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Lyapunov Characterizations on Input-to-State Stability of Nonlinear Systems with Infinite Delays

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Abstract

This paper addresses Lyapunov characterizations on input-to-state stability (ISS) of time-varying nonlinear systems with infinite delays. With novel ISS definitions in the case of nonlinear systems with infinite delays, we present several results on their ISS Lyapunov characterizations in the form of both ISS Lyapunov theorems and converse ISS Lyapunov theorems. It is shown that an infinite-delayed system is (locally) ISS if it has a (local) ISS Lyapunov functional, and conversely, there exists a (local) ISS Lyapunov functional if it is (locally) ISS. To prove the converse ISS Lyapunov theorems, we establish a key technical lemma bridging ISS/LISS and robust asymptotic stability of systems with infinite delays and two converse Lyapunov theorems concerning *robust* asymptotic stability of systems with infinite delays. Two distinctive advantages of this work are that a large class of infinite dimensional spaces are allowed and the results are established based on a more general Lipschitz condition, i.e., the right hand side Lipschitz (RS-L) condition. An example is provided for illustration of the obtained results.

Key words: Infinite delays, nonlinear systems, (local) input-to-state stability, Lyapunov characterizations.

1 Introduction

The well-known concept of input-to-state stability (ISS), which is often recognized as a milestone in the field of nonlinear systems and control, is first proposed in later 1980s by Eduardo Sontag (Sontag, 1989). Since then, the ISS concept has received increasing attention, and many significant results have been reported, see, for example, Sontag (1990); Jiang et al. (1994); Sontag & Wang (1995); Jiang et al. (1996); Jiang & Wang (2001); Sontag (2008). ISS is an important description for properties of nonlinear systems which not only implies the asymptotic stability of corresponding zero input systems but also the boundedness of the states when inputs are bounded. In fact, ISS has become an important tool in studying various control problems, such as networked control, quantization control and mobile robots formation control (Nešić & Teel, 2004; Liu & Jiang, 2013; Liberzon, 2006).

Several important results have been obtained to extend the ISS concept to delayed nonlinear systems in the past years (Teel, 1998; Pepe & Jiang, 2006; Yeganefar et al., 2008; Karafyllis et al., 2008). In Teel (1998), a Razumikhin-type theorem that guarantees input-to-state stability of time delayed systems is established using the nonlinear small-gain theorem. In Pepe & Jiang (2006), a Lyapunov-Krasovskii methodology for ISS and iISS (integral ISS) of time delayed systems is proposed. In Yeganefar et al. (2008), the authors establish links between ISS of time delayed systems and exponential stability of their corresponding zero input systems. In Karafyllis et al. (2008), the input-to-output stability (IOS), which can be seen as an extension of ISS, of time-varying systems with time delays is discussed. Compared with systems without delays, study of ISS of delayed systems is more challenging mainly because it involves infinite dimensional systems. More recently, in Mironchenko (2016); Mironchenko & Wirth (2017), the authors obtain a series of ISS results based on a more general model of infinite dimensional systems. However, all of these aforementioned ISS results (Teel, 1998; Pepe & Jiang, 2006; Yeganefar et al., 2008; Karafyllis et al., 2008; Mironchenko, 2016; Mironchenko & Wirth, 2017) only consider bounded delays.

Infinite delays, also called unbounded delays, are more general than bounded delays, and include bounded delays as their special cases. In practical systems, there are many scenarios where infinite delays exist or modeling of such infinite delays is needed. For instance, infinite delays are used to model the transcription of genetic mate-

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rials (Josić et al., 2011), the cell dynamics (Djema et al., 2018), the car following systems (Michiels et al., 2009; Sipahi et al., 2007), coupled oscillators (Atay, 2003), neural networks (Gopalsamy & He, 1994), wireless communication networks (Roesch & Roth, 2005) and so on. The existence of infinite delays brings several additional challenges in analysis and synthesis of infinite-delayed systems, as pointed out in Xu et al. (2018, 2019, 2020b); Hale & Kato (1978). These challenges include limitations of available analysis tools, the sensitivity of solutions to initial conditions and mathematical complexity caused by the lack of delay bounds. In the study of infinite-delayed systems, the choice of phase spaces is critical while it is not in the case of bounded delays, as pointed out in Hale & Kato (1978). This is mainly because in the case of infinite-delayed systems, the state will always contain a part of initial conditions (Hale & Kato, 1978). In 1970s and 1980s, some mathematicians propose a hypothesis for phase spaces of infinite-delayed systems, which is also adopted in this work. Based on such hypothesis, some basic theories for infinite-delayed systems, including existence and uniqueness of solutions, continuous dependence and stability theorems, are proved, see, for example, Hale (1974); Hale & Kato (1978); Kato (1978); Sawano (1979); Hino (1983). Some more advanced stability results on infinite-delayed systems are reported in Zhang (1990, 2002). More recently, more general Lyapunov theorems for asymptotic and exponential stabilities of systems with infinite delays are presented in our work (Xu et al., 2020b). However, to our best knowledge, there are no works on ISS of infinite-delayed systems in open literature, which motivates this study.

In this paper, we characterize ISS of time-varying nonlinear systems with infinite delays. To this end, ISS of infinite-delayed systems needs to be newly defined first. Based on the new definitions, we obtain several results on Lyapunov characterizations on ISS of systems with infinite delays. To our best knowledge, this work is the first one on ISS/LISS of systems with infinite delays in open literature. Contributions of this work can be summarized as follows.

First, new ISS and local ISS definitions of infinite-delayed systems are established. Our definitions include the existing ISS/LISS definitions of systems with no delays and bounded delays as their special cases. A distinctive advantage of our definitions is that a large class of infinite dimensional spaces are allowed.

Second, Lyapunov characterizations on ISS of systems with infinite delays are obtained. Both ISS Lyapunov theorems and converse ISS Lyapunov theorems are included. It is shown that an infinite-delayed system is (locally) ISS if it has a (local) ISS Lyapunov functional, and conversely, there exists a (local) ISS Lyapunov functional if it is (locally) ISS.

Third, two converse Lyapunov theorems are also obtained for *robust* asymptotic stability of infinite-delayed systems. A key technical lemma is further proved to bridge ISS/LISS and *robust* asymptotic stability of infinite-delayed systems. Moreover, our results are established based on a more general Lipschitz condition, i.e., the right hand side Lipschitz (RS-L) condition, which is proved to be less stringent than the bounded set Lipschitz condition used in Mironchenko & Wirth (2017); Xu et al. (2020b) (see Xu et al. (2020a) for detailed proof).

The rest of this paper is organized as follows. In Section 2, some preliminaries are provided, including hypothesises of phase spaces, system dynamics and new ISS definitions. In Section 3, Lyapunov characterizations for LISS/ISS of infinite-delayed systems are presented. Conclusions are drawn in Section 4. Some necessary technical lemmas can be found in Appendix.

Notation: Throughout this paper, the following notations are used. The notation $|\cdot|$ represents the absolute value for real numbers, the module for complex numbers, the Euclidean norm for vectors or the induced 2-norm for matrices. The notations $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{Z}^+$ represent the sets of real numbers, nonnegative real numbers, integers and positive integers, respectively. For any $T \in \mathbb{R}$, the floor function $\lfloor T \rfloor$ denotes the largest integer less than or equal to T. A function $\gamma : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\lim_{r\to+\infty} \gamma(r) = +\infty$ with its inverse function denoted by γ^{-1}. For any γ_1 and γ_2 of class $\mathcal{K}$, $\gamma_1 \circ \gamma_2$ is the composed function $\gamma_1(\gamma_2(s))$, $s \in [0, +\infty)$. A function $\sigma : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{KL}$ if it is continuous and for each fixed r, the mapping $\sigma(\cdot, r)$ is of class $\mathcal{K}$ and for each fixed s, the mapping $\sigma(s, \cdot)$ is decreasing and $\lim_{r\to+\infty} \sigma(s, r) = 0$. The symbol $(\mathbb{B}, \|\cdot\|)$ denotes a vector space $\mathbb{B}$ with norm $\|\cdot\|$. A functional $f : \mathbb{B} \to \mathbb{R}^n$ is completely continuous if it is continuous and maps any bounded set of $\mathbb{B}$ into a bounded set of $\mathbb{R}^n$. For two normed vector spaces $(\mathbb{B}_1, \|\cdot\|_1)$ and $(\mathbb{B}_2, \|\cdot\|_2)$, $\mathbb{B}_1 \times \mathbb{B}_2$ denotes the product space with the product norm $\|\cdot\| = \|\cdot\|_1 + \|\cdot\|_2$.

2 Preliminaries

2.1 Phase Spaces and System Dynamics

Suppose $+\infty \geq r \geq 0$ and $[-r, 0] = (-\infty, 0]$ if $r = +\infty$. Let $x_t : [-r, 0] \to \mathbb{R}^n$ be defined as $x_t(\theta) = x(t + \theta), \forall \theta \in [-r, 0]$. In this paper, we consider the following nonlinear system with infinite delays,

$$\dot{x} = f(t, x_t, u(t)), \tag{1}$$

where $x_t \in \mathbb{B}_x$, $u(t) \in \mathbb{B}_u$ and $f : \mathbb{R} \times \mathbb{B}_x \times \mathbb{B}_u \to \mathbb{R}^n$ is completely continuous and $f(t, 0, 0) = 0$. The state space $\mathbb{B}_x$ and the input space $\mathbb{B}_u$ are assumed to be equipped with norms $\|\cdot\|_x$ and $\|\cdot\|_u$, respectively. In this work, the state space $\mathbb{B}_x$ is assumed to satisfy the following hypothesis.

Hypothesis 2.1 *(Kato, 1978; Hale & Kato, 1978; Hino et al., 1991) Let* $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ *denote a vector space* *of functions mapping from $[-r,0]$ into $\mathbb{R}^n$. For any $+\infty \geq A > \sigma \geq 0$, if $x(t)$ is defined on $[\sigma - r, A)$ and continuous on $[\sigma, A)$, $x_t : [-r,0] \to \mathbb{R}^n$ is defined by $x_t(\theta) = x(t+\theta), \theta \in [-r,0]$, then for any $t \in [\sigma, A)$,*

1) if $x_\sigma \in \mathbb{B}$, then $x_t \in \mathbb{B}$;

2) x_t is continuous in t with respect to $\|\cdot\|_{\mathbb{B}}$;

3) there exist constants $M_0 > 0$, $K > 0$ and a function $M(t)$ continuous on $[\sigma, +\infty)$ such that $\lim_{t\to+\infty} M(t) = 0$ and

$$
\begin{aligned}
&|x(t)| \leq M_0 \|x_t\|_{\mathbb{B}},\\
&\|x_t\|_{\mathbb{B}} \leq K \sup_{\sigma \leq s \leq t} |x(s)| + M(t-\sigma)\|x_\sigma\|_{\mathbb{B}}. \qquad (2)
\end{aligned}
$$

Remark 2.1 *It is noted that r can be chosen as zero, i.e., $r = 0$, and thus finite dimensional spaces such as the Euclidean space $\mathbb{R}^n$ can be included by Hypothesis 2.1. Therefore, our results include those for systems with no delays as special cases. Moreover, the space frequently used in bounded delayed systems, i.e., $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ consisting of continuous functions mapping from $[-\tau,0]$ into $\mathbb{R}^n$ and equipped with sup norm, also satisfies this hypothesis. Thus our results also include the results of those existing works on ISS of bounded-delayed systems under the space $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ as special cases. For infinite-delayed systems, there exist many such spaces, see Kato (1978); Hale & Kato (1978); Sawano (1979) for more details. One example of such spaces is the Banach space $\mathcal{C}_r$ for some real $r > 0$ consisting of continuous functions $\phi : (-\infty, 0] \to \mathbb{R}^n$ having the limit $\lim_{s\to-\infty} e^{rs}\phi(s)$ and equipped with norm $\|\phi\|_{\mathcal{C}_r} = \sup_{s\leq 0} e^{rs}|\phi(s)|$. Another example is the Banach space $\mathbb{B}$ of functions measurable on $(-\infty,-\tau]$ and continuous on $[-\tau,0], \tau \geq 0$ with norm*

$$
\|\phi\|_{\mathbb{B}} = \Big[\sup_{0\leq\eta\leq\tau} |\phi(-\eta)|^p + \int_0^{+\infty} g(\eta)|\phi(-\eta)|^p d\eta\Big]^{\frac{1}{p}} < +\infty, \qquad (3)
$$

where $1 \leq p < +\infty$, g is positive, integrable and non-increasing on $[0,+\infty)$ and $g(\eta_1+\eta_2) \leq g(\eta_1)g(\eta_2)$ for $\eta_1, \eta_2 \in [0,+\infty)$.

Remark 2.2 *Hypothesis 2.1 ensures existence of solutions to system* (1) *(Hino et al., 1991). It is shown in Haddock (1985); Seifert (1982) that if Hypothesis 2.1 is not satisfied, the system may have no solution. Moreover, similar hypotheses have also been used in some more recent works on infinite-delayed systems, see for example, Adimy et al. (2001, 2014); Ndambomve & Ezzinbi (2019); Druzhinina & Sedova (2014); while some other recent works have adopted spaces satisfying Hypothesis 2.1, see for example, Liu et al. (2018); Liu & Caraballo (2019).*

Moreover, in this work, f is assumed to satisfy the right hand side Lipschitz condition, whose definition is given as follows.

Definition 2.1 *The function f of system* (1) *is said to be right-hand side Lipschitz (RS-L) continuous if for any bounded set $\Omega \subseteq \mathbb{B}_x \times \mathbb{B}_u$, there exists a positive constant $L(\Omega) > 0$ such that for all $(\phi_1, v_1), (\phi_2, v_2) \in \Omega$ and $t \geq 0$,*

$$
\begin{aligned}
&(\phi_1(0) - \phi_2(0))^T (f(t,\phi_1,v_1) - f(t,\phi_2,v_2))\\
&\qquad \leq L(\Omega)(\|\phi_1 - \phi_2\|_x^2 + \|v_1 - v_2\|_u^2). \qquad (4)
\end{aligned}
$$

The right hand side Lipschitz condition, also called one-sided Lipschitz condition, has been widely adopted in the study of differential equations (Donchev, 1991; Karafyllis, 2006). In Xu et al. (2020a), we have adopted it for the case of infinite-delayed systems and proved that the right hand side Lipschitz condition is more general than the bounded set Lipschitz condition in this case.

2.2 ISS Definitions

A measurable input $u : \mathbb{R}^+ \to \mathbb{B}_u$ is said to be essentially bounded if $\operatorname{ess\,sup}_{t\in\mathbb{R}^+} \|u(t)\|_u = \inf\{a \geq 0 : \|u(t)\|_u > a, t \geq 0 \text{ almost everywhere}\} < +\infty$. It is said to be locally essentially bounded if for any bounded set I of $\mathbb{R}^+$, $\operatorname{ess\,sup}_{t\in I} \|u(t)\|_u = \inf\{a \geq 0 : \|u(t)\|_u > a, t \in I \text{ almost everywhere}\} < +\infty$. In what follows, for notation convenience, we use 'sup' to represent 'ess sup' if there is no confusion.

Let $\mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$ be the space of all locally essentially bounded and Lebesgue measurable mappings $u : \mathbb{R}^+ \to \mathbb{B}_u$. Suppose the initial condition $\phi \in \mathbb{B}_x$, disturbance $u \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$ and initial condition $t_0 \geq 0$. A solution of system (1) on an interval I of $\mathbb{R}$ is a function $x(t_0,\phi,u) : \cup_{t\in I}[-r,t] \to \mathbb{R}^n$ such that (t, x_t) is in $\mathbb{R} \times \mathbb{B}_x$ for all $t \in I$, and $x(t)$ is locally absolutely continuous on I and satisfies the derivative condition $\dot{x}(s) = f(s, x_s, u(s))$ almost everywhere in I. The segment $x_t(t_0,\phi,u)$ is an element of $\mathbb{B}_x$ defined as $x_t(t_0,\phi,u)(\theta) = x(t_0,\phi,u)(t+\theta), \forall\theta \in [-r,0]$. It follows from Hino et al. (1991) that solutions to (1) can be ensured by Hypothesis 2.1. In our previous work (Xu et al., 2020a), we have further shown that uniqueness of solutions to (1) can be ensured by the RS-L condition. Therefore, for any given $\phi \in \mathbb{B}_x$, $u \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$ and $t_0 \geq 0$, the solution $x(t_0,\phi,u)$ exists in a maximal interval $[t_0, b), t_0 < b \leq +\infty$ and is unique. We say system (1) is forward complete if $b = +\infty$. Moreover, if $b < +\infty$, then the solution is unbounded in $[t_0, b)$ (Druzhinina & Sedova, 2014). Due to this property, the forward completeness can be easily verified in this paper.

One can notice that in most of the discussions, there is no confusion if we denote $x(\sigma,\phi,u)(t)$, $x_t(\sigma,\phi,u)$ and ϕ by $x(t)$, x_t and x_{t_0}, respectively. Therefore, in what follows, for notational convenience, we will use the notations $x(t)$, x_t and x_{t_0} if no confusion will arise in the context.

Based on these discussions, we can introduce a new definition of input-to-state stability for infinite-delayed systems. This definition, when the case of infinite delays reduces to the case of no delays and the case of bounded delays, includes those in Sontag & Wang (1995); Teel

(1998); Pepe & Jiang (2006); Yeganefar et al. (2008) as its special cases.

Definition 2.2 *Let $t_0 \geq 0$, $x_{t_0} \in \mathbb{B}_x$ and $u \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$ be the initial time, the initial condition and the input of system* (1)*, respectively. Then system* (1) *is said to be*

1) locally input-to-state stable (LISS) if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and two positive constants k_x, k_u such that for all $\phi : \|\phi\|_x < k_x$, $u : \sup_{s\geq 0} \|u(s)\|_u < k_u$ and $t \geq t_0$,

$$\|x_t\|_x \leq \beta(\|x_{t_0}\|_x, t-t_0) + \gamma(\sup_{t_0\leq s\leq t} \|u(s)\|_u); \qquad (5)$$

2) input-to-state stable (ISS) if inequality (5) *holds for all $t \geq t_0$, $x_{t_0} \in \mathbb{B}_x$ and $u \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$.*

One can also find that in Definition 2.2, the left hand side of inequality (5) is the functional norm $\|\cdot\|_x$. It follows from condition 3) of Hypothesis 2.1 that if we replace this functional norm $\|\cdot\|_x$ by the vector norm $|\cdot|$, the definition seems to be more general. However, we will show by the following lemma that these two definitions, under Hypothesis 2.1, are equivalent.

Lemma 2.1 *System* (1) *is LISS in the sense of Definition 2.2 if and only if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ and two positive constants k_x, k_u such that for all $\phi : \|\phi\|_x < k_x$, $u : \sup_{s\geq 0} \|u(s)\|_u < k_u$ and $t \geq t_0$,*

$$|x(t)| \leq \beta(\|x_{t_0}\|_x, t-t_0) + \gamma(\sup_{t_0\leq s\leq t} \|u(s)\|_u). \qquad (6)$$

Moreover, it is ISS if (6) *holds for all $t \geq t_0$, $x_{t_0} \in \mathbb{B}_x$ and $u \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$.*

Proof: "Only if" part: It is straightforward by considering the property $|x(t)| \leq M_0\|x_t\|_x$.

"If" part: It follows from Hypothesis 2.1 that

$$\begin{aligned}\|x_t\|_x \leq& K \sup_{\frac{t+t_0}{2}\leq s\leq t} |x(s)| + M(\frac{t-t_0}{2})\|x_{\frac{t+t_0}{2}}\|_x \\ \leq& K \sup_{\frac{t+t_0}{2}\leq s\leq t} \beta(\|x_{t_0}\|_x, s-t_0) + K\gamma(\sup_{t_0\leq s\leq t} \|u(s)\|_u) \\ &+ M(\frac{t-t_0}{2})\|x_{\frac{t+t_0}{2}}\|_x \\ \leq& K\beta(\|x_{t_0}\|_x, \frac{t-t_0}{2}) + K\gamma(\sup_{t_0\leq s\leq t} \|u(s)\|_u) \\ &+ M(\frac{t-t_0}{2})\|x_{\frac{t+t_0}{2}}\|_x, \end{aligned} \qquad (7)$$

where K and $M(t)$ are the constant and function given in condition 3) of Hypothesis 2.1. Following the similar arguments, one can also obtain

$$\begin{aligned}\|x_{\frac{t+t_0}{2}}\|_x \leq& K\beta(\|x_{t_0}\|_x, 0) + K\gamma(\sup_{t_0\leq s\leq t} \|u(s)\|_u) \\ &+ M(\frac{t-t_0}{2})\|x_{t_0}\|_x. \end{aligned} \qquad (8)$$

Since $\lim_{t\to+\infty} M(t) = 0$, one can define a constant $\overline{M} = \sup_{s\geq 0} M(s)$ and a strictly decreasing function $\overline{M}(t) = (1 + e^{-t}) \sup_{s\geq t} M(s)$. Therefore, by choosing class $\mathcal{KL}$ function $\beta_0(s,r) = K\beta(s, \frac{r}{2}) + \overline{M}(\frac{r}{2})K\beta(s,0) + \overline{M}^2(\frac{r}{2})s$ and class $\mathcal{K}$ function $\gamma_0(s) = (\overline{M}+1)K\gamma(s)$, one obtains that

$$\|x_t\|_x \leq \beta_0(\|x_{t_0}\|_x, t-t_0) + \gamma_0(\sup_{t_0\leq s\leq t} \|u(s)\|_u). \qquad (9)$$

Noting that these arguments are valid for both LISS and ISS, one can conclude that the lemma is thus proved. □

It should be further emphasized that in our paper, the input space $\mathbb{B}_u$ is not limited to $\mathbb{R}^n$, while in Pepe & Jiang (2006); Yeganefar et al. (2008), the input space $\mathbb{B}_u$ is limited to Euclidean spaces. This shows that our system model and ISS definitions are more general even when the infinite-delayed case reduces to bounded delayed case compared with those existing works.

3 Lyapunov Characterizations for LISS/ISS of Infinite-Delayed Systems

In this section, we present Lyapunov characterizations for LISS/ISS of system (1) in the form of both Lyapunov theorems and converse Lyapunov theorems.

3.1 ISS Lyapunov Theorems

Recall the notations $x_t(t_0, \phi, u)$ and $x(t_0, \phi, u)$ in Section 2. Let $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$ be a continuous functional, then the derivative of V along system (1) is given as

$$\dot{V}_{(1,v)}(t,\phi) = \limsup_{h\to 0^+} \frac{V(t+h, x_{t+h}(t,\phi,\hat{u})) - V(t,\phi)}{h}, \qquad (10)$$

where $v \in \mathbb{B}_u$ and $\hat{u} \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$ is a functional such that for some $\hat{h} > 0$, $\hat{u}(t+s) \equiv v, \forall s \in [0, \hat{h})$. Such a definition of $\dot{V}$ is borrowed from Kankanamalage et al. (2017); Lin & Wang (2018). When $v = 0$, it reduces to the frequently used notation in the study of functional differential equations, see, for example, Hale & Kato (1978); Sawano (1979); Hino et al. (1991); Hale & Lunel (2013). Moreover, V is assumed to be locally Lipschitz in the sense of the following definition.

Definition 3.1 *A functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$ is said to be locally Lipschitz if it is continuous and $\forall (t,\phi) \in \mathbb{R}^+ \times \mathbb{B}_x$, there exist a neighborhood of (t,ϕ), denoted as Ω_0, and a positive real $L(\Omega_0) > 0$ such that $\forall (\hat{t}, \phi_1), (\hat{t}, \phi_2) \in \Omega_0$,*

$$|V(\hat{t},\phi_1) - V(\hat{t},\phi_2)| \leq L(\Omega_0)\|\phi_1 - \phi_2\|_x. \qquad (11)$$

Based on the local Lipschitz property of V, we can obtain the following result for the derivative (10).

Lemma 3.1 *Assume the continuous functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$ is locally Lipschitz. Moreover, let x_t be the segment constructed by the solution of system* (1) *and $u(\cdot) \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$ be the input. Then one has that for almost all $t \geq t_0$,*

$$\dot{V}_{(1,u(t))}(t, x_t) = D^+V(t, x_t), \qquad (12)$$

where $D^+V(t,x_t)$ is the right hand derivative of $V(t,x_t)$. Moreover, if $u(\cdot)$ is piecewise and right continuous, then (12) holds for all $t \geq t_0$.

Proof: For notation convenience, we denote $\hat{x}_s = x_s(t,x_t,\hat{u}), \forall s \geq t$, $x_s = x_s(t,x_t,u), \forall s \geq t$, $\hat{x} = x(t,x_t,\hat{u})$ and $x = x(t,x_t,u)$. It can be obtained from the definitions that when $s = t$, one has $\hat{x}_t = x_t$. It follows from Hypothesis 2.1 that

$$\begin{aligned}
&\|\hat{x}_{t+h} - x_{t+h}\|_x \\
\leq &K \sup_{\theta\in[-h,0]} |\hat{x}_{t+h}(\theta) - x_{t+h}(\theta)| \\
= &K \sup_{\theta\in[-h,0]} |\hat{x}(t+h+\theta) - x(t+h+\theta)| \\
= &K \sup_{\theta\in[-h,0]} |\int_t^{t+h+\theta} \big(f(s,\hat{x}_s,\hat{u}(s)) - f(s,x_s,u(s))\big)ds| \\
\leq &K \int_t^{t+h} |f(s,\hat{x}_s,\hat{u}(s)) - f(s,x_s,u(s))|ds. \qquad (13)
\end{aligned}$$

Note that

$$\dot{V}_{(1,u(t))}(t,x_t) = \limsup_{h\to 0^+} \frac{V(t+h,\hat{x}_{t+h}) - V(t,x_t)}{h}, \qquad (14)$$

and

$$D^+V(t,x_t) = \limsup_{h\to 0^+} \frac{V(t+h,x_{t+h}) - V(t,x_t)}{h}. \qquad (15)$$

It follows from the local Lipschitz property and Hypothesis 2.1 that

$$\begin{aligned}
&|\dot{V}_{(1,u(t))}(t,x_t) - D^+V(t,x_t)| \\
\leq &L \limsup_{h\to 0^+} \frac{1}{h}\|\hat{x}_{t+h} - x_{t+h}\|_x \\
\leq &LK \limsup_{h\to 0^+} \frac{1}{h} \int_t^{t+h} |f(s,\hat{x}_s,\hat{u}(s)) - f(s,x_s,u(s))|ds, \qquad (16)
\end{aligned}$$

where L is a Lipschitz constant of $V(t,\phi)$ in a neighborhood of (t,ϕ) and K is the constant defined in Hypothesis 2.1. Since $\hat{u}(t) = u(t)$, it further follows from the well known Lebesgue differentiation theorem that

$$\begin{aligned}
&|\dot{V}_{(1,u)}(t,x_t) - D^+V(t,x_t)| \\
\leq &LK|f(t,\hat{x}_t,\hat{u}(t)) - f(t,x_t,u(t))| = 0, \ a.e.. \qquad (17)
\end{aligned}$$

Moreover, it can be derived from (16) that if $u(\cdot)$ is piecewise and right continuous, then the description 'almost everywhere' in (17) can be removed. The lemma is thus proved. □

Lemma 3.2 *Assume the continuous functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$ is locally Lipschitz. Then we have that*

$$\dot{V}_{(1,v)}(t,\phi) = \limsup_{h\to 0^+} \frac{V(t+h,\phi_{h,v}) - V(t,\phi)}{h}, \qquad (18)$$

where $\phi_{h,v}$ is defined as

$$\phi_{h,v}(s) = \begin{cases} \phi(s+h), & -r \leq s \leq -h, \\ \phi(0) + (s+h)f(t,\phi,v), & -h \leq s \leq 0. \end{cases} \qquad (19)$$

Proof: The proof of this lemma is similar to that of Lemma 3.1. For notation convenience, we denote $\hat{x}_s = x_s(t,\phi,\hat{u}), \forall s \geq t$ and $\hat{x} = x(t,\phi,\hat{u})$. It follows from Hypothesis 2.1 that $\phi_{h,v} \in \mathbb{B}_x$ and

$$\begin{aligned}
&\|\hat{x}_{t+h} - \phi_{h,v}\|_x \\
\leq &K \sup_{\theta\in[-h,0]} |\hat{x}_{t+h}(\theta) - \phi_{h,v}(\theta)| \\
= &K \sup_{\theta\in[-h,0]} |\hat{x}(t+h+\theta) - \phi_{h,v}(\theta)| \\
= &K \sup_{\theta\in[-h,0]} |\int_t^{t+h+\theta} \big(f(s,\hat{x}_s,\hat{u}(s)) - f(t,\phi,v)\big)ds| \\
\leq &K \int_t^{t+h} |f(s,\hat{x}_s,\hat{u}(s)) - f(t,\phi,v)|ds. \qquad (20)
\end{aligned}$$

It follows from the local Lipschitz property and Hypothesis 2.1 that

$$\begin{aligned}
&|\dot{V}_{(1,u)}(t,\phi) - \limsup_{h\to 0^+} \frac{V(t+h,\phi_{h,v}) - V(t,\phi)}{h}| \\
\leq &L \limsup_{h\to 0^+} \frac{1}{h}\|\hat{x}_{t+h} - \phi_{h,v}\|_x \\
\leq &LK \limsup_{h\to 0^+} \frac{1}{h} \int_t^{t+h} |f(s,\hat{x}_s,\hat{u}(s)) - f(t,\phi,v)|ds, \qquad (21)
\end{aligned}$$

where L is a Lipschitz constant of $V(t,\phi)$ in a neighborhood of (t,ϕ) and K is the constant defined in Hypothesis 2.1. It further follows from the well known Lebesgue differentiation theorem that

$$\begin{aligned}
&\limsup_{h\to 0^+} \frac{1}{h} \int_t^{t+h} |f(s,\hat{x}_s,\hat{u}(s)) - f(t,\phi,v)|ds \\
= &|f(t,\phi,\hat{u}(t)) - f(t,\phi,v)| = 0. \qquad (22)
\end{aligned}$$

It is noted that since $\hat{u}$ is right continuous at t, then (22) holds everywhere, instead of 'almost everywhere'. The lemma is thus proved. □

Equation (18) provides another definition of derivative, which is called the Driver's derivative. It is shown in Lin & Wang (2018) that for bounded-delayed systems, if V is locally Lipschitz, then the two definitions (10) and (18) are equivalent. In this work, we show that for infinite-delayed cases, they are also equivalent.

Remark 3.1 *Lemma 3.1 can be seen as a counterpart of Theorem 2 in Pepe (2007a) for the infinite-delayed case. Compared with Theorem 2 in Pepe (2007a), Lemma 3.1 in this paper considers more general delays, i.e., infinite delays while Theorem 2 in Pepe (2007a) considers more general dynamics, i.e., dynamics described by a retarded functional differential equation coupled with a continuous time difference equation. Moreover, Lemma 3.2 shows that the derivative definition used in this paper and the derivative definition used in Pepe (2007a) are equivalent. The proofs of both Lemmas are partly inspired by Theorem 2 in Pepe (2007a). However, since we consider more general phase spaces and norms, the proof in Pepe (2007a) cannot be directly applied in this work.*

Throughout the paper, the following hypothesis is assumed for any involved Lyapunov functional V.

Hypothesis 3.1 *The time functional $V(t, x_t)$ is locally absolutely continuous.*

Remark 3.2 *Hypothesis 3.1 is necessary for all the theorems presented in this paper although we will not emphasize it subsequently. It is well known that under the condition of local absolute continuity, if $D^+V(t, x_t) \leq 0$, a.e., then $V(t, x_t)$ is nonincreasing. Without Hypothesis 3.1, the decreasing properties of $V(t, x_t)$ cannot be guaranteed. When our work reduces to the bounded-delayed case, it follows from Pepe (2007b) that Hypothesis 3.1 can be ensured by the local Lipschitz property. However, for the infinite-delayed case, similar results are difficult to be proved as initial conditions could be discontinuous and unbounded. It also follows from Lemma 3.1 that Hypothesis 3.1 is not needed when the input function $u(\cdot)$ is piecewise and right continuous. It is because in this case, the concept of "almost everywhere" is no longer involved.*

Then we can establish the following theorems for characterizations of LISS and ISS, respectively.

Theorem 3.1 *For system (1), if there exist a locally Lipschitz functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$, positive constants k_x, k_u, functions α_1, α_2 of class $\mathcal{K}_\infty$, and functions α_3, ρ of class $\mathcal{K}$ such that for all $v : \|v\|_u \leq k_u$,*

1) $\alpha_1(|\phi(0)|) \leq V(t, \phi) \leq \alpha_2(\|\phi\|_x), \forall \phi \in \mathbb{B}_x$,

2) $\forall t \geq t_0, \forall \phi : \|\phi\|_x < k_x$,

$$\dot{V}_{(1,v)}(t, \phi) \leq -\alpha_3(\|\phi\|_x) \tag{23}$$

whenever $\|\phi\|_x \geq \rho(\|v\|_u)$, then system (1) is LISS in the sense of Definition 2.2 with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Theorem 3.2 *For system (1), if there exist a locally Lipschitz functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, and functions α_3, ρ of class $\mathcal{K}$ such that*

1) $\alpha_1(|\phi(0)|) \leq V(t, \phi) \leq \alpha_2(\|\phi\|_x), \forall \phi \in \mathbb{B}_x$,

2) $\forall t \geq t_0, \forall \phi \in \mathbb{B}_x$,

$$\dot{V}_{(1,v)}(t, \phi) \leq -\alpha_3(\|\phi\|_x) \tag{24}$$

whenever $\|\phi\|_x \geq \rho(\|v\|_u)$, then system (1) is ISS in the sense of Definition 2.2 with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

We will provide the proof of Theorem 3.2. The proof of Theorem 3.1 is similar and thus omitted.

Proof of Theorem 3.2: Let $u(\cdot)$ be the input. Define $c(t) = \alpha_2 \circ \rho(\sup_{t_0 \leq s \leq t} \|u(s)\|_u)$. We first prove the following claim.

Claim 1: If for some $\sigma \geq t_0$, $V(\sigma, x_\sigma) \leq c(\sigma)$, then $V(t, x_t) \leq c(t)$ for all $t \geq \sigma$.

Denote $\omega(t) = V(t, x_t) - c(t)$. Then the claim is established if we can prove that $\omega(\sigma) \leq 0$ implies $\omega(t) \leq 0, \forall t \geq \sigma$. The remainder of the proof can be completed by contradiction. Suppose there exists $t_1 > \sigma$ such that $\omega(t_1) > 0$. There exists $\varepsilon_0 > 0$ such that $\omega(t_1) \geq \varepsilon_0$. There exists the smallest value $t_1^* \geq \sigma$ such that $\omega(t_1^*) \geq \varepsilon_0$, i.e., $t_1^* = \inf_{t \geq \sigma}\{t : \omega(t) \geq \varepsilon_0\}$. Then we have that $V(t_1^*, x_{t_1^*}) \geq c(t_1^*) + \varepsilon_0$. We need to first prove that t_1^* is strictly greater than σ. Since $V(t, x_t)$ is continuous and $\omega(\sigma) = V(\sigma, x_\sigma) - c(\sigma) \leq 0$, then there exists $\delta_0 > 0$ such that $V(t, x_t) \leq c(\sigma) + \frac{\varepsilon_0}{2}, \forall t \in [\sigma, \sigma + \delta_0]$. Furthermore, since $c(t)$ is nondecreasing, then one has that $V(t, x_t) \leq c(t) + \frac{\varepsilon_0}{2}, \forall t \in [\sigma, \sigma + \delta_0]$, which implies $\omega(t) \leq \frac{\varepsilon_0}{2} < \varepsilon_0$. Therefore, one has that $t_1^* \geq \sigma + \delta_0 > \sigma$.

Since $V(t, x_t)$ is continuous, then there exists $\delta > 0$ such that $V(t, x_t) > c(t_1^*), \forall t \in [t_1^* - \delta, t_1^*]$, where $t_1^* - \delta \geq \sigma$. Furthermore, since $c(t)$ is nondecreasing, then one has that $V(t, x_t) > c(t_1^*) \geq c(t)$, which implies $\omega(t) > 0, \forall t \in [t_1^* - \delta, t_1^*]$. Therefore, one can find that $\|x_t\|_x \geq \alpha_2^{-1}(V(t, x_t)) > \alpha_2^{-1}(c(t)) \geq \rho(\|u(t)\|_u)$. Then it follows from condition 2) that $\dot{V}_{(1,u)}(t, x_t) \leq 0, \forall t \in [t_1^* - \delta, t_1^*]$. It follows from Lemma 3.1 that $D^+V(t, x_t) \leq 0$, *a.e.*. Since V is locally absolutely continuous, then $V(t, x_t)$ is nonincreasing in $[t_1^* - \delta, t_1^*]$. Moreover, since $c(t)$ is nondecreasing, then $\omega(t) = V(t, x_t) - c(t)$ is nonincreasing in $[t_1^* - \delta, t_1^*]$. Thus $\omega(t_1^* - \delta) \geq \omega(t_1^*) \geq \varepsilon_0$, which is a contradiction to the definition of t_1^*. Claim 1 is thus proved.

Next, we consider the case where $\sup_{t \geq t_0} c(t) = 0$. By considering the fact that $c(t)$ is nondecreasing and nonnegative, we can obtain that $u(t) = 0$ *a.e.*. We have that for almost all $t \geq t_0$,

$$D^+V(t, x_t) \leq -\alpha_3 \circ \alpha_2^{-1}(V(t, x_t)). \tag{25}$$

Since $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function, there exists a class $\mathcal{KL}$ function β_0 such that $|V(t, x_t)| \leq \beta_0(|V(t_0, x_{t_0})|, t - t_0), \forall t \geq t_0$ (refer to Lemma 4.4 in Lin et al. (1996)). Therefore, one has that $|x(t)| \leq \alpha_1^{-1}(\beta_0(\alpha_2(\|x_{t_0}\|_x, t - t_0))) \triangleq \beta(\|x_{t_0}\|_x, t - t_0), \forall t \geq t_0$, and it can be verified that β is a class $\mathcal{KL}$ function.

Moreover, in the case of $\sup_{t \geq t_0} c(t) > 0$. The following claim can be further proved.

Claim 2: If $V(t_0, x_{t_0}) > c(t_0)$, then there exists $t' > t_0$ and a class $\mathcal{KL}$ function β such that

i) $|x(t)| \leq \beta(\|x_{t_0}\|_x, t - t_0), \forall t \in [t_0, t')$,

ii) $V(t, x_t) \leq c(t), \forall t \geq t'$.

We first prove that $\omega(t) = V(t, x_t) - c(t)$ will decrease to zero in finite time, i.e., there exists $T > 0$ such that $V(t', x_{t'}) \leq c(t')$ for some $t' \in [t_0, t_0 + T]$. One has that for any $V(t, x_t) > c(t)$,

$$\begin{aligned} D^+V(t, x_t) &= \dot{V}_{(1,u(t))}(t, x_t) \\ &\leq -\alpha_3(\|x_t\|_x) \leq -\alpha_3 \circ \alpha_2^{-1}(V(t, x_t)) \\ &\leq -k(t) \ a.e., \end{aligned} \tag{26}$$

where $k(t) = \alpha_3 \circ \alpha_2^{-1}(c(t))$. This implies that $\omega(t) = V(t, x_t) - c(t) \leq V(t_0, x_{t_0}) - \int_{t_0}^t k(s)ds - c(t)$. Since $\sup_{t \geq t_0} c(t) > 0$ and $c(t)$ is nondecreasing, then there exists $t^* \geq t_0$ such that $k(t) > 0, \forall t \geq t^*$ and $k(t)$ is nondecreasing, which yield that $\int_{t_0}^t k(s)ds$ goes to infinity as t goes to infinity. Suppose $\omega(t) > 0$ for all

$t \geq t_0$. Then we have $\omega(t) \leq V(t_0, x_{t_0}) - \int_{t_0}^{t} k(s)ds - c(t)$ for all $t \geq t_0$. However, since $\int_{t_0}^{t} k(s)ds$ goes to infinity as t goes to infinity, there exists $T > 0$ such that $\omega(t) \leq V(t_0, x_{t_0}) - \int_{t_0}^{t} k(s)ds - c(t) < 0$ for all $t \geq t_0 + T$, which is a contradiction. Therefore, there exists $T > 0$ such that $V(t', x_{t'}) \leq c(t')$ for some $t' \in [t_0, t_0 + T]$.

Let t' be the first time $\omega(t)$ decreases to zero, i.e., $t' = \inf_{t \geq t_0} \{t : \omega(t) \leq 0\}$. By Claim 1, one can find that $V(t, x_t) \leq c(t), \forall t \geq t'$. For $t \in [t_0, t')$, one has that

$$D^+V(t, x_t) \leq -\alpha_3 \circ \alpha_2^{-1}(V(t, x_t)) \text{ a.e. in } [t_0, t'), \quad (27)$$

which implies that $|V(t, x_t)| \leq \beta_0(|V(t_0, x_{t_0})|, t - t_0), \forall t \in [t_0, t')$. Therefore, by the similar argument, one has that $|x(t)| \leq \beta(\|x_{t_0}\|_x, t - t_0), \forall t \in [t_0, t')$. Claim 2 is thus proved.

One can then conclude from these claims that

$$\begin{aligned} |x(t)| &\leq \beta(\|x_{t_0}\|_x, t - t_0) + \alpha_1^{-1}(c(t)) \\ &= \beta(\|x_{t_0}\|_x, t - t_0) + \gamma(\sup_{t_0 \leq s \leq t} \|u(s)\|_u), \end{aligned} \quad (28)$$

where $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$. Then ISS can be established and the theorem is thus proved.

Remark 3.3 *The proof of Theorem 3.2 is inspired by the work in Pepe & Jiang (2006) which considers ISS of bounded-delayed systems. However, compared to Pepe & Jiang (2006), our proof is new even when it reduces to bounded-delayed cases. In Pepe & Jiang (2006), the authors define a constant $c = \alpha_2 \circ \rho(\sup_{t_0 \leq s} |u(s)|)$ with which one first obtains that*

$$|x(t)| \leq \beta(|x_{t_0}|, t - t_0) + \gamma(\sup_{t_0 \leq s} |u(s)|). \quad (29)$$

Note that the supreme of $u(s)$ in (29) is from t_0 to infinity. Then based on (29), the authors further obtain the following ISS property

$$|x(t)| \leq \beta(|x_{t_0}|, t - t_0) + \gamma(\sup_{t_0 \leq s \leq t} |u(s)|). \quad (30)$$

Note that the supreme of $u(s)$ in (30) is from t_0 to t, which precisely satisfies the ISS definition and also implies (29). In contrary, we define a time-varying function $c(t) = \alpha_2 \circ \rho(\sup_{t_0 \leq s \leq t} |u(s)|)$ instead of a constant c. And with new techniques, we can directly obtain (30) instead of (29) first. Thus our proof is an improved version even when it reduces to bounded-delayed cases.

The following corollary can be derived from Theorems 3.1 and 3.2.

Corollary 3.1 *For system (1), if there exist a locally Lipschitz functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$, functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of class $\mathcal{K}_\infty$ and positive constants k_x, k_u such that for all $v : \|v\|_u \leq k_u$,*

1) $\alpha_1(|\phi(0)|) \leq V(t, \phi) \leq \alpha_2(\|\phi\|_x), \forall \phi \in \mathbb{B}_x$,

2) $\forall t \geq t_0$, $\forall \phi : \|\phi\|_x < k_x$,

$$\dot{V}_{(1,v)}(t, \phi) \leq -\alpha_3(\|\phi\|_x) + \alpha_4(\|v\|_u), \quad (31)$$

then system (1) is LISS in the sense of Definition 2.2. Moreover, it is ISS if the two conditions are satisfied for all $t \geq t_0$, $\phi \in \mathbb{B}_x$ and $v \in \mathbb{B}_u$.

Example 3.1 *We provide an example to illustrate the results in this subsection. Consider a vector-valued system as follows,*

$$\dot{x}(t) = A(t)x(t) + \int_0^{+\infty} g(t, \eta, x(t - \eta) + d(t - \eta))d\eta, \quad (32)$$

where $x \in \mathbb{R}^n$ is the state, d is the disturbance, $A(t) \in \mathbb{R}^{n \times n}$ is a continuous matrix function, $g(t, \eta, r)$ is continuous in (t, η, r). Let $\mathbb{B}$ be a space of all continuous functions ϕ mapping from $(-\infty, 0]$ to $\mathbb{R}^n$ with the following norm,

$$\|\phi\|_{\mathbb{B}} = |\phi(0)| + \int_0^{+\infty} p(\eta)|\phi(-\eta)|d\eta, \quad (33)$$

where p is a positive, nonincreasing and Lebesgue integrable function satisfying

$$p(u + v) \leq p(u)p(v), \forall u, v \geq 0. \quad (34)$$

Assume $x_t, d_t \in \mathbb{B}$. Also assume that

$$|g(t, \eta, s)| \leq p(\eta)|s|, \forall t \geq 0, \eta \geq 0, s \in \mathbb{R}, \quad (35)$$

and there exist a symmetric and continuously differentiable matrix function $P(t)$ and positive constants ρ_1, ρ_2, δ such that

$$\begin{aligned} &\rho_1 I \leq P(t) \leq \rho_2 I, \\ &A^T(t)P(t) + P(t)A(t) + \dot{P}(t) \leq -2\delta\rho_2 I, \\ &\delta > b = \int_0^{+\infty} p(\eta)d\eta. \end{aligned} \quad (36)$$

Furthermore, one can define the following functional $V : \mathbb{R}^+ \times \mathbb{B} \to \mathbb{R}^+$ for system (32),

$$V(t, \phi) = |P^{\frac{1}{2}}(t)\phi(0)| + c\int_0^{+\infty} |\phi(-\eta)| \int_\eta^{+\infty} p(s)dsd\eta, \quad (37)$$

where $c > 0$ is to be determined and $P^{\frac{1}{2}}(t)$ is the unique positive definite square root of $P(t)$. The condition 1) in Theorem 3.2 is usually easy to be satisfied for bounded-delayed systems while it may fail to be satisfied for infinite-delayed systems. In this example, it can be verified that $\sqrt{\rho_1}|\phi(0)| \leq V(t, \phi)$ and

$$\begin{aligned} V(t, \phi) &\leq \sqrt{\rho_2}|\phi(0)| + c\int_0^{+\infty} |\phi(-\eta)| \int_0^{+\infty} p(s + \eta)dsd\eta \\ &\leq \sqrt{\rho_2}|\phi(0)| + c\int_0^{+\infty} p(\eta)|\phi(-\eta)|d\eta \int_0^{+\infty} p(s)ds \\ &\leq \max\{\sqrt{\rho_2}, cb\}\|\phi\|_{\mathbb{B}}. \end{aligned} \quad (38)$$

Next, it can be obtained that

$$V(t, x_t) = |P^{\frac{1}{2}}(t)x(t)| + c\int_{-\infty}^{t} |x(\eta)| \int_{t-\eta}^{+\infty} p(s)dsd\eta. \quad (39)$$

One thus has that for $x(t) \neq 0$,

$$\begin{aligned}\dot V(t,x_t) =&\frac{x^T\dot P(t)x+2x^TP(t)\dot x}{2|P^{\frac12}(t)x(t)|}\\ =&\frac{1}{2|P^{\frac12}(t)x(t)|}\big(x^T(A^T(t)P(t)+P(t)A(t)+\dot P(t))x\\ &+2x^TP(t)\int_0^{+\infty}g(t,\eta,x(t-\eta)+d(t-\eta))d\eta\big)\\ &+cb|x(t)|-c\int_0^{+\infty}p(\eta)|x(t-\eta)|d\eta\\ \le&-\delta\sqrt{\rho_2}|x(t)|\\ &+\sqrt{\rho_2}\int_0^{+\infty}p(\eta)|x(t-\eta)+d(t-\eta)|d\eta\\ &+cb|x(t)|-c\int_0^{+\infty}p(\eta)|x(t-\eta)|d\eta\\ \le&-(\delta\sqrt{\rho_2}-cb)|x(t)|+\sqrt{\rho_2}\int_0^{+\infty}p(\eta)|d(t-\eta)|d\eta\\ &+(\sqrt{\rho_2}-c)\int_0^{+\infty}p(\eta)|x(t-\eta)|d\eta\\ \le&-(\delta\sqrt{\rho_2}-cb+\sqrt{\rho_2}-c)|x(t)|\\ &+(\sqrt{\rho_2}-c)\|x_t\|_{\mathbb B}+\sqrt{\rho_2}\|d_t\|_{\mathbb B},\end{aligned}\tag{40}$$

and for $x(t)=0$,

$$\begin{aligned}\dot V(t,x_t)\le&|\frac{dP^{\frac12}}{dt}x+P^{\frac12}(t)\frac{dx}{dt}|\\ &+cb|x(t)|-c\int_0^{+\infty}p(\eta)|x(t-\eta)|d\eta\\ =&-c\int_0^{+\infty}p(\eta)|x(t-\eta)|d\eta+|P^{\frac12}(t)\big(A(t)x\\ &+\int_0^{+\infty}g(t,\eta,x(t-\eta)+d(t-\eta))d\eta\big)|\\ \le&\sqrt{\rho_2}\int_0^{+\infty}p(\eta)|x(t-\eta)+d(t-\eta)|d\eta\\ &-c\int_0^{+\infty}p(\eta)|x(t-\eta)|d\eta\\ \le&\sqrt{\rho_2}\int_0^{+\infty}p(\eta)|d(t-\eta)|d\eta\\ &+(\sqrt{\rho_2}-c)\int_0^{+\infty}p(\eta)|x(t-\eta)|d\eta\\ \le&(\sqrt{\rho_2}-c)\|x_t\|_{\mathbb B}+\sqrt{\rho_2}\|d_t\|_{\mathbb B}\\ =&-(\delta\sqrt{\rho_2}-cb+\sqrt{\rho_2}-c)|x(t)|\\ &+(\sqrt{\rho_2}-c)\|x_t\|_{\mathbb B}+\sqrt{\rho_2}\|d_t\|_{\mathbb B}.\end{aligned}\tag{41}$$

Since $\delta > b$, one can choose that $\frac{\delta+1}{b+1}\sqrt{\rho_2} > c > \sqrt{\rho_2}$. Then one has $\delta\sqrt{\rho_2}-cb+\sqrt{\rho_2}-c>0$ and $c-\sqrt{\rho_2}>0$, which yields

$$\dot V(t,x_t)\le-(c-\sqrt{\rho_2})\|x_t\|_{\mathbb B}+\sqrt{\rho_2}\|d_t\|_{\mathbb B}.\tag{42}$$

Therefore, we have that

$$\dot V_{(32,d_t)}(t,\phi)\le-(c-\sqrt{\rho_2})\|\phi\|_{\mathbb B}+\sqrt{\rho_2}\|d_t\|_{\mathbb B}.\tag{43}$$

It thus follows from Corollary 3.1 that system (32) is ISS in the sense of Definition 2.2 by considering x as its state and d_t as its input, i.e., there exist a class $\mathcal{KL}$ function β and a class $\mathcal K$ function γ such that

$$\|x_t\|_{\mathbb B}\le\beta(\|x_{t_0}\|_{\mathbb B},t-t_0)+\gamma(\sup_{t_0\le s\le t}\|d_s\|_{\mathbb B}).\tag{44}$$

Example 3.2 *Consider a wheeled inverted pendulum system with time delays borrowed from Zhou & Wang (2016) but with infinite delays, described as follows,*

$$\begin{aligned}\dot x=&Ax+\int_0^{+\infty}B(\eta)\xi(t-\eta)d\eta,\\ \dot\xi=&u+d,\end{aligned}\tag{45}$$

where

$$A=\begin{bmatrix}0&1&0&0\\ \frac{140}{17}&0&0&0\\ 0&0&0&1\\ -\frac{140}{17}&0&0&0\end{bmatrix},B(\eta)=\begin{bmatrix}0\\ -\frac{15}{34}\Gamma_1(\eta)\\ 0\\ \frac{14}{17}\Gamma_2(\eta)\end{bmatrix},\tag{46}$$

with $\Gamma_1(\eta)=3e^{-3\eta}$ and $\Gamma_2(\eta)=9\eta e^{-3\eta}$. In this example, u is the control signal, d is the disturbance, η is the integrator signal which control the wheeled inverted pendulum. The infinite delay is caused by viscoelasticity. With the same controller as in Xu et al. (2021), the ISS of the closed loop system by considering (x,ξ) as state and d as input can be established.

3.2 Converse ISS Lyapunov Theorems

In this subsection, we establish converse ISS Lyapunov theorems for system (1). Before presenting our main theorems, we will consider the following system first,

$$\dot x=g(t,x_t,d(t)),\tag{47}$$

where $x_t\in\mathbb B_x$, $d(t)\in\mathcal D$ with $\mathcal D$ being a bounded set of $\mathbb B_u$, $g:\mathbb R\times\mathbb B_x\times\mathcal D$ is completely continuous, and satisfies the following conditions. It is noted that $\mathbb B_x$ and $\mathbb B_u$ are the same as those of system (1).

1) For any bounded set $\Omega\subseteq\mathbb B_x$, there exists a positive constant $L(\Omega)>0$ such that for all $\phi_1,\phi_2\in\Omega$ and $v\in\mathcal D$,

$$\begin{aligned}(\phi_1(0)-\phi_2(0))^T&(g(t,\phi_1,v)-g(t,\phi_2,v))\\ &\le L(\Omega)\|\phi_1-\phi_2\|_x^2.\end{aligned}\tag{48}$$

2) $g(t,0,d)=0$ for all $(t,d)\in\mathbb R^+\times\mathcal D$.

3) For any $\varepsilon>0$ and any $t\in\mathbb R^+$, there exists $\delta=\delta(t,\varepsilon)>0$ such that

$$|s-t|+\|\phi\|_x<\delta\text{ implies }\sup_{d\in\mathcal D}|g(s,\phi,d)|<\varepsilon.\tag{49}$$

In system (47), x is the state and d is the disturbance or uncertainty. Let $\mathcal M_{\mathcal D}=\mathcal M(\mathbb R^+,\mathcal D)$ be the set of all Lebesgue measurable and locally essentially bounded functions mapping from $\mathbb R^+$ to $\mathcal D$. By noting $d(\cdot)\in\mathcal M_{\mathcal D}$, we refer to a Lebesgue measurable and locally essentially bounded function $d:\mathbb R^+\to\mathcal D$, that is, $d(t)\in\mathcal D$ for all $t\in\mathbb R^+$. We recall the notations $x(t_0,\phi,d)$ and

$x_t(t_0, \phi, d)$ defined in Section 2. Then we can introduce the following concepts of *robust* stability concerned with system (47).

Definition 3.2 *The system* (47) *is said to be*

1) uniformly robustly asymptotically stable (URAS) if there exist β of class $\mathcal{KL}$ and $k > 0$ such that for all $\phi : \|\phi\|_x < k$,

$$\sup_{d \in \mathcal{M}_\mathcal{D}} \|x_t(t_0, \phi, d)\|_x \le \beta(\|\phi\|_x, t - t_0). \tag{50}$$

2) uniformly robustly globally asymptotically stable (URGAS) if (50) *holds for all $\phi \in \mathbb{B}_x$.*

Then we are ready to present the following converse Lyapunov theorems for URAS and URGAS, respectively.

Theorem 3.3 *Denote $\Omega_k = \{\phi \in \mathbb{B}_x : \|\phi\|_x < k\}$ for any $k > 0$. If system* (47) *is URAS, then for some positive constant $k > 0$, there exist a continuous functional $V : \mathbb{R}^+ \times \Omega_k \to \mathbb{R}^+$ and functions α_1, α_2 of class $\mathcal{K}_\infty$ such that for all $t \ge t_0$, $\phi \in \Omega_k$ and $d \in \mathcal{D}$,*

1) $\alpha_1(\|\phi\|_x) \le V(t, \phi) \le \alpha_2(\|\phi\|_x)$,

2) $\dot{V}_{(47,d)}(t, \phi) \le -V(t, \phi)$.

Furthermore, the functional V is Lipschitz in Ω_k, i.e., there exists $L(k) > 0$ such that for all $\phi_1, \phi_2 \in \Omega_k$,

$$|V(t, \phi_1) - V(t, \phi_2)| \le L(k)\|\phi_1 - \phi_2\|_x. \tag{51}$$

Theorem 3.4 *If system* (47) *is URGAS, then there exist a continuous functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$ and functions α_1, α_2 of class $\mathcal{K}_\infty$ such that for all $t \ge t_0$, $\phi \in \mathbb{B}_x$ and $d \in \mathcal{D}$,*

1) $\alpha_1(\|\phi\|_x) \le V(t, \phi) \le \alpha_2(\|\phi\|_x)$,

2) $\dot{V}_{(47,d)}(t, \phi) \le -V(t, \phi)$.

Furthermore, the functional V is bounded set Lipschitz in the sense that for any bounded set Ω of $\mathbb{B}_x$, there exists $L(\Omega) > 0$ such that for all $\phi_1, \phi_2 \in \Omega$,

$$|V(t, \phi_1) - V(t, \phi_2)| \le L(\Omega)\|\phi_1 - \phi_2\|_x. \tag{52}$$

The proofs of these two theorems will be presented in next subsection. Before presenting converse LISS and ISS Lyapunov theorems, we introduce the following lemma, which can be seen as a bridge between *robust* asymptotic stability of system (47) and (L)ISS of system (1).

Lemma 3.3 *If system* (1) *is LISS in the sense of Definition 2.2, then there exist a class $\mathcal{K}_\infty$ function ρ, a class $\mathcal{KL}$ function β and positive constants k_x, k_u such that for all $x_{t_0} : \|x_{t_0}\|_x < k_x$ and $u : \sup_{s \ge 0} \|u(s)\|_u < k_u$, $\|u(t)\|_u \le \rho(\|x_t\|_x), \forall t \ge t_0$ implies*

$$\|x_t\|_x \le \beta(\|x_{t_0}\|_x, t - t_0). \tag{53}$$

Moreover, if system (1) *is ISS, then there exist a class $\mathcal{K}_\infty$ function ρ and a class $\mathcal{KL}$ function β such that for all $x_{t_0} \in \mathbb{B}_x$ and $u(\cdot) \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$, $\|u(t)\|_u \le \rho(\|x_t\|_x), \forall t \ge t_0$ implies* (53).

Proof: We only provide the proof of the ISS case. The proof of the LISS case is similar and thus omitted.

Since system (1) is ISS, then there exist β_1 of class $\mathcal{KL}$ and γ_1 of class $\mathcal{K}$ such that for all $t \ge \sigma \ge t_0$,

$$\|x_t\|_x \le \beta_1(\|x_\sigma\|_x, t - \sigma) + \gamma_1(\sup_{\sigma \le s \le t} \|u(s)\|_u). \tag{54}$$

Choose a class $\mathcal{K}_\infty$ function ρ such that $\gamma_1 \circ \rho(s) < \frac{1}{2}s, \forall s > 0$. It follows from the condition $\|u(t)\|_u \le \rho(\|x_t\|_x), \forall t \ge t_0$ that

$$\|x_t\|_x \le \beta_1(\|x_\sigma\|_x, t - \sigma) + \frac{1}{2} \sup_{\sigma \le s \le t} \|x_s\|_x. \tag{55}$$

Define $S[\sigma, t] = \sup_{\sigma \le s \le t} \|x_s\|_x$. Then we have that for any $t \ge t' \ge \sigma$,

$$S[t', t] \le \beta_1(\|x_\sigma\|_x, t' - \sigma) + \frac{1}{2}S[\sigma, t], \tag{56}$$

which implies that $S[\sigma, t] \le 2\beta_1(\|x_\sigma\|_x, 0)$ by considering $t' = \sigma$. Furthermore, let n be a nonnegative integer and T be a nonnegative real number such that $t \ge \sigma + nT$. It can be obtained that

$$\begin{aligned}
S[t, t] \le& \beta_1(\|x_{t-T}\|_x, T) + \frac{1}{2}S[t - T, t] \\
\le& \beta_1(2\beta_1(\|x_\sigma\|_x, 0), T) \\
&+ \frac{1}{2}(\beta_1(\|x_{t-2T}\|_x, T) + \frac{1}{2}S[t - 2T, t]) \\
\le& (1 + \frac{1}{2})\beta_1(2\beta_1(\|x_\sigma\|_x, 0), T) + \frac{1}{2^2}S[t - 2T, t] \\
&\cdots \\
\le& (1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}})\beta_1(2\beta_1(\|x_\sigma\|_x, 0), T) \\
&+ \frac{1}{2^n}S[t - nT, t] \\
\le& 2\beta_1(2\beta_1(\|x_\sigma\|_x, 0), T) + \frac{2\beta_1(\|x_\sigma\|_x, 0)}{2^n}.
\end{aligned} \tag{57}$$

We recall the floor function here, that is, for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Choose $\sigma = t_0$, $n = \lfloor \sqrt{t - t_0} \rfloor + 1$ and $T = \frac{t - t_0}{\lfloor \sqrt{t - t_0} \rfloor + 1}$. Then we have $t = t_0 + nT$. It follows from inequality (57) that

$$\begin{aligned}
\|x_t\|_x =& S[t, t] \le 2\beta_1(2\beta_1(\|x_{t_0}\|_x, 0), T) + \frac{\beta_1(\|x_{t_0}\|_x, 0)}{2^{n-1}} \\
\le& 2\beta_1(2\beta_1(\|x_{t_0}\|_x, 0), \frac{t - t_0}{\sqrt{t - t_0} + 1}) + \frac{\beta_1(\|x_{t_0}\|_x, 0)}{2^{\sqrt{t - t_0} - 1}} \\
\triangleq& \beta(\|x_{t_0}\|_x, t - t_0),
\end{aligned} \tag{58}$$

where $\beta(s, r) = 2\beta_1(2\beta_1(s, 0), \frac{r}{\sqrt{r} + 1}) + \frac{\beta_1(s, 0)}{2^{\sqrt{r} - 1}}$ is a class $\mathcal{KL}$ function. The lemma is thus proved. □

Remark 3.4 *In the proof of Lemma 3.3, ρ is chosen such that it is of class $\mathcal{K}_\infty$ and $\gamma_1 \circ \rho(s) < \frac{1}{2}s, \forall s > 0$. It should be emphasized that for any γ_1 of class $\mathcal{K}$, such a function ρ always exists. In fact, define $\bar{\gamma}_1$ as $\bar{\gamma}_1(s) = (1 + s)\gamma_1(s), \forall s \ge 0$. It can be verified that $\bar{\gamma}_1$ is of class $\mathcal{K}_\infty$ and $\bar{\gamma}_1(s) > \gamma_1(s), \forall s > 0$. Choose $\rho(s) = \bar{\gamma}_1^{-1}(\frac{1}{2}s), \forall s \ge 0$, then we have that ρ is of class $\mathcal{K}_\infty$ and $\gamma_1 \circ \rho(s) < \bar{\gamma}_1 \circ \rho(s) = \frac{1}{2}s, \forall s > 0$.*

Now based on Theorems 3.3-3.4 and Lemma 3.3, we can obtain the following converse LISS and ISS Lyapunov theorems, respectively.

Theorem 3.5 *Denote $\Omega_k = \{\phi \in \mathbb{B}_x : \|\phi\|_x < k\}$ for any $k > 0$. If system (1) is LISS in the sense of Definition 2.2, then for some $k_x > 0$ and $k_u > 0$, there exist a continuous functional $V : \mathbb{R}^+ \times \Omega_{k_x} \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, function ρ of class $\mathcal{K}$ such that for all $\|v\|_u < k_u$,*

1) $\alpha_1(\|\phi\|_x) \le V(t,\phi) \le \alpha_2(\|\phi\|_x), \forall \phi \in \Omega_{k_x}$,

2) $\forall t \ge t_0, \forall \phi \in \Omega_{k_x}$,

$$\dot{V}_{(1,v)}(t,\phi) \le -V(t,\phi) \tag{59}$$

whenever $\|\phi\|_x \ge \rho(\|v\|_u)$. Furthermore, the functional V is Lipschitz in Ω_{k_x}, i.e., there exists $L(k_x) > 0$ such that for all $\phi_1, \phi_2 \in \Omega_{k_x}$,

$$|V(t,\phi_1) - V(t,\phi_2)| \le L(k)\|\phi_1 - \phi_2\|_x. \tag{60}$$

Theorem 3.6 *If system (1) is ISS in the sense of Definition 2.2, then there exist a continuous functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$ and function ρ of class $\mathcal{K}$ such that*

1) $\alpha_1(\|\phi\|_x) \le V(t,\phi) \le \alpha_2(\|\phi\|_x), \forall \phi \in \mathbb{B}_x$,

2) $\forall t \ge t_0, \forall \phi \in \mathbb{B}_x$,

$$\dot{V}_{(1,v)}(t,\phi) \le -V(t,\phi) \tag{61}$$

whenever $\|\phi\|_x \ge \rho(\|v\|_u)$. Furthermore, the functional V is bounded set Lipschitz in the sense that for any bounded set Ω of $\mathbb{B}_x$, there exists $L(\Omega) > 0$ such that for all $\phi_1, \phi_2 \in \Omega$,

$$|V(t,\phi_1) - V(t,\phi_2)| \le L(\Omega)\|\phi_1 - \phi_2\|_x. \tag{62}$$

The proofs of these two theorems are very similar, and we thus only show the proof of Theorem 3.6.

Proof of Theorem 3.6: It follows from Lemma 3.3 that there exists ρ of class $\mathcal{K}_\infty$ such that following system is URAS if $\|v\|_u \le \rho(\|x_t\|_x)$,

$$\dot{x} = f(t, x_t, v). \tag{63}$$

It is also noted that $\dot{V}_{(1,v)}(t,\phi) = \dot{V}_{(63,v)}(t,\phi)$. It is without loss of generality to assume that ρ is globally Lipschitz with the unit constant, i.e., $|\rho(s_1) - \rho(s_2)| \le |s_1 - s_2|$. Otherwise, we only need to replace ρ by a new function $\tilde{\rho}(s) = \inf_{\tau \ge 0} \left\{ \min\{\frac{1}{2}\tau, \rho(\tau)\} + |\tau - s| \right\}$ which is globally Lipschitz with the unit constant, of class $\mathcal{K}_\infty$ and satisfies $\tilde{\rho}(s) \le \rho(s)$ (Karafyllis, 2006). And Lemma 3.3 can also be established by replacing ρ with $\tilde{\rho}$. Next, we consider the following system,

$$\dot{y} = f(t, y_t, d\rho(\|y_t\|_x)) \triangleq g(t, y_t, d), \tag{64}$$

where f is defined in (1), $y_t \in \mathbb{B}_x$ and $d \in \mathbb{B}_u$ with $\|d\|_u \le 1$. Define $\mathcal{D} = \{d \in \mathbb{B}_u : \|d\|_u \le 1\}$. We can verify that g is completely continuous since f is completely continuous. Furthermore, it can be shown that g satisfies conditions 1)-3) of system (47) by the following reasoning.

1) Since f is RS-L, for any bounded set $\Omega \subseteq \mathbb{B}_x$, there exists $L(\Omega) > 0$ such that for any $\phi_1, \phi_2 \in \mathbb{B}_x$ and $d \in \mathcal{D}$,

$$\begin{aligned}
&(\phi_1(0) - \phi_2(0))^T (g(t,\phi_1,d) - g(t,\phi_2,d)) \\
=&(\phi_1(0) - \phi_2(0))^T \times \\
&(f(t,\phi_1, d\rho(\|\phi_1\|_x)) - f(t,\phi_2, d\rho(\|\phi_2\|_x))) \\
\le& L(\Omega)(\|\phi_1 - \phi_2\|_x^2 + \|d\|_u^2 |\rho(\|\phi_1\|_x) - \rho(\|\phi_2\|_x)|^2) \\
\le& 2L(\Omega)\|\phi_1 - \phi_2\|_x^2.
\end{aligned} \tag{65}$$

2) $g(t, 0, d) = f(t, 0, 0) = 0$.

3) Condition 3) directly follows from complete continuity of f and the fact that $g(t,\phi,d) = f(t,\phi,d\rho(\|\phi\|_x))$.

It then follows from Lemma 3.3 that system (64) is UGRAS if system (1) is ISS. Let $y(\sigma,\phi,d)$ and $x(\sigma,\phi,u)$ denote the solutions of system (64) and (1), respectively. It follows from Theorem 3.4 that there exist a continuous functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$ and functions α_1, α_2 of class $\mathcal{K}_\infty$ such that for all $\phi \in \mathbb{B}_x$ and $d \in \mathcal{D}$,

$$\begin{aligned}
&\alpha_1(\|\phi\|_x) \le V(t,\phi) \le \alpha_2(\|\phi\|_x) \\
&\dot{V}_{(64,d)}(t,\phi) = \limsup_{h \to 0^+} \frac{V(t+h, y_{t+h}(t,\phi,\hat{d})) - V(t,\phi)}{h} \\
&\qquad \le -V(t,\phi),
\end{aligned} \tag{66}$$

where $\hat{d} \in \mathcal{M}(\mathbb{R}^+, \mathbb{B}_u)$ is a functional such that for some $\hat{h} > 0$, $\hat{d}(t+s) \equiv d, \forall s \in [0, \hat{h})$. Moreover, it follows from Theorem 3.4 that V is bounded set Lipschitz, i.e., there exists $L(\cdot)$ such that for all $\phi_1, \phi_2 \in \mathbb{B}_x$,

$$|V(t,\phi_1) - V(t,\phi_2)| \le L(\|\phi_1\|_x + \|\phi_2\|_x)\|\phi_1 - \phi_2\|_x. \tag{67}$$

It then follows from the similar discussions in the proof of Lemma 3.1 that for all $t \ge 0$, $\phi \in \mathbb{B}_x$, $v \in \mathbb{B}_u$ and $d \in \mathcal{D}$,

$$\begin{aligned}
&\dot{V}_{(1,v)}(t,\phi) - \dot{V}_{(64,d)}(t,\phi) \\
\le& L(2\|\phi\|_x)K \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |f(s, x_s(t,\phi,\hat{u}), \hat{u}(s)) \\
&- g(s, y_s(t,\phi,\hat{d}), \hat{d}(s))| ds.
\end{aligned} \tag{68}$$

Since $\hat{u}$ and $\hat{d}$ are right continuous on an interval $[t, t+\hat{h})$, then we have that $\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |f(s, x_s(t,\phi,\hat{u}), \hat{u}(s)) - g(s, y_s(t,\phi,\hat{d}), \hat{d}(s))| ds = |f(t,\phi,v) - g(t,\phi,d)|$, which implies that

$$\begin{aligned}
&\dot{V}_{(1,v)}(t,\phi) - \dot{V}_{(64,d)}(t,\phi) \\
\le& |f(t,\phi,v) - g(t,\phi,d)|.
\end{aligned} \tag{69}$$

It is noted that (69) holds for all $d \in \mathcal{D}$, i.e., $\|d\|_u \le 1$. For any $(t,\phi) \in \mathbb{R}^+ \times \mathbb{B}_x$, if $\|v\|_u \le \rho(\|\phi\|_x)$, then there exists $d_v \in \mathcal{D}$ such that $v = d_v \rho(\|\phi\|_x)$ and $\|d_v\|_u \le 1$, which implies that $f(t,\phi,v) = f\big(t,\phi,d_v\rho(\|\phi\|_x)\big) = g(t,\phi,d_v)$. Therefore, one has that $\forall (t,\phi) \in \mathbb{R}^+ \times \mathbb{B}_x$,

$$\begin{aligned}
&\dot{V}_{(1,v)}(t,\phi) \\
\le& \dot{V}_{(64,d_v)}(t,\phi) + |f(t,\phi,v) - g(t,\phi,d_v)| \\
=& \dot{V}_{(64,d_v)}(t,\phi) \le -V(t,\phi)
\end{aligned} \tag{70}$$

whenever $\|\phi\|_x \geq \rho^{-1}(\|v\|_u)$. The theorem is thus proved. ■

Based on the Lyapunov theorems and converse Lyapunov theorems, we can obtain the following characterizations for LISS and ISS.

Theorem 3.7 *For system* (1)*, the following conditions are equivalent:*

a) *The system is LISS in the sense of Definition 2.2;*
b) *There exist a locally Lipschitz functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$, positive constants k_x, k_u, functions α_1, α_2 of class $\mathcal{K}_\infty$, and functions α_3, ρ of class $\mathcal{K}$ such that for all $v : \|v\|_u \leq k_u$,*
   1) $\alpha_1(|\phi(0)|) \leq V(t,\phi) \leq \alpha_2(\|\phi\|_x), \forall \phi \in \mathbb{B}_x$,
   2) $\forall t \geq t_0, \forall \phi : \|\phi\|_x < k_x$,
$$\dot{V}_{(1,v)}(t,\phi) \leq -\alpha_3(\|\phi\|_x) \tag{71}$$
   whenever $\|\phi\|_x \geq \rho(\|v\|_u)$;
c) *For some $k_x > 0$, $k_u > 0$ and $\Omega_{k_x} = \{\phi \in \mathbb{B}_x : \|\phi\|_x < k_x\}$, there exist a continuous functional $V : \mathbb{R}^+ \times \Omega_{k_x} \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, function ρ of class $\mathcal{K}$ such that for all $v : \|v\|_u < k_u$,*
   1) $\alpha_1(\|\phi\|_x) \leq V(t,\phi) \leq \alpha_2(\|\phi\|_x), \forall \phi \in \Omega_{k_x}$,
   2) $\forall t \geq t_0, \forall \phi \in \Omega_{k_x}$,
$$\dot{V}_{(1,v)}(t,\phi) \leq -V(t,\phi) \tag{72}$$
   whenever $\|\phi\|_x \geq \rho(\|v\|_u)$. Furthermore, the functional V is Lipschitz in Ω_{k_x}, i.e., there exists $L(k_x) > 0$ such that for all $\phi_1, \phi_2 \in \Omega_{k_x}$,
$$|V(t,\phi_1) - V(t,\phi_2)| \leq L(k)\|\phi_1 - \phi_2\|_x. \tag{73}$$

Theorem 3.8 *For system* (1)*, the following conditions are equivalent:*

a) *The system is ISS in the sense of Definition 2.2;*
b) *There exist a locally Lipschitz functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, and functions α_3, ρ of class $\mathcal{K}$ such that*
   1) $\alpha_1(|\phi(0)|) \leq V(t,\phi) \leq \alpha_2(\|\phi\|_x), \forall \phi \in \mathbb{B}_x$,
   2) $\forall t \geq t_0, \forall \phi \in \mathbb{B}_x$,
$$\dot{V}_{(1,v)}(t,\phi) \leq -\alpha_3(\|\phi\|_x) \tag{74}$$
   whenever $\|\phi\|_x \geq \rho(\|v\|_u)$;
c) *There exist a continuous functional $V : \mathbb{R}^+ \times \mathbb{B}_x \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$ and function ρ of class $\mathcal{K}$ such that*
   1) $\alpha_1(\|\phi\|_x) \leq V(t,\phi) \leq \alpha_2(\|\phi\|_x), \forall \phi \in \mathbb{B}_x$,
   2) $\forall t \geq t_0, \forall \phi \in \mathbb{B}_x$,
$$\dot{V}_{(1,v)}(t,\phi) \leq -V(t,\phi) \tag{75}$$
   whenever $\|\phi\|_x \geq \rho(\|v\|_u)$. Furthermore, the functional V is bounded set Lipschitz in the sense that for any bounded set Ω of $\mathbb{B}_x$, there exists $L(\Omega) > 0$ such that for all $\phi_1, \phi_2 \in \Omega$,
$$|V(t,\phi_1) - V(t,\phi_2)| \leq L(\Omega)\|\phi_1 - \phi_2\|_x. \tag{76}$$

We only show the proof of Theorem 3.8.

Proof of Theorem 3.8: It follows from Theorem 3.2 that b)⇒a). It also follows from Theorem 3.6 that a)⇒c). Moreover, it is not difficult to prove that c)⇒ b). The theorem can be thus established.

3.3 Proofs of Theorems 3.3 and 3.4

In this subsection, we provide the proofs of Theorems 3.3 and 3.4. The proofs of these two theorems are quite similar and thus we only show the proof of Theorem 3.4. Before presenting our proof, the following lemma is needed.

Lemma 3.4 *If system* (47) *is URAS, then there exists a positive constant $k > 0$ such that for any $\|\phi_1\|_x < k$ and $\|\phi_2\|_x < k$, the following inequality holds for all $d \in \mathcal{M}_\mathcal{D}$ and any $t \geq \sigma \geq t_0$,*
$$\begin{aligned}&\|x_t(\sigma,\phi_1,d) - x_t(\sigma,\phi_2,d)\|_x\\ \leq& N(\|\phi_1\|_x + \|\phi_2\|_x, t-\sigma)\|\phi_1 - \phi_2\|_x,\end{aligned} \tag{77}$$
where $N(r,t)$ is continuous, positive and nondecreasing on both arguments. Furthermore, if system (47) *is URGAS, then inequality* (77) *holds for all $\phi_1, \phi_2 \in \mathbb{B}_x$.*

Proof: We only provide the proof for the UGRAS case. The proof for the URAS case is similar and thus omitted. If system (47) is URGAS, then there exists a class $\mathcal{KL}$ function β, such that for any $\phi \in \mathbb{B}_x$ and $d \in \mathcal{M}_\mathcal{D}$,
$$\|x_t(\sigma,\phi,d)\|_x \leq \beta(\|\phi\|_x, t-\sigma). \tag{78}$$
For any $\phi_1, \phi_2 \in \mathbb{B}_x$, define $h(t) = |x(\sigma,\phi_1,d)(t) - x(\sigma,\phi_2,d)(t)|^2$. For notation convenience, we denote $x_t(\sigma,\phi_1,d)$, $x_t(\sigma,\phi_2,d)$, $x(\sigma,\phi_1,d)(t)$ and $x(\sigma,\phi_2,d)(t)$ by $y_t, z_t, y(t)$ and $z(t)$, respectively. It can be thus obtained that $h(t) = |y(t) - z(t)|^2$. Then it follows from the RS-L condition that there exists a positive function $L(\cdot)$ such that
$$\begin{aligned}\dot{h}(t) =& 2(y(t)-z(t))^T(g(t,y_t,d(t)) - g(t,z_t,d(t)))\\ \leq& L(\|y_t\|_x + \|z_t\|_x)\|y_t - z_t\|_x^2\\ \leq& \bar{L}(\|\phi_1\|_x + \|\phi_2\|_x)\|y_t - z_t\|_x^2 \quad a.e.,\end{aligned} \tag{79}$$
where $\bar{L}(\|\phi_1\|_x + \|\phi_2\|_x) = L(2\beta(\|\phi_1\|_x + \|\phi_2\|_x, 0))$. Therefore, by denoting $\bar{L} = \bar{L}(\|\phi_1\|_x + \|\phi_2\|_x)$, we have that
$$\begin{aligned}h(t) \leq& h(\sigma) + \bar{L}\int_\sigma^t \|y_s - z_s\|_x^2 ds\\ =& |\phi_1(0) - \phi_2(0)|^2 + \bar{L}\int_\sigma^t \|y_s - z_s\|_x^2 ds.\end{aligned} \tag{80}$$
Moreover, since $y_\sigma = \phi_1$ and $z_\sigma = \phi_2$, then it follows from Hypothesis 2.1 that
$$\begin{aligned}&\|y_t - z_t\|_x^2\\ \leq&(K\sup_{\sigma\leq s\leq t}|y(s) - z(s)| + M(t-\sigma)\|\phi_1 - \phi_2\|_x)^2\\ \leq& 2K^2\sup_{\sigma\leq s\leq t} h(s) + 2M^2(t-\sigma)\|\phi_1 - \phi_2\|_x^2\\ \leq& 2K^2|\phi_1(0) - \phi_2(0)|^2 + 2M^2(t-\sigma)\|\phi_1 - \phi_2\|_x^2\\ &+ 2K^2\bar{L}\int_\sigma^t \|y_s - z_s\|_x^2 ds\\ \leq&(2K^2M_0^2 + 2\bar{M}^2)\|\phi_1 - \phi_2\|_x^2 + 2K^2\bar{L}\int_\sigma^t \|y_s - z_s\|_x^2 ds,\end{aligned}$$

where $\bar{M} = \sup_{s\geq 0} M(s)$. By applying the well known Gronwall's inequality, we have that

$$\|y_t - z_t\|_x \leq \sqrt{2K^2M_0^2 + 2\bar{M}^2} e^{K^2\bar{L}(t-\sigma)} \|\phi_1 - \phi_2\|_x. \tag{81}$$

This lemma can thus be established by choosing

$$\begin{aligned} &N(\|\phi_1\|_x + \|\phi_2\|_x, t-\sigma) \\ =&\sqrt{2K^2M_0^2 + \bar{M}^2} e^{K^2\bar{L}(\|\phi_1\|_x + \|\phi_2\|_x)(t-\sigma)}. \end{aligned} \tag{82}$$

□

Then we are ready to present the proof of Theorem 3.4 as follows.

Proof of Theorem 3.4: It follows from the URGAS property that for any $\phi \in \mathbb{B}_x$ and any $d \in \mathcal{M}_{\mathcal{D}}$,

$$\|x_t(\sigma, \phi, d)\|_x \leq \beta(\|\phi\|_x, t-\sigma), \forall t \geq \sigma \geq t_0, \tag{83}$$

where β is a class $\mathcal{KL}$ function. It can be further obtained by Lemma A.1 that there exists two class $\mathcal{K}_\infty$ functions α_1, α_2 such that

$$\begin{aligned} &\beta(s,t) \leq \alpha_1^{-1}(\alpha_2(s)e^{-2t}), \\ &|\alpha_1(s_1) - \alpha_1(s_2)| \leq |s_1 - s_2|, \forall s_1, s_2 \geq 0. \end{aligned} \tag{84}$$

Then for any positive integer $q \in \mathbb{Z}^+$ and any $\phi \in \mathbb{B}_x$, we can define the following functional,

$$W_q(t,\phi) = \sup_{\substack{\tau\geq 0 \\ d\in\mathcal{M}_{\mathcal{D}}}} \big\{ \max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d)\|_x) - q^{-1}\} e^{\tau} \big\}.$$

Next, we will prove the following properties of $W_q(t,\phi)$.

Property 1: For any $\phi \in \mathbb{B}_x$, $\max\{0, \alpha_1(\|\phi\|_x) - q^{-1}\} \leq W_q(t,\phi) \leq \alpha_2(\|\phi\|_x)$.

Since $x_{t+\tau}(t,\phi,d) = \phi$ when $\tau = 0$, the left hand side of the inequality holds by considering the case $\tau = 0$. It follows from (83) that

$$\|x_{t+\tau}(t,\phi,d)\|_x \leq \beta(\|\phi\|_x, \tau), \forall \tau \geq 0. \tag{85}$$

Then the right hand side holds since for all $\phi \in \mathbb{B}_x$ and $\tau \geq 0$,

$$\begin{aligned} &\alpha_1(\|x_{t+\tau}(t,\phi,d)\|_x)e^{\tau} \\ \leq&\alpha_1(\beta(\|\phi\|_x,\tau))e^{\tau} \leq \alpha_2(\|\phi\|_x)e^{-\tau} \leq \alpha_2(\|\phi\|_x). \end{aligned} \tag{86}$$

Property 2: There exists $T(\|\phi\|_x, q) \geq 0$ such that $W_q(t,\phi) = \sup_{\substack{T(\|\phi\|_x,q)\geq\tau\geq 0 \\ d\in\mathcal{M}_{\mathcal{D}}}} \big\{ \max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d)\|_x) - q^{-1}\} e^{\tau} \big\}$.

Choose $T(r,q) = \max\{0, \frac{1}{2}\ln(q\alpha_2(r+1))\}$. One has that for any $\tau \geq T(\|\phi\|_x, q)$,

$$\begin{aligned} &\alpha_1(\|x_{t+\tau}(t,\phi,d)\|_x) - q^{-1} \\ \leq&\alpha_1(\beta(\|\phi\|_x, T(\|\phi\|_x,q))) - q^{-1} \\ \leq&\alpha_2(\|\phi\|_x)e^{-2T(\|\phi\|_x,q)} - q^{-1} < 0, \end{aligned} \tag{87}$$

which implies this property.

The Property 2 shows that the value of $W_q(t,\phi)$ is only determined by $\tau \in [0, T(\|\phi\|_x, q)]$. Based on this property, we can prove the next one.

Property 3: There exists a continuous function $L(r,q)$ such that for all $\phi_1, \phi_2 \in \mathbb{B}_x$,

$$|W_q(t,\phi_1) - W_q(t,\phi_2)| \leq L(\|\phi_1\|_x + \|\phi_2\|_x, q)\|\phi_1 - \phi_2\|_x.$$

It can be obtained from Property 2 that $W_q(t,\phi_i) = \sup_{\substack{T(\|\phi_1\|_x+\|\phi_2\|_x,q)\geq\tau\geq 0 \\ d\in\mathcal{M}_{\mathcal{D}}}} \big\{ \max\{0, \alpha_1(\|x_{t+\tau}(t,\phi_i,d)\|_x) - q^{-1}\} e^{\tau} \big\}$, for $i = 1, 2$.

It further follows from the facts $|\max\{0, r_1\} - \max\{0, r_2\}| \leq |r_1 - r_2|$ and $|\alpha_1(s_1) - \alpha_2(s_2)| \leq |s_1 - s_2|$ that

$$\begin{aligned} &|W_q(t,\phi_1) - W_q(t,\phi_2)| \\ \leq& \sup_{\substack{T(\|\phi_1\|_x+\|\phi_2\|_x,q)\geq\tau\geq 0 \\ d\in\mathcal{M}_{\mathcal{D}}}} \|x_{t+\tau}(t,\phi_1,d) - x_{t+\tau}(t,\phi_2,d)\|_x e^{\tau}. \end{aligned}$$

Moreover, it follows from Lemma 3.4 that

$$\begin{aligned} &\|x_{t+\tau}(t,\phi_1,d) - x_{t+\tau}(t,\phi_2,d)\|_x \\ \leq&N(\|\phi_1\|_x + \|\phi_2\|_x, \tau)\|\phi_1 - \phi_2\|_x, \end{aligned} \tag{88}$$

where $N(r,t)$ is a continuous, nonnegative and nondecreasing on both arguments. Therefore, Property 3 can be established by defining

$$L(r,q) = N(r, T(r,q))e^{T(r,q)}. \tag{89}$$

It can be verified that $L(r,q)$ is continuous, nonnegative and nondecreasing on both arguments.

Property 4: For any $h \geq 0$, any $d \in \mathcal{M}_{\mathcal{D}}$ and any $\phi \in \mathbb{B}_x$, there holds

$$W_q(t+h, x_{t+h}(t,\phi,d)) \leq e^{-h}W_q(t,\phi). \tag{90}$$

It can be first found that for any $d, \bar{d} \in \mathcal{M}_{\mathcal{D}}$, there exists $\tilde{d} \in \mathcal{M}_{\mathcal{D}}$ such that

$$x_{t+h+\tau}(t+h, x_{t+h}(t,\phi,d), \bar{d}) = x_{t+h+\tau}(t,\phi,\tilde{d}), \tag{91}$$

where $\tilde{d}$ is defined as

$$\tilde{d}(s) = \begin{cases} d(s), \ s \in [t, t+h), \\ \bar{d}(s), \ s \geq t+h. \end{cases} \tag{92}$$

Therefore, we can obtain that

$$\begin{aligned} &\max\{0, \alpha_1(\|x_{t+h+\tau}(t+h, x_{t+h}(t,\phi,d), \bar{d})\|_x) - q^{-1}\}e^{\tau} \\ =&\max\{0, \alpha_1(\|x_{t+h+\tau}(t,\phi,\tilde{d})\|_x) - q^{-1}\}e^{\tau} \\ \leq& \sup_{\substack{\tau\geq 0 \\ \tilde{d}\in\mathcal{M}_{\mathcal{D}}}} \{\max\{0, \alpha_1(\|x_{t+h+\tau}(t,\phi,\tilde{d})\|_x) - q^{-1}\}e^{\tau}\} \\ =&e^{-h} \sup_{\substack{\tau\geq h \\ \tilde{d}\in\mathcal{M}_{\mathcal{D}}}} \{\max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,\tilde{d})\|_x) - q^{-1}\}e^{\tau}\} \\ \leq&e^{-h}W_q(t,\phi). \end{aligned} \tag{93}$$

In fact, $W_q(t+h, x_{t+h}(t,\phi,d))$ is nothing but the supremum of the first term of inequality (93) when $\tau \geq 0$ and $\bar{d} \in \mathcal{M}_{\mathcal{D}}$. The property is thus proved.

Property 5: There exists a nonnegative function $G : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{Z}^+ \to \mathbb{R}^+$ such that for any $\varepsilon > 0$, any $\phi \in \mathbb{B}_x$ and any $t \in \mathbb{R}^+$, there exists $\delta = \delta(t,\phi,\varepsilon) > 0$, which is independent of q, such that $|h| < \delta$ implies

$$|W_q(t+h,\phi) - W_q(t,\phi)| \leq (\alpha_2(\|\phi\|_x) + G(t, \|\phi\|_x, q))\varepsilon.$$

This property, together with Property 3, implies continuity of $W_q(t, \phi)$.

We first consider the case when $h \geq 0$. It follows from Lemma A.2 that there exist a class $\mathcal{K}_\infty$ function ζ and a positive and nondecreasing function χ such that

$$|g(t,\phi,d)| \leq \zeta(\chi(t)\|\phi\|_x), \forall (t,\phi,d) \in \mathbb{R}^+ \times \mathbb{B}_x \times \mathcal{D}. \tag{94}$$

It can be obtained from equation (47) that $\forall \theta \in [-h, 0]$,

$$x_{t+h}(t,\phi,d)(\theta) = \phi(0) + \int_t^{t+\theta+h} g(s, x_s(t,\phi,d), d(s))ds.$$

Next, for any $h \geq 0$ and $\phi \in \mathbb{B}_x$, we define ϕ_h as follows,

$$\phi_h(\theta) = \begin{cases} \phi(\theta+h), & \theta \leq -h, \\ \phi(0), & \theta \in [-h, 0]. \end{cases} \tag{95}$$

It follows from condition 1) of Hypothesis 2.1 that $\phi_h \in \mathbb{B}_x$ and from condition 2) of Hypothesis 2.1 that $\lim_{h\to 0^+} \|\phi_h - \phi\|_x = 0$. It can also be verified that for any $\theta \leq -h$, $x_{t+h}(t,\phi,d)(\theta) = x(t,\phi,d)(t+h+\theta) = \phi(\theta+h) = \phi_h(\theta)$. Therefore, we have that for any $\phi \in \mathbb{B}_x$ and any $d \in \mathcal{M}_\mathcal{D}$,

$$\begin{aligned}
&\|x_{t+h}(t,\phi,d) - \phi\|_x \\
\leq&\|x_{t+h}(t,\phi,d) - \phi_h\|_x + \|\phi_h - \phi\|_x \\
\leq&K \sup_{\theta\in[-h,0]} |x_{t+h}(t,\phi,d)(\theta) - \phi_h(\theta)| + \|\phi_h - \phi\|_x \\
=&K \sup_{\theta\in[-h,0]} |x_{t+h}(t,\phi,d)(\theta) - \phi(0)| + \|\phi_h - \phi\|_x \\
\leq&K \int_t^{t+h} |g(s, x_s(t,\phi,d), d(s))|ds + \|\phi_h - \phi\|_x \\
\leq&Kh\zeta(\chi(t+h)\beta(\|\phi\|_x, 0)) + \|\phi_h - \phi\|_x \\
\leq&Kh\zeta(\chi(t+h)\alpha_1^{-1}(\alpha_2(\|\phi\|_x))) + \|\phi_h - \phi\|_x \\
\triangleq&KhG_1(t+h, \|\phi\|_x) + G_2(h,\phi),
\end{aligned} \tag{96}$$

where $G_1(t+h, \|\phi\|_x) = \zeta(\chi(t+h)\alpha_1^{-1}(\alpha_2(\|\phi\|_x)))$ and $G_2(h,\phi) = \|\phi_h - \phi\|_x$. It can be found that G_1 is nonnegative and nondecreasing on both arguments and $\lim_{h\to 0^+} G_2(h,\phi) = 0$. Thus there exists $\delta_1 = \delta_1(t,\phi) \in (0,1]$ such that for all $h \in [0,\delta_1)$, $\|x_{t+h}(t,\phi,d) - \phi\|_x \leq 1$ which implies $\|x_{t+h}(t,\phi,d)\|_x \leq \|\phi\|_x + 1$.

It follows from the definition of $W_q(t,\phi)$ that $\forall \mu > 0$, there exists $d_\mu \in \mathcal{M}_\mathcal{D}$ such that

$$W_q(t,\phi) \leq \mu + \sup_{\tau\geq 0}\{\max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d_\mu)\|_x) - q^{-1}\}e^\tau\}.$$

In addition, it follows from Properties 3 and 4 that for all $h \in [0,\delta_1)$ and all $d \in \mathcal{M}_\mathcal{D}$,

$$\begin{aligned}
&|W_q(t+h,\phi) - W_q(t,\phi)| \\
\leq&|W_q(t+h,\phi) - W_q(t+h, x_{t+h}(t,\phi,d_\mu))| \\
&+ |e^{-h}W_q(t,\phi) - W_q(t+h, x_{t+h}(t,\phi,d_\mu))| \\
&+ (1-e^{-h})W_q(t,\phi) \\
\leq&L(2\|\phi\|_x + 1, q)\|x_{t+h}(t,\phi,d_\mu) - \phi\|_x \\
&+ |e^{-h}W_q(t,\phi) - W_q(t+h, x_{t+h}(t,\phi,d_\mu))| \\
&+ (1-e^{-h})W_q(t,\phi) \\
\leq&L(2\|\phi\|_x + 1, q)(KhG_1(t+1, \|\phi\|_x) + G_2(h,\phi)) \\
&+ (e^{-h}W_q(t,\phi) - W_q(t+h, x_{t+h}(t,\phi,d_\mu))) \\
&+ (1-e^{-h})W_q(t,\phi).
\end{aligned} \tag{97}$$

Furthermore, there holds that

$$\begin{aligned}
&e^{-h}W_q(t,\phi) - W_q(t+h, x_{t+h}(t,\phi,d_\mu)) \\
\leq&e^{-h}\mu + e^{-h}\sup_{\tau\geq 0}\{\max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d_\mu)\|_x) \\
&- q^{-1}\}e^\tau\} - \sup_{\tau\geq 0}\{\max\{0, \\
&\alpha_1(\|x_{t+\tau+h}(t, x_{t+h}(t,\phi,d_\mu), d_\mu)\|_x) - q^{-1}\}e^\tau\} \\
=&e^{-h}\mu + e^{-h}\sup_{\tau\geq 0}\{\max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d_\mu)\|_x) \\
&- q^{-1}\}e^\tau\} - \sup_{\tau\geq 0}\{\max\{0, \\
&\alpha_1(\|x_{t+\tau+h}(t,\phi,d_\mu)\|_x) - q^{-1}\}e^\tau\} \\
=&e^{-h}\mu + e^{-h}\sup_{\tau\geq 0}\{\max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d_\mu)\|_x) \\
&- q^{-1}\}e^\tau\} - e^{-h}\sup_{\tau\geq h}\{\max\{0, \\
&\alpha_1(\|x_{t+\tau}(t,\phi,d_\mu)\|_x) - q^{-1}\}e^\tau\}.
\end{aligned} \tag{98}$$

For notation convenience, let

$$\omega(t,\tau,\phi) = \{\max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d_\mu)\|_x) - q^{-1}\}e^\tau. \tag{99}$$

Then we have that

$$\sup_{\tau\geq 0}\omega(t,\tau,\phi) = \max\{\sup_{\tau\geq h}\omega(t,\tau,\phi), \sup_{h\geq\tau\geq 0}\omega(t,\tau,\phi)\}. \tag{100}$$

We can thus proceed by considering the following two cases.

Case 1: If $\sup_{\tau\geq 0}\omega(t,\tau,\phi) = \sup_{\tau\geq h}\omega(t,\tau,\phi)$, then it can be obtained that

$$e^{-h}W_q(t,\phi) - W_q(t+h, x_{t+h}(t,\phi,d_\mu) \leq e^{-h}\mu. \tag{101}$$

Case 2: If $\sup_{\tau\geq 0}\omega(t,\tau,\phi) = \sup_{h\geq\tau\geq 0}\omega(t,\tau,\phi)$, then it can be obtained that

$$\begin{aligned}
&e^{-h}W_q(t,\phi) - W_q(t+h, x_{t+h}(t,\phi,d_\mu) \\
\leq&e^{-h}\mu + e^{-h}\sup_{h\geq\tau\geq 0}\omega(t,\tau,\phi) - e^{-h}\sup_{\tau\geq h}\omega(t,\tau,\phi) \\
\leq&e^{-h}\mu + e^{-h}\sup_{h\geq\tau\geq 0}\omega(t,\tau,\phi) - e^{-h}\omega(t,h,\phi) \\
\leq&e^{-h}\mu + \sup_{h\geq\tau\geq 0}\{\max\{0, \alpha_1(\|x_{t+\tau}(t,\phi,d_\mu)\|_x) - q^{-1}\} \\
&- \{\max\{0, \alpha_1(\|x_{t+h}(t,\phi,d_\mu)\|_x) - q^{-1}\} \\
\leq&e^{-h}\mu + \sup_{h\geq\tau\geq 0}\|x_{t+\tau}(t,\phi,d_\mu) - x_{t+h}(t,\phi,d_\mu)\|_x,
\end{aligned} \tag{102}$$

where the last inequality holds since α_1 and $\max\{0, s - q^{-1}\}$ are globally Lipschitz with the unit constant. Furthermore, it follows from (96) that

$$\begin{aligned}
&e^{-h}W_q(t,\phi)-W_q(t+h,x_{t+h}(t,\phi,d_\mu)) \\
\leq& e^{-h}\mu+\sup_{h\geq\tau\geq 0}\|x_{t+\tau}(t,\phi,d_\mu)-\phi\|_x \\
&+\|x_{t+h}(t,\phi,d_\mu)-\phi\|_x \\
\leq& e^{-h}\mu+\sup_{h\geq\tau\geq 0}\left\{K\tau G_1(t+\tau,\|\phi\|_x)+G_2(\tau,\phi)\right\} \\
&+KhG_1(t+h,\|\phi\|_x)+G_2(h,\phi) \\
\leq& e^{-h}\mu+2KhG_1(t+h,\|\phi\|_x)+2\sup_{h\geq\tau\geq 0}G_2(\tau,\phi).
\end{aligned} \tag{103}$$

It can be concluded from both cases that $e^{-h}W_q(t,\phi)-W_q(t+h,x_{t+h}(t,\phi,d_\mu))\leq e^{-h}\mu+2KhG_1(t+h,\|\phi\|_x)+2\sup_{h\geq\tau\geq 0}G_2(\tau,\phi)$. Therefore, it follows from (97) that for all $h\in[0,\delta_1)$,

$$\begin{aligned}
&|W_q(t+h,\phi)-W_q(t,\phi)| \\
\leq& L(2\|\phi\|_x+1,q)\big(KhG_1(t+1,\|\phi\|_x)+G_2(h,\phi)\big) \\
&+e^{-h}\mu+2KhG_1(t+1,\|\phi\|_x)+2\sup_{h\geq\tau\geq 0}G_2(\tau,\phi) \\
&+(1-e^{-h})W_q(t,\phi) \\
\leq& e^{-h}\mu+h\alpha_2(\|\phi\|_x)+(2+L(2\|\phi\|_x+1,q))\times \\
&(KhG_1(t+1,\|\phi\|_x)+\sup_{h\geq\tau\geq 0}G_2(\tau,\phi)).
\end{aligned} \tag{104}$$

Since inequality (104) holds for all $\mu>0$ and its first term is independent of μ, then we have that

$$\begin{aligned}
&|W_q(t+h,\phi)-W_q(t,\phi)| \\
\leq& h\alpha_2(\|\phi\|_x)+(2+L(2\|\phi\|_x+1,q))\times \\
&(KhG_1(t+1,\|\phi\|_x)+\sup_{h\geq\tau\geq 0}G_2(\tau,\phi)).
\end{aligned} \tag{105}$$

Next, we consider the case $h\leq 0$. An inequality similar to (105) can be obtained by noting the fact that $t^*=t+h$ and $t=t^*+|h|$. Then it can be obtained that for all $|h|<\delta_1$,

$$\begin{aligned}
&|W_q(t+h,\phi)-W_q(t,\phi)| \\
\leq& |h|\alpha_2(\|\phi\|_x)+(2+L(2\|\phi\|_x+1,q))\times \\
&(K|h|G_1(t+1,\|\phi\|_x)+\sup_{|h|\geq\tau\geq 0}G_2(\tau,\phi)).
\end{aligned} \tag{106}$$

Moreover, since $\lim_{h\to 0^+}G_2(h,\phi)=0$, then for any $\varepsilon>0$ and $\phi\in\mathbb{B}_x$, there exists $\delta_2=\delta_2(\varepsilon,\phi)>0$ such that $\sup_{|h|\geq\tau\geq 0}G_2(\tau,\phi)<\varepsilon,\forall|h|<\delta_2$. Choosing $\delta=\min\{\delta_1,\delta_2,\varepsilon\}$, we can obtain from (106) that for all $|h|<\delta$,

$$|W_q(t+h,\phi)-W_q(t,\phi)|\leq\varepsilon(\alpha_2(\|\phi\|_x)+G(t,\|\phi\|_x,q)),$$

where $G(t,r,q)=(2+L(2r+1,q))(KG_1(t+1,r)+1)$. It can be verified that δ is independent of q. Property 5 is thus proved.

Based on Properties 1-5, we can construct the following Lyapunov functional,

$$V(t,\phi)=\sum_{q=1}^{+\infty}\frac{2^{-q}W_q(t,\phi)}{1+L(2q,q)+G(q,q,q)}, \tag{107}$$

where L and G are given in Properties 3 and 5, respectively. Next, we will prove that V is a continuous functional satisfying conditions in Theorem 3.4.

Define $\tilde{\alpha}_1:\mathbb{R}^+\to\mathbb{R}^+$ as follows,

$$\tilde{\alpha}_1(s)=\sum_{q=1}^{+\infty}\frac{2^{-q}\max\{0,\alpha_1(s)-q^{-1}\}}{1+L(2q,q)+G(q,q,q)}. \tag{108}$$

It can be verified that $\tilde{\alpha}_1$ is a class $\mathcal{K}_\infty$ function. Then it follows from Property 1 of W_q that $\forall\phi\in\mathbb{B}_x$,

$$\tilde{\alpha}_1(\|\phi\|_x)\leq V(t,\phi)\leq\alpha_2(\|\phi\|_x), \tag{109}$$

which means that V satisfies condition 1) of Theorem 3.4.

For any bounded set $\Omega\subseteq\mathbb{B}_x$, there exists $R>0$ such that $\|\phi\|_x\leq R,\forall\phi\in\Omega$. It follows from Property 3 of W_q that for all $\phi_1,\phi_2\in\Omega$,

$$\begin{aligned}
&|V(t,\phi_1)-V(t,\phi_2)| \\
\leq&\sum_{q=1}^{+\infty}\frac{2^{-q}L(2R,q)}{1+L(2q,q)+G(q,q,q)}\|\phi_1-\phi_2\|_x \\
\leq&\sum_{q=1}^{\lfloor R\rfloor}\frac{2^{-q}L(2R,q)}{1+L(2q,q)+G(q,q,q)}\|\phi_1-\phi_2\|_x \\
&+\sum_{q=\lfloor R\rfloor+1}^{+\infty}\frac{2^{-q}L(2q,q)}{1+L(2q,q)+G(q,q,q)}\|\phi_1-\phi_2\|_x \\
\leq&(1+\sum_{q=1}^{\lfloor R\rfloor}\frac{2^{-q}L(2R,q)}{1+L(2q,q)+G(q,q,q)})\|\phi_1-\phi_2\|_x \\
\triangleq&\tilde{L}(R)\|\phi_1-\phi_2\|_x,
\end{aligned} \tag{110}$$

which shows that V is bounded set Lipschitz.

For any $\varepsilon>0$, any $\phi\in\mathbb{B}_x$ and any $t\geq t_0$, it follows from Property 5 of W_q that there exists $\delta=\delta(t,\phi,\varepsilon)>0$ such that

$$\begin{aligned}
&|V(t+h,\phi)-V(t,\phi)| \\
\leq&\varepsilon\sum_{q=1}^{+\infty}\frac{2^{-q}(\alpha_2(\|\phi\|_x)+G(t,\|\phi\|_x,q))}{1+L(2q,q)+G(q,q,q)} \\
\leq&\alpha_2(\|\phi\|_x)\varepsilon+\tilde{G}(t,\|\phi\|_x)\varepsilon,
\end{aligned} \tag{111}$$

where $\tilde{G}(t,r)=1+\sum_{q=1}^{[t]+[\|\phi\|_x]}\frac{2^{-q}G(t,\|\phi\|_x,q)}{1+L(2q,q)+G(q,q,q)}$. By combining inequality (111) with the bounded set Lipschitz property, it can be concluded that V is continuous on $\mathbb{R}^+\times\mathbb{B}_x$.

It follows from Property 4 of W_q that for any $(t,\phi,d)\in\mathbb{R}^+\times\mathbb{B}_x\times\mathcal{M}_{\mathcal{D}}$ and any $h\geq 0$,

$$V(t+h,x_{t+h}(t,\phi,d))\leq e^{-h}V(t,\phi). \tag{112}$$

Therefore, one has that

$$\dot{V}_{(47,d)}(t,\phi) = \limsup_{h\to 0^+} \frac{V(t+h, x_{t+h}(t,\phi,\hat{d})) - V(t,\phi)}{h} \leq \limsup_{h\to 0^+} \frac{e^{-h}-1}{h} V(t,\phi) = -V(t,\phi). \tag{113}$$

Hence, V satisfies condition 2) of Theorem 3.4. The proof is thus completed. ∎

Remark 3.5 *The proof of Theorem 3.4 is inspired by Karafyllis (2006) where converse Lyapunov theorems for robust global asymptotic stability of systems with bounded delays are obtained. However, compared with Karafyllis (2006), many additional challenges arise due to the existence of infinite delays as systems with infinite delays involve more general but more complicated phase spaces and norms. For instance, the proof of Lemma 3.4 is more challenging and more complex. Moreover, to prove Property 5, we introduce a newly defined function ϕ_h in* (95) *based on Hypothesis 2.1. In the case of bounded delayed systems, ϕ_h is not needed since the norm $\|\cdot\|_x$ has a specific expression, i.e., $\|\phi\|_x = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|$ and the term $\|x_{t+h}(t,\phi,d) - \phi\|_x$ can be directly calculated. However, in the case of infinite-delayed systems, there are many different spaces and norms, see, for example, our previous works Xu et al. (2018, 2019, 2020a,b). By applying such a function ϕ_h, we can cover all those spaces and norms and also include the phase space for bounded delayed systems as a special case. It is thus noted that ϕ_h defined in* (95) *is critical and can thus be recognized as an additional contribution.*

3.4 Relation to Existing Works on ISS of Non-Delayed Systems and Bounded-Delayed Systems

The Lyapunov characterizations given in this section include some existing results on systems with no delays (Sontag & Wang, 1995) or bounded delays (Fridman et al., 2008) as their special cases. Moreover, compared with some other works (Pepe & Jiang, 2006; Mironchenko & Wirth, 2017; Mironchenko, 2016; Zhu & Hu, 2010) on ISS Lyapunov characterizations of bounded delayed systems, our work also has its distinctions when it reduces to the case of bounded delays. First, our work considers time-varying nonlinear systems while those works focus on time-invariant nonlinear systems. Second, a more general Lipschitz condition, i.e., the right hand side Lipschitz (RS-L) condition is adopted in our work. Last but not least, a new estimation on the upper bound of solutions is develoed for the proof of ISS Lyapunov functional theorem.

4 Conclusion

In this paper, we have considered Lyapunov characterizations on input-to-state stability of time-varying nonlinear systems with infinite delays. Novel definitions for both LISS and ISS are given, which include the ISS definitions of non delayed systems and bounded delayed systems as their special cases. We then provide Lyapunov characterizations for LISS and ISS of systems with infinite delays. It is shown that an infinite-delayed system is (locally) ISS if it has a (local) ISS Lyapunov functional, and conversely, there exists a (local) ISS Lyapunov functional if it is (locally) ISS. In order to obtain the converse ISS Lyapunov theorems, a key lemma and two converse Lyapunov theorems for *robust* asymptotic stability of infinite-delayed systems are also established. To our best knowledge, our work is the first one on ISS/LISS of infinite delayed systems. Our future research will focus on finding applications of ISS results obtained in this work.

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Appendix

In this Appendix, we provide some technical lemmas. The first lemma is a combination of the results in Karafyllis (2006) and Sontag (1998).

Lemma A.1 *Assume that β is of class $\mathcal{KL}$. Then there exists class $\mathcal{K}_\infty$ functions a, b such that*

$$\beta(s,t) \leq a^{-1}(b(s)e^{-2t}). \tag{.1}$$

Moreover, function a can be chosen to satisfy global Lipschitz condition with the unit constant, i.e., for all $s_1, s_2 \geq 0$,

$$|a(s_1) - a(s_2)| \leq |s_1 - s_2|. \tag{.2}$$

Lemma A.2 *(Karafyllis, 2004) Let $(\mathbb{B}_1, \|\cdot\|_1), (\mathbb{B}_2, \|\cdot\|_2), (\mathbb{B}_3, \|\cdot\|_3)$ be three normed vector spaces satisfying Hypothesis 2.1. Let $\mathcal{D}$ be a bounded subset of $\mathbb{B}_2$. If $H : \mathbb{R}^+ \times \mathbb{B}_1 \times \mathcal{D} \to \mathbb{B}_3$ satisfies the following conditions,*

1) for every bounded set $S \subseteq \mathbb{R}^+ \times \mathbb{B}_1$, the function H maps $S \times \mathcal{D}$ into a bounded set;

2) $H(t, 0, d) = 0$ for all $(t, d) \in \mathbb{R}^+ \times \mathcal{D}$; and

3) for any $\varepsilon > 0$ and any $t \in \mathbb{R}^+$, there exists $\delta = \delta(t,\varepsilon) > 0$ such that

$$|s - t| + \|\phi\|_1 < \delta \text{ implies } \sup_{d\in\mathcal{D}} \|H(s,\phi,d)\|_3 < \varepsilon; \tag{.3}$$

then there exist a class $\mathcal{K}_\infty$ function ζ and a positive and nondecreasing function χ such that

$$\|H(t,\phi,d)\|_3 \leq \zeta(\chi(t)\|\phi\|_1), \quad \forall (t,\phi,d) \in \mathbb{R}^+ \times \mathbb{B}_1 \times \mathcal{D}. \tag{.4}$$

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