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Gap Probability for the Hard Edge Pearcey Process

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Gap probability for the hard edge Pearcey process

Dan Dai*, Shuai-Xia Xu† and Lun Zhang‡

November 27, 2022

Abstract

The hard edge Pearcey process is universal in random matrix theory and many other stochastic models. This paper deals with the gap probability for the thinned/unthinned hard edge Pearcey process over the interval $(0, s)$ by working on a 3×3 matrix-valued Riemann-Hilbert problem for the relevant Fredholm determinants. We establish an integral representation of the gap probability via the Hamiltonian related to a new system of coupled differential equations. Together with some remarkable differential identities for the Hamiltonian, we derive the large gap asymptotics for the thinned hard edge Pearcey process, including the explicit evaluation of the constant factor in terms of the Barnes G-function. As an application, we also obtain the asymptotic statistical properties of the counting function for the hard edge Pearcey process.

Contents

1	Introduction	2
2	Main results	4
3	A RH formulation	7
3.1	RH characterizations of K_α and its resolvent kernel	7
3.2	A RH problem related to $\frac{d}{ds}F$	12
4	The Lax pair and differential identities	16
4.1	Lax pair equations	16
4.2	Differential identities for the Hamiltonian	20
5	Asymptotic analysis of the RH problem for Φ as $s \rightarrow +\infty$	21
5.1	First transformation: $\Phi \rightarrow T$	21
5.2	Second transformation: $T \rightarrow S$	23
5.3	Global parametrix	24
5.4	Local parametrix near the origin	27
5.5	Local parametrix near 1	29
5.6	Final transformation and the small norm RH problem	31

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6	Asymptotic analysis of the RH problem for Φ as $s \rightarrow 0^+$	37
6.1	Global parametrix	37
6.2	Local parametrix	37
6.3	Final transformation	39
7	Asymptotics of the functions $p_{i,k}$ and $q_{i,k}$	40
8	Proofs of main results	48
8.1	Proof of Theorem 2.1	48
8.2	Proof of Theorem 2.2	50
Appendix A Confluent hypergeometric parametrix		53
Appendix B Proof of Proposition 7.2		55

1 Introduction

A fundamental question in random matrix theory is to understand distributions of the eigenvalues for large random matrices in different regimes. For the local eigenvalue statistics of random Hermitian matrices, it is well-known that they are encoded in the Fredholm determinants of some integral operators [33, 37, 54]. Moreover, the associated determinantal point processes [47, 58] are believed to be universal [36, 49] for a large family of interacting particle systems, which serves as the cornerstone of widespread applications of random matrix theory in different areas ranging from theoretical to practical sciences.

It comes out that many canonical point processes arising from random matrix theory are characterized by integral kernels of the form

$$K(x, y) = \frac{1}{(2\pi i)^2} \int_{\gamma_u} \int_{\gamma_t} \frac{e^{\mathcal{V}(u;x) - \mathcal{V}(t;y)}}{t - u} du dt \quad (1.1)$$

for $x, y \in \mathbb{R}$ or $(0, +\infty)$, where $\mathcal{V}(u; x)$ takes a simple form in u , the contours γ_u and γ_t depend on the precise form of \mathcal{V} . The most prominent example is the Airy kernel [54], which corresponds to $\mathcal{V}(u; x) = u^3/3 - xu$. The Airy determinant gives us the celebrated Tracy-Widom distribution [62], which describes the limiting distributions of extreme eigenvalues for many random matrix ensembles and share an inherent connection with statistics of other stochastic models beyond the realm of random matrices. One can generalize the Airy kernel by setting \mathcal{V} to be an odd polynomial of the form

$$(-1)^{n+1} \left(\frac{u^{2n+1}}{2n+1} + \sum_{j=1}^{n-1} \rho_j u^{2j+1} \right) - xu, \quad \rho_j \in \mathbb{R}, \quad n = 1, 2, \dots$$

The associated point processes appear in multicritical edge statistics for the momenta of fermions in nonharmonic traps [32] and other statistical physics model [1]. The classical Pearcey kernel corresponds to $\mathcal{V}(u; x) = u^4/4 - \rho u^2/2 + xu$, $\rho \in \mathbb{R}$, i.e., a quartic polynomial in u . Near the points of the spectrum where the density exhibits a cusp-like singularity, the local statistics is usually modeled by the Pearcey process. It originates from Gaussian random matrices with an external source [3, 9, 10] and appears in many other random matrix ensembles [35, 39, 41]. It is also possible that the function \mathcal{V} admits some singularities near the origin. By choosing $\mathcal{V}(u; x) = -xu - \alpha \ln u + 1/(4u)$, $\alpha > -1$, we obtain, up to a conjugation term $(y/x)^{\alpha/2}$, the classical Bessel kernel [38, 61]. The Bessel point process typically describes the smallest

eigenvalue distribution near the hard edge 0 for the Wishart-Laguerre ensemble and its unitary invariant generalizations [52, 63].

Let \mathcal{K} be the integral operator acting on a proper interval $J \in \mathbb{R}$ with kernel K . The Fredholm determinant

$$D(J; \gamma) := \det(I - \gamma \mathcal{K}), \quad 0 < \gamma \leq 1, \quad (1.2)$$

has attracted great interest over the past few years. On one hand, $D(J; \gamma)$ is essential in understanding the point process determined by K . Indeed, $D(J; 1)$ can be interpreted as the probability of finding no particles (also known as the gap probability) on the interval I [47, 58]—a basic object in the theory, while $D(J; \gamma)$, $0 < \gamma < 1$, is the gap probability for the associated thinned process. Thinning is a classical operation in the studies of point processes [42], which is obtained from the original one by removing each particle independently with probability $1 - \gamma$; we refer to [4, 18] for more information about thinning in random matrix theory. It is also worth noticing that one can deduce global rigidity upper bounds for determinantal point processes from $D(J; \gamma)$ by using a general result established in [17]. On the other hand, the Fredholm determinant $D(J; \gamma)$ alone has rich structures and connections. It is well-known that for the universal sine, Airy and Bessel kernels, the associated determinants admit integral representations in terms of Painlevé transcendents or the associated Hamiltonians [45, 61, 62]. Moreover, the relevant asymptotic expansions as $|J| \rightarrow \infty$ usually have a simple and elegant form. To derive these large gap asymptotics, however, one needs a deep and strong effort [48]. For the aforementioned various kernels in the form (1.1), the asymptotics of $D(J; \gamma)$ with J being a single interval can be found in [2, 7, 25] for the Airy point process, in [11] for the generalized Airy point process, in [8, 15, 27, 34] for the Bessel point process, and in [20, 22, 23] for the Pearcey process.

In this paper, we are concerned with the hard edge Pearcey process characterized by the kernel (see [50, Equation (1.19)])

$$K_\alpha(x, y; \rho) := \frac{1}{(2\pi i)^2} \int_{\gamma_u} \int_{\gamma_t} \frac{e^{\rho/s + 1/(2s^2) - \rho/t - 1/(2t^2) + xs - yt}}{t - s} \left(\frac{s}{t}\right)^\alpha du dt, \quad x, y > 0, \quad (1.3)$$

that is, $\mathcal{V}(u; x) = xu + \alpha \ln u + \rho/u + 1/(2u^2)$ in (1.1), where the parameters $\alpha > -1$, $\rho \in \mathbb{R}$, and the contours γ_u and γ_t are illustrated in Figure 1. This process was first introduced by Desrosiers and Forrester for a chiral Gaussian unitary ensemble with a source term [31], where K_α is given in a double integral form involving the Bessel function of the first kind of order α . In the context of non-intersecting squared Bessel paths, Kuijlaars, Martínez-Finkelshtein and Wielonsky obtained the one shown in (1.3). More precisely, consider n non-intersecting squared Bessel paths with the same positive starting point and the same ending point taken to be 0. As $n \rightarrow \infty$, all paths, after proper scaling, initially stay away from the hard edge 0, but at a certain critical time $t = t^*$, the lowest paths hit the hard edge and are stuck to it from then on. If $t \neq t^*$, the local correlations are described by the classical sine, Airy, and Bessel kernels in random matrix theory [51], and one obtains the kernel K_α at the hard edge after proper scaling around t^* [50]. The two different formulations of K_α in [31] and [50] are later proved to be equivalent by Delvaux and Vető in [29]. Since the kernel K_α appears near the cusp of the non-intersecting squared Bessel paths model and bears a resemblance to the Pearcey kernel, we call it the hard edge Pearcey kernel, following the terminology initiated in [29]. The hard edge Pearcey process is also universal as can be seen from its emergence in various different stochastic models, which include the random surface model [6, 12], hard edge asymptotics of the Jacobi growth model [13], critical behavior in Angelesco ensembles [30] and a model of non-intersecting Brownian bridges between reflecting or absorbing walls [53]; see also [29, 40] for multi-time extensions of the hard edge Pearcey process.

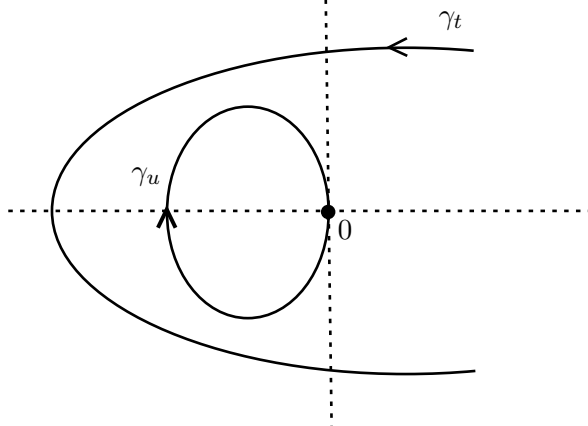


Figure 1: The contours γ_u and γ_t in the definition of K_α (1.3).

The purpose of this paper to add to the collection of formulas for the very few basic gap probabilities by working on the hard edge Pearcey determinant. Our main results will be stated in the next section.

2 Main results

Let $\mathcal{K}_{s,\alpha}$ be the trace class operator acting on $L^2(0, s)$, $s \geq 0$, with the kernel K_α shown in (1.3). As aforementioned, the Fredholm determinant $\det(I - \gamma\mathcal{K}_{s,\alpha})$, $0 < \gamma \leq 1$ gives us the gap probability for the thinned/unthinned hard edge Pearcey process over the interval $(0, s)$, which corresponds to $0 < \gamma < 1$ and $\gamma = 1$, respectively. By setting

$$F(s; \alpha, \gamma, \rho) := \ln \det(I - \gamma\mathcal{K}_{s,\alpha}), \quad (2.1)$$

our first result establishes an integral representation of F via a Hamiltonian related a system of 12 coupled differential equations.

The system of differential equations involves 12 unknown functions denoted by

$$p_{i,k}(s), \quad q_{i,k}(s), \quad i = 0, 1, \quad k = 1, 2, 3, \quad (2.2)$$

and reads as follows:

$$\begin{cases} q'_{0,k}(s) = \frac{1}{s}q_{1,k}(s) \sum_{j=1}^3 p_{1,j}(s)q_{0,j}(s), \\ q'_{1,k}(s) = q_{1,k+1}(s) + \frac{1}{s}q_{0,k}(s) \sum_{j=1}^3 p_{0,j}(s)q_{1,j}(s), \\ p'_{0,k}(s) = -\frac{1}{s}p_{1,k}(s) \sum_{j=1}^3 p_{0,j}(s)q_{1,j}(s), \\ p'_{1,k}(s) = -p_{1,k-1}(s) - \frac{1}{s}p_{0,k}(s) \sum_{j=1}^3 p_{1,j}(s)q_{0,j}(s), \end{cases} \quad (2.3)$$

where it is assumed that $p_{1,0} = q_{1,4} \equiv 0$. By further imposing the conditions

$$\sum_{k=1}^3 q_{0,k}(s)p_{0,k}(s) = -\alpha, \quad \sum_{k=1}^3 q_{1,k}(s)p_{1,k}(s) = 0, \quad (2.4)$$

it is readily seen that the function

$$H(s) := (p_{1,1}(s)q_{1,2}(s) + p_{1,2}(s)q_{1,3}(s)) + \frac{1}{s} \left(\sum_{j=1}^3 p_{1,j}(s)q_{0,j}(s) \right) \left(\sum_{j=1}^3 p_{0,j}(s)q_{1,j}(s) \right) \quad (2.5)$$

satisfies the relations

$$\frac{\partial H}{\partial p_{i,k}} = q'_{i,k}(s), \quad \frac{\partial H}{\partial q_{i,k}} = -p'_{i,k}(s), \quad i = 0, 1, \quad k = 1, 2, 3. \quad (2.6)$$

Thus, H in (2.5) is a Hamiltonian for the system (2.3) and (2.4). The following theorem reveals an elegant relation between F and H .

Theorem 2.1. *With the function $F(s; \alpha, \gamma, \rho)$ defined in (2.1), we have*

$$F(s; \alpha, \gamma, \rho) = \int_0^s H(\tau) d\tau, \quad s \in (0, +\infty), \quad (2.7)$$

where $H(s)$ given in (2.5) is the Hamiltonian associated with a special solution to the system of equations (2.3) and (2.4). Moreover, $H(s)$ satisfies the following asymptotics: as $s \rightarrow 0^+$,

$$H(s) = \mathcal{O}(s^\alpha), \quad 0 < \gamma \leq 1, \quad (2.8)$$

and as $s \rightarrow +\infty$,

$$H(s) = \frac{\sqrt{3}\beta i}{s^{1/3}} - \frac{\sqrt{3}\rho\beta i}{3s^{2/3}} - \frac{2\beta^2}{3s} - \frac{2\beta i}{3\sqrt{3}s} \cos(2\psi(s)) + \mathcal{O}(s^{-4/3}) \quad (2.9)$$

for $0 < \gamma < 1$, where

$$\beta = \frac{1}{2\pi i} \ln(1 - \gamma), \quad (2.10)$$

and

$$\psi(s) = \frac{\alpha\pi}{3} + \arg \Gamma(1 - \beta) - \beta i \left(\frac{2}{3} \ln s + \ln 9 \right) + \frac{\sqrt{3}}{2} \left(\rho s^{1/3} - \frac{3}{2} s^{2/3} \right) \quad (2.11)$$

with Γ being Euler's gamma function.

The local behavior of H in (2.8) ensures the integral (2.7) is well-defined. The existence of a special solution to the system of equations (2.3) and (2.4) follows from the Lax pair for a Riemann-Hilbert (RH) problem. Asymptotic properties of this solution is collected in Proposition 7.1 below, which might be of independent interest. We also note that Girotti found a connection between $F(s; 1, \rho)$ and a Lax pair which might be related to a higher order Painlevé III hierarchy in [40]. It would be interesting to compare (2.7) with the result therein and to work out the precise relationship.

The complicated form of H in $p_{i,k}(s)$ and $q_{i,k}(s)$ implies that one cannot evaluate F explicitly for a given value of s , it is then natural to approximate it by establishing the asymptotic formula. We next show the large- s asymptotics of $F(s; \alpha, \gamma, \rho)$ for $0 < \gamma < 1$, up to and including the notoriously difficult constant term.

Theorem 2.2. *With the function $F(s; \alpha, \gamma, \rho)$ defined in (2.1), we have, as $s \rightarrow +\infty$,*

$$F(s; \alpha, \gamma, \rho) = \frac{3\sqrt{3}\beta i}{2} s^{2/3} - \sqrt{3}\rho\beta i s^{1/3} - \frac{2\beta^2}{3} \ln s + \ln(G(1 + \beta)G(1 - \beta)) - 2\beta^2 \ln 3 - \frac{2\alpha\beta\pi i}{3} + \mathcal{O}(s^{-1/3}), \quad 0 \leq \gamma < 1, \quad (2.12)$$

uniformly for γ and ρ in any compact subset of $[0, 1)$ and \mathbb{R} , respectively, where β is given in (2.10) and $G(z)$ is the Barnes G -function.

If $\gamma \rightarrow 0^+$, we have $\beta \rightarrow 0$. Since $G(1) = 0$, it is readily seen from (2.12) that $F(s; \alpha, 0, \rho) = \mathcal{O}(s^{-1/3})$, which is consistent with the fact that $F(s; \alpha, 0, \rho) = 0$. The asymptotics of $F(s; \alpha, 1, \rho)$, however, should exhibit a significantly different asymptotic behavior. Indeed, according to the Forrester-Chen-Eriksen-Tracy conjecture [21, 38], it is expected that $F(s; \alpha, 1, \rho) = \mathcal{O}(s^{4/3})$ for large s . Recently, the large gap asymptotics of $F(s; \alpha, 1, \rho)$ has been worked out in [64] confirming the Forrester-Chen-Eriksen-Tracy conjecture in the situation of hard-edge Pearcey process.

As an application of Theorem 2.2, we are able to extract the asymptotic statistical properties of $N(s)$ – the counting function of the number of particles in the hard edge Pearcey process falling into the interval $(0, s)$.

Corollary 2.3. *As $s \rightarrow +\infty$, we have*

$$\mathbb{E}(N(s)) = \mu(s) - \frac{\alpha}{3} + \left(\frac{\ln s}{s^{1/3}} \right), \quad (2.13)$$

$$\text{Var}(N(s)) = \sigma(s)^2 + \frac{1 + \ln 9 + \gamma_E}{2\pi^2} + \left(\frac{(\ln s)^2}{s^{1/3}} \right), \quad (2.14)$$

where $\gamma_E = -\Gamma'(1) \approx 0.57721$ is Euler's constant,

$$\mu(s) = \frac{3\sqrt{3}}{4\pi} s^{2/3} - \frac{\sqrt{3}\rho}{4\pi} s^{1/3}, \quad \sigma(s)^2 = \frac{1}{3\pi^2} \ln s. \quad (2.15)$$

Moreover, the random variable $\frac{N(s) - \mu(s)}{\sqrt{\sigma(s)^2}}$ converges in distribution to the normal law $\mathcal{N}(0, 1)$ as $s \rightarrow +\infty$ and the following upper bound for the maximum fluctuation of the counting function $N(s)$ holds:

$$\lim_{a \rightarrow \infty} \mathbb{P} \left(\sup_{s > a} \left| \frac{N(s) - \left(\frac{3\sqrt{3}}{4\pi} s^{2/3} - \frac{\sqrt{3}\rho}{4\pi} s^{1/3} \right)}{\ln s} \right| \leq \frac{2}{3\pi} + \epsilon \right) = 1, \quad \epsilon > 0. \quad (2.16)$$

Proof. It is well-known that the moment generating function of $N(s)$ is equal to a specified deformed hard edge Pearcey determinant:

$$\mathbb{E} \left(e^{-2\pi\nu N(s)} \right) = \sum_{k=0}^{\infty} \mathbb{P}(N(s) = k) e^{-2\pi\nu k} = \det \left(I - (1 - e^{-2\pi\nu}) \mathcal{K}_{s,\alpha} \right), \quad \nu \geq 0; \quad (2.17)$$

cf. [47, 58]. Since

$$\mathbb{E} \left(e^{-2\pi\nu N(s)} \right) = 1 - 2\pi\mathbb{E}(N(s))\nu + 2\pi^2\mathbb{E}(N(s)^2)\nu^2 + \mathcal{O}(\nu^3), \quad \nu \rightarrow 0, \quad (2.18)$$

one has

$$\begin{aligned} F(s; \alpha, 1 - e^{-2\pi\nu}, \rho) &= \log \mathbb{E} \left(e^{-2\pi\nu N(s)} \right) \\ &= -2\pi\mathbb{E}(N(s))\nu + 2\pi^2\text{Var}(N(s))\nu^2 + \mathcal{O}(\nu^3), \quad \nu \rightarrow 0, \end{aligned} \quad (2.19)$$

where F is defined in (2.1). In view of (2.12), we have, as $s \rightarrow +\infty$,

$$\begin{aligned} F(s; \alpha, 1 - e^{-2\pi\nu}, \rho) &= -2\pi \left(\mu(s) - \frac{\alpha}{3} \right) \nu + 2\pi^2 \left(\sigma(s)^2 + \frac{\ln 3}{\pi^2} \right) \nu^2 \\ &\quad + \ln(G(1 + \nu i)G(1 - \nu i)) + \mathcal{O}(s^{-1/3}), \end{aligned} \quad (2.20)$$

where the functions $\mu(s)$ and $\sigma(s)^2$ are defined in (2.15). Note that

$$G(1+z) = 1 + \frac{\ln(2\pi) - 1}{2}z + \left(\frac{(\ln(2\pi) - 1)^2}{8} - \frac{1 + \gamma_E}{2} \right) z^2 + \mathcal{O}(z^3), \quad z \rightarrow 0, \quad (2.21)$$

where γ_E is Euler's constant, we then obtain (2.13) and (2.14) from (2.19), (2.20) and (5.75).

Finally, since $\sigma(s)^2 \rightarrow +\infty$ for large positive s , it is easily seen from (2.17) and (2.20) that

$$\mathbb{E} \left(e^{t \frac{N(s) - \mu(s)}{\sqrt{\sigma(s)^2}} \right) \rightarrow e^{\frac{t^2}{2}}, \quad s \rightarrow +\infty, \quad (2.22)$$

which implies the convergence of $\frac{N(s) - \mu(s)}{\sqrt{\sigma(s)^2}}$ in distribution to the normal law $\mathcal{N}(0, 1)$. The upper bound (2.16) follows directly by combining [17, Lemma 2.1 and Theorem 1.2], (2.19) and (2.20).

This completes the proof of Corollary 2.3. \square

According to [59, 60], it is believed that $\text{Var}(N(s))$ takes a universal asymptotic form $c_0 \ln s + c_1 + o(1)$ for many determinantal point processes in random matrix theory, where the constants c_0 and c_1 vary for different processes. This has been confirmed for the classical sine, Airy, Bessel and Pearcey processes; cf. [14, 15, 22, 57]. By (2.14), the same is true for the hard edge Pearcey process. In addition, the upper bound (2.16) particularly means that, for large s , the counting function of hard edge Pearcey process lies in the interval $(\mu(s) - (2/(3\pi) + \epsilon) \ln s, \mu(s) + (2/(3\pi) + \epsilon) \ln s)$ with high probability. The rigidity upper bound of the Airy and Bessel processes has been established in [17], and that of the classical Pearcey process has been obtained recently in [16].

The rest of this paper is devoted to the proofs of our main results. The idea follows from a general framework established in [5, 25, 26, 43], which gives a connection between the Fredholm determinants of integrable kernels and RH problems. The precise relationship in our case is discussed in Section 3, where we connects $dF(s; \alpha, \gamma, \rho)/ds$ to a 3×3 RH problem for Φ with constant jumps by using the RH problem for the hard edge Pearcey established in [50]. We then derive a Lax pair for Φ and several useful differential identities for the Hamiltonian in Section 4. After performing a Deift-Zhou steepest descent analysis [28] on the RH problem for Φ as $s \rightarrow \infty$ and $s \rightarrow 0^+$ in Sections 5 and 6, we present asymptotics of the relevant solution to the system of equations (2.3) and (2.4), and proofs of Theorems 2.1 and 2.2 in Sections 7 and 8, respectively.

3 A RH formulation

3.1 RH characterizations of K_α and its resolvent kernel

As illustrated in [50, RH problem 1.3 and Lemma 5.3], the hard edge Pearcey kernel K_α in (1.3) is also characterized by the following 3×3 RH problem.

RH problem 3.1.

(1) $\Psi_\alpha(z) = \Psi_\alpha(z; \rho)$ is analytic in $\mathbb{C} \setminus \{\cup_{j=0}^5 \Sigma_j \cup \{0\}\}$, where

$$\begin{aligned} \Sigma_0 &= (0, +\infty), & \Sigma_1 &= e^{\frac{\pi i}{4}}(0, +\infty), & \Sigma_2 &= e^{\frac{3\pi i}{4}}(0, +\infty), \\ \Sigma_3 &= (-\infty, 0), & \Sigma_4 &= e^{-\frac{3\pi i}{4}}(0, +\infty), & \Sigma_5 &= e^{-\frac{\pi i}{4}}(0, +\infty), \end{aligned} \quad (3.1)$$

with the orientations as shown in Figure 2.

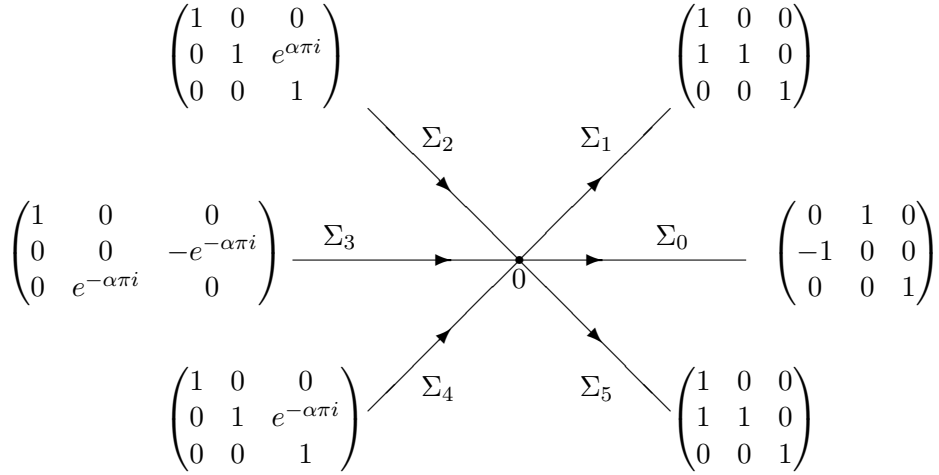


Figure 2: The jump contours Σ_k , $k = 0, \dots, 5$, and the corresponding jump matrix J_{Ψ_α} in the RH problem for Ψ_α .

(2) For $z \in \Sigma_k$, $k = 0, 1, \dots, 5$, the limiting values of Ψ_α exist and satisfy the jump condition

$$\Psi_{\alpha,+}(z) = \Psi_{\alpha,-}(z)J_{\Psi_\alpha}(z), \quad (3.2)$$

where $J_{\Psi_\alpha}(z)$ is depicted in Figure 2.

(3) As $z \rightarrow \infty$ and $\pm \text{Im } z > 0$, we have

$$\Psi_\alpha(z) = \frac{iz^{-\frac{\alpha}{3}}}{\sqrt{3}} C_\Psi (I + \mathcal{O}(z^{-1})) \text{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) L_\pm \text{diag} \left(e^{\pm \frac{\alpha \pi i}{3}}, e^{\mp \frac{\alpha \pi i}{3}}, 1 \right) e^{\Theta(z)}, \quad (3.3)$$

where the constant matrix C_Ψ is defined in (3.18) below, L_\pm are given by

$$L_+ = \begin{pmatrix} \omega & \omega^2 & 1 \\ 1 & 1 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad L_- = \begin{pmatrix} \omega^2 & -\omega & 1 \\ 1 & -1 & 1 \\ \omega & -\omega^2 & 1 \end{pmatrix}, \quad (3.4)$$

with $\omega = e^{\frac{2\pi i}{3}}$, and

$$\Theta(z) = \Theta(z; \rho) = \begin{cases} \text{diag}(\theta_1(z; \rho), \theta_2(z; \rho), \theta_3(z; \rho)), & \text{Im } z > 0, \\ \text{diag}(\theta_2(z; \rho), \theta_1(z; \rho), \theta_3(z; \rho)), & \text{Im } z < 0, \end{cases} \quad (3.5)$$

with

$$\theta_k(z; \rho) = \frac{3}{2} \omega^{2k} z^{\frac{2}{3}} + \rho \omega^k z^{\frac{1}{3}}, \quad k = 1, 2, 3. \quad (3.6)$$

(4) As $z \rightarrow 0$, we have

$$\Psi_\alpha(z) = \Psi_\alpha^{(0)}(z) \begin{cases} \begin{pmatrix} 1 & \frac{i}{2\pi} \ln z & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{(-1)^{\alpha+1}}{2\pi} i \ln z & 1 \end{pmatrix}, & \text{if } \alpha \in \mathbb{N} \cup \{0\}, \\ \begin{pmatrix} 1 & \frac{e^{-\alpha \pi i}}{2 \sin(\alpha \pi)} i & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & -\frac{i}{2 \sin(\alpha \pi)} & 1 \end{pmatrix}, & \text{if } \alpha \notin \mathbb{Z}, \end{cases} \quad (3.7)$$

for $\pi/4 < \arg z < 3\pi/4$, where $\Psi_\alpha^{(0)}(z)$ is analytic at $z = 0$. The local behaviors of $\Psi_\alpha(z)$ near $z = 0$ in other regions can be determined through the above formula and the jump condition (3.2).

It is shown in [50, Theorem 1.4 and Proposition 5.2] that the above RH problem has a unique solution with

$$\det \Psi_\alpha(z) = z^{-\alpha}, \quad z \in \mathbb{C} \setminus \{\cup_{j=0}^5 \Sigma_j \cup \{0\}\}. \quad (3.8)$$

Moreover, the solution is explicitly given by

$$\Psi_\alpha(z) = \frac{e^{\rho^2/6}}{\sqrt{2\pi}} \begin{pmatrix} -p_4(z) & p_3(z) & p_1(z) \\ -p_4'(z) & p_3'(z) & p_1'(z) \\ -p_4''(z) & p_3''(z) & p_1''(z) \end{pmatrix}, \quad 0 < \arg z < \frac{\pi}{4}, \quad (3.9)$$

$$\Psi_\alpha(z) = \frac{e^{\rho^2/6}}{\sqrt{2\pi}} \begin{pmatrix} p_2(z) & p_3(z) & p_1(z) \\ p_2'(z) & p_3'(z) & p_1'(z) \\ p_2''(z) & p_3''(z) & p_1''(z) \end{pmatrix}, \quad \frac{\pi}{4} < \arg z < \frac{3\pi}{4}, \quad (3.10)$$

$$\Psi_\alpha(z) = \frac{e^{\rho^2/6}}{\sqrt{2\pi}} \begin{pmatrix} p_2(z) & p_3(z) & -e^{-\alpha\pi i} p_4(z) \\ p_2'(z) & p_3'(z) & -e^{-\alpha\pi i} p_4'(z) \\ p_2''(z) & p_3''(z) & -e^{-\alpha\pi i} p_4''(z) \end{pmatrix}, \quad \frac{3\pi}{4} < \arg z < \pi, \quad (3.11)$$

$$\Psi_\alpha(z) = \frac{e^{\rho^2/6}}{\sqrt{2\pi}} \begin{pmatrix} p_2(z) & p_4(z) & e^{\alpha\pi i} p_3(z) \\ p_2'(z) & p_4'(z) & e^{\alpha\pi i} p_3'(z) \\ p_2''(z) & p_4''(z) & e^{\alpha\pi i} p_3''(z) \end{pmatrix}, \quad -\pi < \arg z < -\frac{3\pi}{4}, \quad (3.12)$$

$$\Psi_\alpha(z) = \frac{e^{\rho^2/6}}{\sqrt{2\pi}} \begin{pmatrix} p_2(z) & p_4(z) & p_1(z) \\ p_2'(z) & p_4'(z) & p_1'(z) \\ p_2''(z) & p_4''(z) & p_1''(z) \end{pmatrix}, \quad -\frac{3\pi}{4} < \arg z < -\frac{\pi}{4}, \quad (3.13)$$

$$\Psi_\alpha(z) = \frac{e^{\rho^2/6}}{\sqrt{2\pi}} \begin{pmatrix} p_3(z) & p_4(z) & p_1(z) \\ p_3'(z) & p_4'(z) & p_1'(z) \\ p_3''(z) & p_4''(z) & p_1''(z) \end{pmatrix}, \quad -\frac{\pi}{4} < \arg z < 0. \quad (3.14)$$

In the above expressions, $p_1(z)$ and $p_2(z)$ are entire functions in z , while $p_3(z)$ and $p_4(z)$ are analytic for $z \in \mathbb{C} \setminus i\mathbb{R}_-$ and $z \in \mathbb{C} \setminus i\mathbb{R}_+$, respectively. They are solutions to the third-order differential equation

$$xy'''(x) + \alpha y''(x) - \rho y'(x) - y(x) = 0, \quad (3.15)$$

and any three of $p_k(z)$, $k = 1, \dots, 4$, are linearly independent. In addition, we have the following integral representations for p_k :

$$p_k(z) = \begin{cases} \int_{\Gamma_k} t^{-1} e^{zt+V(t)} dt, & k = 1, \\ e^{-\alpha\pi i} \int_{\Gamma_k} t^{-1} e^{zt+V(t)} dt, & k = 2, 3, \\ e^{\alpha\pi i} \int_{\Gamma_k} t^{-1} e^{zt+V(t)} dt, & k = 4, \end{cases} \quad (3.16)$$

where

$$V(t) = (\alpha - 2) \ln t + \frac{\rho}{t} + \frac{1}{2t^2}. \quad (3.17)$$

Note that $t^{-1} e^{zt+V(t)} = t^{\alpha-3} e^{zt + \frac{\rho}{t} + \frac{1}{2t^2}}$, we choose t^α with $-\frac{\pi}{2} < \arg t < \frac{\pi}{2}$ in the definition of p_1 , with $\frac{\pi}{2} < \arg t < \frac{3\pi}{2}$ in the definition of p_2 , with $0 < \arg t < \pi$ in the definition of p_3 and with $-\pi < \arg t < 0$ in the definition of p_4 . The contours Γ_k , $k = 1, 2, 3, 4$, in (3.16) are defined as follows: Γ_1 is a simple closed contour lying in the right half plane, which is tangent to the imaginary axis at the origin, Γ_2 is the reflection of Γ_1 with respect to the imaginary axis, Γ_3 lies

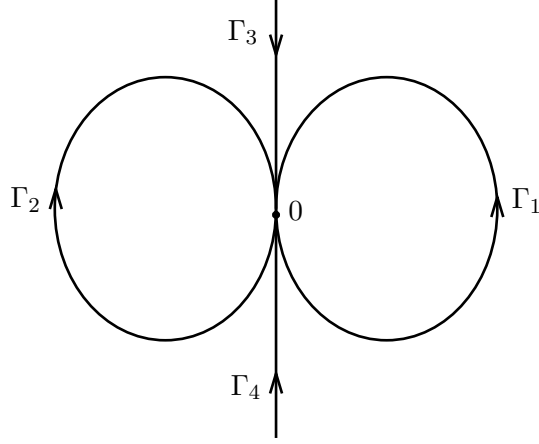


Figure 3: The contour of integration Γ_k in the definition of $p_k(z)$, $k = 1, 2, 3, 4$.

in the upper half plane which starts at infinity with an angle where $\text{Re}(zt) < 0$ as $t \rightarrow \infty$ and ends at the origin along the positive imaginary axis and Γ_4 lies in the lower half plane which starts at infinity with an angle where $\text{Re}(zt) < 0$ as $t \rightarrow \infty$ and ends at the origin along the negative imaginary axis; see Figure 3 for an illustration.

Remark 3.2. Comparing the above RH problem with [50, RH problem 1.3], the large- z behavior in item (3) is slightly modified for brevity, while the local behavior near the origin in item (4) is expressed in details to facilitate our future discussions. The constant matrix C_Ψ in (3.3) is given by

$$C_\Psi = \begin{pmatrix} 1 & \pi_3(\rho) & \pi_6(\rho) \\ 0 & 1 & \pi_3(\rho) + \frac{\rho}{3} \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.18)$$

where

$$\pi_3(\rho) = \frac{\rho(\rho^2 + 9\alpha - 18)}{27}, \quad (3.19)$$

$$\pi_6(\rho) = \frac{\rho^6 + (18\alpha - 45)\rho^4 + (81\alpha^2 - 405\alpha + 405)\rho^2 - 243\alpha^2 + 729\alpha - 405}{2 \cdot 3^6}, \quad (3.20)$$

are two polynomials in ρ . The derivation of C_Ψ follows directly from the explicit solution of Ψ_α and an asymptotic analysis of the relevant integrals.

By [50, Equation (8.20)], it follows that

$$K_\alpha(x, y; \rho) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \tilde{\Psi}_\alpha(y; \rho)^{-1} \tilde{\Psi}_\alpha(x; \rho) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x, y > 0, \quad (3.21)$$

where

$$\tilde{\Psi}_\alpha(z) = \tilde{\Psi}_\alpha(z; \rho) := \frac{e^{\frac{\rho^2}{6}}}{\sqrt{2\pi}} \begin{pmatrix} p_2(z) & p_3(z) & p_1(z) \\ p_2'(z) & p_3'(z) & p_1'(z) \\ p_2''(z) & p_3''(z) & p_1''(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus i\mathbb{R}_-, \quad (3.22)$$

is the analytic extension of the restriction of Ψ_α on the region bounded by Σ_1 and Σ_2 in (3.10) to $\mathbb{C} \setminus i\mathbb{R}_-$. Then, one readily sees that

$$\gamma K_\alpha(x, y; \rho) = \frac{\mathbf{f}(x)^t \mathbf{h}(y)}{x-y}, \quad (3.23)$$

where

$$\mathbf{f}(x) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} := \tilde{\Psi}_\alpha(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{h}(y) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} := \frac{\gamma}{2\pi i} \tilde{\Psi}_\alpha(y)^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (3.24)$$

It is clear that the kernel $\gamma K_\alpha(x, y; \rho)$ is integrable for all $0 < \gamma \leq 1$ in the sense of Its et al. [43], which is crucial for the subsequent analysis of F defined in (2.1).

First, it is known that the derivative of F satisfies the following relation:

$$\begin{aligned} \frac{d}{ds} F(s; \alpha, \gamma, \rho) &= \frac{d}{ds} \ln \det(I - \gamma \mathcal{K}_{s, \alpha}) = -\text{tr} \left((I - \gamma \mathcal{K}_{s, \alpha})^{-1} \gamma \frac{d}{ds} \mathcal{K}_{s, \alpha} \right) \\ &= -R(s, s), \end{aligned} \quad (3.25)$$

where $R(u, v)$ stands for the kernel of the resolvent operator, that is, the operator defined by

$$(I - \gamma \mathcal{K}_{s, \alpha})^{-1} - I = \gamma \mathcal{K}_{s, \alpha} (I - \gamma \mathcal{K}_{s, \alpha})^{-1} = \gamma (I - \gamma \mathcal{K}_{s, \alpha})^{-1} \mathcal{K}_{s, \alpha}.$$

As $\gamma K_\alpha(x, y; \rho)$ is integrable, its resolvent kernel is integrable as well; cf. [26, 43]. More precisely, let \mathbf{F} and \mathbf{H} be defined as

$$\mathbf{F}(u) = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} := (I - \gamma \mathcal{K}_{s, \alpha})^{-1} \mathbf{f}, \quad \mathbf{H}(v) = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} := (I - \gamma \mathcal{K}_{s, \alpha})^{-1} \mathbf{h}, \quad (3.26)$$

we then have

$$R(u, v) = \frac{\mathbf{F}(u)^t \mathbf{H}(v)}{u - v}. \quad (3.27)$$

Next, due to the integrable structure of the kernel $\gamma K_\alpha(x, y; \rho)$, the following fundamental RH problem relates \mathbf{f}, \mathbf{h} to \mathbf{F}, \mathbf{H} , respectively; cf. [26].

RH problem 3.3.

(1) $Y(z)$ is defined and analytic in $\mathbb{C} \setminus [0, s]$, where the orientation is taken from the left to the right.

(2) For $x \in (0, s)$, we have

$$Y_+(x) = Y_-(x)(I - 2\pi i \mathbf{f}(x) \mathbf{h}(x)^t), \quad (3.28)$$

with the functions \mathbf{f} and \mathbf{h} defined in (3.24).

(3) As $z \rightarrow \infty$, we have

$$Y(z) = I + \frac{Y_1}{z} + \mathcal{O}(z^{-2}). \quad (3.29)$$

(4) As $z \rightarrow 0$, we have

$$Y(z) = \begin{cases} \mathcal{O}(\ln z), & \alpha \geq 0, \\ \mathcal{O}(z^\alpha), & -1 < \alpha < 0. \end{cases} \quad (3.30)$$

(5) As $z \rightarrow s$, we have $Y(z) = \mathcal{O}(\ln(z - s))$.

The unique solution to the above RH problem is given by

$$Y(z) = I - \int_0^s \frac{\mathbf{F}(w) \mathbf{h}(w)^t}{w - z} dw, \quad (3.31)$$

and one has

$$\mathbf{F}(z) = Y(z) \mathbf{f}(z), \quad \mathbf{H}(z) = Y(z)^{-t} \mathbf{h}(z). \quad (3.32)$$

Remark 3.4. Since $\det(I - \gamma \mathcal{K}_{s, \alpha})$ stands for the gap probability for the thinned/unthinned hard edge Pearcey process, it must be strictly positive and invertible. This guarantees the solvability of the above RH problem for Y .

3.2 A RH problem related to $\frac{d}{ds}F$

With the aid of RH problems 3.1 and 3.3 for Ψ_α and Y , we will construct a new one with constant jumps and establish its connection with $\frac{d}{ds}F$. This is a standard procedure known as the undressing transformation in the study of Fredholm determinants associated with integrable kernels. The idea is to incorporate the jump condition of Y on $(0, s)$ in the RH problem 3.1 for Ψ_α , and replace the rays Σ_1 and Σ_5 by their parallel rays $\Sigma_1^{(s)}$ and $\Sigma_5^{(s)}$ emanating from the point s ; see Figure 4.

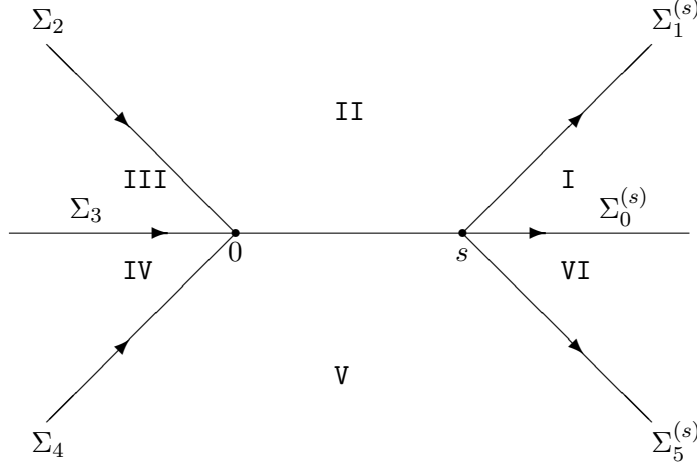


Figure 4: Regions I-VI and the jump contours for the RH problem for Φ .

Let I–VI be six regions in the complex plane illustrated in Figure 4, we now define $\Phi(z) = \Phi(z; s)$ as follows:

$$\Phi(z) = \begin{cases} Y(z)\Psi_\alpha(z), & z \in \text{I} \cup \text{III} \cup \text{IV} \cup \text{VI}, \\ Y(z)\tilde{\Psi}_\alpha(z), & z \in \text{II}, \\ Y(z)\hat{\Psi}_\alpha(z), & z \in \text{V}, \end{cases} \quad (3.33)$$

where $\tilde{\Psi}_\alpha(z)$ is given in (3.22) and

$$\hat{\Psi}_\alpha(z) := \frac{e^{\frac{\rho^2}{6}}}{\sqrt{2\pi}} \begin{pmatrix} p_2(z) & p_4(z) & p_1(z) \\ p_2'(z) & p_4'(z) & p_1'(z) \\ p_2''(z) & p_4''(z) & p_1''(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus i\mathbb{R}_+, \quad (3.34)$$

i.e., the analytic extension of the restriction of Ψ_α on the region bounded by Σ_4 and Σ_5 in (3.13) to $\mathbb{C} \setminus i\mathbb{R}_+$. Then, Φ satisfies the following RH problem.

Proposition 3.5. *The function $\Phi(z) = \Phi(z; s)$ defined in (3.33) has the following properties:*

- (1) $\Phi(z)$ is defined and analytic in $\mathbb{C} \setminus \Sigma_\Phi$, where

$$\Sigma_\Phi := \cup_{i=2}^4 \Sigma_i \cup \Sigma_0^{(s)} \cup \Sigma_1^{(s)} \cup \Sigma_5^{(s)} \cup [0, s], \quad (3.35)$$

with

$$\Sigma_0^{(s)} = (s, +\infty), \quad \Sigma_1^{(s)} = s + e^{\frac{\pi i}{4}}(0, +\infty), \quad \Sigma_5^{(s)} = s + e^{-\frac{\pi i}{4}}(0, +\infty), \quad (3.36)$$

and Σ_j , $j = 2, 3, 4$, given in (3.1); see Figure 4 for an illustration and the orientations.

(2) For $z \in \Sigma_\Phi$, Φ satisfies the jump condition

$$\Phi_+(z) = \Phi_-(z)J_\Phi(z), \quad (3.37)$$

where

$$J_\Phi(z) := \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_0^{(s)}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1^{(s)}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_5^{(s)}, \\ \begin{pmatrix} 1 & 1 - \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (0, s), \\ J_{\Psi_\alpha}(z), & z \in \cup_{i=2}^4 \Sigma_i, \end{cases} \quad (3.38)$$

with $J_{\Psi_\alpha}(z)$ given in (3.2).

(3) As $z \rightarrow \infty$ and $\pm \text{Im } z > 0$, we have

$$\Phi(z) = \frac{iz^{-\frac{\alpha}{3}}}{\sqrt{3}} C_\Psi (I + \mathcal{O}(z^{-1})) \text{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) L_\pm \text{diag} \left(e^{\pm \frac{\alpha\pi i}{3}}, e^{\mp \frac{\alpha\pi i}{3}}, 1 \right) e^{\Theta(z)}, \quad (3.39)$$

where C_Ψ , L_\pm and $\Theta(z)$ are given in (3.18), (3.4) and (3.5), respectively.

(4) As $z \rightarrow 0$, we have, if $\alpha \in \mathbb{N} \cup \{0\}$,

$$\Phi(z) = \Phi^{(L)}(z) \begin{pmatrix} 1 & \frac{\gamma-1}{2\pi i} \ln z & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{e^{\alpha\pi i}}{2\pi i} \ln z & 1 \end{pmatrix} \begin{cases} I, & z \in II, \\ \begin{pmatrix} 1 & \gamma-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in V, \end{cases} \quad (3.40)$$

if $\alpha \notin \mathbb{Z}$,

$$\Phi(z) = \Phi^{(L)}(z) \begin{cases} \begin{pmatrix} 1 & c_+ & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{1}{2\sin(\alpha\pi i)} & 1 \end{pmatrix}, & z \in II, \\ \begin{pmatrix} 1 & c_- & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{1}{2\sin(\alpha\pi i)} & 1 \end{pmatrix}, & z \in V, \end{cases} \quad (3.41)$$

where we take principal branch for $\ln z$ and z^α ,

$$c_\pm = \frac{\gamma-1}{2i\sin(\alpha\pi)} e^{\mp \alpha\pi i}, \quad (3.42)$$

and $\Phi^{(L)}(z)$ is analytic at $z=0$ satisfying the expansion

$$\Phi^{(L)}(z) = \Phi_0^{(0)}(s) \left(I + \Phi_1^{(0)}(s) z + \mathcal{O}(z^2) \right), \quad z \rightarrow 0, \quad (3.43)$$

for some functions $\Phi_0^{(0)}(s)$ and $\Phi_1^{(0)}(s)$. The local behavior of Φ near $z=0$ in other regions can be determined through (3.40), (3.41) and the jump condition (3.37) and (3.38).

(5) As $z \rightarrow s$, we have

$$\Phi(z) = \Phi^{(R)}(z) \begin{pmatrix} 1 & -\frac{\gamma}{2\pi i} \ln(z-s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{cases} I, & z \in II, \\ \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in V, \end{cases} \quad (3.44)$$

where the principal branch is taken for $\ln(z-s)$, and $\Phi^{(R)}(z)$ is analytic at $z = s$ satisfying the expansion

$$\Phi^{(R)}(z) = \Phi_0^{(1)}(s) \left(I + \Phi_1^{(1)}(s)(z-s) + \mathcal{O}(z-s)^2 \right), \quad z \rightarrow s, \quad (3.45)$$

for some functions $\Phi_0^{(1)}(s)$ and $\Phi_1^{(1)}(s)$. The local behavior of Φ near $z = s$ in other regions can be determined through (3.44) and the jump condition (3.37) and (3.38).

Proof. It is clear that $\Phi(z)$ defined in (3.33) is analytic in $\mathbb{C} \setminus \Sigma_\Phi$. To show the jump condition (3.37) and (3.38), we only need to check the jump over $(0, s)$, while the other jumps follow directly from (3.33) and (3.2). By (3.24) and (3.28), we have, for $0 < x < s$,

$$Y_-(x)^{-1}Y_+(x) = I - 2\pi i f(x) \mathbf{h}(x)^t = \tilde{\Psi}_\alpha(x) \begin{pmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\Psi}_\alpha(x)^{-1}. \quad (3.46)$$

This, together with (3.33), implies that for $x \in (0, s)$,

$$\begin{aligned} J_\Phi(x) &= \Phi_-(x)^{-1}\Phi_+(x) = \hat{\Psi}_\alpha(x)^{-1}Y_-(x)^{-1}Y_+(x)\tilde{\Psi}_\alpha(x) \\ &= \hat{\Psi}_\alpha(x)^{-1}\tilde{\Psi}_\alpha(x) \begin{pmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (3.47)$$

Recall that both $\tilde{\Psi}_\alpha(x)$ and $\hat{\Psi}_\alpha(x)$ are analytic continuation of $\Psi_\alpha(z)$, it then follows from the jump condition of Ψ_α in (3.2) that

$$\hat{\Psi}_\alpha(x)^{-1}\tilde{\Psi}_\alpha(x) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.48)$$

A combination of the above two formulas gives us the jump of $\Phi(z)$ over $(0, s)$.

We next verify the local behavior of $\Phi(z)$ near $z = 0$ given in (3.40) and (3.41), and it suffices to show that they satisfy the jump condition of Φ . If $\alpha \in \mathbb{N} \cup \{0\}$, we see from the jumps of $\Phi(z)$ on Σ_2 and Σ_4 that, as $z \rightarrow 0$,

$$\Phi(z) = \Phi^{(L)}(z) \begin{pmatrix} 1 & \frac{\gamma-1}{2\pi i} \ln z & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{e^{\alpha\pi i}}{2\pi i} \ln z & 1 \end{pmatrix} \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \text{III}, \\ \begin{pmatrix} 1 & \gamma-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{-\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \text{IV}. \end{cases} \quad (3.49)$$

Thus, for $x < 0$, it follows from (3.37), (3.38) and (3.49) that

$$\begin{aligned} \Phi_-(x)^{-1}\Phi_+(x) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{-\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\gamma-1}{2\pi i}(\ln|x| - \pi i)|x|^\alpha e^{-\alpha\pi i} & 0 \\ 0 & |x|^\alpha e^{-\alpha\pi i} & 0 \\ 0 & -\frac{e^{\alpha\pi i}}{2\pi i}(\ln|x| - \pi i)|x|^\alpha e^{-\alpha\pi i} & 1 \end{pmatrix} \\ &\quad \times \Phi_-^{(L)}(x)^{-1}\Phi_+^{(L)}(x) \begin{pmatrix} 1 & \frac{\gamma-1}{2\pi i}(\ln|x| + \pi i) & 0 \\ 0 & |x|^{-\alpha} e^{-\alpha\pi i} & 0 \\ 0 & \frac{e^{\alpha\pi i}}{2\pi i}(\ln|x| + \pi i) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -e^{-\alpha\pi i} \\ 0 & e^{-\alpha\pi i} & 0 \end{pmatrix}, \end{aligned}$$

as required. Similarly, if $\alpha \notin \mathbb{Z}$, we have, as $z \rightarrow 0$,

$$\Phi(z) = \Phi^{(L)}(z) \begin{cases} \begin{pmatrix} 1 & c_+ & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{1}{2i \sin(\alpha\pi)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \text{III}, \\ \begin{pmatrix} 1 & c_- & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{1}{2i \sin(\alpha\pi)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{-\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \text{IV}. \end{cases} \quad (3.50)$$

Thus, it follows from (3.37), (3.38) and (3.50) that, for $x < 0$,

$$\begin{aligned} \Phi_-(x)^{-1}\Phi_+(x) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{-\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -c_-|x|^\alpha e^{-\alpha\pi i} & 0 \\ 0 & |x|^\alpha e^{-\alpha\pi i} & 0 \\ 0 & -\frac{1}{2i \sin(\alpha\pi)}|x|^\alpha e^{-\alpha\pi i} & 1 \end{pmatrix} \\ &\quad \times \Phi_-^{(L)}(x)^{-1}\Phi_+^{(L)}(x) \begin{pmatrix} 1 & c_+ & 0 \\ 0 & |x|^{-\alpha} e^{-\alpha\pi i} & 0 \\ 0 & \frac{1}{2i \sin(\alpha\pi)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{\alpha\pi i} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -e^{-\alpha\pi i} \\ 0 & e^{-\alpha\pi i} & 0 \end{pmatrix}, \end{aligned}$$

and from (3.37), (3.38) and (3.41) that, for $x \in (0, s)$,

$$\begin{aligned} \Phi_-(x)^{-1}\Phi_+(x) &= \begin{pmatrix} 1 & -c_-x^\alpha & 0 \\ 0 & x^\alpha & 0 \\ 0 & -\frac{1}{2i \sin(\alpha\pi)}x^\alpha & 1 \end{pmatrix} \Phi_-^{(L)}(x)^{-1}\Phi_+^{(L)}(x) \begin{pmatrix} 1 & c_+ & 0 \\ 0 & x^{-\alpha} & 0 \\ 0 & \frac{1}{2i \sin(\alpha\pi)} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1-\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

which is consistent with the jump condition satisfied by Φ .

Finally, the local behavior of $\Phi(z)$ at $z = s$ can be verified in a manner similar to that at $z = 0$, we omit the details here.

This completes the proof of Proposition 3.5. \square

The connection between the above RH problem and the derivative of $F(s; \gamma, \rho)$ is revealed in the following proposition.

Proposition 3.6. *With F defined in (2.1), we have*

$$\frac{d}{ds}F(s; \alpha, \gamma, \rho) = \frac{d}{ds} \ln \det (I - \gamma \mathcal{K}_{s, \alpha}) = -\frac{\gamma}{2\pi i} \left(\Phi_1^{(1)}(s) \right)_{21}, \quad (3.51)$$

where $\Phi_1^{(1)}(s)$ is given in (3.45) and $(M)_{ij}$ stands for the (i, j) -th entry of a matrix M .

Proof. For $z \in \mathbb{II}$, we see from (3.24), (3.32) and (3.33) that

$$\mathbf{F}(z) = Y(z)\mathbf{f}(z) = Y(z)\tilde{\Psi}_\alpha(z) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Phi(z) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (3.52)$$

and

$$\mathbf{H}(z) = Y(z)^{-t}\mathbf{h}(z) = \Phi(z)^{-t}\tilde{\Psi}_\alpha(z)^t \cdot \frac{\gamma}{2\pi i} \tilde{\Psi}_\alpha(z)^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\gamma}{2\pi i} \Phi(z)^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (3.53)$$

Combining the above formulas and (3.27), we obtain

$$R(z, z) = \frac{\gamma}{2\pi i} \left(\Phi(z)^{-1} \Phi'(z) \right)_{21}, \quad z \in \mathbb{II}. \quad (3.54)$$

This, together with (3.25), implies that

$$\frac{d}{ds}F(s; \alpha, \gamma, \rho) = -\frac{\gamma}{2\pi i} \lim_{z \rightarrow s} \left(\Phi(z)^{-1} \Phi'(z) \right)_{21}, \quad z \in \mathbb{II}. \quad (3.55)$$

With the aid of the local behavior of $\Phi(z)$ as $z \rightarrow s$ given in (3.44) and (3.45), we arrive at (3.51).

This completes the proof of Proposition 3.6. \square

4 The Lax pair and differential identities

Based on the RH problem for Φ in Proposition 3.5, we will construct the Lax pair for $\Phi(z; s)$, which leads to the system of equations (2.3) and (2.4). Several useful differential identities for the associated Hamiltonian (2.5) will be also derived for later use.

4.1 Lax pair equations

The Lax pair for $\Phi(z; s)$ is given as follows.

Proposition 4.1. *For the function $\Phi(z) = \Phi(z; s)$ defined in (3.33), we have*

$$\frac{\partial}{\partial z} \Phi(z; s) = L(z; s) \Phi(z; s), \quad \frac{\partial}{\partial s} \Phi(z; s) = U(z; s) \Phi(z; s), \quad (4.1)$$

where

$$L(z; s) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{A_0(s)}{z} + \frac{A_1(s)}{z-s} \quad (4.2)$$

and

$$U(z; s) = -\frac{A_1(s)}{z-s}, \quad (4.3)$$

with

$$A_i(s) = \begin{pmatrix} q_{i,1}(s) \\ q_{i,2}(s) \\ q_{i,3}(s) \end{pmatrix} (p_{i,1}(s) \quad p_{i,2}(s) \quad p_{i,3}(s)), \quad i = 0, 1. \quad (4.4)$$

Moreover, the functions $p_{i,k}$ and $q_{i,k}$, $i = 0, 1$, $k = 1, 2, 3$, in (4.4) satisfy (2.3), (2.4) and

$$p_{0,1}(s)q_{0,3}(s) + p_{1,1}(s)q_{1,3}(s) = 1, \quad (4.5)$$

$$p_{0,1}(s)q_{0,2}(s) + p_{0,2}(s)q_{0,3}(s) + p_{1,1}(s)q_{1,2}(s) + p_{1,2}(s)q_{1,3}(s) = \rho. \quad (4.6)$$

Proof. Since all the jump matrices of Φ given in (3.38) are independent of z and s , then the coefficient matrices

$$L(z; s) := \frac{\partial}{\partial z} \Phi(z; s) \cdot \Phi(z; s)^{-1}, \quad U(z; s) := \frac{\partial}{\partial s} \Phi(z; s) \cdot \Phi(z; s)^{-1} \quad (4.7)$$

are analytic in the complex z -plane except for possible isolated singularities at $z = 0$, $z = s$ and $z = \infty$.

To evaluate L and U , we need to know their local behaviors near these points. As $z \rightarrow \infty$, we have from (3.39) that

$$L(z; s) = A + \mathcal{O}(z^{-1}), \quad (4.8)$$

where

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.9)$$

On the other hand, in view of the local behaviors of Φ near $z = z_0 = 0$ and $z = z_1 = s$ as shown in (3.40)–(3.45), it is readily seen that

$$L(z; s) \sim \frac{A_i(s)}{z - z_i}, \quad z \rightarrow z_i, \quad i = 0, 1,$$

where

$$A_0(s) = \begin{cases} -\alpha \Phi_0^{(0)}(s) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi_0^{(0)}(s)^{-1}, & \alpha \neq 0, \\ \frac{1}{2\pi i} \Phi_0^{(0)}(s) \begin{pmatrix} 0 & \gamma - 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Phi_0^{(0)}(s)^{-1}, & \alpha = 0, \end{cases} \quad (4.10)$$

$$A_1(s) = -\frac{\gamma}{2\pi i} \Phi_0^{(1)}(s) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi_0^{(1)}(s)^{-1}, \quad (4.11)$$

and where $\Phi_0^{(i)}(s)$, $i = 0, 1$, are given in (3.43) and (3.45), respectively. Thus, if we define

$$\begin{pmatrix} q_{0,1}(s) \\ q_{0,2}(s) \\ q_{0,3}(s) \end{pmatrix} = \begin{cases} \Phi_0^{(0)}(s) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \alpha \neq 0, \\ \frac{1}{2\pi i} \Phi_0^{(0)}(s) \begin{pmatrix} \gamma - 1 \\ 0 \\ 1 \end{pmatrix}, & \alpha = 0, \end{cases} \quad (4.12)$$

$$\begin{pmatrix} p_{0,1}(s) \\ p_{0,2}(s) \\ p_{0,3}(s) \end{pmatrix} = \begin{cases} -\alpha \Phi_0^{(0)}(s)^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \alpha \neq 0, \\ \Phi_0^{(0)}(s)^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \alpha = 0, \end{cases} \quad (4.13)$$

and

$$\begin{pmatrix} q_{1,1}(s) \\ q_{1,2}(s) \\ q_{1,3}(s) \end{pmatrix} = \Phi_0^{(1)}(s) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} p_{1,1}(s) \\ p_{1,2}(s) \\ p_{1,3}(s) \end{pmatrix} = -\frac{\gamma}{2\pi i} \Phi_0^{(1)}(s)^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (4.14)$$

we see the expressions of $A_0(s)$ and $A_1(s)$ in (4.4) from (4.10) and (4.11). Moreover, the equations (4.10) and (4.11) also imply that

$$\mathrm{Tr} A_0(s) = \sum_{k=1}^3 q_{0,k}(s) p_{0,k}(s) = -\alpha, \quad \mathrm{Tr} A_1(s) = \sum_{k=1}^3 q_{1,k}(s) p_{1,k}(s) = 0, \quad (4.15)$$

which are the relations (2.4). Similarly, by (3.39), it follows that

$$U(z; s) = \mathcal{O}(z^{-1}), \quad z \rightarrow \infty. \quad (4.16)$$

A further appeal to (3.40)–(3.45) shows that

$$U(z; s) = \mathcal{O}(1), \quad z \rightarrow 0, \quad U(z; s) \sim -\frac{A_1(s)}{z-s}, \quad z \rightarrow s, \quad (4.17)$$

where $A_1(s)$ is given in (4.11). Combining (4.16) and (4.17) gives us (4.3).

We next come to the differential equations satisfied by the functions $p_{i,k}$ and $q_{i,k}$, $i = 0, 1$, $k = 1, 2, 3$. From (4.1)–(4.3), one has

$$\lim_{z \rightarrow 0} \frac{\partial}{\partial s} \Phi(z; s) \Phi(z; s)^{-1} = \lim_{z \rightarrow 0} U(z; s) = \frac{A_1(s)}{s}, \quad (4.18)$$

and

$$\lim_{z \rightarrow s} \left(\frac{\partial}{\partial z} \Phi(z; s) + \frac{\partial}{\partial s} \Phi(z; s) \right) \Phi(z; s)^{-1} = \lim_{z \rightarrow s} (L(z; s) + U(z; s)) = A + \frac{A_0(s)}{s}. \quad (4.19)$$

Substituting (3.43) and (3.45) into the left-hand side of (4.18) and (4.19), respectively, it is easily seen that

$$\frac{d}{ds} \Phi_0^{(0)}(s) = \frac{A_1(s)}{s} \Phi_0^{(0)}(s), \quad (4.20)$$

and

$$\frac{d}{ds} \Phi_0^{(1)}(s) = \left(A + \frac{A_0(s)}{s} \right) \Phi_0^{(1)}(s). \quad (4.21)$$

Recall the definitions of $q_{i,k}(s)$, $i = 0, 1$, $k = 1, 2, 3$, given in (4.12) and (4.14), we then obtain from the above two formulas that

$$(q'_{0,1}(s) \quad q'_{0,2}(s) \quad q'_{0,3}(s))^t = \frac{A_1(s)}{s} (q_{0,1}(s) \quad q_{0,2}(s) \quad q_{0,3}(s))^t, \quad (4.22)$$

$$(q'_{1,1}(s) \quad q'_{1,2}(s) \quad q'_{1,3}(s))^t = \left(A + \frac{A_0(s)}{s} \right) (q_{1,1}(s) \quad q_{1,2}(s) \quad q_{1,3}(s))^t. \quad (4.23)$$

By (4.4) and (4.9), equations (4.22) and (4.23) are equivalent to the first two equations in (2.3). To show the equations for $p'_{0,k}(s)$ and $p'_{1,k}(s)$ in (2.3), we note that for an invertible matrix-valued function $\mathcal{M}(s)$,

$$\frac{d}{ds}\mathcal{M}(s)^{-t} = -\mathcal{M}(s)^{-t} \cdot \frac{d}{ds}\mathcal{M}(s)^t \cdot \mathcal{M}(s)^{-t}.$$

Thus, we see from (4.20), (4.21) and the definitions of $p_{i,k}(s)$, $i = 0, 1$, $k = 1, 2, 3$, given in (4.13) and (4.14) that

$$(p'_{0,1}(s) \ p'_{0,2}(s) \ p'_{0,3}(s))^t = -\frac{A_1(s)^t}{s} (p_{0,1}(s) \ p_{0,2}(s) \ p_{0,3}(s))^t, \quad (4.24)$$

$$(p'_{1,1}(s) \ p'_{1,2}(s) \ p'_{1,3}(s))^t = -\left(A^t + \frac{A_0(s)^t}{s}\right) (p_{1,1}(s) \ p_{1,2}(s) \ p_{1,3}(s))^t, \quad (4.25)$$

which gives us the last two equations in (2.3).

Finally, to show (4.5) and (4.6), we note that, after a direct calculation, the $\mathcal{O}(z^{-1})$ term in (4.8) reads

$$\begin{pmatrix} * & * & * \\ \pi_3(\rho) + \frac{2\rho}{3} & * & * \\ 1 & -\pi_3(\rho) + \frac{\rho}{3} & * \end{pmatrix} \frac{1}{z}, \quad (4.26)$$

where $*$ stands for some unimportant entry and $\pi_3(\rho)$ is defined in (3.19). To that end, the following observation is helpful:

$$M(z)\Theta'(z)M(z)^{-1} = \begin{pmatrix} 0 & 1 & \frac{\rho}{3} \\ \frac{\rho}{3z} & 0 & 1 \\ \frac{1}{z} & \frac{\rho}{3z} & 0 \end{pmatrix}, \quad (4.27)$$

where $\Theta(z)$ is defined in (3.5) and $M(z) = \text{diag}\left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}}\right) L_{\pm} \text{diag}\left(e^{\pm\frac{\alpha\pi i}{3}}, e^{\mp\frac{\alpha\pi i}{3}}, 1\right)$ with L_{\pm} given in (3.4). In view of (4.2), it is readily seen from (4.8) and (4.26) that

$$(A_0)_{31} + (A_1)_{31} = 1, \quad (A_0)_{21} + (A_0)_{32} + (A_1)_{21} + (A_1)_{32} = \rho. \quad (4.28)$$

Combining the above equations with the expressions of A_0 and A_1 in (4.4) gives us (4.5) and (4.6).

This completes the proof of the Proposition 4.1. \square

Remark 4.2. From the general theory of Jimbo-Miwa-Ueno [46], it follows that

$$H(s) = -\frac{\gamma}{2\pi i} \text{Tr} \left(\Phi_1^{(1)}(s) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = -\frac{\gamma}{2\pi i} \left(\Phi_1^{(1)}(s) \right)_{21}, \quad (4.29)$$

where $\Phi_1^{(1)}(s)$ given in (3.45) appears in the expansion of $\Phi(z)$ near $z = s$. In view of the first equation in the Lax pair (4.1), by sending $z \rightarrow s$, we obtain from (4.2) and (3.44) that the $\mathcal{O}(1)$ term in the expansion yields

$$\Phi_1^{(1)}(s) = \frac{\gamma}{2\pi i} \left[\Phi_1^{(1)}(s), \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] + \Phi_0^{(1)}(s)^{-1} \left(A + \frac{A_0(s)}{s} \right) \Phi_0^{(1)}(s), \quad (4.30)$$

where $[A, B] := AB - BA$ denotes the commutator of two matrices A and B . Since the $(2, 1)$ -entry of $\left[\Phi_1^{(1)}(s), \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$ equals 0, we obtain from (4.29), (4.30), the expression of $A_0(s)$ in (4.4), and the definitions of $p_{i,k}(s)$ and $q_{i,k}(s)$, $i = 0, 1$, $k = 1, 2, 3$, in (4.12)–(4.14) that

$$\begin{aligned} sH(s) &= -\frac{s\gamma}{2\pi i} \left(\Phi_1^{(1)}(s) \right)_{21} = -\frac{s\gamma}{2\pi i} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \Phi_1^{(1)}(s) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= -\frac{\gamma}{2\pi i} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \Phi_0^{(1)}(s)^{-1} (sA + A_0(s)) \Phi_0^{(1)}(s) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p_{1,1}(s) & p_{1,2}(s) & p_{1,3}(s) \end{pmatrix} \\ &\quad \times \left(\begin{pmatrix} 0 & s & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} q_{0,1}(s) \\ q_{0,2}(s) \\ q_{0,3}(s) \end{pmatrix} \begin{pmatrix} p_{0,1}(s) & p_{0,2}(s) & p_{0,3}(s) \end{pmatrix} \right) \begin{pmatrix} q_{1,1}(s) \\ q_{1,2}(s) \\ q_{1,3}(s) \end{pmatrix} \\ &= s(p_{1,1}(s)q_{1,2}(s) + p_{1,2}(s)q_{1,3}(s)) + \left(\sum_{k=1}^3 p_{1,k}(s)q_{0,k}(s) \right) \left(\sum_{k=1}^3 p_{0,k}(s)q_{1,k}(s) \right), \end{aligned}$$

which is equivalent to (2.5).

4.2 Differential identities for the Hamiltonian

Besides the Hamiltonian H defined in (2.5), the following differential identities are crucial in the derivation of the asymptotics for $F(s; \alpha, \gamma, \rho)$ in Theorem 2.2, especially for the constant term.

Proposition 4.3. *With the Hamiltonian H defined in (2.5), we have*

$$\frac{d}{ds}(sH(s)) = p_{1,1}(s)q_{1,2}(s) + p_{1,2}(s)q_{1,3}(s), \quad (4.31)$$

and H is related to the action differential by

$$\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s)q'_{i,k}(s) - H(s) = H(s) - \frac{d}{ds}(sH(s)). \quad (4.32)$$

Moreover, we also have the following differential identities with respect to the parameters β and ρ :

$$\frac{\partial}{\partial \beta} \left(\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s)q'_{i,k}(s) - H(s) \right) = \frac{d}{ds} \left(\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \beta} q_{i,k}(s) \right), \quad (4.33)$$

$$\frac{\partial}{\partial \rho} \left(\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s)q'_{i,k}(s) - H(s) \right) = \frac{d}{ds} \left(\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \rho} q_{i,k}(s) \right), \quad (4.34)$$

where the parameter $\beta = \frac{1}{2\pi i} \ln(1 - \gamma)$, $0 \leq \gamma < 1$ as defined in (2.10).

Proof. Applying the Hamiltonian equations (2.6), we have

$$\frac{d}{ds}(sH(s)) = \sum_{i=0}^1 \sum_{k=1}^3 \left(s \frac{\partial H}{\partial p_{i,k}} p'_{i,k}(s) + s \frac{\partial H}{\partial q_{i,k}} q'_{i,k}(s) \right) + \frac{\partial}{\partial s}(sH(s)) = \frac{\partial}{\partial s}(sH(s)). \quad (4.35)$$

This, together with (2.5), gives us (4.31). A further combination of (2.5), (4.31) and the differential equations (2.3) implies (4.32). To obtain the differential identity with respect to the parameter γ , it is readily seen from (2.6) that

$$\begin{aligned} \frac{\partial}{\partial \gamma} H(s) &= \sum_{i=0}^1 \sum_{k=1}^3 \left(\frac{\partial H}{\partial p_{i,k}} \frac{\partial}{\partial \gamma} p_{i,k}(s) + \frac{\partial H}{\partial q_{i,k}} \frac{\partial}{\partial \gamma} q_{i,k}(s) \right) \\ &= \sum_{i=0}^1 \sum_{k=1}^3 \left(q'_{i,k}(s) \frac{\partial}{\partial \gamma} p_{i,k}(s) - p'_{i,k}(s) \frac{\partial}{\partial \gamma} q_{i,k}(s) \right), \end{aligned} \quad (4.36)$$

which gives us (4.33) recalling the definition of β in (2.10). The differential identity with respect to ρ in (4.34) can be proved in a similar manner and we omit the details here.

This completes the proof of Proposition 4.3. \square

In the next two sections, we will investigate asymptotics of the RH problem for Φ as $s \rightarrow +\infty$ and $s \rightarrow 0^+$ by using Deift-Zhou nonlinear steepest descent method [28], respectively. The main idea is to convert the original RH problem into a small-norm one via a series of explicit and invertible transformations.

5 Asymptotic analysis of the RH problem for Φ as $s \rightarrow +\infty$

Throughout this section, it is assumed that $0 < \gamma < 1$.

5.1 First transformation: $\Phi \rightarrow T$

Note that the jump contour Σ_Φ in (3.35) is unbounded when $s \rightarrow +\infty$, our first transformation is to rescale the problem as follows:

$$T(z) = -i\sqrt{3} s^{\frac{\alpha}{3}} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) C_\Psi^{-1} \Phi(sz) e^{-\Theta(sz)}, \quad (5.1)$$

where we also take this opportunity to get rid of the exponential factor in the large- z expansion in (3.39). In the above formula, the function $\Theta(z)$ and the constant matrix C_Ψ are given in (3.5) and (3.18), respectively. Then, $T(z)$ satisfies the following RH problem.

Proposition 5.1. *The function T defined in (5.1) has the following properties:*

- (1) $T(z)$ is defined and analytic in $\mathbb{C} \setminus \Sigma_T$, where

$$\Sigma_T := \cup_{i=2}^4 \Sigma_i \cup \Sigma_0^{(1)} \cup \Sigma_1^{(1)} \cup \Sigma_5^{(1)} \cup [0, 1],$$

and where the contours $\Sigma_j^{(1)}$, $j = 0, 1, 5$, are defined in (3.36) with $s = 1$.

- (2) For $z \in \Sigma_T$, T satisfies the jump condition

$$T_+(z) = T_-(z) J_T(z), \quad (5.2)$$

where

$$J_T(z) := \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_0^{(1)}, \\ \begin{pmatrix} 1 & 0 & 0 \\ e^{\theta_2(sz)-\theta_1(sz)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1^{(1)}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{\alpha\pi i} e^{\theta_2(sz)-\theta_3(sz)} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -e^{-\alpha\pi i} \\ 0 & e^{-\alpha\pi i} & 0 \end{pmatrix}, & z \in \Sigma_3, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{-\alpha\pi i} e^{\theta_1(sz)-\theta_3(sz)} \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_4, \\ \begin{pmatrix} 1 & 0 & 0 \\ e^{\theta_1(sz)-\theta_2(sz)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_5^{(1)}, \\ \begin{pmatrix} e^{\theta_2(sz)-\theta_1(sz)} & 1-\gamma & 0 \\ 0 & e^{\theta_1(sz)-\theta_2(sz)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (0, 1), \end{cases} \quad (5.3)$$

with $\theta_k(z) = \theta_k(z; \rho)$, $k = 1, 2, 3$, being defined in (3.6).

(3) As $z \rightarrow \infty$ and $\pm \text{Im } z > 0$, we have

$$T(z) = z^{-\frac{\alpha}{3}} (I + \mathcal{O}(z^{-1})) \text{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) L_{\pm} \text{diag} \left(e^{\pm \frac{\alpha\pi i}{3}}, e^{\mp \frac{\alpha\pi i}{3}}, 1 \right), \quad (5.4)$$

where L_{\pm} is given in (3.4).

(4) As $z \rightarrow 1$, we have $T(z) = \mathcal{O}(\ln(z-1))$.

(5) As $z \rightarrow 0$, we have

$$T(z) = \begin{cases} \mathcal{O}(z^{-\alpha}), & \alpha > 0, \\ \mathcal{O}(\ln z), & \alpha = 0, \\ \mathcal{O}(1), & -1 < \alpha < 0. \end{cases} \quad (5.5)$$

Proof. The above problem is a straightforward result from the transformation (5.1) and the RH problem for Φ given in Proposition 3.5, except for the jump on $\Sigma_3 = (-\infty, 0)$ where the function $\Theta(z)$ is not analytic. For $x \in (-\infty, 0)$, it is readily seen from (3.37), (3.38) and (5.1) that

$$\begin{aligned} J_T(x) &= T_-(x)^{-1} T_+(x) \\ &= \text{diag} \left(e^{\theta_{2,-}(sx)}, e^{\theta_{1,-}(sx)}, e^{\theta_{3,-}(sx)} \right) \Phi_-(sx)^{-1} \Phi_+(sx) \\ &\quad \times \text{diag} \left(e^{-\theta_{1,+}(sx)}, e^{-\theta_{2,+}(sx)}, e^{-\theta_{3,+}(sx)} \right) \\ &= \begin{pmatrix} e^{\theta_{2,-}(sx)-\theta_{1,+}(sx)} & 0 & 0 \\ 0 & 0 & -e^{-\alpha\pi i} e^{\theta_{1,-}(sx)-\theta_{3,+}(sx)} \\ 0 & e^{-\alpha\pi i} e^{\theta_{3,-}(sx)-\theta_{2,+}(sx)} & 0 \end{pmatrix}. \end{aligned} \quad (5.6)$$

In view of the definitions of θ_k , $k = 1, 2, 3$, in (3.6), we have

$$\theta_{1,+}(x) = \theta_{2,-}(x), \quad \theta_{2,+}(x) = \theta_{3,-}(x), \quad \theta_{3,+}(x) = \theta_{1,-}(x), \quad x < 0. \quad (5.7)$$

Combining the above two formulas, we obtain the jump J_T on $\Sigma_3 = (-\infty, 0)$ as shown in (5.3).

This completes the proof of Proposition 5.1. \square

5.2 Second transformation: $T \rightarrow S$

From the definitions of θ_k , $k = 1, 2, 3$, in (3.6), one can see that all the entries containing $\theta_k(sz)$ in $J_T(z)$ (5.3) are exponentially small as $s \rightarrow +\infty$, except $e^{\theta_2(sz) - \theta_1(sz)}$ when $x \in (0, 1)$. Indeed, we have

$$\theta_2(sz) - \theta_1(sz) = \sqrt{3}is^{\frac{2}{3}} \left(\frac{3}{2}x^{\frac{2}{3}} - \frac{\rho}{s^{\frac{1}{3}}}x^{\frac{1}{3}} \right), \quad x \in (0, 1), \quad (5.8)$$

which implies that the (1, 1) and (2, 2) entries of $J_T(z)$ are highly oscillatory for $x \in (0, 1)$ as $s \rightarrow +\infty$. Based on the factorizations

$$\begin{aligned} & \begin{pmatrix} e^{\theta_2(sz) - \theta_1(sz)} & 1 - \gamma & 0 \\ 0 & e^{\theta_1(sz) - \theta_2(sz)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_1(sz) - \theta_2(sz)}}{1 - \gamma} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 - \gamma & 0 \\ \frac{1}{\gamma - 1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_2(sz) - \theta_1(sz)}}{1 - \gamma} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (0, 1), \quad (5.9) \end{aligned}$$

a standard lens opening transformation can be introduced to remove the highly oscillatory entries of J_T on $(0, 1)$. Let Ω_{\pm} be the lens-shaped regions in the neighbourhood of $[0, 1]$; see Figure 5. The second transformation is set to be

$$S(z) = T(z) \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_1(sz) - \theta_2(sz)}}{1 - \gamma} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Omega_-, \\ \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_2(sz) - \theta_1(sz)}}{\gamma - 1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Omega_+, \\ I, & \text{elsewhere,} \end{cases} \quad (5.10)$$

Then, it is straightforward to verify that $S(z)$ satisfies the following RH problem.

RH problem 5.2.

(1) $S(z)$ is defined and analytic in $\mathbb{C} \setminus \Sigma_S$, where

$$\Sigma_S := \cup_{j=2}^4 \Sigma_j \cup [0, 1] \cup \partial\Omega_{\pm} \cup \Sigma_0^{(1)} \cup \Sigma_1^{(1)} \cup \Sigma_5^{(1)}; \quad (5.11)$$

see Figure 5 for an illustration.

(2) For $z \in \Sigma_S$, $S(z)$ satisfies the jump condition

$$S_+(z) = S_-(z)J_S(z), \quad (5.12)$$

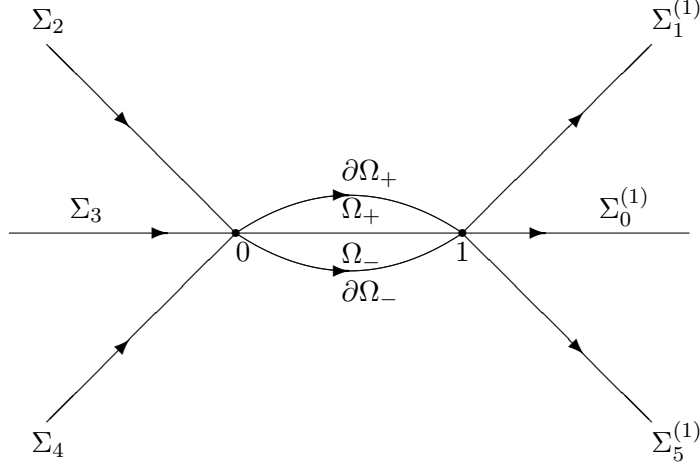


Figure 5: Regions Ω_{\pm} and the jump contours for the RH problem for S .

where

$$J_S(z) := \begin{cases} J_T(z), & z \in \cup_{j=2}^4 \Sigma_j \cup \Sigma_0^{(1)} \cup \Sigma_1^{(1)} \cup \Sigma_5^{(1)}, \\ \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_1(sz) - \theta_2(sz)}}{1-\gamma} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \partial\Omega_-, \\ \begin{pmatrix} 1 & 0 & 0 \\ \frac{e^{\theta_2(sz) - \theta_1(sz)}}{1-\gamma} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \partial\Omega_+, \\ \begin{pmatrix} 0 & 1-\gamma & 0 \\ \frac{1}{\gamma-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (0, 1), \end{cases} \quad (5.13)$$

and where J_T is defined in (5.3).

- (3) As $z \rightarrow \infty$ and $\pm \text{Im } z > 0$, $S(z)$ has the same behavior as $T(z)$ given in (5.4).
- (4) As $z \rightarrow 1$ or $z \rightarrow 0$ outside the lens, $S(z)$ has the same local behaviors as $T(z)$.

5.3 Global parametrix

As all the exponential entries in $J_S(z)$ tend to 0 exponentially fast as $s \rightarrow +\infty$, we are led to consider the following global parametrix.

RH problem 5.3.

- (1) $N(z)$ is defined and analytic in $\mathbb{C} \setminus \mathbb{R}$.
- (2) For $x \in \mathbb{R}$, N satisfies the jump condition

$$N_+(x) = N_-(x)J_N(x), \quad (5.14)$$

where

$$J_N(x) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -e^{-\alpha\pi i} \\ 0 & e^{-\alpha\pi i} & 0 \end{pmatrix}, & x \in (-\infty, 0), \\ \begin{pmatrix} 0 & 1-\gamma & 0 \\ \frac{1}{\gamma-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & x \in (0, 1), \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & x \in (1, +\infty). \end{cases} \quad (5.15)$$

(3) As $z \rightarrow \infty$ and $\pm \text{Im } z > 0$, we have

$$N(z) = z^{-\frac{\alpha}{3}} (I + \mathcal{O}(z^{-1})) \text{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) L_{\pm} \text{diag} \left(e^{\pm \frac{\alpha\pi i}{3}}, e^{\mp \frac{\alpha\pi i}{3}}, 1 \right). \quad (5.16)$$

The above RH problem can be solved explicitly in terms of the functions $d_k(z)$, $k = 1, 2, 3$, defined as follows. Recall that $\omega = e^{\frac{2}{3}\pi i}$, we set

$$\lambda(\xi) = \left(\frac{\xi - \omega}{\xi - 1} \right)^{\beta}, \quad \xi \in \mathbb{C} \setminus \{(0, 1) \cup (0, 1)\omega\}, \quad (5.17)$$

where β is given in (2.10) and the branch is chosen such that $\lambda(\xi) \rightarrow 1$ as $\xi \rightarrow \infty$; see Figure 6 for an illustration. The functions $d_k(z)$, $k = 1, 2, 3$, are then defined by

$$\begin{aligned} d_1(z) &= \begin{cases} \lambda(z^{\frac{1}{3}}), & \text{Im } z > 0, \\ \lambda(\omega z^{\frac{1}{3}}), & \text{Im } z < 0, \end{cases} \\ d_2(z) &= \begin{cases} \lambda(\omega z^{\frac{1}{3}}), & \text{Im } z > 0, \\ \lambda(z^{\frac{1}{3}}), & \text{Im } z < 0, \end{cases} \\ d_3(z) &= \lambda(\omega^{-1} z^{\frac{1}{3}}), \end{aligned} \quad (5.18)$$

where we choose $\arg z \in (-\pi, \pi)$; see [22] for a similar construction of the global parametrix.

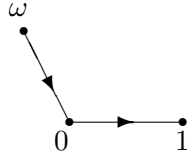


Figure 6: The branch cut of the function λ .

Proposition 5.4. *The functions $d_k(z)$, $k = 1, 2, 3$, in (5.18) satisfy the following properties.*

(i) For $x \in \mathbb{R}$, we have

$$d_{1,+}(x) = \frac{d_{2,-}(x)}{1-\gamma}, \quad d_{2,+}(x) = d_{1,-}(x)(1-\gamma), \quad x \in (0, 1), \quad (5.19)$$

$$d_{3,+}(x) = d_{3,-}(x), \quad x \in (0, +\infty), \quad (5.20)$$

$$d_{1,+}(x) = d_{2,-}(x), \quad d_{2,+}(x) = d_{1,-}(x), \quad x \in (1, +\infty), \quad (5.21)$$

$$d_{1,+}(x) = d_{1,-}(x), \quad x \in (-\infty, 0), \quad (5.22)$$

$$d_{2,+}(x) = d_{3,-}(x), \quad d_{3,+}(x) = d_{2,-}(x), \quad x \in (-\infty, 0). \quad (5.23)$$

(ii) As $z \rightarrow \infty$, we have

$$\begin{aligned} d_1(z) &= 1 + (1 - \omega)\beta z^{-\frac{1}{3}} + \left(-\frac{3}{2}\omega\beta^2 + \frac{1-\omega^2}{2}\beta\right)z^{-\frac{2}{3}} + \mathcal{O}(z^{-1}), \\ d_2(z) &= 1 + (1 - \omega)\omega^2\beta z^{-\frac{1}{3}} + \left(-\frac{3}{2}\omega\beta^2 + \frac{1-\omega^2}{2}\beta\right)\omega z^{-\frac{2}{3}} + \mathcal{O}(z^{-1}), \\ d_3(z) &= 1 + (1 - \omega)\omega\beta z^{-\frac{1}{3}} + \left(-\frac{3}{2}\omega\beta^2 + \frac{1-\omega^2}{2}\beta\right)\omega^2 z^{-\frac{2}{3}} + \mathcal{O}(z^{-1}). \end{aligned} \quad (5.24)$$

(iii) As $z \rightarrow 1$, we have, for $\text{Im } z > 0$,

$$\begin{aligned} d_1(z) &= (3\sqrt{3})^\beta e^{-\frac{\beta\pi i}{6}} (z-1)^{-\beta} \left(1 + \frac{\beta}{3\sqrt{3}}(\sqrt{3} + e^{\frac{\pi i}{6}})(z-1) + \mathcal{O}((z-1)^2)\right), \\ d_2(z) &= (3\sqrt{3})^{-\beta} e^{-\frac{\beta\pi i}{6}} (z-1)^\beta \left(1 - \frac{\beta}{3\sqrt{3}}(\sqrt{3} + e^{-\frac{\pi i}{6}})(z-1) + \mathcal{O}((z-1)^2)\right), \\ d_3(z) &= e^{\frac{\beta\pi i}{3}} \left(1 - \frac{\beta i}{3\sqrt{3}}(z-1) + \mathcal{O}((z-1)^2)\right). \end{aligned} \quad (5.25)$$

(iv) As $z \rightarrow 0$, we have, for $\text{Im } z > 0$,

$$\begin{aligned} d_1(z) &= e^{-\frac{4\beta\pi i}{3}} \left(1 + \sqrt{3}e^{\frac{\pi i}{6}}\beta z^{\frac{1}{3}} + \frac{\sqrt{3}}{2}\beta(e^{-\frac{\pi i}{6}} - \sqrt{3}\beta\omega^2)z^{\frac{2}{3}} + \mathcal{O}(z)\right), \\ d_2(z) &= e^{\frac{2\beta\pi i}{3}} \left(1 - \sqrt{3}e^{-\frac{\pi i}{6}}\beta z^{\frac{1}{3}} - \frac{\sqrt{3}}{2}\beta(e^{\frac{\pi i}{6}} + \sqrt{3}\beta\omega)z^{\frac{2}{3}} + \mathcal{O}(z)\right), \\ d_3(z) &= e^{\frac{2\beta\pi i}{3}} \left(1 - \sqrt{3}\beta i z^{\frac{1}{3}} + \frac{\sqrt{3}}{2}\beta(i - \sqrt{3}\beta)z^{\frac{2}{3}} + \mathcal{O}(z)\right). \end{aligned} \quad (5.26)$$

Proof. It follows from the definition of $\lambda(\xi)$ in (5.17) that

$$\lambda_+(\xi) = e^{-2\beta\pi i} \lambda_-(\xi) = \frac{\lambda_-(\xi)}{1 - \gamma}, \quad \xi \in (0, 1) \cup (0, 1)\omega, \quad (5.27)$$

where the orientation of the contour is shown in Figure 6, and as $\xi \rightarrow \infty$,

$$\lambda(\xi) = 1 + (1 - \omega)\beta \frac{1}{\xi} + \left(-\frac{3}{2}\omega\beta^2 + \frac{1 - \omega^2}{2}\beta\right) \frac{1}{\xi^2} + \mathcal{O}(\xi^{-3}). \quad (5.28)$$

The above formulas, together with the definitions of d_k , $k = 1, 2, 3$, in (5.18), give us the jump condition (5.19)–(5.23) and the large- z behavior (5.24). Similarly, we obtain the asymptotics of d_k near the points 0 and 1 via a straightforward computation.

This completes the proof of Proposition 5.4. \square

As a consequence of the above proposition, we have the following result.

Lemma 5.5. *The solution of RH problem 5.3 is given by*

$$N(z) = z^{-\frac{\alpha}{3}} C_N \text{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) L_\pm \text{diag} \left(e^{\pm \frac{\alpha\pi i}{3}} d_1(z), e^{\mp \frac{\alpha\pi i}{3}} d_2(z), d_3(z) \right) \quad (5.29)$$

for $\pm \text{Im } z > 0$, where

$$C_N = \begin{pmatrix} 1 & -\sqrt{3}\beta i & -\frac{3}{2}\beta^2 + \frac{\sqrt{3}}{2}\beta i \\ 0 & 1 & -\sqrt{3}\beta i \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.30)$$

the functions $d_k(z)$, $k = 1, 2, 3$ and the constant matrices L_\pm are given in (5.18) and (3.4), respectively.

Proof. By setting

$$N_\infty(z) = z^{-\frac{\alpha}{3}} \operatorname{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) L_\pm \operatorname{diag} \left(e^{\pm \frac{\alpha\pi i}{3}}, e^{\mp \frac{\alpha\pi i}{3}}, 1 \right), \quad \pm \operatorname{Im} z > 0, \quad (5.31)$$

it is readily seen that we could rewrite $N(z)$ in (5.29) as

$$N(z) = C_N N_\infty(z) \operatorname{diag} (d_1(z), d_2(z), d_3(z)). \quad (5.32)$$

First, one could check directly that

$$N_{\infty,+}(x) = N_{\infty,-}(x) \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -e^{-\alpha\pi i} \\ 0 & e^{-\alpha\pi i} & 0 \end{pmatrix}, & x \in (-\infty, 0), \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & x \in (0, +\infty). \end{cases} \quad (5.33)$$

Next, with the help of the relations of $d_k(z)$ established in (5.19)–(5.23), one can see $N(z)$ in (5.32) indeed satisfies the desired jump conditions (5.14) and (5.15). Finally, substituting the large- z asymptotics of $d_k(z)$ (5.24) into (5.29), we obtain the asymptotic behavior (5.16).

This completes the proof of Lemma 5.5. \square

Although $J_S(z) \rightarrow I$ as $s \rightarrow +\infty$ for $z \in \Sigma_S \setminus \mathbb{R}$, the convergence is not uniform near $z = 0$ and $z = 1$. The next two subsections are then devoted to the construction of local parametrices near these two points.

5.4 Local parametrix near the origin

Let $D(z_0, \delta)$ be a fixed open disc centered at z_0 with radius $\delta > 0$ and denote by $\partial D(z_0, \delta)$ its boundary. We look for a function $P^{(0)}(z)$ satisfying the following RH problem in $D(0, \delta)$.

RH problem 5.6.

- (1) $P^{(0)}(z)$ is defined and analytic in $D(0, \delta) \setminus \Sigma_S$, where δ is a small positive number and Σ_S is defined in (5.11).
- (2) $P^{(0)}(z)$ satisfies the jump condition

$$P_+^{(0)}(z) = P_-^{(0)}(z) J_S(z), \quad z \in D(0, \delta) \cap \Sigma_S, \quad (5.34)$$

where $J_S(z)$ is defined in (5.13).

- (3) As $s \rightarrow +\infty$, we have the matching condition

$$P^{(0)}(z) = \left(I + \mathcal{O}(s^{-\frac{1}{3}}) \right) N(z), \quad z \in \partial D(0, \delta), \quad (5.35)$$

where $N(z)$ is given in (5.29).

We solve the above RH problem explicitly in terms of the solution $\Psi_\alpha(z)$ of the RH problem 3.1 in the following lemma.

Lemma 5.7. *The solution to the RH problem 5.6 is given by*

$$P^{(0)}(z) = s^{\frac{\alpha}{3}} E_0(z) \operatorname{diag}(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}) \Psi_\alpha(sz) e^{-\Theta(sz)} \operatorname{diag}\left((1-\gamma)^{-\frac{2}{3}}, (1-\gamma)^{\frac{1}{3}}, (1-\gamma)^{\frac{1}{3}}\right), \quad (5.36)$$

where

$$E_0(z) = -\sqrt{3} i N(z) \operatorname{diag}\left((1-\gamma)^{\frac{2}{3}}, (1-\gamma)^{-\frac{1}{3}}, (1-\gamma)^{-\frac{1}{3}}\right) N_\infty(z)^{-1}, \quad (5.37)$$

with $\Theta(z)$, $N(z)$ and $N_\infty(z)$ defined in (3.5), (5.29) and (5.31), respectively.

Proof. We see from (5.33) that $N_\infty(z) \operatorname{diag}\left((1-\gamma)^{-\frac{2}{3}}, (1-\gamma)^{\frac{1}{3}}, (1-\gamma)^{\frac{1}{3}}\right)$ satisfies the same jump relation (5.15) as $N(z)$ for $z \in (-\delta, 0) \cup (0, \delta)$. This means that $E_0(z)$ given in (5.37) is analytic for $z \in D(0, \delta) \setminus \{0\}$. It follows from the definitions of $N(z)$ and $N_\infty(z)$ in (5.29) and (5.31) that

$$\begin{aligned} N(z) \operatorname{diag}\left((1-\gamma)^{\frac{2}{3}}, (1-\gamma)^{-\frac{1}{3}}, (1-\gamma)^{-\frac{1}{3}}\right) N_\infty(z)^{-1} &= C_N \operatorname{diag}\left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}}\right) \\ &\times L_\pm \operatorname{diag}\left((1-\gamma)^{\frac{2}{3}} d_1(z), (1-\gamma)^{-\frac{1}{3}} d_2(z), (1-\gamma)^{-\frac{1}{3}} d_3(z)\right) L_\pm^{-1} \operatorname{diag}\left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}\right). \end{aligned}$$

On account of the local behaviors of $d_k(z)$, $k = 1, 2, 3$, given in (5.26), it is readily seen that $E_0(z) = \mathcal{O}(z^{-2/3})$ as $z \rightarrow 0$, which implies that $E_0(z)$ is actually analytic in $D(0, \delta)$. Recalling the jump of Ψ_α in (3.2), it is then straightforward to see that $P^{(0)}(z)$ in (5.36) satisfies the required jump condition (5.34). Finally, we observe that one may rewrite the large- z asymptotics of Ψ_α in (3.3) as

$$\begin{aligned} \Psi_\alpha(z) &= \frac{iz^{-\frac{\alpha}{3}}}{\sqrt{3}} \operatorname{diag}\left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}}\right) \\ &\times \left(I + \frac{\Psi_1(\rho)}{z^{1/3}} + \frac{\Psi_2(\rho)}{z^{2/3}} + \mathcal{O}(z^{-1})\right) L_\pm \operatorname{diag}\left(e^{\pm \frac{\alpha\pi i}{3}}, e^{\mp \frac{\alpha\pi i}{3}}, 1\right) e^{\Theta(z)}, \quad (5.38) \end{aligned}$$

with

$$\Psi_1(\rho) = \begin{pmatrix} 0 & \pi_3(\rho) & 0 \\ 0 & 0 & \pi_3(\rho) + \frac{\rho}{3} \\ \pi_3(\rho) + \frac{2\rho}{3} & 0 & 0 \end{pmatrix}, \quad (5.39)$$

$$\Psi_2(\rho) = \begin{pmatrix} 0 & 0 & \pi_6(\rho) \\ \pi_6(\rho) + \frac{\rho}{3}\pi_3(\rho) + \frac{1-\alpha}{3} & 0 & 0 \\ 0 & \pi_6(\rho) + \frac{2\rho}{3}\pi_3(\rho) + \frac{\rho^2}{9} + \frac{1-2\alpha}{3} & 0 \end{pmatrix}, \quad (5.40)$$

where $\pi_3(\rho)$ and $\pi_6(\rho)$ are polynomials defined in (3.19) and (3.20). This implies that, as $z \rightarrow \infty$,

$$\Psi_\alpha(z) = \frac{i}{\sqrt{3}} N_\infty(z) \left(I + \mathcal{O}(z^{-\frac{1}{3}})\right) e^{\Theta(z)}$$

with $N_\infty(z)$ given in (5.31). A combination of (5.36), (5.37) and the above formula gives us

$$\begin{aligned} P^{(0)}(z) N(z)^{-1} &= N(z) \operatorname{diag}\left((1-\gamma)^{\frac{2}{3}}, (1-\gamma)^{-\frac{1}{3}}, (1-\gamma)^{-\frac{1}{3}}\right) \left(I + \mathcal{O}(s^{-\frac{1}{3}})\right) \\ &\times \operatorname{diag}\left((1-\gamma)^{-\frac{2}{3}}, (1-\gamma)^{\frac{1}{3}}, (1-\gamma)^{\frac{1}{3}}\right) N(z)^{-1}, \quad s \rightarrow +\infty, \end{aligned}$$

for $z \in \partial D(0, \delta)$. As $N(z)$ is independent of s , the above formula is indeed equivalent to the matching condition (5.35).

This completes the proof of Lemma 5.7. \square

In view of (5.26), the following proposition is immediate.

Proposition 5.8. *With the function $E_0(z)$ defined in (5.37), we have*

$$E_0(z) = E_0(0) \left(I + E_0(0)^{-1} E_0'(0) z + \mathcal{O}(z^2) \right), \quad z \rightarrow 0, \quad (5.41)$$

where

$$E_0(0) = -\sqrt{3} i C_N C_N^t \quad (5.42)$$

with C_N given in (5.30), and

$$E_0(0)^{-1} E_0'(0) \quad (5.43)$$

$$= \frac{\beta}{2} \begin{pmatrix} \beta(3 + \sqrt{3}\beta i) & \sqrt{3} i(1 + \sqrt{3}\beta i) & -2\sqrt{3} i \\ -\frac{1}{4}(9\beta^3 - 6\sqrt{3}\beta^2 i + 3\beta + 2\sqrt{3} i) & -2\sqrt{3}\beta^2 i & \sqrt{3} i(1 - \sqrt{3}\beta i) \\ \beta(\beta^2 + 1) - \frac{2\sqrt{3} i}{15}(6\beta^4 + 15\beta^2 - 1) & \frac{1}{4}(9\beta^3 + 6\sqrt{3}\beta^2 i + 3\beta - 2\sqrt{3} i) & -\beta(3 - \sqrt{3}\beta i) \end{pmatrix}.$$

To derive (5.43), we actually need more terms than those given in (5.26). Since all the calculations are straightforward, we omit the details here.

5.5 Local parametrix near 1

Near $z = 1$, we intend to find a function $P^{(1)}(z)$ satisfying the following RH problem.

RH problem 5.9.

- (1) $P^{(1)}(z)$ is defined and analytic in $D(1, \delta) \setminus \Sigma_S$, where Σ_S is defined in (5.11).
- (2) For $z \in \Sigma_S \cap D(1, \delta)$, $P^{(1)}(z)$ satisfies the jump condition

$$P_+^{(1)}(z) = P_-^{(1)}(z) J_S(z), \quad (5.44)$$

where $J_S(z)$ is given in (5.13).

- (3) As $s \rightarrow +\infty$, we have the matching condition

$$P^{(1)}(z) = \left(I + \mathcal{O}(s^{-\frac{2}{3}}) \right) N(z), \quad z \in \partial D(1, \delta), \quad (5.45)$$

where $N(z)$ is given in (5.29).

As in [22, 44], we can construct $P^{(1)}(z)$ explicitly by using the confluent hypergeometric parametrix. To proceed, we introduce the following local conformal mapping near $z = 1$:

$$f(z) := -is^{-\frac{2}{3}} [(\theta_2(sz) - \theta_1(sz)) - (\theta_2(s) - \theta_1(s))] \\ = \frac{3\sqrt{3}}{2} \left(z^{\frac{2}{3}} - 1 \right) - \frac{\sqrt{3}\rho}{s^{1/3}} \left(z^{\frac{1}{3}} - 1 \right), \quad z \in D(1, \delta). \quad (5.46)$$

It is easily seen that

$$f(1) = 0, \quad f'(1) = \sqrt{3} - \frac{\sqrt{3}}{3s^{1/3}}\rho, \quad f''(1) = -\frac{1}{\sqrt{3}} + \frac{2\rho}{3\sqrt{3}s^{1/3}}, \quad (5.47)$$

where $f'(1)$ is positive when s is large enough. Let $\Phi^{(\text{CHF})}(z; \beta)$ be the confluent hypergeometric parametrix given in Appendix A with the parameter β given in (2.10). We then define, for $z \in D(1, \delta) \setminus \Sigma_S$,

$$P^{(1)}(z) = E_1(z) \text{diag} \left(\Phi^{(\text{CHF})} \left(s^{\frac{2}{3}} f(z); \beta \right) e^{-\frac{\beta}{2} \pi i \sigma_3}, 1 \right) \\ \times \begin{cases} \text{diag} \left(e^{\frac{\theta_2(sz) - \theta_1(sz)}{2} \sigma_3}, 1 \right), & \text{Im } z > 0, \\ \text{diag} \left(e^{\frac{\theta_1(sz) - \theta_2(sz)}{2} \sigma_3}, 1 \right), & \text{Im } z < 0, \end{cases} \quad (5.48)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f(z)$ is given in (5.46) and

$$E_1(z) = \begin{cases} N(z) \text{diag} \left(\left(e^{\frac{\theta_1(s) - \theta_2(s) + \beta \pi i}{2}} s^{\frac{2\beta}{3}} f(z) \right)^{\sigma_3}, 1 \right), & \text{Im } z > 0, \\ N(z) \text{diag} \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(e^{\frac{\theta_1(s) - \theta_2(s) + \beta \pi i}{2}} s^{\frac{2\beta}{3}} f(z) \right)^{\sigma_3}, 1 \right), & \text{Im } z < 0, \end{cases} \quad (5.49)$$

with $N(z)$ given in (5.29). Here, we use the notation

$$\text{diag}(A, 1) = \begin{pmatrix} (A)_{11} & (A)_{12} & 0 \\ (A)_{21} & (A)_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for a 2×2 matrix A .

From the RH problem for $\Phi^{(\text{CHF})}(z; \beta)$ and our construction of $P^{(1)}(z)$ in (5.48), the following lemma is immediate. We omit the proof but refer to [22, Proposition 5.8] for a similar situation.

Lemma 5.10. *The local parametrix $P^{(1)}(z)$ defined in (5.48) solves the RH problem 5.9 for large positive s . Moreover, the prefactor $E_1(z)$ therein is analytic in $D(1, \delta)$.*

We conclude this section with the local behavior of $E_1(z)$ near $z = 1$.

Proposition 5.11. *With the function $E_1(z)$ defined in (5.49), we have*

$$E_1(z) = E_1(1) \left(I + E_1(1)^{-1} E_1'(1) (z - 1) + \mathcal{O}((z - 1)^2) \right), \quad z \rightarrow 1, \quad (5.50)$$

where

$$E_1(1) = C_N L_+ \text{diag} \left(e^{\frac{\beta \pi i}{3}} c_1(s), e^{-\frac{2\beta \pi i}{3}} c_1(s)^{-1}, e^{\frac{\beta \pi i}{3}} \right), \quad (5.51)$$

and

$$E_1(1)^{-1} E_1'(1) \\ = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{\sqrt{3}\beta f''(1)}{2f'(1)} - \frac{\beta}{3} e^{-\frac{1}{6}\pi i} + \frac{2\beta}{\sqrt{3}} & \frac{i}{3} e^{-\beta \pi i} c_1(s)^{-2} & -\frac{i}{3} c_1(s)^{-1} \\ -\frac{i}{3} e^{\beta \pi i} c_1(s)^2 & -\frac{\sqrt{3}\beta f''(1)}{2f'(1)} + \frac{\beta}{3} e^{\frac{1}{6}\pi i} - \frac{2\beta}{\sqrt{3}} & \frac{i}{3} e^{\beta \pi i} c_1(s) \\ \frac{i}{3} c_1(s) & -\frac{i}{3} e^{-\beta \pi i} c_1(s)^{-1} & -\frac{i}{3} \beta \end{pmatrix} - \frac{\alpha}{3} I \quad (5.52)$$

with

$$c_1(s) := (3\sqrt{3})^\beta f'(1)^\beta e^{\frac{\alpha \pi i}{3}} e^{\frac{\theta_1(s) - \theta_2(s)}{2}} s^{\frac{2\beta}{3}}, \quad (5.53)$$

C_N , L_+ and f given in (5.30), (3.4) and (5.46), respectively.

Proof. The coefficients in (5.50) follow directly from (5.25), (5.49) and straightforward calculations. In particular, to derive (5.52), one needs to use the fact that

$$L_+^{-1} \text{diag}(1, 0, -1) L_+ = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

This completes the proof of Proposition 5.11. \square

5.6 Final transformation and the small norm RH problem

The final transformation is defined by

$$R(z) = \begin{cases} S(z)N(z)^{-1}, & z \in \mathbb{C} \setminus \{D(0, \delta) \cup D(1, \delta) \cup \Sigma_S\}, \\ S(z)P^{(0)}(z)^{-1}, & z \in D(0, \delta), \\ S(z)P^{(1)}(z)^{-1}, & z \in D(1, \delta). \end{cases} \quad (5.54)$$

The RH problems for N , $P^{(0)}$ and $P^{(1)}$ particularly show that R is analytic for $z \in D(0, \delta) \cup D(1, \delta) \setminus \{0, 1\}$. One can then check that both 0 and 1 are actually removable singularities of R . Then, R satisfies the following RH problem.

RH problem 5.12.

(1) $R(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$, where

$$\Sigma_R := \Sigma_S \cup \partial D(0, \delta) \cup \partial D(1, \delta) \setminus \{\mathbb{R} \cup D(0, \delta) \cup D(1, \delta)\}. \quad (5.55)$$

(2) $R(z)$ satisfies the jump condition

$$R_+(z) = R_-(z)J_R(z), \quad z \in \Sigma_R,$$

where

$$J_R(z) = \begin{cases} P^{(0)}(z)N(z)^{-1}, & z \in \partial D(0, \delta), \\ P^{(1)}(z)N(z)^{-1}, & z \in \partial D(1, \delta), \\ N(z)J_S(z)N(z)^{-1}, & z \in \Sigma_R \setminus \{D(0, \delta) \cup D(1, \delta)\}, \end{cases} \quad (5.56)$$

and where the orientations of $\partial D(0, \delta)$ and $\partial D(1, \delta)$ are taken in a clockwise manner.

(3) As $z \rightarrow \infty$, we have $R(z) = I + \mathcal{O}(z^{-1})$.

For $z \in \Sigma_R \setminus \{\partial D(0, \delta) \cup \partial D(1, \delta)\}$, it is readily seen from (5.13), (5.29) and (5.56) that there exists a positive constant $c > 0$ such that

$$J_R(z) = I + \mathcal{O}\left(e^{-cs^{2/3}}\right), \quad s \rightarrow \infty. \quad (5.57)$$

For $z \in \partial D(0, \delta) \cup \partial D(1, \delta)$, it follows from (5.35), (5.45) and (5.56) that

$$J_R(z) = I + \frac{J_1(z)}{s^{1/3}} + \frac{J_2(z)}{s^{2/3}} + \mathcal{O}(s^{-1}), \quad s \rightarrow \infty, \quad (5.58)$$

for some matrix-valued functions $J_1(z)$ and $J_2(z)$. The above approximations show that R satisfies a small norm RH problem when s is large, which leads to the following approximation of R .

Proposition 5.13. *As $s \rightarrow \infty$, we have*

$$R(z) = I + \frac{R_1(z)}{s^{1/3}} + \frac{R_2(z)}{s^{2/3}} + \mathcal{O}(s^{-1}), \quad (5.59)$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, where

$$R_1(z) = \frac{1}{z} \operatorname{Res}_{\zeta=0} J_1^{(0)}(\zeta) - \begin{cases} J_1^{(0)}(z), & z \in D(0, \delta), \\ 0, & z \in \mathbb{C} \setminus D(0, \delta), \end{cases} \quad (5.60)$$

and

$$R_2(z) = \frac{1}{z} \operatorname{Res}_{\zeta=0} \left[J_2^{(0)}(\zeta) + R_1(\zeta) J_1^{(0)}(\zeta) \right] + \frac{1}{z-1} \operatorname{Res}_{\zeta=1} J_2^{(1)}(\zeta) - \begin{cases} J_2^{(0)}(z) + R_1(z) J_1^{(0)}(z), & z \in D(0, \delta), \\ J_2^{(1)}(z), & z \in D(1, \delta), \\ 0, & \text{otherwise.} \end{cases} \quad (5.61)$$

In (5.60) and (5.61), the functions $J_k^{(0)}(z)$, $k = 1, 2$, and $J_1^{(1)}(z)$ are given by

$$J_k^{(0)}(z) = z^{-\frac{k}{3}} E_0(z) \operatorname{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) \Psi_k(\rho) \times \operatorname{diag} \left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}} \right) E_0(z)^{-1}, \quad z \in D(0, \delta) \setminus \{0\}, \quad (5.62)$$

and

$$J_2^{(1)}(z) = \frac{2}{3\sqrt{3}(z^{\frac{2}{3}} - 1)} E_1(z) \operatorname{diag} \left(\Phi_1^{(\text{CHF})}, 0 \right) E_1(z)^{-1}, \quad z \in D(1, \delta) \setminus \{1\}, \quad (5.63)$$

where E_0 , Ψ_k , f , E_1 and $\Phi_1^{(\text{CHF})}$ are defined in (5.37), (5.39), (5.40), (5.46), (5.49) and (A.3), respectively.

Proof. By a standard argument for the small norm RH problem (cf. [24, 28]), the expansion (5.75) follows directly from asymptotics of the jump $J_R(z)$ given in (5.57) and (5.58).

To obtain explicit expressions of $R_1(z)$ and $R_2(z)$, we need more information about the functions $J_1(z)$ and $J_2(z)$ in the expansion (5.58). For $z \in \partial D(0, \delta)$ and $\operatorname{Im} z > 0$, combining the large- z expansion of Ψ_α in (5.38) and the definitions of $P^{(0)}(z)$ and $E_0(z)$ in (5.36) and (5.37), we have, as $s \rightarrow \infty$

$$\begin{aligned} & P^{(0)}(z) N(z)^{-1} \\ &= \frac{iz^{-\alpha/3}}{\sqrt{3}} E_0(z) \operatorname{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) \left(I + \frac{\Psi_1(\rho)}{(sz)^{1/3}} + \frac{\Psi_2(\rho)}{(sz)^{2/3}} + \mathcal{O}(s^{-1}) \right) L_+ \\ & \quad \times \operatorname{diag} \left(e^{\frac{\alpha\pi i}{3}}, e^{-\frac{\alpha\pi i}{3}}, 1 \right) \operatorname{diag} \left((1-\gamma)^{-\frac{2}{3}}, (1-\gamma)^{\frac{1}{3}}, (1-\gamma)^{\frac{1}{3}} \right) N(z)^{-1} \\ &= E_0(z) \operatorname{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) \left(I + \frac{\Psi_1(\rho)}{(sz)^{1/3}} + \frac{\Psi_2(\rho)}{(sz)^{2/3}} + \mathcal{O}(s^{-1}) \right) \operatorname{diag} \left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}} \right) E_0(z)^{-1}, \end{aligned}$$

where $\Psi_1(\rho)$ and $\Psi_2(\rho)$ are given in (5.39) and (5.40), respectively. It is easy to verify that the above expansion also holds for $z \in \partial D(0, \delta)$ and $\operatorname{Im} z < 0$. As $E_0(z)$ is independent of s , the above formula gives us

$$P^{(0)}(z) N(z)^{-1} = I + \frac{J_1^{(0)}(z)}{s^{1/3}} + \frac{J_2^{(0)}(z)}{s^{2/3}} + \mathcal{O}(s^{-1}), \quad s \rightarrow \infty, \quad (5.64)$$

for $z \in \partial D(0, \delta)$ with $J_k^{(0)}(z)$, $k = 1, 2$, given in (5.62).

Similarly, for $z \in \partial D(1, \delta)$ and $\operatorname{Im} z > 0$, we have from (5.48) and (A.2) that

$$P^{(1)}(z) N(z)^{-1} = E_1(z) \left(I + \frac{\operatorname{diag} \left(\Phi_1^{(\text{CHF})}, 0 \right)}{s^{2/3} f(z)} + \mathcal{O}(s^{-\frac{4}{3}}) \right) \times \operatorname{diag} \left(\left(e^{-\frac{\beta\pi i}{2} - \frac{s^{2/3} f(z) i}{2}} s^{-\frac{2\beta}{3}} f(z)^{-\beta} \right)^{\sigma_3}, 1 \right) \operatorname{diag} \left(e^{\frac{\theta_2(s z) - \theta_1(s z)}{2} \sigma_3}, 1 \right) N(z)^{-1}. \quad (5.65)$$

Recalling the definitions of $f(z)$ and $E_1(z)$ in (5.46) and (5.49), we have

$$P^{(1)}(z)N(z)^{-1} = E_1(z) \left(I + \frac{2 \operatorname{diag} \left(\Phi_1^{(\text{CHF})}, 0 \right)}{3\sqrt{3}(z^{2/3} - 1)s^{2/3}} + \mathcal{O}(s^{-1}) \right) E_1(z)^{-1}. \quad (5.66)$$

Again, the above formula holds for $z \in \partial D(1, \delta)$ and $\operatorname{Im} z < 0$ as well. Although $E_1(z)$ depends on s , it is bounded when $s \rightarrow \infty$ since both $\theta_1(s) - \theta_2(s) = (-\frac{3\sqrt{3}}{2}s^{2/3} + \sqrt{3}\rho s^{1/3})i$ and $\beta = \frac{1}{2\pi i} \ln(1 - \gamma)$ are purely imaginary. Then, we have from the above formula that

$$P^{(1)}(z)N(z)^{-1} = I + \frac{J_2^{(1)}(z)}{s^{2/3}} + \mathcal{O}(s^{-1}), \quad s \rightarrow \infty, \quad (5.67)$$

for $z \in \partial D(1, \delta)$ with $J_2^{(1)}(z)$ given in (5.63). Therefore, it follows from (5.56), (5.58), (5.64) and (5.67) that

$$J_1(z) = \begin{cases} J_1^{(0)}(z), & z \in \partial D(0, \delta), \\ 0, & z \in \partial D(1, \delta), \end{cases} \quad J_2(z) = \begin{cases} J_2^{(0)}(z), & z \in \partial D(0, \delta), \\ J_2^{(1)}(z), & z \in \partial D(1, \delta). \end{cases} \quad (5.68)$$

Now, we are ready to derive explicit expressions of $R_1(z)$ and $R_2(z)$ in (5.75). From the RH problem 5.12 for R , it is easy to see from (5.57)–(5.75) that $R_1(z)$ satisfies a RH problem as follows:

RH problem 5.14.

- (1) $R_1(z)$ is analytic in $\mathbb{C} \setminus \partial D(0, \delta)$.
- (2) For $z \in \partial D(0, \delta)$, we have

$$R_{1,+}(z) - R_{1,-}(z) = J_1(z) = J_1^{(0)}(z), \quad (5.69)$$

where $J_1^{(0)}(z)$ is given in (5.62).

- (3) As $z \rightarrow \infty$, we have $R_1(z) = \mathcal{O}(z^{-1})$.

By Cauchy's theorem, we have

$$R_1(z) = \frac{1}{2\pi i} \int_{\partial D(0, \delta)} \frac{J_1^{(0)}(\zeta)}{\zeta - z} d\zeta. \quad (5.70)$$

To evaluate this integral, we observe from (5.39) that

$$\begin{aligned} z^{-\frac{1}{3}} \operatorname{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) \Psi_1(\rho) \operatorname{diag} \left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}} \right) \\ = \begin{pmatrix} 0 & \pi_3(\rho) & 0 \\ 0 & 0 & \pi_3(\rho) + \frac{\rho}{3} \\ \left(\pi_3(\rho) + \frac{2\rho}{3} \right) / z & 0 & 0 \end{pmatrix}, \end{aligned} \quad (5.71)$$

where $\pi_3(\rho)$ is given in (3.19). This, together with (5.62) and the fact that $E_0(z)$ is analytic near $z = 0$, implies that $J_1(z)$ admits a simple pole at $z = 0$. We then obtain (5.60) from (5.70) and the residue theorem.

The RH problem for R_2 reads as follows.

RH problem 5.15.

- (1) $R_2(z)$ is analytic in $\mathbb{C} \setminus \{\partial D(0, \delta) \cup \partial D(1, \delta)\}$.
(2) For $z \in \partial D(0, \delta) \cup \partial D(1, \delta)$, we have

$$\begin{aligned} R_{2,+}(z) - R_{2,-}(z) &= J_2(z) + R_{1,-}(z)J_1(z) \\ &= \begin{cases} J_2^{(0)}(z) + R_{1,-}(z)J_1^{(0)}(z), & z \in \partial D(0, \delta), \\ J_2^{(1)}(z), & z \in \partial D(1, \delta), \end{cases} \end{aligned} \quad (5.72)$$

where the functions $J_k^{(0)}(z)$, $k = 1, 2$, and $J_2^{(1)}(z)$ are given in (5.62) and (5.63).

- (3) As $z \rightarrow \infty$, we have $R_2(z) = \mathcal{O}(z^{-1})$.

Similar to the calculation of R_1 , we note that, due to (5.40), the function

$$\begin{aligned} & z^{-\frac{2}{3}} \text{diag} \left(z^{\frac{1}{3}}, 1, z^{-\frac{1}{3}} \right) \Psi_2(\rho) \text{diag} \left(z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}} \right) \\ &= \begin{pmatrix} 0 & 0 & \pi_6(\rho) \\ \left(\pi_6(\rho) + \frac{\rho}{3}\pi_3(\rho) + \frac{1-\alpha}{3} \right) / z & 0 & 0 \\ 0 & \left(\pi_6(\rho) + \frac{2\rho}{3}\pi_3(\rho) + \frac{\rho^2}{9} + \frac{1-2\alpha}{3} \right) / z & 0 \end{pmatrix} \end{aligned} \quad (5.73)$$

is analytic in a neighborhood of the origin except for a simple pole at $z = 0$ in the (2,1) and (3,2) entries. This implies $J_2^{(0)}(z) + R_{1,-}(z)J_1^{(0)}(z)$ only has a simple pole at $z = 0$ in $D(0, \delta)$. As $z^{2/3} - 1$ is analytic at $z = 1$, we have $J_2^{(1)}(z)$ is analytic in $D(1, \delta) \setminus \{1\}$. Thus, we obtain (5.61) from

$$R_2(z) = \frac{1}{2\pi i} \oint_{\partial D(0, \delta)} \frac{J_2^{(0)}(\zeta) + R_{1,-}(\zeta)J_1^{(0)}(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \oint_{\partial D(1, \delta)} \frac{J_2^{(1)}(\zeta)}{\zeta - z} d\zeta \quad (5.74)$$

and the residue theorem.

This finishes the proof of Proposition 5.13. \square

It is noted that from (5.64)-(5.66) we also have the expansion of the derivatives of $R(z)$ with respect to the parameter γ

$$\frac{d^k}{d\gamma^k} R(z) = \frac{\frac{d^k}{d\gamma^k} R_1(z)}{s^{1/3}} + \frac{\frac{d^k}{d\gamma^k} R_2(z)}{s^{2/3}} + \mathcal{O}\left(\frac{(\ln s)^k}{s}\right), \quad s \rightarrow \infty, \quad k \geq 1, \quad (5.75)$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, with $R_1(z)$ and $R_2(z)$ given in (5.60) and (5.61); see [19, Section 3.5] where a similar analysis is done in detail.

To facilitate our future calculations, it is convenient to rewrite the large- s expansion of $R(z)$ in (5.75) as

$$R(z) = C_N \widehat{R}(z) C_N^{-1} = C_N \left(I + \frac{\widehat{R}_1(z)}{s^{1/3}} + \frac{\widehat{R}_2(z)}{s^{2/3}} + \mathcal{O}(s^{-1}) \right) C_N^{-1}, \quad (5.76)$$

where C_N is given in (5.30) and

$$\widehat{R}_i(z) = C_N^{-1} R_i(z) C_N, \quad i = 1, 2. \quad (5.77)$$

We conclude this section by calculating $\widehat{R}_1(0)$ and $\widehat{R}_2(0)$ explicitly.

We start with the observation that if

$$\mathcal{J}(z) = \mathcal{E}(z) \left(\mathcal{C}_0 + \frac{\mathcal{C}_1}{z} \right) \mathcal{E}(z)^{-1},$$

where \mathcal{C}_0 and \mathcal{C}_1 are two constant matrices and $\mathcal{E}(z)$ is analytic near the origin, then

$$\lim_{z \rightarrow 0} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} \mathcal{J}(\zeta) - \mathcal{J}(z) \right) = \mathcal{E}(0) \left([\mathcal{C}_1, \mathcal{E}(0)^{-1} \mathcal{E}'(0)] - \mathcal{C}_0 \right) \mathcal{E}(0)^{-1}. \quad (5.78)$$

Thus, by (5.60), (5.62), (5.71) and the above formula, it follows that

$$R_1(0) = E_0(0) \left(\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pi_3(\rho) + \frac{2\rho}{3} & 0 & 0 \end{pmatrix}, E_0(0)^{-1} E_0'(0) \right] - \begin{pmatrix} 0 & \pi_3(\rho) & 0 \\ 0 & 0 & \pi_3(\rho) + \frac{\rho}{3} \\ 0 & 0 & 0 \end{pmatrix} \right) E_0(0)^{-1}.$$

Inserting (5.42) and (5.43) into the above formula, we obtain after a straightforward calculation that

$$\widehat{R}_1(0) = C_N^{-1} R_1(0) C_N = \begin{pmatrix} \frac{2\sqrt{3}}{3} \rho \beta i & -\pi_3(\rho) & 0 \\ \frac{\sqrt{3}}{2} \rho \beta (\sqrt{3}\beta - \frac{i}{3}) & -\frac{\sqrt{3}}{3} \rho \beta i & -\pi_3(\rho) - \frac{\rho}{3} \\ -\frac{\sqrt{3}}{2} \rho \beta^2 (\sqrt{3} + \beta i) & \frac{\sqrt{3}}{3} \rho \beta i & -\frac{\sqrt{3}}{3} \rho \beta i \end{pmatrix}. \quad (5.79)$$

To evaluate $\widehat{R}_2(0)$, it is readily seen from (5.60), (5.61) and (5.77) that

$$\widehat{R}_2(0) = C_N^{-1} \left(-\operatorname{Res}_{\zeta=1} J_2^{(1)}(\zeta) + \lim_{z \rightarrow 0} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} J_2^{(0)}(\zeta) - J_2^{(0)}(z) \right) + R_1(0)^2 \right) C_N. \quad (5.80)$$

For later use, it suffices to find the first row and the last column of $\widehat{R}_2(0)$. We next calculate three terms on the right hand side of the above formula separately. First, it follows from (5.63), (5.51) and (A.3) that

$$\begin{aligned} \operatorname{Res}_{\zeta=1} C_N^{-1} J_2^{(1)}(\zeta) C_N &= \frac{1}{\sqrt{3}} C_N^{-1} E_1(1) \operatorname{diag} \left(\Phi_1^{(\text{CHF})}, 0 \right) E_1(1)^{-1} C_N \\ &= \frac{1}{3\sqrt{3}} \\ &\times \begin{pmatrix} 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2}{\Gamma(\beta)\omega} \right) & -\sqrt{3}\beta^2 + 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2\omega}{\Gamma(\beta)} \right) & \sqrt{3}\beta^2 + 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2}{\Gamma(\beta)} \right) \\ \sqrt{3}\beta^2 + 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2\omega}{\Gamma(\beta)} \right) & 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2}{\Gamma(\beta)} \right) & -\sqrt{3}\beta^2 + 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2}{\Gamma(\beta)\omega} \right) \\ -\sqrt{3}\beta^2 + 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2}{\Gamma(\beta)} \right) & \sqrt{3}\beta^2 + 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2}{\Gamma(\beta)\omega} \right) & 2\operatorname{Im} \left(\frac{\Gamma(1-\beta)c_1(s)^2\omega}{\Gamma(\beta)} \right) \end{pmatrix}, \end{aligned} \quad (5.81)$$

where we have also made use of the fact that β is purely imaginary. From the definition of $c_1(s)$ in (5.53), it is easily seen that

$$\Gamma(1-\beta)c_1(s) = |\Gamma(1-\beta)| e^{i\vartheta(s)}, \quad (5.82)$$

where

$$\vartheta(s) = \frac{\alpha\pi}{3} + \arg \Gamma(1-\beta) - \beta i \left(\frac{2}{3} \ln s + \frac{3}{2} \ln 3 + \ln f'(1) \right) - i \frac{\theta_1(s) - \theta_2(s)}{2}. \quad (5.83)$$

Moreover, from [56, Formula 5.5.3], we have

$$\Gamma(1-\beta)\Gamma(\beta) = \frac{\pi}{\sin(\beta\pi)},$$

and thus,

$$\Gamma(1 + \beta)\Gamma(1 - \beta) = |\Gamma(1 - \beta)^2| = \beta\Gamma(\beta)\Gamma(1 - \beta) = \frac{\beta\pi}{\sin(\beta\pi)}. \quad (5.84)$$

This gives us

$$\operatorname{Im} \frac{\Gamma(1 - \beta)c_1(s)^2}{\Gamma(\beta)} = \operatorname{Im} \frac{\Gamma(1 - \beta)^2 c_1(s)^2}{\Gamma(1 - \beta)\Gamma(\beta)} = \operatorname{Im} \left(-\beta i e^{2i\vartheta(s) + \frac{\pi i}{2}} \right) = -\beta i \cos(2\vartheta(s)). \quad (5.85)$$

Therefore, (5.81) can be rewritten as

$$\begin{aligned} & \operatorname{Res}_{\zeta=1} C_N^{-1} J_2^{(1)}(\zeta) C_N \quad (5.86) \\ &= \frac{1}{3\sqrt{3}} \begin{pmatrix} -2\beta i \cos(2\vartheta(s) - \frac{2\pi}{3}) & -\sqrt{3}\beta^2 - 2\beta i \cos(2\vartheta(s) + \frac{2\pi}{3}) & \sqrt{3}\beta^2 - 2\beta i \cos(2\vartheta(s)) \\ \sqrt{3}\beta^2 - 2\beta i \cos(2\vartheta(s) + \frac{2\pi}{3}) & -2\beta i \cos(2\vartheta(s)) & -\sqrt{3}\beta^2 - 2\beta i \cos(2\vartheta(s) - \frac{2\pi}{3}) \\ -\sqrt{3}\beta^2 - 2\beta i \cos(2\vartheta(s)) & \sqrt{3}\beta^2 - 2\beta i \cos(2\vartheta(s) - \frac{2\pi}{3}) & -2\beta i \cos(2\vartheta(s) + \frac{2\pi}{3}) \end{pmatrix}. \end{aligned}$$

Next, by (5.62), (5.73) and (5.78), it is easily seen that

$$\begin{aligned} & \lim_{z \rightarrow 0} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} J_0^{(2)}(\zeta) - J_0^{(2)}(z) \right) \\ &= E_0(0) \left(\left[\begin{pmatrix} 0 & 0 & 0 \\ \mathcal{A} + \pi_6(\rho) & 0 & 0 \\ 0 & \mathcal{B} + \pi_6(\rho) & 0 \end{pmatrix}, E_0(0)^{-1} E_0'(0) \right] - \begin{pmatrix} 0 & 0 & \pi_6(\rho) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) E_0(0)^{-1}, \end{aligned}$$

where

$$\mathcal{A} = \frac{\rho}{3}\pi_3(\rho) + \frac{1 - \alpha}{3}, \quad \mathcal{B} = \frac{2\rho}{3}\pi_3(\rho) + \frac{\rho^2}{9} + \frac{1 - 2\alpha}{3}. \quad (5.87)$$

This, together with (5.42) and (5.43), implies that

$$\lim_{z \rightarrow 0} \left(C_N^{-1} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} J_0^{(2)}(\zeta) - J_0^{(2)}(z) \right) C_N \right)_{11} = \frac{\beta}{2} \left(\sqrt{3}\mathcal{A}(-i + \sqrt{3}\beta) - 6\mathcal{B}\beta \right), \quad (5.88)$$

$$\lim_{z \rightarrow 0} \left(C_N^{-1} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} J_0^{(2)}(\zeta) - J_0^{(2)}(z) \right) C_N \right)_{12} = \sqrt{3}\beta\mathcal{B}i, \quad (5.89)$$

$$\lim_{z \rightarrow 0} \left(C_N^{-1} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} J_0^{(2)}(\zeta) - J_0^{(2)}(z) \right) C_N \right)_{13} = -\pi_6(\rho), \quad (5.90)$$

$$\lim_{z \rightarrow 0} \left(C_N^{-1} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} J_0^{(2)}(\zeta) - J_0^{(2)}(z) \right) C_N \right)_{23} = -\sqrt{3}\beta\mathcal{A}i, \quad (5.91)$$

$$\lim_{z \rightarrow 0} \left(C_N^{-1} \left(\frac{1}{z} \operatorname{Res}_{\zeta=0} J_0^{(2)}(\zeta) - J_0^{(2)}(z) \right) C_N \right)_{33} = \frac{\beta}{2} \left(\sqrt{3}\mathcal{B}(i + \sqrt{3}\beta) - 6\mathcal{A}\beta \right). \quad (5.92)$$

We also observe from (5.79) that

$$\begin{aligned} & \widehat{R}_1(0)^2 = C_N^{-1} R_1(0)^2 C_N = \\ &= \begin{pmatrix} -\frac{4}{3}\rho^2\beta^2 - \frac{\sqrt{3}}{2}\rho\pi_3(\rho)\beta(\sqrt{3}\beta - \frac{i}{3}) & -\frac{\sqrt{3}}{3}\rho\pi_3(\rho)\beta i & \pi_3(\rho)(\pi_3(\rho) + \frac{\rho}{3}) \\ * & * & \frac{2\sqrt{3}}{3}\rho\beta(\pi_3(\rho) + \frac{\rho}{3})i \\ * & * & -\frac{\rho^2\beta^2}{3} - \frac{\sqrt{3}}{3}\rho\beta(\pi_3(\rho) + \frac{\rho}{3})i \end{pmatrix}. \quad (5.93) \end{aligned}$$

Thus, inserting (5.86)–(5.93) into (5.80), we finally obtain that

$$\begin{aligned} \left(\widehat{R}_2(0)\right)_{11} &= \frac{\beta}{2} \left(\sqrt{3}\mathcal{A}(-i + \sqrt{3}\beta) - 6\mathcal{B}\beta \right) - \frac{4}{3}\rho^2\beta^2 \\ &\quad - \frac{\sqrt{3}}{2}\rho\pi_3(\rho)\beta \left(\sqrt{3}\beta - \frac{i}{3} \right) + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) - \frac{2\pi}{3}) \\ &= -\beta^2 \left(3\rho\pi_3(\rho) + \frac{5}{3}\rho^2 - \frac{3\alpha - 1}{2} \right) + \frac{\beta i}{2\sqrt{3}}(\alpha - 1) + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) - \frac{2\pi}{3}), \end{aligned} \quad (5.94)$$

$$\begin{aligned} \left(\widehat{R}_2(0)\right)_{12} &= \sqrt{3}\beta\mathcal{B}i - \frac{\sqrt{3}}{3}\rho\pi_3(\rho)\beta i + \frac{\beta^2}{3} + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) + \frac{2\pi}{3}) \\ &= \frac{\beta^2}{3} + \frac{\beta i}{\sqrt{3}} \left(\rho\pi_3(\rho) + \frac{1}{3}\rho^2 - 2\alpha + 1 \right) + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) + \frac{2\pi}{3}), \end{aligned} \quad (5.95)$$

$$\left(\widehat{R}_2(0)\right)_{13} = -\pi_6(\rho) + \pi_3(\rho) \left(\pi_3(\rho) + \frac{\rho}{3} \right) - \frac{\beta^2}{3} + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s)), \quad (5.96)$$

$$\begin{aligned} \left(\widehat{R}_2(0)\right)_{23} &= -\sqrt{3}i\beta\mathcal{A} + \frac{2i\beta}{\sqrt{3}}\rho \left(\pi_3(\rho) + \frac{\rho}{3} \right) + \frac{\beta^2}{3} + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) - \frac{2\pi}{3}) \\ &= \frac{\beta^2}{3} + \frac{\beta i}{\sqrt{3}} \left(\rho\pi_3(\rho) + \frac{2}{3}\rho^2 + \alpha - 1 \right) + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) - \frac{2\pi}{3}), \end{aligned} \quad (5.97)$$

$$\begin{aligned} \left(\widehat{R}_2(0)\right)_{33} &= \frac{\beta}{2} \left(\sqrt{3}\mathcal{B}(i + \sqrt{3}\beta) - 6\mathcal{A}\beta \right) - \frac{\rho^2\beta^2}{3} \\ &\quad - \frac{\sqrt{3}}{3}\rho\beta \left(\pi_3(\rho) + \frac{\rho}{3} \right) i + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) + \frac{2\pi}{3}) \\ &= -\frac{\beta^2}{2} \left(1 + \frac{\rho^2}{3} \right) - \frac{\beta i}{2\sqrt{3}} \left(\frac{\rho^2}{3} + 2\alpha - 1 \right) + \frac{2\beta i}{3\sqrt{3}} \cos(2\vartheta(s) + \frac{2\pi}{3}), \end{aligned} \quad (5.98)$$

where $\pi_3(\rho)$, $\pi_6(\rho)$, \mathcal{A} and \mathcal{B} are given in (3.19), (3.20) and (5.87), respectively.

6 Asymptotic analysis of the RH problem for Φ as $s \rightarrow 0^+$

Throughout this section, it is assumed that $0 < \gamma \leq 1$.

6.1 Global parametrix

As $s \rightarrow 0^+$, the interval $(0, s)$ vanishes, and a comparison of the RH problems Ψ_α and Φ invokes us to define the global parametrix \widetilde{N} for $|z| > \delta > s$ by

$$\widetilde{N}(z) := \Psi_\alpha(z) \begin{cases} J_{\Psi_\alpha,1}, & \arg z < \frac{\pi}{4} \text{ and } \arg(z-s) > \frac{\pi}{4}, \\ J_{\Psi_\alpha,5}^{-1}, & \arg z > -\frac{\pi}{4} \text{ and } \arg(z-s) < -\frac{\pi}{4}, \\ I, & \text{elsewhere,} \end{cases} \quad (6.1)$$

where $J_{\Psi_\alpha,k}$, $k = 0, 1, \dots, 5$, stands for the jump matrix J_{Ψ_α} in (3.2) restricted on the contour Σ_k .

6.2 Local parametrix

Near $z = 0$, we intend to find a function $\widetilde{P}^{(0)}(z)$ satisfying the following RH problem.

RH problem 6.1.

(1) $\tilde{P}^{(0)}(z)$ is defined and analytic in $D(0, \delta) \setminus \Sigma_\Phi$, where Σ_Φ is defined in (3.35).

(2) For $z \in \Sigma_\Phi \cap D(0, \delta)$, $\tilde{P}^{(0)}(z)$ satisfies the jump condition

$$\tilde{P}_+^{(0)}(z) = \tilde{P}_-^{(0)}(z)J_\Phi(z), \quad (6.2)$$

where $J_\Phi(z)$ is given in (3.38).

(3) As $s \rightarrow 0^+$, we have the matching condition

$$\tilde{P}^{(0)}(z) = (I + \mathcal{O}(s^{\alpha+1}))\tilde{N}(z), \quad z \in \partial D(0, \delta), \quad (6.3)$$

where $\tilde{N}(z)$ is given in (6.1).

Recall the analytic function $\Psi_\alpha^{(0)}(z)$ in (3.7) which describes the local behavior of Ψ_α near the origin, we look for a solution of the above RH problem in the form

$$\tilde{P}^{(0)}(z) = \Psi_\alpha^{(0)}(z) \begin{pmatrix} 1 & \eta(z/s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left\{ \begin{array}{l} \begin{pmatrix} 1 & \frac{i}{2\pi} \ln z & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{(-1)^{\alpha+1}}{2\pi} i \ln z & 1 \end{pmatrix}, & \text{if } \alpha \in \mathbb{N} \cup \{0\} \\ \begin{pmatrix} 1 & \frac{e^{-\alpha\pi i}}{2\sin(\alpha\pi)} i & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & -\frac{i}{2\sin(\alpha\pi)} & 1 \end{pmatrix}, & \text{if } \alpha \notin \mathbb{Z} \end{array} \right\} \\ \times \begin{cases} J_{\Phi,1}^{-1}, & z \in D(0, \delta) \cap \text{I}, \\ I, & z \in D(0, \delta) \cap \text{II}, \\ J_{\Phi,2}^{-1}, & z \in D(0, \delta) \cap \text{III}, \\ J_{\Phi,1}^{-1} J_{\Phi,0}^{-1} J_{\Phi,5}^{-1} J_{\Phi,4}, & z \in D(0, \delta) \cap \text{IV}, \\ J_{\Phi,1}^{-1} J_{\Phi,0}^{-1} J_{\Phi,5}^{-1}, & z \in D(0, \delta) \cap \text{V}, \\ J_{\Phi,1}^{-1} J_{\Phi,0}^{-1}, & z \in D(0, \delta) \cap \text{VI}, \end{cases} \quad (6.4)$$

where $\eta(z)$ is a function to be determined later, $J_{\Phi,k}$, $k = 0, 1, \dots, 5$, denote the jump matrix J_Φ in (3.38) restricted on the contour $\Sigma_0^{(s)}, \Sigma_1^{(s)}, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5^{(s)}$, respectively, and the regions I–VI are shown in Figure 4. By choosing the principal branch cut for z^α and $\ln z$, it is straightforward to check that if $z < 0$,

$$\begin{pmatrix} 1 & \frac{i}{2\pi} \ln z & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{(-1)^{\alpha+1}}{2\pi} i \ln z & 1 \end{pmatrix}_+ J_{\Phi,2}^{-1} = \begin{pmatrix} 1 & \frac{i}{2\pi} \ln z & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{(-1)^{\alpha+1}}{2\pi} i \ln z & 1 \end{pmatrix}_- \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & (-1)^\alpha & 1 \end{pmatrix} J_{\Phi,2}^{-1} \\ = \begin{pmatrix} 1 & \frac{i}{2\pi} \ln z & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & \frac{(-1)^{\alpha+1}}{2\pi} i \ln z & 1 \end{pmatrix}_- J_{\Phi,1}^{-1} J_{\Phi,0}^{-1} J_{\Phi,5}^{-1} J_{\Phi,4} J_{\Phi,3} \quad (6.5)$$

for $\alpha \in \mathbb{N} \cup \{0\}$, and

$$\begin{pmatrix} 1 & \frac{e^{-\alpha\pi i}}{2\sin(\alpha\pi)} i & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & -\frac{i}{2\sin(\alpha\pi)} & 1 \end{pmatrix}_+ J_{\Phi,2}^{-1} = \begin{pmatrix} 1 & \frac{e^{-\alpha\pi i}}{2\sin(\alpha\pi)} i & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & -\frac{i}{2\sin(\alpha\pi)} & 1 \end{pmatrix}_- \begin{pmatrix} 1 & -e^{-2\alpha\pi i} & 0 \\ 0 & e^{-2\alpha\pi i} & 0 \\ 0 & e^{-\alpha\pi i} & 1 \end{pmatrix} J_{\Phi,2}^{-1} \\ = \begin{pmatrix} 1 & \frac{e^{-\alpha\pi i}}{2\sin(\alpha\pi)} i & 0 \\ 0 & z^{-\alpha} & 0 \\ 0 & -\frac{i}{2\sin(\alpha\pi)} & 1 \end{pmatrix}_- J_{\Phi,1}^{-1} J_{\Phi,0}^{-1} J_{\Phi,5}^{-1} J_{\Phi,4} J_{\Phi,3} \quad (6.6)$$

for $\alpha \notin \mathbb{Z}$. This, together with the fact that

$$J_{\Phi,1}^{-1} J_{\Phi,0}^{-1} J_{\Phi,5}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.7)$$

implies that $\tilde{P}^{(0)}(z)$ in (6.4) satisfies the jump condition (6.2) if the function $\eta(z)$ therein solves the following scalar RH problem.

RH problem 6.2.

- (1) $\eta(z)$ is defined and analytic in $\mathbb{C} \setminus [0, 1]$.
- (2) For $x \in (0, 1)$, η satisfies the jump condition

$$\eta_+(x) = \eta_-(x) - \gamma(sx)^\alpha, \quad (6.8)$$

where the orientation of $(0, 1)$ is taken from the left to the right.

- (3) As $z \rightarrow \infty$, we have $\eta(z) = \mathcal{O}(z^{-1})$.

By the Sokhotske-Plemelj formula, it is easily seen that

$$\eta(z) = -\frac{\gamma s^\alpha}{2\pi i} \int_0^1 \frac{x^\alpha}{x-z} dx. \quad (6.9)$$

Since $\eta(z/s) = \mathcal{O}(s^{\alpha+1})$ as $s \rightarrow 0^+$ for $z \in \partial D(0, \delta)$, we then obtain the matching condition (6.3) from (6.1), (6.4) and (3.7).

6.3 Final transformation

The final transformation is defined by

$$\tilde{R}(z) = \begin{cases} \Phi(z)\tilde{N}(z)^{-1}, & z \in \mathbb{C} \setminus D(0, \delta), \\ \Phi(z)\tilde{P}^{(0)}(z)^{-1}, & z \in D(0, \delta). \end{cases} \quad (6.10)$$

By the RH problems for Φ , Ψ_α and $\tilde{P}^{(0)}$, it follows that \tilde{R} is analytic for $z \in D(0, \delta) \setminus \{0, s\}$. To this end, we obtain from (6.9) and [55, Section 30] that as $z \rightarrow 0$,

$$\eta(z/s) = \begin{cases} \frac{\gamma}{2\sin(\alpha\pi)} e^{\mp\alpha\pi i} z^\alpha + \mathcal{O}(1), & \text{if } -1 < \alpha < 0 \text{ and } \pm\text{Im } z > 0, \\ \mathcal{O}(\ln z), & \text{if } \alpha = 0, \\ \mathcal{O}(1), & \text{if } \alpha > 0, \end{cases} \quad (6.11)$$

and as $z \rightarrow s$,

$$\eta(z/s) = \mathcal{O}(\ln(z-s)). \quad (6.12)$$

A combination of the above two estimates, (6.4), and items (4) and (5) of Proposition 3.5 implies that both 0 and s are removable singularities of \tilde{R} defined in (6.10). Thus, we arrive at the following RH problem for \tilde{R} .

RH problem 6.3.

- (1) $\tilde{R}(z)$ is defined and analytic in $\mathbb{C} \setminus D(0, \delta)$.

(2) For $z \in \partial D(0, \delta)$, $\tilde{R}(z)$ satisfies the jump condition

$$\tilde{R}_+(z) = \tilde{R}_-(z)J_{\tilde{R}}(z), \quad (6.13)$$

where

$$J_{\tilde{R}}(z) = \tilde{P}^{(0)}(z)\tilde{N}(z)^{-1}$$

and the orientation of $\partial D(0, \delta)$ is taken in a clockwise manner.

(3) As $z \rightarrow \infty$, we have $\tilde{R}(z) = I + \mathcal{O}(z^{-1})$.

In view of (6.3), it is readily seen that $J_{\tilde{R}}(z) = I + \mathcal{O}(s^{\alpha+1})$, as $s \rightarrow 0^+$, uniformly for the parameters γ and ρ in any compact subsets of $(0, 1]$ and \mathbb{R} . As a consequence, we have

$$\tilde{R}(z) = I + \mathcal{O}(s^{\alpha+1}), \quad \frac{d}{dz}\tilde{R}(z) = \mathcal{O}(s^{\alpha+1}), \quad s \rightarrow 0^+, \quad (6.14)$$

uniformly for $z \in \mathbb{C} \setminus \partial D(0, \delta)$.

7 Asymptotics of the functions $p_{i,k}$ and $q_{i,k}$

By Proposition 4.1, it is readily seen that there exists at least one solution to the system of equations (2.3) and (2.4). As an outcome of asymptotic analysis of the RH problem for Φ , we are able to obtain asymptotics of this special solution, which is also needed in the proof of Theorem 2.2. To that end, it is convenient to introduce the functions $\hat{p}_{i,k}$ and $\hat{q}_{i,k}$, $i = 0, 1$, $k = 1, 2, 3$, by

$$\begin{pmatrix} q_{i,1}(s) \\ q_{i,2}(s) \\ q_{i,3}(s) \end{pmatrix} = C_{\Psi} \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) C_N \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) \begin{pmatrix} \hat{q}_{i,1}(s) \\ \hat{q}_{i,2}(s) \\ \hat{q}_{i,3}(s) \end{pmatrix} \quad (7.1)$$

and

$$\begin{pmatrix} p_{i,1}(s) \\ p_{i,2}(s) \\ p_{i,3}(s) \end{pmatrix} = C_{\Psi}^{-t} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) C_N^{-t} \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) \begin{pmatrix} \hat{p}_{i,1}(s) \\ \hat{p}_{i,2}(s) \\ \hat{p}_{i,3}(s) \end{pmatrix}, \quad (7.2)$$

where the constant matrices C_{Ψ} and C_N are given in (3.18) and (5.30), respectively.

Proposition 7.1. *There exists one solution $\{p_{i,k}(s), q_{i,k}(s)\}_{i=0,1,k=1,2,3}$ to the system of equations (2.3) and (2.4) such that the following asymptotic behaviors hold. As $s \rightarrow +\infty$, we have, with $\{\hat{p}_{i,k}(s), \hat{q}_{i,k}(s)\}_{i=0,1,k=1,2,3}$ defined in (7.1) and (7.2) and $0 < \gamma < 1$,*

$$\begin{aligned} \hat{q}_{0,1}(s) &= \frac{(1-\gamma)^{1/3} e^{\rho^2/6} c(\alpha)}{\sqrt{2\pi}} \\ &\times \left(-\pi_6(\rho) + \pi_3(\rho) \left(\pi_3(\rho) + \frac{\rho}{3} \right) - \frac{\beta^2}{3} + \frac{2\beta i}{3\sqrt{3}} \cos(2\psi(s)) + \mathcal{O}(s^{-\frac{1}{3}}) \right), \end{aligned} \quad (7.3)$$

$$\begin{aligned} \hat{q}_{0,2}(s) &= \frac{(1-\gamma)^{1/3} e^{\rho^2/6} c(\alpha)}{\sqrt{2\pi}} \left(-\pi_3(\rho) - \frac{\rho}{3} \right. \\ &\left. + \left(\frac{\beta^2}{3} + \frac{\beta i}{\sqrt{3}} \left(\rho\pi_3(\rho) + \frac{2}{3}\rho^2 + \alpha - 1 \right) + \frac{2\beta i}{3\sqrt{3}} \cos(2\psi(s) - \frac{2\pi}{3}) \right) s^{-\frac{1}{3}} + \mathcal{O}(s^{-\frac{2}{3}}) \right), \end{aligned} \quad (7.4)$$

$$\begin{aligned} \hat{q}_{0,3}(s) &= \frac{(1-\gamma)^{1/3} e^{\rho^2/6} c(\alpha)}{\sqrt{2\pi}} \left(1 - \frac{\sqrt{3}}{3} \rho \beta i s^{-\frac{1}{3}} \right. \\ &\left. - \left(\frac{\beta^2}{2} \left(1 + \frac{\rho^2}{3} \right) + \frac{\beta i}{2\sqrt{3}} \left(\frac{\rho^2}{3} + 2\alpha - 1 \right) - \frac{2\beta i}{3\sqrt{3}} \cos(2\psi(s) + \frac{2\pi}{3}) \right) s^{-\frac{2}{3}} \right. \\ &\left. + \mathcal{O}(s^{-1}) \right), \end{aligned} \quad (7.5)$$

$$\begin{aligned}\hat{p}_{0,1}(s) &= \frac{\sqrt{2\pi}e^{-\rho^2/6}}{(1-\gamma)^{1/3}c(\alpha)} \left(1 + \frac{i\beta}{\sqrt{3}}\rho s^{-\frac{1}{3}} + \left(\frac{1}{2}\beta^2(3\rho\pi_3(\rho) + \frac{5}{3}\rho^2 + 1) \right. \right. \\ &\quad \left. \left. + \frac{i\beta}{2\sqrt{3}}(\rho\pi_3(\rho) + \rho^2 + 2\alpha + 1 - \frac{4}{3}\cos(2\psi(s) - \frac{2}{3}\pi)) \right) s^{-\frac{2}{3}} + \mathcal{O}(s^{-1}) \right),\end{aligned}\quad (7.6)$$

$$\begin{aligned}\hat{p}_{0,2}(s) &= \frac{\sqrt{2\pi}e^{-\rho^2/6}}{(1-\gamma)^{1/3}c(\alpha)} \left(\pi_3(\rho) + \rho \right. \\ &\quad \left. + \left(-\frac{1}{3}\beta^2 + \frac{i\beta}{3\sqrt{3}}(3\rho\pi_3(\rho) + 2\rho^2 - 3\alpha - 3 - 2\cos(2\psi(s) + \frac{2}{3}\pi)) \right) s^{-\frac{1}{3}} \right. \\ &\quad \left. + \mathcal{O}(s^{-\frac{2}{3}}) \right),\end{aligned}\quad (7.7)$$

$$\begin{aligned}\hat{p}_{0,3}(s) &= \frac{\sqrt{2\pi}e^{-\rho^2/6}}{(1-\gamma)^{1/3}c(\alpha)} \left(-\alpha + \pi_6(\rho) + \rho\pi_3(\rho) + \frac{\rho^2}{3} + \frac{\beta^2}{3} - \frac{2\beta i}{3\sqrt{3}}\cos(2\psi(s)) \right. \\ &\quad \left. + \mathcal{O}(s^{-\frac{1}{3}}) \right),\end{aligned}\quad (7.8)$$

and

$$\hat{q}_{1,1}(s) = \frac{2e^{-(\frac{1}{2}\theta_3(s) + \frac{2}{3}\beta\pi i)i}}{\sqrt{3}} s^{-\frac{\alpha-1}{3}} |\Gamma(1-\beta)| \left(\cos(\psi(s) + \frac{2\pi}{3}) + \mathcal{O}(s^{-\frac{1}{3}}) \right),\quad (7.9)$$

$$\hat{q}_{1,2}(s) = \frac{2e^{-(\frac{1}{2}\theta_3(s) + \frac{2}{3}\beta\pi i)i}}{\sqrt{3}} s^{-\frac{\alpha}{3}} |\Gamma(1-\beta)| \left(\cos(\psi(s)) + \mathcal{O}(s^{-\frac{1}{3}}) \right),\quad (7.10)$$

$$\hat{q}_{1,3}(s) = \frac{2e^{-(\frac{1}{2}\theta_3(s) + \frac{2}{3}\beta\pi i)i}}{\sqrt{3}} s^{-\frac{\alpha+1}{3}} |\Gamma(1-\beta)| \left(\cos(\psi(s) - \frac{2\pi}{3}) + \mathcal{O}(s^{-\frac{1}{3}}) \right),\quad (7.11)$$

$$\hat{p}_{1,1}(s) = \frac{\gamma e^{\frac{1}{2}\theta_3(s) - \frac{1}{3}\beta\pi i}}{\sqrt{3}\pi i} s^{\frac{\alpha-1}{3}} |\Gamma(1-\beta)| \left(\sin(\psi(s) - \frac{\pi}{3}) + \mathcal{O}(s^{-\frac{1}{3}}) \right),\quad (7.12)$$

$$\hat{p}_{1,2}(s) = -\frac{\gamma e^{\frac{1}{2}\theta_3(s) - \frac{1}{3}\beta\pi i}}{\sqrt{3}\pi i} s^{\frac{\alpha}{3}} |\Gamma(1-\beta)| \left(\sin(\psi(s)) + \mathcal{O}(s^{-\frac{1}{3}}) \right),\quad (7.13)$$

$$\hat{p}_{1,3}(s) = \frac{\gamma e^{\frac{1}{2}\theta_3(s) - \frac{1}{3}\beta\pi i}}{\sqrt{3}\pi i} s^{\frac{\alpha+1}{3}} |\Gamma(1-\beta)| \left(\sin(\psi(s) + \frac{\pi}{3}) + \mathcal{O}(s^{-\frac{1}{3}}) \right),\quad (7.14)$$

where

$$c(\alpha) := \begin{cases} 1, & \alpha = 0, \\ -\Gamma(\alpha), & \alpha \neq 0, \end{cases}\quad (7.15)$$

β , $\psi(s)$, $\pi_3(\rho)$ and $\theta_3(s)$ are given in (2.10), (2.11), (3.19) and (3.6), respectively. As $s \rightarrow 0^+$, we have

$$q_{0,k}(s) = \mathcal{O}(1), \quad p_{0,k}(s) = \mathcal{O}(1), \quad q_{1,k}(s) = \mathcal{O}(1), \quad p_{1,k}(s) = \mathcal{O}(s^\alpha), \quad k = 1, 2, 3 \quad (7.16)$$

for $0 < \gamma \leq 1$.

The rest of this section is devoted to the proof of Proposition 7.1.

Proofs of (7.3)–(7.8)

In view of (4.12) and (4.13), it is readily seen that the functions $p_{0,k}$ and $q_{0,k}$, $k = 1, 2, 3$, are related to the function $\Phi_0^{(0)}(s)$. From (3.40), (3.41) and (3.43), it follows that

$$\Phi_0^{(0)}(s) = \lim_{\substack{z \rightarrow 0, \\ z \in \text{II}}} \Phi(z) \begin{cases} \begin{pmatrix} 1 & -\frac{\gamma-1}{2\pi i} z^\alpha \ln z & 0 \\ 0 & z^\alpha & 0 \\ 0 & -\frac{e^{\alpha\pi i}}{2\pi i} z^\alpha \ln z & 1 \end{pmatrix}, & \text{if } \alpha \in \mathbb{N} \cup \{0\}, \\ \begin{pmatrix} 1 & -c_+ z^\alpha & 0 \\ 0 & z^\alpha & 0 \\ 0 & -\frac{z^\alpha}{2i \sin(\alpha\pi)} & 1 \end{pmatrix}, & \text{if } \alpha \notin \mathbb{Z}. \end{cases} \quad (7.17)$$

By tracing back the transformations $\Phi \mapsto T \mapsto S \mapsto R$ in (5.1), (5.10), (5.54), we observe that

$$\Phi(sz) = \frac{i}{\sqrt{3}} s^{-\frac{\alpha}{3}} C_\Psi \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) R(z) P^{(0)}(z) e^{\Theta(sz)}, \quad z \in D(0, \delta) \cap \text{II}. \quad (7.18)$$

This, together with (7.17), (3.42) and $P^{(0)}(z)$ in (5.36), implies that

$$\begin{aligned} \Phi_0^{(0)}(s) &= (1 - \gamma)^{\frac{1}{3}} \frac{i}{\sqrt{3}} C_\Psi \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) R(0) E_0(0) \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) \\ &\times \lim_{\substack{z \rightarrow 0, \\ \frac{\pi}{4} < \arg z < \frac{3\pi}{4}}} \Psi_\alpha(z) \begin{cases} \begin{pmatrix} \frac{1}{1-\gamma} & \frac{z^\alpha}{2\pi i} \ln z & 0 \\ 0 & z^\alpha & 0 \\ 0 & -\frac{e^{\alpha\pi i}}{2\pi i} z^\alpha \ln z & 1 \end{pmatrix}, & \text{if } \alpha \in \mathbb{N} \cup \{0\}, \\ \begin{pmatrix} \frac{1}{1-\gamma} & \frac{e^{-\alpha\pi i} z^\alpha}{2i \sin(\alpha\pi)} & 0 \\ 0 & z^\alpha & 0 \\ 0 & -\frac{z^\alpha}{2i \sin(\alpha\pi)} & 1 \end{pmatrix}, & \text{if } \alpha \notin \mathbb{Z}. \end{cases} \end{aligned} \quad (7.19)$$

To study the above limit, we recall from (3.10) that, if $\pi/4 < \arg z < 3\pi/4$, $\Psi_\alpha(z)$ admits an explicit representation in terms of $p_k(z)$, $k = 1, 2, 3$, defined in (3.16), and the functions $p_1(z)$ and $p_2(z)$ are entire functions. We then introduce the following auxiliary functions:

$$\mathcal{W}(z) := \mathcal{W}(z; \rho) = \mathcal{W}[p_1, p_2](z) = p_1(z)p_2'(z) - p_1'(z)p_2(z), \quad (7.20)$$

and

$$q(z) := p_3(z) - \frac{1}{2i \sin(\alpha\pi)} p_1(z) + \frac{e^{-\pi i \alpha}}{2i \sin(\alpha\pi)} p_2(z), \quad \alpha \notin \mathbb{Z}, \quad z \in \mathbb{C} \setminus i\mathbb{R}_-. \quad (7.21)$$

The proposition below will play an important role for us to determine the limit in (7.19). Since the proof is lengthy, it is postponed to Appendix B.

Proposition 7.2. *Let $p_k(z)$, $k = 1, 2, 3$, $\mathcal{W}(z)$ and $q(z)$ be functions defined in (3.16), (7.20) and (7.21), respectively. We have*

$$\mathcal{W}(z) = -\frac{(2\pi)^{3/2}}{e^{\rho^2/2}} \sum_{k=0}^{\infty} w_k z^k, \quad \alpha > -1, \quad (7.22)$$

for some constants w_k , $k = 0, 1, 2, \dots$, with

$$w_0 = \frac{1}{\Gamma(\alpha)}, \quad w_1 = \frac{\rho}{\Gamma(\alpha + 1)}; \quad (7.23)$$

if $\alpha = 0$,

$$p_1(z) - p_2(z) = \sum_{k=0}^{\infty} b_k z^{2+k}, \quad (7.24)$$

for some constants b_k , $k = 0, 1, 2, \dots$, with

$$b_0 = \pi i; \quad (7.25)$$

and

$$q(z) = \sum_{k=0}^{\infty} c_k z^{2+k-\alpha}, \quad \alpha \notin \mathbb{Z}, \quad (7.26)$$

for some constants c_k , $k = 0, 1, 2, \dots$, with

$$c_0 = -\frac{\pi}{\sin(\alpha\pi)\Gamma(3-\alpha)}. \quad (7.27)$$

Moreover, we have

$$\lim_{z \rightarrow 0} z^\alpha p_3''(z) = -\Gamma(\alpha), \quad \alpha \in \mathbb{N}. \quad (7.28)$$

With the aid of the above proposition, we are now ready to derive the asymptotics of $\hat{q}_{0,k}$ and $\hat{p}_{0,k}$, $k = 1, 2, 3$. To show (7.3)–(7.5), we first have from (3.10), (7.24) and (7.25) that, if $\alpha = 0$,

$$\lim_{\substack{z \rightarrow 0, \\ \frac{\pi}{4} < \arg z < \frac{3\pi}{4}}} \frac{\sqrt{2\pi}}{e^{\rho^2/6}} \Psi_0(z) \begin{pmatrix} \frac{1}{1-\gamma} & \frac{1}{2\pi i} \ln z & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2\pi i} \ln z & 1 \end{pmatrix} \begin{pmatrix} \gamma - 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_1(0) - p_2(0) \\ p_1'(0) - p_2'(0) \\ p_1''(0) - p_2''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2\pi i \end{pmatrix}. \quad (7.29)$$

If $\alpha \in \mathbb{N}$, we note from the fact $p_1(z)$ and $p_2(z)$ are entire functions that

$$\lim_{z \rightarrow 0} (\ln z) z^\alpha p_k(z) = 0, \quad k = 1, 2.$$

Thus, it is readily seen from (3.10) and (7.28) that

$$\begin{aligned} \lim_{\substack{z \rightarrow 0, \\ \frac{\pi}{4} < \arg z < \frac{3\pi}{4}}} \frac{\sqrt{2\pi}}{e^{\rho^2/6}} \Psi_\alpha(z) \begin{pmatrix} \frac{1}{1-\gamma} & \frac{z^\alpha}{2\pi i} \ln z & 0 \\ 0 & z^\alpha & 0 \\ 0 & -\frac{e^{\alpha\pi i}}{2\pi i} z^\alpha \ln z & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ = \lim_{\substack{z \rightarrow 0, \\ \frac{\pi}{4} < \arg z < \frac{3\pi}{4}}} \begin{pmatrix} z^\alpha p_3(z) \\ z^\alpha p_3'(z) \\ z^\alpha p_3''(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\Gamma(\alpha) \end{pmatrix}. \end{aligned} \quad (7.30)$$

If $\alpha \notin \mathbb{Z}$, we obtain from (7.26) and (7.27) that

$$\begin{aligned} \lim_{\substack{z \rightarrow 0, \\ \frac{\pi}{4} < \arg z < \frac{3\pi}{4}}} \frac{\sqrt{2\pi}}{e^{\rho^2/6}} \Psi_\alpha(z) \begin{pmatrix} \frac{1}{1-\gamma} & \frac{e^{-\alpha\pi i} z^\alpha}{2i \sin(\alpha\pi)} & 0 \\ 0 & z^\alpha & 0 \\ 0 & -\frac{z^\alpha}{2i \sin(\alpha\pi)} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ = \lim_{\substack{z \rightarrow 0, \\ \frac{\pi}{4} < \arg z < \frac{3\pi}{4}}} \begin{pmatrix} z^\alpha q(z) \\ z^\alpha q'(z) \\ z^\alpha q''(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\Gamma(\alpha) \end{pmatrix}. \end{aligned} \quad (7.31)$$

Therefore, a combination of the above three formulas, (4.12) and (7.19) yields

$$\begin{pmatrix} q_{0,1}(s) \\ q_{0,2}(s) \\ q_{0,3}(s) \end{pmatrix} = (1-\gamma)^{\frac{1}{3}} \frac{ie^{\rho^2/6}}{\sqrt{6\pi}} C_{\Psi} \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) R(0)E_0(0) \operatorname{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) \begin{pmatrix} 0 \\ 0 \\ c(\alpha) \end{pmatrix}, \quad (7.32)$$

where the function $c(\alpha)$ is defined in (7.15).

We next substitute (7.1) into (7.32) and obtain from (5.76) that

$$\begin{pmatrix} \hat{q}_{0,1}(s) \\ \hat{q}_{0,2}(s) \\ \hat{q}_{0,3}(s) \end{pmatrix} = (1-\gamma)^{\frac{1}{3}} \frac{ie^{\rho^2/6}}{\sqrt{6\pi}} \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) \widehat{R}(0)C_N^{-1}E_0(0) \operatorname{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) \begin{pmatrix} 0 \\ 0 \\ c(\alpha) \end{pmatrix}. \quad (7.33)$$

From $E_0(0)$ in (5.42), it is easily seen that

$$\begin{aligned} & \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) \widehat{R}(0)C_N^{-1}E_0(0) \operatorname{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) \begin{pmatrix} 0 \\ 0 \\ c(\alpha) \end{pmatrix} \\ &= -\sqrt{3}i \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) \left(C_N^t + \widehat{R}_1(0)C_N^t s^{-\frac{1}{3}} + \widehat{R}_2(0)C_N^t s^{-\frac{2}{3}} + \mathcal{O}(s^{-1})\right) \\ & \quad \times \operatorname{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) \begin{pmatrix} 0 \\ 0 \\ c(\alpha) \end{pmatrix} \\ &= -\sqrt{3}i c(\alpha) \begin{pmatrix} \left(\widehat{R}_1(0)\right)_{13} s^{\frac{1}{3}} + \left(\widehat{R}_2(0)\right)_{13} + \mathcal{O}(s^{-\frac{1}{3}}) \\ \left(\widehat{R}_1(0)\right)_{23} + \left(\widehat{R}_2(0)\right)_{23} s^{-\frac{1}{3}} + \mathcal{O}(s^{-\frac{2}{3}}) \\ 1 + \left(\widehat{R}_1(0)\right)_{33} s^{-\frac{1}{3}} + \left(\widehat{R}_2(0)\right)_{33} s^{-\frac{2}{3}} + \mathcal{O}(s^{-1}) \end{pmatrix}. \end{aligned} \quad (7.34)$$

Inserting the above formula into (7.33), we finally arrive at (7.3)–(7.5) from $\widehat{R}_1(0)$ in (5.79), the first row and the last column of $\widehat{R}_2(0)$ in (5.94)–(5.98) and the fact that

$$\vartheta(s) = \psi(s) + \mathcal{O}(s^{-\frac{1}{3}}), \quad (7.35)$$

where $\vartheta(s)$ and $\psi(s)$ are defined in (5.83) and (2.11), respectively.

The asymptotics of $\hat{p}_{0,k}(s)$, $k = 1, 2, 3$, in (7.6)–(7.8) can be proved in a similar manner. From the definitions of $p_{0,k}(s)$ in (4.13), we need to study $\Phi_0^{(0)}(s)^{-t}$, which, by (7.19), is

$$\begin{aligned} \Phi_0^{(0)}(s)^{-t} &= -\frac{\sqrt{3}i}{(1-\gamma)^{1/3}} C_{\Psi}^{-t} \operatorname{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) R(0)^{-t} E_0(0)^{-t} \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) \\ & \quad \times \lim_{\substack{z \rightarrow 0, \\ \frac{\pi}{4} < \arg z < \frac{3\pi}{4}}} \Psi_{\alpha}(z)^{-t} \begin{cases} \begin{pmatrix} 1-\gamma & 0 & 0 \\ -\frac{1-\gamma}{2\pi i} \ln z & z^{-\alpha} & \frac{e^{\alpha\pi i}}{2\pi i} \ln z \\ 0 & 0 & 1 \end{pmatrix}, & \text{if } \alpha \in \mathbb{N} \cup \{0\}, \\ \begin{pmatrix} 1-\gamma & 0 & 0 \\ -\frac{(1-\gamma)e^{-\alpha\pi i}}{2i \sin(\alpha\pi)} & z^{-\alpha} & \frac{1}{2i \sin(\alpha\pi)} \\ 0 & 0 & 1 \end{pmatrix}, & \text{if } \alpha \notin \mathbb{Z}. \end{cases} \end{aligned} \quad (7.36)$$

Recall that $\det \Psi_\alpha(z) = z^{-\alpha}$, we then obtain from (3.10) and the definition of $\mathcal{W}(z)$ in (7.20) that

$$\begin{aligned} \Psi_\alpha(z)^{-t} \begin{pmatrix} 0 \\ z^{-\alpha} \\ 0 \end{pmatrix} &= \frac{e^{\rho^2/3}}{2\pi} \begin{pmatrix} p_1'(z)p_2''(z) - p_1''(z)p_2'(z) \\ -(p_1(z)p_2''(z) - p_1''(z)p_2(z)) \\ p_1(z)p_2'(z) - p_1'(z)p_2(z) \end{pmatrix} \\ &= \frac{e^{\rho^2/3}}{2\pi} \begin{pmatrix} p_1'(z)p_2''(z) - p_1''(z)p_2'(z) \\ -\mathcal{W}'(z) \\ \mathcal{W}(z) \end{pmatrix}. \end{aligned} \quad (7.37)$$

Using (7.22), (7.23), (B.17) and (B.18), we further have

$$\lim_{z \rightarrow 0} \Psi_\alpha(z)^{-t} \begin{pmatrix} 0 \\ z^{-\alpha} \\ 0 \end{pmatrix} = \frac{\sqrt{2\pi}e^{-\rho^2/6}}{\Gamma(\alpha+1)} \begin{pmatrix} 1 \\ \rho \\ -\alpha \end{pmatrix}. \quad (7.38)$$

A combination of (4.13), (7.36) and the above formula gives us that

$$\begin{aligned} \begin{pmatrix} p_{0,1}(s) \\ p_{0,2}(s) \\ p_{0,3}(s) \end{pmatrix} &= \frac{\sqrt{6\pi}e^{-\rho^2/6}}{i(1-\gamma)^{1/3}c(\alpha)} C_\Psi^{-t} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) R(0)^{-t} E_0(0)^{-t} \\ &\quad \times \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) \begin{pmatrix} 1 \\ \rho \\ -\alpha \end{pmatrix}, \end{aligned} \quad (7.39)$$

where $c(\alpha)$ is given in (7.15). By inserting (7.2) into the above formula, we see from (5.76) that

$$\begin{aligned} \begin{pmatrix} \hat{p}_{0,1}(s) \\ \hat{p}_{0,2}(s) \\ \hat{p}_{0,3}(s) \end{pmatrix} &= \frac{\sqrt{6\pi}e^{-\rho^2/6}}{i(1-\gamma)^{1/3}c(\alpha)} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) \hat{R}(0)^{-t} C_N^t E_0(0)^{-t} \\ &\quad \times \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) \begin{pmatrix} 1 \\ \rho \\ -\alpha \end{pmatrix}. \end{aligned} \quad (7.40)$$

Since

$$\begin{aligned} \hat{R}(0)^{-t} &= \left(I + \frac{\hat{R}_1(0)}{s^{1/3}} + \frac{\hat{R}_2(0)}{s^{2/3}} + \mathcal{O}(s^{-1}) \right)^{-t} \\ &= I - \frac{\hat{R}_1(0)^t}{s^{1/3}} + \frac{(\hat{R}_1(0)^2 - \hat{R}_2(0))^t}{s^{2/3}} + \mathcal{O}(s^{-1}), \quad s \rightarrow +\infty, \end{aligned}$$

it follows from the explicit formula of $E_0(0)$ in (5.42) that

$$\begin{aligned} &\text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) \hat{R}(0)^{-t} C_N^t E_0(0)^{-t} \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) \\ &= \frac{i}{\sqrt{3}} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) \left(C_N^{-1} - \frac{\hat{R}_1(0)^t C_N^{-1}}{s^{1/3}} + \frac{(\hat{R}_1(0)^2 - \hat{R}_2(0))^t C_N^{-1}}{s^{2/3}} + \mathcal{O}(s^{-1}) \right) \\ &\quad \times \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) = \frac{i}{\sqrt{3}} \mathcal{P}(s), \end{aligned} \quad (7.41)$$

where

$$\begin{aligned}
(\mathcal{P}(s))_{11} &= 1 - (\widehat{R}_1(0))_{11} s^{-\frac{1}{3}} + (\widehat{R}_1(0)^2 - \widehat{R}_2(0))_{11} s^{-\frac{2}{3}} + \mathcal{O}(s^{-1}), \\
(\mathcal{P}(s))_{12} &= \sqrt{3}\beta i s^{-\frac{1}{3}} - (\sqrt{3}\beta i (\widehat{R}_1(0))_{11} + (\widehat{R}_1(0))_{21}) s^{-\frac{2}{3}} + \mathcal{O}(s^{-1}), \\
(\mathcal{P}(s))_{13} &= -\left(\frac{3}{2}\beta^2 + \frac{\sqrt{3}\beta i}{2}\right) s^{-\frac{2}{3}} + \mathcal{O}(s^{-1}), \\
(\mathcal{P}(s))_{21} &= -(\widehat{R}_1(0))_{12} + (\widehat{R}_1(0)^2 - \widehat{R}_2(0))_{12} s^{-\frac{1}{3}} + \mathcal{O}(s^{-\frac{2}{3}}), \\
(\mathcal{P}(s))_{22} &= 1 - (\sqrt{3}\beta i (\widehat{R}_1(0))_{12} + (\widehat{R}_1(0))_{22}) s^{-\frac{1}{3}} + \mathcal{O}(s^{-\frac{2}{3}}), \\
(\mathcal{P}(s))_{23} &= \sqrt{3}\beta i s^{-\frac{1}{3}} + \mathcal{O}(s^{-\frac{2}{3}}), \\
(\mathcal{P}(s))_{31} &= -(\widehat{R}_1(0))_{13} s^{\frac{1}{3}} + (\widehat{R}_1(0)^2 - \widehat{R}_2(0))_{13} + \mathcal{O}(s^{-\frac{1}{3}}), \\
(\mathcal{P}(s))_{32} &= -\sqrt{3}\beta i (\widehat{R}_1(0))_{13} - (\widehat{R}_1(0))_{23} + \mathcal{O}(s^{-\frac{1}{3}}), \\
(\mathcal{P}(s))_{33} &= 1 + \mathcal{O}(s^{-\frac{1}{3}}).
\end{aligned}$$

By further evaluating each entry in \mathcal{P} with the aid of explicit formulas of $\widehat{R}_1(0)$, $\widehat{R}_1(0)^2$ and $(\widehat{R}_2(0))_{13}$ in (5.79), (5.93) and (5.96), we then obtain (7.6)–(7.8) from (7.40) and (7.41).

Proofs of (7.9)–(7.14)

From the definitions of $p_{1,k}$ and $q_{1,k}$, $k = 1, 2, 3$, in (4.14), one needs to study $\Phi_0^{(1)}(s)$ which appears in the local behavior of $\Phi(z)$ as $z \rightarrow s$. By (3.45), it follows that

$$\Phi_0^{(1)}(s) = \lim_{\substack{z \rightarrow s, \\ z \in \Pi}} \Phi(z) \begin{pmatrix} 1 & \frac{\gamma}{2\pi i} \ln(z-s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7.42)$$

To evaluate the above limit, we trace back the transformations $\Phi \mapsto T \mapsto S \mapsto R$ in (5.1), (5.10) (5.54), and obtain

$$\Phi(sz) = \frac{i}{\sqrt{3}} s^{-\frac{\alpha}{3}} C_\Psi \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) R(z) P^{(1)}(z) e^{\Theta(sz)}, \quad z \in D(1, \delta) \cap \Pi. \quad (7.43)$$

In view of $P^{(1)}(z)$ in (5.48), we see from (7.42), (7.43) and the fact $\theta_1(z) + \theta_2(z) + \theta_3(z) = 0$ that

$$\begin{aligned}
\Phi_0^{(1)}(s) &= \frac{is^{-\frac{\alpha}{3}}}{\sqrt{3}} C_\Psi \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) R(1) E_1(1) \\
&\quad \times \lim_{\substack{z \rightarrow 1, \\ z \in \Pi}} \left[e^{-\frac{1}{2}\theta_3(sz)} \operatorname{diag}\left(\Phi^{(\text{CHF})}(s^{\frac{2}{3}} f(z)) e^{-\frac{\beta}{2}\pi i \sigma_3}, e^{\frac{3}{2}\theta_3(sz)}\right) \begin{pmatrix} 1 & \frac{\gamma}{2\pi i} \ln(sz-s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right],
\end{aligned}$$

where we have made use of fact that $R(z)$ and $E_1(z)$ are analytic at $z = 1$. Since $f(z)$ is analytic near $z = 1$ and $f(1) = 0$ (see (5.47)), we have from the local behavior of $\Phi^{(\text{CHF})}(z)$ near $z = 0$ in (A.4) that

$$\begin{aligned}
\Phi_0^{(1)}(s) &= \frac{is^{-\frac{\alpha}{3}}}{\sqrt{3}} e^{-\frac{1}{2}\theta_3(s)} C_\Psi \operatorname{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) R(1) E_1(1) \operatorname{diag}(\Upsilon_0, 1) \\
&\quad \times \begin{pmatrix} 1 & \frac{\gamma}{2\pi i} \left(\ln s - \ln(e^{-\frac{\pi i}{2}} s^{\frac{2}{3}} f'(1))\right) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{3}{2}\theta_3(s)} \end{pmatrix}, \quad (7.44)
\end{aligned}$$

where Υ_0 is the constant matrix given in (A.5). We are now ready to derive the asymptotics of $\hat{q}_{1,k}$ and $\hat{p}_{1,k}$, $k = 1, 2, 3$.

To show (7.9)–(7.11), we see from (7.44), the definitions of $q_{1,k}$ in (4.14), $R(z)$ in (5.76) and Υ_0 in (A.5) that

$$\begin{aligned} \begin{pmatrix} q_{1,1}(s) \\ q_{1,2}(s) \\ q_{1,3}(s) \end{pmatrix} &= \Phi_0^{(1)}(s) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{i}{\sqrt{3}} s^{-\frac{\alpha}{3}} e^{-\frac{1}{2}\theta_3(s)} C_\Psi \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) C_N \widehat{R}(1) C_N^{-1} E_1(1) \begin{pmatrix} \Gamma(1-\beta)e^{-\beta\pi i} \\ \Gamma(1+\beta) \\ 0 \end{pmatrix}. \end{aligned} \quad (7.45)$$

On account of $E_1(1)$ in (5.51), we note that

$$E_1(1) \begin{pmatrix} \Gamma(1-\beta)e^{-\beta\pi i} \\ \Gamma(1+\beta) \\ 0 \end{pmatrix} = e^{-\frac{2\beta\pi i}{3}} C_N \begin{pmatrix} \omega\Gamma(1-\beta)c_1(s) + \omega^2\Gamma(1+\beta)c_1(s)^{-1} \\ \Gamma(1-\beta)c_1(s) + \Gamma(1+\beta)c_1(s)^{-1} \\ \omega^2\Gamma(1-\beta)c_1(s) + \omega\Gamma(1+\beta)c_1(s)^{-1} \end{pmatrix}, \quad (7.46)$$

with $c_1(s)$ given in (5.53). As $\text{Re } \beta = 0$ and $\text{Re}(\theta_1(s) - \theta_2(s)) = 0$, each entry in the above column matrix is a sum of two complex conjugates. This observation gives us

$$E_1(1) \begin{pmatrix} \Gamma(1-\beta)e^{-\beta\pi i} \\ \Gamma(1+\beta) \\ 0 \end{pmatrix} = 2e^{-\frac{2}{3}\beta\pi i} C_N \text{Re} \begin{pmatrix} \omega\Gamma(1-\beta)c_1(s) \\ \Gamma(1-\beta)c_1(s) \\ \omega^2\Gamma(1-\beta)c_1(s) \end{pmatrix}. \quad (7.47)$$

Inserting the above formula into (7.45), it is readily seen from (7.1) that

$$\begin{pmatrix} \hat{q}_{1,1}(s) \\ \hat{q}_{1,2}(s) \\ \hat{q}_{1,3}(s) \end{pmatrix} = \frac{i2e^{-(\frac{1}{2}\theta_3(s) + \frac{2}{3}\beta\pi i)}}{\sqrt{3}} s^{-\frac{\alpha}{3}} \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) \widehat{R}(1) \text{Re} \begin{pmatrix} \omega\Gamma(1-\beta)c_1(s) \\ \Gamma(1-\beta)c_1(s) \\ \omega^2\Gamma(1-\beta)c_1(s) \end{pmatrix}. \quad (7.48)$$

The asymptotic formulas (7.9)–(7.11) then follow from the above formula, (5.82), (7.35) and asymptotics of $\widehat{R}(z)$ in (5.76).

The asymptotics of $\hat{p}_{1,k}(s)$, $k = 1, 2, 3$, in (7.12)–(7.14) can be derived in a similar manner. We have from (4.14), (5.76) and (7.44) that

$$\begin{aligned} \begin{pmatrix} p_{1,1}(s) \\ p_{1,2}(s) \\ p_{1,3}(s) \end{pmatrix} &= -\frac{\gamma}{2\pi i} \Phi_0^{(1)}(s)^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{\sqrt{3}\gamma e^{\theta_3(s)/2}}{2\pi} s^{\frac{\alpha}{3}} C_\Psi^{-t} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) C_N^{-t} \widehat{R}(1)^{-t} C_N^t E_1(1)^{-t} \text{diag} \left(\Upsilon_0^{-t}, 1 \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \quad (7.49)$$

Again, using $E_1(1)$ and Υ_0 in (5.51) and (A.5), it follows that

$$\begin{aligned} E_1(1)^{-t} \text{diag} \left(\Upsilon_0^{-t}, 1 \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ = \frac{1}{3} e^{-\frac{1}{3}\beta\pi i} C_N^{-t} \begin{pmatrix} -e^{-\frac{\pi}{3}i}\Gamma(1-\beta)c_1(s) + e^{\frac{\pi}{3}i}\Gamma(1+\beta)c_1(s)^{-1} \\ \Gamma(1-\beta)c_1(s) - \Gamma(1+\beta)c_1(s)^{-1} \\ -e^{\frac{\pi}{3}i}\Gamma(1-\beta)c_1(s) + e^{-\frac{\pi}{3}i}\Gamma(1+\beta)c_1(s)^{-1} \end{pmatrix}. \end{aligned} \quad (7.50)$$

Inserting (7.50) into (7.49), we obtain from (7.2) that

$$\begin{pmatrix} \hat{p}_{1,1}(s) \\ \hat{p}_{1,2}(s) \\ \hat{p}_{1,3}(s) \end{pmatrix} = -\frac{\gamma e^{\frac{1}{2}\theta_3(s) - \frac{1}{3}\beta\pi i}}{\sqrt{3}\pi i} s^{\frac{\alpha}{3}} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) \widehat{R}(1)^{-t} \text{Im} \begin{pmatrix} -e^{-\frac{\pi}{3}i}\Gamma(1-\beta)c_1(s) \\ \Gamma(1-\beta)c_1(s) \\ -e^{\frac{\pi}{3}i}\Gamma(1-\beta)c_1(s) \end{pmatrix}. \quad (7.51)$$

A combination of (5.82), (7.35), (5.76) and the above formula gives us (7.12)–(7.14).

Proof of (7.16)

By (7.42), we obtain from (4.14), (6.10), (6.14) and (6.4) that, as $s \rightarrow 0^+$,

$$\begin{aligned} \begin{pmatrix} q_{1,1}(s) \\ q_{1,2}(s) \\ q_{1,3}(s) \end{pmatrix} &= \Phi_0^{(1)}(s) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \lim_{\substack{z \rightarrow s, \\ z \in \mathbb{H}}} \Phi(z) \begin{pmatrix} 1 & \frac{\gamma}{2\pi i} \ln(z-s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \lim_{\substack{z \rightarrow s, \\ z \in \mathbb{H}}} \widetilde{R}(z) \widetilde{P}^{(0)}(z) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (I + \mathcal{O}(s^{\alpha+1})) (\Psi_\alpha^{(0)}(0) + \mathcal{O}(s)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{e^{\rho^2/6}}{\sqrt{2\pi}} (I + \mathcal{O}(s^{\alpha+1})) \begin{pmatrix} p_2(0) + \mathcal{O}(s) \\ p_2'(0) + \mathcal{O}(s) \\ p_2''(0) + \mathcal{O}(s) \end{pmatrix}, \end{aligned} \quad (7.52)$$

where $p_2(z)$ defined in (3.16) is an entire function. Hence, it is readily seen that, as $s \rightarrow 0^+$,

$$q_{1,k}(s) = \mathcal{O}(1), \quad k = 1, 2, 3. \quad (7.53)$$

The other estimates in (7.16) can be proved in a similar manner, and we omit the details here. This completes the proof of Proposition 7.1. \square

8 Proofs of main results

8.1 Proof of Theorem 2.1

By (3.51) and (4.29), it follows that

$$\frac{d}{ds} F(s; \alpha, \gamma, \rho) = H(s), \quad s > 0. \quad (8.1)$$

We then obtain the integral representation of F in (2.7) after integrating (8.1).

To show the asymptotics of $H(s)$, we make use of (4.29), which establishes a relation between $H(s)$ and the function $\Phi_1^{(1)}(s)$ in (3.45). It is then easily seen from (3.44) that

$$\begin{aligned} H(s) &= -\frac{\gamma}{2\pi i} \left(\Phi_1^{(1)}(s) \right)_{21} \\ &= -\frac{\gamma}{2\pi i} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \Phi_0^{(1)}(s)^{-1} \lim_{\substack{z \rightarrow 1, \\ z \in \mathbb{H}}} \left[\Phi(sz) \begin{pmatrix} 1 & \frac{\gamma}{2\pi i} \ln(sz-s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (8.2)$$

where $'$ denotes the derivative with respect to z .

The small- s asymptotics of $H(s)$ in (2.8) then follows directly from (6.4), (6.10), (6.14) and the above formula.

The large- s asymptotics of $H(s)$ is more involved. We start with the calculation of the product outside the limit in (8.2). From (7.44), we have

$$\begin{aligned} & (0 \ 1 \ 0) \Phi_0^{(1)}(s)^{-1} \\ &= -\sqrt{3}ie^{\frac{1}{2}\theta_3(s)}s^{\frac{\alpha}{3}}(0 \ 1 \ 0) \operatorname{diag}(\Upsilon_0^{-1}, 1) E_1(1)^{-1}R(1)^{-1} \operatorname{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) C_{\Psi}^{-1}. \end{aligned} \quad (8.3)$$

For the limit in (8.2), we observe from (5.48) and (7.43) that

$$\begin{aligned} & -\sqrt{3}is^{\frac{\alpha}{3}}C_{\Psi}^{-1} \operatorname{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) \lim_{\substack{z \rightarrow 1, \\ z \in \mathbb{H}}} \left[\Phi(sz) \begin{pmatrix} 1 & \frac{\gamma}{2\pi i} \ln(sz-s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \\ &= \lim_{\substack{z \rightarrow 1, \\ z \in \mathbb{H}}} \left[R(z)E_1(z)e^{-\frac{1}{2}\theta_3(sz)} \operatorname{diag}\left(\Phi^{(\text{CHF})}(s^{\frac{2}{3}}f(z))e^{-\frac{\beta}{2}\pi i\sigma_3}, 1\right) \right. \\ & \quad \times \begin{pmatrix} 1 & \frac{\gamma}{2\pi i} \ln(sz-s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{3}{2}\theta_3(s)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \left. \right]' \\ &= \lim_{\substack{z \rightarrow 1, \\ z \in \mathbb{H}}} \left[R(z)E_1(z)e^{-\frac{1}{2}\theta_3(sz)} \operatorname{diag}\left(\Phi^{(\text{CHF})}(s^{\frac{2}{3}}f(z))e^{-\frac{\beta}{2}\pi i\sigma_3}, 1\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]'. \end{aligned} \quad (8.4)$$

By (5.47), we can calculate the above limit with the aid of (A.4) and the result reads

$$\begin{aligned} & e^{-\frac{1}{2}\theta_3(s)}(R'(1)E_1(1) \operatorname{diag}(\Upsilon_0, 1) + R(1)E_1'(1) \operatorname{diag}(\Upsilon_0, 1) \\ & \quad + s^{\frac{2}{3}}f'(1)R(1)E_1(1) \operatorname{diag}(\Upsilon_0\Upsilon_1, 0) - \frac{s\theta_3'(s)}{2}R(1)E_1(1) \operatorname{diag}(\Upsilon_0, 1)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (8.5)$$

A combination of (8.2)–(8.5) gives us

$$\begin{aligned} H(s) &= -\frac{\gamma}{2\pi si} \left(\operatorname{diag}(\Upsilon_0^{-1}, 1) E_1(1)^{-1}R(1)^{-1}R'(1)E_1(1) \operatorname{diag}(\Upsilon_0, 1) \right. \\ & \quad \left. + \operatorname{diag}(\Upsilon_0^{-1}, 1) E_1(1)^{-1}E_1'(1) \operatorname{diag}(\Upsilon_0, 1) + s^{\frac{2}{3}}f'(1) \operatorname{diag}(\Upsilon_1, 0) - \frac{s\theta_3'(s)}{2}I \right)_{21}. \end{aligned} \quad (8.6)$$

Next, we estimate the quantities contained in the above round bracket term by term. We note from (5.75) that $R(1) = I + \mathcal{O}(s^{-1/3})$ and $R'(1) = \mathcal{O}(s^{-1/3})$. As $\operatorname{Re} \beta = 0$ (see (2.10)), it then follows from (5.51) that $E_1(1) = \mathcal{O}(1)$. Combining all these estimates, we obtain

$$\left(\operatorname{diag}(\Upsilon_0^{-1}, 1) E_1(1)^{-1}R(1)^{-1}R'(1)E_1(1) \operatorname{diag}(\Upsilon_0, 1) \right)_{21} = \mathcal{O}(s^{-\frac{1}{3}}). \quad (8.7)$$

For the second term in the bracket, we have from the explicit expression of Υ_0 in (A.5) that

$$\begin{aligned} & \left(\operatorname{diag}(\Upsilon_0^{-1}, 1) E_1(1)^{-1}E_1'(1) \operatorname{diag}(\Upsilon_0, 1) \right)_{21} \\ &= \Gamma(1+\beta)\Gamma(1-\beta)e^{-\beta\pi i} \left((E_1(1)^{-1}E_1'(1))_{22} - (E_1(1)^{-1}E_1'(1))_{11} \right) \\ & \quad - \Gamma(1+\beta)^2 (E_1(1)^{-1}E_1'(1))_{12} + \Gamma(1-\beta)^2 e^{-2\beta\pi i} (E_1(1)^{-1}E_1'(1))_{21}. \end{aligned} \quad (8.8)$$

In view of (5.47), (5.82), (5.84) and $E_1(1)^{-1}E_1'(1)$ in (5.52), it follows from straightforward calculations that

$$\begin{aligned} & \Gamma(1+\beta)\Gamma(1-\beta)e^{-\beta\pi i} \left((E_1(1)^{-1}E_1'(1))_{22} - (E_1(1)^{-1}E_1'(1))_{11} \right) \\ &= -\frac{\beta^2\pi}{\sin(\beta\pi)} e^{-\beta\pi i} \left(\frac{f''(1)}{f'(1)} + 1 \right) = -\frac{\beta^2\pi}{\sin(\beta\pi)} e^{-\beta\pi i} \left(\frac{2}{3} + \mathcal{O}(s^{-\frac{1}{3}}) \right), \end{aligned} \quad (8.9)$$

and

$$\begin{aligned}
& -\Gamma(1+\beta)^2 (E_1(1)^{-1}E_1'(1))_{12} + \Gamma(1-\beta)^2 e^{-2\beta\pi i} (E_1(1)^{-1}E_1'(1))_{21} \\
& = -\frac{\sqrt{3}i}{9} e^{-\beta\pi i} \left(\Gamma(1+\beta)^2 c_1(s)^{-2} + \Gamma(1-\beta)^2 c_1(s)^2 \right) \\
& = -\frac{2\sqrt{3}i}{9} \frac{\beta\pi}{\sin(\beta\pi)} e^{-\beta\pi i} \cos(2\vartheta(s)), \tag{8.10}
\end{aligned}$$

where $\vartheta(s)$ is given in (5.83). For the last two terms in the bracket, we see from (5.47) and (A.6) that

$$\left(s^{\frac{2}{3}} f'(1) \operatorname{diag}(\Upsilon_1, 0) \right)_{21} = \frac{\beta\pi i e^{-\beta\pi i}}{\sin(\beta\pi)} \left(\sqrt{3}s^{\frac{2}{3}} - \frac{\sqrt{3}}{3} \rho s^{\frac{1}{3}} \right), \tag{8.11}$$

and obviously

$$\left(\frac{s\theta_3'(s)}{2} I \right)_{21} = 0. \tag{8.12}$$

On account of the fact that $\gamma = -2i \sin(\beta\pi) e^{\beta\pi i}$ (see (2.10)), we finally obtain the desired large- s asymptotics of $H(s)$ in (2.9) by combining (8.6)–(8.12) with (7.35).

This completes the proof of Theorem 2.1. \square

8.2 Proof of Theorem 2.2

The proof is based on the differential identities established in Proposition 4.3. First, we have from (2.7), (4.32) and the small- s behavior of $H(s)$ in (2.8) that

$$F(s; \alpha, \gamma, \rho) = \int_0^s H(\tau) d\tau = \int_0^s \left(\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(\tau) q'_{i,k}(\tau) - H(\tau) \right) d\tau + sH(s). \tag{8.13}$$

By an easy calculation, one can see that the differential identity with respect to γ in (4.33) still holds if we replace γ by β . Integrating both sides of (4.33) with respect to s , we have

$$\frac{\partial}{\partial\beta} \int_0^s \left(\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(\tau) q'_{i,k}(\tau) - H(\tau) \right) d\tau = \sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial\beta} q_{i,k}(s). \tag{8.14}$$

Here, we have used the fact that

$$\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(0) \frac{\partial}{\partial\beta} q_{i,k}(0) = 0,$$

which follows from the small- s asymptotics of $p_{i,k}(s)$, $q_{i,k}(s)$, $i = 0, 1$, $k = 1, 2, 3$, given in (7.16) and (7.52).

We next rewrite the second integral in (8.13) by integrating both sides of (8.14) with respect to β . Since $\gamma = 1 - e^{2\beta\pi i}$, it follows that $\beta = 0$ if $\gamma = 0$. This, together with (4.14) and (4.29), implies that $p_{1,k}(s) = 0$ and $H(s) = 0$ for $\beta = 0$. In addition, from the first equation in (2.3), we also have $q'_{0,k}(s) = 0$ as $p_{1,k}(s) = 0$. These identities particularly give us

$$\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) q'_{i,k}(s) - H(s) \equiv 0, \quad \beta = 0. \tag{8.15}$$

Integrating both sides of (8.14), we then obtain from the above formula that

$$\int_0^s \left(\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(\tau) q'_{i,k}(\tau) - H(\tau) \right) d\tau = \int_0^\beta \sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \beta'} q_{i,k}(s) d\beta'. \quad (8.16)$$

A combination of (8.16) and (8.13) yields

$$F(s; \alpha, \gamma, \rho) = \int_0^\beta \sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \beta'} q_{i,k}(s) d\beta' + sH(s). \quad (8.17)$$

Since the large- s asymptotics of $p_{i,k}(s)$, $q_{i,k}(s)$ and $H(s)$ established in Proposition 7.1 and (2.9) are uniform in β , we are able to obtain the asymptotic expansion of $F(s)$ by estimating each term in the above integrand. To that end, we see from the definitions of $\hat{p}_{i,k}$ and $\hat{q}_{i,k}$ in (7.1) and (7.2) that

$$\begin{aligned} \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \beta} q_{i,k}(s) &= \begin{pmatrix} \hat{p}_{i,1}(s) \\ \hat{p}_{i,2}(s) \\ \hat{p}_{i,3}(s) \end{pmatrix}^t \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) C_N^{-1} \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) C_\Psi^{-1} \\ &\quad \times \frac{\partial}{\partial \beta} \left(C_\Psi \text{diag} \left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}} \right) C_N \text{diag} \left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}} \right) \begin{pmatrix} \hat{q}_{i,1}(s) \\ \hat{q}_{i,2}(s) \\ \hat{q}_{i,3}(s) \end{pmatrix} \right). \end{aligned} \quad (8.18)$$

Note that only C_N depends on β , but it is not the case for C_Ψ ; see their definitions in (3.18) and (5.30). By (8.18) and the fact that

$$C_N^{-1} \frac{\partial}{\partial \beta} C_N = \begin{pmatrix} 0 & -\sqrt{3}i & \frac{\sqrt{3}i}{2} \\ 0 & 0 & -\sqrt{3}i \\ 0 & 0 & 0 \end{pmatrix}, \quad (8.19)$$

it follows

$$\begin{aligned} \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \beta} q_{i,k}(s) &= \begin{pmatrix} \hat{p}_{i,1}(s) \\ \hat{p}_{i,2}(s) \\ \hat{p}_{i,3}(s) \end{pmatrix}^t \frac{\partial}{\partial \beta} \begin{pmatrix} \hat{q}_{i,1}(s) \\ \hat{q}_{i,2}(s) \\ \hat{q}_{i,3}(s) \end{pmatrix} \\ &\quad + \begin{pmatrix} \hat{p}_{i,1}(s) \\ \hat{p}_{i,2}(s) \\ \hat{p}_{i,3}(s) \end{pmatrix}^t \begin{pmatrix} 0 & -\sqrt{3}is^{\frac{1}{3}} & \frac{\sqrt{3}is^{\frac{2}{3}}}{2} \\ 0 & 0 & -\sqrt{3}is^{\frac{1}{3}} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{q}_{i,1}(s) \\ \hat{q}_{i,2}(s) \\ \hat{q}_{i,3}(s) \end{pmatrix}. \end{aligned}$$

Thus, it is easily seen that

$$\begin{aligned} &\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \beta} q_{i,k}(s) \\ &= \sum_{i=0}^1 \sum_{k=1}^3 \hat{p}_{i,k}(s) \frac{\partial}{\partial \beta} \hat{q}_{i,k}(s) - \sqrt{3}is^{\frac{1}{3}} \sum_{i=0}^1 (\hat{p}_{i,1}(s) \hat{q}_{i,2}(s) + \hat{p}_{i,2}(s) \hat{q}_{i,3}(s)) \\ &\quad + \frac{\sqrt{3}}{2} is^{\frac{2}{3}} \sum_{i=0}^1 \hat{p}_{i,1}(s) \hat{q}_{i,3}(s). \end{aligned} \quad (8.20)$$

It comes out that we can calculate the last two terms in the above formula explicitly. Note that the matrix $C_\Psi \text{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) C_N \text{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right)$ commutes with the matrix A^2 , while

$$\begin{aligned} & \text{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) C_N^{-1} \text{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) C_\Psi^{-1} \cdot A \cdot C_\Psi \text{diag}\left(s^{\frac{1}{3}}, 1, s^{-\frac{1}{3}}\right) C_N \text{diag}\left(s^{-\frac{1}{3}}, 1, s^{\frac{1}{3}}\right) \\ &= A + \frac{\rho}{3} A^2, \end{aligned} \quad (8.21)$$

where A is defined in (4.9). It is then easily seen from (7.1), (7.2) and the relations (4.5), (4.6) that

$$\sum_{i=0}^1 \hat{p}_{i,1}(s) \hat{q}_{i,3} = \sum_{i=0}^1 p_{i,1}(s) q_{i,3} = 1, \quad (8.22)$$

$$\sum_{i=0}^1 \left(\hat{p}_{i,1}(s) \hat{q}_{i,2} + \hat{p}_{i,2}(s) \hat{q}_{i,3} \right) = \sum_{i=0}^1 \left(p_{i,1}(s) q_{i,2} + p_{i,2}(s) q_{i,3} \right) - \frac{\rho}{3} \sum_{i=0}^1 \hat{p}_{i,1}(s) \hat{q}_{i,3} = \frac{2}{3} \rho. \quad (8.23)$$

Substituting the above two formulas into (8.20), we have

$$\sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s) \frac{\partial}{\partial \beta} q_{i,k}(s) = \sum_{i=0}^1 \sum_{k=1}^3 \hat{p}_{i,k}(s) \frac{\partial}{\partial \beta} \hat{q}_{i,k}(s) - \frac{2\sqrt{3}}{3} \rho i s^{\frac{1}{3}} + \frac{\sqrt{3}}{2} i s^{\frac{2}{3}}. \quad (8.24)$$

We now compute large- s asymptotics of $\hat{p}_{i,k}(s) \partial \hat{q}_{i,k}(s) / \partial \beta$. By rewriting $\partial \hat{q}_{1,k}(s) / \partial \beta$ as $\hat{q}_{1,k}(s) \partial \ln \hat{q}_{1,k}(s) / \partial \beta$, it follows from (7.9)–(7.14) that

$$\begin{aligned} \hat{p}_{1,1}(s) \frac{\partial}{\partial \beta} \hat{q}_{1,1}(s) &= -\frac{\gamma e^{-\beta \pi i}}{3\pi} |\Gamma(1-\beta)|^2 \left(\sin(2\psi(s) - \frac{2\pi}{3}) + \mathcal{O}(s^{-\frac{1}{3}}) \right) \\ &\quad \times \left(-\frac{2}{3} \pi i + \frac{\partial}{\partial \beta} |\Gamma(1-\beta)| - \tan(\psi(s) - \frac{\pi}{3}) \frac{\partial}{\partial \beta} \psi(s) + \mathcal{O}(s^{-\frac{1}{3}} \ln s) \right), \end{aligned} \quad (8.25)$$

$$\begin{aligned} \hat{p}_{1,2}(s) \frac{\partial}{\partial \beta} \hat{q}_{1,2}(s) &= -\frac{\gamma e^{-\beta \pi i}}{3\pi} |\Gamma(1-\beta)|^2 \left(\sin(2\psi(s)) + \mathcal{O}(s^{-\frac{1}{3}}) \right) \\ &\quad \times \left(-\frac{2}{3} \pi i + \frac{\partial}{\partial \beta} |\Gamma(1-\beta)| - \tan(\psi(s)) \frac{\partial}{\partial \beta} \psi(s) + \mathcal{O}(s^{-\frac{1}{3}} \ln s) \right), \end{aligned} \quad (8.26)$$

$$\begin{aligned} \hat{p}_{1,3}(s) \frac{\partial}{\partial \beta} \hat{q}_{1,3}(s) &= -\frac{\gamma e^{-\beta \pi i}}{3\pi} |\Gamma(1-\beta)|^2 \left(\sin(2\psi(s) + \frac{2\pi}{3}) + \mathcal{O}(s^{-\frac{1}{3}}) \right) \\ &\quad \times \left(-\frac{2}{3} \pi i + \frac{\partial}{\partial \beta} |\Gamma(1-\beta)| - \tan(\psi(s) + \frac{\pi}{3}) \frac{\partial}{\partial \beta} \psi(s) + \mathcal{O}(s^{-\frac{1}{3}} \ln s) \right). \end{aligned} \quad (8.27)$$

Note that, there is a factor s^β in error estimates $\mathcal{O}(s^{-\frac{1}{3}})$ in (7.9)–(7.14). Then, an additional $\ln s$ factor appears in the error estimates after differentiating with respect to β in the above formulas. Adding (8.25)–(8.27) together, we have from (5.84) and the facts $1 - \gamma = e^{2\beta \pi i}$,

$$\begin{aligned} \sin(2\psi) + \sin(2\psi - \frac{2\pi}{3}) + \sin(2\psi + \frac{2\pi}{3}) &= 0, \\ \sin(\psi)^2 + \sin(\psi - \frac{\pi}{3})^2 + \sin(\psi + \frac{\pi}{3})^2 &= \frac{3}{2}, \end{aligned}$$

that

$$\begin{aligned} \sum_{k=1}^3 \hat{p}_{1,k}(s) \frac{\partial}{\partial \beta} \hat{q}_{1,k}(s) &= -2\beta i \frac{\partial}{\partial \beta} \psi(s) + \mathcal{O}(s^{-\frac{1}{3}} \ln s) \\ &= -2\beta i \frac{\partial}{\partial \beta} \arg \Gamma(1-\beta) - 2\beta \left(\frac{2}{3} \ln s + 2 \ln 3 \right) + \mathcal{O}(s^{-\frac{1}{3}} \ln s). \end{aligned} \quad (8.28)$$

Similarly, since $\partial(1 - \gamma)^{1/3}/\partial\beta = 2\pi i(1 - \gamma)^{1/3}/3$, it is readily seen from (7.3)–(7.5) and (7.6)–(7.8) that

$$\begin{aligned}\hat{p}_{0,2}(s)\frac{\partial}{\partial\beta}\hat{q}_{0,2}(s) &= (\pi_3(\rho) + \rho + \mathcal{O}(s^{-\frac{1}{3}}))\left(\frac{2\pi i}{3} + \frac{\partial}{\partial\beta}\right)\left(-\pi_3(\rho) - \frac{\rho}{3} + \mathcal{O}(s^{-\frac{1}{3}})\right) \\ &= -\frac{2\pi i}{3}(\pi_3(\rho) + \rho)(\pi_3(\rho) + \frac{\rho}{3}) + \mathcal{O}(s^{-\frac{1}{3}}\ln s),\end{aligned}\quad (8.29)$$

and

$$\begin{aligned}\hat{p}_{0,1}(s)\frac{\partial}{\partial\beta}\hat{q}_{0,1}(s) + \hat{p}_{0,3}(s)\frac{\partial}{\partial\beta}\hat{q}_{0,3}(s) &= (1 + \mathcal{O}(s^{-\frac{1}{3}}))\left(\frac{2\pi i}{3} + \frac{\partial}{\partial\beta}\right)\left(-\pi_6(\rho) + \pi_3(\rho)\left(\pi_3(\rho) + \frac{\rho}{3}\right) - \frac{\beta^2}{3} + \frac{2\beta i}{3\sqrt{3}}\cos(2\psi(s)) + \mathcal{O}(s^{-\frac{1}{3}})\right) \\ &\quad + \left(-\alpha + \pi_6(\rho) + \rho\pi_3(\rho) + \frac{\rho^2}{3} + \frac{\beta^2}{3} - \frac{2\beta i}{3\sqrt{3}}\cos(2\psi(s)) + \mathcal{O}(s^{-\frac{1}{3}})\right)\left(\frac{2\pi i}{3} + \mathcal{O}(s^{-\frac{1}{3}})\right) \\ &= \frac{2\pi i}{3}\left(-\alpha + (\pi_3(\rho) + \rho)(\pi_3(\rho) + \frac{\rho}{3})\right) + \frac{\partial}{\partial\beta}\left(\frac{2\beta i}{3\sqrt{3}}\cos(2\psi(s)) - \frac{\beta^2}{3}\right) + \mathcal{O}(s^{-\frac{1}{3}}\ln s).\end{aligned}\quad (8.30)$$

Adding the above two formulas together, we obtain

$$\sum_{k=1}^3 \hat{p}_{0,k}(s)\frac{\partial}{\partial\beta}\hat{q}_{0,k}(s) = -\frac{2\alpha\pi i}{3} + \frac{\partial}{\partial\beta}\left(\frac{2\beta i}{3\sqrt{3}}\cos(2\psi(s)) - \frac{\beta^2}{3}\right) + \mathcal{O}(s^{-\frac{1}{3}}\ln s).\quad (8.31)$$

Inserting (8.28) and (8.31) into (8.24), we have, after integrating with respect to β and a straightforward calculation, that

$$\begin{aligned}&\int_0^\beta \sum_{i=0}^1 \sum_{k=1}^3 p_{i,k}(s)\frac{\partial}{\partial\beta'} q_{i,k}(s) d\beta' \\ &= -2i \int_0^\beta \beta' \frac{\partial}{\partial\beta} \arg \Gamma(1 - \beta') d\beta' - \left(\frac{2}{3}\ln s + 2\ln 3\right)\beta^2 \\ &\quad - \frac{2\alpha\beta\pi i}{3} + \frac{2\beta i}{3\sqrt{3}}\cos(2\psi(s)) - \frac{\beta^2}{3} - \frac{2\sqrt{3}}{3}\rho\beta i s^{\frac{1}{3}} + \frac{\sqrt{3}}{2}\beta i s^{\frac{2}{3}} + \mathcal{O}(s^{-\frac{1}{3}}\ln s) \\ &= \frac{\sqrt{3}}{2}\beta i s^{\frac{2}{3}} - \frac{2\sqrt{3}}{3}\rho\beta i s^{\frac{1}{3}} - \frac{2\beta^2}{3}\ln s + \frac{2\beta i}{3\sqrt{3}}\cos(2\psi(s)) \\ &\quad + \ln\left(G(1 + \beta)G(1 - \beta)\right) + \left(\frac{2}{3} - 2\ln 3\right)\beta^2 - \frac{2\alpha\beta\pi i}{3} + \mathcal{O}(s^{-\frac{1}{3}}\ln s),\end{aligned}\quad (8.32)$$

where $G(z)$ is the Barnes G-function; see a similar integral computation in [22, Equation (7.50)].

Finally, we obtain (2.12) with the error estimate $\mathcal{O}(s^{-\frac{1}{3}}\ln s)$ by substituting the above formula and (2.9) into (8.17). Integrating on both sides of (2.9), we see that the error estimate $\mathcal{O}(s^{-\frac{1}{3}}\ln s)$ can be replaced by $\mathcal{O}(s^{-\frac{1}{3}})$ eventually.

This completes the proof of Theorem 2.2. \square

Appendix A Confluent hypergeometric parametrix

The confluent hypergeometric parametrix $\Phi^{(\text{CHF})}(z) = \Phi^{(\text{CHF})}(z; \beta)$ with β being a parameter is a solution of the following RH problem.

RH problem for $\Phi^{(\text{CHF})}$

- (a) $\Phi^{(\text{CHF})}(z)$ is analytic in $\mathbb{C} \setminus \{\cup_{j=1}^6 \widehat{\Sigma}_j \cup \{0\}\}$, where the contours $\widehat{\Sigma}_j$, $j = 1, \dots, 6$, are indicated in Fig. 7.

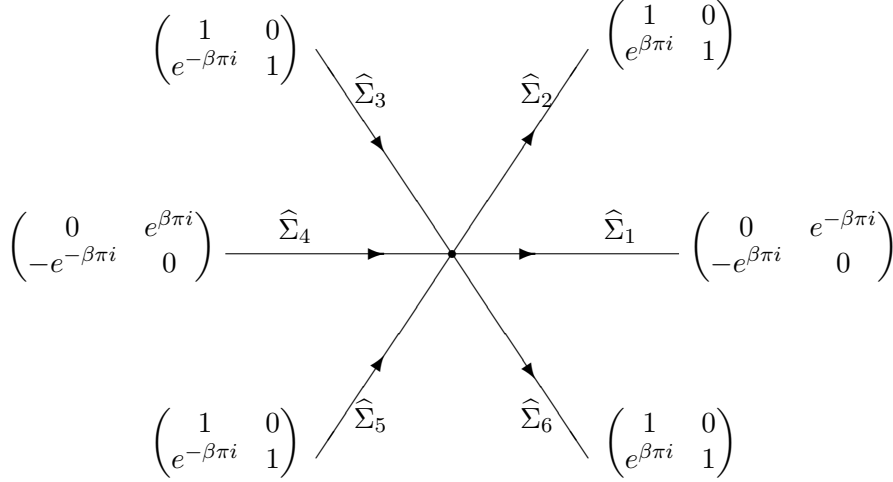


Figure 7: The jump contours $\widehat{\Gamma}_i$ and the corresponding jump matrices \widehat{J}_i , $i = 1, \dots, 6$, in the RH problem for $\Phi^{(\text{CHF})}$.

- (b) $\Phi^{(\text{CHF})}$ satisfies the following jump condition:

$$\Phi_+^{(\text{CHF})}(z) = \Phi_-^{(\text{CHF})}(z) \widehat{J}_i(z), \quad z \in \widehat{\Sigma}_i, \quad j = 1, \dots, 6, \quad (\text{A.1})$$

where the constant matrices J_i are shown in Figure 7.

- (c) $\Phi^{(\text{CHF})}(z)$ satisfies the following asymptotic behavior at infinity:

$$\Phi^{(\text{CHF})}(z) = \left(I + \frac{\Phi_1^{(\text{CHF})}}{z} + \mathcal{O}(z^{-2}) \right) z^{-\beta\sigma_3} e^{-\frac{iz}{2}\sigma_3} \times \begin{cases} I, & 0 < \arg z < \pi, \\ \begin{pmatrix} 0 & -e^{\beta\pi i} \\ e^{-\beta\pi i} & 0 \end{pmatrix}, & \pi < \arg z < \frac{3\pi}{2}, \\ \begin{pmatrix} 0 & -e^{-\beta\pi i} \\ e^{\beta\pi i} & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg z < 0, \end{cases} \quad (\text{A.2})$$

where

$$\Phi_1^{(\text{CHF})} = \begin{pmatrix} \beta^2 i & -\frac{\Gamma(1-\beta)}{\Gamma(\beta)} e^{-\beta\pi i} i \\ \frac{\Gamma(1+\beta)}{\Gamma(-\beta)} e^{\beta\pi i} i & -\beta^2 i \end{pmatrix}. \quad (\text{A.3})$$

- (d) As $z \rightarrow 0$, we have $\Phi^{(\text{CHF})}(z) = \mathcal{O}(\ln |z|)$.

From [44], it follows that the above RH problem can be solved explicitly in terms of the confluent hypergeometric functions. Moreover, as $z \rightarrow 0$, we have (cf. [22, Equation (A.10)])

$$\Phi^{(\text{CHF})}(z) e^{-\frac{\beta\pi i}{2}\sigma_3} = \Upsilon_0 (I + \Upsilon_1 z + \mathcal{O}(z^2)) \begin{pmatrix} 1 & -\frac{\gamma}{2\pi i} \ln(e^{-\frac{\pi i}{2}} z) \\ 0 & 1 \end{pmatrix}, \quad (\text{A.4})$$

for z belonging to the region bounded by the rays $\widehat{\Sigma}_2$ and $\widehat{\Sigma}_3$, where $\gamma = 1 - e^{2\beta\pi i}$,

$$\Upsilon_0 = \begin{pmatrix} \Gamma(1-\beta)e^{-\beta\pi i} & \frac{1}{\Gamma(\beta)} \left(\frac{\Gamma'(1-\beta)}{\Gamma(1-\beta)} + 2\gamma_E \right) \\ \Gamma(1+\beta) & -\frac{e^{\beta\pi i}}{\Gamma(-\beta)} \left(\frac{\Gamma'(-\beta)}{\Gamma(-\beta)} + 2\gamma_E \right) \end{pmatrix} \quad (\text{A.5})$$

with γ_E being the Euler's constant, and

$$(\Upsilon_1)_{21} = \frac{\beta\pi i e^{-\beta\pi i}}{\sin(\beta\pi)}. \quad (\text{A.6})$$

Appendix B Proof of Proposition 7.2

Since the functions $p_1(z)$ and $p_2(z)$ are entire, it is clear that their Wronskian $\mathcal{W}(z)$ admits an expansion as shown in (7.22). To establish the first two coefficients therein, we start with the special case that $\rho = 0$. In this case, it is easily seen from (3.16) that

$$p_1(z) = \int_{\Gamma_1} t^{\alpha-3} e^{zt + \frac{1}{2i^2}} dt, \quad p_2(-z) = e^{-\alpha\pi i} \int_{\Gamma_2} t^{\alpha-3} e^{-zt + \frac{1}{2i^2}} dt, \quad (\text{B.1})$$

where the contours Γ_1 and Γ_2 are illustrated in Figure 3. We then obtain the relation $p_1(z) = -p_2(-z)$ after a change of variable $t = e^{\pi i} \tau$ in the above integral representation of $p_2(-z)$. By inserting the expansion of e^{tz} into the integral representation of $p_1(z)$, we find that

$$p_1(z) = \sum_{k=0}^{\infty} \int_{\Gamma_1} t^{\alpha+k-3} e^{\frac{1}{2i^2}} dt \cdot \frac{z^k}{k!}. \quad (\text{B.2})$$

To evaluate the integral in (B.2), we see after a change of variable $\tau = \frac{1}{2t^2}$ that

$$\int_{\Gamma_1} t^{\alpha+k-3} e^{\frac{1}{2i^2}} dt = 2^{-\frac{\alpha+k}{2}} \int_{\mathcal{H}} \tau^{-\frac{\alpha+k}{2}} e^{\tau} d\tau, \quad (\text{B.3})$$

where \mathcal{H} is the Hankel's loop which starts at $-\infty$, circles the origin once in the positive direction and returns to $-\infty$. Using the Hankel's loop integral for $1/\Gamma(z)$ (see [56, Formula 5.9.2]), we have

$$\int_{\Gamma_1} t^{\alpha+k-3} e^{\frac{1}{2i^2}} dt = 2^{-\frac{\alpha+k}{2}} \frac{2\pi i}{\Gamma((\alpha+k)/2)}. \quad (\text{B.4})$$

This, together with (B.2), gives us

$$p_1(z) = -p_2(-z) = 2\pi i \sum_{k=0}^{\infty} 2^{-\frac{\alpha+k}{2}} \frac{z^k}{k! \Gamma((\alpha+k)/2)}, \quad (\text{B.5})$$

for $\rho = 0$. As a consequence, we get

$$\mathcal{W}(0;0) = 2p_1(0)p_1'(0) = -\frac{\pi^2 2^{-\alpha+5/2}}{\Gamma(\alpha/2)\Gamma((\alpha+1)/2)} = -\frac{(2\pi)^{3/2}}{\Gamma(\alpha)}, \quad (\text{B.6})$$

$$\mathcal{W}'(0;0) = 0, \quad (\text{B.7})$$

which agree with those given in (7.22) and (7.23) for $\rho = 0$. The third equality in (B.6) follows from the following duplication formula for Gamma function:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \quad 2z \neq 0, -1, -2, \dots;$$

cf. [56, Formula 5.5.5].

We next deal with the case for general ρ . By (3.16), it is easily seen that

$$\frac{d}{d\rho} p_i^{(k+1)}(0) = p_i^{(k)}(0), \quad i = 1, 2, \quad k = 0, 1, 2, \dots, \quad (\text{B.8})$$

where $p_i^{(k)}$ indicates the k -th derivative with respect to z . This, together with the definition of \mathcal{W} in (7.20), implies that

$$\frac{d}{d\rho} \mathcal{W}(0; \rho) = \frac{d}{d\rho} p_1(0) \cdot p_2'(0) - \frac{d}{d\rho} p_2(0) \cdot p_1'(0). \quad (\text{B.9})$$

Note that

$$\begin{aligned} \frac{d}{d\rho} p_1(0) &= \int_{\Gamma_1} t^{\alpha-4} e^{\frac{\rho}{t} + \frac{1}{2t^2}} dt = - \int_{\Gamma_1} t^{\alpha-1} e^{\frac{\rho}{t}} \frac{d}{dt} \left(e^{\frac{1}{2t^2}} \right) \\ &= (\alpha - 1) p_1'(0) - \rho p_1(0), \end{aligned} \quad (\text{B.10})$$

and similarly,

$$\frac{d}{d\rho} p_2(0) = (\alpha - 1) p_2'(0) - \rho p_2(0). \quad (\text{B.11})$$

We then obtain from (B.9)–(B.11) that

$$\frac{d}{d\rho} \mathcal{W}(0; \rho) = -\rho \mathcal{W}(0; \rho). \quad (\text{B.12})$$

Solving this first order differential equation with the initial condition (B.6) gives us

$$\mathcal{W}(0; \rho) = -\frac{(2\pi)^{3/2}}{\Gamma(\alpha)} e^{-\rho^2/2}, \quad \alpha > -1. \quad (\text{B.13})$$

The function $\mathcal{W}'(0; \rho)$ satisfies a similar first order differential equation as well. From (B.8) and (B.10), we have

$$\begin{aligned} \frac{d}{d\rho} \mathcal{W}'(0; \rho) &= \frac{d}{d\rho} p_1(0) \cdot p_2''(0) + p_1(0) \cdot \frac{d}{d\rho} p_2''(0) - \frac{d}{d\rho} p_1''(0) \cdot p_2(0) - p_1''(0) \cdot \frac{d}{d\rho} p_2(0) \\ &= -\rho \mathcal{W}'(0; \rho) + \mathcal{W}(0; \rho) + (\alpha - 1) \left(p_1'(0) p_2''(0) - p_1''(0) p_2'(0) \right). \end{aligned} \quad (\text{B.14})$$

In view of (B.5) and (B.13), we obtain the following initial condition after setting $\rho = 0$ in (B.14)

$$\left. \frac{d}{d\rho} \mathcal{W}'(0; \rho) \right|_{\rho=0} = -\frac{(2\pi)^{3/2}}{\Gamma(\alpha + 1)}. \quad (\text{B.15})$$

Recall that both p_1 and p_2 satisfy the equation (3.15), by setting $x = 0$ in (3.15), we have

$$\alpha p_i''(0) = \rho p_i'(0) + p_i(0), \quad i = 1, 2. \quad (\text{B.16})$$

The above formula gives us that, if $\alpha \neq 0$

$$p_1'(0) p_2''(0) - p_1''(0) p_2'(0) = \frac{1}{\alpha} \left(p_1'(0) p_2(0) - p_1(0) p_2'(0) \right) = -\frac{1}{\alpha} \mathcal{W}(0; \rho), \quad (\text{B.17})$$

and if $\alpha = 0$,

$$p_1'(0) p_2''(0) - p_1''(0) p_2'(0) = -\frac{1}{\rho} \left(p_1(0) p_2''(0) - p_1''(0) p_2(0) \right) = -\frac{1}{\rho} \mathcal{W}'(0; \rho). \quad (\text{B.18})$$

Substituting the above two formulas into (B.14), we obtain the following differential equations for $\mathcal{W}'(0; \rho)$:

$$\frac{d}{d\rho}\mathcal{W}'(0; \rho) = \begin{cases} -\rho\mathcal{W}'(0; \rho) + \frac{1}{\alpha}\mathcal{W}(0; \rho), & \text{if } \alpha \neq 0, \\ -(\rho - \frac{1}{\rho})\mathcal{W}'(0; \rho) + \mathcal{W}(0; \rho), & \text{if } \alpha = 0, \end{cases} \quad (\text{B.19})$$

Solving the above equations with $\mathcal{W}(0; \rho)$ in (B.13), the initial conditions (B.7) and (B.15), we obtain

$$\mathcal{W}'(0; \rho) = -\frac{(2\pi)^{3/2}}{\Gamma(\alpha + 1)}\rho e^{-\rho^2/2}, \quad \alpha > -1. \quad (\text{B.20})$$

A combination of (7.22), (B.13) and (B.20) yields (7.23).

To prove (7.24) and (7.25), we introduce

$$p(z; \alpha) = e^{\alpha\pi i}p_1(z) - p_2(z). \quad (\text{B.21})$$

First, we differentiate (B.10) with respect to ρ once again and obtain

$$\begin{aligned} \frac{d^2}{d\rho^2}p_i(0) &= (\alpha - 1)\frac{d}{d\rho}p'_i(0) - p_i(0) - \rho\frac{d}{d\rho}p_i(0) \\ &= (\alpha - 2)p_i(0) - \rho\frac{d}{d\rho}p_i(0), \quad i = 1, 2, \end{aligned} \quad (\text{B.22})$$

where (B.8) is used in the last step. This, together with the definition of $p(z; \alpha)$ in (B.21), implies that $p(0; 0)$ satisfies the following second order differential equation

$$\frac{d^2}{d\rho^2}p(0; 0) = -2p(0; 0) - \rho\frac{d}{d\rho}p(0; 0). \quad (\text{B.23})$$

Similar to the derivation of (B.10), we also note that

$$\frac{d}{d\rho}p'_i(0) = \alpha p''_i(0) - \rho p'_i(0), \quad i = 1, 2. \quad (\text{B.24})$$

We then obtain from (B.8), (B.10) and the above differential relations the following differential equations for $p'(0; 0)$ and $p''(0; 0)$:

$$\frac{d^2}{d\rho^2}p'(0; 0) = -p'(0; 0) - \rho\frac{d}{d\rho}p'(0; 0), \quad (\text{B.25})$$

$$\frac{d^2}{d\rho^2}p''(0; 0) = -\rho\frac{d}{d\rho}p''(0; 0). \quad (\text{B.26})$$

We next establish the initial conditions for these differential equations. By setting $\alpha = 0$ in (B.5), we have, if $\rho = 0$,

$$p(0; 0) = 2p_1(0) = 0, \quad p'(0; 0) = 0, \quad p''(0; 0) = 2p''_1(0) = 2\pi i. \quad (\text{B.27})$$

This, together with (B.8) and (B.10), implies that, for $\rho = 0$,

$$\frac{d}{d\rho}p(0; 0) = 0, \quad \frac{d}{d\rho}p'(0; 0) = 0, \quad \frac{d}{d\rho}p''(0; 0) = 0. \quad (\text{B.28})$$

Solving the differential equations (B.23), (B.25) and (B.26) with the above initial conditions, we have

$$p(0; 0) = 0, \quad p'(0; 0) = 0, \quad p''(0; 0) = 2\pi i, \quad (\text{B.29})$$

for general ρ . Since $p(z; \alpha)$ is an entire function in z , the above formulas give us (7.24) and (7.25).

To show (7.26) and (7.27), we see from (7.21) that

$$q(z) = -\frac{1}{2 \sin(\alpha\pi)i} (p_1(z) - e^{-\alpha\pi i} p_2(z) - (e^{\alpha\pi i} - e^{-\alpha\pi i}) p_3(z)). \quad (\text{B.30})$$

We intend to rewrite right-hand side of the above formula as a single integral. Note that, although the integrands of $p_k(z)$, $k = 1, 2, 3$, in (3.16) appear to be the same, the branches of t^α are chosen differently. In (B.30), we maintain the branches of t^α in the integrals of $p_1(z)$ and $e^{\alpha\pi i} p_3(z)$, while modify the branches of t^α in the integrals of $e^{-\alpha\pi i} p_2(z)$ and $e^{-\alpha\pi i} p_3(z)$ by making a change of variables $t \rightarrow te^{2\pi i}$, that is,

$$e^{-\alpha\pi i} p_2(z) = \int_{\Gamma_2} t^{\alpha-3} e^{zt + \frac{\rho}{t} + \frac{1}{2i^2}} dt, \quad \text{with } -\frac{3\pi}{2} < \arg t < -\frac{\pi}{2}, \quad (\text{B.31})$$

$$e^{-\alpha\pi i} p_3(z) = \int_{\Gamma_3} t^{\alpha-3} e^{zt + \frac{\rho}{t} + \frac{1}{2i^2}} dt, \quad \text{with } -2\pi < \arg t < -\pi. \quad (\text{B.32})$$

Furthermore, we require that the integral contour Γ_3 of $e^{\alpha\pi i} p_3(z)$ and $e^{-\alpha\pi i} p_3(z)$ lies in the right/left half plane, respectively. Thus, the argument in (B.32) is eventually set to be $-3\pi/2 < \arg t < -\pi$. By (3.16) and (B.31)–(B.32), the contribution near $t = 0$ cancels in (B.30). Therefore, we can deform the contour of integral in (B.30) to be a single Hankel loop $\mathcal{H}e^{-\frac{\pi}{2}i}$, i.e., for $\pi/4 < \arg z < 3\pi/4$, we have

$$q(z) = -\frac{1}{2 \sin(\alpha\pi)i} \int_{\mathcal{H}e^{-\frac{\pi}{2}i}} t^{\alpha-3} e^{zt + \frac{\rho}{t} + \frac{1}{2i^2}} dt, \quad \text{with } -\frac{3\pi}{2} < \arg t < \frac{\pi}{2}. \quad (\text{B.33})$$

A further change of variable $zt \rightarrow t$ yields

$$q(z) = -\frac{z^{2-\alpha}}{2 \sin(\alpha\pi)i} \int_{\mathcal{H}} t^{\alpha-3} e^{t + \frac{z\rho}{t} + \frac{z^2}{2i^2}} dt. \quad (\text{B.34})$$

Expanding $\exp\left(\frac{z\rho}{t} + \frac{z^2}{2i^2}\right)$ in the above formula, we obtain (7.26) with

$$c_0 = -\frac{1}{2 \sin(\alpha\pi)i} \int_{\mathcal{H}} t^{\alpha-3} e^t dt = -\frac{\pi}{\sin(\alpha\pi)\Gamma(3-\alpha)}, \quad (\text{B.35})$$

as shown in (7.27).

Finally, to show (7.28), we see from the integral representation for $p_3(z)$ in (3.16) that

$$p_3''(z) = e^{-\alpha\pi i} \int_{\Gamma_3} t^{\alpha-1} e^{zt + \frac{\rho}{t} + \frac{1}{2i^2}} dt, \quad (\text{B.36})$$

where the integral contour Γ_3 goes along the imaginary axis from $\infty e^{\frac{\pi}{2}i}$ to the origin for $\pi/4 < \arg z < 3\pi/4$. If $\alpha > 0$, we obtain after a change of variable $zt \rightarrow t$ that

$$p_3''(z) = e^{-\alpha\pi i} z^{-\alpha} \int_{-\infty}^0 t^{\alpha-1} e^{t + \frac{z\rho}{t} + \frac{z^2}{2i^2}} dt. \quad (\text{B.37})$$

As $z \rightarrow 0$, it follows that

$$\lim_{z \rightarrow 0} z^\alpha p_3''(z) = e^{-\pi i \alpha} \int_{-\infty}^0 t^{\alpha-1} e^t dt = -\Gamma(\alpha), \quad (\text{B.38})$$

as required.

This completes the proof of Proposition 7.2. \square

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