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Xu Zhang[1,a] and Guanrong Chen[2,b]

AFFILIATIONS

[1]Department of Mathematics, Shandong University, Weihai 264209, Shandong, China
[2]Department of Electrical Engineering, City University of Hong Kong, Hong Kong SAR, China

Note: This article is part of the Focus Issue, Non-smooth Dynamics.
[a]**Author to whom correspondence should be addressed:** xu_zhang_sdu@mail.sdu.edu.cn
[b]**Email:** eegchen@cityu.edu.hk

ABSTRACT

A simple topological model describing the chaotic dynamics of two coupled neurons is established and analyzed based on the Smale horseshoe theory.

A non-smooth model is introduced to describe the interaction of two neurons, where the non-smoothness is caused by the impulse effects. This model is described by a modified Smale horseshoe and represented by a one-sided symbolic dynamical system. It is found that a simple neuronal model can present extremely complex chaotic dynamics, which nevertheless can be rigorously analyzed by mathematical chaos theory.

I. INTRODUCTION

Many biological systems can be modeled and explained by nonlinear systems. One well-known example is the Hodgkin–Huxley equation defined on a four-dimensional vector field.[1] Another familiar example is the brain, which contains billions of neurons with complex functions.[2]

The impulse function is a useful mathematical tool in the study of the neuronal dynamics, where it describes an effect causing sudden changes that only last for a very short time or instantaneously. Dynamical systems with impulse functions are typical non-smooth systems, existing in various fields of applications such as engineering, physics, biology, finance, and economics.

There are many progressive results in the study of non-smooth systems. A comprehensive review on one-parameter non-smooth bifurcations for continuous-time piecewise-smooth dynamical systems was presented in Ref. 3. A detailed review on border-collision bifurcations for piecewise-smooth maps was given in Ref. 4. Moreover, a variational analysis of non-smooth systems was reported in Ref. 5. In fact, there is a considerable number of works on bifurcation, periodic orbits, attractors, and applications in many fields, such as circuits and systems, mechanics, and so on.[6–11] Some of these studies investigate chaos, for instance, a class of non-smooth Chen systems with impulsive forces.[12] More importantly, for the brain and neural networks,[13] the existence of Smale horseshoe in the Hodgkin–Huxley model was studied, e.g., in Ref. 14. The dynamical behavior of a class of horseshoe-like non-smooth maps is shown to be topologically conjugate to a symbolic dynamical system in the forward direction.[15]

If the impulsive effects are dependent on time, namely, if the systems are non-autonomous, some promising results have been obtained. The homoclinic structure of non-autonomous systems was found and analyzed in Ref. 16. The Melnikov method with non-autonomous perturbation was applied to a Hamiltonian system having a transverse homoclinic point such that the perturbed system has a horseshoe.[17] The attractors of non-autonomous systems with impulsive perturbations were studied in Ref. 18. Several types of attractors were discussed for non-autonomous systems Refs. 19–21.

The artificial neural network is a simulation and computational model for the connecting neurons and the biological processes in the human brain. The first neural network model using electrical circuits was suggested by McCulloch and Pitts in 1943,[22] which inspired the application of neural networks to artificial intelligence. The computational model based on biological neural networks can learn and describe the relationships between inputs and outputs of many nonlinear and complicated systems, reveal hidden relationships, patterns, and predictions, and has been applied to computer vision, speech recognition, machine translation, social network filtering, medical diagnosis, solving equations, deep learning toward big data problems, and so on.[23–29]

The chaotic dynamics of the brain were studied from different aspects based on different models, including experiments, and data analysis.[30,31] Even the possibility of controlling the chaotic dynamics in the brain was considered.[32] In fact, there are various models and numerical evidence on the existence of chaotic dynamics in the brain.[33–35]

Noticeably, the simplest model of a single neuron with chaotic dynamics was studied in Ref. 36. In this article, a theoretical model of two interacting neurons is proposed and investigated, which is defined by a non-smooth system, where the existence of Smale horseshoe is verified and analyzed. This model may help explain the high instability in the brain.

The new topological model is established spatiotemporal impulsive effects, which is found to have horseshoe that is somewhat different from the classical Smale horseshoe. This kind of new horseshoe can be thought of as a non-autonomous horseshoe since the timing of the impulse might not be periodic. Furthermore, this kind of horseshoe is only one-directional, where the system orbits can be obtained through one direction, while the horseshoe in other directions might be empty. Similarly to the classical Smale horseshoe, the dynamics of the new horseshoe can also be presented by a one-sided symbolic dynamical system.

The rest of the article is organized as follows. In Sec. II, some basic concepts are introduced. In Sec. III, the non-smooth system is formulated and the chaotic dynamical behavior is analyzed.

II. PRELIMINARY

In this section, some basic concepts are introduced.

Definition 2.1: (Convex Set) A subset C of $\mathbb{R}^2$ is said to be convex if, for any $x, y \in C$ and any $\lambda \in [0,1]$, one has $\lambda x + (1-\lambda)y \in C$; that is, the closed line segment containing x and y is contained in C whenever $x, y \in C$.

The convex hull of a set S, denoted by $\overline{co}(S)$, is the smallest convex subset of $\mathbb{R}^n$ containing S.

Definition 2.2: Given a vector field f: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, the Filippov set-valued map $\mathscr{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, with $x \to \mathscr{F}(t,x)$, is defined by

$$\mathscr{F}(t,x) = \bigcap_{\varepsilon>0} \bigcap_{\substack{N\subset\mathbb{R}^n;\\ \mu(N)=0}} \overline{co}[f(t, \mathbb{B}(x,\varepsilon)) \setminus N)],$$

where $\overline{co}$ denotes the convex hull, μ is the Lebesgue measure on $\mathbb{R}^n$, and $\mathbb{B}(x,\varepsilon) = x + \varepsilon\mathbb{B}$, with $\mathbb{B}$ being the unit ball centered at the origin.

Filippov's problem is formulated by replacing the discontinuous Cauchy problem

$$\dot{x}(t) = f(t, x(t)),\ x(0) = x_0,$$

with the differential inclusion

$$\dot{x}(t) \in \mathscr{F}(t, x(t)),\ x(0) = x_0,$$

where $\mathscr{F}$ is introduced above. A Filippov solution on $[0, T] \subset \mathbb{R}$ is an absolutely continuous function satisfying the above inclusion for almost all $t \in [0, T]$. Under certain conditions, solutions exist (Chaps. 2 and 3 of Ref. 5).

Impulses defined by the Dirac delta function[37] have been introduced to dynamical systems, which is a generalized function or a distribution, denoted by $\delta(\cdot)$, with $\delta(t) = 0$ for any $t \neq 0$ and $\int_{\mathbb{R}} \delta(t)dt = 1$.[38–40]

One definition of chaos is introduced, which will be useful in the sequel.

Definition 2.3 (Li–Yorke chaos): Let (X, d) be a metric space, $f: X \to X$ a map, and S a subset of X with at least two points. Then, S is a scrambled set of f if, for any two distinct points $x, y \in S$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

The map f is said to be chaotic in the sense of Li–Yorke if there exists an uncountable scrambled set S of f in (X, d).

Next, the one-sided symbolic dynamical system is introduced.[41]

Let $S_0 := \{0, 1\}$. The one-sided sequence space

$$\textstyle\sum_2^+ := \{\alpha = (a_0, a_1, a_2, \ldots) : \ a_i \in S_0,\ i \geq 0\}$$

is a metric space equipped with the distance

$$d(\alpha, \beta) = \sum_{i=0}^{\infty} \frac{d(a_i, b_i)}{2^i}, \tag{2.1}$$

where $\alpha = (a_0, a_1, a_2, \ldots)$, $\beta = (b_0, b_1, b_2, \ldots) \in \sum_2^+$, $d(a_i, b_i) = 1$ if $a_i \neq b_i$, and $d(a_i, b_i) = 0$ if $a_i = b_i$, $i \geq 0$. Then, $(\sum_2^+, d)$ is a complete metric space and also a Cantor set.

Define the shift map $\sigma : \sum_2^+ \to \sum_2^+$ by $\sigma(\alpha) = (a_1, a_2, \ldots)$, where $\alpha = (a_0, a_1, \ldots)$. Then, $(\sum_2^+, \sigma)$ is called the one-sided symbolic dynamical system on two symbols.

The dynamical system $(\sum_2^+, \sigma)$ is chaotic in the sense of Li–Yorke.[41,42]

III. TOPOLOGICAL HORSESHOE

In this section, a specific non-smooth differential equation is introduced, and its complex dynamics are analyzed.

Consider the following subsets of $\mathbb{R}^2$:

$$U = [-3, 3] \times [-3, 3],\ V = [6, 8] \times [-1, 1],\ W = [-4, 4] \times [1, 2].$$

Construct a linear map from V to W, as illustrated by Fig. 4, where V is represented by the red rectangle, W is represented by the blue rectangle, and the set W is the image of V under the linear map. Since V and W are rectangles, it suffices to consider the following linear functions defined on the boundaries of V and W:

(i) $g_1 : [6, 8] \times \{1\} \to [-4, 4] \times \{2\}$

$$g_1(x, y) = (4x - 28,\ 2);$$

(ii) $g_2 : \{8\} \times [-1, 1] \to \{4\} \times [1, 2]$

$$g_2(x, y) = \left(4,\ \frac{y+3}{2}\right);$$

(iii) $g_3 : [6,8] \times \{-1\} \to [-4,4] \times \{1\}$

$$g_3(x,y) = (4x - 28,\ 1);$$

(iv) $g_4 : \{6\} \times [-1,1] \to \{-4\} \times [1,2]$

$$g_4(x,y) = \left(-4,\ \frac{y+3}{2}\right).$$

For any $(x,y) \in V$, pick two different points on the boundary of V, denoted as (x_1,y_1) and (x_2,y_2), such that these three points are on the same line. Without loss of generality, assume that $(x_1,y_1) \in [6,8] \times \{1\}$ and $(x_2,y_2) \in [6,8] \times \{-1\} \to [-4,4] \times \{1\}$ and $(x,y) = \lambda(x_1,y_1) + (1-\lambda)(x_2,y_2)$ with some $\lambda \in [0,1]$. Then, define

$$g_0(x,y) = \lambda g_1(x_1,y_1) + (1-\lambda) g_3(x_2,y_2).$$

This g_0 is a linear continuous function, since V and W are convex sets.

Remark 3.1: The use of this linear function is only for the convenience of discussion. In real models, this function g_0 could be a nonlinear one.

Define a function

$$G(x,y) = \begin{cases} g_0(x,y) - (x,y) & \text{if } (x,y) \in V, \\ 0 & \text{other.} \end{cases}$$

Remark 3.2: The function G is not continuous, but it can be easily extended to a continuous one. For simplicity, only consider this simple expression below. Note that its extension might bring some "noise" orbits different from the ones discussed below, where by "noise" orbits, it means that some of their dynamical behavior might be uncontrollable.

For clarity, the function G is written as $G = (G_1, G_2)$.

Given a sequence of real numbers, $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, with $t_{k+1} > t_k$ and $\lim_{k\to+\infty} t_k = +\infty$, denote

$$x(t_k^{\pm}) = \lim_{t\to t_k\pm 0} x(t) \text{ and } y(t_k^{\pm}) = \lim_{t\to t_k\pm 0} y(t).$$

Given a sequence of subsets $\{V_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$, a sequence of functions $G^k = (G_1^k, G_2^k) : V_k \to \mathbb{R}^2$, $k \in \mathbb{N}$, and a continuous function $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$, a class of impulse systems depending on both the position (state) and time can be defined as follows:

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} & \text{if } (x,y) \notin V_k \text{ or } t \neq t_k, \\ \begin{pmatrix} x(t_k^+) \\ y(t_k^+) \end{pmatrix} = \begin{pmatrix} x(t_k^-) \\ y(t_k^-) \end{pmatrix} + \begin{pmatrix} G_1^k(x(t_k^-), y(t_k^-)) \\ G_2^k(x(t_k^-), y(t_k^-)) \end{pmatrix} & \text{if } (x(t_k^-), y(t_k^-)) \in V_k \text{ and } t = t_k. \end{cases}$$

For the model to be studied in this paper, $V = V_k$ and $G^k = G = (G_1, G_2)$, $k \in \mathbb{N}$. Specifically, the following impulse system is considered, where the impulse is dependent on both the position (state) and time,

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } (x,y) \notin V \text{ or } t \neq t_k, \\ \begin{pmatrix} x(t_k^+) \\ y(t_k^+) \end{pmatrix} = \begin{pmatrix} x(t_k^-) \\ y(t_k^-) \end{pmatrix} + \begin{pmatrix} G_1(x(t_k^-), y(t_k^-)) \\ G_2(x(t_k^-), y(t_k^-)) \end{pmatrix} & \text{if } (x(t_k^-), y(t_k^-)) \in V \text{ and } t = t_k. \end{cases}$$

Remark 3.3: The subsets U and V can be thought of as two neurons. Thus, this simple model can be regarded as two interacting neurons if the impulsive functions are appropriately defined. For example, for biological composition and neuronal function, the impulsive function plays an important role in the bio-system.[2]

Remark 3.4: The effects of the impulse function are written in this form such that one can understand the action of impulses clearly. Actually, there are several different forms for the expressions of impulse systems.[43–46]

Remark 3.5: The well-posedness, or the existence and possible uniqueness of the solutions, can be derived following the approach in Ref. 5. Here, only the geometric aspects of the solutions, namely, the formation of the horseshoe, will be discussed.

Denote the solution by ϕ and set

$$f_k(\cdot) := \phi(t_k^+, \cdot),\ k \geq 1.$$

The inverse of $\phi(t_k^+, \cdot)$ is denoted by $f_k^{-1}(\cdot)$. Consider a non-autonomous system induced by the solution ϕ, expressed as

$$f = (f_1, f_2, f_3, \ldots).$$

Theorem 3.1: For the above system with any given time sequence $\{t_k\}_{k\in\mathbb{N}}$, if $t_1 \geq \frac{1}{2}$ and $t_{k+1} - t_k \geq \frac{1}{2}$, $k \in \mathbb{N}$, then there is a Cantor set Λ, on which f is topologically semi-conjugate to the one-sided symbolic dynamical system on two symbols such that the relations shown in Fig. 1 hold,

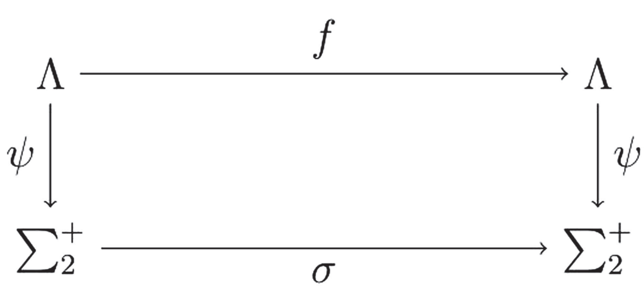

FIG. 1. The semi-conjugate relationship of maps.

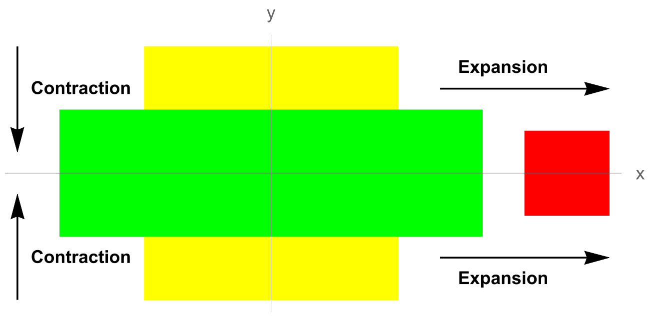

FIG. 2. Illustration of the formation of the horseshoe, where the yellow rectangle represents the set U, the red rectangle represents the set V, and the green rectangle is the image of U under the flow within a short time, the contraction is along the vertical axes, and the expansion is along the horizon axes.

In Fig. 1, ψ is a continuous surjective map from Λ to $\sum_2^+$ and for any point $z \in \Lambda$ and any positive integer k, one has $f^k(z) = f_k(z)$. The relation in Fig. 1 is interpreted as

$$\psi \circ f_k(z) = \sigma^k \circ \psi(z), \ \forall k \in \mathbb{N}.$$

Therefore, the system is chaotic in the sense of Li–Yorke.[47]

Figures 2–4 illustrate the formulation of the horseshoe, where the yellow rectangle represents the set U, the red rectangle represents the set V, and the blue rectangle represents the set W. The green rectangle is the evolution of the set U through the equation. Note that the expanding direction is along the x-axes, and the contracting direction is along the y-axes.

In Fig. 2, the green rectangle is the image of U through the equation within a short time, where the set U does not intersect the set V. In Fig. 3, the green rectangle is the image of U through the equation such that it intersects the set V, where the impulse does not appear. In Fig. 4, the impulse brings the set of points in V back to the set U. Figure 5 shows the effect of Figs. 2–4.

The assumptions of $t_1 \geq \frac{1}{2}$ and $t_{k+1} - t_k \geq \frac{1}{2}$, $k \in \mathbb{N}$ are made so as to allow the formation of the horseshoe; in other words, the neurons need some time to process the information. In Figs. 2 and 3, the formation of the horseshoe needs to satisfy the following

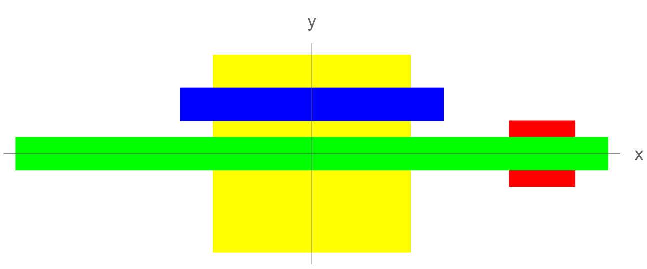

FIG. 3. Illustration of the formation of the horseshoe, where the yellow rectangle represents the set U, the red rectangle represents the set V, the blue rectangle represents the set W, and the green rectangle is the image of U under the flow such that the image intersects the set V before the effects of the impulses.

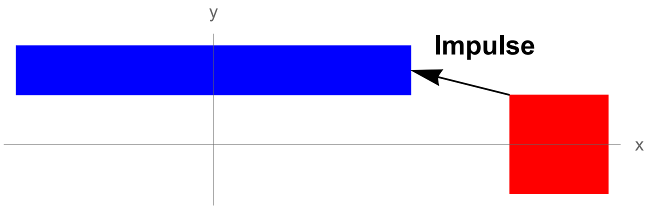

FIG. 4. Illustration of the formation of the horseshoe, where the red rectangle represents the set V, the blue rectangle represents the set W, and the set W is the image of V through the flow with the effects of the impulses.

inequalities:

$$\begin{cases} 3e^{2t_0} \geq 8, \\ 2e^{-2t_0} \leq 1, \end{cases}$$

where the time t_0 makes the formation of the horseshoe; in other words, the points in $W \cap U$ need time t_0 such that they can move into V through the flow. The above inequalities hold if $t_0 \geq \frac{1}{2}$, since $e \approx 2.718\,281\,828\,459$, and

$$\begin{cases} 3e^{2t_0} \geq 3e^1 \geq 3 \times 2.7 = 8.1 > 8, \\ 2e^{-2t_0} \leq 2e^{-1} < \frac{2}{2.7} < 1. \end{cases}$$

Remark 3.6: The horseshoe constructed above is different from the well-known Smale horseshoe. The above equation might be a non-autonomous system since $t_{k+1} - t_k$ might not be a fixed constant for $k \in \mathbb{N}$. The above horseshoe is an autonomous system if and only if $t_{k+1} - t_k$ is a fixed constant.

Next, set

$$D := U \cap W.$$

Remark 3.7: The set $\bigcap_{n=0}^{+\infty} f_n(D)$ might be empty, where $f_0(\cdot) = \text{id}$ is the identity map. Note that the choice of t_k might lead this set to be empty. Nevertheless, the interesting part is the set $\bigcap_{n=0}^{+\infty} f_n^{-1}(D)$, as further discussed below.

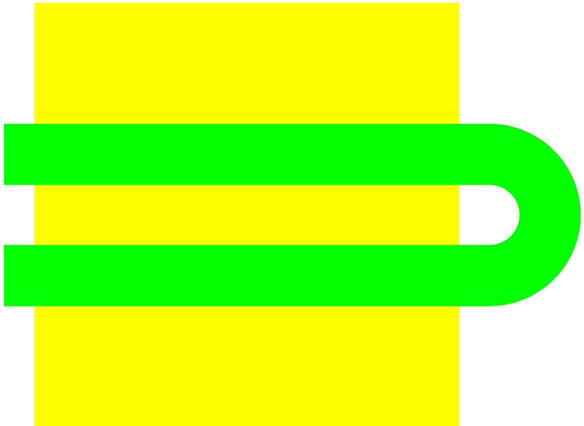

FIG. 5. Illustration of the formation of the horseshoe.

Now, construct the set $\bigcap_{n=0}^{+\infty} f_n^{-1}(D)$ inductively. For the classical Smale horseshoe, the construction of the invariant set can be found in, for example, Ref. 48. There exist a uniform contraction factor and a uniform expansion factor, where these two factors together establish the topological conjugacy between the invariant set and the two-sided symbolic dynamical systems. However, the arguments here can only be used to establish the topological semi-conjugacy between the invariant set and one-sided symbolic dynamical system, which fortunately suffices to show the existence of chaos in the system.

Specifically, consider $D \cap f_1^{-1}(D)$. With the formation of the horseshoe shown in Figs. 2–4, this set consists of the two horizontal rectangles H_0 and H_1, denoted as follows:

$$D \cap f_1^{-1}(D) = \bigcup_{s_0 \in S} H_{s_0} = \{p \in D : \ p \in H_{s_0},\ s_0 \in S\}.$$

Consider the set $D \cap f_1^{-1}(D) \cap f_2^{-1}(D)$. Note that the inverse of f might not be well defined at some points; nevertheless, here only the points that have meaningful properties are considered.

Under the above assumptions, one can see that this is well defined when the interest is only in the set $U \cup V$, and the time here is restricted only to the impulse instants t_k.

By the definition of f, when restricted to the set $V \cup U$, the following identity holds:

$$D \cap f_1^{-1}(D) \cap f_2^{-1}(D) = D \cap f_1^{-1}(D \cap f_1 \circ f_2^{-1}(D)).$$

Remark 3.8: In the above expression, $f_2 \circ f_1^{-1}$ represents the formation of the horseshoe from instant t_1^+ to instant t_2^+, as illustrated in Figs. 2–5, where $f_1 \circ f_2^{-1}$ is the inverse of $f_2 \circ f_1^{-1}$ restricted to $U \cup V$. Similarly, in the following discussions, $f_{i+1} \circ f_i^{-1}$ represents the formation of the horseshoe from instant t_i^+ to instant t_{i+1}^+, where $f_i \circ f_{i+1}^{-1}$ is the inverse of $f_{i+1} \circ f_i^{-1}$ restricted to $U \cup V$.

Note that $D \cap f_1 \circ f_2^{-1}(D)$ contains two disjoint components. This, taking into account also the above discussions on $D \cap f_1^{-1}(D)$, leads to the conclusion that $D \cap f_1^{-1}(D \cap f_1 \circ f_2^{-1}(D))$ contains 2^2 disjoint components. It is evident that

$$D \cap f_1^{-1}(D \cap f_1 \circ f_2^{-1}(D)) \subset D \cap f_1^{-1}(D) = H_0 \cup H_1.$$

Furthermore, one has

$$\begin{aligned}
&\{p \in D : \ p \in H_{s_0},\ f_2 \circ f_1^{-1}(p) \in H_{s_1},\ s_i \in S,\ i = 0, 1\} \\
&\quad = \bigcup_{\substack{s_i \in S \\ i=0,1}} (f_1^{-1} \circ f_2(H_{s_1}) \cap H_{s_0}) = \bigcup_{\substack{s_i \in S, \\ i=0,1}} H_{s_0 s_1} \\
&\quad = D \cap f_1^{-1}(D \cap f_1 \circ f_2^{-1}(D)) = D \cap f_1^{-1}(D) \cap f_2^{-1}(D).
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
&D \cap f_1^{-1}(D) \cap f_2^{-1}(D) \cap f_3^{-1}(D) \\
&\quad = D \cap f_1^{-1}(D) \cap f_2^{-1}(D \cap f_2 \circ f_3^{-1}(D)) \\
&\quad = D \cap f_1^{-1}(D \cap f_1 \circ f_2^{-1}(D \cap f_2 \circ f_3^{-1}(D)))
\end{aligned}$$

and

$$\begin{aligned}
&\{p \in D : \ p \in H_{s_0},\ f_2 \circ f_1^{-1}(p) \in H_{s_1},\ f_3 \circ f_2^{-1}(p) \in H_{s_2}, \\
&\quad s_i \in S,\ i = 0, 1, 2\} \\
&\quad = \bigcup_{\substack{s_i \in S \\ i=0,1,2}} (f_2 \circ f_3^{-1}(H_{s_1 s_2}) \cap H_{s_0}) = \bigcup_{\substack{s_i \in S \\ i=0,1,2}} H_{s_0 s_1 s_2} \\
&\quad = D \cap f_1^{-1}(D) \cap f_2^{-1}(D) \cap f_3^{-1}(D).
\end{aligned}$$

Inductively, one obtains

$$\begin{aligned}
&\{p \in D : \ p \in H_{s_0},\ f_2 \circ f_1^{-1}(p) \in H_{s_1}, \ldots, f_i \circ f_{i-1}^{-1}(p) \in H_{s_i}, \\
&\quad s_i \in S,\ i = 0, 1, \ldots, k-1\} \\
&\quad = \bigcup_{\substack{s_i \in S \\ i=0,1,2,\ldots,k-1}} (f_{k-1} \circ f_k^{-1}(H_{s_1 s_2 \cdots s_{k-1}}) \cap H_{s_0}) = \bigcup_{\substack{s_i \in S, \\ i=0,1,2,\ldots,k-1}} H_{s_0 s_1 \cdots s_{k-1}} \\
&\quad = D \cap f_1^{-1}(D) \cap f_2^{-1}(D) \cap f_3^{-1}(D) \cap \cdots \cap f_k^{-1}(D).
\end{aligned}$$

Let $s_0 \cdots s_k \cdots$ be any infinite sequence of 0′s and 1′s. Then, $H_{s_0 \cdots s_k \cdots}$ corresponds to a nonempty set. The topological semi-conjugacy is defined for all $p \in H_{s_0 \cdots s_k \cdots}$, so the image of p in the one-sided symbolic space is

$$\psi : p \to s_0 \cdots s_k \cdots .$$

By the assumption of $f_k(p) \in H_{s_k}$, one has the topological semi-conjugacy.

Denote

$$\Lambda = \bigcap_{n=0}^{+\infty} f_n^{-1}(D).$$

Now, it can be shown that ψ is continuous and surjective.

Recall that d in (2.1) is a metric in the symbolic system. For simplicity of discussion, assume that d_E is the Euclidean metric on the plane. For any two subsets $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$, the distance between A and B is given by

$$d_E(A, B) = \inf_{a \in A,\ b \in B} d_E(a, b).$$

To show that the conjugacy map is continuous, for any $z \in \Lambda$, suppose that A_z is a connected component in Λ containing z. By the construction above, $\psi(A_z)$ is the same element in $\sum_2^+$. Given any $z_0 \in \Lambda$, assume that $\psi(A_{z_0}) = \alpha_0 \in \sum_2^+$.

Now, it can be shown that for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $d(A_{z_0}, A_z) < \delta$ with $z \in \Lambda$, one has $|\psi(A_{z_0}) - \psi(A_z)| < \varepsilon$. For this ε, by using the distance in (2.1), there is a positive integer N, such that for any $0 \le i \le N$, the first N symbols of $\psi(A_{z_0})$ and $\psi(A_z)$ are the same. Note that the system without the impulsive effects is contracting along the vertical direction and is expanding along the horizontal direction. This, together with the construction above, implies the existence of such a δ. Hence, the conjugacy map is continuous. The surjectivity of ψ can be deduced directly from the above construction.

IV. CONCLUSION

In this article, a topological model for the coupling of two neurons is constructed to illustrate the existence of chaotic dynamics of neuron systems. The mechanism illustrated by this model might be used to establish practical multi-neuron models depending on both position (state) and time, where the position can also be time-varying, which will be useful for real applications.

This model can also be extended to a general neural network depending on both state and time. Since the classical neural network is very useful in many fields, the generalized impulsive neural network should provide a powerful tool for further applications.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Xu Zhang: Writing – original draft (equal). **Guanrong Chen:** Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] A. L. Hodgkin and A. F. Huxley, "A quantitative description of membrane current and its applications to conduction and excitation in nerve," J. Physiol. (Lond.) **116**, 500–544 (1952).
[2] W. Gerstner, W. M. Kistler, R. Naud, and L. Paninski, *Neuronal Dynamics: From Single Neurons to Networks and Models of Cognition* (Cambridge University Press, Cambridge, 2014).
[3] M. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk, A. B. Nordmark, G. O. Tost, and P. T. Piiroinen, "Bifurcations in nonsmooth dynamical systems," SIAM Rev. **50**, 629–701 (2008).
[4] D. Simpson, "Border-collision bifurcations in $\mathbb{R}^n$," SIAM Rev. **58**(2), 177–226 (2016).
[5] S. Adly, *A Variational Approach to Nonsmooth Dynamics: Applications in Unilateral Mechanics and Electronics* (Springer, Berlin, 2017).
[6] F. Angulo, C. Ocampo, G. Olivar, and R. Ramos, "Nonlinear and nonsmooth dynamics in a DC-DC buck converter: Two experimental set-ups," Nonlinear Dyn. **46**, 239–257 (2006).
[7] S. Banerjee, M. S. Karthik, G. Yuan, and J. A. Yorke, "Bifurcations in one-dimensional piecewise smooth maps–theory and applications in switching circuits," IEEE Trans. Circuits Syst. I **47**, 389–394 (2000).
[8] M. Dutta, H. E. Nusse, E. Ott, J. A. Yorke, and G. Yuan, "Multiple attractor bifurcations: A source of unpredictability in piecewise smooth systems," Phys. Rev. Lett. **83**, 4281–4284 (1999).
[9] G. Li, Y. Yue, J. Xie, and C. Grebogi, "Strange nonchaotic attractors in a nonsmooth dynamical system," Commun. Nonlinear Sci. Numer. Simul. **78**, 104858 (2019).
[10] D. Simpson, "Sequences of periodic solutions and infinitely many coexisting attractors in the border-collision normal form," Int. J. Bifurcation Chaos **24**, 1430018 (2014).
[11] D. Weiss, T. Kpper, and H. A. Hosham, "Invariant manifolds for nonsmooth systems," Physica D **241**(22), 1895–1902 (2012).
[12] H. Jiang, L. Zhang, Z. Chen, and Q. Bi, "Non-smooth bifurcation analysis of Chen system via impulsive force," Acta. Phys. Sin. **61**(8), 82–89 (2012).
[13] *The Handbook of Brain Theory and Neural Networks*, 2nd ed., edited by M. A. Arbib (The MIT Press, Cambridge, 2003).
[14] J. Guckenheimer and R. A. Oliva, "Chaos in the Hodgkin-Huxley model," SIAM J. Appl. Dyn. Syst. **1**, 105–114 (2002).
[15] J. J. B. Biemond, A. P. S. de Moura, C. Grebogi, N. van de Wouw, and H. Nijmeijer, "Dynamical collapse of trajectories, part II: Limit set of a horseshoe-like map," in *Proceedings of the 8th European Nonlinear Dynamics Conference (ENOC 2014)* (Vienna University of Technology, Vienna, 2014).
[16] L. M. Lerman and L. P. Shil'nikov, "Homoclinical structures in nonautonomous systems: Nonautonomous chaos," Chaos **2**, 447–454 (1992).
[17] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics and Chaos* (CRC Press, Florida, 1999).
[18] B. Schmalfuss, "Attractors for nonautonomous and random dynamical systems perturbed by impulses," Discret. Contin. Dyn. Syst. **9**, 727–744 (2003).
[19] D. N. Cheban, P. E. Kloeden, and B. Schmalfuss, "The relationship between pullback, forward and global attractors of nonautonomous dynamical systems," Nonlinear Dyn. Syst. Theor. **2**, 125–144 (2002).
[20] V. V. Chepyzhov, "Uniform attractors of dynamical processes and non-autonomous equations of mathematical physics," Russ. Math. Surv. **68**, 159–196 (2013).
[21] X. Zhang, "Regular nonchaotic attractors with positive plural," Int. J. Bifurcation Chaos **26**, 1650241 (2016).
[22] W. McCulloch, and W. Pitts, "A logical calculus of the ideas immanent in nervous activity," Bull. Math. Biophys. **5**, 115–133 (1943).
[23] J. J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," Proc. Natl. Acad. Sci. U.S.A. **81**, 3088–3092 (1984).
[24] J. J. Hopfield and D. W. Tank, "Neural computation of decisions optimization problems," Biol. Cybernet **52**, 141–152 (1985).
[25] I. E. Lagaris, A. Likas, and D. I. Fotiadis, "Artificial neural networks for solving ordinary and partial differential equations," IEEE Trans. Neural Netw. **9**, 987–1000 (1998).
[26] W. Liu, Z. Wang, X. Liu, N. Zeng, Y. Liu, and F. E. Alsaadi, "A survey of deep neural network architectures and their applications," Neurocomputing **234**, 11–26 (2017).
[27] R. Rojas, *Neural Networks: A Systematic Introduction* (Springer, Berlin, 1996).
[28] J. Schmidhuber, "Deep learning in neural networks: An overview," Neural Netw. **61**, 85–117 (2015).
[29] T. P. Vogels, K. Rajan, and L. F. Abbott, "Neural network dynamics," Annu. Rev. Neurosci. **28**, 357–376 (2005).
[30] *Brain Dynamics: Progress and Perspectives*, Springer Series in Brain Dynamics, edited by E. Başar and T. H. Bullock (Springer, Berlin, 1989).
[31] H. Korn and P. Faure, "Is there chaos in the brain? II. Experimental evidence and related models," C. R. Biologies **326**, 787–840 (2003).
[32] S. J. Schiff, K. Jerger, D. H. Duong, T. Chang, M. L. Spano, and W. L. Ditto, "Controlling chaos in the brain," Nature **370**, 615–620 (1994).
[33] H. Preissl, W. Lutzenberger, and F. Pulvermüller, "Is there chaos in the brain?," Behav. Brain Sci. **19**, 307–308 (1996).
[34] *Emergent Neural Computational Architectures Based on Neuroscience: Towards Neuroscience-Inspired Computing*, edited by S. Wermter, J. Austin, and D. Willshaw (Springer-Verlag, Berlin, 2001).
[35] J. J. Wright and D. T. J. Liley, "Dynamics of the brain at global and microscopic scales: Neural networks and the EEG," Behav. Brain Sci. **19**, 285–320 (1996).
[36] K. Aihara, T. Takabe, and M. Toyoda, "Chaotic neural networks," Phys. Lett. A **144**, 333–340 (1990).
[37] P. Dirac, *The Principles of Quantum Mechanics* (Oxford Press, Oxford, 1958).
[38] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists* (Academic Press, Boston, 2000).

[39] I. M. Gel'fand and G. E. Shilov, *Generalized Functions, 1–5* (Academic Press, Boston, 1966–1968).
[40] L. Schwartz, *Théorie des Distribution 1* (Hermann, 1950).
[41] B. P. Kitchens, *Symbolic Dynamics, One-sided, Two-sided and Countable State Markov Shifts* (Springer-Verlag, New York, 1998).
[42] F. Blanchard, E. Glasner, S. Kolyada, and A. Maass, "On Li-Yorke pairs," J. Reine Angew. Math. **547**, 51–68 (2002).
[43] A. N. Churilov, A. Medvedev, and Z. T. Zhusubaliyev, "Impulsive Goodwin oscillator with large delay: Periodic oscillations, bistability, and attractors," Nonlin. Anal. Hybrid Syst. **21**, 171–183 (2016).
[44] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides* (Kluwer, Dordrecht, 1988).
[45] P. Kowalczyk, M. di Bernardo, A. R. Champneys, S. J. Hogan, M. Homer, Y. A. Kuznetsov, A. Nordmark, and P. T. Piiroinen, "Two-parameter non-smooth bifurcations of limit cycles: Classification and open problems," Int. J. Bifurcation Chaos **16**, 601–629 (2006).
[46] A. P. Kuznetsov, L. V. Turukina, and E. Mosekilde, "Dynamical systems of different classes as models of the kicked nonlinear oscillator," Int. J. Bifurcation Chaos **11**, 1065–1077 (2001).
[47] X. Zhang and G. Chen, "A geometric criterion for the existence of chaos based on periodic orbits in continuous-time autonomous systems," J. Dyn. Contr. Syst. (published online 2022).
[48] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer-Verlag, New York, 1990).