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GLOBAL SUBSONIC JET WITH STRONG TRANSONIC SHOCK OVER A CONVEX CORNERED WEDGE FOR THE TWO-DIMENSIONAL STEADY FULL EULER EQUATIONS*

YU PEI† AND WEI XIANG†

Abstract. For a uniformly supersonic flow past a convex cornered wedge with the pressure being given for the surrounding quiescent gas at the downstream, as shown in experimental results, it is expected to form a shock followed by a contact discontinuity, which is also called the jet flow. By the shock polar analysis, it is well-known that there are two possible shocks, one a strong shock and the other one a weak shock. The strong shock is always transonic, while the weak shock could be transonic or supersonic. In this paper, we prove the global existence, asymptotic behaviors, uniqueness, and stability of the subsonic jet with a strong transonic shock under the perturbation of the upstream flow and the pressure of the surrounding quiescent gas, for the two-dimensional steady full Euler equations. We first formulate the problem into a nonlinear problem with two free boundaries meeting at the wedge corner, and formulate the boundary conditions on them. Then we introduce a modified Lagrange coordinates transformation to straighten the two free boundaries at the same time, and study the elliptic estimate with proper weighted Hölder norms to deal with the wedge corner singularity and the asymptotic behaviors for the Euler equations in the Lagrangian coordinates carefully, and then design an iteration scheme based on the estimates.

Key words. transonic shock, strong shock, jet flow, contact discontinuity, free boundary, steady Euler equations, compressible flow, conservation laws

AMS subject classifications. Primary, 35R35, 35M12, 35L65, 76H05, 35B45, 35B35, 35B40, 35B36; Secondary, 35J67, 76N10, 76L05, 76J20, 76N20, 76G25

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1. Introduction.

When a supersonic flow passes a convex cornered wedge with the given pressure at the downstream and being smaller than the pressure at the upstream, a shock and a contact discontinuity attached to the wedge corner are expected to be generated (see Courant and Friedrichs [18]). Due to the entropy condition, the slop of the shock is bigger than the slop of the contact discontinuity. Hence as shown in Figure 1, the transonic shock separating the supersonic incoming flow and the subsonic flow ahead, together with the contact discontinuity, forms a subsonic jet. The goal of this paper is to study this structure by proving the global existence, uniqueness, and stability of such steady full Euler flows in two dimensions.

In $\mathbb{R}^2$, the governing equations are the two-dimensional steady full Euler equations:

$$
\begin{cases}
\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0,\\
\partial_{x_1}(\rho u_1^2 + p) + \partial_{x_2}(\rho u_1 u_2) = 0,\\
\partial_{x_1}(\rho u_1 u_2) + \partial_{x_2}(\rho u_2^2 + p) = 0,\\
\partial_{x_1}(\rho u_1(E + \frac{p}{\rho})) + \partial_{x_2}(\rho u_2(E + \frac{p}{\rho})) = 0,
\end{cases}
\tag{1.1}
$$

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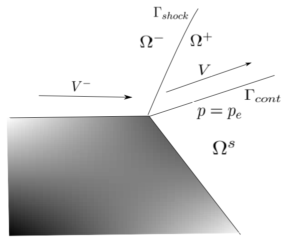

FIG. 1. *Subsonic jet with strong transonic shock.*

where $\rho = \rho(\mathbf{x}), \mathbf{u} = (u_1, u_2)(\mathbf{x}), p = p(\mathbf{x})$, and $E = E(\mathbf{x})$ are the density, velocity, pressure, and total energy, respectively, and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Moreover, E satisfies

$$E = \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{(\gamma-1)\rho}, \tag{1.2}$$

for a constant $\gamma > 1$, which is called the adiabatic exponent.

Now let us introduce the concerned problem mathematically. Let $\Omega^w \subset \mathbb{R}^2$ be an open and connected set, that is, the subdomain in $\mathbb{R}^2$ outside of the convex cornered wedge. As shown in Figure 1, suppose that two non-self-intersecting C^1-curves Γ_{shock} and Γ_{cont}, where Γ_{shock} is the shock and Γ_{cont} is the contact discontinuity, divide Ω^w into three disjoint open subdomains $\Omega^\pm$ and Ω^s, such that $\Omega^w = \Omega^- \cup \Gamma_{shock} \cup \Omega^+ \cup \Gamma_{cont} \cup \Omega^s$. We introduce two functions s and g such that

$$\Gamma_{shock} := \{\mathbf{x} : x_1 = s(x_2), \mathbf{x} \in \Omega^w\}, \tag{1.3}$$

$$\Gamma_{cont} := \{\mathbf{x} : x_2 = g(x_1), \mathbf{x} \in \Omega^w\}. \tag{1.4}$$

It is well-known that along the two discontinuities, we have the following Rankine–Hugoniot conditions: First, on the shock front Γ_{shock},

$$\begin{cases} [\rho u_1]_{\Gamma_{shock}} = s'(x_2)[\rho u_2]_{\Gamma_{shock}}, \\ [\rho u_1^2 + p]_{\Gamma_{shock}} = s'(x_2)[\rho u_1 u_2]_{\Gamma_{shock}}, \\ [\rho u_1 u_2]_{\Gamma_{shock}} = s'(x_2)[\rho u_2^2 + p]_{\Gamma_{shock}}, \\ [\rho u_1(E + \frac{p}{\rho})]_{\Gamma_{shock}} = s'(x_2)[\rho u_2(E + \frac{p}{\rho})]_{\Gamma_{shock}}, \end{cases} \tag{1.5}$$

where $[\cdot]_{\Gamma_{shock}}$ denotes the jump of the concerned quantity across the shock front, i.e.,

$$[w(\mathbf{x})]_{\Gamma_{shock}} = w(\mathbf{x})|_{\Omega^+} - w(\mathbf{x})|_{\Omega^-} \qquad \text{for } \mathbf{x} \in \Gamma_{shock}. \tag{1.6}$$

The discontinuity Γ_{shock} is called a shock wave if Rankine–Hugoniot conditions (1.5) and the entropy condition $[\rho(\mathbf{x})]_{\Gamma_{shock}} > 0$ hold. Second, on the contact discontinuity Γ_{cont},

$$\begin{cases} g'(x_1)[\rho u_1]_{\Gamma_{cont}} = [\rho u_2]_{\Gamma_{cont}}, \\ g'(x_1)[\rho u_1^2 + p]_{\Gamma_{cont}} = [\rho u_1 u_2]_{\Gamma_{cont}}, \\ g'(x_1)[\rho u_1 u_2]_{\Gamma_{cont}} = [\rho u_2^2 + p]_{\Gamma_{cont}}, \\ g'(x_1)[\rho u_1(E + \frac{p}{\rho})]_{\Gamma_{cont}} = [\rho u_2(E + \frac{p}{\rho})]_{\Gamma_{cont}}, \end{cases} \tag{1.7}$$

where $[\cdot]_{\Gamma_{cont}}$ denotes the jump of the concerned quantity across the contact discontinuity, i.e.,

$$[w(\mathbf{x})]_{\Gamma_{cont}} = w(\mathbf{x})|_{\Omega^s} - w(\mathbf{x})|_{\Omega^+} \qquad \text{for } \mathbf{x} \in \Gamma_{cont}. \tag{1.8}$$

One of the key differences between the shock and the contact discontinuity is that the contact discontinuity is a characteristic discontinuity, that is, $\mathbf{u}\cdot\mathbf{n}_g = 0$ on Γ_{cont}, where $\mathbf{n}_g$ denotes a unit normal vector field on Γ_{cont}. Hence, by the straightforward computation, conditions (1.7) are equivalent to

$$p = p|_{\Omega^+} \quad \text{and} \quad \mathbf{u}\cdot\mathbf{n}_g = 0 \qquad \text{on } \Gamma_{cont}. \tag{1.9}$$

In this paper, we mainly construct weak solutions with discontinuities Γ_{shock} and Γ_{cont} in the following sense.

DEFINITION 1.1. $V = (\mathbf{u}, \rho, p)$ *is a piecewisely smooth weak solution in the* Ω *with a shock* Γ_{shock} *and a contact discontinuity* Γ_{cont} *if* V *satisfies the following properties:*

(1) V *is smooth and satisfies Euler equations* (1.1) *classically in* Ω^-, Ω^+, *and* Ω^s.

(2) Γ_{shock} *and* Γ_{cont} *are two non-self-intersecting* C^1*-curve, on which* (1.5) *and* (1.9) *hold, respectively.*

(3) *Entropy condition holds on* Γ_{shock}.

Especially, as shown in Figure 1, we will study the following problem.

PROBLEM 1.1. *Given the incoming flow* V^- *on* Ω^- *and the pressure* p_e *on* Ω^s, *we want to find a classic solution* V *on* Ω^+ *with free boundaries* Γ_{shock} *and* Γ_{cont}, *which together with* V^- *on* Ω^- *and* $p = p_e$ *on* Ω^s *is the weak solution in the sense of Definition* 1.1.

We will prove the well-posedness and asymptotic behaviors of the global subsonic jet with a strong transonic shock governed by the two-dimensional steady Euler equations, in the sense of Definition 1.1. As far as we known, all the attention on the supersonic Euler flow over a convex cornered wedge is on the supersonic jet with a supersonic shock (cf. [10, 11, 29]) or a rarefaction wave (cf. [28]), so this paper is the first mathematical result on the global subsonic Euler jet with a transonic shock. In order to make it, we introduce a coordinates transformation to flatten the free boundaries, and then formulate the Euler system into a system with two transport equations for the Bernoulli's function and the entropy, which will be introduced in (3.28) and (3.30), and an elliptic system of two equations of first order for the pressure and the flow direction $w = \frac{u_2}{u_1}$. Because the pressure on the contact discontinuity is given a priori, we design an iteration by first solving the pressure via a linearized elliptic equation of second order with an oblique-Dirichlet mixed boundary condition in a cornered domain, and then solving the flow direction as well as the Bernoulli's function and the entropy by linearized transport equations. After carefully deriving estimates for the solutions one by one, we finally show the existence of solutions of the nonlinear problem by the Schauder fixed point theorem.

Since the Schauder fixed point theorem cannot give the uniqueness, we establish the asymptotic behavior of solutions via the weighted Hölder norm with decay at infinity to obtain the uniqueness. We remark that because of the transport equations for the Bernoulli's function and the entropy, the solutions decay at a different rate in the different directions. Therefore, the weighted Hölder norms contain two parts. One is the decay in the radius direction, and the other one is the decay in the normal direction of the contact discontinuity. More precisely, we establish that the decay rate of (w, p) is $|\mathbf{x}|^{-1-\beta}$, and the decay rate of (u_1, ρ) is $|x_2|^{-1-\beta}$. That is, the asymptotic behaviors of (u_1, ρ) converge to the background state only when $x_2 \to \infty$.

Up to now, there have been many works in the literature on the steady transonic flow past a wedge and the stability of the steady contact discontinuity separately. First, for the two dimensional attached transonic shock, Chen and Fang in [14] show the stability for the potential flow equations. For the Euler flow, Fang in [19] introduced a weighted Soblev space, and Yin and Zhou in [32] used the Hölder norms to show the stability for a supersonic Euler flow past an infinitely long two-dimensional wedge with a sharp angle. Then Chen, Chen, and Feldman in [4] studied this problem for both strong and weak shocks, whose different asymptotic behaviors were established. Recently, Bae and Xiang in [3] established the global existence of the detached shock past a blunt body for the potential flow. For other related results, one can refer to [5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 20, 21, 22, 23, 26, 27, 30, 31]. Second, for the two-dimensional subsonic contact discontinuity, Bae in [1] studied the stability of piecewise constant subsonic contact discontinuity in two-dimensional nozzles. Then Chen et al. in [9] established the global existence of subsonic Euler flows with large vorticity and contact discontinuities in two-dimensional nozzles, which are not necessarily a perturbation of piecewise constant solutions. Finally, in [2], Bae and Park use Helmholtz decomposition to decompose the Rankine–Hugoniot conditions on the contact discontinuity to prove the existence of a subsonic weak solution in a semi-infinitely long nozzle. Recently, the stability of supersonic or transonic contact discontinuity in a finitely long nozzle has been successfully established in [24, 25].

The rest of this paper is organized as follow. In section 2 we state the problem and main theorem as well as define the weighted Hölder norm. In section 3, we reformulate the problem and main theorem in the Lagrangian coordinates, then the iteration scheme is introduced. In section 4, estimates and existence of $\delta\tilde{p}$ are established. Then we establish the existence and estimates of $(\delta\tilde{w}, \delta\tilde{\rho}, \delta\tilde{u}_1)$ in section 5. In section 6, we show the existence of solutions of the free boundary problem by the Schauder fixed point theorem. Finally, the proof of the main theorem is concluded by establishing the asymptotic behaviors and uniqueness of the solutions in section 7.

2. Free boundary problem and main theorem.

Let W_w be the set of points lying in the convex cornered wedge. In the domain $\Omega^w := \{\mathbf{x} : x_1 > 0\}\backslash W_w$, as shown in Figure 1, the shock and contact discontinuity of the background solution are given separately by

$$\Gamma^0_{shock} := \{\mathbf{x} : x_1 = s_0(x_2) = k_0 x_2\} \qquad \text{and} \qquad \Gamma^0_{cont} := \{\mathbf{x} : x_2 = g_0(x_1) = b_0 x_1\}. \tag{2.1}$$

From the standard shock polar analysis, which is based on the Rankine–Hugoniot conditions (1.5) and (1.9) and the entropy condition (cf. see [18]), it is easy to see that $0 < b_0 < \frac{1}{k_0}$.

For notational simplicity, we rotate the coordinates axis anticlockwise by an angle so that $b_0 = 0$ and still denote the shock as $x_1 = k_0 x_2$. That is,

$$\Gamma^0_{shock} := \{\mathbf{x} : x_1 = s_0(x_2) = k_0 x_2\} \qquad \text{and} \qquad \Gamma^0_{cont} := \{\mathbf{x} : x_2 = 0\}. \tag{2.2}$$

Then the piecewise constant background solution V_0 in Ω^w is given as

$$V_0 = V_0^- = (\mathbf{u}_0^-, \rho_0^-, p_0^-) \qquad \text{on } \Omega_0^- := \{\mathbf{x} : 0 < x_1 < k_0 x_2\}, \tag{2.3}$$

$$V_0 = V_0^+ = (\mathbf{u}_0^+, \rho_0^+, p_0^+) \qquad \text{on } \Omega_0^+ := \{\mathbf{x} : x_1 > k_0 x_2, x_2 > 0\}, \tag{2.4}$$

$$V_0 = V_0^s = ((0,0), \rho_0^s, p_0^+) \qquad \text{on } \Omega_0^s := \{\mathbf{x} : x_1 > 0, x_2 < 0\}. \tag{2.5}$$

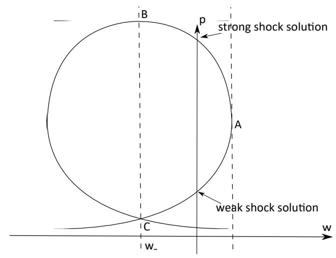

FIG. 2. *Shock polar analysis.*

Moreover, by (1.9), the vertical velocity of $\mathbf{u}_0^+$ vanishes on Ω_0^s in the new coordinates, i.e., $\mathbf{u}_0^+ = (|\mathbf{u}_0^+|, 0)$. Let $\mathbf{u}_0^- = (u_1^-, u_2^-)$. Let

$$w = \frac{u_2}{u_1} \qquad \text{and} \qquad w_- = \frac{u_2^-}{u_1^-}. \tag{2.6}$$

As shown in Figure 2, it is well-known that for any given incoming flow V_0^- with w_- given by (2.6), all the possible solutions of Rankine–Hugoniot conditions (1.5) form a shock polar, which can be written as $w = w(p)$ with

$$w'(p) < 0 \qquad \text{on curve AB.} \tag{2.7}$$

For this problem after the coordinates rotation, we know that $w = 0$ for the state behind the shock. So as shown in Figure 2, there are two entropy shock solutions, one of which is called the strong shock (lying on the curve AB), while the other one is called the weak shock (lying on the curve AC).

In this paper, we will consider the strong shock case and show such a background solution is globally stable with respect to perturbations on the upstream flow in Ω_0^- and the downstream pressure in the Ω_0^s, i.e., when V_0^- and p_0^+ are replaced by V^- and p_e, respectively. Now, Γ_{shock} and Γ_{cont} are defined by (1.3) and (1.4), respectively. And domains Ω^w, Ω^-, Ω^+, and Ω^s are defined as follows:

$$\Omega^w := \{\mathbf{x} : x_1 > 0\}, \tag{2.8}$$

$$\Omega^- := \{\mathbf{x} : 0 < x_1 < s(x_2)\}, \tag{2.9}$$

$$\Omega^+ := \{\mathbf{x} : x_1 \geq s(x_2), x_2 \geq g(x_1)\}, \tag{2.10}$$

$$\Omega^s := \{\mathbf{x} : x_1 > 0, x_2 < g(x_1)\}. \tag{2.11}$$

Finally, on the contact discontinuity Γ_{cont}, we have boundary condition (1.9) with $p|_{\Omega^+} = p_e$.

Hence, we can introduce Problem 1.1 more rigorously.

PROBLEM 2.1. *Suppose that the upstream V^- and downstream pressure p_e are small perturbations of the background solutions V_0^- and p_0^+, respectively. We want to establish the existence and uniqueness of global subsonic jet flow V, which are closed to V_0^+, with a transonic shock Γ_{shock} and a contact discontinuity Γ_{cont} as free boundaries, which are closed to Γ_{shock}^0 and Γ_{cont}^0 respectively. More precisely,*

(1) *V satisfies the full Euler equations* (1.1) *in Ω^+,*

(2) *Rankine–Hugoniot conditions* (1.5) *and* (1.9) *hold along Γ_{shock} and Γ_{cont}, respectively.*

We will answer Problem 2.1, which is the main theorem of this paper. Before giving it, we introduce two weighted Hölder norms. The first one concerns the singularity coming from the corner **O** and the contact discontinuity.

Definition 2.1. *Let Ω be a two-dimensional domain. Let u be a function defined in Ω, and let $\mathbf{x} = (x_1, x_2) \in \Omega$. For $\mathbf{x}, \mathbf{y} \in \Omega$, define*

$$d_0^{\mathbf{x}} := \min\{|\mathbf{x}|, 1\}, \qquad d_{0,\Gamma_{cont}}^{\mathbf{x}} := \min\{dist\{\mathbf{x}, \Gamma_{cont}\}, 1\}, \tag{2.12}$$

$$d_0^{\mathbf{x},\mathbf{y}} := \min\{d_0^{\mathbf{x}}, d_0^{\mathbf{y}}\}, \qquad d_{0,\Gamma_{cont}}^{\mathbf{x},\mathbf{y}} := \min\{d_{0,\Gamma_{cont}}^{\mathbf{x}}, d_{0,\Gamma_{cont}}^{\mathbf{y}}\}. \tag{2.13}$$

Let $\mathbf{I} = (i_1, i_2)$ be an index vector with $i_1, i_2 \geq 0$ and $|\mathbf{I}| = i_1 + i_2$. Let $D^{\mathbf{I}} = D_{x_1}^{i_1} D_{x_2}^{i_2}$. $\alpha \in (0,1)$, and $\chi_1, \chi_2 \in \mathbb{R}$ with $\chi_1 \geq \chi_2$. Define

$$\|u\|_{k;\Omega}^{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})} := \sum_{j=0}^{k} \sup_{\substack{\mathbf{x}\in\Omega \\ |\mathbf{I}|=j}} (d_0^{\mathbf{x}})^{\max\{\chi_1+\min\{j,-\chi_2\},0\}} (d_{0,\Gamma_{cont}}^{\mathbf{x}})^{\max\{j+\chi_2,0\}} |D^{\mathbf{I}} u| \tag{2.14}$$

and

$$\begin{aligned} \|u\|_{k,\alpha;\Omega}^{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})} &:= \|u\|_{k;\Omega}^{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})} \\ &+ \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega \\ \mathbf{x}\neq\mathbf{y} \\ |\mathbf{I}|=k}} (d_0^{\mathbf{x},\mathbf{y}})^{\max\{\chi_1+\min\{k+\alpha,-\chi_2\},0\}} (d_{0,\Gamma_{cont}}^{\mathbf{x},\mathbf{y}})^{\max\{k+\alpha+\chi_2,0\}} \frac{|D^{\mathbf{I}}u(\mathbf{x}) - D^{\mathbf{I}}u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}. \end{aligned} \tag{2.15}$$

The second weighted Hölder norm concerns the decay at the infinity or from the contact discontinuity.

Definition 2.2. *Let Ω be a two-dimensional domain. Let u be a function defined in Ω, and $\mathbf{x} = (x_1, x_2) \in \Omega$. Define*

$$d_\infty^{\mathbf{x}} := |\mathbf{x}| + 1, \qquad d_{\infty,\Gamma_{cont}}^{\mathbf{x}} := dist\{\mathbf{x}, \Gamma_{cont}\} + 1, \tag{2.16}$$

$$d_\infty^{\mathbf{x},\mathbf{y}} := \min\{d_\infty^{\mathbf{x}}, d_\infty^{\mathbf{y}}\}, \qquad d_{\infty,\Gamma_{cont}}^{\mathbf{x},\mathbf{y}} := \min\{d_{\infty,\Gamma_{cont}}^{\mathbf{x}}, d_{\infty,\Gamma_{cont}}^{\mathbf{y}}\}. \tag{2.17}$$

Suppose $\alpha \in (0,1)$, and $\chi_3, \chi_4 \in \mathbb{R}$. Define

$$\|u\|_{k;(\chi_3,\chi_4);\Omega} = \sum_{j=0}^{k} \sup_{\substack{\mathbf{x}\in\Omega \\ |\mathbf{I}|=j}} (d_\infty^{\mathbf{x}})^{j+\chi_3} (d_{\infty,\Gamma_{cont}}^{\mathbf{x}})^{\chi_4} |D^{\mathbf{I}}u|, \tag{2.18}$$

$$\|u\|_{k;(\chi_3,\chi_4);\Omega}^{*} = \sum_{j=0}^{k} \sup_{\substack{\mathbf{x}\in\Omega \\ |\mathbf{I}|=j}} (d_\infty^{\mathbf{x}})^{\chi_3} (d_{\infty,\Gamma_{cont}}^{\mathbf{x}})^{j+\chi_4} |D^{\mathbf{I}}u|, \tag{2.19}$$

$$|u|_{k,\alpha;(\chi_3,\chi_4);\Omega} = \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega \\ \mathbf{x}\neq\mathbf{y} \\ |\mathbf{I}|=k}} (d_\infty^{\mathbf{x},\mathbf{y}})^{k+\alpha+\chi_3} (d_{\infty,\Gamma_{cont}}^{\mathbf{x},\mathbf{y}})^{\chi_4} \frac{|D^{\mathbf{I}}u(\mathbf{x}) - D^{\mathbf{I}}u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}, \tag{2.20}$$

$$|u|_{k,\alpha;(\chi_3,\chi_4);\Omega}^{*} = \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega \\ \mathbf{x}\neq\mathbf{y} \\ |\mathbf{I}|=k}} (d_\infty^{\mathbf{x},\mathbf{y}})^{\chi_3} (d_{\infty,\Gamma_{cont}}^{\mathbf{x},\mathbf{y}})^{k+\alpha+\chi_4} \frac{|D^{\mathbf{I}}u(\mathbf{x}) - D^{\mathbf{I}}u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}, \tag{2.21}$$

$$\|u\|_{k,\alpha;(\chi_3,\chi_4);\Omega} = \|u\|_{k;(\chi_3,\chi_4);\Omega} + |u|_{k,\alpha;(\chi_3,\chi_4);\Omega}, \tag{2.22}$$

$$\|u\|_{k,\alpha;(\chi_3,\chi_4);\Omega}^{*} = \|u\|_{k;(\chi_3,\chi_4);\Omega}^{*} + |u|_{k,\alpha;(\chi_3,\chi_4);\Omega}^{*}. \tag{2.23}$$

Then combining the norms together, we have the following.

Definition 2.3. *Let* Ω *be a two-dimensional domain. Let* u *be a function defined in* Ω*, and* $\mathbf{x} = (x_1, x_2) \in \Omega$*. Define the weighted Hölder norm*

$$\|u\|^{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}_{k,\alpha;(\chi_3,\chi_4);\Omega} := \|u\|^{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}_{k,\alpha;\Omega} + \|u\|_{k,\alpha;(\chi_3,\chi_4);\Omega}, \tag{2.24}$$

$$\|u\|^{*(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}_{k,\alpha;(\chi_3,\chi_4);\Omega} := \|u\|^{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}_{k,\alpha;\Omega} + \|u\|^{*}_{k,\alpha;(\chi_3,\chi_4);\Omega}, \tag{2.25}$$

and the weighted functional space

$$C^{k,\alpha;(\chi_3,\chi_4)}_{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}(\Omega) := \{u : \|u\|^{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}_{k,\alpha;(\chi_3,\chi_4);\Omega} \le \infty\}, \tag{2.26}$$

$$C^{*k,\alpha;(\chi_3,\chi_4)}_{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}(\Omega) := \{u : \|u\|^{*(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}_{k,\alpha;(\chi_3,\chi_4);\Omega} \le \infty\}. \tag{2.27}$$

It is easy to check that $C^{k,\alpha;(\chi_3,\chi_4)}_{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}(\Omega)$ and $C^{*k,\alpha;(\chi_3,\chi_4)}_{(\chi_1,\mathbf{O})(\chi_2,\Gamma_{cont})}(\Omega)$ are Banach spaces.

For the three definitions introduced above, we have several remarks.

Remark 2.1. For the flow $V = (u_1, u_2, \rho, p)$ with $w = \frac{u_2}{u_1}$ and $w_0^+ = 0$, let $U = (p, w, \rho, u_1)$ and $U_0^+ = (p_0^+, 0, \rho_0^+, u_{10}^+)$. In order to shorten the expressions, we use the notation below for a norm of U (then for V),

$$\begin{aligned}\|U - U_0^+\|_{\mathcal{F}^+} =& \|p - p_0^+\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\Omega^+} + \|w\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\Omega^+}\\ &+\|\rho - \rho_0^+\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\Omega^+} + \|(\rho - \rho_0^+)_{x_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\beta,0);\Omega^+}\\ &+\|u_1 - u_{10}^+\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\Omega^+} + \|(u_1 - u_{10}^+)_{x_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\beta,0);\Omega^+}.\end{aligned} \tag{2.28}$$

We remark that in (2.28), the weighted Hölder norms used for $(\rho-\rho_0^+, u_1-u_{10}^+)$ are different from the ones used for $(p-p_0^+, w)$ and $((\rho-\rho_0^+)_{x_1}, (u_1-u_{10}^+)_{x_1})$. The reason is that $(p-p_0^+, w)$ are governed by elliptic equations and $((\rho-\rho_0^+)_{x_1}, (u_1-u_{10}^+)_{x_1})$ rely on $((p-p_0^+)_{x_1}, w_{x_1})$, while $(\rho-\rho_0^+, u_1-u_{10}^+)$ are governed by transport equations, so the decay estimates are different in different directions.

Remark 2.2. In order to iterate the shock position, we need to introduce a weighted Hölder norm for function u defined on $\mathbb{R}^+ = (0,\infty)$. That is, $\|u\|^{(\chi_1,0)}_{k,\alpha;(\chi_4);\mathbb{R}^+}$, with $\chi_1 = \chi_2$, $\chi_3 = \chi_4$, $d_0^{\mathbf{x}} = d_{0,\Gamma_{cont}}^{\mathbf{x}} = \min\{x, 1\}$, and $d_\infty^{\mathbf{x}} = d_{\infty,\Gamma_{cont}}^{\mathbf{x}} = x + 1$, and Ω and $\mathbf{x}$ being replaced by $\mathbb{R}^+$ and x, respectively, in (2.24) for any $x \in \mathbb{R}^+$.

Remark 2.3. Condition $\chi_1 \ge \chi_2$ means that the singularity to the contact discontinuity is no worse than the vertex singularity. If $\chi_1 = \chi_2$, then the weights in (2.14) and (2.15) can be replaced by $(d_0^{\mathbf{x}})^{\max\{j+\chi_1,0\}}$ and $(d_0^{\mathbf{x},\mathbf{y}})^{\max\{k+\alpha+\chi_1,0\}}$, because these two singularities are equivalent.

Then we can introduce the main theorem in this paper.

Theorem 2.1. *For the background states as given in* (2.2)–(2.5)*, which is a strong transonic shock solution with a subsonic contact discontinuity, there exist constants* α*,* $\beta \in (0,1)$*, and a sufficiently small constant* ε_0 *such that if* $\varepsilon \in (0, \varepsilon_0)$ *and the upstream flow* V^- *and the downstream pressure* p_e *satisfy*

$$\|V^- - V_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|p_e - p_0^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \le \varepsilon, \tag{2.29}$$

then Problem 2.1 *admits a unique subsonic jet flow* $V = (\mathbf{u}, \rho, p)$ *in* Ω^+ *with* w *being defined by* (2.6) *and two free boundaries* Γ_{shock} *and* Γ_{cont} *defined by* (1.3)–(1.4)*, satisfying the estimate that*

$$\|U - U_0^+\|_{\mathcal{F}^+} + \|s' - k_0\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + \|g\|^{(-1-\alpha;0)}_{3,\alpha;(\beta);\mathbb{R}^+} \leq C\varepsilon, \tag{2.30}$$

where constant C *depends only on* (U_0^-, U_0^+), α, β, *and* ε_0, *and* U *and* U_0^+ *are given in Remark* 2.1.

3. Mathematical formulation of Problem 2.1.

In this section, we will formulate Problem 2.1 by straightening the two free boundaries Γ_{cont} and Γ_{shock}, and decomposing the Euler equations. Then the iteration scheme will be introduced.

3.1. Straighten the contact discontinuity.

To overcome the difficulty of the free boundary Γ_{cont}, first, we will apply the Lagrangian coordinates transformation to straighten the contact discontinuity.

By the first equation in (1.1), there exists a unique function φ in the region Ω^- and Ω^+ such that

$$\varphi_{x_1} = -\rho u_2 \qquad \text{and} \qquad \varphi_{x_2} = \rho u_1$$

with condition

$$\varphi(0,0) = 0. \tag{3.1}$$

Then we can introduce the following Lagrangian coordinates transformation:

$$\begin{cases} y_1 = x_1, \\ y_2 = \varphi(x_1, x_2). \end{cases} \tag{3.2}$$

Denote $\mathbf{y} = (y_1, y_2)$. It follows from the second condition in (1.9) that φ is a constant on Γ_{cont}. So by (3.1), $\varphi \equiv 0$ on Γ_{cont}. Therefore, in the new $\mathbf{y}$-coordinates, the constant discontinuity Γ_{cont} becomes

$$\bar{\Gamma}_{cont} := \{\mathbf{y} : y_1 > 0, y_2 = 0\}. \tag{3.3}$$

And the first condition in (1.9) is

$$p = p_e \qquad \text{on } \bar{\Gamma}_{cont}. \tag{3.4}$$

If ε_0 is sufficiently small, then for any U satisfying (2.30), obviously, $|\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}| = \rho u_1 > 0$. So the coordinates transformation (3.2) is invertible. Moreover, let $\mathbb{D}^+$ be the image of Ω^+ via the coordinates transformation (3.2). Then the Hölder norms defined on Ω^+ and $\mathbb{D}^+$ by (2.24) are controlled by each other with positive constants, which depend only on U_0^+ and ε_0. Hence, we will use the same notation on the Hölder norms and the Hölder space as in Defintion 2.3 and Remark 2.1.

In $\mathbf{y}$-coordinates, Euler equations (1.1) become

$$\partial_{y_1}\left(\frac{1}{\rho u_1}\right) - \partial_{y_2}\left(\frac{u_2}{u_1}\right) = 0, \tag{3.5}$$

$$\partial_{y_1}\left(u_1 + \frac{1}{\rho u_1}\right) - \partial_{y_2}\left(\frac{p u_2}{u_1}\right) = 0, \tag{3.6}$$

$$\partial_{y_1}(u_2) + \partial_{y_2} p = 0, \tag{3.7}$$

$$\partial_{y_1}\left(\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\right) = 0. \tag{3.8}$$

Let

$$\bar{\Gamma}_{shock} := \{\mathbf{y} : y_1 > 0, y_1 = \bar{s}(y_2)\} \tag{3.9}$$

be the image of shock front Γ_{shock} via the coordinates transformation (3.2). Then in the $\mathbf{y}$-coordinates, the R-H conditions (1.5) along Γ_{shock} become the following ones along $\bar{\Gamma}_{shock}$:

$$\left[\frac{1}{\rho u_1}\right]_{\bar{\Gamma}_{shock}} = -\left[\frac{u_2}{u_1}\right]_{\bar{\Gamma}_{shock}} \bar{s}'(y_2), \tag{3.10}$$

$$\left[u_1 + \frac{p}{\rho u_1}\right]_{\bar{\Gamma}_{shock}} = -\left[\frac{p u_2}{u_1}\right]_{\bar{\Gamma}_{shock}} \bar{s}'(y_2), \tag{3.11}$$

$$[u_2]_{\bar{\Gamma}_{shock}} = [p]_{\bar{\Gamma}_{shock}} \bar{s}'(y_2), \tag{3.12}$$

$$\left[\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\right]_{\bar{\Gamma}_{shock}} = 0. \tag{3.13}$$

For notational simplicity, from now on, we will denote $[\cdot]_{\bar{\Gamma}_{shock}}$, $\bar{\Gamma}_{cont}, \bar{\Gamma}_{shock}$, and $\bar{s}$ by $[\cdot]$, Γ_{cont}, Γ_{shock}, and s, respectively, without further notification. Let

$$\Gamma_0 := \{\mathbf{y} : y_1 > 0,\ y_1 = k_1 y_2\}, \tag{3.14}$$

where $k_1 = \frac{k_0}{\rho_0^+ u_0^+}$. Let

$$\mathbb{D}^w := \{\mathbf{y} : y_1 > 0\}, \tag{3.15}$$

$$\mathbb{D}^- := \{\mathbf{y} : 0 < y_1 < s(y_2)\}, \tag{3.16}$$

$$\mathbb{D}^+ := \{\mathbf{y} : y_1 \geq s(y_2),\ y_2 \geq 0\}, \tag{3.17}$$

$$\mathbb{D}_0^+ := \{\mathbf{y} : y_1 \geq k_1 y_2,\ y_2 \geq 0\}, \tag{3.18}$$

$$\mathbb{D}^s := \{\mathbf{y} : y_1 > 0,\ y_2 < 0\}. \tag{3.19}$$

It is easy to see that Γ_0, $\mathbb{D}^w$, $\mathbb{D}^-$, $\mathbb{D}^+$, $\mathbb{D}_0^+$, and $\mathbb{D}^s$ are the images of Γ^0_{shock}, Ω^w, Ω^-, Ω^+, Ω_0^+, and Ω^s via the coordinates transformation (3.2), respectively.

Then in order to show Theorem 2.1, we only need to show the following theorem.

THEOREM 3.1. *Let assumptions be given as in Theorem* 2.1 *on the background solutions* (2.2)–(2.5) *for* $\varepsilon \in (0, \varepsilon_0)$ *such that*

$$\|V^- - V_0^-\|_{2,\alpha;(1+\beta,0);\mathbb{D}^-} + \|p_e - p_0^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \leq \varepsilon. \tag{3.20}$$

We have the following:

(a) *(Existence and uniqueness) There exists a unique solution* $V = (\mathbf{u}, \rho, p)$ *to* (3.5)–(3.8) *in* $\mathbb{D}^s$ *with boundary conditions* (3.4) *and* (3.10)–(3.13) *satisfying*

$$\|U - U_0^+\|_{\mathcal{F}_L^+} + \|s' - k_1\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \leq C\varepsilon, \tag{3.21}$$

where C *is a constant only depending on* (V_0^-, V_0^+), α, β, *and* ε_0, *and* $\|\cdot\|_{\mathcal{F}_L^+}$ *is the Hölder norm defined in* (2.28) *with* Ω^+ *being replaced by* $\mathbb{D}^+$, *and* U *and* U_0^+ *are given in Remark* 2.1.

(b) *(Asymptotic behavior) For* p *and* w, *as* $|\mathbf{y}| \to \infty$,

$$|p - p_0^+| + |w| = O(|\mathbf{y}|^{-(1+\beta)}). \tag{3.22}$$

For ρ *and* u_1, *as* $y_2 \to \infty$,

$$|\rho - \rho_0^+| + |u_1 - |\mathbf{u}_0^+|| = O(y_2^{-(1+\beta)}). \tag{3.23}$$

Remark 3.1. In Theorem 3.1, (u_1, ρ) has different asymptotic behaviors from the ones of (p, u_2). The key reason is that, from the next subsection (see (3.27) and (3.29) below), there are two tranport equations related to (u_1, ρ) in Euler equations (3.5)–(3.8) such that it is impossible to have the decay behaviors of (u_1, ρ) as $y_1 \to \infty$.

We can show Theorem 2.1 once we have Theorem 3.1.

Proof of Theorem 2.1. The existence of solutions $V(\mathbf{x})$ is obvious if we define $V(\mathbf{x}) = V(\mathbf{y}(\mathbf{x}))$, where $V(\mathbf{y})$ is obtained in Theorem 3.1. The uniqueness and estimates of V and s in (2.30) come from (3.21) and the fact that the coordinates transformation is invertible and its Jacobian is uniformly bounded and away from 0. Moreover, it follows from (3.21) that $\|w\|^{(-\alpha,\mathbf{O})}_{2,\alpha;(1+\beta,0);\mathbb{D}^s} \leq C\varepsilon$. Because of the second R-H condition in (1.9), we have

$$g(x_1) = \int_0^{x_1} w(z_1, 0)dz_1.$$

Hence, (2.30) follows easily. ∎

3.2. Decomposition of (3.5)–(3.8). In this subsection, we will decompose the Euler equations (3.5)–(3.8) into two transport equations and two nonlinear elliptic equations.

By straightforward computation, we can see that when the flow is subsonic, i.e., when

$$q = \sqrt{u_1^2 + u_2^2} < c := \sqrt{\frac{\gamma p}{\rho}}, \tag{3.24}$$

there are two complex eigenvalues and two real eigenvalues, which are

$$\lambda_{\pm} = \lambda_R + i\lambda_I \qquad \text{and} \qquad \lambda_{01} = \lambda_{02} = 0, \tag{3.25}$$

where

$$\lambda_R = -\frac{c^2 \rho u_2}{c^2 - u_1^2} \qquad \text{and} \qquad \lambda_I = \frac{c\rho u_1 \sqrt{c^2 - q^2}}{c^2 - u_1^2}, \tag{3.26}$$

with the following left eigenvectors:

$$\begin{aligned}
l_{01} &= \begin{pmatrix} 0, & 0, & 0, & 1 \end{pmatrix}, \\
l_{02} &= \begin{pmatrix} -pu_1, & u_1, & u_2, & -1 \end{pmatrix}, \\
l_R &= \Bigg(\left(\frac{\gamma p^2 u_2}{(\gamma-1)u_1} - \frac{pu_1}{\gamma-1} \right) \lambda_R + \frac{\gamma p^2 u_2}{(\gamma-1)u_1}, \quad -\left(u_1 + \frac{\gamma p}{(\gamma-1)u_1} \right) \lambda_R - \frac{\gamma p u_2}{(\gamma-1)u_1}, \\
&\qquad\qquad \frac{\gamma p}{\gamma-1} - u_2\lambda_R, \quad \lambda_R \Bigg), \\
l_I &= \left(\left(\frac{\gamma p^2 u_2}{(\gamma-1)u_1} - \frac{pu_1}{\gamma-1} \right) \lambda_I, \quad -\left(u_1 + \frac{\gamma p}{(\gamma-1)u_1} \right) \lambda_I, \quad -u_2\lambda_I, \quad \lambda_I \right).
\end{aligned}$$

Then we can multiply the left eigenvectors on the Euler equations (3.5)–(3.8) to diagonalize and then decompose the Euler equations (3.5)–(3.8). So there are two transport equations corresponding to the real eigenvalues λ_{01} and λ_{02},

$$\left(\frac{1}{2}q^2 + \frac{\gamma p}{(\gamma-1)\rho} \right)_{y_1} = 0, \tag{3.27}$$

which means

$$\frac{1}{2}q^2 + \frac{\gamma p}{(\gamma-1)\rho} = B(y_2), \tag{3.28}$$

where B is the Bernoulli function, and

$$\left(\frac{p}{\rho^\gamma}\right)_{y_1} = 0, \tag{3.29}$$

that is,

$$\frac{p}{\rho^\gamma} = S(y_2), \tag{3.30}$$

where S is the entropy.

Moreover, there are two elliptic equations corresponding to the complex eigenvalues $\lambda_\pm$:

$$\begin{cases} \partial_R w + \Lambda \partial_I p = 0, \\ \partial_I w - \Lambda \partial_R p = 0, \end{cases} \tag{3.31}$$

where $\Lambda = \frac{\sqrt{c^2-q^2}}{c\rho u_1^2}$, $w = \frac{u_2}{u_1}$, $\partial_I = \lambda_I \partial y_2$, and $\partial_R = \partial y_1 + \lambda_R \partial y_2$.

3.3. Iteration scheme for solving Theorem 3.1. First, in order to define the iteration set, we introduce the following norms and functional spaces.

DEFINITION 3.1. *Denote*

$$\begin{aligned} &\|\cdot\|_{\mathcal{F}_1} := \|\cdot\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}_0^+}, \\ &\|\cdot\|_{\mathcal{F}_2} := \|\cdot\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\mathbb{D}_0^+} + \|\cdot\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\beta,0);\mathbb{D}_0^+}, \\ &\|\cdot\|_{\mathcal{F}_3} := \|\cdot\|^{(-1-\alpha;0)}_{3,\alpha;(\beta);\mathbb{R}^+}. \end{aligned}$$

For any given constant $\xi > 0$ and for each $i = 1$, 2, and 3, define

$$\mathcal{F}_i^\xi := \{v : \|v\|_{\mathcal{F}_i} \le \xi\} \qquad \text{and} \qquad \mathcal{F}^\xi := \mathcal{F}_1^\xi \times \mathcal{F}_1^\xi \times \mathcal{F}_2^\xi \times \mathcal{F}_2^\xi \times \mathcal{F}_3^\xi. \tag{3.32}$$

Finally, for $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5) \in \mathcal{F}^\xi$, we define the norm

$$\|\mathbf{v}\|_{\mathcal{F}} = \|v_1\|_{\mathcal{F}_1} + \|v_2\|_{\mathcal{F}_1} + \|v_3\|_{\mathcal{F}_2} + \|v_4\|_{\mathcal{F}_2} + \|v_5\|_{\mathcal{F}_3}. \tag{3.33}$$

We remark that in Definition 3.1, the domain concerned for $\mathcal{F}_3$ is $\mathbb{R}^+$, which is different from the one for $\mathcal{F}_1$ and $\mathcal{F}_2$.

For the given incoming flow V^- and the downstream pressure p_e, which satisfy

$$\|V^- - V_0^-\|_{2,\alpha;(1+\beta,0);\mathbb{D}^-} + \|p_e - p_0^+\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \le \varepsilon, \tag{3.34}$$

now we are ready to give the iteration scheme for Thorem 3.1 by solving (3.28), (3.30), and (3.31) with (3.10)–(3.13) for U, which is given in Remark 2.1, on the free boundary Γ_{shock}:

(1) Let $(\delta p, \delta w, \delta\rho, \delta u_1, \delta s) = (\delta U, \delta s) := (U, s) - (U_0^+, k_1 y_2)$, where k_1 is given in (3.14) and U and U_0^+ are given in Remark 2.1. Given

$$(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}, \tag{3.35}$$

where the value of $\mathcal{A}\varepsilon$ will be given in Lemma 6.1 below, we first introduce another coordinates transformation to make the free boundary Γ_{shock} into a fixed boundary by

$$\begin{cases} z_1 = y_1 - \delta s(y_2), \\ z_2 = y_2. \end{cases} \tag{3.36}$$

Then we have

$$\begin{cases} \partial_{z_1} = \partial_{y_1}, \\ \partial_{z_2} = (\delta s)'(y_2)\partial_{y_1} + \partial_{y_2}. \end{cases} \tag{3.37}$$

Obviously, it transforms domain $\mathbb{D}^+$ into the fixed domain $\mathbb{D}_0^+$, and Γ_{shock} in the new coordinates is fixed, i.e.,

$$\Gamma_{shock} \equiv \{\mathbf{z} : z_1 > 0,\ z_1 = k_1 z_2\}. \tag{3.38}$$

The coordinates transformation is invertiable if $\mathcal{A}\varepsilon$ is sufficiently small.
Let $\mathbf{z} = (z_1, z_2)$. In the $\mathbf{z}$-coordinates, define $U_s(\mathbf{z}) = U(\mathbf{y}(\mathbf{z}))$, where $\mathbf{y}(\mathbf{z})$ is the inverse of the coordinates transformation (3.36), i.e.,

$$\begin{cases} y_1 = z_1 + \delta s(z_2), \\ y_2 = z_2. \end{cases} \tag{3.39}$$

Moreover, for the incoming flow, $V_s^-(\mathbf{z}) = V^-(\mathbf{y}(\mathbf{z}))$ i.e., $V_s^-(\mathbf{z}) = V^-(z_1 + \delta s(z_2), z_2)$. Therefore, in the $\mathbf{z}$-coordinates, it is easy to see that $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$.
In $\mathbf{z}$-coordinates, system (3.31) for w and p becomes

$$\begin{cases} \widetilde{\partial_R} w + \Lambda \widetilde{\partial_I} p = 0, \\ \widetilde{\partial_I} w - \Lambda \widetilde{\partial_R} p = 0, \end{cases} \tag{3.40}$$

where

$$\widetilde{\partial_R} = (1 - \delta s' \lambda_R)\partial_{z_1} + \lambda_R \partial_{z_2}, \qquad \widetilde{\partial_I} = \lambda_I(-\delta s' \partial_{z_1} + \partial_{z_2}). \tag{3.41}$$

Finally, the Rankine–Hugoniot conditions (3.10)–(3.13) keep the same form, except that $s(y_2)$ is replaced by $s(z_2)$ and V^- is replaced by V_s^-.
Because $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$ for sufficiently small $\mathcal{A}\varepsilon$, we know (3.24) holds. So

$$\Lambda > 0, \qquad \lambda_I > 0, \tag{3.42}$$

and system (3.40) is elliptic.

(2) Now we are going to introduce the map $\mathcal{I} : \mathcal{F}^{\mathcal{A}\varepsilon} \to \mathcal{F}^{\mathcal{A}\varepsilon}$ such that $\mathcal{I}(\delta U, \delta s) = (\delta\tilde{U}, \delta\tilde{s})$, where $(\delta\tilde{U}, \delta\tilde{s}) = (\delta\tilde{p}, \delta\tilde{w}, \delta\tilde{\rho}, \delta\tilde{u}_1, \delta\tilde{s})$. First, we linearize system (3.40) by the following system:

$$\begin{cases} \widetilde{\partial_R} \tilde{w} + \Lambda \widetilde{\partial_I} \tilde{p} = 0, \\ \widetilde{\partial_I} \tilde{w} - \Lambda \widetilde{\partial_R} \tilde{p} = 0, \end{cases} \tag{3.43}$$

where the coefficient Λ and differential operators $\widetilde{\partial_R}$ and $\widetilde{\partial_I}$ are given by (3.31) and (3.41) with taking value on (U, s') and $s' = k_1 + (\delta s)'$. Note that $\partial_{z_i}\tilde{w} = \partial_{z_i}\delta\tilde{w}$ and $\partial_{z_i}\tilde{p} = \partial_{z_i}\delta\tilde{p}$. (3.43) is equivalent to the following elliptic system:

$$\begin{cases} \widetilde{\partial_R}\delta\tilde{w} + \Lambda\widetilde{\partial_I}\delta\tilde{p} = 0, \\ \widetilde{\partial_I}\delta\tilde{w} - \Lambda\widetilde{\partial_R}\delta\tilde{p} = 0. \end{cases} \tag{3.44}$$

(3) Then we solve $\delta\tilde{p}$ by eliminating $\delta\tilde{w}$ in (3.44) to obtain an elliptic equation of second order for $\delta\tilde{p}$ with appreciate boundary conditions derived in the next section below, and by establishing estimates of $\delta\tilde{p}$.
(4) Based on $\delta\tilde{p}$, we obtain $\delta\tilde{w}$ by solving a transport equation derived from (3.44) with initial condition on Γ_{shock} derived from the Rankine–Hugoniot conditions (3.10)–(3.13).
(5) Then we will obtain $\delta\tilde{\rho}$ and $\delta\tilde{u}_1$ by solving (3.28) and (3.30).
(6) Update the free boundary Γ_{shock} by obtaining $\delta\tilde{s}$ based on $\delta\widetilde{U}$.
(7) Finally, we will show $\mathcal{I}(\delta U, \delta s) = (\delta\widetilde{U}, \delta\tilde{s}) \in \mathcal{F}^{\mathcal{A}\varepsilon}$. Then we can apply the Schauder fixed point theorem to establish the existence of transonic solutions as given in Theorem 3.1.

Once the existence has been established, in the last section of this paper, we will establish the asymptotic behaviors of these solutions and then prove the uniqueness as given in Theorem 3.1.

4. Estimates and existence of $\delta\tilde{p}$.

In this section, we will consider step (3) in the iteration scheme given in section 3.3 to establish the estimates and existence of $\delta\tilde{p}$. For simplicity, we write $(\delta s)'$ as $\delta s'$.

4.1. Second order elliptic equation of $\delta\tilde{p}$.

It follows from (3.44) that

$$(\delta\tilde{w})z_1 = -\frac{\Lambda}{\lambda_I}[\lambda_R - \delta s'({\lambda_R}^2 + {\lambda_I}^2)](\delta\tilde{p})_{z_1} - \frac{\Lambda}{\lambda_I}({\lambda_R}^2 + {\lambda_I}^2)(\delta\tilde{p})_{z_2}, \tag{4.1}$$

$$(\delta\tilde{w})z_2 = \frac{\Lambda}{\lambda_I}[(1 - \delta s'\lambda_R)^2 + (\delta s'\lambda_I)^2](\delta\tilde{p})_{z_1} + \frac{\Lambda}{\lambda_I}[\lambda_R - \delta s'({\lambda_R}^2 + {\lambda_I}^2)](\delta\tilde{p})_{z_2}, \tag{4.2}$$

where $\Lambda = \frac{\sqrt{c^2 - q^2}}{c\rho u_1^2}$. Then we use $\partial_{z_1}(4.2) - \partial_{z_2}(4.1)$ to eliminate $\delta\tilde{w}$ to obtain

$$\sum_{i,j=1}^{2} a_{ij}\partial_{z_i z_j}(\delta\tilde{p}) = f, \tag{4.3}$$

where

$$a_{11} = \frac{\Lambda}{\lambda_I}[(1 - \delta s'\lambda_R)^2 + (\delta s'\lambda_I)^2], \tag{4.4}$$

$$a_{12} = a_{21} = \frac{\Lambda}{\lambda_I}[\lambda_R - \delta s'({\lambda_R}^2 + {\lambda_I}^2)], \tag{4.5}$$

$$a_{22} = \frac{\Lambda}{\lambda_I}({\lambda_R}^2 + {\lambda_I}^2), \tag{4.6}$$

$$f = \sum_{i,j=1}^{2} -(a_{ij})_{z_j}(\delta p)_{z_i}. \tag{4.7}$$

Linearize (4.3) by

$$\sum_{i,j=1}^{2} a_{ij}^0 \partial_{z_i z_j}(\delta\tilde{p}) = f + \sum_{i,j=1}^{2} (a_{ij}^0 - a_{ij})\partial_{z_i z_j}(\delta p), \tag{4.8}$$

where $a_{ij}^0 = a_{ij}(\mathbf{0})$, i.e., a_{ij} takes value at $(\delta U, \delta s') = \mathbf{0}$. More precisely, $a_{11}^0 = \frac{\Lambda^0}{\lambda_I^0}$, $a_{12}^0 = 0$, and $a_{22}^0 = \Lambda^0\lambda_I^0$ with $\lambda_I^0 = \lambda_I(\mathbf{0})$ and $\Lambda^0 = \Lambda(\mathbf{0})$.

In order to derive estimates of $\delta\tilde{p}$, based on (3.42), we introduce another coordinates transformation as follows:

$$\begin{cases} \hat{z}_1 = \sqrt{\frac{\lambda_I^0}{\Lambda^0}} z_1, \\ \hat{z}_2 = \sqrt{\frac{1}{\Lambda^0\lambda_I^0}} z_2. \end{cases} \tag{4.9}$$

It is easy to see that constants

$$\lambda_I^0 = \frac{c_0^+\rho_0^+ u_{10}^+}{\sqrt{(c_0^+)^2 - (u_{10}^+)^2}} > 0 \qquad \text{and} \qquad \Lambda^0 = \frac{\sqrt{(c_0^+)^2 - (u_{10}^+)^2}}{c_0^+\rho_0^+(u_{10}^+)^2} > 0, \tag{4.10}$$

where $c_0^+ = \sqrt{\frac{\gamma p_0^+}{\rho_0^+}}$. So coordinates transformation (4.9) is invertable and smooth.

In $\hat{\mathbf{z}}$-coordinates, it is easy to see that the transonic shock becomes $\hat{\Gamma}_{shock} := \{\hat{\mathbf{z}} : \hat{z}_1 = k_2\hat{z}_2\}$ where $k_2 = k_1\lambda_I^0$. And then $\mathbb{D}_0^+$ becomes $\mathbb{D} := \{\hat{\mathbf{z}} : \hat{z}_1 > k_2\hat{z}_2,\ \hat{z}_2 > 0\}$. Besides, we write Γ_{cont} as $\hat{\Gamma}_{cont}$ in the new coordinates.

Let $\delta\tilde{\hat{p}}(\hat{\mathbf{z}}) = \delta\tilde{p}(\mathbf{z}(\hat{\mathbf{z}}))$. In $\hat{\mathbf{z}}$-coordinates, we linearize (4.8) such that

$$\Delta(\delta\tilde{\hat{p}}) = \hat{f}, \tag{4.11}$$

where Δ denotes the Laplace operator and

$$\hat{f} = \sum_{i,j=1}^{2} -(\hat{a}_{ij})_{\hat{z}_j}(\delta\hat{p})_{\hat{z}_i} - (\hat{a}_{ij} - \delta_{ij})(\delta\hat{p})_{\hat{z}_i\hat{z}_j} \tag{4.12}$$

with

$$\hat{a}_{11}(\hat{\mathbf{z}}) = \frac{\lambda_I^0}{\Lambda^0} a_{11}\left(\sqrt{\frac{\Lambda^0}{\lambda_I^0}}\hat{z}_1, \sqrt{\Lambda^0\lambda_I^0}\hat{z}_2\right), \tag{4.13}$$

$$\hat{a}_{12}(\hat{\mathbf{z}}) = \hat{a}_{21}(\hat{\mathbf{z}}) = \frac{1}{\Lambda^0} a_{12}\left(\sqrt{\frac{\Lambda^0}{\lambda_I^0}}\hat{z}_1, \sqrt{\Lambda^0\lambda_I^0}\hat{z}_2\right), \tag{4.14}$$

$$\hat{a}_{22}(\hat{\mathbf{z}}) = \frac{1}{\Lambda^0\lambda_I^0} a_{22}\left(\sqrt{\frac{\Lambda^0}{\lambda_I^0}}\hat{z}_1, \sqrt{\Lambda^0\lambda_I^0}\hat{z}_2\right). \tag{4.15}$$

For (4.11), we have the following two lemmas.

LEMMA 4.1. *Suppose $\hat{a}_{ij}$ are given in* (4.13)–(4.15). *λ_R, λ_I, and Λ are given in* (3.26) *and* (3.31). *For any given $\mathcal{A} > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, for $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$, we have*

$$\|\hat{a}_{ij} - \delta_{ij}\|^{*(-\alpha;\hat{\Gamma}_{cont})}_{1,\alpha;(0,1+\beta);\mathbb{D}} \leq C_0(\mathcal{A}+1)\varepsilon, \tag{4.16}$$

where C_0 is a constant depending only on $\mathcal{A}$, ε, and the background solution V_0^+.

Proof. Because the expression of $\hat{a}_{ij}$ contains λ_R and λ_I, we need to estimate them one by one first. Notice that $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$ and $u_{20}^+ = 0$, therefore, it follows from the expression of λ_R and λ_I that

$$\begin{aligned}
|\lambda_R| &= \left|-\frac{c^2\rho u_2}{c^2 - u_1^2}\right| \\
&\leq (|\delta\rho| + \rho_0^+)|(\delta u_2 + u_{20}^+)| \\
&\leq C(|\delta\rho| + \rho_0^+)|(\delta u_1 + \delta w)| \\
&\leq C(|\delta\rho| + 1)(|\delta u_1| + |\delta w|) \\
&\leq C\mathcal{A}\varepsilon(\mathcal{A}\varepsilon + 1)(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-1-\beta},
\end{aligned}$$

$$\begin{aligned}
|\lambda_R|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} &\leq C((d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{1+\beta+\alpha}(|\delta\rho|_{0,\alpha,\mathbb{D}} + |\delta u_1|_{0,\alpha,\mathbb{D}} + |\delta w|_{0,\alpha,\mathbb{D}}) \\
&\leq C((d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{1+\beta+\alpha}(|\delta\rho|_{0,\alpha,\mathbb{D}} + |\delta u_1|_{0,\alpha,\mathbb{D}}) + (d^{\hat{\mathbf{z}}}_{\infty})^{1+\beta+\alpha}|\delta w|_{0,\alpha,\mathbb{D}}) \\
&\leq C(|\delta\rho|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} + |\delta u_1|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} + |\delta w|_{0,\alpha,(1+\beta,0),\mathbb{D}}) \\
&\leq C\mathcal{A}\varepsilon,
\end{aligned}$$

$$\begin{aligned}
|\lambda_I| &= \left|\frac{c\rho u_1\sqrt{c^2 - u_1^2 - u_2^2}}{c^2 - u_1^2}\right| \\
&\leq C(|\rho| + |u_1|) \\
&\leq C(|\delta\rho| + \rho_0^+ + |\delta u_1| + u_{10}^+) \\
&\leq C[(\mathcal{A}\varepsilon)(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-1-\beta} + 1],
\end{aligned}$$

$$|\lambda_I|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} \leq C(|\delta\rho|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} + |\delta u_1|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} + |\delta w|_{0,\alpha,(1+\beta,0),\mathbb{D}} + 1) \leq C(\mathcal{A}\varepsilon + 1),$$

$$\begin{aligned}
|(\lambda_R)_{\hat{z}_1}| &= \left|-\frac{c^2}{(c^2 - u_1^2)^2}\{(\rho u_2)_{\hat{z}_1}(c^2 - u_1^2) + 2u_1(u_1)_{\hat{z}_1}(\rho u_2)\}\right| \\
&\leq C(|(\rho u_2)_{\hat{z}_1}| + |(\delta u_1)_{\hat{z}_1}||\rho u_2|) \\
&\leq C(|\{\rho(\delta u_1 + \delta w)_{\hat{z}_1} + (\delta\rho)_{\hat{z}_1}(\delta u_1 + \delta w)\}| + |(\delta u_1)_{\hat{z}_1}(\delta u_1 + \delta w)|\rho) \\
&\leq C(1 + |\delta\rho| + |\delta u_1| + |\delta w|)(|(\delta\rho)_{\hat{z}_1}| + |(\delta u_1)_{\hat{z}_1}| + |(\delta w)_{\hat{z}_1}|) \\
&\leq C(\mathcal{A}\varepsilon + 1)\mathcal{A}\varepsilon(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta},
\end{aligned}$$

$$|(\lambda_R)_{\hat{z}_1}|^*_{0,\alpha,(0,2+\beta),\mathbb{D}} \leq C(|\delta\rho|^*_{1,\alpha,(0,1+\beta),\mathbb{D}} + |\delta u_1|^*_{1,\alpha,(0,1+\beta),\mathbb{D}} + |\delta w|_{1,\alpha,(1+\beta,0),\mathbb{D}}) \leq C\mathcal{A}\varepsilon,$$

$$
\begin{aligned}
|(\lambda_I)_{\hat{z}_1}| =& \left|\frac{c}{c^2-u_1^2}\{(\rho u_1)_{\hat{z}_1}\sqrt{c^2-u_1^2-u_2^2}-(\rho u_1)(u_1(u_1)_{\hat{z}_1}+u_2(u_2)_{\hat{z}_1})(c^2-u_1^2-u_2^2)^{-\frac{1}{2}}\right. \\
&\left.+2(\rho u_1)u_1(u_1)_{\hat{z}_1}(c^2-u_1^2-u_2^2)^{\frac{1}{2}}(c^2-u_1^2)^{-1}\}\right| \\
\le& \frac{c}{c^2-u_1^2}\left\{\left|(\rho u_1)_{\hat{z}_1}\sqrt{c^2-u_1^2-u_2^2}-\frac{(\rho u_1)u_2(u_2)_{\hat{z}_1}}{\sqrt{c^2-u_1^2-u_2^2}}\right|+\left|\frac{(\rho u_1)u_1(u_1)_{\hat{z}_1}}{\sqrt{c^2-u_1^2-u_2^2}}\right|\right\} \\
\le& C(1+|\delta\rho|+|\delta u_1|+|\delta w|)(|(\delta\rho)_{\hat{z}_1}|+|(\delta u_1)_{\hat{z}_1}|+|(\delta w)_{\hat{z}_1}|) \\
\le& C(\mathcal{A}\varepsilon+1)\mathcal{A}\varepsilon(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta},
\end{aligned}
$$

$$
|(\lambda_I)_{\hat{z}_1}|^*_{0,\alpha,(0,2+\beta),\mathbb{D}} \le C(|\delta\rho|^*_{1,\alpha,(0,1+\beta),\mathbb{D}}+|\delta u_1|^*_{1,\alpha,(0,1+\beta),\mathbb{D}}+|\delta w|^*_{1,\alpha,(0,1+\beta),\mathbb{D}}) \le C\mathcal{A}\varepsilon.
$$

Similarly, we also have

$$
\begin{aligned}
&|(\lambda_R)_{\hat{z}_2}| \le C\mathcal{A}\varepsilon(\mathcal{A}\varepsilon+1)(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta}, \\
&|(\lambda_I)_{\hat{z}_2}| \le C\mathcal{A}\varepsilon(\mathcal{A}\varepsilon+1)(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta}, \\
&|(\lambda_R)_{\hat{z}_2}|^*_{0,\alpha,(0,2+\beta),\mathbb{D}} \le C\mathcal{A}\varepsilon, \\
&|(\lambda_I)_{\hat{z}_2}|^*_{0,\alpha,(0,2+\beta),\mathbb{D}} \le C\mathcal{A}\varepsilon.
\end{aligned}
$$

Therefore, to summarize all the inequalities above, we have

$$
\begin{aligned}
&|\lambda_R|^*_{k,\alpha,(0,1+\beta),\mathbb{D}} \le C\mathcal{A}\varepsilon, \quad k=0,1, \\
&|\lambda_I|^*_{k,\alpha,(0,1+\beta),\mathbb{D}} \le C\mathcal{A}\varepsilon, \quad k=0,1,
\end{aligned}
$$

where C is a positive constatnt. Because $(\delta U,\delta s)\in\mathcal{F}^{\mathcal{A}\varepsilon}$,

$$
\|\delta s'\|^{(-\alpha;\mathbf{O})}_{2,\alpha;(1+\beta,0);\mathbb{D}} \le \mathcal{A}\varepsilon. \tag{4.17}
$$

Then for the coefficients $\hat{a}_{ij}$, we have

$$
\begin{aligned}
|\hat{a}_{11}-1| &= C|-2\delta s'\lambda_R+(\delta s'\lambda_R)^2+(\delta s'\lambda_I)^2| \\
&\le C[|\delta s'||\lambda_R|+(\delta s'\lambda_R)^2+(\delta s'\lambda_I)^2] \\
&\le C(\mathcal{A}\varepsilon)^2(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-1-\beta},
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
|\hat{a}_{11}-1|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} &\le C\{|\delta s'|_{0,\alpha,(1+\beta,0);\mathbb{D}}(|\lambda_R|+|\lambda_I|)+|\delta s'|(|\lambda_R|^*_{0,\alpha,(0,1+\beta);\mathbb{D}} \\
&\quad +|\lambda_I|^*_{0,\alpha,(0,1+\beta);\mathbb{D}})\} \\
&\le C\mathcal{A}\varepsilon
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
|\hat{a}_{12}\| &= |\hat{a}_{21}| \le C\left|\frac{1}{\lambda_I}[\lambda_R-\delta s'({\lambda_R}^2+{\lambda_I}^2)]\right|, \\
&\le C\left(|\delta s'||\lambda_I|+\left|\frac{\lambda_R}{\lambda_I}\right||(1-\delta s'\lambda_R)|\right) \\
&\le C\left(|\delta s'||\lambda_I|+\left|\frac{\lambda_R}{\lambda_I}\right|\right) \\
&\le C(\mathcal{A}\varepsilon)(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-1-\beta},
\end{aligned} \tag{4.20}
$$

$$
\begin{aligned}
|a_{12}|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} &\le C\{|\delta s'|_{0,\alpha,(1+\beta,0),\mathbb{D}}|\lambda_I|+|\delta s'||\lambda_I|^*_{0,\alpha,(0,1+\beta),\mathbb{D}}+(|\delta u_1|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} \\
&\quad +|\delta w|_{0,\alpha,(1+\beta,0)\mathbb{D}})\} \\
&\le C\mathcal{A}\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
|\hat{a}_{22}-1| &= \frac{{\lambda_R}^2}{{\lambda_I}^2} \le C(|\delta u_1|^2+|\delta w|^2) \le C(\mathcal{A}\varepsilon)^2(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-1-\beta},\\
|\hat{a}_{22}-1|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} &\le C(|\delta u_1|^*_{0,\alpha,(0,1+\beta),\mathbb{D}}+|\delta w|_{0,\alpha,(1+\beta,0)\mathbb{D}})^2\\
&\le C\mathcal{A}\varepsilon\\
|(\hat{a}_{11})_{\hat{z}_1}| &= C|(1-\delta s'\lambda_R)(\delta s'\lambda_R)_{\hat{z}_1}+(\delta s'\lambda_I)(\delta s'\lambda_I)_{\hat{z}_1}|\\
&\le C\{|\delta s'||(\lambda_R)_{\hat{z}_1}|+|(\delta s')_{\hat{z}_1}||\lambda_R|+|\delta s'||\lambda_I|(|\delta s'||(\lambda_I)_{\hat{z}_1}|+|(\delta s')_{\hat{z}_1}||\lambda_I|)\}\\
&\le C(\mathcal{A}\varepsilon)^2\{(d^{\hat{\mathbf{z}}}_{\infty})^{-1-\beta}(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta}+(d^{\hat{\mathbf{z}}}_{\infty})^{-2-\beta}(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-1-\beta}\}\\
&\le C(\mathcal{A}\varepsilon)^2(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta},\\
|(\hat{a}_{11})_{\hat{z}_1}|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} &\le C\{\|\delta s'\|_{1,\alpha,(1+\beta,0),\mathbb{D}}(\|\lambda_R\|^*_{1,\alpha,(0,1+\beta),\mathbb{D}}+\|\lambda_I\|^*_{1,\alpha,(0,1+\beta),\mathbb{D}})\}\\
&\le C\mathcal{A}\varepsilon,\\
|(\hat{a}_{12})_{\hat{z}_1}| &= C\left|\left(\frac{\lambda_R}{\lambda_I}\right)_{\hat{z}_1}(1-\delta s'\lambda_R)+\left(\frac{\lambda_R}{\lambda_I}\right)(1-\delta s'\lambda_R)_{\hat{z}_1}-(\delta s'\lambda_I)_{\hat{z}_1}\right|\\
&\le C|\{(\delta u_1+\delta w)_{\hat{z}_1}+(\delta u_1+\delta w)(\delta s'\lambda_R)_{\hat{z}_1}+(\delta s'\lambda_I)_{\hat{z}_1}\}|\\
&\le C|\{(|(\delta u_1)_{\hat{z}_1}|+|(\delta w)_{\hat{z}_1}|)+(|\delta u_1|+|\delta w|)|(\delta s'\lambda_R)_{\hat{z}_1}|+|(\delta s'\lambda_I)_{\hat{z}_1}|\}\\
&\le C\mathcal{A}\varepsilon(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta},\\
|(\hat{a}_{12})_{\hat{z}_1}|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} &\le C\{(\|\delta u_1\|^*_{1,\alpha,(0,1+\beta),\mathbb{D}}+\|\delta w\|_{1,\alpha,(1+\beta,0)\mathbb{D}}+1)(|\delta s'|_{1,\alpha,(1+\beta,0),\mathbb{D}}\\
&\quad+|\lambda_R|^*_{1,\alpha,(0,1+\beta),\mathbb{D}})\}\\
&\le C\mathcal{A}\varepsilon,\\
|(\hat{a}_{22})_{\hat{z}_1}| &= C\left\{\left(\frac{\lambda_R}{\lambda_I}\right)_{\hat{z}_1}\right\}^2 \le C(|(\delta u_1)_{\hat{z}_1}|^2+|(\delta w)_{\hat{z}_1}|^2 \le C(\mathcal{A}\varepsilon)^2(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta},
\end{aligned}
$$

(4.21)

$$
|(\hat{a}_{22})_{\hat{z}_1}|^*_{0,\alpha,(0,1+\beta),\mathbb{D}} \le C\{|\delta u_1|^*_{1,\alpha,(0,1+\beta),\mathbb{D}}+|\delta w|_{1,\alpha,(1+\beta,0),\mathbb{D}}\}^2 \le C\mathcal{A}\varepsilon.
$$

Similarly, we can get the estimate of $(a_{ij}-\delta_{ij})_{\hat{z}_2}$. Therefore, we have

$$
\|\hat{a}_{ij}-\delta_{ij}\|^*_{1,\alpha;(0,1+\beta);\mathbb{D}} \le C(\mathcal{A}+1)\varepsilon. \tag{4.22}
$$

Repeating the same process again, we can also get the corner estimates

$$
\|\hat{a}_{ij}-\delta_{ij}\|^{(-\alpha;\hat{\Gamma}_{cont})}_{1,\alpha;\mathbb{D}} \le C(\mathcal{A}+1)\varepsilon. \tag{4.23}
$$

Hence, we have the conclusion that

$$
\|\hat{a}_{ij}-\delta_{ij}\|^{*(-\alpha;\hat{\Gamma}_{cont})}_{1,\alpha;(0,1+\beta);\mathbb{D}} \le C_0(\mathcal{A}+1)\varepsilon. \tag{4.24}
$$

□

LEMMA 4.2. *Suppose $\hat{a}_{ij}$ satisfy (4.16). For any given $\mathcal{A}>0$, there exists $\varepsilon_0>0$ such that if $\varepsilon\in(0,\varepsilon_0)$, for $(\delta U,\delta s)\in\mathcal{F}^{\mathcal{A}\varepsilon}$, we have*

$$
\|\hat{f}\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}} \le K_1[(\mathcal{A}\varepsilon)^2+(\mathcal{A}\varepsilon^2)], \tag{4.25}
$$

where K_1 is a nonnegative constant depending only on C_0, provided that $\mathcal{A}\varepsilon$ is sufficiently small.

Proof. By Definition 3.1, (4.12), and (4.16), we have when $|\hat{\mathbf{z}}| \geq 1$

$$|\hat{f}| = |-(\hat{a}_{ij})_{\hat{z}_j}(\delta\hat{p})_{\hat{z}_i} - (\hat{a}_{ij} - \delta_{ij})(\delta\hat{p})_{\hat{z}_i\hat{z}_j}|$$

$$\leq |(\hat{a}_{ij})_{\hat{z}_j}||(\delta\hat{p})_{\hat{z}_i}| + |(\hat{a}_{ij} - \delta_{ij})||(\delta\hat{p})_{\hat{z}_i\hat{z}_j}| \tag{4.26}$$

$$\leq C(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta}(d^{\hat{\mathbf{z}}}_{\infty})^{-2-\beta}[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)] + (d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-1-\beta}(d^{\hat{\mathbf{z}}}_{\infty})^{-3-\beta}[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)] \tag{4.27}$$

$$\leq C(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\beta}(d^{\hat{\mathbf{z}}}_{\infty})^{-2-\beta}[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)], \tag{4.28}$$

and

$$\|\hat{f}\|^{*}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}} \leq C_0(\mathcal{A}+1)\varepsilon[(\mathcal{A}\varepsilon) + (\mathcal{A}\varepsilon)] \leq C[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)]. \tag{4.29}$$

Repeating the same process again we can also get the corner estimate when $|\hat{\mathbf{z}}| < 1$, that is

$$\|\hat{f}\|^{(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;\mathbb{D}} \leq C[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)]. \tag{4.30}$$

Therefore, we have

$$\|\hat{f}\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}} \leq C_0(\mathcal{A}+1)\varepsilon[(\mathcal{A}\varepsilon) + (\mathcal{A}\varepsilon)] \leq K_1[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)], \tag{4.31}$$

where K_1 is a nonnegative constant depending only on C_0, provided that $\mathcal{A}\varepsilon$ is sufficiently small. ∎

4.2. Boundary conditions for $\delta\tilde{p}$, $\delta\tilde{w}$, $\delta\tilde{u}_1$, and $\delta\tilde{\rho}$ on Γ_{shock}. In this subsection, we will derive equivalent boundary conditions of $\delta\tilde{w}$, $\delta\tilde{u}_1$, $\delta\tilde{\rho}$, and $\delta\tilde{p}$ in the $\mathbf{z}$-coordinates from (3.10)–(3.13), which will be applied in sections below. First, it follows from (3.12) along Γ_{shock}, on which $z_1 = k_1 z_2$, that

$$\tilde{s}'(z_2) = \frac{[\tilde{u}_1\tilde{w}]}{[\tilde{p}]}(k_1 z_2, z_2). \tag{4.32}$$

Plugging (4.32) into (3.10), (3.11), and (3.13), we obtain

$$H_1(U_s^-, \widetilde{U}) = [\tilde{p}]\left[\frac{1}{\tilde{\rho}\tilde{u}_1}\right] + [\tilde{w}][\tilde{u}_1\tilde{w}] = 0, \tag{4.33}$$

$$H_2(U_s^-, \widetilde{U}) = [\tilde{p}]\left[\tilde{u}_1 + \frac{\tilde{p}}{\tilde{\rho}\tilde{u}_1}\right] + [\tilde{p}\tilde{w}][\tilde{u}_1\tilde{w}] = 0, \tag{4.34}$$

$$H_3(U_s^-, \widetilde{U}) = \left[\frac{1}{2}|\tilde{\mathbf{u}}|^2 + \frac{\gamma\tilde{p}}{(\gamma-1)\tilde{\rho}}\right] = 0, \tag{4.35}$$

where $\tilde{U} = (\tilde{p}, \tilde{w}, \tilde{\rho}, \tilde{u}_1)$ and $U_s^- = (p^-, w^-, \rho^-, u_1^-)$ with $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_1\tilde{w})$ and $w^- = \frac{u_2^-}{u_1^-}$.

By (4.35), we have the formulas of $\tilde{u}_1$ and $\delta\tilde{u}_1$ as

$$\tilde{u}_1 = \frac{\sqrt{2B(z_2) - \frac{2\gamma\tilde{p}}{(\gamma-1)\rho}}}{\sqrt{1+\tilde{w}^2}}, \tag{4.36}$$

$$\delta\tilde{u}_1 = \tilde{u}_1 - u_0^+, \tag{4.37}$$

where the Bernoulli function $B(z_2)$ satisfies

$$B(z_2) = \left(\frac{1}{2}\left|\mathbf{u}^-\right|^2 + \frac{\gamma p^-}{(\gamma-1)\rho^-}\right)(k_1 z_2, z_2). \tag{4.38}$$

Hence we can treat $\tilde{u}_1$ as a function of $(\tilde{p}, \tilde{w}, \tilde{\rho})$ through $B(z_2)$ (i.e., through U_s^-) by (4.36). Therefore, we can regard H_i for $i = 1, 2$ as a function of $\tilde{\mathcal{U}} = (\tilde{p}, \tilde{w}, \tilde{\rho})$. That is, $\tilde{U} = (\tilde{\mathcal{U}}, \tilde{u}_1)$ and

$$H_i(\tilde{\mathcal{U}}; U_s^-) = 0 \qquad \text{on } \Gamma_{shock} \qquad \text{for } i = 1, 2. \tag{4.39}$$

Linearizing (4.39), we can get that for $i = 1, 2$

$$\begin{aligned} 0 = H_i(\tilde{w}, \tilde{p}, \tilde{\rho}) - H_i(0, p_0^+, \rho_0^+) &= H_{i\tilde{p}}(0, p_0^+, \rho_0^+)\delta\tilde{p} + H_{i\tilde{w}}(0, p_0^+, \rho_0^+)\delta\tilde{w} \\ &+ H_{i\tilde{\rho}}(0, p_0^+, \rho_0^+)\delta\tilde{\rho} + O(\delta\tilde{\mathcal{U}}) \quad \text{on } \Gamma_{shock}, \end{aligned} \tag{4.40}$$

where $O(\delta\tilde{\mathcal{U}})$ is a smooth function with respect to vector function $\delta\tilde{\mathcal{U}}$ and $|O(\delta\tilde{\mathcal{U}})| \le C|\delta\tilde{\mathcal{U}}|^2$ for some constant C depending only on $\mathcal{A}\varepsilon$ and the background solution U_0^+. To be convenient, (4.40) can be rewritten as

$$e_{i1}\delta\tilde{w} + e_{i2}\delta\tilde{p} + e_{i3}\delta\tilde{\rho} = \mathcal{H}_i(\delta\tilde{\mathcal{U}}), \quad i = 1, 2, \tag{4.41}$$

where

$$(e_{i1}, e_{i2}, e_{i3}) := (H_{i\tilde{w}}(0, p_0^+, \rho_0^+), H_{i\tilde{p}}(0, p_0^+, \rho_0^+), H_{i\tilde{\rho}}(0, p_0^+, \rho_0^+)) \text{ and } \mathcal{H}_i(\delta\tilde{\mathcal{U}}) := -O(\delta\tilde{\mathcal{U}}).$$

Using (4.41) to eliminate $\delta\tilde{\rho}$, we have

$$(e_{11}e_{23} - e_{21}e_{13})\delta\tilde{w} + (e_{12}e_{23} - e_{22}e_{13})\delta\tilde{p} = e_{23}\mathcal{H}_1 - e_{13}\mathcal{H}_2. \tag{4.42}$$

Because $w_0^- < 0$, and

$$e_{11} = [u_{10}w_0] + [w_0]u_{10}^+, \tag{4.43}$$

$$e_{13} = [p_0]\left\{\frac{-1}{(\rho_0^+)^2 u_{10}^+} - \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^3}\right\}, \tag{4.44}$$

$$e_{21} = [u_{10}w_0]p_0^+ + [p_0 w_0]u_{10}^+, \tag{4.45}$$

$$e_{23} = [p]\left\{\frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^2 u_{10}^+} + p_0^+\left(\frac{-1}{(\rho_0^+)^2 u_{10}^+} - \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^3}\right)\right\}, \tag{4.46}$$

the coefficient of $\delta\tilde{w}$ is

$$\begin{aligned} e_{11}e_{23} - e_{21}e_{13} &= -[p_0]w_0^-\left\{[p_0]u_{10}^+\left(\frac{-1}{(\rho_0^+)^2 u_{10}^+} - \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^3}\right)\right. \\ &\qquad \left. + \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^2 u_{10}^+}(u_{10}^- + u_{10}^+)\right\} \\ &= -[p_0]w_0^-\left\{\frac{2\gamma p_0^+}{(\gamma-1)(\rho_0^+)^2} - \frac{p_0^+}{{\rho_0^+}^2} + \frac{p_0^-}{{\rho_0^+}^2}\right\} \\ &= -[p_0]\frac{w_0^-}{\rho_0^+}\left\{\frac{(\gamma+1)p_0^+}{(\gamma-1)\rho_0^+} + \frac{p_0^-}{\rho_0^+}\right\} > 0. \end{aligned} \tag{4.47}$$

Therefore, (4.42) becomes

$$\delta\tilde{w} + e\delta\tilde{p} = h_1, \tag{4.48}$$

where

$$e = \frac{e_{12}e_{23} - e_{22}e_{13}}{e_{11}e_{23} - e_{21}e_{13}}, \qquad h_1 = \frac{e_{23}\mathcal{H}_1 - e_{13}\mathcal{H}_2}{e_{11}e_{23} - e_{21}e_{13}}. \tag{4.49}$$

Obviously, there exists a nonnegative constant K_2 depending only on $\mathcal{A}\varepsilon$ and the background solution V_0^- and U_0^+ such that

$$\|h_1\|_{2,\alpha;(1+\beta);\mathbb{R}^+}^{(-\alpha;0)} \le K_2(\mathcal{A}\varepsilon)^2, \tag{4.50}$$

provided that $(\delta\tilde{U}, \delta\tilde{s}) \in \mathcal{F}^{\mathcal{A}\varepsilon}$ and (3.34) hold.

Moreover, on Γ_{shock}, we can also express $\delta\tilde{w}$ as

$$\delta\tilde{w} = h_1 - e\delta\tilde{p} \qquad \text{on } \Gamma_{shock}. \tag{4.51}$$

Next, let us consider the boundary condition for $\delta\tilde{\rho}$. Because

$$e_{13} = -[p_0]\left(\frac{1}{(\rho_0^+)^2 u_{10}^+} + \frac{\gamma p_0^+}{(\gamma-1)(\rho_0^+)^3(u_{10}^+)^3}\right) < 0, \tag{4.52}$$

the condition (4.41) for $i = 1$ can be rewritten as

$$\delta\tilde{\rho} = h_\rho - e_2\delta\tilde{w} - e_3\delta\tilde{p}, \tag{4.53}$$

where $h_\rho = \frac{\mathcal{H}_1}{e_{13}}, e_2 = \frac{e_{11}}{e_{13}}, e_3 = \frac{e_{12}}{e_{13}}$. Then, the boundary condition of the entropy S on boundary Γ_{shock} is

$$S = \frac{\delta\tilde{p} + p_0^+}{(\delta\tilde{\rho} + \rho_0^+)^\gamma} = \frac{\frac{h_1 - \delta\tilde{w}}{e} + p_0^+}{(h_\rho - e_2\delta\tilde{w} - e_3\delta\tilde{p} + \rho_0^+)^\gamma}. \tag{4.54}$$

Finally, we remark that all the argument above can be reversed such that boundary conditions (4.38), (4.48), and (4.54) are equivalent to the Rankine–Hugoniot conditions (3.10)–(3.13).

4.3. Boundary conditions for $\boldsymbol{\delta\tilde{\hat{p}}}$. Now, we continue to derive the boundary conditions for $\delta\tilde{\hat{p}}$, which we will study in this section. First, it follows from (3.4) that

$$\delta\tilde{p} = h_0 \qquad \text{on } \bar{\Gamma}_{cont}, \tag{4.55}$$

where $h_0(z_1) = p_e(y_1 + \delta s(0)) - p_0^+ = p_e(y_1) - p_0^+$ with $z_1 = y_1$ on $\bar{\Gamma}_{cont}$. Then in $\hat{\mathbf{z}}$-coordinates,

$$\delta\tilde{\hat{p}} = \hat{h}_0 \qquad \text{on } \hat{\Gamma}_{cont}, \tag{4.56}$$

where

$$\hat{h}_0(\hat{z}_1) = h_0\left(\sqrt{\frac{\Lambda^0}{\lambda_I^0}}\hat{z}_1\right). \tag{4.57}$$

Next, in the $\hat{\mathbf{z}}$-coordinates, let $\delta\tilde{\hat{w}}(\hat{\mathbf{z}}) = \delta\tilde{w}(\mathbf{z}(\hat{\mathbf{z}}))$ and $\hat{h}_1(\hat{\mathbf{z}}) = h_1(\mathbf{z}(\hat{\mathbf{z}}))$; then (4.48) becomes

$$\delta\tilde{\hat{w}} + e\delta\tilde{\hat{p}} = \hat{h}_1 \qquad \text{on } \hat{\Gamma}_{shock}. \tag{4.58}$$

Differentiating (4.58) along $\hat{\Gamma}_{shock}$ and using (4.1)–(4.2) with coordinates transformation (4.9) to eliminate $\delta\tilde{\hat{w}}$, we have the following boundary condition:

$$\mu_1(\delta\tilde{\hat{p}})_{\hat{z}_1} + \mu_2(\delta\tilde{\hat{p}})_{\hat{z}_2} = \hat{h}_2, \tag{4.59}$$

where

$$\mu_1 = ek_2 + \Lambda^0, \qquad \mu_2 = e - \Lambda^0 k_2, \tag{4.60}$$

$$\hat{h}_2 = \hat{h}_1' - \Lambda^0((\hat{a}_{11}-1) - k_2\hat{a}_{12})(\delta\tilde{\hat{p}})_{\hat{z}_1} + \Lambda^0(k_2(\hat{a}_{22}-1) - \hat{a}_{12})(\delta\tilde{\hat{p}})_{\hat{z}_2}. \tag{4.61}$$

We remark that in $\mathbf{z}$-coordinates, boundary condition (4.59) becomes

$$\mu_1\sqrt{\frac{\Lambda^0}{\lambda_I^0}}(\delta\tilde{p})_{z_1} + \mu_2\sqrt{\Lambda^0\lambda_I^0}(\delta\tilde{p})_{z_2} = h_2 \qquad \text{on } \Gamma_{shock}, \tag{4.62}$$

where $h_2(\mathbf{z}) = \hat{h}_2(\hat{\mathbf{z}}(\mathbf{z}))$.

Now, we linearize boundary condition (4.59) via replacing $\delta\tilde{\hat{p}}$ by $\delta\hat{p}$ in (4.61) with $\delta\hat{p}(\hat{\mathbf{z}}) = \delta p(\mathbf{z}(\hat{\mathbf{z}}))$ and replacing $(\delta\tilde{U}, \delta\tilde{s})$ in (4.61) by $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$, that is,

$$\hat{h}_3 = \hat{h}_1'(\delta U, \delta s) - \Lambda^0((\hat{a}_{11}-1) - k_2\hat{a}_{12})(\delta\hat{p})_{\hat{z}_1} + \Lambda^0(k_2(\hat{a}_{22}-1) - \hat{a}_{12})(\delta\hat{p})_{\hat{z}_2}. \tag{4.63}$$

Then, we have the following boundary condition for $\delta\tilde{\hat{p}}$:

$$\mu_1(\delta\tilde{\hat{p}})_{\hat{z}_1} + \mu_2(\delta\tilde{\hat{p}})_{\hat{z}_2} = \hat{h}_3. \tag{4.64}$$

We remark that boundary conditions (4.59) and (4.64) are equivalent if $\tilde{p} = p$.

Finally, we normalize the boundary operator by defining $\boldsymbol{\nu} = \frac{(\mu_1,\mu_2)}{\sqrt{\mu_1^2+\mu_2^2}}$. Then

$$\boldsymbol{\nu} \cdot \bigtriangledown(\delta\tilde{\hat{p}}) = \hat{h}_4 \qquad \text{on } \hat{\Gamma}_{shock}, \tag{4.65}$$

where

$$\hat{h}_4 = \frac{\hat{h}_3}{\sqrt{\mu_1^2+\mu_2^2}}. \tag{4.66}$$

For the direction of unit vector $\boldsymbol{\nu}$, we have the following lemma.

LEMMA 4.3. *Let background solution* (V_0^-, V_0^+) *be a strong shock solution. Let* U_0^+ *be defined as in Remark* 2.1. *Let* $\mathbf{n}_f = \frac{(-1,k_2)}{\sqrt{k_2^2+1}}$ *and* $\boldsymbol{\tau}_f = \frac{(k_2,1)}{\sqrt{k_2^2+1}}$ *be the unit outer normal and tangent of* $\hat{\Gamma}_{shock}$. *Then*

$$\boldsymbol{\nu} \cdot \mathbf{n}_f < 0 \qquad \text{and} \qquad \boldsymbol{\nu} \cdot \boldsymbol{\tau}_f > 0. \tag{4.67}$$

Proof. First, we discuss the sign of e in (4.48) and (4.58). Based on the shock polar analysis (see Figure 2), $\tilde{w}$ can be regarded as a function of $\tilde{p}$, and $\tilde{w}'(\tilde{p}) < 0$ for the strong shock solution. Let $\tilde{\mathfrak{U}} = (\tilde{p}, \tilde{w})$ and $\delta\tilde{\mathfrak{U}} = (\delta\tilde{p}, \delta\tilde{w})$. h_1 can be treated as $O(\delta\tilde{\mathfrak{U}})$, which is a smooth function with respect to vector function $\delta\tilde{\mathfrak{U}}$ and $|O(\delta\tilde{\mathfrak{U}})| \leq C|\delta\tilde{\mathfrak{U}}|^2$

for some constant C depending only on $\mathcal{A}\varepsilon$ and the background solution U_0^+, since $\delta\tilde{\rho}$ can be expressed as a function of $\delta\tilde{p}$ and $\delta\tilde{\rho}$ via (4.53), and $|O(\delta\tilde{\mathcal{U}})| \leq C|\delta\tilde{\mathcal{U}}|^2$. Therefore, combined with (4.51), we have

$$e = -\tilde{w}'(\tilde{p})\Big|_{\tilde{p}=p_0^+,\, \tilde{w}=0} > 0. \tag{4.68}$$

By straightforward computation, we have

$$\boldsymbol{\nu}\cdot\mathbf{n}_f = -\frac{\Lambda^0\sqrt{1+k_2^2}}{\sqrt{\mu_1^2+\mu_2^2}} < 0 \qquad \text{and} \qquad \boldsymbol{\nu}\cdot\boldsymbol{\tau}_f = \frac{e\sqrt{1+k_2^2}}{\sqrt{\mu_1^2+\mu_2^2}} > 0. \tag{4.69}$$

Here we use (4.10) and (4.68). ∎

Next, for $\hat{h}_4$, we have the following estimate.

LEMMA 4.4. *Suppose $\hat{a}_{ij}$ satisfy* (4.16) *and $\hat{h}_4$ is defined in* (4.66). *For any given $\mathcal{A} > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, for $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$, we have*

$$\|\hat{h}_4\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} \leq K_3[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)], \tag{4.70}$$

where K_3 is a nonnegative constant, depending only on $C_0, \mathcal{A}, \varepsilon, \Lambda^0, U_0^-$, and U_0^+.

Proof. Because μ_1 and μ_2 are constants defined in (4.60), it suffices to show $\hat{h}_3$ has the same estimate as $\hat{h}_4$, that is,

$$\|\hat{h}_3\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} \leq K_3[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)]. \tag{4.71}$$

It follows from (4.16), (4.50), (4.63), and the fact that $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$ that

$$\begin{aligned}
&\|\hat{h}_3\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+}\\
\leq&\|\hat{h}_1'\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} + \|-\Lambda^0((\hat{a}_{11}-1) - k_2\hat{a}_{12})(\delta\hat{p})_{\hat{z}_1} + \Lambda^0(k_2(\hat{a}_{22}-1) - \hat{a}_{12})(\delta\hat{p})_{\hat{z}_2}\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+}\\
\leq&\|\hat{h}_1'\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} + K_4(\mathcal{A}\varepsilon)^2 + K_5(\mathcal{A}\varepsilon^2)\\
\leq&K_3[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)],
\end{aligned}$$

where K_4, K_5, and then K_3 are nonnegative constants, depending only on $C_0, \mathcal{A}$, $\varepsilon, \Lambda^0, U_0^-$, and U_0^+. ∎

4.4. Existence and uniqueness of $\boldsymbol{\delta\tilde{\tilde{p}}}$. For notational convenience to derive the estimate, in this subsection, we denote $v = \delta\tilde{\tilde{p}}$. Then v satisfies the elliptic equation

$$\Delta v = \hat{f} \qquad \text{for } \hat{\mathbf{z}} \in \mathbb{D} \tag{4.72}$$

with boundary conditions

$$v = \hat{h}_0 \qquad \textit{on } \hat{\Gamma}_{cont}, \tag{4.73}$$

$$\boldsymbol{\nu}\cdot\nabla v = \hat{h}_4 \qquad \textit{on } \hat{\Gamma}_{shock}. \tag{4.74}$$

Let $w_0 \in (0, \frac{\pi}{2})$ be the angle between Γ_{cont} and Γ_{shock} from domain $\mathbb{D}$. Then $\boldsymbol{\tau}_f$ and $\mathbf{n}_f$ can be expressed as

$$\mathbf{n}_f = (-\sin w_0, \cos w_0) \qquad \text{and} \qquad \boldsymbol{\tau}_f = (\cos w_0, \sin w_0). \tag{4.75}$$

Define

$$\mathcal{G}_1 := \{g_1, \ \|g_1\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} < \infty\}, \tag{4.76}$$

$$\mathcal{G}_2 := \{g_2, \ \|g_2\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} < \infty\}, \tag{4.77}$$

and

$$\mathcal{G}_3 := \{g_3, \ \|g_3\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}} < \infty\}. \tag{4.78}$$

Then in this subsection, we will establish the following theorem.

THEOREM 4.1. *Suppose there exist constants $\alpha, \beta \in (0,1)$ such that*

$$\hat{h}_0 \in \mathcal{G}_1, \quad \hat{h}_4 \in \mathcal{G}_2, \quad \hat{f} \in \mathcal{G}_3. \tag{4.79}$$

Suppose (4.67) *holds. Then there exists a unique solution v satisfying* (4.72)–(4.74). *Moreover, v satisfies the following estimate:*

$$\|v\|^{(-\alpha;\mathbf{O})(-1-\alpha;\hat{\Gamma}_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}} \leq C(\|\hat{h}_0\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + \|\hat{h}_4\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} + \|\hat{f}\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}}), \tag{4.80}$$

where $C > 0$ is a constant, depending only on α, β, w_0, and $\boldsymbol{\nu}$.

Once Theorem 4.1 is proved, we can complete step (3) in section 3.3 because it follows from (3.34), (4.25), and (4.70) that

$$\hat{h}_0 = \hat{p}_e - \hat{p}_0^+ \in \mathcal{G}_1, \quad \hat{h}_4 \in \mathcal{G}_2, \quad \hat{f} \in \mathcal{G}_3. \tag{4.81}$$

Actually, we can easily establish the theorem for the existence, uniqueness, and regularity of $\delta\tilde{p}$ by replacing v in Theorem 4.1 by $\delta\tilde{p}$.

THEOREM 4.2. *Assume* (3.34) *holds. For any given $\mathcal{A} > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, for $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$, there exists $\delta\tilde{\hat{p}}$, which is the unique solution of* (4.11), (4.56), *and* (4.65), *that satisfies*

$$\|\delta\tilde{\hat{p}}\|^{(-\alpha;\mathbf{O})(-1-\alpha;\hat{\Gamma}_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}} \leq C_{\hat{p}} N_1, \tag{4.82}$$

where

$$N_1 := \|V^- - V_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|\hat{p}_e - \hat{p}_1\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + (\mathcal{A}\varepsilon)^2 + \mathcal{A}\varepsilon^2,$$

and $C_{\hat{p}} > 0$ is a constant, depending only on α, β, and $\boldsymbol{\mu}$.

Then going back to the $\boldsymbol{z}$-coordinates, $\delta\tilde{p}$ is the unique solution of (4.8), (4.55), *and* (4.62), *satisfying the following estimate:*

$$\|\delta\tilde{p}\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}_0^+} \leq C_p N_1, \tag{4.83}$$

where C_p is a positive constant, depending only on $C_{\hat{p}}$, Λ^0, and λ_I^0.

The proof of Theorem 4.2 is trivial since it is only the rephrasing of Theorem 4.1 by applying Lemmas 4.2 and 4.4. So we omit it for brevity. The rest of this subsection is devoted to proving Theorem 4.1.

Proof of Theorem 4.1. The proof is divided into five steps.

1. Because domain $\mathbb{D}$ is unbounded, a natural way to deal with it is to truncate the domain by defining a seqence of approximating bounded domains

$$\mathbb{D}^T := \{\hat{\mathbf{z}} \in \mathbb{D} :\ \hat{z}_1 < T\} \tag{4.84}$$

for any given $T > T_0$ as $T \to \infty$, where T_0 is a sufficiently large constant determined later. Let

$$\Gamma_T = \{\hat{\mathbf{z}} \in \mathbb{D} :\ \hat{z}_1 = T\}. \tag{4.85}$$

Let

$$\hat{\Gamma}^T_{shock} := \hat{\Gamma}_{shock} \cap \overline{\mathbb{D}^T} \qquad \text{and} \qquad \hat{\Gamma}^T_{cont} := \hat{\Gamma}_{cont} \cap \overline{\mathbb{D}^T}. \tag{4.86}$$

Then it is easy to see that $\partial\mathbb{D}^T = \hat{\Gamma}^T_{shock} \cup \hat{\Gamma}^T_{cont} \cup \Gamma_T$. There are two more corners, P^1_T and P^2_T, which are the endpoints of Γ_T, compared to the unbounded domain $\mathbb{D}$. In order to study the problem in $\mathbb{D}^T$ with the additional corners and in order to pass the limit $T \to \infty$, different from Definition 2.1, we introduce the following weighted Hölder norm on $\mathbb{D}^T$ by also adding the weight to P^1_T, which is compatible to the weight to the infinity via scaling.

Definition 4.1. *Fix a constant $T > 1$, let u be a function defined in $\mathbb{D}^T$. Let $\hat{\mathbf{z}} = (\hat{z}_1, \hat{z}_2) \in \mathbb{D}^T$ and $\hat{\mathbf{z}}' = (\hat{z}'_1, \hat{z}'_2) \in \mathbb{D}^T$. Around corner P^1_T, for $\hat{\mathbf{z}}, \hat{\mathbf{z}}' \in \mathbb{D}^T$, define*

$$d^{\hat{\mathbf{z}}}_{P^1_T} := \min\{|\hat{\mathbf{z}} - P^1_T|, 1 + \hat{z}_1\}, \qquad d^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}_{P^1_T} := \min\{d^{\hat{\mathbf{z}}}_{P^1_T}, d^{\hat{\mathbf{z}}'}_{P^1_T}\}.$$

Suppose $\mathbf{I} = (I_1, I_2)$ is an integer-valued vector, with $I_1, I_2 \geq 0$, $|\mathbf{I}| = I_1 + I_2$, and $D^{\mathbf{I}} = \partial^{I_1}_{\hat{z}_1}\partial^{I_2}_{\hat{z}_2}$. Let $\alpha \in (0,1)$, with $m, k \in \{0\} \cup \mathbb{N}$, and $k \leq m$. Let (χ_3, χ_4) be given as in Definition 2.2. For $k_1 < T$, letting

$$\Gamma_{k_1} = \{\hat{\mathbf{z}} \in \mathbb{D} | \hat{z}_1 = k_1\} \tag{4.87}$$

and $\mathbb{D}^{k_1}$ be the triangular domain with boundaries Γ_{cont}, Γ_{shock} and Γ_{k_1}. $\hat{\mathbf{z}} = (\hat{z}_1, \hat{z}_2)$, and $\hat{\mathbf{z}}' = (\hat{z}'_1, \hat{z}'_2)$, define

$$\begin{aligned} \|u\|^{(-(k+\alpha), P^1_T)}_{m;(\chi_3,\chi_4);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} &:= \|u\|_{m,\alpha;(\chi_3,\chi_4);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} + \sum_{j>k}^{m} \sup_{\substack{\hat{\mathbf{z}}\in\mathbb{D}^T\setminus\mathbb{D}^{k_1}\\ |\mathbf{I}|=j}} (1+\hat{z}_1)^{j+\chi_3}(1+\hat{z}_2)^{\chi_4} \\ &\quad \times \left(\frac{d^{\hat{\mathbf{z}}}_{P^1_T}}{1+\hat{z}_1}\right)^{j-(k+\alpha)} |D^{\mathbf{I}}u|, \end{aligned} \tag{4.88}$$

$$\begin{aligned} [u]^{(-(k+\alpha), P^1_T)}_{m,\alpha;(\chi_3,\chi_4);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} &:= \sup_{\substack{\hat{\mathbf{z}},\hat{\mathbf{z}}'\in\mathbb{D}^T\setminus\mathbb{D}^{k_1}\\ \hat{\mathbf{z}}\neq\hat{\mathbf{z}}'\\ |\mathbf{I}|=m}} (1+\min\{\hat{z}_1,\hat{z}'_1\})^{m+\alpha+\chi_3}(1+\min\{\hat{z}_2,\hat{z}'_2\})^{\chi_4} \\ &\quad \times \left(\frac{d^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}_{P^1_T}}{1+\max\{\hat{z}_1,\hat{z}'_1\}}\right)^{m-k} \\ &\quad \times \frac{|D^{\mathbf{I}}u(\hat{\mathbf{z}}) - D^{\mathbf{I}}u(\hat{\mathbf{z}}')|}{|\hat{\mathbf{z}}-\hat{\mathbf{z}}'|^{\alpha}}, \end{aligned} \tag{4.89}$$

(4.90)
$$\|u\|^{(-(k+\alpha),P_T^1)}_{m,\alpha;(\chi_3,\chi_4);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} := \|u\|^{(-(k+\alpha),P_T^1)}_{m;(\chi_3,\chi_4);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} + [u]^{(-(k+\alpha),P_T^1)}_{m,\alpha;(\chi_3,\chi_4);\mathbb{D}^T\setminus\mathbb{D}^{k_1}}.$$

Similarly as Definition 2.3*, we define*

$$\|u\|^{(\chi_1,\mathbf{O})(\chi_2,\hat{\Gamma}_{cont})(-(k+\alpha),P_T^1)}_{m,\alpha;(\chi_3,\chi_4);\mathbb{D}^T} := \|u\|^{(\chi_1,\mathbf{O})(\chi_2,\hat{\Gamma}_{cont})}_{m,\alpha;\mathbb{D}^{k_1}} + \|u\|^{(-(k+\alpha),P_T^1)}_{m,\alpha;(\chi_3,\chi_4);\mathbb{D}^T\setminus\mathbb{D}^{k_1}}, \tag{4.91}$$

where (χ_1,χ_2) *are given as in Definition* 2.2*.*

Remark 4.1. In domain $\mathbb{D}^T$, for any $\hat{\mathbf{z}}\in\mathbb{D}^T$, we have $0\le\hat{z}_2\le\hat{z}_1\tan w_0$. Therefore,

$$0\le\hat{z}_1\le|\hat{\mathbf{z}}|\le\bar{C}\hat{z}_1, \tag{4.92}$$

where $\bar{C}$ is a constant satisfying $\bar{C}=\sqrt{1+\tan^2 w_0}<\infty$. So in Definition 4.1, we replace $|\hat{\mathbf{z}}|$ by $\hat{z}_1$ in the weight for the notational and computational simplicity.

In $\mathbb{D}^T$, we have the following theorem.

THEOREM 4.3. *Let* $h_5\in\mathcal{G}_1$, $h_6\in\mathcal{G}_2$, *and* $f\in\mathcal{G}_3$. *The linear boundary value problem*

$$\begin{cases} Lv_T\equiv\Delta v_T=f & \text{in } \mathbb{D}^T,\\ v_T=h_5(\hat{z}_1) & \text{on } \hat{\Gamma}_{cont},\\ v_T=h_5(T) & \text{on } \Gamma_T,\\ Mv_T=h_6(\hat{z}_2) & \text{on } \hat{\Gamma}_{shock}, \end{cases} \tag{4.93}$$

where $Mv_T:=\boldsymbol{\nu}\cdot\nabla v_T$ *with* $\boldsymbol{\nu}$ *being a unit constant vector,* $\nu\cdot\mathbf{n}_f<0$*, and* $\nu\cdot\tau_f>0$*, admits a unique solution* $v_T\in C^0(\overline{\mathbb{D}^T})\cap C^{2,\alpha}(\mathbb{D}^T)$*, satisfying that*
(4.94)
$$\|v_T\|^{(-\alpha;O)(-1-\alpha;\hat{\Gamma}_{cont})(-1-\alpha;P_T^1)}_{2,\alpha;(1+\beta,0);\mathbb{D}^T}\le C(\|h_5\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+}+\|h_6\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+}+\|f\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}}).$$

The proof for the existence, uniqueness, and regularity away from the corners is standard in the theory of the elliptic partial differential equations of second order, so we omit it for brevity. The solution near corner P_T^2 can be treated by a simple odd extension across Γ_T for function $v_T-h_5(T)$, so we omit it for brevity too. Then in order to conclude the proof of Theorem 4.3, in the following three steps, we will consider the estimate near origin $\mathbf{O}$, the decay estimate in $\mathbb{D}^T\setminus B_{\frac{1}{2}}(P_T^1)$, and the corner estimate near corner P_T^1 one by one.

2. In this step, let us consider the corner estimate near origin $\mathbf{O}$. Obviously, we only need to show that for a fixed constant $k_1<T$, we have the following conclusion in domain $\mathbb{D}^{k_1}$:

$$\|v_T\|^{(-\alpha;O)(-1-\alpha;\hat{\Gamma}_{cont})}_{2,\alpha;\mathbb{D}^{k_1}}\le CN, \tag{4.95}$$

where

$$N:=(\|h_5\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+}+\|h_6\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+}+\|f\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}}). \tag{4.96}$$

Let $\bar{v}_T(\hat{\mathbf{z}}):=v_T(\hat{\mathbf{z}})-h_5(0)$. We first consider the following barrier function:

$$v_1=C_1N(r^\alpha\sin\bar{\theta}+\hat{z}_2^\alpha), \tag{4.97}$$

where (r,θ) are the polar coordinates such that $\hat{\mathbf{z}} = r(\cos\theta, \sin\theta)$, and $\bar{\theta} := (\alpha+\tau)\theta + \theta_0$. Choose τ and θ_0 small enough such that $0 < (\alpha+\tau)w_0 + \theta_0 < \frac{\pi}{2}$.

Through straightforward computation, we have

$$\Delta v_1 = C_1 N[(\alpha^2 - (\alpha+\tau)^2) r^{-2+\alpha} \sin\bar{\theta} + \alpha(\alpha-1)\hat{z}_2^{-2+\alpha}] \le -C_1 c_1 N r^{-2+\alpha} (\sin\theta)^{-2+\alpha}. \tag{4.98}$$

It follows from (4.96) that if C_1 is large, then

$$f \ge -C_1 c_2 N r^{-2+\alpha} (\sin\theta)^{-2+\alpha}. \tag{4.99}$$

Therefore, when C_1 is chosen sufficiently large,

$$\Delta v_1 \le f. \tag{4.100}$$

Similarly, on the boundaries, when choosing constant C_1 large enough, we have

$$v_1 = C_1 N r^{\alpha} \sin\theta_0 \ge r^{\alpha} \|h_5\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \ge h_5(\hat{z}_1) - h_5(0) \qquad \text{on } \hat{\Gamma}_{cont},$$
$$v_1 \ge C_1 N T^{\alpha} \sin\theta_0 \ge T^{\alpha} \|h_5\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \ge h_5(T) - h_5(0) \qquad \text{on } \Gamma_T,$$
$$M v_1 = C_1 N r^{\alpha-1} [(\alpha+\tau)\cos\bar{\theta}(-\sin\theta\nu_1 + \cos\theta\nu_2) + \alpha\sin\bar{\theta}(\cos\theta\nu_1 + \sin\theta\nu_2) + \alpha(\sin\theta)^{\alpha-1}\nu_2]$$
$$\le r^{\alpha-1} \|h_6\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} \le h_6(\hat{z}_2) \qquad \text{on } \hat{\Gamma}_{shock}.$$

We remark that to derive the inequalities above, we should choose α sufficiently small.

Thus, by the maximum principle, we have

$$|v_T(\hat{\mathbf{z}})| \le CN|\hat{\mathbf{z}}|^{\alpha} \quad \forall \hat{\mathbf{z}} \in \mathbb{D}^{k_1}. \tag{4.101}$$

Then it follows from the standard weighted Schauder estimates that

$$\|v_T\|^{(-\alpha,O)}_{1,\alpha,\mathbb{D}^{k_1}} \le CN, \tag{4.102}$$

where C is a constant.

Finally, let us consider the $C^{2,\alpha}$-regularity of solution v_T. Before making it, we introduce the following weighted Hölder norm $\|\cdot\|'$ first.

Let Ω be a domain, let u be a function defined in Ω, and define $d := diam\Omega$. Then define

$$[u]_{j,0;\Omega} = |D^k u|_{0;\Omega} = \sup_{|\beta|=k} \sup_{\Omega} |D^{\beta} u|, \quad k = 0,1,2,\ldots, \tag{4.103}$$

$$[u]_{\alpha;\Omega} = [u]_{0,\alpha;\Omega} = \sup_{x,y\in\Omega} \frac{|Du(x) - Du(y)|}{|x-y|^{\alpha}}, \tag{4.104}$$

$$[u]_{j,\alpha;\Omega} = |D^k u|_{\alpha;\Omega} = \sup_{|\beta|=k} [D^{\beta} u]_{\alpha;\Omega}, \tag{4.105}$$

$$\|u\|'_{k;\Omega} = \sum_{j=0}^{k} d^j [u]_{j,0;\Omega}, \tag{4.106}$$

$$\|u\|'_{k,\alpha;\Omega} = \|u\|'_{k;\Omega} + d^{j+\alpha} [u]_{j,\alpha;\Omega}. \tag{4.107}$$

For $\hat{\mathbf{z}}^0 \in \mathbb{D}^{k_1}$ near corner O with $\hat{z}_2^0 < 1$, set

$$\bar{\hat{z}}^0 = (\hat{z}_1^0, \frac{\hat{z}_2^0}{6}) \qquad \text{and} \qquad \bar{v}(\hat{\mathbf{z}}) = v_T(\hat{\mathbf{z}}) - v_T(\bar{\hat{\mathbf{z}}}^{\mathbf{0}}) - \bigtriangledown v_T(\bar{\hat{\mathbf{z}}}^{\mathbf{0}}) \cdot (\hat{\mathbf{z}} - \bar{\hat{\mathbf{z}}}^{\mathbf{0}}). \tag{4.108}$$

Let $R = \frac{\hat{z}_2^0}{6}$, and let $B_{nR}(\hat{z}^0)$ be a ball with radius nR and center $\hat{z}^0$. For any $\hat{\mathbf{z}} \in B_{2R}$,

$$|\bar{v}(\hat{\mathbf{z}})| \leq CR^{1+\alpha}[\bigtriangledown v_R]_{0,\alpha;B_{3R}}. \tag{4.109}$$

When $\hat{z}_2^0 < 1$, combining the Schauder estimate, (4.109), and (4.102), we have

$$\|\bar{v}\|'_{2,\alpha;B_R} \leq C\|\bar{v}\|_{0,0;B_{2R}} \leq CN(\hat{z}_2^0)^{1+\alpha}. \tag{4.110}$$

Next, for the other cases, by the corner estimate (4.102), we have

$$[\bigtriangledown v_R]_{0,\alpha;B_{3R}} \leq CN|\hat{\mathbf{z}}^0|^{-1}. \tag{4.111}$$

Therefore, we also have

$$\|\bar{v}\|'_{2,\alpha;B_R} \leq C\|\bar{v}\|_{0,0;B_{2R}} \leq CN|\hat{\mathbf{z}}^0|^{-1}(\hat{z}_2^0)^{1+\alpha}. \tag{4.112}$$

According to (4.110) and (4.112), we can obtain (4.95).

3. In this step, we will derive the following decay estimate in domain $\mathbb{D}^T \setminus B_{\frac{1}{2}}(P_T^1)$.

LEMMA 4.5. *Assume $\nu \cdot \mathbf{n}_f < 0$, $\nu \cdot \tau_f > 0$ and the conditions listed in Theorem* 4.3 *hold. Then*

$$\|v_T\|_{2,\alpha;(1+\beta,0);\mathbb{D}^*} \leq CN, \tag{4.113}$$

where $\mathbb{D}^* := \mathbb{D}^T \setminus (\mathbb{D}^{k_1} \cup B_{\frac{1}{2}}(P_T^1))$.

Proof of Lemma 4.5. Denote $\bar{\theta} := (1+\beta+\tau)\theta + \theta_0$. Select τ and θ_0 suitably small, so that $(1+\beta+\tau)w_0 + \theta_0 < \frac{\pi}{2}$. We construct the barrier function as follows:

$$v_2 = Nr^{-1-\beta}[C_2 \sin\bar{\theta} + C_{2*}(\sin\theta)^\alpha], \tag{4.114}$$

From straightforward calculation, we have

$$\begin{aligned}\Delta v_2 = Nr^{-\beta-3}\{&C_2[(\beta+1)^2 - (1+\beta+\tau)^2]\sin\bar{\theta}\\ &+ C_{2*}[-\alpha(1-\alpha)(\sin\theta)^{-2+\alpha} + ((1+\beta)^2 - \alpha^2)(\sin\theta)^\alpha]\}\\ \leq -C_2&c_4Nr^{-\beta-3}(\sin\theta)^{-2+\alpha}\end{aligned} \tag{4.115}$$

when $\frac{C_2}{C_{2*}}$ is sufficiently large. Hence when δ is chosen sufficiently small,

$$\Delta v_2 \leq f. \tag{4.116}$$

On the boundaries, we have

$$\begin{aligned}Mv_2 &= \bigtriangledown v_2 \cdot \nu\\ &= Nr^{-2-\beta}\{C_2[(1+\beta+\tau)\cos\bar{\theta}(-\sin\theta\nu_1 + \cos\theta\nu_2) - (\beta+1)\sin\bar{\theta}(\cos\theta\nu_1 + \sin\theta\nu_2)]\\ &\qquad + C_{2*}[\alpha(\sin\theta)^{-1+\alpha}\cos\theta(-\sin\theta\nu_1 + \cos\theta\nu_2) - (1+\beta)(\sin\theta)^\alpha(\cos\theta\nu_1 + \sin\theta\nu_2)]\}\\ &\leq C_2c_6Nr^{-2-\beta}\\ &\leq h_6(\hat{z}_2) \qquad on\ \hat{\Gamma}_{shock},\end{aligned} \tag{4.117}$$

$$v_2 \geq v_T \qquad on\ \hat{\Gamma}_{cont} \cup \Gamma_T \cup \Gamma_{k_1}. \tag{4.118}$$

Thus, by the comparison principle, we conclude

$$|v_T(\hat{\mathbf{z}})| \le CN|\hat{\mathbf{z}}|^{-1-\beta} \quad \forall \hat{\mathbf{z}} \in \mathbb{D}^T \setminus \mathbb{D}^{k_1}. \tag{4.119}$$

It is equivalent to the following C^0-estimate:

$$\|v_T\|_{0,0;(1+\beta,0);\mathbb{D}^T} \le CN. \tag{4.120}$$

Now we are going to consider the $C^{1,\alpha}$-estimate and $C^{2,\alpha}$-estimate of v_T in $\mathbb{D}^*$.

Set $R = \frac{T}{4\sqrt{1+k_1^2}}$. For any $\hat{\mathbf{z}}^* = (\hat{z}_1^*, \hat{z}_2^*) \in \mathbb{D}^*$, we divide the situation into three cases: $\hat{z}_2^* < R$, $R \le \hat{z}_2^* \le 3R$, and $3R < \hat{z}_2^* < 4R$.

Case 1, $\hat{z}_2^* < R$: Set $B_{nR}(\hat{\mathbf{z}}^*)$ to be a ball with radius $r = nR$ and center at $\hat{\mathbf{z}}^*$. Set $B_{nR}^+ = B_{nR} \cap \mathbb{D}^T$ and $\Gamma_1 = B_{2R} \cap \Gamma_{cont}$. By the standard Schauder estimate, we have

$$\|v_T\|'_{1,\alpha;B_R^+} \le C(\|v_T\|_{0,0,B_{2R}^+} + \|h_5\|'_{1,\alpha;\Gamma_1} + R^2\|f\|'_{0,\alpha,B_{2R}^+}) \le CN|\hat{\mathbf{z}}|^{-1-\beta}, \tag{4.121}$$

where we use the fact that when δ is sufficiently small, $\|f\|_{0,\alpha,B_{2R}^+} \to 0$, and then combine with (4.94) and (4.120) to get (4.121).

Case 2, $R \le \hat{z}_2^* \le 3R$: Set $B_{nR}(\hat{\mathbf{z}}^*)$ and $B_{nR}^+ = B_{nR/4} \cap \mathbb{D}^T$ as before. Then choosing δ sufficiently small, and by (4.120) and the Schauder estimate, we have

$$\|v_T\|'_{1,\alpha;B_R^+} \le C(\|v_T\|_{0,0,B_{2R}^+} + R^2\|f\|'_{0,\alpha,B_{2R}^+}) \le CN|\hat{\mathbf{z}}|^{-1-\beta}. \tag{4.122}$$

Case 3, $3R < \hat{z}_2^* < 4R$: Set $B_{nR}(\hat{\mathbf{z}}^*)$ and $B_{nR}^+ = B_{nR} \cap \mathbb{D}^T$ as before. Let $\Gamma_2 = B_{2R} \cap \Gamma_{shock}$. Then choosing δ sufficiently small, and by (4.120) and the Schauder estimate, we have

$$\|v_T\|'_{1,\alpha;B_R^+} \le C(\|v_T\|_{0,0,B_{2R}^+} + R\|h_6\|'_{0,\alpha;\Gamma_2} + R^2\|f\|'_{0,\alpha,B_{2R}^+}) \le CN|\hat{\mathbf{z}}|^{-1-\beta}. \tag{4.123}$$

By (4.121)–(4.123), we have

$$\|v_T\|_{1,\alpha;(1+\beta,0);\mathbb{D}^*} \le CN. \tag{4.124}$$

The $C^{2,\alpha}$-estimate can be derived via a similar process, so we have

$$\|v_T\|_{2,\alpha;(1+\beta,0);\mathbb{D}^*} \le CN. \tag{4.125}$$ □

4. Now let us finally consider the corner estimates near P_T^1 to conclude the proof of Theorem 4.3. Actually, we will show the following lemma in this step.

LEMMA 4.6. *Assume the assumptions listed in Theorem* 4.3 *hold. Then the solution $v_T \in C^0(\overline{\mathbb{D}^T}) \cap C^{2,\alpha}(\mathbb{D}^T)$ given in Theorem* 4.3 *satisfies the following estimate:*

$$\|v_T\|^{(-(1+\alpha),P_T^1)}_{2,\alpha;(1+\beta,0);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} \le C(\|h_5\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + \|h_6\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} + \|f\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}}). \tag{4.126}$$

For $\hat{\mathbf{z}} \in \mathbb{D}^T$, let $\bar{v}_P(\hat{\mathbf{z}}) = v_T(\hat{\mathbf{z}}) - v_T(P_T^1) - \mathbf{d}(\hat{\mathbf{z}} - P_T^1)$, where vector $\mathbf{d} = (d_1, 0)$ satisfies $\boldsymbol{\nu} \cdot \mathbf{d} = h_6(T\tan w_0)$, i.e., the definition of $\mathbf{d}$ is derived from the last two boundary conditions of v_T, (4.93), at P_T^1. Let $B_{\frac{1}{2}}(P_T^1)$ be a ball with radius $\frac{1}{2}$ and center P_T^1. Then in $(\mathbb{D}^T\setminus\overline{\mathbb{D}^{\frac{1}{2}T}}) \cap B_{\frac{1}{2}}(P_T^1)$, $\bar{v}_P(\hat{\mathbf{z}})$ satisfies the following linear boundary value problem:

$$\begin{cases} L\bar{v}_P \equiv \Delta\bar{v}_P = f & in\ (\mathbb{D}^T\setminus\overline{\mathbb{D}^{\frac{1}{2}T}}) \cap B_{\frac{1}{2}}(P_T^1), \\ \bar{v}_P = 0 & on\ \Gamma_T \cap B_{\frac{1}{2}}(P_T^1), \\ M\bar{v}_P = h_6(\hat{z}_2) - h_6(T\tan w_0) & on\ \hat{\Gamma}_{shock} \cap B_{\frac{1}{2}}(P_T^1). \end{cases} \tag{4.127}$$

Then to show Lemma 4.6, we will show that $\bar{v}_P(\hat{\mathbf{z}})$ satisfies the following estimate:
(4.128)

$$\|\bar{v}_P\|^{(-(1+\alpha),P_T^1)}_{2,\alpha;(1+\beta,0);(\mathbb{D}^T\setminus\overline{\mathbb{D}^{\frac{1}{2}T}})\cap B_{\frac{1}{2}}(P_T^1)} \leq C(\|h_5\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+}+\|h_6\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+}+\|f\|^{*(1;O)(1-\alpha,\hat{\Gamma}_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}}).$$

To show it, first let us define the scaling by introducing $Q_R = \{\boldsymbol{\eta} = (\eta_1,\eta_2) \in \mathbb{R}^2,\ \frac{1}{2} < \eta_1 < 1, R\boldsymbol{\eta} \in \mathbb{D}^T \cap B_{\frac{1}{2}}(P_T^1)\}$. Define $U(\boldsymbol{\eta}) := T^{1+\beta}\bar{v}_P(T\boldsymbol{\eta})$, where $\boldsymbol{\eta} \in Q_T$. In Q_T, we introduce the following weighted Hölder norm.

DEFINITION 4.2. *Suppose* $\mathbf{I} = (I_1, I_2)$ *is an integer-valued vector, where* $I_1, I_2 \geq 0$, $|\mathbf{I}| = I_1 + I_2$, *and* $D^{\mathbf{I}} = \partial^{I_1}_{\hat{z}_1}\partial^{I_2}_{\hat{z}_2}$. *Fix* $m, k \in \{0\} \cup \mathbb{N}$ *with* $k \leq m$, *and* $\alpha \in (0,1)$. *Let* Q *be an open bounded and connected domain in* $\mathbb{R}^2$, *and let* $\mathbf{q}$ *be a fixed point on* ∂Q. *For* $\boldsymbol{\eta} = (\eta_1,\eta_2), \boldsymbol{\eta}' = (\eta_1',\eta_2') \in Q$, *set*

$$d_{\boldsymbol{\eta}} = |\boldsymbol{\eta} - \mathbf{q}|, \qquad d_{\boldsymbol{\eta},\boldsymbol{\eta}'} := \min\{d_{\boldsymbol{\eta}}, d_{\boldsymbol{\eta}'}\}.$$

For a function $u : Q \to \mathbb{R}^2$, *define*

$$\|u\|^{(-(k+\alpha),\{\mathbf{q}\})}_{m,Q} := \|u\|_{k,\alpha,Q} + \sum_{j>k}^{m} \sup_{\substack{\boldsymbol{\eta}\in Q\\ |\mathbf{I}|=j}} (d_{\boldsymbol{\eta}})^{j-(k+\alpha)}|D^{\mathbf{I}}u|,$$

$$[u]^{(-(k+\alpha),\{\mathbf{q}\})}_{m,\alpha,Q} := \sup_{\substack{\boldsymbol{\eta}\neq\boldsymbol{\eta}'\in Q\\ |\mathbf{I}|=m}} (d_{\boldsymbol{\eta},\boldsymbol{\eta}'})^{m-k}\frac{|D^{\mathbf{I}}u(\boldsymbol{\eta}) - D^{\mathbf{I}}u(\boldsymbol{\eta}')|}{|\boldsymbol{\eta}-\boldsymbol{\eta}'|^{\alpha}},$$

$$\|u\|^{(-(k+\alpha),\{\mathbf{q}\})}_{m,\alpha,Q} := \|u\|^{(-(k+\alpha),\{\mathbf{q}\})}_{m,Q} + [u]^{(-(k+\alpha),\{\mathbf{q}\})}_{m,\alpha,Q}.$$

Now by (4.127), we know U satisfies

$$\begin{cases} \Delta_{\boldsymbol{\eta}} U = T^{3+\beta} f & \text{in } Q_T, \\ U = 0 & \text{on } \Gamma_T \cap B_{\frac{1}{2}}(P_T^1), \\ M_{\boldsymbol{\eta}} U = T^{2+\beta}(h_6(T\eta_2) - h_6(T\tan w_0)) & \text{on } \hat{\Gamma}_{shock} \cap B_{\frac{1}{2}}(P_T^1), \end{cases} \tag{4.129}$$

where $\Delta_{\boldsymbol{\eta}} = \partial^2_{\eta_1} + \partial^2_{\eta_2}$ and $M_{\boldsymbol{\eta}} = \boldsymbol{\nu} \cdot \nabla_{\boldsymbol{\eta}}$.

Let (r,θ) be polar coordinates centered at P_T^1, such that $\Gamma_T \subset \{\theta = -\frac{\pi}{2}\}$, and $\Gamma_{shock} \subset \{-\pi + w_0 < \theta < -\frac{\pi}{2}\}$. In the (r,θ)-coordinates, we have

$$\mathbf{n}'_{sh} = (-\sin w_0, \cos w_0) \qquad \text{and} \qquad \boldsymbol{\tau}'_{sh} = (-\cos w_0, -\sin w_0), \tag{4.130}$$

$$\boldsymbol{\nu}\cdot\mathbf{n}'_{sh} < 0 \qquad \text{and} \qquad \boldsymbol{\nu}\cdot\boldsymbol{\tau}'_{sh} < 0. \tag{4.131}$$

Now, for $N > 0$ defined by (4.96), we construct a barrier function as

$$v_3 = C_3 N r^{1+\alpha} \sin\bar{\theta}, \tag{4.132}$$

where $\bar{\theta} := (1+\alpha+\tau)\theta + \theta_0$. Choose τ, θ_0 small enough to satisfy $(1+\alpha+\tau)w_0 + \theta_0 < \frac{\pi}{2}$.

By straightforward computation, we have

$$\Delta v_3 = C_3 N[(1+\alpha)^2 - (1+\alpha+\tau)^2]r^{-1+\alpha}\sin\bar{\theta} \leq -C_3 c_7 N r^{-1+\alpha}\sin\bar{\theta}. \tag{4.133}$$

Moreover, in Q_T,

$$T^{3+\beta} f(\boldsymbol{\eta}) \geq -\|f\|^{*(1;O)(1-\alpha,\Gamma_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}} \geq -C_3 c_8 N\delta r^{-1+\alpha}\sin\bar{\theta}.$$

Hence, when δ is chosen sufficiently small,

$$\Delta v_3 \le T^{3+\beta} f = \Delta_{\boldsymbol{\eta}} U \qquad \text{in } Q_T. \tag{4.134}$$

Next, on $\Gamma_T \cap B_{\frac{1}{2}}(P_T^1)$, $v_3 \ge 0 = U$.

On $\Gamma_{shock} \cap B_{\frac{1}{2}}(P_T^1)$, we know that

$$MU = T^{2+\beta}(h_6(T\eta_2) - h_6(T\tan w_0)) \le C_4 \|h_6\|_{1,\alpha;(2+\beta);\mathbb{R}} (\tan w_0)^{2+\beta} r^\alpha = C_4 c_9 N r^\alpha. \tag{4.135}$$

Therefore, we obtain

$$\begin{aligned} Mv_3 &= \triangledown v_3 \cdot \boldsymbol{\nu} \\ &= C_3 N r^\alpha [(1+\alpha+\tau)\cos\bar{\theta}(-\sin\theta\nu_1 + \cos\theta\nu_2) + (1+\alpha)\sin\bar{\theta}(-\cos\theta\nu_1 - \sin\theta\nu_2)] \\ &\le C_3 c_{10} N r^\alpha \\ &= M_{\boldsymbol{\eta}} U \qquad \textit{on } \hat{\Gamma}_{shock} \cap B_{\frac{1}{2}}(P_T^1). \end{aligned} \tag{4.136}$$

Thus, by the comparison principle, we conclude

$$|U(\eta_1, \eta_2)| \le CNr^{1+\alpha} \qquad \forall \boldsymbol{\eta} \in \overline{Q_T}. \tag{4.137}$$

Therefore,

$$\|U\|_{2,\alpha;Q_T}^{(-(1+\alpha),P_T^1)} \le CN. \tag{4.138}$$

Define $\tilde{Q}_T := \{T\boldsymbol{\eta} : \boldsymbol{\eta} \in Q_T\}$.

Let us set

$$\mathcal{A}_2(\bar{v}_P) := (1+|\hat{z}_1|)^{3+\beta} \left(\frac{d_{P_T^1}^{\hat{\mathbf{z}}}}{1+|\hat{z}_1|} \right)^{1-\alpha} \sum_{l=0}^{2} |\partial_{\hat{z}_1}^l \partial_{\hat{z}_2}^{2-l} \bar{v}_P|, \tag{4.139}$$

$$\mathcal{A}_{2+\alpha}(\bar{v}_P) := (1+|\hat{z}_1|)^{3+\beta+\alpha} \left(\frac{d_{P_T^1}^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}}{1+\max\{|\hat{z}_1|, |\hat{z}_1'|\}} \right) \sum_{l=0}^{2} \left(\frac{|\partial_{\hat{z}_1}^l \partial_{\hat{z}_2}^{2-l} \bar{v}_P(\hat{\mathbf{z}}) - \partial_{\hat{z}_1}^l \partial_{\hat{z}_2}^{2-l} \bar{v}_P(\hat{\mathbf{z}}')|}{|\hat{\mathbf{z}} - \hat{\mathbf{z}}'|^\alpha} \right). \tag{4.140}$$

In Q_T, for every points $\hat{\mathbf{z}}, \hat{\mathbf{z}}'$, we can find $\boldsymbol{\eta}, \boldsymbol{\eta}'$ such that $\hat{\mathbf{z}} = T\boldsymbol{\eta}$, and $\hat{\mathbf{z}}' = T\boldsymbol{\eta}'$. Therefore, we obtain

$$\frac{d_{P_T^1}^{\hat{\mathbf{z}}}}{1+|\hat{z}_1|} \le 4d_{\boldsymbol{\eta}} \tag{4.141}$$

and

$$\frac{d_{P_T^1}^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}}{1+\max\{|\hat{z}_1|, |\hat{z}_1'|\}} \le \min\left\{ \frac{d_{P_T^1}^{\hat{\mathbf{z}}}}{1+|\hat{z}_1|}, \frac{d_{P_T^1}^{\hat{\mathbf{z}}'}}{1+|\hat{z}_1'|} \right\} \le 4d_{\boldsymbol{\eta},\boldsymbol{\eta}'}. \tag{4.142}$$

Then, (4.139) and (4.140) become

$$\mathcal{A}_2(\bar{v}_P) \le (4d_{\boldsymbol{\eta}})^{1-\alpha} |\partial_{\eta_1}^l \partial_{\eta_2}^{2-l} U| \le CN, \tag{4.143}$$

$$\mathcal{A}_{2+\alpha}(\bar{v}_P) \le 4d_{\boldsymbol{\eta},\boldsymbol{\eta}'} \sum_{l=0}^{2} \frac{|\partial_{\eta_1}^l \partial_{\eta_2}^{2-l} U(\boldsymbol{\eta}) - \partial_{\eta_1}^l \partial_{\eta_2}^{2-l} U(\boldsymbol{\eta}')|}{|\boldsymbol{\eta} - \boldsymbol{\eta}'|^\alpha} \le CN. \tag{4.144}$$

In conclusion, we have

$$\|\bar{v}_P\|^{(-(1+\alpha),P_T^1)}_{2,\alpha;(1+\beta,0);\tilde{Q}_T} \leq CN. \tag{4.145}$$

Now we are going to estimate v_T in the new weighted Hölder norm. From (4.128), we obtain

$$\|v_T\|^{(-(1+\alpha),P_T^1)}_{2,\alpha;(1+\beta,0);\tilde{Q}_T} \leq CN. \tag{4.146}$$

Besides, in $(\mathbb{D}^T \setminus \mathbb{D}^{k_1}) \setminus \tilde{Q}_T$, we have

$$\|v_T\|_{2,\alpha;(1+\beta,0);(\mathbb{D}^T\setminus\mathbb{D}^{k_1})\setminus\tilde{Q}_T} \leq CN \tag{4.147}$$

and

$$\frac{d^{\hat{\mathbf{z}}}_{P_T^1}}{1+|\hat{z}_1|} \leq 1. \tag{4.148}$$

By using (4.147) and (4.148), we can obtain

$$\|v_T\|^{(-(1+\alpha),P_T^1)}_{2,0;(1+\beta,0);(\mathbb{D}^T\setminus\mathbb{D}^{k_1})\setminus\tilde{Q}_T} \leq CN. \tag{4.149}$$

Combined with (4.146), we have

$$\|v_T\|^{(-(1+\alpha),P_T^1)}_{2,0;(1+\beta,0);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} \leq CN. \tag{4.150}$$

Finally, let us discuss $[v_T]^{(-(1+\alpha),P_T^1)}_{2,\alpha;(1+\beta,0);\mathbb{D}^T\setminus\mathbb{D}^{k_1}}$. For $\hat{\mathbf{z}}, \hat{\mathbf{z}}' \in \mathbb{D}^T \setminus \mathbb{D}^{k_1}$, which satisfy $|\hat{z}_1| < |\hat{z}_1'|$, we set

$$\mathcal{B}_{2+\alpha}(\hat{\mathbf{z}},\hat{\mathbf{z}}') := (1+|\hat{z}_1|)^{3+\alpha+\beta}\left(\frac{d^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}_{P_T^1}}{1+|\hat{z}_1'|}\right)\sum_{l=0}^{2}\frac{|\partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2}v_T(\hat{\mathbf{z}}) - \partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2}v_T(\hat{\mathbf{z}}')|}{|\hat{\mathbf{z}}-\hat{\mathbf{z}}'|^\alpha}. \tag{4.151}$$

By (4.142) and (4.148), we have

$$\frac{d^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}_{P_T^1}}{1+|\hat{z}_1'|} \leq \min\left\{\frac{d^{\hat{\mathbf{z}}}_{P_T^1}}{1+|\hat{z}_1|}, \frac{d^{\hat{\mathbf{z}}'}_{P_T^1}}{1+|\hat{z}_1'|}\right\} \leq \begin{cases} 4d_{\boldsymbol{\eta},\boldsymbol{\eta}'} & in\ \tilde{Q}_T, \\ 1 & in\ (\mathbb{D}^T\setminus\mathbb{D}^{k_1})\setminus\tilde{Q}_T. \end{cases} \tag{4.152}$$

Case 1. If $|\hat{\mathbf{z}}-\hat{\mathbf{z}}'| \leq \frac{|\hat{z}_1|}{40}$, then by (4.146), and (4.149), we obtain that

$$\mathcal{B}_{2+\alpha}(\hat{\mathbf{z}},\hat{\mathbf{z}}') \leq CN. \tag{4.153}$$

Case 2. If $|\hat{\mathbf{z}}-\hat{\mathbf{z}}'| \geq \frac{|\hat{z}_1'|}{40}$, then (4.151) becomes

$$\mathcal{B}_{2+\alpha}(\hat{\mathbf{z}},\hat{\mathbf{z}}')] \leq \left(\frac{1+|\hat{z}_1|}{\frac{|\hat{z}_1'|}{40}}\right)^\alpha [(1+|\hat{z}_1|)^{3+\beta}\left(\frac{d^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}_{P_T^1}}{1+|\hat{z}_1'|}\right)^{1-\alpha}\sum_{l=0}^{2}|\partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2}v_T(\hat{\mathbf{z}})| + |\partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2}v_T(\hat{\mathbf{z}}')|]. \tag{4.154}$$

By (4.150), we obtain

$$\mathcal{B}_{2+\alpha}(\hat{\mathbf{z}},\hat{\mathbf{z}}') \leq CN. \tag{4.155}$$

Combining Case 1 and Case 2 together, we have

$$[v_T]^{(-(1+\alpha),P_T^1)}_{2,\alpha;(1+\beta,0);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} < CN. \tag{4.156}$$

In conclusion, we obtain

$$\|v_T\|^{(-(1+\alpha),P_T^1)}_{2,\alpha;(1+\beta,0);\mathbb{D}^T\setminus\mathbb{D}^{k_1}} \leq CN. \tag{4.157}$$

Hence, combining the results in steps 2, 3, and 4, we establish estimate (4.94) to conclude the proof of Theorem 4.3.

5. In this step, we will complete the proof of Theorem 4.1 by passing the limit $T \to \infty$ and showing the uniqueness.

Define $T_n = 4(n + \frac{2}{1+tan^2 w_0})$ for any fixed $n \in \mathbb{N}$. For any $\hat{\mathbf{z}} \in \mathbb{D}^{\frac{T_n}{2}}$, we have

$$\frac{\min\{1+|\hat{z}_1|, |\hat{\mathbf{z}} - P^1_{T_n}|\}}{1+|\hat{z}_1|} \geq \frac{1}{2}. \tag{4.158}$$

Therefore,

$$\frac{d^{\hat{\mathbf{z}}}_{P^1_T}}{1+|\hat{z}_1|} \geq \frac{1}{2}.$$

Since $1 < 1+\alpha < 2$,

$$\left(\frac{d^{\hat{\mathbf{z}}}_{P^1_T}}{1+|\hat{z}_1|}\right)^{2-(1+\alpha)} \geq \left(\frac{1}{2}\right)^{1-\alpha}. \tag{4.159}$$

Hence

$$2 \times \left(\frac{d^{\hat{\mathbf{z}}}_{P^1_T}}{1+|\hat{z}_1|}\right)^{1-\alpha} \geq 2^\alpha > 1. \tag{4.160}$$

By Definition 4.1, we have

$$\|v_{T_n}\|_{2;(1+\beta,0);\mathbb{D}^{\frac{T_n}{2}}} \leq 2\|v_{T_n}\|^{(-(1+\alpha),P^1_{T_n})}_{2;(1+\beta,0);\mathbb{D}^{T_n}} \leq 2CN. \tag{4.161}$$

For $\hat{\mathbf{z}}, \hat{\mathbf{z}}' \in \mathbb{D}^{\frac{T_n}{2}}$ with $0 < |\hat{z}_1| < |\hat{z}'_1| < \frac{T_n}{2}$, we have

$$\frac{d^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}_{P^1_{T_n}}}{1+\max\{|\hat{z}_1|, |\hat{z}'_1|\}} = \frac{\min\{|\hat{\mathbf{z}} - P^1_{T_n}|, |\hat{\mathbf{z}}' - P^1_{T_n}|, 1+|\hat{z}_1|\}}{1+|\hat{z}'_1|}. \tag{4.162}$$

Moreover, for $0 \leq l \leq 2$, define

(4.163)

$$g_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}') := (1+|\hat{z}_1|)^{3+\alpha+\beta} \left(\frac{d^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}_{P^1_{T_n}}}{1+\max\{|\hat{z}_1|, |\hat{z}'_1|\}}\right) \frac{|\partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2} v_{T_n}(\hat{\mathbf{z}}') - \partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2} v_{T_n}(\hat{\mathbf{z}})|}{|\hat{\mathbf{z}}' - \hat{\mathbf{z}}|^\alpha},$$

(4.164)

$$g_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}') := (1+|\hat{z}_1|)^{3+\alpha+\beta} \frac{|\partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2} v_{T_n}(\hat{\mathbf{z}}') - \partial^l_{\hat{z}_1}\partial^{2-l}_{\hat{z}_2} v_{T_n}(\hat{\mathbf{z}})|}{|\hat{\mathbf{z}}' - \hat{\mathbf{z}}|^\alpha}.$$

If $|\hat{\mathbf{z}}' - \hat{\mathbf{z}}| \geq \frac{1}{4}(1+|\hat{z}_1|)$, then we have

$$\begin{aligned} g_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}') \leq& 4^\alpha [(1+|\hat{z}_1|)^{3+\beta} |\partial_{\hat{z}_1}^l \partial_{\hat{z}_2}^{2-l} v_{T_n}(\hat{\mathbf{z}}') - \partial_{\hat{z}_1}^l \partial_{\hat{z}_2}^{2-l} v_{T_n}(\hat{\mathbf{z}})|] \\ \leq& 4^\alpha [(1+|\hat{\mathbf{z}}|)^{3+\beta} |\partial_{\hat{z}_1}^l \partial_{\hat{z}_2}^{2-l} v_{T_n}(\hat{\mathbf{z}}')| + (1+|\hat{\mathbf{z}}|)^{3+\beta} |\partial_{\hat{z}_1}^l \partial_{\hat{z}_2}^{2-l} v_{T_n}(\hat{\mathbf{z}})|] \\ \leq& 2^{2\alpha+1} \|v_{T_n}\|_{2;(1+\beta,0);\mathbb{D}^{\frac{T_n}{2}}}. \end{aligned} \tag{4.165}$$

Next, let us discuss the case that $|\hat{\mathbf{z}}' - \hat{\mathbf{z}}| \leq \frac{1}{4}(1+|\hat{z}_1|)$.

If $0 \leq |\hat{z}_1| \leq \frac{T_n}{12}$, then combining with the assumption that $|\hat{\mathbf{z}}' - \hat{\mathbf{z}}| \leq \frac{1}{4}(1+|\hat{z}_1|)$, we obtain $0 < |\hat{z}_1| < |\hat{z}_1'| < \frac{T_n}{4}$. Then

$$\begin{aligned} \frac{d_{P_{T_n}^1}^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}}{1+\max\{|\hat{z}_1|, |\hat{z}_1'|\}} &= \frac{\min\{|\hat{\mathbf{z}} - P_{T_n}^1|, |\hat{\mathbf{z}}' - P_{T_n}^1|, 1+|\hat{z}_1|\}}{1+|\hat{z}_1'|} \\ &= \frac{1+|\hat{z}_1|}{1+|\hat{z}_1'|} \\ &= \frac{1+|\hat{z}_1'| - (|\hat{z}_1'| - |\hat{z}_1|)}{1+|\hat{z}_1|} \\ &> 1 - \frac{|\hat{z}_1' - \hat{z}_1|}{1+|\hat{z}_1'|} \\ &> \frac{3}{4}. \end{aligned} \tag{4.166}$$

Therefore, we have

$$g_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}') = g_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}') \left(\frac{d_{P_{T_n}^1}^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}}{1+\max\{|\hat{z}_1|, |\hat{z}_1'|\}} \right)^{-1} < \frac{4}{3} \|v_{T_n}\|_{2,\alpha;(1+\beta,0);\mathbb{D}^{T_n}}^{(-(1+\alpha),P_{T_n}^1)}. \tag{4.167}$$

If $\frac{T_n}{12} \leq |\hat{z}_1| \leq |\hat{z}_1'| \leq \frac{T_n}{2}$, then

$$\frac{d_{P_{T_n}^1}^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}}{1+\max\{|\hat{\mathbf{z}}|, |\hat{\mathbf{z}}'|\}} \geq \frac{\min\{\frac{T_n}{2}, 1+\frac{T_n}{12}\}}{1+\frac{T_n}{2}} \geq \frac{1}{12}. \tag{4.168}$$

Hence, we obtain

$$g_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}') = g_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}') (\frac{d_{P_{T_n}^1}^{\hat{\mathbf{z}},\hat{\mathbf{z}}'}}{1+\max\{|\hat{z}_1|, |\hat{z}_1'|\}})^{-1} \leq 12 \|v_{T_n}\|_{2,\alpha;(1+\beta,0);\mathbb{D}^{T_n}}^{(-(1+\alpha),P_{T_n}^1)}. \tag{4.169}$$

In conclusion, by (4.161), (4.165), (4.167), and (4.169), we have

$$\|v_{T_n}\|_{2,\alpha;(1+\beta,0);\mathbb{D}^{\frac{T_n}{2}}} \leq 36CN. \tag{4.170}$$

Now, by the Arzelà–Ascoli theorem and the standard diagonal argument, there exists a function v and a subsequence of v_{T_n}, which we still denote as v_{T_n} for notational simplicity, such that $v(\hat{\mathbf{z}}) = \lim_{n\to\infty} v_{T_n}(\hat{\mathbf{z}})$ for any $\hat{\mathbf{z}} \in \mathbb{D}$, with the estimate that

$$\|v\|_{2,\alpha;(1+\beta,0);\mathbb{D}\backslash\mathbb{D}^{k_1}} \leq CN \tag{4.171}$$

for any $k_1 > 0$, where C is a constant.

Combining it with the corner estimate at O, we finally have

(4.172)

$$\|v\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}} \leq C(\|h_5\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + \|h_6\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} + \|f\|^{*(1;O)(1-\alpha,\Gamma_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{D}}).$$

Finally, we will prove the uniqueness of v in Theorem 4.1.

Suppose that there exists another solution $\tilde{v} \in C^{2,\alpha;(1+\beta)}_{(-1-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}(\mathbb{D})$. Define $\hat{v} := v - \tilde{v}$. Since $\hat{v} \in C^{2,\alpha;(1+\beta)}_{(-1-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}(\mathbb{D})$, we know

$$\hat{v} \to 0 \qquad as\ |\hat{\mathbf{z}}| \to \infty.$$

Then for any $\varepsilon > 0$, there exists a $T > 0$ such that $|\hat{v}(\hat{\mathbf{z}})| < \varepsilon$ on Γ_T. So $\hat{v}$ satisfies

$$\begin{cases} \Delta\hat{v} = 0 & in\ \mathbb{D}^T, \\ \hat{v} = 0 & on\ \Gamma_{cont}, \\ |\hat{v}| < \varepsilon & on\ \Gamma_T, \\ M\hat{v} = 0 & on\ \Gamma_{shock}. \end{cases} \tag{4.173}$$

It easily follows from the maximum principle that $|\hat{v}| \leq \varepsilon$ in $\mathbb{D}^T$. Because ε is arbitary, let $\varepsilon \to 0$, then $\hat{v} \equiv 0$ in $\mathbb{D}$. □

5. Existence and estimates of $(\boldsymbol{\delta\tilde{w}}, \boldsymbol{\delta\tilde{\rho}}, \boldsymbol{\delta\tilde{u_1}})$ and $\boldsymbol{\delta\tilde{s}}$. Based on the existence and regularity of $\delta\tilde{p}$ obtained in Theorem 4.2, we will introduce the existence and estimates of $\delta\tilde{w}$, $\delta\tilde{\rho}$, $\delta\tilde{u_1}$ and $\delta\tilde{s}$ in $\mathbf{z}$-coordinates in $\mathbb{D}_0^+$ in this section.

THEOREM 5.1. *Assume* (3.34) *holds. There exists a unique function $\delta\tilde{w}$ satisfying the equation*

$$(\delta\tilde{w})_{z_2} = a^0_{11}(\delta\tilde{p} - \delta p)_{z_1} + a^0_{12}(\delta\tilde{p} - \delta p)_{z_2} + a_{11}(\delta p)_{z_1} + a_{12}(\delta p)_{z_2}, \tag{5.1}$$

and boundary condition (4.48) *(i.e.,* (4.51)*), with the following estimate:*

$$\|\delta\tilde{w}\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}_0^+} \leq C_w N_1, \tag{5.2}$$

where C_w is a constant depending only on C_p and the data.

Remark 5.1. For the fixed point, i.e., if $\delta\tilde{p} = \delta p$, then (5.1) is equivalent to (4.2).

Proof of Theorem 5.1. For any point $\mathbf{z} = (z_1, z_2) \in \mathbb{D}$, let $\mathbf{z}^L = (z_1, z_2^L)$ be the intersection point of Γ_{shock} and the vertical line passing through $\mathbf{z}$. It is easy to see that $z_2^L = z_1/k_1$. Then it follows from (5.1) and (4.51) that $\delta\tilde{w}(\mathbf{z})$ given by the formula

(5.3)

$$\delta\tilde{w} = h_1\left(\frac{z_1}{k_1}\right) - e\delta\tilde{p}\left(z_1, \frac{z_1}{k_1}\right) + \int_{z_2^L}^{z_2} (a^0_{11}(\delta\tilde{p}-\delta p)_{z_1} + a^0_{12}(\delta\tilde{p}-\delta p)_{z_2} + a_{11}(\delta p)_{z_1} + a_{12}(\delta p)_{z_2})(z_1, s)ds$$

satisfies (5.1) and (4.48).

Next, it follows from (4.83) that

$$|(\delta\tilde{p})_{z_1}| \leq C_5 N_1 r^{-2-\beta}, \tag{5.4}$$

where C_5 is a constant depending only on C_p and the data. Then, by (3.35) and (5.4),

$$
\begin{aligned}
|\delta\tilde{w}| &\leq |h_1 - e\delta\tilde{p}|(\mathbf{z}^L) + \left|\int_{z_2^L}^{z_2} (a_{11}^0(\delta\tilde{p}-\delta p)_{z_1} + a_{12}^0(\delta\tilde{p}-\delta p)_{z_2} + a_{11}(\delta\tilde{p})_{z_1} + a_{12}(\delta\tilde{p})_{z_2})(z_1,s)ds\right| \\
&\leq C_6 N_1 |\mathbf{z}|^{-1-\beta} + C_5(N_1 + \mathcal{A}\varepsilon)\int_0^{z_2^L} |\mathbf{z}|^{-2-\beta}(z_1,s)ds \\
&\leq C_7 N_1 |\mathbf{z}|^{-1-\beta},
\end{aligned}
$$

where constants C_5, C_6, and C_7 depend only on C_p and the data. For the weight at the corner, it is easy to check that for any $\mathbf{z} \in \mathbb{D}^{k_1}$, by (5.3),

$$
|\delta\tilde{w}(\mathbf{z}) - \delta\tilde{w}(0,0)| \leq C_8 N_1 |\mathbf{z}|^{\alpha}. \tag{5.5}
$$

Now, we are going to consider $(\delta\tilde{w})_{z_1}$. According to (4.8), we have

$$
\begin{aligned}
&(a_{11}^0(\delta\tilde{p}-\delta p)_{z_1} + a_{12}^0(\delta\tilde{p}-\delta p)_{z_2} + a_{11}(\delta p)_{z_1} + a_{12}(\delta p)_{z_2})_{z_1} \\
&= -(a_{12}^0(\delta\tilde{p}-\delta p)_{z_1} + a_{22}^0(\delta\tilde{p}-\delta p)_{z_2} + a_{12}(\delta p)_{z_1} + a_{22}(\delta p)_{z_2})_{z_2}. \qquad (5.6)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\delta\tilde{w})_{z_1} =& \frac{1}{k_1} h_1'\left(\frac{z_1}{k_1}\right) - e((\delta\tilde{p})_{z_1} + \frac{1}{k_1}(\delta\tilde{p})_{z_2})(\mathbf{z}^L) \\
&+ \int_{z_2^L}^{z_2} (a_{11}^0(\delta\tilde{p}-\delta p)_{z_1} + a_{12}^0(\delta\tilde{p}-\delta p)_{z_2} + a_{11}(\delta p)_{z_1} + a_{12}(\delta p)_{z_2})_{z_1}(z_1,s)ds \qquad (5.7) \\
=& \frac{1}{k_1} h_1'\left(\frac{z_1}{k_1}\right) - e\left((\delta\tilde{p})_{z_1} + \frac{1}{k_1}(\delta\tilde{p})_{z_2}\right)(\mathbf{z}^L) \qquad (5.8) \\
&- \int_{z_2^L}^{z_2} (a_{12}^0(\delta\tilde{p}-\delta p)_{z_1} + a_{22}^0(\delta\tilde{p}-\delta p)_{z_2} + a_{12}(\delta p)_{z_1} + a_{22}(\delta p)_{z_2})_{z_2}(z_1,s)ds \\
=& \frac{1}{k_1} h_1'\left(\frac{z_1}{k_1}\right) - e\left((\delta\tilde{p})_{z_1} + \frac{1}{k_1}(\delta\tilde{p})_{z_2}\right)(\mathbf{z}^L) \qquad (5.9) \\
&+ [(a_{12}^0(\delta\tilde{p}-\delta p)_{z_1} + a_{22}^0(\delta\tilde{p}-\delta p)_{z_2} + a_{12}(\delta p)_{z_1} + a_{22}(\delta p)_{z_2})]|_{\mathbf{z}}^{\mathbf{z}^L} \\
\leq& (2C_p N_1 + 4\mathcal{A}\varepsilon) r^{\alpha-1}. \qquad (5.10)
\end{aligned}
$$

Hence, by straightforward calculation, we have

$$
\|\delta\tilde{w}\|_{2,\alpha;(1+\beta,0);\mathbb{D}_0^+}^{(-\alpha;\mathbf{O})(-1-\alpha;\mathcal{L}_1)} \leq C_w N_1, \tag{5.11}
$$

where constant C_w depends only on C_p and the data. ◻

Up to now, we have obtained $\delta\tilde{p}$ and $\delta\tilde{w}$. Then, we can use the algebraic equations (3.28) and (3.30) with $y_2 = z_2$ given in coordinates transormation (3.36) to derive estimates of $\delta\tilde{\rho}$ and $\delta\tilde{u}_1$. Moreover, from the algebraic equations, we can also obtain the decay estimate of $\delta\tilde{\rho}$ and $\delta\tilde{u}_1$ only in the z_2-direction. That is the reason that we use the norm defined in (2.23) for $\delta\tilde{p}$ and $\delta\tilde{w}$. However, note that the expression of $(\delta\rho)_{z_1}$ and $(\delta u_1)_{z_1}$ is related to $(\delta p)_{z_1}$ and $(\delta w)_{z_1}$. Therefore, we can use the norm defined in (2.22) for $(\delta\rho)_{z_1}$ and $(\delta u_1)_{z_1}$. More precisely, by (4.38) and (4.54),

$$
S(z_2) = \frac{\frac{h_1 - \delta\tilde{w}}{e} + p_0^+}{(h_\rho - e_2\delta\tilde{w} - e_3\delta\tilde{p} + \rho_0^+)^\gamma}(k_1 z_2, z_2) \tag{5.12}
$$

and

$$
B(z_2) = \left(\frac{1}{2}\left|\mathbf{u}^-\right|^2 + \frac{\gamma p^-}{(\gamma-1)\rho^-}\right)(k_1 z_2, z_2). \tag{5.13}
$$

Then we have the following theorem.

THEOREM 5.2. *Assume* (3.34) *holds. There exists a unique* $\delta\tilde{\rho}$ *with* $\delta\tilde{\rho} = \tilde{\rho} - \rho_0^+$ *satisfying*

$$\frac{\tilde{p}}{\tilde{\rho}^{\gamma}}(\mathbf{z}) = S(z_2), \tag{5.14}$$

and the estimate

$$\|\delta\tilde{\rho}\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\mathbb{D}_0^+} + \|(\delta\tilde{\rho})_{z_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\beta,0);\mathbb{D}_0^+} \leq C_\rho N_1, \tag{5.15}$$

where C_ρ *is a constant depending only on* C_p.

Moreover, there exists a unique $\delta\tilde{u}_1$ *with* $\delta\tilde{u}_1 = \tilde{u}_1 - u_{10}^+$ *satisfying*

$$\tilde{u}_1 = \frac{\sqrt{2B(z_2) - \dfrac{2\gamma\tilde{p}}{(\gamma-1)\tilde{\rho}}}}{\sqrt{1+\tilde{w}^2}}, \tag{5.16}$$

and the estimate

$$\|\delta\tilde{u}_1\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\mathbb{D}_0^+} + \|(\delta\tilde{u}_1)_{z_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\beta,0);\mathbb{D}_0^+} \leq C_u N_1, \tag{5.17}$$

where C_u *is a constant depending only on* C_p.

Proof. For any $\mathbf{z} = (z_1, z_2) \in \mathbb{D}$, let $\mathbf{z}^H = (z_1^H, z_2^H)$ be the coordinates of the intersection point between Γ_{shock} and the horizontal line passing through $\mathbf{z}$, i.e., $(z_1^H, z_2^H) = (k_1 z_2, z_2)$. According to equation $(\frac{p}{\rho^\gamma})_{z_1} = 0$, $\tilde{p}$ and $\tilde{\rho}$ satisfy the following formula:

$$\frac{\tilde{p}}{\tilde{\rho}^{\gamma}}(\mathbf{z}) = \frac{\tilde{p}}{\tilde{\rho}^{\gamma}}(\mathbf{z}^H) = S(z_2). \tag{5.18}$$

By (4.53), we have on Γ_{shock},

$$\delta\tilde{\rho}(k_1 z_2, z_2) = (h_\rho - e_2\delta\tilde{w} - e_3\delta\tilde{p})(k_1 z_2, z_2), \tag{5.19}$$

where $h_\rho = \frac{\mathcal{H}_1}{e_{13}}, e_2 = \frac{e_{11}}{e_{13}}$, and $e_3 = \frac{e_{12}}{e_{13}}$. Then, it follows from (5.18) that

$$\begin{aligned}
\delta\tilde{\rho}(\mathbf{z}) &= \tilde{\rho}(\mathbf{z}) - \rho_0^+ \\
&= \left[\frac{\tilde{p}(\mathbf{z})}{\tilde{p}(\mathbf{z}^H)}\right]^{\frac{1}{\gamma}} \tilde{\rho}(\mathbf{z}^H) - \rho_0^+ \qquad (5.20)\\
&= \left[\frac{\tilde{p}(\mathbf{z})}{\tilde{p}(\mathbf{z}^H)}\right]^{\frac{1}{\gamma}} (\tilde{\rho}(\mathbf{z}^H) - \rho_0^+) + \left[\left(\frac{\tilde{p}(\mathbf{z})}{\tilde{p}(\mathbf{z}^H)}\right)^{\frac{1}{\gamma}} - 1\right]\rho_0^+ \\
&= \left[\frac{\tilde{p}(\mathbf{z})}{\tilde{p}(\mathbf{z}^H)}\right]^{\frac{1}{\gamma}} \delta\tilde{\rho}(k_1 z_2, z_2) + \left[\left(\frac{\tilde{p}(\mathbf{z})}{\tilde{p}(\mathbf{z}^H)}\right)^{\frac{1}{\gamma}} - 1\right]\rho_0^+ \\
&= \left[\frac{\tilde{p}(\mathbf{z})}{\tilde{p}(\mathbf{z}^H)}\right]^{\frac{1}{\gamma}} (h_\rho - e_2\delta\tilde{w} - e_3\delta\tilde{p})(\mathbf{z}^H) + \left[\left(\frac{\delta\tilde{p}(\mathbf{z}) + p_0^+}{\delta\tilde{p}(\mathbf{z}^H) + p_0^+}\right)^{\frac{1}{\gamma}} - 1\right]\rho_0^+. \qquad (5.21)
\end{aligned}$$

From the expression above, we can see that the $\delta\tilde{\rho}$ share the same decay in the z_2-direction and the regularity of the $\delta\tilde{p}$ and $\delta\tilde{w}$. Therefore, we have

$$\|\delta\tilde{\rho}\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\mathbb{D}} \leq C_9 N_1. \tag{5.22}$$

Moreover, it follows from (5.18) that $\delta\tilde{\rho}$ satisfies

$$\delta\tilde{\rho}(\mathbf{z}) = S(z_2)\tilde{p}(\mathbf{z})^{\frac{1}{\gamma}} - \rho_0^+. \tag{5.23}$$

Taking derivative on $\delta\tilde{\rho}$ with respect to z_1, we have

$$(\delta\tilde{\rho})_{z_1}(\mathbf{z}) = \frac{1}{\gamma}S(z_2)\tilde{p}(\mathbf{z})^{\frac{1}{\gamma}-1}(\delta\tilde{p})_{z_1}. \tag{5.24}$$

Then

$$\|(\delta\tilde{\rho})_{z_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\beta,0);\mathbb{D}_0^+} \leq C_{10}N_1. \tag{5.25}$$

So inequality (5.15) is established with constant C_ρ depending only on C_p.

The uniqueness of $\delta\tilde{\rho}$ is obvious, because of the uniqueness of $\delta\tilde{p}$ and $\delta\tilde{w}$, and (5.14).

The uniqueness and estimate of $\delta\tilde{u}_1$ in (5.17) can be derived by the same way as done for $\delta\tilde{\rho}$ by (5.16). ◻

Finally, we can use $\delta\tilde{U}$ to update the shock front by solving $\delta\tilde{s}'$.

THEOREM 5.3. *Assume* (3.34) *holds. There exists a unique* $\delta\tilde{s}$ *with* $\delta\tilde{s} = \tilde{s} - s_0 = \tilde{s} - k_1 z_2$ *satisfying*

$$\tilde{s}'(z_2) = \frac{[\tilde{u}_1\tilde{w}]}{[\tilde{p}]}(k_1z_2, z_2) \tag{5.26}$$

with $\delta\tilde{s}(0) = 0$*, and the estimate*

$$\|\delta\tilde{s}'\|^{(-\alpha;\mathbf{O})}_{2,\alpha;(1+\beta,0);\mathbb{R}^+} \leq C_sN_1, \tag{5.27}$$

where C_s *is a constant depending only on* C_p.

Proof. The unique existence of $\delta\tilde{s}$ of (5.26) is obvious by the unique existence of $\delta\tilde{U}$. Moreover, it follows from (5.26) and the fact that $w_0^+ = 0$ that

$$\begin{aligned}
\delta\tilde{s}'(z_2) &= \tilde{s}'(z_2) - s_0'(z_2)\\
&= \frac{\tilde{u}_1^+\tilde{w}^+ - \tilde{u}_1^-\tilde{w}^-}{\tilde{p}^+ - \tilde{p}^-} + \frac{u_{10}^-w_0^-}{p_0^+ - p_0^-}\\
&= \frac{(\tilde{u}_1^+\tilde{w}^+ - \tilde{u}_1^-\tilde{w}^-)(p_0^+ - p_0^-) + u_{10}^-w_0^-(\tilde{p}^+ - \tilde{p}^-)}{(\tilde{p}^+ - \tilde{p}^-)(p_0^+ - p_0^-)}\\
&= \frac{(\delta\tilde{u}\tilde{w}^+ + \delta\tilde{w}u_{10}^+ + T(U_s^-, U_0^-))(p_0^+ - p_0^-) + u_{10}^-w_0^-[\delta\tilde{p} + (p_0^- - \tilde{p}^-)]}{(\tilde{p}^+ - \tilde{p}^-)(p_0^+ - p_0^-)},
\end{aligned} \tag{5.28}$$

where

$$T(U_s^-, U_0^-) = u_{10}^-w_0^- - \tilde{u}_1^-\tilde{w}^-. \tag{5.29}$$

Therefore, we have

$$\begin{aligned}
&\|\delta\tilde{s}'\|^{(-\alpha;\mathbf{O})}_{2,\alpha;(1+\beta,0);\mathbb{R}^+} \leq C_{11}(\|\delta\tilde{u}_1\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\mathbb{D}} + \|\delta\tilde{u}_1\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\beta,0);\mathbb{D}} +\\
&\quad \|\delta\tilde{p}\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}} + \|\delta\tilde{w}\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}} + \|V^- - V_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-})\\
&\leq C_sN_1.
\end{aligned} \tag{5.30}$$

◻

6. Existence of solutions of the free boundary problem. We have already established the unique existence and estimates of $\delta\tilde{U}$ and $\delta\tilde{s}$, i.e., we have constructed the iteration map $\mathcal{I}(\delta U, \delta s) = (\delta\tilde{U}, \delta\tilde{s})$. Now we will prove the existence of solutions of the free boundary problem by the Schauder fixed point theorem as follows.

THEOREM 6.1. *Assume* $(\delta U, \delta s) = (\delta u_1, \delta\rho, \delta p, \delta w, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$. *For the iteration map* $\mathcal{I}(\delta U, \delta s) = (\delta\tilde{U}, \delta\tilde{s})$ *given in sections* 4.4 *and* 5, *if* $\mathcal{I}$ *is a compact continuous operator, and* $\mathcal{I}(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$, *then there exists a fixed point* $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$ *such that* $\mathcal{I}(\delta U, \delta s) = (\delta U, \delta s)$.

Now, we are going to show the following three facts one by one: $\mathcal{I}(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$, $\mathcal{I}$ is compact, and $\mathcal{I}$ is continuous. First, we will prove the following lemma.

LEMMA 6.1. *Let* $\mathcal{I}$ *be given as in Theorem* 6.1. *Let* $\widetilde{C} := \max\{C_p, C_w, C_\rho, C_u, C_s\}$, *and let* $C^* := \max\{1, K_1, K_3\}$. *Suppose* $\mathcal{A} > 7\widetilde{C}C^*$ *and* $\varepsilon < \frac{1}{\mathcal{A}^2+1}$; *then we have*

$$\mathcal{I}(\delta U, \delta s) = (\delta\tilde{U}, \delta\tilde{s}) \in \mathcal{F}^{\mathcal{A}\varepsilon}. \tag{6.1}$$

Proof. Based on the estimates in Theorems 4.2, 5.1, 5.2, and 5.3, we only need to prove that $\widetilde{C}N_1 \le \mathcal{A}\varepsilon$, where

$$N_1 := \|p_e - p_1\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} + \|V^- - V_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|\hat{h}_4\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} + \|f\|^{*(1;O)(1-\alpha,\Gamma_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{R}^+}. \tag{6.2}$$

It follows from (3.34), (4.25), and (4.70) that

$$\|V^- - V_0^-\|_{2,\alpha;(1+\beta,0);\Omega^-} + \|p_e - p_1\|^{(-\alpha;0)}_{2,\alpha;(1+\beta);\mathbb{R}^+} \le \varepsilon, \tag{6.3}$$

$$\|\hat{h}_4\|^{(1-\alpha;0)}_{1,\alpha;(2+\beta);\mathbb{R}^+} \le K_3[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)], \tag{6.4}$$

$$\|f\|^{*(1;O)(1-\alpha,\Gamma_{cont})}_{0,\alpha;(2+\beta,2+\beta);\mathbb{R}^+} \le K_1[(\mathcal{A}\varepsilon)^2 + (\mathcal{A}\varepsilon^2)]. \tag{6.5}$$

Therefore, let $C^* := \max\{1, K_1, K_3\}$; then

$$\widetilde{C}N_1 \le \widetilde{C}C^*[\varepsilon + \mathcal{A}\varepsilon^2 + (\mathcal{A}\varepsilon)^2].$$

Choosing $\mathcal{A} > 7\widetilde{C}C^*$ and $\varepsilon < \frac{1}{\mathcal{A}^2+1}$, we have

$$\widetilde{C}N_1 \le 3\widetilde{C}C^*\varepsilon < \mathcal{A}\varepsilon. \tag{6.6}$$

So we have $\mathcal{I}(\delta U, \delta s) = (\delta\tilde{U}, \delta\tilde{s}) \in \mathcal{F}^{\mathcal{A}\varepsilon}$. ∎

Next, we will prove the mapping $\mathcal{I}$ is compact. Define the Banach space $\mathcal{F}^*$:

$$\mathcal{F}^* := \{(f_1, f_2, f_3, f_4, f_5) : \|(f_1, f_2)\|_{\mathcal{F}_1^*} + \|(f_3, f_4)\|_{\mathcal{F}_2^*} + \|f_5\|_{\mathcal{F}_3^*} < \infty\},$$

where $\|\cdot\|_{\mathcal{F}_i^*}$ is defined in the same way as done in (3.32), except that α is replaced by α', where $0 < \alpha' < \alpha$. Then we have the following.

LEMMA 6.2. *For* $\mathcal{I}$ *given in Theorem* 6.1, *and for* $\mathcal{A}$ *and* ε *defined in Lemma* 6.1, *if* $0 < \alpha' < \alpha$, *the mapping* $\mathcal{I}$ *is well-defined in* $\mathcal{F}^{\mathcal{A}\varepsilon}$ *and* $\mathcal{F}^*$, *where* $\mathcal{F}^{\mathcal{A}\varepsilon}$ *is a nonempty, convex, and compact subset of* $\mathcal{F}^*$.

Proof. Because $0 < \alpha' < \alpha$, $\mathcal{F}^{\mathcal{A}\varepsilon}$ is a nonempty, convex, and compact subset of $\mathcal{F}^*$. Moreover, it is easy to see that $\mathcal{I}$ is well-defined in $\mathcal{F}^{\mathcal{A}\varepsilon}$ and $\mathcal{F}^*$. ∎

Finally, we prove that $\mathcal{I}$ is continuous in $\mathcal{F}^*$.

LEMMA 6.3. *Let $\mathcal{I}$ be given in Theorem* 6.1. *Let $\mathcal{A}$, ε, and $\mathcal{F}^*$ be defined in Lemmas* 6.1 *and* 6.2. *Then the mapping $\mathcal{I}$ is continuous in $\mathcal{F}^*$.*

Proof. Suppose that $\mathcal{I}$ is not continuous; then there exists an $\varepsilon_0 > 0$ for a sequence of $\{(\delta U_n, \delta s_n)\}_{n\in\mathbb{N}}$ and $(\delta U, \delta s)$ such that $(\delta U_n, \delta s_n) \to (\delta U, \delta s)$ in $\mathcal{F}^*$ as $n \to \infty$, and

$$\|\mathcal{I}(\delta U_n, \delta s_n) - \mathcal{I}(\delta U, \delta s)\|_{\mathcal{F}^*} > \varepsilon_0.$$

Since $\{\mathcal{I}(\delta U_n, \delta s_n)\}_{n\in\mathbb{N}} \in \mathcal{F}^*$, which is compact in a Banach space $\mathcal{F}$, which is defined in the same way as done for $\mathcal{F}^*$ except that α' is replaced by α'' with $0 < \alpha'' < \alpha'$, there exists a subsequence $\{\mathcal{I}(\delta U_{n_k}, \delta s_{n_k})\}_{k\in\mathbb{N}}$ of $\{\mathcal{I}(\delta U_n, \delta s_n)\}_{n\in\mathbb{N}}$ such that

$$\mathcal{I}(\delta U_{n_k}, \delta s_{n_k}) \to (\delta U_\kappa, \delta s_\kappa) \qquad \text{as } n_k \to \infty$$

in $\mathcal{F}$ with $(\delta U_\kappa, \delta s_\kappa) \in \mathcal{F}^*$.

According to the definition of $\mathcal{I}$ and the uniqueness of solutions $(\delta\tilde{U}, \delta\tilde{s})$, we know that

$$(\delta U_\kappa, \delta s_\kappa) = \mathcal{I}(\delta U, \delta s),$$

which is a contradiction. Therefore, $\mathcal{I}$ is continuous. ☐

Hence, by applying the Schauder fixed point theorem, Theorem 6.1 holds, i.e., there exists a fixed point $(\delta U, \delta s) \in \mathcal{F}^{\mathcal{A}\varepsilon}$ such that $\mathcal{I}(\delta U, \delta s) = (\delta U, \delta s)$.

Moreover, we can prove that the fixed point $(\delta U, \delta s)$ is actually a solution of the free boundary problem governed by (3.27), (3.29), and (3.31) with boundary conditions (3.4) and (3.10)–(3.13).

THEOREM 6.2. *Let $\mathcal{I}$ be given in Theorem* 6.1. *Let $\mathcal{A}$, ε, and $\mathcal{F}^*$ be defined in Lemmas* 6.1 *and* 6.2. *Then the fixed point $(\delta U, \delta s)$ is a solution of the free boundary problem governed by* (3.5)–(3.8) *with boundary conditions* (3.4) *and* (3.10)–(3.13).

Proof. Based on the construction, it is easy to see that the fixed point satisfies boundary conditions (4.38), (4.48), and (4.54), which are equivalent to the R-H conditions (3.10)–(3.13), and also satisfies (3.27) and (3.29) with boundary condition (3.4). For the fixed point, (3.31) is equivalent to (4.1) and (4.2). By Remark 5.1 and (5.3), it is easy to know that (4.2) holds. Because (3.27), (3.29), and (3.31) are equivalent to (3.5)–(3.8), the remaining task is to show (4.1) holds.

Note that (4.8) comes from $\partial_{z_1}(4.2) - \partial_{z_2}(4.1)$. Therefore, it follows from (4.8) and (4.2) that

$$\partial_{z_2}((\delta w)_{z_1} + a_{12}(\delta p)_{z_1} + a_{22}(\delta p)_{z_2}) = 0. \tag{6.7}$$

Define

$$\mathcal{W}(x_1) = (\delta w)_{z_1} + a_{12}(\delta p)_{z_1} + a_{22}(\delta p)_{z_2}. \tag{6.8}$$

Then (4.1) holds if and only if $\mathcal{W}(x_1) = 0$. Hence the rest of the proof shows for the fixed point, i.e., $(\delta U, \delta s) = (\delta\tilde{U}, \delta\tilde{s})$, we have

$$\mathcal{W}(x_1) = 0. \tag{6.9}$$

Let $h_3(\mathbf{z}) = \hat{h}_3(\hat{\mathbf{z}}(\mathbf{z}))$. By (4.63), we can obtain on Γ_{shock}

$$
\begin{aligned}
h_3(z_2) =& h_1'(z_2)\frac{dz_2}{d\hat{z}_2} - \Lambda^0\left[\left(\frac{\lambda_I^0}{\Lambda^0}a_{11}-1\right) - \frac{k_1\lambda_I^0}{\Lambda^0}a_{12}\right](\delta p)_{z_1}\frac{dz_1}{d\hat{z}_1}\\
&+\Lambda^0\left[k_1\lambda_I^0\left(\frac{1}{\Lambda^0\lambda_I^0}a_{22}-1\right)-\frac{1}{\Lambda^0}a_{12}\right](\delta p)_{z_2}\frac{dz_2}{d\hat{z}_2}\\
=& h_1'(z_2)\sqrt{\Lambda^0\lambda_I^0} - \Lambda^0\left[\left(\frac{\lambda_I^0}{\Lambda^0}a_{11}-1\right) - \frac{k_1\lambda_I^0}{\Lambda^0}a_{12}\right](\delta p)_{z_1}\sqrt{\frac{\Lambda^0}{\lambda_I^0}}\\
&+\Lambda^0\left[k_1\lambda_I^0\left(\frac{1}{\Lambda^0\lambda_I^0}a_{22}-1\right)-\frac{1}{\Lambda^0}a_{12}\right](\delta p)_{z_2}\sqrt{\Lambda^0\lambda_I^0}.
\end{aligned}
\tag{6.10}
$$

By (4.64), we can obtain that on Γ_{shock}, if $\delta\tilde{p} = \delta p$, then

$$
\begin{aligned}
h_3(z_2) &= (e_1k_1\lambda_I^0 + \Lambda^0)(\delta p)_{z_1}\frac{dz_1}{d\hat{z}_1} + (e_1 - \Lambda^0k_1\lambda_I^0)(\delta p)_{z_2}\frac{dz_2}{d\hat{z}_2}\\
&= (e_1k_1\lambda_I^0 + \Lambda^0)(\delta p)_{z_1}\sqrt{\frac{\Lambda^0}{\lambda_I^0}} + (e_1 - \Lambda^0k_1\lambda_I^0)(\delta p)_{z_2}\sqrt{\Lambda^0\lambda_I^0}.
\end{aligned}
\tag{6.11}
$$

Then it follows from (6.10) and (6.11) that on Γ_{shock}

$$
\begin{aligned}
h_1'(z_2) =& \frac{\Lambda^0}{\lambda_I^0}\left[\left(\frac{\lambda_I^0}{\Lambda^0}a_{11}-1\right) - \frac{k_1\lambda_I^0}{\Lambda^0}a_{12}\right](\delta p)_{z_1} - \Lambda^0\left[k_1\lambda_I^0\left(\frac{1}{\Lambda^0\lambda_I^0}a_{22}-1\right)-\frac{1}{\Lambda^0}a_{12}\right](\delta p)_{z_2}\\
&+\frac{1}{\lambda_I^0}(e_1k_1\lambda_I^0+\Lambda^0)(\delta p)_{z_1} + (e_1-\Lambda^0k_1\lambda_I^0)(\delta p)_{z_2}\\
=& (a_{11}+e_1k_1-k_1a_{12})(\delta p)_{z_1} + (e_1+a_{12}-k_1a_{22})(\delta p)_{z_2}
\end{aligned}
\tag{6.12}
$$

According to (4.48), we have

$$
\delta\tilde{w} = h_1 - e_1\delta\tilde{p} \qquad on\ \Gamma_{shock}. \tag{6.13}
$$

So if $(\delta\tilde{w}, \delta\tilde{p}) = (\delta w, \delta p)$, then we differentiate (6.13) along Γ_{shock} to obtain

$$
k_1(\delta w)_{z_1} + (\delta w)_{z_2} = h_1'(z_2) - e_1k_1(\delta p)_{z_1} - e_1(\delta p)_{z_2}. \tag{6.14}
$$

By (6.12), (6.14), and (4.2) we have

$$
\begin{aligned}
k_1(\delta w)_{z_1} &= h_1'(z_2) - e_1k_1(\delta p)_{z_1} - e_1(\delta p)_{z_2} - (\delta w)_{z_2}\\
&= h_1'(z_2) - (e_1k_1 + a_{11})(\delta p)_{z_1} - (e_1 + a_{12})(\delta p)_{z_2}\\
&= -k_1a_{12}(\delta p)_{z_1} - k_1a_{22}(\delta p)_{z_2}.
\end{aligned}
\tag{6.15}
$$

Therefore, we can obtain

$$
\mathcal{W}(x_1) = (\delta w)_{z_1} + a_{12}(\delta p)_{z_1} + a_{22}(\delta p)_{z_2} = 0.
$$

So for the fixed point, (6.9) holds. It completes the proof of this theorem. ∎

7. Proof of Theorem 3.1.

In this section, we are ready to prove the main theorem of this paper, Theorem 3.1.

Proof of Theorem 3.1. The existence result stated in Theorem 3.1 has already been established in Theorem 6.2, so we only need to show the asymptotic behaviors and uniqueness of the solutions. It will be devided into two steps.

1. In this step, we will show the asymptotic behaviors given in Theorem 3.1.

From the estimates of $(\delta\tilde{U},\delta\tilde{s})$, we can obtain the asymptotic behavior of the subsonic solution via the estimates we obtained above.

For p and u_2, by Theorems 4.2 and 5.1, we have

$$\|p-p_0^+\|^{(-\alpha;\mathbf{O}),(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}_0^+}+\|w\|^{(-\alpha;\mathbf{O}),(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\beta,0);\mathbb{D}_0^+}\leq CN_1. \tag{7.1}$$

So $p\to p_0^+$ and $u_2=u_1w\to 0$ with the decay rate $|\mathbf{z}|^{-1-\beta}$ as $|\mathbf{z}|\to\infty$.

For ρ and u_1, as state in Remark 3.1, (u_1,ρ) does not converge only in the z_2 direction. It can be seen by Theorem 5.2 that

$$\|\rho-\rho_0^+\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\mathbb{D}_0^+}+\|u_1-|\mathbf{u}_0^+|\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\beta);\mathbb{D}_0^+}\leq CN_1. \tag{7.2}$$

So $\rho\to\rho_0^+$ and $u_1\to|\mathbf{u}_0^+|$ with the decay rate $z_2^{-1-\beta}$ as $z_2\to\infty$.

2. In this step, we will show the uniqueness of the obtained solutions.

Suppose $(\delta U^1,\delta s^1),(\delta U^2,\delta s^2)\in\mathcal{F}^{\mathcal{A}\varepsilon}$ are two fixed points of the mapping $\mathcal{I}$. Define

$$(\delta U^E,\delta s^E)=(\delta U^1,\delta s^1)-(\delta U^2,\delta s^2) \tag{7.3}$$

with $(\delta U^E,\delta s^E)=(\delta u_1^E,\delta p^E,\delta w^E,\delta\rho^E,\delta s^E)$.

Let $\delta\hat{p}^i(\hat{\mathbf{z}})=\delta p^i(\mathbf{z}(\hat{\mathbf{z}}))$, and $(\delta\hat{U}^i(\hat{\mathbf{z}}),\delta\hat{s}^i(\hat{\mathbf{z}}))=(\delta U^i(\mathbf{z}(\hat{\mathbf{z}}),\delta s^i(\mathbf{z}(\hat{\mathbf{z}}))$ for $i=1,2$. Moreover, we define $\hat{a}^k_{ij}(\hat{\mathbf{z}}):=\hat{a}_{ij}((\delta\hat{U}^k(\hat{\mathbf{z}}),(\delta\hat{s}^k)'(\hat{\mathbf{z}})))$ for $k=1,2$, and $\hat{h}^i_j(\hat{\mathbf{z}}):=\hat{h}_j((\delta\hat{U}^i(\hat{\mathbf{z}}),(\delta\hat{s}^i)'(\hat{\mathbf{z}})))$, $j=0,1,2,3,4$.

Based on the construction of the iteration mapping $\mathcal{I}$, we know that $\delta\hat{p}^i$ satisfies (4.11) for $i=1,2$. Therefore, we have

$$\Delta(\delta\hat{p}^E)=\hat{f}, \tag{7.4}$$

where $\hat{f}=-[\hat{a}^1_{ij}(\hat{\mathbf{z}})-\hat{a}^2_{ij}(\hat{\mathbf{z}})]_{\hat{z}_j}(\delta\hat{p}^1)_{\hat{z}_i}+(\hat{a}^2_{ij}(\hat{\mathbf{z}}))_{\hat{z}_j}(\delta\hat{p}^E)_{\hat{z}_i}-[\hat{a}^1_{ij}(\hat{\mathbf{z}})-\hat{a}^2_{ij}(\hat{\mathbf{z}})](\delta\hat{p}^1)_{z_iz_j}+[\hat{a}^2_{ij}(\hat{\mathbf{z}})-\delta_{ij}](\delta\hat{p}^E)_{z_iz_j}$.

On Γ_{cont}, $\delta\hat{p}^E$ satisfies

$$\delta\hat{p}^E=0. \tag{7.5}$$

Moreover, on Γ_{shock}, it follows from (4.65) that $\delta\hat{p}^E$ satisfies

$$\begin{aligned}M\delta\hat{p}^E=\bigtriangledown(\delta\hat{p}^E)\cdot\nu=\frac{1}{\sqrt{\mu_1^2+\mu_2^2}}\{[(\hat{h}^1_1)'(\hat{\mathbf{z}})-(\hat{h}^2_1)'(\hat{\mathbf{z}})]+g((\delta\hat{U}^1,(\delta\hat{s}^1)'),(\delta\hat{p}^1)_{z_i})\\ -g((\delta\hat{U}^2,(\delta\hat{s}^2)'),(\delta\hat{p}^2)_{z_i})\},\end{aligned} \tag{7.6}$$

where

$$g((\delta\hat{U}^j,(\delta\hat{s}^j)'),(\delta\hat{p}^j)_{z_j})=-\Lambda^0((\hat{a}^j_{11}-1)-k_2\hat{a}^j_{12})(\delta\hat{p}^j)_{\hat{z}_1}+\Lambda^0(k_2(\hat{a}^j_{22}-1)-\hat{a}^j_{12})(\delta\hat{p}^j)_{\hat{z}_2}. \tag{7.7}$$

Denote the right-hand side of (7.6) as $\hat{h}_7$, i.e., $\hat{h}_7=\hat{h}^1_4(\hat{\mathbf{z}})-\hat{h}^2_4(\hat{\mathbf{z}})$. Then (7.6) can be rewritten as

$$M\delta\hat{p}^E=\bigtriangledown(\delta\hat{p}^E)\cdot\nu=\hat{h}_7. \tag{7.8}$$

Let norm $\|\cdot\|_{\hat{\mathcal{F}}}$ be defined in (3.32) with β being replaced by $\frac{\beta}{3}$. Set

$$N_2 = \|(\delta U^E, \delta s^E)\|_{\hat{\mathcal{F}}}.$$

Obviously, we can conclude the proof of uniqueness if we can show $N_2 = 0$.
On the contrary, assume that $N_2 > 0$. When $|\hat{\mathbf{z}}| \geq 1$, we have

$$\begin{aligned}
|\hat{f}| =& |-[\hat{a}^1_{ij}(\hat{\mathbf{z}}) - \hat{a}^2_{ij}(\hat{\mathbf{z}})]_{\hat{z}_j}(\delta\hat{p}^1)_{\hat{z}_i} + (\hat{a}^2_{ij}(\hat{\mathbf{z}}))_{\hat{z}_j}(\delta\hat{p}^E)_{\hat{z}_i} - [\hat{a}^1_{ij}(\hat{\mathbf{z}}) - \hat{a}^2_{ij}(\hat{\mathbf{z}})](\delta\hat{p}^1)_{z_iz_j} \\
&+ [\hat{a}^2_{ij}(\hat{\mathbf{z}}) - \delta_{ij}](\delta\hat{p}^E)_{z_iz_j}| \\
\leq& |[\hat{a}^1_{ij}(\hat{\mathbf{z}}) - \hat{a}^2_{ij}(\hat{\mathbf{z}})]_{\hat{z}_j}||(\delta\hat{p}^1)_{\hat{z}_i}| + |(\hat{a}^2_{ij}(\hat{\mathbf{z}}))_{\hat{z}_j}||(\delta\hat{p}^E)_{\hat{z}_i}| + |[\hat{a}^1_{ij}(\hat{\mathbf{z}}) - \hat{a}^2_{ij}(\hat{\mathbf{z}})]||(\delta\hat{p}^1)_{z_iz_j}| \\
&+ |[\hat{a}^2_{ij}(\hat{\mathbf{z}}) - \delta_{ij}]||(\delta\hat{p}^E)_{z_iz_j}| \\
\leq& K_7|[(\delta U^E, \delta s'^E)]_{\hat{z}_j}||(\delta\hat{p}^1)_{\hat{z}_i}| + |(\hat{a}^2_{ij}(\hat{\mathbf{z}}))_{\hat{z}_j}||(\delta\hat{p}^E)_{\hat{z}_i}| + K_8|(\delta U^E, \delta s'^E)||(\delta\hat{p}^1)_{z_iz_j}| \\
&+ |[\hat{a}^2_{ij}(\hat{\mathbf{z}}) - \delta_{ij}]||(\delta\hat{p}^E)_{z_iz_j}| \\
\leq& K_7\mathcal{A}\varepsilon\|(\delta U^E, \delta s'^E)\|_{\hat{\mathcal{F}}}(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\frac{\beta}{3}}(d^{\hat{\mathbf{z}}}_{\infty})^{-2-\frac{\beta}{3}} \\
\leq& CN_2\varepsilon(d^{\hat{\mathbf{z}}}_{\infty,\Gamma_{cont}})^{-2-\frac{\beta}{3}}(d^{\hat{\mathbf{z}}}_{\infty})^{-2-\frac{\beta}{3}}
\end{aligned} \tag{7.9}$$

and

$$\|\hat{f}\|^*_{0,\alpha,(2+\frac{\beta}{3},2+\frac{\beta}{3});\mathbb{D}\cap\{|\hat{\mathbf{z}}|\geq 1\}} \leq CN_2\varepsilon \tag{7.10}$$

where K_7, K_8, and C are constants.

Similarly, we can get

$$\|g((\delta\hat{U}^1, (\delta\hat{s}^1)'), (\delta\hat{p}^1)_{z_i}) - g((\delta\hat{U}^2, (\delta\hat{s}^2)'), (\delta\hat{p}^2)_{z_i})\}\|_{1,\alpha,(2+\frac{\beta}{3}),\mathbb{R}^+\cap\{|\hat{\mathbf{z}}|\geq 1\}} \leq CN_2\varepsilon \tag{7.11}$$

and

$$\|\hat{h}^1_1 - \hat{h}^2_1\|_{2,\alpha;(1+\frac{\beta}{3});\mathbb{R}^+\cap\{|\hat{\mathbf{z}}|\geq 1\}} \leq CN_2\varepsilon. \tag{7.12}$$

So

$$\|\hat{h}_7\|_{1,\alpha,(2+\frac{\beta}{3}),\mathbb{R}^+\cap\{|\hat{\mathbf{z}}|\geq 1\}} \leq CN_2\varepsilon. \tag{7.13}$$

By the same argument, when $|\hat{\mathbf{z}}| < 1$, we have

$$\|\hat{f}\|^{*(1,O)(1-\alpha,\Gamma_{const})}_{0,\alpha,\mathbb{D}\cap\{|\hat{\mathbf{z}}|<1\}} \leq CN_2\varepsilon \tag{7.14}$$

and

$$\|\hat{h}_7\|^{(1-\alpha;0)}_{1,\alpha;\mathbb{R}^+\cap\{|\hat{\mathbf{z}}|<1\}} \leq CN_2\varepsilon. \tag{7.15}$$

Therefore, we obtain

$$\|\hat{f}\|^{*(1,O)(1-\alpha,\Gamma_{const})}_{0,\alpha,(2+\frac{\beta}{3},2+\frac{\beta}{3});\mathbb{D}} + |\hat{h}_7\|^{(1-\alpha;0)}_{1,\alpha;(2+\frac{\beta}{3});\mathbb{R}^+} \leq CN_2\varepsilon. \tag{7.16}$$

Now Theorem 4.1 yields that $\|\delta\hat{p}^E\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\frac{\beta}{3},0);\mathbb{D}} < CN_2\varepsilon$.

Based on the estimate of $\delta\hat{p}^E$, we will consider the estimates of $(\delta U^E, \delta s^E)$.

First, when $|\hat{\mathbf{z}}| > 1$, it follows from (5.3) that

$$\begin{aligned}|\delta\hat{w}^E| =& \Big|(\hat{h}_1^1 - \hat{h}_1^2)\left(\frac{\hat{z}_1}{k_1}\right) - e(\delta\hat{p}^E)\left(\hat{z}_1, \frac{\hat{z}_1}{k_1}\right) + \int_{\hat{z}_2^L}^{\hat{z}_2} [(\hat{a}_{11}^1(\delta\hat{p}^1)_{\hat{z}_1} - \hat{a}_{11}^2(\delta\hat{p}^2)_{\hat{z}_1}) \\ & + (\hat{a}_{12}^1(\delta\hat{p}^1)_{\hat{z}_2} - \hat{a}_{12}^2(\delta\hat{p}^2)_{\hat{z}_2})](\hat{z}_1, s)ds\Big| \\ =& \Big|(\hat{h}_1^1 - \hat{h}_1^2)\left(\frac{\hat{z}_1}{k_1}\right) - e(\delta\hat{p}^E)\left(\hat{z}_1, \frac{\hat{z}_1}{k_1}\right) + \int_{\hat{z}_2^L}^{\hat{z}_2} \sum_{j=1}^{2} [(\hat{a}_{1j}^1 - \hat{a}_{1j}^2)(\delta\hat{p}^1)_{\hat{z}_j} \\ & - |\hat{a}_{1j}^2|(\delta\hat{p}^E)_{\hat{z}_j}](\hat{z}_1, s)ds\Big| \\ \le& |\hat{h}_1^1 - \hat{h}_1^2| + e|\delta\hat{p}^E| + K_9|(\delta U^E, \delta s'^E)| \int_0^{\hat{z}_2^L} \sum_{j=1}^{2} |(\delta\hat{p}^1)_{\hat{z}_j}|(\hat{z}_1, s)ds + K_{10}\int_0^{\hat{z}_2^L} \\ & \times \sum_{j=1}^{2} |(\delta\hat{p}^E)_{\hat{z}_j}|(\hat{z}_1, s)ds. \end{aligned} \tag{7.17}$$

Therefore,

$$\|\delta\hat{w}^E\|_{2,\alpha;(1+\frac{\beta}{3},0);\mathbb{D}\cap\{|\hat{\mathbf{z}}|>1\}} < CN_2\varepsilon. \tag{7.18}$$

Next, by (5.21),

$$\begin{aligned}|\delta\hat{\rho}^E| =& \Bigg|\left[\frac{\hat{p}^1(\hat{\mathbf{z}})}{\hat{p}^1(\hat{\mathbf{z}}^H)}\right]^{\frac{1}{\gamma}}(\hat{h}_\rho^1 - e_2\delta\hat{w}^1 - e_3\delta\hat{p}^1)(\hat{\mathbf{z}}^H) - \left[\frac{\hat{p}^2(\hat{\mathbf{z}})}{\hat{p}^2(\hat{\mathbf{z}}^H)}\right]^{\frac{1}{\gamma}}(\hat{h}_\rho^2 - e_2\delta\hat{w}^2 - e_3\delta\hat{p}^2)(\hat{\mathbf{z}}^H) \\ & + \left[\left(\frac{\delta\hat{p}^1(\mathbf{z}) + \hat{p}_0^+}{\delta\hat{p}^1(\mathbf{z}^H) + \hat{p}_0^+}\right)^{\frac{1}{\gamma}} - \left(\frac{\delta\hat{p}^2(\mathbf{z}) + \hat{p}_0^+}{\delta\hat{p}^2(\mathbf{z}^H) + \hat{p}_0^+}\right)^{\frac{1}{\gamma}}\right]\hat{\rho}_0^+\Bigg| \\ \le& |[\frac{\hat{p}^1(\hat{\mathbf{z}})}{\hat{p}^1(\hat{\mathbf{z}}^H)}]^{\frac{1}{\gamma}}[(\hat{h}_\rho^1 - \hat{h}_\rho^2) - e_2\delta\hat{w}^E - e_3\delta\hat{p}^E](\hat{\mathbf{z}}^H) \\ & + K_{11}[(\hat{h}_\rho^2 - e_2\delta\hat{w}^2 - e_3\delta\hat{p}^2)(\hat{\mathbf{z}}^H) + \hat{\rho}_0^+][\hat{p}^1(\hat{\mathbf{z}}^H)\delta p^E(\hat{\mathbf{z}}) - \hat{p}^1(\hat{\mathbf{z}})\delta p^E(\hat{\mathbf{z}^H})]| \\ \le& K_{12}[|\hat{h}_\rho^1 - \hat{h}_\rho^2| + |\delta\hat{w}^E| + |\delta\hat{p}^E|] + K_{13}|\delta\hat{p}^E|, \end{aligned} \tag{7.19}$$

where $K_9 - K_{13}$ are constants.

Note that

$$\|\hat{h}_\rho^1 - \hat{h}_\rho^2\|_{2,\alpha;(1+\frac{\beta}{3});\mathbb{R}^+\cap\{|\hat{\mathbf{z}}|\ge1\}} \le CN_2\varepsilon. \tag{7.20}$$

Then we have

$$\|\delta\hat{p}^E\|^*_{2,\alpha;(0,1+\frac{\beta}{3});\mathbb{D}\cap\{|\hat{\mathbf{z}}|>1\}} < CN_2\varepsilon. \tag{7.21}$$

Moreover, it follows from (5.24) that

$$|(\delta\hat{\rho}^E)_{\hat{z}_1}| \le C|(\delta\hat{p}^E)_{\hat{z}_1}|. \tag{7.22}$$

So

$$\|(\delta\hat{\rho}^E)_{\hat{z}_1}\|_{1,\alpha;(2+\frac{\beta}{3},0);\mathbb{D}\cap\{|\hat{\mathbf{z}}|>1\}} < CN_2\varepsilon. \tag{7.23}$$

Then by combining (7.21) and (7.23), we have

$$\|\delta\hat{p}^E\|^*_{2,\alpha;(0,1+\frac{\beta}{3});\mathbb{D}\cap\{|\hat{\mathbf{z}}|>1\}} + \|(\delta\hat{\rho}^E)_{\hat{z}_1}\|_{1,\alpha;(2+\frac{\beta}{3},0);\mathbb{D}\cap\{|\hat{\mathbf{z}}|>1\}} < CN_2\varepsilon. \tag{7.24}$$

Finally, for $\delta\hat{u}_1^E$, similarly to (5.16), we have

$$\|\delta\hat{u}_1^E\|^*_{2,\alpha;(0,1+\frac{\beta}{3});\mathbb{D}\cap\{|\hat{\mathbf{z}}|>1\}} + \|(\delta\hat{u}_1^E)_{\hat{z}_1}\|_{1,\alpha;(2+\frac{\beta}{3},0);\mathbb{D}\cap\{|\hat{\mathbf{z}}|>1\}} < CN_2\varepsilon. \tag{7.25}$$

In addition, when $|\hat{\mathbf{z}}| < 1$, following the same argument as above, we have

$$\begin{aligned}
&\|\delta\hat{w}^E\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;\mathbb{D}\cap\{|\hat{\mathbf{z}}|<1\}} < CN_2\varepsilon,\\
&\|\delta\hat{\rho}^E\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,\mathbb{D}\cap\{|\hat{\mathbf{z}}|<1\}} + \|(\delta\hat{\rho}^E)_{\hat{z}_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,\mathbb{D}\cap\{|\hat{\mathbf{z}}|<1\}} \le CN_2\varepsilon,\\
&\|\delta\hat{u}_1^E\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,\mathbb{D}\cap\{|\hat{\mathbf{z}}|<1\}} + \|(\delta\hat{u}_1^E)_{\hat{z}_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,\mathbb{D}\cap\{|\hat{\mathbf{z}}|<1\}} \le CN_2\varepsilon.
\end{aligned} \tag{7.26}$$

In conclusion, it follows from (7.18)–(7.26) that

$$\begin{aligned}
&\|\delta\hat{w}^E\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\frac{\beta}{3},0);\mathbb{D}} < CN_2\varepsilon,\\
&\|\delta\hat{\rho}^E\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\frac{\beta}{3});\mathbb{D}} + \|(\delta\hat{\rho}^E)_{\hat{z}_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\frac{\beta}{3},0);\mathbb{D}} \le CN_2\varepsilon,\\
&\|\delta\hat{u}_1^E\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\frac{\beta}{3});\mathbb{D}} + \|(\delta\hat{u}_1^E)_{\hat{z}_1}\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\frac{\beta}{3},0);\mathbb{D}} \le CN_2\varepsilon.
\end{aligned} \tag{7.27}$$

Now, by (5.26) and (5.28), it is easy to see that

$$\begin{aligned}
\|(\delta\hat{s}^E)'\|^{(-\alpha;\mathbf{O})}_{2,\alpha;(1+\frac{\beta}{3},0);\mathbb{R}^+} &\le K_{12}\big(\|\delta\hat{u}_1^E\|^{*(-\alpha,\Gamma_{cont})}_{2,\alpha,(0,1+\frac{\beta}{3});\mathbb{D}} + \|\delta\hat{u}_1^E\|^{(1-\alpha,\Gamma_{cont})}_{1,\alpha,(2+\frac{\beta}{3},0);\mathbb{D}}\\
&\quad + \|\delta\hat{p}^E\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\frac{\beta}{3},0);\mathbb{D}} + \|\delta\hat{w}^E\|^{(-\alpha;\mathbf{O})(-1-\alpha;\Gamma_{cont})}_{2,\alpha;(1+\frac{\beta}{3},0);\mathbb{D}}\big)\\
&\le CN_2\varepsilon.
\end{aligned} \tag{7.28}$$

Hence we obtain $\|(\delta U^E, \delta s^E)\|_{\hat{\mathcal{F}}} \le C\varepsilon\|(\delta U^E, \delta s^E)\|_{\hat{\mathcal{F}}}$. That is, $N_2 \le C\varepsilon N_2$.

Choose ε sufficiently small, so that $C\varepsilon = \frac{1}{2}$. Then $N_2 = \|(\delta U^E, \delta s'^E)\|_{\hat{\mathcal{F}}} = 0$, which contradicts the assumption that $N_2 > 0$.

It completes the proof of this theorem. ∎

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