Formation-Containment Tracking for Heterogeneous Linear Multi-Agent Systems Under Unbounded Distributed Transmission Delays

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Abstract—This paper considers the formation-containment tracking problem of heterogeneous linear multi-agent systems with unbounded distributed transmission delays. The multi-agent system under consideration consists of a reference leader with a desired trajectory, a group of formation-leaders with a prescribed formation, and a group of followers. Novel distributed control laws are proposed for the formation-leaders and followers, respectively. It is shown that under the proposed control laws, the outputs of the formation-leaders not only achieve a prescribed time-varying formation but also track the desired trajectory generated by the reference leader, while the output of each follower converges to the convex hull spanned by the outputs of those formation-leaders. It should be pointed out that many existing results on consensus tracking, formation tracking, and containment control of linear multi-agent systems with constant or bounded distributed transmission delays are included as special cases of our results. Finally, a numerical example is given to illustrate the effectiveness of the proposed controllers.

Index Terms—Formation-containment tracking, heterogeneous, multi-agent systems, unbounded distributed delays, transmission delays.

I. INTRODUCTION

Cooperative control of multi-agent systems (MASs), such as consensus, containment, and formation, has attracted increasing research interests over the past decade from different research fields, see, for example, [1–16]. Leader-following consensus control, as a typical class of consensus control, is mainly focused on a single leader, while in the context of multiple leaders, containment control problems often arise. The objective of containment control is to make all followers converge to the convex hull spanned by the trajectories of multiple leaders. Some relevant works on containment control of different MASs can be found in [9–12]. However, it should be noted that no interaction and coordination among leaders are considered in the aforementioned works on containment control.

On the other hand, formation control, whose aim is to steer a group of agents to achieve a prescribed formation, has been studied a lot during the past decades, see, for example, [13–15] and references therein. Formation tracking control of MASs have also been investigated in [17–19], where besides forming the prescribed formation configuration, the entire MASs should also track the desired reference trajectory generated by a real or virtual leader.

In some applications with complex and difficult tasks, the so-called formation-containment problem arises, where a group of leaders cooperatively achieve desired formation while a group of followers are driven to the convex hull spanned by the trajectories of those leaders. Various formation-containment control strategies have been proposed for identical MASs, referred to as homogeneous MASs [20, 21]. However, agents’ dynamics are often different in practice [22, 23]. Such MASs are called heterogeneous MASs. Recently, some effort has been devoted to formation-containment tracking problems of heterogeneous linear MASs [24–26].

Time-delays are often inevitable in practice and have been made considerable effort to the control community for the past decades, see, for example, [27–29]. As one particular type among many others, transmission delays often occurring during the information exchange between agents, have been widely studied for cooperative control of MASs, such as consensus problems [30, 31], containment control problems [32], formation control problems [33] and formation-containment control problems [34], respectively. It should be noted that only bounded delays are considered in these works. However, unbounded delays do exist in some practical systems. Typical examples include neural networks [35–37], biological networks [38, 39], the modelling of oscillators [40], the modelling of traffic flow [41, 42], to name a few. In particular, consensus of MASs with unbounded distributed delays is considered for the car following system in [41, 42], where unbounded distributed delays are used to model the memory effects of drivers. More recently, cooperative control of linear MASs under unbounded distributed transmission delays has been studied for homogeneous MASs [43] and heterogeneous MASs [44, 45], respectively. In particular, the containment control problem of heterogeneous linear MASs with unbounded transmission delays was studied in [44] where the multiple leaders are assumed to be homogeneous and no cooperation among them is required. In fact, to the best of our knowledge, a more general and more challenging problem, that is, the formation-containment tracking problem of MASs with unbounded distributed transmission delays is yet to be addressed, which motivates this work.

In this paper, we study the formation-containment tracking problem of heterogeneous linear MASs with unbounded distributed transmission delays. Novel distributed controllers are designed for the formation-leaders and followers, respectively.
It is shown that the formation-leaders not only achieve a prescribed formation but also track the desired trajectory generated by the reference leader, while the output of each follower converges to the convex hull spanned by the outputs of those formation-leaders. It should be pointed out that many existing results on consensus tracking, formation tracking, and containment control of linear MASs with constant or bounded distributed transmission delays are included as special cases of our results.

Compared with those relevant existing works [32–34, 43, 44], the following distinctions/advantages can be obtained in this work. First, compared with [32–34], this work considers more general transmission delays where constant/bounded distributed delays can be included as special cases. Moreover, unlike [32–34], the prior information of transmission delays is not needed in this work. Second, this work considers heterogeneous agent dynamics, which contain homogeneous agent dynamics studied in [33, 34] as special cases. Third, compared with our recent works on consensus or containment control of MASs with unbounded distributed transmission delays in [43, 44], the formation-containment control problem is more challenging since the formation-leaders and followers are required to be simultaneously controlled with unbounded distributed transmission delays to achieve formation tracking and containment, respectively.

**Notation:** $\mathbb{R}^{n \times m}$ and $\mathbb{C}^{n \times m}$ denote the sets of $n \times m$ real and complex matrices, respectively. $\mathbb{R}_{0}^{+}$ denotes the set of non-negative real numbers $[0, +\infty)$. For $H_{i} \in \mathbb{R}^{n \times m}$, $i = 1, \ldots, m$, $\text{diag}(H_{1}, \ldots, H_{m})$ represents a block matrix with its $i$-th diagonal entry being $H_{i}$ and all other entries being zero. $\delta$ denotes the Dirac delta function. $I$ and $0$ denote the identity matrix and the zero matrix with compatible dimensions, respectively. $\det(\cdot)$ stands for the determinant of a matrix. $\mathbf{1}_{n}$ denotes an $N \times 1$ column vector whose elements are all $1$. $\sigma(A)$ represents the spectrum of square matrix $A$. $j\mathbb{R}$ represents the imaginary axis. $\otimes$ and $\| \cdot \|$ denotes the Kronecker product of matrices and the Euclidean norm, respectively. For matrices $A, B, C, D, (A \otimes B)(C \otimes D) = (AC \otimes BD)$, and for square invertible matrices $A, B, (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$. $\text{dist}(x, C) = \inf_{y \in C} \| x - y \|$ denotes the distance from $x \in \mathbb{R}^{n}$ to the set $C \subseteq \mathbb{R}^{n}$. The convex hull $Co(\Omega)$ of a finite set of points $\Omega = \{\omega_{1}, \omega_{2}, \ldots, \omega_{q}\}$ is defined as $Co(\Omega) = \{\sum_{i=1}^{q} \rho_{i} \omega_{i} | \rho_{i} \in \Omega, \rho_{i} \in [0, 1], \sum_{i=1}^{q} \rho_{i} = 1\}$.

**II. Preliminaries and Problem Formulation**

**A. Algebraic Graph Theory**

A directed graph is defined as $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ consisting of a finite set of nodes $\mathcal{N} = \{1, 2, \ldots, N\}$, a set of edges $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ and a weighted adjacency matrix $A = [a_{ij}]_{N \times N}$, where $a_{ij} > 0 \iff (j, i) \in \mathcal{E}$ and otherwise $a_{ij} = 0$ for all $i, j \in \mathcal{N}$. For any agent $i$, $\mathcal{N}_{i} = \{ j \in \mathcal{N} | (j, i) \in \mathcal{E} \}$ denotes its neighbor set. A spanning tree is a graph that every node has exactly one neighbor except the root which has no neighbor. The Laplacian matrix $L = [l_{ij}]_{N \times N}$ associated with graph $G$ is defined as $l_{ij} = -a_{ij}, i \neq j$, and $l_{ii} = -\sum_{j=1}^{N} a_{ij}$.

**B. Problem Statement**

Consider a heterogeneous linear multi-agent system consisting of $M$ formation-leaders and $N$ followers with their dynamics described by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t),$$

$$y_i(t) = C_i x_i(t), \quad i = 1, \ldots, M + N,$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^{r}$, $u_i \in \mathbb{R}^{p_i}$ are the state, output and input of the $i$-th formation-leader or follower, respectively, $A_i$, $B_i$, and $C_i$ are matrices of compatible dimensions.

The dynamics of the reference leader indexed by $0$ are described by

$$\dot{x}_0(t) = A_0 x_0(t),$$

$$y_0(t) = C_0 x_0(t),$$

where $x_0 \in \mathbb{R}^{m}$ and $y_0 \in \mathbb{R}^{q}$ are the state and output of the reference leader, respectively, $A_0$ and $C_0$ are matrices of compatible dimensions.

The reference leader is predefined to provide the desired trajectory for the entire group. As mentioned in [23–25], a subset of the formation-leaders are supposed to have access to the information of the desired trajectory generated by the reference leader.

As described in [23, 25], in the context of formation-containment control, the system composed of (1) and (2) can be regarded as a multi-agent system composed of $M$ formation-leaders, $N$ followers and one reference leader. The sets of the formation-leaders and followers are denoted by $\mathcal{L} = \{1, \ldots, M\}$ and $\mathcal{F} = \{1, \ldots, M\}$ with $M_i = M + i$ for $i = 1, \ldots, N$, respectively. The corresponding Laplacian matrix can be partitioned as follows,

$$L = \begin{pmatrix}
L_{11} & L_{12} & L_{13} & L_{21} & L_{22} & L_{23} & L_{31} & L_{32} & L_{33}
\end{pmatrix},$$

where $L_{ij} = [a_{i1}, \ldots, a_{i0}, \ldots, a_{iM}]^T \in \mathbb{R}^{M \times 1}$, $a_{i0} > 0$ if the $i$-th formation-leader can get the information from the reference leader, otherwise $a_{i0} = 0$. $L_{11} \in \mathbb{R}^{M \times M}$ represents the communication topology among the formation-leaders. $L_{F1} \in \mathbb{R}^{N \times N}$ represents the communication topology among the followers. $L_{F2} \in \mathbb{R}^{N \times M}$ represents the communication topology between the formation-leaders and the followers.

The prescribed time-varying formation shape for the formation-leaders is specified by a vector $h(t) = [h_1(t)^T, \ldots, h_M(t)^T]^T$ with $h_i(t) \in \mathbb{R}^{l_i}$, $i \in \mathcal{L}$ being piecewise continuously differentiable and satisfying $h_i(t) = A_{h,i} h_i(t), h_0(t) = C_{h,i} h_0(t)$, where $h_0, A_{h,i}$ and $C_{h,i}$ are matrices of compatible dimensions.

In this work, the delayed information that agent $i$ receives from its neighboring agent $j \in N_i$ is described by

$$\int_{0}^{\infty} k_{ij}(\theta) \zeta_j(t - \theta) d\theta,$$

where $k_{ij}(\theta) : \mathbb{R}_{0}^{+} \rightarrow \mathbb{R}_{0}^{+}$ is the delay kernel representing unbounded distributed delays and satisfying $\int_{0}^{\infty} k_{ij}(\theta) d\theta = 1$, and $\zeta_j(t)$ is the information to be transmitted from its neighboring agent $j$.

The objective of this paper is to solve the problem formulated as follows.

**Definition 1 (Formation-Containment Tracking Problem):** Given a multi-agent system consisting of (1) and (2) under
a directed graph represented by its Laplacian matrix (3) with unbounded distributed transmission delays, find a distributed control law for each formation-leader so that

$$\lim_{t \to +\infty} \| y_i(t) - y_0(t) - h_{oi}(t) \| = 0, \forall i \in \mathbb{L},$$

and find a distributed control law for each follower so that

$$\lim_{t \to +\infty} \text{dist}(y_i(t), Co(y_j(t), j \in \mathbb{L})) = 0, \forall i \in \mathbb{F}. \quad (5)$$

Equation (4) implies that the formation-leaders not only accomplish the desired formation shape but also track the desired trajectory generated by the reference leader (2). Equation (5) implies that the outputs of the followers tend to the convex hull spanned by the outputs of the formation-leaders. If both (4) and (5) are achieved, then the formation-containment tracking problem is said to be solved.

To proceed, the following assumptions are made.

**Assumption 1:** The communication topology among the reference leader and formation-leaders contains a spanning tree with the reference leader as its root.

**Assumption 2:** For each follower $i, j \in \mathbb{F}$, there exists at least one formation-leader $j, j \in \mathbb{L}$, that has a directed path to the follower.

**Assumption 3:** The matrix pairs $(A_i, B_i)$ and $(C_i, A_i)$, $i \in \mathbb{L} \cup \mathbb{F}$, are stabilizable and detectable, respectively.

**Assumption 4:** The following linear matrix equations,

$$\Pi_i A_0 = A_i \Pi_i + B_i \Gamma_i,$$

$$0 = C_i \Pi_i - C_0,$$  \quad (6)

have solutions $(\Pi_i, \Gamma_i)$ for all $i \in \mathbb{L} \cup \mathbb{F}$.

**Assumption 5:** The following linear matrix equations,

$$\Pi_{hi} A_h = A_i \Pi_{hi} + B_i \Gamma_{hi},$$

$$0 = C_i \Pi_{hi} - C_h,$$  \quad (7)

have solutions $(\Pi_{hi}, \Gamma_{hi})$ for all $i \in \mathbb{L} \cup \mathbb{F}$.

**Assumption 6:** The delay kernel functions $k_{ij}(\theta), i \in \mathbb{L} \cup \mathbb{F}, j \in \mathcal{N}_i$, satisfy (8) and (9) for $i \in \mathbb{L}$ or (8) and (10) for $i \in \mathbb{F},$

$$k_{ij}(\theta) \leq e^{-c\theta} p(\theta),$$

$$\int_0^{+\infty} p(\theta) x_0(-\theta) d\theta < +\infty, \quad (8)$$

$$\int_0^{+\infty} p(\theta) h_j(-\theta) d\theta < +\infty, \quad (9)$$

$$\int_0^{+\infty} k_{ij}(\theta) e^{A_0\theta} \eta_j(t-\theta) d\theta < +\infty, \quad (10)$$

where $c > 0$ is a constant and $p : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a non-increasing and Lebesgue integrable function, $x_0$ is the state of the reference leader described by (2), and $h_j$ is the state of the time-varying formation shape for the $j$-th formation-leader.

**Remark 1:** Assumptions 1-4 are common in formation-containment control of heterogeneous MASs even without time delays [24, 25]. Assumption 5 is similar to the corresponding assumption in [24, 25] but more general. In fact, the formation shape adopted in this work includes those in [24, 25] as its special cases, and also includes the time-invariant formation shape when $A_h \equiv 0$ as its special case. Assumption 6 is common in dealing with unbounded distributed delays [43, 44, 46, 47]. Condition (8) is a restriction on the delay kernels. Inequality (9) and inequality (10) are the restrictions on the initial conditions about $x_0$ and $h_j$, respectively.

### III. Formation-Containment Tracking Control

In this section, the distributed formation tracking controllers for the formation-leaders will be presented, and then the distributed containment controllers for the followers will be given.

#### A. Formation Tracking Control of Formation-Leaders

For each formation-leader, the following distributed output feedback formation tracking controller is proposed,

$$u_i = K_{i1} \hat{x}_i + K_{i2} \eta_i + K_{i3} h_i,$$  \quad (11a)

$$\hat{x}_i = A_i \hat{x}_i + B_i u_i + L_i (y_i - C_i \hat{x}_i),$$  \quad (11b)

$$\eta_i = A_0 \eta_i - \mu a_{i0} (\eta_i(t) - \int_0^{+\infty} k_{i0}(\theta) e^{A_0\theta} x_0(t-\theta) d\theta) - \mu \sum_{j=1}^M a_{ij} (\eta_i(t) - \int_0^{+\infty} k_{ij}(\theta) e^{A_0\theta} \eta_j(t-\theta) d\theta),$$ \quad (11c)

where $\hat{x}_i \in \mathbb{R}^{n_i}$ is the state of the Luenberger observer to estimate the agent’s own unmeasurable state, $\eta_i \in \mathbb{R}^m$ is the state of the distributed observer to estimate the state $x_0$ of the reference leader, $\mu > 0$ is a real number, and $K_{i1}, K_{i2}, K_{i3}, L_i$ are constant matrices to be determined later.

Since the state $x_0$ of the reference leader is only available to a subset of the formation-leaders, the distributed observer (11c) is thus adopted to estimate $x_0$. It should be pointed out that $\int_0^{+\infty} k_{i0}(\theta) e^{A_0\theta} x_0(t-\theta) d\theta$ or $\int_0^{+\infty} k_{ij}(\theta) e^{A_0\theta} \eta_j(t-\theta) d\theta$ in (11c) is the signal to be received by formation-leader $i$ from its neighboring agent, the reference leader or agent $j, j = 1, \cdots, M$, without prior knowledge of delay kernels under the transmission framework illustrated in [43, 44]. Then the following technical lemma on the distributed observer (11c) will be presented.

**Lemma 2:** Consider the reference leader (2) and the distributed observer (11c) under Assumptions 1 and 6 with inequalities (8) and (9). Assume that $\sigma(A_0) \in j \mathbb{R}$, then

$$\lim_{t \to +\infty} \| \eta_i - x_0 \| = 0, \forall i \in \mathbb{L},$$

**Proof:** Letting $\eta_L = (\eta_1^T, \cdots, \eta_M^T)^T$, it follows from (2) and (11c) that

$$\dot{x}_0(t) = A_0 x_0,$$

$$\dot{\eta}_L(t) = (I_M \otimes A_0) \eta_L - \mu (D_L \eta_L + \int_0^{+\infty} (L_{1\theta} \otimes e^{A_0\theta}) \eta_L(t - \theta) d\theta + \int_0^{+\infty} (L_{2\theta} \otimes e^{A_0\theta}) x_0(t-\theta) d\theta),$$  \quad (12)
where $\bar{D}_L = D_L \otimes I_m$, $D_L = \text{diag}\{d_1, \ldots, d_M\}$ with $d_i = \sum_{j=0}^{M} a_{ij}$, $i \in \mathbb{L}$, $L_{1i} = [l_{ij}]_{M \times M}$ with $l_{ij} = -a_{ij}k_{ij}(\theta)$ for $i \neq j$ and $l_{ii} = 0$ for $i = j$, $i \in \mathbb{L}$, and $L_{2i} = [l_{ij}]_{1 \times M}$ with $l_{ij} = -a_{ij}k_0(\theta)$, $i \in \mathbb{L}$. It can be observed that equation (12) has the same form as equation (41) in Appendix B by letting $\phi_L = x_0$, $\psi_F = \eta_L$ and $S = A_0$. Then by Lemma 6 in Appendix B, one has that
\[ \lim_{t \to +\infty} \| \eta_L - (1_M \otimes x_0) \| = 0 \] exponentially. That is,
\[ \lim_{t \to +\infty} \| \eta_i - x_0 \| = 0, i = 1, \ldots, M, \text{ exponentially.} \]

Now, the main result in this subsection is ready to be presented.

**Theorem 1:** Consider the reference leader (2) and $M$ formation-leaders described in (1) under Assumptions 1, 3-5 and 6 with inequalities (8) and (9). Let $\lambda_0 \in \sigma(A_0)$ and $\lambda_i \in \sigma(A_i)$ be on the imaginary axis. Let $K_{1i}, l_i \in \mathbb{L}_c$ be chosen such that $A_1 + B_1K_{1i}$ and $A_i + L_i C_i$ are Hurwitz. Let $K_{2i} = \Gamma_i - K_{1i} \Pi_i$ and $K_{hi} = \Gamma_{hi} - K_i \Pi_{hi}, i \in \mathbb{L}$, where the matrix pairs $(\Pi_i, \Gamma_i)$ and $(\Pi_{hi}, \Gamma_{hi})$ satisfy equation (6) and equation (7), respectively. Then the formation tracking problem described by equation (4) is solved by the distributed formation tracking controller (11).

**Proof:** Define $\gamma_i = y_i - y_0 - h_i$, $i \in \mathbb{L}$. From (1), (2), and (11), one can obtain the following closed-loop system consisting of the reference leader and formation-leaders,
\[
\begin{align*}
\dot{x}_0 &= A_0x_0, \\
\dot{x}_i &= A_ix_i + B_iK_{1i}x_i + B_iK_{2i}\eta_i + B_iK_{3i}h_i, \\
\dot{\eta}_i &= A_i\eta_i - \mu a_{ii}(\eta_i(t) - \int_{0}^{\infty} k_{ii}(\theta) \, d\theta) \\
&\quad \times e^{A_i\theta}x_0(t - \theta) \, d\theta - \mu \sum_{j=1}^{M} a_{ij}(\eta_j(t) - \int_{0}^{\infty} k_{ij}(\theta) e^{A_i\theta}\eta_j(t - \theta) \, d\theta), \\
\dot{h}_i(t) &= A_h h_i, \\
e_i(t) &= C_i x_i - C_0x_0 - C_h h_i.
\end{align*}
\]

Letting $\eta_L = (\eta_1^T, \ldots, \eta_M^T)^T$, it follows from the fourth equation of (13) that the dynamic of $\eta_L$ is the same as in equation (12). Define $\tilde{\eta}_L = \eta_L - (1_M \otimes x_0)$. By Lemma 2, one has that
\[ \lim_{t \to +\infty} \| \tilde{\eta}_L \| = 0 \] exponentially.

Let $x_L = (x_1^T, \ldots, x_M^T)^T$, $\dot{x}_L = (\dot{x}_1^T, \ldots, \dot{x}_M^T)^T$, $y_L = (y_1^T, \ldots, y_M^T)^T$, and $e_L = (e_1^T, \ldots, e_M^T)^T$. Then the compact form of the closed-loop system (13) can be rewritten as follows,
\[
\begin{align*}
\dot{x}_L &= A_Lx_L + B_LK_{1L}\dot{x}_L + B_LK_{2L}\eta_L + B_LK_{3L}h, \\
\dot{\eta}_L &= (A_1 + B_1K_{1L})x_L + B_1K_{2L}\eta_L + B_1K_{3L}h \\
&\quad + L_LC_L(x_L - \dot{x}_L), \\
\dot{h}_L &= (1_M \otimes A_h)h, \\
e_L &= C_Lx_L - (1_M \otimes C_0)x_0 - (1_M \otimes C_h)h,
\end{align*}
\]

where
\[
\begin{align*}
A_L &= \text{diag}\{A_1, A_2, \ldots, A_M\}, \\
B_L &= \text{diag}\{B_1, B_2, \ldots, B_M\}, \\
C_L &= \text{diag}\{C_1, C_2, \ldots, C_M\}, \\
K_{1L} &= \text{diag}\{K_{11}, K_{12}, \ldots, K_{1M}\}, \\
K_{2L} &= \text{diag}\{K_{21}, K_{22}, \ldots, K_{2M}\}, \\
K_{3L} &= \text{diag}\{K_{31}, K_{32}, \ldots, K_{3M}\}, \\
L_L &= \text{diag}\{L_1, L_2, \ldots, L_M\}.
\end{align*}
\]

Define $\tilde{x}_L = x_L - \Pi_L(1_M \otimes x_0) - \Pi_{hL}h$ and $\tilde{x}_L = x_L - \tilde{x}_L$. Note that the following equations hold,
\[
\begin{align*}
\Pi_L(1_M \otimes A_0) &= A_L\Pi_L + B_L\Gamma_L, \\
0 &= C_L\Pi_L - (I_M \otimes C_0), \\
\Pi_{hL}(1_M \otimes A_h) &= A_1\Pi_{hL} + B_1\Gamma_{hL}, \\
0 &= C_L\Pi_{hL} - (I_M \otimes C_h).
\end{align*}
\]
B. Containment Control of Followers

For each follower, the following distributed output feedback containment controller is proposed,

\[ u_i = K_{i1}\dot{x}_i + K_{i2}w_i + K_{i3}\dot{h}_i, \]  
\[ \dot{x}_i = A_1\dot{x}_i + B_1u_i + L_i(y_i - C_1\dot{x}_i), \]  
\[ \dot{w}_i = A_0w_i - \mu \sum_{j=1}^{M} a_{ij}(w_i(t) - \int_{0}^{+\infty} k_{ij}(\theta) \times e^{A_0\theta}\eta_j(t - \theta)d\theta - \mu \sum_{j=M+1}^{M+N} a_{ij}(w_i(t) - \int_{0}^{+\infty} k_{ij}(\theta)\eta_j(t - \theta)d\theta), \]  
\[ \dot{h}_i = A_2\dot{h}_i - \nu \sum_{j=1}^{M} a_{ij}\dot{h}_i(t) - \int_{0}^{+\infty} k_{ij}(\theta)e^{A_2\theta}h_j(t - \theta)d\theta, \]  
\[ \dot{\hat{h}}_i = A_3\dot{\hat{h}}_i - \nu \sum_{j=M+1}^{M+N} a_{ij}\dot{\hat{h}}_i(t) - \int_{0}^{+\infty} k_{ij}(\theta)e^{A_3\theta}\hat{h}_j(t - \theta)d\theta, \]  

where \( \dot{x}_i \in \mathbb{R}^{n_i} \) is the state of the Luenberger observer to estimate the agent’s own unmeasurable state, \( w_i \in \mathbb{R}^{m_i} \) is the state of the distributed observer to estimate the state \( x_0 \) of the reference leader via the information \( \eta_j \) when follower \( i \) has access to formation-leader \( j \). \( h_i \in \mathbb{R}^{m_i} \) is the state of the distributed observer to estimate the convex hull spanned by the formation shape \( h_j, j \in \mathbb{L}, \mu > 0, \nu > 0 \) are real numbers, and the compatible constant matrices \( K_{i1}, K_{i2}, K_{i3}, L_i \) are to be determined later.

Since the state \( x_0 \) of the reference leader is not available to the followers, the distributed observer (18c) is thus adopted to estimate \( x_0 \) via the information \( \eta_j \) when follower \( i \) has access to formation-leader \( j \). Moreover, since the formation shape \( h_j, j \in \mathbb{L} \), is only available to a subset of the followers, the distributed observer (18d) is thus adopted to estimate the convex hull spanned by \( h_j, j \in \mathbb{L} \). Similar to the distributed observer (11c), \( \int_{0}^{+\infty} k_{ij}(\theta)e^{A_0\theta}\eta_j(t - \theta)d\theta \) or \( \int_{0}^{+\infty} k_{ij}(\theta)e^{A_2\theta}h_j(t - \theta)d\theta \) or \( \int_{0}^{+\infty} k_{ij}(\theta)e^{A_3\theta}\hat{h}_j(t - \theta)d\theta \) in (18c), and \( \int_{0}^{+\infty} k_{ij}(\theta)e^{A_0\theta}\eta_j(t - \theta)d\theta \) or \( \int_{0}^{+\infty} k_{ij}(\theta)e^{A_2\theta}h_j(t - \theta)d\theta \) or \( \int_{0}^{+\infty} k_{ij}(\theta)e^{A_3\theta}\hat{h}_j(t - \theta)d\theta \) in (18d) are the signals to be received by follower \( i \) from its neighboring agent \( j, j = M + 1, \cdots, M + N \), without prior knowledge of delay kernels under the transmission framework illustrated in [43, 44]. It should be noted that Theorem 1 shows that \( \lim_{t \rightarrow +\infty} \|y_i(t) - y_0(t) - h_m(t)\| = 0, \forall i \in \mathbb{L} \). In this subsection, the distributed observers in (18c) and (18d) are utilized to estimate \( C_0(y_0(t) + h_m(t), j \in \mathbb{L} \). Next, some technical lemmas on the distributed observer (18c) and the distributed observer (18d) will be presented.

Lemma 3 (Key Technical Lemma): Consider the reference leader (2), the distributed observer (11c) and the distributed observer (18c) under Assumptions 1, 2 and 6 with inequalities (8) and (9). If \( \sigma(A_0) \in j\mathbb{R} \), then \( \lim_{t \rightarrow +\infty} \|w_i(t) - x_0(t)\| = 0, i \in \mathbb{F} \), exponentially.

Proof: Letting \( \eta_L = (\eta_{1L}^T, \cdots, \eta_{M_L}^T)^T \), it follows from (2) and (11c) that the dynamic of \( \eta_L \) is the same as in equation (12). By Lemma 2, one has that \( \lim_{t \rightarrow +\infty} \|\eta_L - (1_M \otimes x_0)\| = 0 \) exponentially. Define \( \xi_L = e^{-(1_M \otimes A_0)t}(\eta_L - (1_M \otimes x_0)) \). Since \( \sigma(A_0) \in j\mathbb{R} \), there exists a polynomial function \( Q_1(t) \) such that \( e^{-(1_M \otimes A_0)t}\|\xi_L\| \leq Q_1(t) \). It then follows that \( \|\xi_L\| \leq Q_1(t)\|\eta_L - (1_M \otimes x_0)\| \), which implies that \( \lim_{t \rightarrow +\infty} \|\dot{\xi}_L\| = 0 \) exponentially.

Letting \( w_F = (w_{M+1}^T, \cdots, w_{M+N}^T)^T \), it follows from (18c) that

\[ \dot{w}_F = (I_M \otimes A_2)w_F - \mu (\hat{D}_Fw_F + \int_{0}^{+\infty}(\mathcal{L}_{F1\theta} \otimes e^{A_0\theta})w_F(t - \theta)d\theta \) 
\[ \times e^{A_0\theta}w_F(t - \theta)d\theta + \int_{0}^{+\infty}(\mathcal{L}_{F2\theta} \otimes e^{A_0\theta})w_F(t - \theta)d\theta + \int_{0}^{+\infty}(\mathcal{L}_{F1\theta} \otimes e^{A_0\theta})w_F(t - \theta)d\theta \),

where the matrices \( \hat{D}_F, \mathcal{L}_{F1\theta} \) and \( \mathcal{L}_{F2\theta} \) are the same as those in equation (41) in Appendix B.

Define \( \tilde{x}_0 = e^{-A_0t}x_0 \) and \( \tilde{w}_F = e^{-(1_N \otimes A_0)t}w_F \). It can be obtained that \( \tilde{x}_0 = 0 \) and

\[ \dot{\tilde{x}}_0 = -\mu (\hat{D}_F\tilde{x}_0 + \int_{0}^{+\infty}(\mathcal{L}_{F1\theta} \otimes I_m)\tilde{x}_0(t - \theta)d\theta \) 
\[ \times \tilde{x}_0(t - \theta)d\theta + \Phi(t) + \Psi(t) \),

\[ e_w = e^{(1_N \otimes A_0)t}\xi_F, \]

where the perturbation terms \( \Phi(t) \) and \( \Psi(t) \) are denoted respectively as follows,

\[ \Phi(t) = \hat{D}_F(1_N \otimes \tilde{x}_0) + \int_{0}^{+\infty}(\mathcal{L}_{F1\theta} \otimes I_m) \] 
\[ \times (1_N \otimes \tilde{x}_0(t - \theta)d\theta + \int_{0}^{+\infty}(\mathcal{L}_{F2\theta} \otimes I_m) \] 
\[ \Psi(t) = \hat{D}_F(1_N \otimes \tilde{x}_0(t - \theta)d\theta, \]

In what follows, we prove via utilizing Lemma 5 in Appendix A that \( \lim_{t \rightarrow +\infty} \|\xi_F(t)\| = 0 \) exponentially.

First, by lemma 7 in Appendix B, it follows that the zero solution of the nominal system of (21a), that is, \( \xi_F = -\mu (\hat{D}_F\xi_F + \int_{0}^{+\infty}(\mathcal{L}_{F1\theta} \otimes I_m)\xi_F(t - \theta)d\theta \) is globally exponentially stable.

Next, we further prove that \( \lim_{t \rightarrow +\infty} \|\Phi(t)\| = 0 \) exponentially and \( \lim_{t \rightarrow +\infty} \|\Psi(t)\| = 0 \) exponentially.
Noting that $\bar{x}_0(t) = 0, \forall t \geq 0$ and $\int_0^{+\infty} k_{ij}(\theta) d\theta = 1$, it follows from (22a) that

$$\Phi(t) = \int_0^{+\infty} (DF_\theta \otimes I_m) d\theta (1_N \otimes \bar{x}_0) + \int_0^{+\infty} (L_{F10} \otimes I_m) d\theta (1_N \otimes \bar{x}_0) + \int_0^{+\infty} (L_{F10} \otimes I_m)(1_N \otimes \bar{x}_0(t - \theta)) d\theta + \int_0^{+\infty} (L_{F20} \otimes I_m) d\theta (1_M \otimes \bar{x}_0) + \int_0^{+\infty} (L_{F20} \otimes I_m)(1_M \otimes \bar{x}_0(t - \theta)) d\theta,$$

where $D_{F\theta} = \text{diag}(d_{\bar{M}_1, 0}, \ldots, d_{\bar{M}_N, 0})$ with $d_{\bar{M}, 0} = \sum_{j=1}^{M+N} a_{\bar{M}, j} \hat{M}_{j, \theta}$. Under Assumption 6, there exist $\varsigma_1 > 0, \varsigma_2 > 0$ such that

$$\|\Phi(t)\| \leq \varsigma_1 e^{-ct} \int_0^{+\infty} p(\theta) d\theta \|\bar{x}_0\| + \varsigma_2 e^{-c(t+\theta)} p(\theta + \theta) \|\bar{x}_0(\theta)\| d\theta.$$  

(24)

Since $p : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is a non-increasing function, it then follows that

$$\|\Phi(t)\| \leq \varsigma_1 e^{-ct} \int_0^{+\infty} p(\theta) d\theta \|\bar{x}_0\| + \varsigma_2 e^{-ct} \int_0^{+\infty} p(\theta) \|\bar{x}_0(\theta)\| d\theta.$$  

(25)

Noting that $\int_0^{+\infty} p(\theta) d\theta \|\bar{x}_0\| < +\infty$ and $\int_0^{+\infty} p(\theta) \|\bar{x}_0(\theta)\| d\theta < +\infty$, it can be obtained that $\lim_{t \rightarrow +\infty} \|\Phi(t)\| = 0$ exponentially.

Now, we consider $\Psi(t)$ in (22b). Under Assumption 6, it follows that there exists a constant $\varsigma_3 > 0$ such that $\|\Psi(t)\| \leq \varsigma_3 \int_0^{+\infty} e^{-\alpha t} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta$.

Define $\kappa(t) = \int_0^{+\infty} e^{-\alpha t} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta$. Then we prove that $\lim_{t \rightarrow +\infty} \kappa(t) = 0$ exponentially. Denote that

$$\kappa(t) = \int_0^{t} e^{-\alpha t} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta + \int_t^{+\infty} e^{-\alpha t} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta = 0 \iff g_1(t) + g_2(t) \leq 0.$$  

Claim 1: $\lim_{t \rightarrow +\infty} \|g_1(t)\| = 0$ exponentially.

Since $\lim_{t \rightarrow +\infty} \|\hat{\xi}_L(t - \theta)\| = 0$ exponentially, there exist $\alpha > 0$ and $\beta > 0$ such that $\|\hat{\xi}_L(t - \theta)\| \leq \alpha e^{-\beta(t-\theta)}$ for $\theta \in [0, t]$.

Then one has that

$$\|g_1(t)\| = \left\| \int_0^{t} e^{-\alpha t} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta \right\| \leq \alpha e^{-\beta t} \int_0^{t} e^{(\beta - \alpha) t} p(\theta) d\theta.$$  

(27)

Noting that $p : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is a non-increasing function, we further consider the term $\int_0^{t} e^{(\beta - \alpha) t} p(\theta) d\theta \leq tp(0)$, and thus it follows from (27) that $\lim_{t \rightarrow +\infty} \|g_1(t)\| = 0$ exponentially;

Claim 2: $\lim_{t \rightarrow +\infty} \|g_2(t)\| = 0$ exponentially.

Since $p : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is a non-increasing function, one has that

$$\|g_2(t)\| = \left\| \int_0^{+\infty} e^{-\alpha t} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta \right\| \leq \alpha e^{-\beta t} \int_0^{+\infty} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta.$$  

(28)

Noting that $\int_0^{+\infty} p(\theta) \|\hat{\xi}_L(t - \theta)\| d\theta < +\infty$, it follows from (28) that $\lim_{t \rightarrow +\infty} \|g_2(t)\| = 0$ exponentially. Claim 2 thus holds.

Combining these two claims, one has that $\lim_{t \rightarrow +\infty} \|\Psi(t)\| = 0$ exponentially.

Then by Lemma 5 in Appendix A, it follows from (21a) that $\lim_{t \rightarrow +\infty} \|\xi_F\| = 0$ exponentially. Since $\sigma(A_0) \in j\mathbb{R}$, there exists a polynomial function $\bar{Q}_2(t)$ such that $e^{(1+\sigma(A_0)) t} \leq \bar{Q}_2(t)$. By (21b), one has that $\lim_{t \rightarrow +\infty} \|w_i - x_0\| = 0, i \in \mathbb{F}$, exponentially. Thus, the proof is completed.

Lemma 4: Consider the distributed observer (18d) under Assumptions 2 and 6 with inequalities (8) and (10). If $\sigma(A_0) \in j\mathbb{R}$ then $\lim_{t \rightarrow +\infty} \|\hat{h}_i(t), Co(h_j(t), j \in \mathbb{F})\| = 0, i \in \mathbb{F}$, exponentially.

Proof: Letting $\vec{h}_F = (\vec{h}_{M+1}^T, \ldots, \vec{h}_{M+N}^T)^T$, it follows from (18d) that

$$\vec{h} = (I_M \otimes A_k) \vec{h},$$

$$\vec{h} = (I_N \otimes A_k) \vec{h} - \nu (DF \vec{h}_F$$

$$+ \int_0^{+\infty} (L_{F10} \otimes e^{h_1, \theta}) \vec{h}_F(t - \theta) d\theta$$

$$+ \int_0^{+\infty} (L_{F20} \otimes e^{h_1, \theta}) \vec{h}_F(t - \theta) d\theta,$$

where the matrices $DF, L_{F10}$ and $L_{F20}$ are the same as in equation (41) in Appendix B. It can be observed that equation (29) has the same form as equation (41) by letting $\phi_L = h, \psi_F = \vec{h}_F$ and $S = A_k$. Then by Lemma 6 in Appendix B, one
has that \( \lim_{t \to +\infty} \| \hat{h}_F - (-L_F^{-1}_1L_F \otimes I_m)h \| = 0 \) exponentially, that is, \( \lim_{t \to +\infty} \text{dist}(\hat{h}_i(t), \text{Co}(h_j(t), j \in L)) = 0, i \in \mathbb{F} \), exponentially.

Now, the main result in this subsection is ready to be presented.

**Theorem 2:** Consider the multi-agent system consisting of (1) and (2) under Assumptions 1-6. Assume that \( \lambda_0 \in j\mathbb{R}, \forall \lambda_0 \in \sigma(A_0) \) and \( \lambda_h \in j\mathbb{R}, \forall \lambda_h \in \sigma(A_h) \). Let \( K_{1i}, L_i \in \mathbb{L} \cup \mathbb{F} \), be chosen such that \( A_1 + B_iK_{1i} \) and \( A_1 + L_iC_i \) are Hurwitz. Let \( K_{2i} = \Gamma_i - K_{1i} \Pi_i \) and \( K_{3i} = \Gamma_{hi} - K_{1i} \Pi_{hi}, i \in \mathbb{L} \cup \mathbb{F} \), where the matrix pairs \((\Pi_i, \Gamma_i)\) and \((\Pi_{hi}, \Gamma_{hi})\) satisfy equation (6) and equation (7), respectively. Then the formation-containment tracking control problem described by Definition 1 is solved by the distributed formation tracking controller (11) for the formation-leaders and the distributed containment controller (18) for the followers.

**Proof:** From (1) and (18), one has the following closed-loop system for the followers,

\[
\dot{x}_i = A_ix_i + B_iK_{1i}\hat{x}_i + B_iK_{2i}w_i + B_iK_{3i}h_i,
\]

\[
\hat{x}_i = (A_1 + B_iK_{1i} - L_iC_i)\hat{x}_i + B_iK_{2i}w_i + B_iK_{3i}\hat{h}_i + L_iC_ix_i,
\]

\[
\dot{w}_i = A_0w_i - \mu \sum_{j=1}^{M} a_{ij}\left( w_i(t) - \int_{t}^{+\infty} k_{ij}(\theta) \right) - e^{A_0\theta} \sum_{j=1}^{M+N} a_{ij}\left( w_i(t) - \int_{t}^{+\infty} k_{ij}(\theta) \right) \times e^{A_0\theta}(t - \theta)d\theta - \mu \sum_{j=M+1}^{M+N} a_{ij}\left( w_i(t) - \int_{t}^{+\infty} k_{ij}(\theta) e^{A_0\theta}(t - \theta)d\theta \right),
\]

\[
\hat{h}_i = A_h\hat{h}_i - \nu \sum_{j=1}^{M} a_{ij}\left( \hat{h}_i(t) - \int_{t}^{+\infty} k_{ij}(\theta) \right) - e^{A_h\theta}\hat{h}_j(t) - \nu \sum_{j=M+1}^{M+N} a_{ij}\left( \hat{h}_i(t) - \int_{t}^{+\infty} k_{ij}(\theta) e^{A_h\theta}\hat{h}_j(t) - \theta)d\theta \right)
\]

\[i \in \mathbb{F}.
\]

Letting \( w_F = (w_{M+1}^T, \ldots, w_{M+N}^T)^T \), it follows from the third equation of (30) that the dynamic of \( w_F \) is the same as in equation (19). Define \( e_w = w_F - (1_N \otimes x_0) \). By Lemma 3, one has that \( \lim_{t \to +\infty} \| e_w \| = 0 \) exponentially.

Letting \( \hat{h}_F = (\hat{h}_{M+1}^T, \ldots, \hat{h}_{M+N}^T)^T \), it follows from the fourth equation of (30) that the dynamic of \( \hat{h}_F \) is the same as in equation (29). Define \( e_h = h_F - (-L_F^{-1}_1L_F \otimes I_m)h \). By Lemma 4, one has that \( \lim_{t \to +\infty} \| e_h \| = 0 \) exponentially.

Let \( x_F = (x_{M+1}^T, \ldots, x_{M+N}^T)^T, \hat{x}_F = (\hat{x}_{M+1}^T, \ldots, \hat{x}_{M+N}^T)^T \), and \( y_F = (y_{M+1}^T, \ldots, y_{M+N}^T)^T \). Then it follows that the compact form of the closed loop system (30) can be rewritten as

\[
\dot{x}_F = A_Fx_F + B_FK_{1F}\hat{x}_F + B_FK_{2F}w_F + B_FK_{3F}\hat{h}_F,
\]

\[\hat{x}_F = (A_F + B_FK_{1F})\hat{x}_F + B_FK_{2F}w_F + B_FK_{3F}\hat{h}_F + L_FC_F(x_F - \hat{x}_F), \quad y_F = C_Fx_F,
\]

where \( A_F = \text{diag}\{A_{M+1}, \ldots, A_{M+N}\}, \quad B_F = \text{diag}\{B_{M+1}, \ldots, B_{M+N}\}, \quad C_F = \text{diag}\{C_{M+1}, \ldots, C_{M+N}\}, \quad K_{1F} = \text{diag}\{K_{1M+1}, \ldots, K_{1M+N}\}, \quad K_{2F} = \text{diag}\{K_{2M+1}, \ldots, K_{2M+N}\}, \quad K_{3F} = \text{diag}\{K_{3M+1}, \ldots, K_{3M+N}\}, \quad L_F = \text{diag}\{L_{M+1}, \ldots, L_{M+N}\}. \]

Define \( \tilde{x}_F = x_F - \Pi_F(1_N \otimes x_0) - \Pi_{Fh}(-L_F^{-1}_1L_F \otimes I_m)h, \)

\[\hat{x}_F = x_F - \tilde{x}_F, \quad e_F = y_F - (-L_F^{-1}_1L_F \otimes I_m)y_FL, \]

with the same \( y_L \) as in the proof of Theorem 1. Note that the following equations hold.

\[\Pi_F(1_N \otimes A_0) = A_F\Pi_F + B_F\Gamma_F, \quad 0 = C_F\Pi_F - (1_N \otimes C_0), \]

\[\Pi_{hF}(1_N \otimes A_h) = A_F\Pi_{hF} + B_F\Gamma_{hF}, \quad 0 = C_F\Pi_{hF} - (1_N \otimes C_h), \]

where \( \Pi_F = \text{diag}\{\Pi_{M+1}, \ldots, \Pi_{M+N}\}, \quad \Pi_{hF} = \text{diag}\{\Pi_{Mh+1}, \ldots, \Pi_{Mh+N}\}, \quad \Gamma_F = \text{diag}\{\Gamma_{M+1}, \ldots, \Gamma_{M+N}\}, \quad \Pi_{hF} = \text{diag}\{\Gamma_{Mh+1}, \ldots, \Gamma_{Mh+N}\}. \)

It then follows from (31) that

\[
\dot{\tilde{x}}_F = (A_F + B_FK_{1F})\tilde{x}_F + \dot{\tilde{x}}_F + B_FK_{2F}e_F + B_FK_{3F}\hat{h}_F,
\]

\[\dot{\hat{x}}_F = (A_F - L_FC_F)e_F, \quad e_F = C_F\tilde{x}_F - (-L_F^{-1}_1L_F \otimes I_q)e_L,
\]

where \( e_L \) is the same as in equation (17c).

Since \( A_i - L_iC_i, i \in \mathbb{F} \) are Hurwitz, it follows from (33b) that \( \lim_{t \to +\infty} \| \tilde{x}_F(t) \| = 0 \) exponentially. By (33a), one has that \( \lim_{t \to +\infty} \| \hat{x}_F(t) \| = 0 \) exponentially, since \( A_1 + B_iK_{1i} \) are Hurwitz. Hence, \( \lim_{t \to +\infty} \| \tilde{x}_F(t) \| = 0 \) exponentially, \( \lim_{t \to +\infty} \| e_h \| = 0 \) exponentially, and \( \lim_{t \to +\infty} \| e_w \| = 0 \) exponentially. It follows from Theorem 1 that equation (4) is guaranteed, that is, \( \lim_{t \to +\infty} \| e_L(t) \| = 0 \) exponentially. Therefore, by (33c), one has that \( \lim_{t \to +\infty} \| e_F(t) \| = 0 \) exponentially, that is, \( \lim_{t \to +\infty} \| y_F - (-L_F^{-1}_1L_F \otimes I_q)y_L \| = 0 \) exponentially. Denote that

\[L_F^{-1}_1L_F \otimes I_q \triangleq \begin{pmatrix} \rho_{M+1} & \cdots & \rho_{M+M} \\ \vdots & \ddots & \vdots \\ \rho_{M+N} & \cdots & \rho_{M+N} \end{pmatrix}.
\]

It follows from Lemma 1 that \( \lim_{t \to +\infty} \| y_i - (\rho_{i+1}y_i + \cdots + \rho_{M+M}y_M) \| = 0 \) exponentially, where \( \rho_{i+1} + \cdots + \rho_{M+M} = 1 \) and \( \rho_{ij} \geq 0, i, j \in \mathbb{F}, j \in \mathbb{L} \). It then follows that \( \lim_{t \to +\infty} \| \text{dist}(y_{i}(t), \text{Co}(y_{j}(t), j \in \mathbb{L})) \| = 0, i \in \mathbb{F}. \) Thus, equation (5) is achieved. Then the formation-containment tracking control problem described by Definition 1 is solved. The proof is thus completed.

**Remark 4:** Compared with [44] where containment control of MASs with unbounded distributed transmission delays was
considered, this work considers more general scenario in the following aspects. Firstly, unlike [44] where autonomous homogeneous formation leaders were assumed, this work considers heterogeneous formation leaders with nonzero control input to achieve the desired time-varying formation shape. Secondly, in the present scenario, it is much more challenging to design distributed controllers for the followers to converge to the convex hull spanned by the trajectories of those formation leaders, in comparison with [44]. To handle the challenges, the distributed observers (18c) and (18d) are proposed for the followers to estimate both the trajectory of the reference leader via the information of their neighboring formation leaders and the convex hull spanned by the formation leaders respectively in this work.

The results in Theorem 1 and Theorem 2 include those results under bounded distributed transmission delays as special cases. Moreover, Theorem 1 and Theorem 2 can be extended to more general cases under both unbounded distributed delays and multiple constant delays by denoting $\hat{k}_{ij}(\theta) = \sum_{\zeta=1}^{\lambda} k_{ij}(\theta - \gamma_{ij}^{\zeta}) + k_{ij}(\theta)$, where $\gamma_{ij}^{\zeta} > 0, \lambda$ denotes the number of constant delays $\gamma_{ij}^{\zeta}$, and $k_{ij}(\theta)$ satisfies Assumption 6. In this case, inequalities (9) and (10) are replaced by $\sup_{\sigma \in [-\tau, 0]} |x_0(\sigma)| + \int_0^{\infty} p(\theta)|x_0(-\theta)|d\theta < +\infty$ and $\sup_{\sigma \in [-\tau, 0]} |h_j(\sigma)| + \int_0^{+\infty} p(\theta)|h_j(-\theta)|d\theta < +\infty$, respectively. In particular, a special case under only multiple constant delays is also included by setting $k_{ij}(\theta) = 0$.

follows,

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (36)$$

The time-varying formation shape $h_i(t) = [h_{i1}^T(t), h_{i2}^T(t)]^T$ is set as follows,

$$h_{i1} = \sin\left(2t + \frac{(i-1)\pi}{2}\right),$$
$$h_{i2} = 2\cos\left(2t + \frac{(i-1)\pi}{2}\right), \quad i = 1, 2, 3, 4,$$

and $h_{i3} = h_i$. It can be verified that $\lim_{t \to +\infty} \sum_{i=1}^{4} h_i(t) = 0$, which means that four formation leaders will rotate around the reference leader. Solving the equations (6) and (7) respectively yields

$$\Pi_i = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{b_i} & 1 & 0 \end{pmatrix},$$
$$\Gamma_i = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{b_i} & 1 & 0 \end{pmatrix}, \quad (38)$$

Figure 1 shows the communication topology. The delay kernels are denoted as $k_{10}(\theta) = 2e^{-2\theta}$, $k_{21}(\theta) = 4e^{-2\theta}$, $k_{23}(\theta) = 2e^{-2\theta}$, $k_{32}(\theta) = 4e^{-2\theta}$, $k_{34}(\theta) = e^{-\theta}$, $k_{41}(\theta) = \theta e^{-\theta}$, $k_{43}(\theta) = 4e^{-2\theta}$, $k_{54}(\theta) = 2e^{-2\theta}$, $k_{56}(\theta) = \theta e^{-\theta}$, $k_{57}(\theta) = 4e^{-2\theta}$, $k_{65}(\theta) = 4e^{-2\theta}$, $k_{67}(\theta) = 2e^{-2\theta}$, $k_{72}(\theta) = \theta e^{-\theta}$, $k_{76}(\theta) = e^{-\theta}$, $k_{83}(\theta) = e^{-\theta}$, $k_{84}(\theta) = 4e^{-2\theta}$, $k_{87}(\theta) = \theta e^{-\theta}$, the initial values are chosen as $\eta_{10}(\theta) = [9, 10]^T$, $\eta_{20}(\theta) = [-2, 5]^T$, $\eta_{30}(\theta) = [-2, 5]^T$, $\eta_{40}(\theta) = [-3, 4]^T$, $\eta_{50}(\theta) = [2, 10]^T$, $\eta_{60}(\theta) = [5, 1]^T$, $\eta_{70}(\theta) = [2, 1]^T$, $\eta_{80}(\theta) = [-1, 2]^T$, $\eta_{50}(\theta) = [5, 10]^T$, $\eta_{60}(\theta) = [3, 9]^T$, $\eta_{70}(\theta) = [3, 3]^T$, and the parameters $\mu = 1$. Choose $K_{11} = [-6, -11, -2]$, $K_{12} = [-6, -11, 4]$, $K_{13} = [-6, -11, -4]$, $K_{14} = [-6, -11, -4]$, $K_{15} = [-6, -11, -4]$, $K_{16} = [-6, -11, 4]$, $K_{17} = [-6, -11, -2]$, $K_{18} = [-6, -11, 2]$ and $L_1 = [2, 1.09:0.48, 0.03: -0.65, 3]$, $L_2 = [2, 1.1: 1; -6: -8, 62]$, $L_3 = [2, 1.1:0,2; 0, -1]$ and $L_4 = [2, 1.0:2; 0, -1]$, $L_5 = [2, 1.0:2; 0, -1]$, $L_6 = [2, 1:0, 2; 0, -1]$, $L_7 = [2, 1.0: 0.03; -0.65, 3]$, $L_8 = [2, 1:0.8, -4; -4.4, 34]$. The matrices $K_{2i}$ and $K_{3i}$ for $i = 1, \ldots, 8$, can be calculated by $K_{2i} = \Gamma_i - K_{1i} \Pi_i$. $K_{3i} = \Gamma_{hi} - K_{hi} \Pi_{hi}$.

Define the formation tracking errors between the formation leaders and the reference leader as $e_L = [e_{i1}^{T}, e_{i2}^{T}, e_{i3}^{T}, e_{i4}^{T}]^T$ with $e_{iL} = [e_{ii1}^{L}, e_{i2}^{L}]^T$, $y_i - y_0 - C_{hi} h_i$, the state errors between the distributed observer (18c) and the reference leader as $e_w = [e_{w1}^{T}, e_{w2}^{T}, e_{w3}^{T}, e_{w4}^{T}]^T = w_F - (I_4 \otimes x_0)$ with $w_F = [w_1^{T}, w_2^{T}, w_3^{T}, w_4^{T}]^T$. Furthermore, define the state errors between the distributed observer (18d) and the formation shape $h_i(t) = [h_{i1}^T(t), h_{i2}^T(t), h_{i3}^T(t), h_{i4}^T(t)]^T$ as $e_h = [e_{h1}^{T}, e_{h2}^{T}, e_{h3}^{T}, e_{h4}^{T}]^T = h_F - (-c_{F1}^{T} c_{F2}^{T} \otimes I_2) h$ with

IV. A SIMULATION EXAMPLE

In this section, a simulation example is provided to illustrate the effectiveness of our proposed controllers. Consider a heterogeneous MAS of the following form [3],

$$\dot{x}_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c_i \end{pmatrix} x_i + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u_i,$$
$$y_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x_i,$$
$$i = 1, \ldots, 8,$$

where the parameters $\{a_i, b_i, c_i, d_i\}$ are set as $\{4, 1, 1, 0\}, \{10, 1, 1, 0\}, \{2, 1, 1, 0\}, \{8, 1, 1, 0\}, \{2, 1, 1, 0\}, \{2, 1, 1, 0\}, \{4, 1, 1, 0\}, \{4, 1, 1, 0\}$, respectively.

The parameters of the reference leader (2) are given as

$$e_{h} = \begin{pmatrix} e_{h1}^{T}, e_{h2}^{T}, e_{h3}^{T}, e_{h4}^{T} \end{pmatrix}^T = h_{F} - (-c_{F1}^{T} c_{F2}^{T} \otimes I_2) h$$
\( \dot{h}_F = [\dot{h}_5^T, \dot{h}_6^T, \dot{h}_7^T, \dot{h}_8^T]^T \) and the output containment errors between the formation-leaders and followers as \( e_F = [\dot{e}_5^T, \dot{e}_6^T, \dot{e}_7^T, \dot{e}_8^T]^T = y_F - (-L_{\hat{F}}^{-1}L_{\hat{F}} \otimes I_2)\theta_L \) with \( y_F = [y_5, y_6, y_7, y_8]^T \) and \( y_L = [y_1^T, y_2^T, y_3^T, y_4^T]^T \). The simulation results are given in Figures 2-6. It can be observed from Figure 2 that \( \lim_{t \to +\infty} e_L(t) = 0 \), which shows that the formation-leaders achieve the desired formation shape and also track the desired trajectory generated by the reference leader as time goes to infinity. Figure 3 shows that \( \lim_{t \to +\infty} e_F(t) = 0 \), that is, the states of the distributed observer (18c) tend to the state of the reference leader as time goes to infinity. The state errors between the distributed observer (18d) and the formation shape \( \hat{h}(t) \) are shown in Figure 4. It can be observed that \( \lim_{t \to +\infty} e_F(t) = 0 \). Figure 5 shows that at \( t = 10s \), the outputs of the followers have not converged to the convex hull spanned by the outputs of the formation-leaders. Figure 6 shows that at \( t = 10s \), the outputs of the followers have converged to the convex hull spanned by those of the formation-leaders.
bounded delays,
\[ \chi(t) = f(\chi_t) + h(t), \]
\[ \chi_0(\theta) = \phi(\theta), \theta \in (-\infty,0], \]
where \( \chi_t \) is defined by \( \chi_t = \chi(t + \theta) \in \mathbb{R}^m, \theta \in (-\infty,0] \),
\( f : \mathcal{B} \to \mathbb{R}^m \) is a linear and continuous function with \( f(0) = 0 \), \( \mathcal{B} \) is a Banach space of functions mapping from \( (-\infty,0] \) to \( \mathbb{R}^m \) equipped with the norm \( \| \cdot \|_{\mathcal{B}} \), the perturbed term \( h(t) \) is a continuous function, and \( \phi = \chi_{t|t=0} \in \mathcal{B} \) is the initial condition satisfying \( \| \phi \|_{\mathcal{B}} < +\infty \). The norm is defined as \( \| \phi \|_{\mathcal{B}} \triangleq \sup_{\sigma \in [-\tau,0]} \| \phi(\sigma) \| + \int_{0}^{+\infty} p(\theta)\| \phi(-\theta) \|d\theta \)
with a non-increasing and positive function \( p : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying \( \int_{0}^{+\infty} p(\theta)d\theta < +\infty \). Suppose the origin of the nominal system \( \chi(t) = f(\chi_t) \) is globally exponentially stable. Then if \( \lim_{t \to +\infty} h(t) = 0 \) exponentially, the solution of system (39) satisfies that \( \lim_{t \to +\infty} \chi(t) = 0 \) exponentially.

APPENDIX B

In this appendix, some preliminary results on the distributed observer for a single leader or multiple leaders under unbounded distributed transmission delays are recalled from [44].

Consider the following multi-agent system with unbounded distributed transmission delays,
\[ \begin{align*}
\dot{\psi}_i &= S\psi_i - \mu \sum_{j=1}^{M} a_{ij} \left( \psi_j(t) - \int_{0}^{+\infty} k_{ij}(\theta) \right. \\
&\quad \times e^{\theta \theta} \phi_j(t - \theta)d\theta \left. - \mu \sum_{j=M+1}^{M+N} a_{ij} \psi_j(t) \right) \\
&\quad - \int_{0}^{+\infty} k_{ij}(\theta)e^{\theta \theta} \psi_j(t - \theta)d\theta, \\
\end{align*} \]
for \( i \in \mathbb{F}, \)
where \( \phi_i \in \mathbb{R}^m, \psi_i \in \mathbb{R}^m, \sigma(S) \in j\mathbb{R}, \mu > 0 \) is a real number, \( k_{ij}(\theta) \) is the delay kernel satisfying inequality (8) and the condition that \( \int_{0}^{+\infty} p(\theta)\phi_i(-\theta)d\theta < +\infty, i \in \mathbb{L}. \)

Letting \( \phi_L = (\phi_1^T, \ldots, \phi_M^T)^T \) and \( \psi_F = (\psi_{M+1}^T, \ldots, \psi_{M+N}^T)^T \), it follows from (40) that
\[ \begin{align*}
\dot{\phi}(t) &= (I_M \otimes S)\phi_L, \\
\dot{\psi}(t) &= (I_N \otimes S)\psi_F - \mu \left( \bar{D}_F \psi_F \right. \\
&\quad + \int_{0}^{+\infty} (L_{F10} \otimes e^{\theta \theta})\psi_F(t - \theta)d\theta \\
&\quad + \int_{0}^{+\infty} (L_{F20} \otimes e^{\theta \theta})\phi_L(t - \theta)d\theta),
\end{align*} \]
where \( \bar{D}_F = D_F \otimes I_m, D_F = diag\{d_{M+1}, \ldots, d_{M+N}\} \) with \( d_i = \sum_{j=1}^{M+N} a_{ij}, i \in \mathbb{L}, L_{F10} = [f_{ij\theta}]_{N \times N} \) with \( f_{ij\theta} = -a_{ij}k_{ij}(\theta) \) for \( i \neq j \) and \( f_{jj\theta} = 0 \) for \( i = j, i \in \mathbb{F}, j \in \mathbb{L}, \) and \( L_{F20} = [f_{j\theta}]_{N \times M} \) with \( f_{j\theta} = -a_{ij}k_{ij}(\theta) \), \( i \in \mathbb{F}, j \in \mathbb{L}. \)

Lemma 6: Consider the multi-agent system (40) with its compact form system (41) under Assumption 2. Then
\[ \lim_{t \to +\infty} \| \psi_F - (-L_{F1}^T L_{F2} \otimes I_m)\phi_L \| = 0, \]

APPENDIX A

In this appendix, a stability result on systems with unbounded delays is introduced.

Lemma 5: ([48]) Consider the following system with unbounded delays,
\[ \chi(t) = f(\chi_t) + h(t), \]
\[ \chi_0(\theta) = \phi(\theta), \theta \in (-\infty,0], \]
where \( \chi_t \) is defined by \( \chi_t = \chi(t + \theta) \in \mathbb{R}^m, \theta \in (-\infty,0] \),
\( f : \mathcal{B} \to \mathbb{R}^m \) is a linear and continuous function with \( f(0) = 0 \), \( \mathcal{B} \) is a Banach space of functions mapping from \( (-\infty,0] \) to \( \mathbb{R}^m \) equipped with the norm \( \| \cdot \|_{\mathcal{B}} \), the perturbed term \( h(t) \) is a continuous function, and \( \phi = \chi_{t|t=0} \in \mathcal{B} \) is the initial condition satisfying \( \| \phi \|_{\mathcal{B}} < +\infty \). The norm is defined as \( \| \phi \|_{\mathcal{B}} \triangleq \sup_{\sigma \in [-\tau,0]} \| \phi(\sigma) \| + \int_{0}^{+\infty} p(\theta)\| \phi(-\theta) \|d\theta \)
with a non-increasing and positive function \( p : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying \( \int_{0}^{+\infty} p(\theta)d\theta < +\infty \). Suppose the origin of the nominal system \( \chi(t) = f(\chi_t) \) is globally exponentially stable. Then if \( \lim_{t \to +\infty} h(t) = 0 \) exponentially, the solution of system (39) satisfies that \( \lim_{t \to +\infty} \chi(t) = 0 \) exponentially.

V. CONCLUSIONS

This paper have considered the formation-containment tracking control problem of heterogeneous linear MASs with unbounded distributed transmission delays. We have proposed distributed output feedback controllers for formation leaders and followers, respectively. It has been shown that the formation-leaders achieve the formation tracking with the reference leader and the outputs of the followers converge to the convex hull spanned by the outputs of those formation leaders under our proposed controllers. One future topic is to extend the results of this paper to more general formation shapes and communication topologies.
exponentially, that is, \( \lim_{t \to +\infty} \text{dist}(\psi_i(t), C_0(\phi_j(t), j \in \mathbb{L})) = 0, \quad i \in \mathbb{F} \), exponentially.

If there exists only one leader, i.e., \( M = 1 \), equation (42) reduces to \( \lim_{t \to +\infty} \| \psi_i - (1_N \otimes \phi_1) \| = 0 \), that is, 
\[
\lim_{t \to +\infty} ||\psi_i(t) - \phi_1|| = 0, \quad i \in \mathbb{F}.
\]

Lemma 7: Consider the following system,
\[
\dot{\xi}(t) = -\mu (\tilde{D}_F \xi(t) + \int_0^{+\infty} (\mathcal{L}_{F10} \otimes I_m) \xi(t - \theta) d\theta),
\]
where \( \xi \in \mathbb{R}^{N \times m} \), \( \mu > 0 \), \( \tilde{D}_F \) and \( \mathcal{L}_{F10} \) are the same as those in Lemma 6. Then the zero solution of system (43) is globally exponentially stable.

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