \frac{\|C\|^2}{4\delta}\right) \|\tilde{v}\|^2 \\ &\quad - \left(\gamma_1 \lambda_2 - \frac{\gamma_2^2}{\delta}\right) \|s\|^2 + 2\delta \|A_c\|^2 \|\tilde{x} - \Psi\tilde{\rho}\|^2. \end{aligned} \quad (22)$$

Since A_c is Hurwitz, there exists a positive definite matrix P_c such that $P_c A_c + A_c^T P_c \leq -I$. Choose another positive definite function $V_2 = (\tilde{x} - \Psi\tilde{\rho})^T P_c (\tilde{x} - \Psi\tilde{\rho})$. It follows from (12a) that the dynamics of $\tilde{x} - \Psi\tilde{\rho}$ satisfies

$$\dot{\tilde{x}} - \Psi\dot{\tilde{\rho}} = A_c (\tilde{x} - \Psi\tilde{\rho}). \quad (23)$$

Thus, the derivative of V_2 along the trajectory of (12) is inferred as

$$\begin{aligned} \dot{V}_2 &= (\tilde{x} - \Psi\tilde{\rho})^T (P_c A_c + A_c^T P_c) (\tilde{x} - \Psi\tilde{\rho}) \\ &\leq -\|\tilde{x} - \Psi\tilde{\rho}\|^2. \end{aligned} \quad (24)$$

Consider the Lyapunov function candidate $V(\tilde{x}, \tilde{\rho}, \tilde{v}) = V_1 + \delta_c V_2$ where $\delta_c = 2\delta \|A_c\|^2 + 1$. Then by combining (22) and (24), the derivative of V along the trajectory of (12) satisfies,

$$\begin{aligned} \dot{V} &\leq -\left(2m - \frac{M^2 + \|C\|^2}{\delta}\right) \|C\tilde{x}\|^2 \\ &\quad - \left(\gamma_2 r_{\min} - \frac{5\delta}{4} - \frac{\|C\|^2}{4\delta}\right) \|\tilde{v}\|^2 \\ &\quad - \left(\gamma_1 \lambda_2 - \frac{\gamma_2^2}{\delta}\right) \|s\|^2 - \|\tilde{x} - \Psi\tilde{\rho}\|^2. \end{aligned} \quad (25)$$

Choose the coupling gains γ_1 and γ_2 such that the following inequalities are satisfied,

$$\begin{cases} 2m - \frac{M^2 + \|C\|^2}{\delta} > 0, \\ \gamma_2 r_{\min} - \frac{5\delta}{4} - \frac{\|C\|^2}{4\delta} > 0, \\ \gamma_1 \lambda_2 - \frac{\gamma_2^2}{\delta} > 0. \end{cases} \quad (26)$$

One then has

$$\dot{V} \leq -\varepsilon (\|C\tilde{x}\|^2 + \|\tilde{v}\|^2 + \|\tilde{x} - \Psi\tilde{\rho}\|^2), \quad (27)$$

where $\varepsilon = \min\left\{2m - \frac{M^2 + \|C\|^2}{\delta}, \gamma_2 r_{\min} - \frac{5\delta}{4} - \frac{\|C\|^2}{4\delta}, 1\right\}$. In fact, by selecting appropriate parameters $\delta > \frac{M^2 + \|C\|^2}{2m - \frac{5\delta^2 + \|C\|^2}{4\delta r_{\min}}}$, we then can successively choose sufficiently large $\gamma_2 > \frac{\gamma_2^2}{\lambda_2 \delta}$ to ensure the inequalities in (26). In other words, (26) can always be satisfied by choosing large enough constants γ_1 and γ_2 .

Note that $V(\tilde{x}, \tilde{\rho}, \tilde{v})$ can be rewritten as

$$V = \begin{pmatrix} C\tilde{x} \\ \tilde{v} \\ \tilde{x} - \Psi\tilde{\rho} \end{pmatrix}^T F \begin{pmatrix} C\tilde{x} \\ \tilde{v} \\ \tilde{x} - \Psi\tilde{\rho} \end{pmatrix}$$

by defining the following positive definite matrix

$$F = \begin{pmatrix} (R \otimes I_q) & \frac{1}{2}(R \otimes I_q) & \mathbf{0} \\ \frac{1}{2}(R \otimes I_q) & \frac{1}{2}(R \otimes I_q) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \delta_c P_c \end{pmatrix}.$$

Let μ denote the maximum eigenvalue of F . One then has $V \leq \mu (\|C\tilde{x}\|^2 + \|\tilde{v}\|^2 + \|\tilde{x} - \Psi\tilde{\rho}\|^2)$ from the definition of $V(\tilde{x}, \tilde{\rho}, \tilde{v})$. Thus, (27) can be rewritten as $\dot{V} \leq -\frac{\varepsilon}{\mu} V$. With this fact, one can claim that $\lim_{t \rightarrow \infty} C\tilde{x} = \mathbf{0}$, $\lim_{t \rightarrow \infty} \tilde{v} = \mathbf{0}$ and $\lim_{t \rightarrow \infty} (\tilde{x} - \Psi\tilde{\rho}) = \mathbf{0}$ exponentially, with a convergence rate no less than ε/μ .

Note that $\tilde{\rho} = C\tilde{x} - C(\tilde{x} - \Psi\tilde{\rho})$. By referring to $\lim_{t \rightarrow \infty} C\tilde{x} = \mathbf{0}$ and $\lim_{t \rightarrow \infty} (\tilde{x} - \Psi\tilde{\rho}) = \mathbf{0}$ exponentially, one then obtains that $\lim_{t \rightarrow \infty} \tilde{\rho} = \mathbf{0}$ exponentially. Thus, using $\lim_{t \rightarrow \infty} (\tilde{x} - \Psi\tilde{\rho}) = \mathbf{0}$, one can obtain that $\lim_{t \rightarrow \infty} \tilde{x} = \mathbf{0}$ exponentially. Therefore, the exponential stability of system (9) is established.

Step 2: The exponential stability of the following system (28) is presented,

$$\dot{\xi} = g(\xi) + \kappa(t, \xi). \quad (28)$$

Note that (28) can be interpreted as the perturbed system of (9), where the perturbation term $\kappa(t, \xi)$ satisfies $\kappa(t, \xi) = \mathbf{0}$. Moreover, $\kappa(t, \xi) \leq \sigma(t) \|\xi - \bar{\xi}\|$ with $\sigma(t) = M \|C\| \max_i |r_i^{-1} - (z_i^i(t))^{-1}|$. Since it is proved that $\lim_{t \rightarrow \infty} Z_N^{-1}(t) = R^{-1}$ exponentially, we have $\lim_{t \rightarrow \infty} \sigma(t) = 0$ exponentially. Then it follows from Lemma 4 that the equilibrium point $\bar{\xi}$ of the perturbed system (28) is exponentially stable.

Step 3: The exponential stability of system (8) at the equilibrium point $\bar{\xi} = \text{col}(\bar{x}, \bar{\rho}, \bar{v})$ is established. Since $\nabla \tilde{f}$ is globally Lipschitz by Assumption 1, we learn that $G(\xi, \omega) = g(\xi) + \kappa(t, \xi) + \omega(t)$ is globally Lipschitz in (ξ, ω) . The boundness of $\nabla \tilde{f}(\bar{Y})$ suggests that $\omega(t)$ is bounded. Then it follows from Lemma 4.6 in [36] that the system (8) is input-to-state stable (ISS), i.e., for any initial state $\xi(t_0)$ and any bounded input $\omega(t)$, there exist a class \mathcal{KL} function φ and a class \mathcal{K} function ψ such that the solution $\xi(t)$ satisfies $\|\xi - \bar{\xi}\| \leq \varphi(\|\xi(t_0) - \bar{\xi}\|, t - t_0) + \psi(\sup_{t_0 \leq \tau \leq t} \omega(\tau))$ for all $t \geq t_0$. Recalling the fact that $\lim_{t \rightarrow \infty} (Z_N^{-1}(t) - R^{-1}) = \mathbf{0}$ exponentially, we have $\lim_{t \rightarrow \infty} \omega(t) = \mathbf{0}$ exponentially. Thus it can be shown that ξ exponentially converges to $\bar{\xi}$ by the property of ISS given in [36]. Therefore, we further claim that Y exponentially converges to $\bar{Y} = \mathbf{1}_N \otimes y^*$, with y^* being the solution of the global cost function. The proof is thus completed. \square

Remark 8. In this paper, a major challenge arises from the asymmetric information transmission caused by the weight-unbalanced directed networks. Inspired by work [18], we deal with this challenge in virtue of the useful results from Kronecker matrix algebra and direct sum operation of vectors, instead of the commonly used orthogonal transformation. This control approach can be adopted to solve the optimal output consensus problem for more general nonlinear multi-agent systems over unbalanced directed networks.

Remark 9. It can be seen from the inequalities in (26) that the choice of control gains γ_1 and γ_2 only depends on the minimum value of the elements in the left eigenvector, instead of the exact value of the left eigenvector as in [17]. Therefore, once we can obtain the lower bound of the minimum value without any global information, it can be directly applied to the controller design in this work. Furthermore, the inequalities in (26) can always be guaranteed as long as control parameters γ_1 and γ_2 are chosen to be large enough.

B. Distributed observer-based output feedback controller

It should be noted that state measurements may be unavailable in practical scenarios. In this subsection, the previous distributed state feedback controller is extended to a distributed observer-based output feedback controller. More specifically, the newly proposed output feedback control law is given as follows,

$$\begin{aligned}
u_i &= -K_i \hat{x}_i + \Upsilon_i \omega_i - (\Phi_i - K_i \Psi_i) \rho_i, \\
\dot{\hat{x}}_i &= A_i \hat{x}_i + B_i u_i + H_i (y_i - C_i \hat{x}_i), \\
\dot{\rho}_i &= \omega_i := -\frac{\nabla f_i(y_i)}{z_i^i} - \gamma_1 \sum_{j=1}^N a_{ij} (y_i - y_j) - \gamma_2 v_i, \\
\dot{v}_i &= \gamma_1 \sum_{j=1}^N a_{ij} (y_i - y_j), \\
\dot{z}_i &= -\sum_{j=1}^N a_{ij} (z_i - z_j),
\end{aligned} \tag{29}$$

where \hat{x}_i is the estimation of state x_i , $H_i \in \mathbb{R}^{n_i \times q}$ is an observer feedback matrix such that $A_i - H_i C_i$ is Hurwitz, and the remaining variables are defined to be the same as those in control law (6).

Substituting the above control law into system dynamics (2) yields the following compact form of the closed-loop system,

$$\begin{aligned}
\dot{x} &= Ax - BK\hat{x} + B\Upsilon\dot{\rho} - B(\Phi - K\Psi)\rho, \\
\dot{\hat{x}} &= (A - BK)\hat{x} + B\Upsilon\dot{\rho} - B(\Phi - K\Psi)\rho + H(Y - C\hat{x}), \\
\dot{\rho} &= -(Z_N^{-1} \otimes I_q) \nabla \tilde{f}(Y) - \gamma_1 (\mathcal{L} \otimes I_q) Y - \gamma_2 v, \\
\dot{v} &= \gamma_1 (\mathcal{L} \otimes I_q) Y, \\
\dot{z} &= -(\mathcal{L} \otimes I_N) z,
\end{aligned} \tag{30}$$

where $\hat{x} = \text{col}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N)$, $H = \text{diag}(H_1, H_2, \dots, H_N)$, and the remaining terms are defined as those in the closed-loop

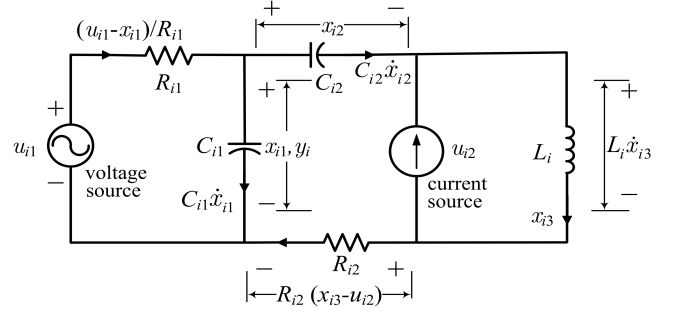


Fig. 1. RLC network.

system (7). Define the estimation error variable $\check{x} = x - \hat{x}$. Then (30) can be rewritten as follows,

$$\begin{aligned}
\dot{x} &= Ax - BK(x - \check{x}) + B\Upsilon\dot{\rho} - B(\Phi - K\Psi)\rho, \\
\dot{\check{x}} &= (A - HC)\check{x}, \\
\dot{\rho} &= -(Z_N^{-1} \otimes I_q) \nabla \tilde{f}(Y) - \gamma_1 (\mathcal{L} \otimes I_q) Y - \gamma_2 v, \\
\dot{v} &= \gamma_1 (\mathcal{L} \otimes I_q) Y, \\
\dot{z} &= -(\mathcal{L} \otimes I_N) z.
\end{aligned}$$

The second main result of this paper is presented below.

Theorem 2. Consider system (2). Suppose Assumptions 1-3 hold and (A_i, C_i) is detectable. Then there exist two positive constants γ_1 and γ_2 such that the output Y exponentially reaches the optimal value $\bar{Y} = \mathbf{1}_N \otimes y^*$ of problem (4) under the output feedback controller (29), with y^* be the optimal solution to problem (3).

Proof. The proof is given in the Appendix. \square

V. ILLUSTRATIVE EXAMPLES

In this section, the effectiveness of the proposed control laws is illustrated by two illustrative examples. We will start with a simplified but practical example in RLC networks.

Example 1. Consider the RLC network depicted in Fig. 1, which is a modification of Figure 2.16 in [39]. It consists of the voltage source u_{i1} , current source u_{i2} , two resistors R_{i1} and R_{i2} , inductor L_i , and two capacitors C_{i1} and C_{i2} . The capacitor voltages and the inductor current will be assigned as state variables x_{i1} , x_{i2} and x_{i3} , respectively. Then we can apply Kirchhoff's current and voltage laws to establish the following equations,

$$\begin{aligned}
x_{i3} &= \frac{u_{i1} - x_{i1}}{R_{i1}} - C_{i1} \dot{x}_{i1} + u_{i2}, \\
x_{i3} &= C_{i2} \dot{x}_{i2} + u_{i2}, \\
x_{i1} &= x_{i2} + L_i \dot{x}_{i3} + R_{i2}(x_{i3} - u_{i2}), \\
y_i &= x_{i1}.
\end{aligned}$$

By defining $x_i = \text{col}(x_{i1}, x_{i2}, x_{i3})$ and $u_i = \text{col}(u_{i1}, u_{i2})$, the state-space description of the RLC network takes the linear

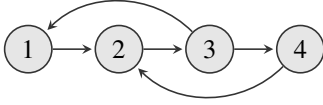


Fig. 2. Communication between RLC networks.

dynamics form (2) with

$$A_i = \begin{bmatrix} -\frac{1}{C_{i1}R_{i1}} & 0 & -\frac{1}{C_{i1}} \\ 0 & 0 & \frac{1}{C_{i2}} \\ \frac{1}{L_i} & -\frac{1}{L_i} & -\frac{R_{i2}}{L_i} \end{bmatrix}, \quad (31)$$

$$B_i = \begin{bmatrix} \frac{1}{C_{i1}R_{i1}} & \frac{1}{C_{i1}} \\ 0 & -\frac{1}{C_{i2}} \\ 0 & \frac{1}{L_i} \end{bmatrix}, \quad C_i = [1 \ 0 \ 0].$$

Consider a multi-agent system of 4 agents described by the RLC network as shown in Fig. 1, the communication topology among agents is given in Fig. 2. Assume that local cost functions of RLC circuits are described by the following quadratic form functions,

$$\begin{aligned} f_1 &= 0.2y^2 - 2y + 1, & f_2 &= 0.4y^2 + y + 2, \\ f_3 &= 0.6y^2 - 3y - 1, & f_4 &= 0.8y^2 + y + 1. \end{aligned}$$

TABLE I
PARAMETERS OF RLC NETWORKS

| i | R_{i1} | R_{i2} | L_i | C_{i1} | C_{i2} |
|-----|----------|----------|-------|----------|----------|
| 1 | 2 | 1 | 3 | 1 | 2 |
| 2 | 1 | 2 | 2 | 3 | 1 |
| 3 | 0.5 | 2 | 1 | 0.5 | 3 |
| 4 | 3 | 0.5 | 2 | 1 | 0.5 |

Then, it is calculated that the optimal value of the global cost function is $y^* = 1.5$. Moreover, it can be verified that Assumptions 1 and 2 are satisfied. Furthermore, to ensure the solvability of linear matrix equations (5) corresponding to dynamics matrices (31), for each agent $i, i = 1, 2, 3, 4$, the parameters of (31) are given in TABLE I so that Assumption 3 is satisfied. Then by Lemma 5, the solution triplets $(\Upsilon_i, \Phi_i, \Psi_i), i = 1, 2, 3, 4$ of linear matrix equations (5) can be chosen as follows,

$$\begin{aligned} \Upsilon_1 &= \begin{bmatrix} 8 \\ -1 \end{bmatrix}, & \Upsilon_2 &= \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, & \Upsilon_3 &= \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}, \\ \Upsilon_4 &= \begin{bmatrix} 7.5 \\ -0.5 \end{bmatrix}, & \Phi_1 &= \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}, & \Phi_2 &= \begin{bmatrix} -1 \\ -1.5 \end{bmatrix}, \\ \Phi_3 &= \begin{bmatrix} -1 \\ 0.2 \end{bmatrix}, & \Phi_4 &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & \Psi_1 &= \begin{bmatrix} 1 \\ 1 \\ -0.5 \end{bmatrix}, \\ \Psi_2 &= \begin{bmatrix} 1 \\ -0.5 \\ 1.5 \end{bmatrix}, & \Psi_3 &= \begin{bmatrix} 1 \\ 1.2 \\ -0.2 \end{bmatrix}, & \Psi_4 &= \begin{bmatrix} 1 \\ 0.5 \\ -1 \end{bmatrix}. \end{aligned}$$

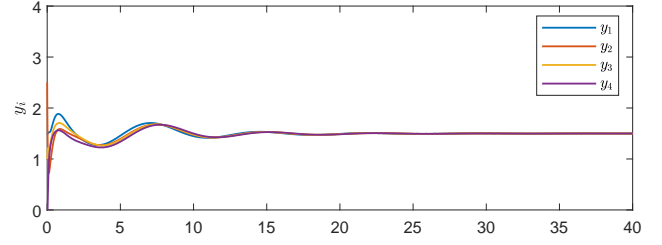


Fig. 3. Convergence performance of RLC networks with the distributed state feedback controller (6) in this work.

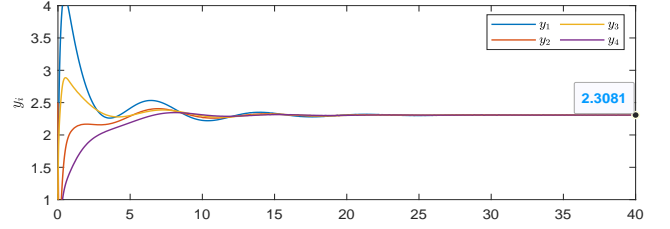


Fig. 4. Convergence performance of RLC networks with the distributed state feedback controller (6) in [28].

Finally, the matrices K_i are chosen as follows such that $A_i - B_i K_i, i = 1, 2, 3, 4$ are Hurwitz,

$$\begin{aligned} K_1 &= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix}, & K_2 &= \begin{bmatrix} 0.5 & 2 & -1 \\ -2 & 0 & 2 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} 2 & -1 & -2 \\ 0 & -3 & 3 \end{bmatrix}, & K_4 &= \begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix}. \end{aligned}$$

Now, we first illustrate the effectiveness of our proposed distributed state feedback controller (6). The initial value $v(0)$ is chosen as $v(0) = \mathbf{0}$. The other initial conditions of the closed-loop system are randomly chosen in the closed interval $[-4, 6]$. The simulation result is shown in Fig. 3. It can be observed that the trajectories of all outputs y_i 's converge to the global minimizer $y^* = 1.5$. Thus the closed-loop system composed of (2) and (6) achieves optimal output consensus eventually.

As a comparison, we will show that the distributed state feedback controller (6) proposed in [28] cannot be applied to solve the optimal output consensus problem of multi-agent systems over unbalanced directed networks considered in this paper, even though the concerned linear agent dynamics are the same. The selection of simulation parameters is the same as the previous setting. The simulation result is shown in Fig. 4. It can be observed that the trajectories of all outputs y_i 's achieve consensus, but unfortunately converge to 2.3081 instead of the global minimizer $y^* = 1.5$. Therefore, it is shown that the distributed state feedback controller (6) proposed in [28] failed to solve the optimal output consensus problem of linear multi-agent systems over unbalanced directed networks. The main reason is that the equilibrium point of the closed-loop system composed of the linear agent dynamics (2) and the controller (6) in [28] is no longer the global optimal solution. \square

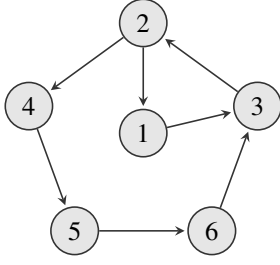


Fig. 5. Weight-unbalanced directed network.

Next, we will provide another example to compare the convergence performance under state feedback control law (6) for different control gains and illustrate the effectiveness of observer-based output feedback control law (29).

Example 2. Consider a group of 6 agents with the unbalanced directed network \mathcal{G} depicted in Fig. 5. For agents $i = 1, 2, \dots, 6$, the local cost functions are respectively given as follows,

$$\begin{aligned} f_1 &= \sin(0.2y - (\pi/2)), & f_2 &= 0.2 \cos(\ln(y^2 + 4) - 0.2), \\ f_3 &= 0.1(y + 0.3)^2 + 0.2(y - 2)^2, & f_4 &= 0.4y^2 \ln(5 + y^2), \\ f_5 &= 0.2y^2(\ln(y^2 + 1) + 1), & f_6 &= 0.3y^2/\sqrt{y^2 + 5}. \end{aligned}$$

It can be verified that Assumptions 1 and 2 are satisfied. Therefore, the strong convexity of the global cost function guarantees that the global minimizer $y^* = 0.286$ is unique.

The dynamics of agents are described by (2) with

$$\begin{aligned} A_1 &= A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & B_1 &= B_2 = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \\ C_1 &= C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, & A_3 &= A_4 = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, \\ B_3 &= B_4 = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}, & C_3 &= C_4 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \\ A_5 &= A_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 1 & -2 \end{bmatrix}, & B_5 &= B_6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ C_5 &= C_6 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}. \end{aligned}$$

Note that Assumption 3 is also satisfied. Then by Lemma 5, the solution triplets $(\Upsilon_i, \Phi_i, \Psi_i), i = 1, 2, \dots, 6$ of linear matrix equations (5) can be chosen as follows,

$$\begin{aligned} \Upsilon_1 &= \Upsilon_2 = \begin{bmatrix} 1.5 & 0.5 \end{bmatrix}^T, & \Upsilon_3 &= \Upsilon_4 = \begin{bmatrix} -0.5 & -2 \end{bmatrix}^T, \\ \Upsilon_5 &= \Upsilon_6 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T, & \Phi_1 &= \Phi_2 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}^T, \\ \Phi_3 &= \Phi_4 = \begin{bmatrix} -0.5 & 0 \end{bmatrix}^T, & \Phi_5 &= \Phi_6 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T, \\ \Psi_1 &= \Psi_2 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T, & \Psi_3 &= \Psi_4 = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}^T, \\ \Psi_5 &= \Psi_6 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T. \end{aligned}$$

Furthermore, the matrices K_i and H_i are respectively chosen as follows such that $A_i - B_i K_i$ and $A_i - H_i C_i, i = 1, 2, \dots, 6$ are Hurwitz,

$$\begin{aligned} K_1 &= K_2 = \begin{bmatrix} 3 & 5 \\ 1.5 & 1 \end{bmatrix}, & K_3 &= K_4 = \begin{bmatrix} 0.75 & -1 \\ 1.25 & -4 \end{bmatrix}, \\ K_5 &= K_6 = \begin{bmatrix} 2.167 & 1 & 0.333 \\ 0 & 3 & 1 \end{bmatrix}, & H_1 &= H_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \end{aligned}$$

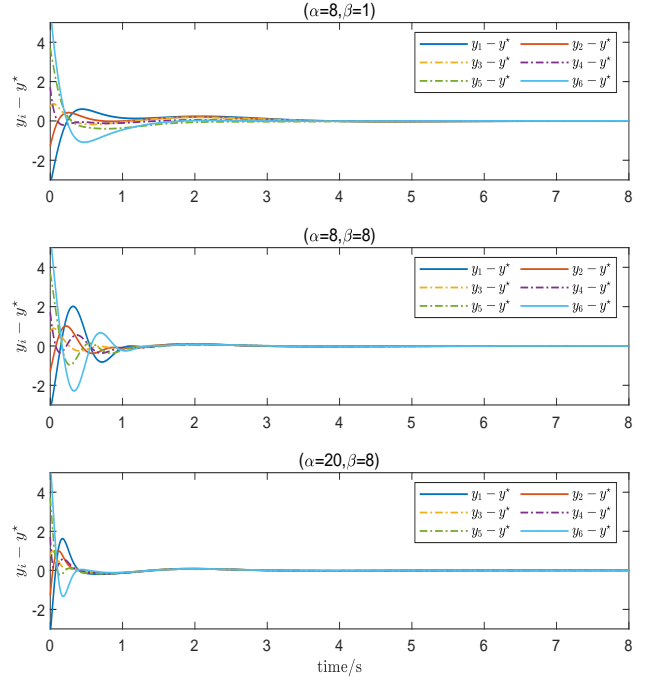


Fig. 6. Convergence performance comparisons under state feedback control law (6).

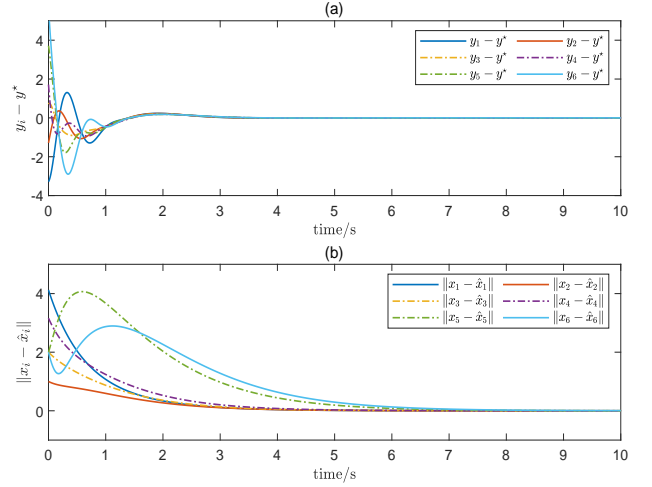


Fig. 7. Convergence performance under output feedback control law (29).

$$H_3 = H_4 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad H_5 = H_6 = \begin{bmatrix} 4 & 3 & 2 \end{bmatrix}^T.$$

The initial values $x(0), \hat{x}(0)$ and $\rho(0)$ can be arbitrarily chosen, while $v(0)$ needs to satisfy $v(0) = \mathbf{0}$. For convergence performance comparisons, three sets of control gains in (6) are given as $(\gamma_1, \gamma_2) = (8, 1)$, $(\gamma_1, \gamma_2) = (8, 8)$, and $(\gamma_1, \gamma_2) = (20, 8)$, respectively. The simulation results of the closed-loop control system via state feedback control law (6) are shown in Fig. 6. One can observe from the figure that the trajectories of outputs y_i converge to the global minimizer $y^* = 0.286$. Moreover, by comparing the convergence performances in Fig. 6(a)-(c), it can be observed that the larger values of γ_1 and γ_2 will bring about faster convergence.

To verify the effectiveness of the observer-based output feedback control law (29), set $(\gamma_1, \gamma_2) = (8, 1)$ and the simulation results of the resulting closed-loop system are shown in Fig. 7. Fig. 7(a) shows that the outputs of all the agents would reach the optimal solution, while Fig. 7(b) indicates that the observer states \hat{x}_i will eventually tend to $x_i, i = 1, 2, \dots, 6$, which is consistent with the theoretical result. \square

VI. CONCLUSION

In this paper, we have studied the distributed optimal output consensus problem of heterogeneous linear multi-agent systems over weight-unbalanced directed networks. We have developed two novel distributed control laws. It is shown that the proposed control laws are able to ensure the agent outputs converge exponentially to the optimal solution under standard assumptions. Our results generalize the optimal output consensus problem of heterogeneous linear multi-agent systems from undirected networks to weight-unbalanced directed networks. Two illustrative examples have also been given to show the effectiveness of the proposed controllers. One of the possible future research topics is to further eliminate the dependence on global information of our current controller design. Another interesting topic is to extend the optimal output consensus problem over weight-unbalanced directed networks to the case that the agent dynamics are uncertain nonlinear systems.

APPENDIX PROOF OF THEOREM 2

Define $\zeta = \text{col}(x, \check{x}, \rho, v)$, then the dynamics of ζ is given as follows,

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{\check{x}} \\ \dot{\rho} \\ \dot{v} \end{pmatrix}}_{\dot{\zeta}} = \underbrace{\begin{pmatrix} Ax - BK(x - \check{x}) + B\Upsilon\dot{\rho} - B(\Phi - K\Psi)\rho \\ (A - HC)\check{x} \\ -(R^{-1} \otimes I_q)\nabla\tilde{f}(Y) - \gamma_1(\mathcal{L} \otimes I_q)Y - \gamma_2v \\ \gamma_1(\mathcal{L} \otimes I_q)Y \end{pmatrix}}_{\bar{g}(\zeta)} + \underbrace{\begin{pmatrix} \mathbf{0}_{Nq} \\ \mathbf{0}_{Nq} \\ ((R^{-1} - Z_N^{-1}) \otimes I_q)(\nabla\tilde{f}(Y) - \nabla\tilde{f}(\bar{Y})) \\ \mathbf{0}_{Nq} \end{pmatrix}}_{\bar{\kappa}(t, \zeta)} + \underbrace{\begin{pmatrix} \mathbf{0}_{Nq} \\ \mathbf{0}_{Nq} \\ ((R^{-1} - Z_N^{-1}) \otimes I_q)\nabla\tilde{f}(\bar{Y}) \\ \mathbf{0}_{Nq} \end{pmatrix}}_{\bar{\omega}(t)}. \quad (32)$$

At first, we show that the output \bar{Y} at the equilibrium point $\bar{\zeta} = (\bar{x}, \bar{\check{x}}, \bar{\rho}, \bar{v})$ of system $\dot{\zeta} = \bar{g}(\zeta)$ is the solution of problem (4). Note that the point $\bar{\zeta} = (\bar{x}, \bar{\check{x}}, \bar{\rho}, \bar{v})$ satisfies

$$\mathbf{0} = (A - BK)(\bar{x} - \Psi\bar{\rho}) + BK\bar{\check{x}}, \quad (33a)$$

$$\mathbf{0} = (A - HC)\bar{\check{x}}, \quad (33b)$$

$$\mathbf{0} = -(R^{-1} \otimes I_q)\nabla\tilde{f}(\bar{Y}) - \gamma_2\bar{v}, \quad (33c)$$

$$\mathbf{0} = \gamma_1(\mathcal{L} \otimes I_q)\bar{Y}. \quad (33d)$$

Since $A - HC$ is Hurwitz, we can obtain from (33b) that $\bar{\check{x}} = 0$. Thus it can be inferred that the component $(\bar{x}, \bar{\rho}, \bar{v})$ at the equilibrium point $\bar{\zeta} = (\bar{x}, 0, \bar{\rho}, \bar{v})$ of system $\dot{\zeta} = \bar{g}(\zeta)$ coincides with that of system (8), i.e., $(\bar{x}, \bar{\rho}, \bar{v})$ satisfies equations (10). Then it can be shown that the output \bar{Y} at the equilibrium point of $\dot{\zeta} = \bar{g}(\zeta)$ is the solution of problem (4) via a similar analysis as in Lemma 6.

In what follows, the exponential stability of the following system is presented,

$$\dot{\zeta} = \bar{g}(\zeta). \quad (34)$$

To this end, transforming the equilibrium point of (34) to the origin by defining $\tilde{x} = x - \bar{x}$, $\tilde{\rho} = \rho - \bar{\rho}$ and $\tilde{v} = v - \bar{v}$ leads to

$$\begin{aligned} \dot{\tilde{x}} &= A_c(\tilde{x} - \Psi\tilde{\rho}) + \Psi\dot{\tilde{\rho}} + BK\tilde{x}, \\ \dot{\tilde{x}} &= A_o\tilde{x}, \\ \dot{\tilde{\rho}} &= -(R^{-1} \otimes I_q)h - \gamma_1(\mathcal{L} \otimes I_q)C\tilde{x} - \gamma_2\tilde{v}, \\ \dot{\tilde{v}} &= \gamma_1(\mathcal{L} \otimes I_q)C\tilde{x}, \end{aligned} \quad (35)$$

where $A_c = A - BK$, $A_o = A - HC$ and $h = \nabla\tilde{f}(C(\bar{x} + \tilde{x})) - \nabla\tilde{f}(C\bar{x})$.

Reconsider the Lyapunov function candidate $V(\tilde{x}, \tilde{\rho}, \tilde{v}) = V_1 + \delta_c V_2$, where V_1, V_2 and δ_c are the same as those defined in the proof of Theorem 1. The derivative of V along the trajectory of (35) is given as follows,

$$\begin{aligned} \dot{V} &= -2(C\tilde{x})^T h - \gamma_1(C\tilde{x})^T(R\mathcal{L} \otimes I_q)(C\tilde{x}) \\ &\quad + 2(C\tilde{x})^T(R \otimes I_q)CA_c(\tilde{x} - \Psi\tilde{\rho}) - \tilde{v}^T h \\ &\quad - 2\gamma_2(C\tilde{x})^T(R \otimes I_q)\tilde{v} - \gamma_2\tilde{v}^T(R \otimes I_q)\tilde{v} \\ &\quad + \tilde{v}^T(R \otimes I_q)CA_c(\tilde{x} - \Psi\tilde{\rho}) \\ &\quad + \delta_c(\tilde{x} - \Psi\tilde{\rho})^T(P_c A_c + A_c^T P_c)(\tilde{x} - \Psi\tilde{\rho}) \\ &\quad + 2(C\tilde{x})^T(R \otimes I_q)CBK\tilde{x} + \tilde{v}^T(R \otimes I_q)CBK\tilde{x}. \end{aligned}$$

Then according to similar arguments in the proof of Theorem 1, one obtains

$$\begin{aligned} \dot{V} &\leq -\left(2m - \frac{M^2 + \|C\|^2}{\delta}\right)\|C\tilde{x}\|^2 \\ &\quad - \left(\gamma_2 r_{\min} - \frac{5\delta}{4} - \frac{\|C\|^2}{4\delta}\right)\|\tilde{v}\|^2 \\ &\quad - \left(\gamma_1 \lambda_2 - \frac{\gamma_2^2}{\delta}\right)\|s\|^2 - \|\tilde{x} - \Psi\tilde{\rho}\|^2 \\ &\quad + 2(C\tilde{x})^T(R \otimes I_q)CBK\tilde{x} + \tilde{v}^T(R \otimes I_q)CBK\tilde{x}. \end{aligned} \quad (36)$$

Note that the following inequalities are satisfied,

$$\begin{aligned} 2(C\tilde{x})^T(R \otimes I_q)CBK\tilde{x} \\ \leq \frac{\|C\|^2}{\delta}\|C\tilde{x}\|^2 + \delta\|BK\|^2\|\tilde{x}\|^2, \end{aligned} \quad (37)$$

$$\begin{aligned} \tilde{v}^T(R \otimes I_q)CBK\tilde{x} \\ \leq \frac{\|C\|^2}{4\delta}\|\tilde{v}\|^2 + \delta\|BK\|^2\|\tilde{x}\|^2. \end{aligned} \quad (38)$$

Then substituting (37) and (38) into (36) leads to

$$\begin{aligned} \dot{V} &\leq -\left(2m - \frac{M^2 + 2\|C\|^2}{\delta}\right)\|C\tilde{x}\|^2 \\ &\quad - \left(\gamma_2 r_{\min} - \frac{5\delta}{4} - \frac{\|C\|^2}{2\delta}\right)\|\tilde{v}\|^2 \end{aligned}$$

$$\begin{aligned}
& - \left(\gamma_1 \lambda_2 - \frac{\gamma_2^2}{\delta} \right) \|s\|^2 - \|\tilde{x} - \Psi \tilde{\rho}\|^2 \\
& + 2\delta \|BK\|^2 \|\tilde{x}\|^2.
\end{aligned} \tag{39}$$

Since A_o is Hurwitz, there exists a positive definite matrix P_o such that $P_o A_o + A_o^T P_o \leq -I$. Consider another positive definite function $V_3 = \tilde{x}^T P_o \tilde{x}$. The derivative of V_3 along the trajectory of (35) satisfies

$$\dot{V}_3 = \tilde{x}^T (P_o A_o + A_o^T P_o) \tilde{x} \leq -\|\tilde{x}\|^2.$$

Consider the Lyapunov function candidate $\bar{V}(\tilde{x}, \tilde{x}, \tilde{\rho}, \tilde{v}) = V_1 + \delta_c V_2 + \delta_o V_3$ with $\delta_o = 2\delta \|BK\|^2 + 1$, then the derivative of \bar{V} along the trajectory of (35) satisfies

$$\begin{aligned}
\dot{\bar{V}} & \leq - \left(2m - \frac{M^2 + 2\|C\|^2}{\delta} \right) \|C\tilde{x}\|^2 \\
& - \left(\gamma_2 r_{\min} - \frac{5\delta}{4} - \frac{\|C\|^2}{2\delta} \right) \|\tilde{v}\|^2 \\
& - \left(\gamma_1 \lambda_2 - \frac{\gamma_2^2}{\delta} \right) \|s\|^2 - \|\tilde{x} - \Psi \tilde{\rho}\|^2 - \|\tilde{x}\|^2.
\end{aligned}$$

The rest of the proof follows arguments similar to those in the proof of Theorem 1, and is thus omitted. \square

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