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On a Class of Multi-Source Distributed Storage With Exact Repair

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ABSTRACT In future communication networks, say 5G networks and Internet of Things, users may need to obtain messages from multiple available sources in a distributed manner, which requires distributed storage in the network. One essential issue in distributed storage is how to repair a failed storage node. In conventional distributed storage model with exact repair, source files are stored in many nodes in a distributed manner and the contents of a failed node are exactly recovered by the surviving nodes. Further, all files can be reconstructed when a decoder has access to a certain number of storage nodes. So, the underlying reconstruction network is equivalent to a single-source multicast problem. This paper considers a variant of the exact repair problem, where the underlying reconstruction network is the independent distributed source coding system, a type of multi-source multicast problem. As the first non-trivial case with two sources and three encoders, the storage-repair tradeoff regions are proved for all the 33 non-isomorphic instances, and it is shown that binary codes are optimal.

INDEX TERMS Independent distributed source coding, exact repair, distributed storage, binary codes.

I. INTRODUCTION

With the rapid development of communication networks and the 5G-enabled Internet of Things, users would like to obtain their desired messages from all available resources. For instance, they may obtain information from multiple neighboring nodes. To achieve this goal, the network may store information on many data servers (e.g., base stations or data-center caches) and/ or other devices nearby in a distributed manner so that one user can get the information from a subset of the available sources. Thus, the distributed storage problem has drawn a lot of attention in recent years.

In [1], they consider a system that stores files coded in a distributed manner on $n$ disks such that any $k$ disks can reconstruct all the files and a failed disk can be repaired by any other $d$ disks. If the repaired disk only needs to work functionally the same as the failed disk, it is called functional repair. If the repaired disk needs to store the same content as the failed disk, it is called exact repair. The measure of interest is the tradeoff between the disk capacity $\alpha$ and the repair bandwidth $\beta$ that limits the information transmitted from each helper in the repair process. The tradeoff regions for functional repair and exact repair have been proved to be different in general [2].

Though both functional and exact repair can be viewed as network coding [3] problems, the exact repair is much more complicate than the functional repair. The functional repair problem can be reduced to single source multicast network coding problem and can be solved by some known codes with easy implementations [4], [5], while the exact repair is not the case. The low-density parity-check (LDPC) codes [6], [7] are also applied to the distributed storage [8]–[11]. Some other progresses on the exact repair problems can be found in [12]–[15]. We will consider the exact repair in this paper as well.

In the conventional setup, the reconstruction part is actually a single source problem, since all decoders require all the source files. It is usually not the case in practice. Therefore, in this paper, we are interested in a variant on the...
reconstruction part, where decoders may decode different subsets of the files. We formulate the underlying reconstruction part as a multi-source multicast problem. Specifically, we regard the reconstruction part as an independent distributed source coding (IDSC) problem [16]–[19], which is a class of multi-source multicast network problem [20]. When the sources have priorities, the IDSC problem can be viewed as the classical multilevel diversity coding system [21]–[23], which was the initial motivating model for network coding and further distributed storage. We denote our problem model as IDSC with exact repair (IDSCER). We are interested in the tradeoff between storage and repair bandwidth in terms of the source entropies. Since this is multi-source problem, we will not normalize the region as in most conventional storage setup. Instead, we will keep the source entropies in the obtained tradeoff regions. One special case was studied in [24] and the inner bound achieved by binary codes was given. We will provide the outer bound on this special case to get its exact region later in this paper.

The main contributions of this paper are:
1) the tradeoff region of general IDSCER is formulated (§II);
2) all 33 two-source three-encoder IDSCER instances are solved and it is shown that binary linear codes are optimal (§III);
3) the results show that the inner bound provided in [24] is actually the exact region of the problem considered in it (§III).

The following notations are adopted in this paper: Normal uppercase letters are usually referred to variables while the bolded cases are vectors and calligraphic ones are sets. Lowercase letters are used as indices. $H(\cdot)$ is used as the entropy function. $\lfloor \cdot \rfloor$ is a vector and $\{ \cdot \}$ is a set. $\lceil \cdot \rceil$ is used for the cardinality of a set. $\lceil \cdot \rceil$ is the ceiling notation. The upper index with parenthesis is a notation for block. For instance, $X^{(N)}$ means a blocks of $N$ outcomes of variable $X$.

II. PROBLEM FORMULATION

We start with an introduction to the model of IDSC with exact repair (IDSCER), and then define its tradeoff region.

A. SYSTEM MODEL

The IDSCER model considered in this paper can be viewed as an IDSC problem [16] with exact repair constraints on the encoders, or equivalently, an exact repair problem in distributed storage [1] with the reconstruction part being an IDSC problem. Specifically, in an IDSCER (as shown in Fig. 1), there are $K$ independent sources $X_1, \ldots, X_K$, which are coded in a distributed manner by $n$ encoders $E_1, \ldots, E_n$ with same capacities $\alpha$. In the reconstruction process, a decoder $D_i$ has access to a subset $\text{Fan}(D_i)$ of the encoders and needs to recover a subset $\tau(D_i)$ of the sources, where $\tau(D_i)$ can be different for different $D_i$. Denote the set of all decoders as $D$. In the repair process, when an encoder fails, every size-$d$ subset of the surviving encoders can exactly repair a failed node, with the repair bandwidth $\beta$ at each helper node. We denote the coded (or stored) messages on encoders as $W_1, \ldots, W_n$ and the transmitted message from the helper $E_i$ to repair a failed encoder, say $E_j$, as $S_{ij}$.

For a fixed $(K, n)$ tuple, there exist many valid IDSC instances. Details about the constraints on possible reconstructions can be found in [17]. The tuple $(K, n, d)$ is used to denote all IDSCER instances with $K$ sources, $n$ encoders, and $d$ helpers in the exact repair process when an encoder fails.

Next, we formulate the tradeoff region of the IDSCER.

B. TRADEOFF REGION FORMULATION

For a $(K, n, d)$ IDSCER instance, we first define an $(N, \omega, \alpha, \beta)$ block code, with $\omega = \{H(X_1), \ldots, H(X_K)\}$ and $N$ as the block length.

(i) The $K$ mutually independent block i.i.d. sources $X_i^{(N)}$, $i \in \{1, \ldots, K\}$ are uniformly distributed in $X_i = \{1, \ldots, 2^{NH(X_i)}\}$. 

---

FIGURE 1. Diagram of an IDSC with exact repair: the reconstruction part is an IDSC problem while the exact repair constraints are added.
(ii) The block encoders, one for each encoder $i$, are functions that map a block of $N$ source observations from all sources to one of $[2^{N\alpha}]$ different descriptions in $\mathcal{W}_i = \{1, \ldots, [2^{N\alpha}]\}$.

$$\phi_i^{(N)}: \prod_{i \in \{1, \ldots, k\}} \mathcal{X}_i \rightarrow \mathcal{W}_j, \quad j \in \{1, \ldots, n\}. \quad (1)$$

(iii) The exact repair constraints indicate that when an encoder $E_j$ fails, any $d$ helpers $E_d$ can exactly repair $E_j$ with transmitted messages $S_d^j, E_d \in E_d$ from the helper nodes. That is, for any $E_d \subseteq \{E_1, \ldots, E_n\} \setminus \{E_j\}, |E_d| = d, j \in \{1, \ldots, n\}$,

$$\psi_j^{(N)}: \prod_{E_i \in E_d^j} S_i^j \rightarrow \mathcal{W}_j, \quad (2)$$

where each transmitted message $S_i^j$ for repair is a mapping from the corresponding stored message,

$$f_i^j: \mathcal{W}_i \rightarrow S_i^j, \quad S_i^j = \{1, \ldots, [2^{N\beta_i}]\}. \quad (3)$$

In addition, it is required that the repaired content should be exactly the same as stored in $E_j$, that is,

$$\psi_j^{(N)}(\prod_{E_i \in E_d^j} S_i^j) = \phi_j^{(N)}(\prod_{i \in \{1, \ldots, k\}} \mathcal{X}_i). \quad (4)$$

(iv) The block decoders are functions that map observations of $\text{Fan}(D_i)$ to the sources $\tau(D_i)$,

$$\mu_i^{(N)}: \prod_{j \in \text{Fan}(D_i)} \mathcal{W}_j \rightarrow \prod_{k \in \tau(D_i)} \mathcal{X}_k, \quad (5)$$

with the reconstructed messages being the same as block observations.

For the code above, we denote the estimate at sink side as $\hat{\mathbf{X}}_i = \mu_i^{(N)}(\text{Fan}(D_i))$ and define the probability of error as

$$p^{(N)} = \max_{i \in \{1, \ldots, |\mathcal{W}|\}} P(\hat{\mathbf{X}}_i \neq \tau(D_i)). \quad (6)$$

Then, we define the achievability of a rate vector as follows.

**Definition 1:** A vector $(\omega, \alpha, \beta)$ is achievable if there exist a sufficiently large $N$, a sequence of encoders $\{\phi_i^{(N)} = [\phi_i^{(N)}]|i = 1, \ldots, n]\}$ satisfying the storage and repair bandwidth limits, a sequence of repair functions $\psi_i^{(N)} = [\psi_i^{(N)}]|i \in \{1, \ldots, n\}$, and decoders $\{\mu_i^{(N)} = [\mu_i^{(N)}]|i = 1, \ldots, |\mathcal{W}|\}$ satisfying the exact repair and reconstruction constraints, with a vanishing error $p^{(N)} \rightarrow 0$, as $N \rightarrow \infty$.

We now define the storage-repair tradeoff region.

**Definition 2:** The storage-repair tradeoff region $\mathcal{R}_{K, n, d}$ of an IDSCER is the closure of the set of all achievable vectors $(\omega, \alpha, \beta)$.

It is not difficult to see that the relations between the entropies of the variables in the network are as follows.

$$H(X_{i,k}^{(N)}) = \sum_{i=1}^{K} NH(X_i), \quad (7)$$

$$H(W_i|X_{i,k}^{(N)}) = 0, \quad i = 1, \ldots, n, \quad (8)$$

$$H(S_d^k|W) = 0, \quad i, k = 1, \ldots, n, \quad i \neq k \quad (9)$$

$$H(W_k^{(N)}|S_d^k, i : E_i \in E_d^k) = 0, \quad k = 1, \ldots, n \quad (10)$$

$$H(\tau(D_i^{(N)}))|W = 0, \quad i = 1, \ldots, |D| \quad (11)$$

$$H(W_i) \leq \alpha, \quad i \in \{1, \ldots, n\} \quad (12)$$

$$H(S_d^k) \leq \beta, \quad i, k \in \{1, \ldots, n\}, \quad i \neq k \quad (13)$$

where (7) represents the independence of sources, (8) represents the encoding constraints for storage, (9) represents the encoding constraints for generating repair messages at each encoder, (10) represents the repair constraints, (11) represents the decoding/ reconstruction constraints, (12) represents the storage capacity constraints, and (13) represents the repair bandwidth constraints.

In the next section, we will fully characterize the tradeoff regions for the first non-trivial IDSCER case with two sources and three encoders.

### III. 2-SOURCE 3-ENCODER IDSC WITH EXACT REPAIR

In this section, we fully characterize the storage-repair tradeoff region of the 2-source 3-encoder IDSCER with $(K, n, d) = (2, 3, 2)$, which is the first non-trivial tuple. For ease of analysis, we will use $X, Y$ to represent the two source variables, and $X_1, X_2, (Y_1, Y_2)$ to indicate the first and second bit of $X$ (or $Y$) respectively, when $H(X) = 2$ (or $H(Y) = 2$).

According to [17], there are 33 non-isomorphic IDSC instances in total. For each of them, we will characterize its exact tradeoff region. Note that, though the reconstruction of the instances are different, the tradeoff regions can be exactly the same or are permutations of each other, since the key constraints determining the characterization can be the same. We will classify the 33 instances into 7 groups and select one representative for each group, since the proofs of the others follow a similar argument. We will use the indices in [25] for the instances proved in this section.

For each group of the network instances, we will use a distinct theorem to present their storage-repair tradeoff regions. Fig.2 may be helpful in understanding the converse proofs by illustrating the repair processes and some cuts. The topologies of the representatives and their achievability proofs can be found in Table 1, where the constructions of codes to achieve each extreme ray in the tradeoff regions are provided. Hence, in the following, we will only show the converse proof following each theorem, where we assume that a point $r$ is selected in the tradeoff region and there exists an $(N, r)$ block code to achieve it.

**Theorem 1:** The tradeoff region of cases 1, 4, 6, 7, 17 of the $(2, 3, 2)$ IDSCER problems contains all rate tuples characterized by the following inequalities:

$$\alpha \geq H(X) + H(Y) \quad (14)$$

$$2\beta \geq H(X) + H(Y) \quad (15)$$
### TABLE 1

<table>
<thead>
<tr>
<th>Cases</th>
<th>Topology</th>
<th>Extreme rays $(\alpha, \beta, H(X), H(Y))$</th>
<th>Achieving Codes $(W_1, W_2, W_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 17</td>
<td>$XY$</td>
<td>$2 1 2 0$</td>
<td>$(X_1X_2, X_1X_2, X_1X_2)$</td>
</tr>
<tr>
<td>Equivalent:</td>
<td>$(1, 4, 6, 7)$</td>
<td>$2 1 0 2$</td>
<td>$(Y_1Y_2, Y_1Y_2, Y_1Y_2)$</td>
</tr>
</tbody>
</table>

| case 25 | $XY$ | $2 1 3 0$ | $(X_1X_2, X_2X_3, X_3X_1)$ |
| Equivalent: | $(2, 3, 15, 16, 20, 23)$ | $2 1 0 3$ | $(Y_1Y_2, Y_2Y_3, Y_3Y_1)$ |
| | | $1 1 2 0$ | $(X_1, X_2, X_1 + X_2)$ |
| | | $1 1 0 2$ | $(Y_1, Y_2, Y_1 + Y_2)$ |

| case 12 | $XY$ | $1 1 0 2$ | $(Y_1, Y_2, Y_1 + Y_2)$ |
| Equivalent: | $(8, 11, 13, 14, 19, 31)$ | $2 1 0 3$ | $(Y_1Y_2, Y_2Y_3, Y_3Y_1)$ |
| | | $2 1 2 0$ | $(X_1X_2, X_1Y, X_2Y)$ |
| | | $2 1 2 1$ | $(X_1X_2, X_1Y, X_2Y)$ |
| | | $1 1 1 1$ | $(X, Y, X + Y)$ |

| case 9 | $XY$ | $2 1 0 2$ | $(Y_1Y_2, Y_1Y_2, Y_1Y_2)$ |
| Equivalent: | $(26)$ | $2 1 2 1$ | $(X_1X_2, X_1Y, X_2Y)$ |
| | | $2 1 2 0$ | $(X_1X_2, X_1Y, X_2Y)$ |

| case 30 | $XY$ | $2 1 0 2$ | $(XY_1, Y_1Y_2, XY_2)$ |
| Equivalent: | $(10)$ | $2 1 1 2$ | $(XY_1, Y_1Y_2, XY_2)$ |
| | | $2 1 2 0$ | $(X_1X_2, X_1Y, X_2Y)$ |
| | | $2 1 2 1$ | $(X_1X_2, X_1Y, X_2Y)$ |
| | | $1 1 1 1$ | $(X, Y, X + Y)$ |

| case 29 | $XY$ | $2 1 0 3$ | $(Y_1Y_2, Y_2Y_3, Y_3Y_1)$ |
| Equivalent: | $(18, 22, 27, 28)$ | $2 1 1 2$ | $(XY_1, XY_2, Y_1Y_2)$ |
| | | $2 1 2 0$ | $(X_1X_2, X_1X_2, X_1X_2)$ |
| | | $1 1 0 2$ | $(Y_1, Y_2, Y_1 + Y_2)$ |

| case 33 | $XY$ | $2 1 0 3$ | $(Y_1Y_2, Y_2Y_3, Y_3Y_1)$ |
| Equivalent: | $(21, 24, 32)$ | $2 1 2 0$ | $(X_1X_2, X_1X_2, X_1X_2)$ |
| | | $1 1 0 2$ | $(Y_1, Y_2, Y_1 + Y_2)$ |

**Proof:** We will show the converse proof for case 17 here. For (14), we have
\[
N \alpha \geq H(W_3) \geq H(X^{(N)}Y^{(N)}) = N(H(X) + H(Y)) \quad (16)
\]
which follows from the capacity constraint on $E_3$, the decoding constraint $H(XY|W_3) = 0$ at $D_4$, and the source independence.
Then, we have some cuts in the repair process used in the converse proof. From the second inequality in (16), the repair constraints of (17) follow from the repair bandwidth constraints and the fundamental property of joint entropy, and the fact that $W_1$ can be repaired by $E_1$, and the source independence.

Then, from (21), we have

\[ 2N\alpha \geq H(W_1) + H(W_2) \]
\[ \geq H(W_1W_2) \]
\[ = H(W_1W_2W_3) \]
\[ \geq H(X^{(N)}Y^{(N)}) \]
\[ = N(H(X) + H(Y)) \]

which follows from the storage capacity constraints, the fundamental property of joint entropy, and the fact that $W_3$ can be repaired by $W_1$ and $W_2$ together, i.e., $H(W_3|W_1W_2) = 0$. Then, the rest follows from the decoding constraints at $D_1$, and the source independence.

For (18), we have

\[ 2N\alpha \geq H(W_1) + H(W_2) \]
\[ \geq H(W_1W_2) \]
\[ = H(W_1W_2W_3) \]
\[ \geq H(X^{(N)}Y^{(N)}) \]
\[ = N(H(X) + H(Y)) \]

which follows from the storage capacity constraints, the fundamental property of joint entropy, and the fact that $W_3$ can be repaired by $W_1$ and $W_2$ together, i.e., $H(W_3|W_1W_2) = 0$. Then, the rest follows from (21).

For (20), we have

\[ N(\alpha + \beta) \geq H(W_3) + H(S_1^1) \]
\[ \geq H(S_3^1S_2^1S_1^1) \]
\[ \geq H(W_1S_3^1S_2^1) \]
\[ \geq H(W_1W_2). \]  

Here, (23) follows from the storage and repair bandwidth constraints, repair coding constraints at $E_3$, fundamental property of joint entropy, and the repair processes of $W_1$ and $W_2$, i.e., $H(W_1|S_3^1S_2^1) = 0$ and $H(W_2|S_3^1S_2^1) = 0$, and the rest follows from (21). □

Note that, the problem considered in [24] actually is the case 16 in [17] with exact repair. Then, from Theorem 2, we will have the following corollary.

**Corollary 1.** In [24], the inner bound provided in Theorem 6 is the exact storage-repair tradeoff region of the problem considered therein.

**Theorem 3.** The tradeoff region of cases 8, 11, 12, 13, 14, 19, 31 of the (2, 3, 2) IDSCER problems contains all rate tuples characterized by (18), (19), (20), and the following inequalities:

\[ \alpha \geq H(X) \]  
\[ 2\beta \geq H(X) \]

**Proof:** We will show the converse proof for the new inequalities (24)–(25) for case 12 here, since the others can be proved similarly as (19)–(20).

(24) can be derived by the storage capacity constraint on $E_1$ and then the decoding constraint at $D_1$. (25) can be derived by considering repairing $E_1$, that is

\[ 2N\beta \geq H(S_1^1) + H(S_2^1) \]
\[ \geq H(W_1) \]
\[ \geq H(U_1) \]
\[ \geq NH(X). \]  

**Theorem 4.** The tradeoff region of cases 9 of the (2, 3, 2) IDSCER problems contains all rate tuples characterized by (24), (25), and the following inequalities:

\[ 2\alpha \geq H(X) + 2H(Y) \]
\[ 4\beta \geq H(X) + 2H(Y) \]

**Proof:** Similarly, we will only show the proofs for (27) and (28).

For (27), we have

\[ 2N\alpha \geq H(W_2) + H(W_3) \]
\[ = H(W_2Y^{(N)}) + H(W_3Y^{(N)}) \]
\[ \geq H(W_2W_3Y^{(N)}) + H(Y^{(N)}) \]
\[ \geq H(W_1W_2W_3Y^{(N)}) + H(Y^{(N)}) \]
\[ \geq NH(XY) + NH(Y) \]
\[ = NH(X) + 2NH(Y), \]

which follows from the storage capacity constraints on $E_2$, $E_3$, decoding abilities of $D_2$, $D_3$, non-negativity of conditional mutual information, the fact that $W_1$ can be repaired...
by $W_2$, $W_3$, decoding abilities of all decoders, and then the source independence.

For (28), we consider cut 2 and cut 3 in Fig. 2. Then, we will have

$$4N \beta \geq H(S_2^1) + H(S_1^2) + H(S_2^3) + H(S_3^2) \geq H(W_2) + H(W_3).$$

Then, the rest follows from (29). Here, (30) simply follows from the repair bandwidth constraints and repair processes of $W_2$, $W_3$. □

**Theorem 5:** The tradeoff region of cases 10, 30 of the (2, 3, 2) IDSCER problems contains all rate tuples characterized by (24), (25), (19), (20), and the following inequalities:

$$\alpha \geq H(Y)$$

$$2\beta \geq H(Y)$$

**Proof:** We will show the proofs for (31) and (32) for case 30. (31) can be derived by considering the storage capacity constraint on $E_2$ and the decoding ability at $D_3$. Similarly, (32) can be derived by considering the repair of $W_2$ and then the decoding ability at $D_3$. □

**Theorem 6:** The tradeoff region of cases 18, 22, 27, 28, 29 of the (2, 3, 2) IDSCER problems contains all rate tuples characterized by (27), (28), (19), and (20).

**Proof:** Though we select case 29 for the achievability proof in Table 1, the converse proof is omitted here, since it can be derived from similar arguments above, with a permutation of variables $X$, $Y$. □

**Theorem 7:** The tradeoff region of cases 21, 24, 32, 33 of the (2, 3, 2) IDSCER problems contains all rate tuples characterized by the following inequalities:

$$2\alpha \geq 2H(X) + H(Y)$$

$$6\beta \geq 3H(X) + 2H(Y)$$

$$2\alpha + 2\beta \geq 3H(X) + 2H(Y)$$

**Proof:** We will show the proofs for case 33. (33) can be derived similarly as (27) with permuting $X$, $Y$.

For (34), we will consider cut 1, cut 2, and cut 3 in Fig. 2. Then, we have

$$12N \beta \geq 2(H(S_1^2) + H(S_1^3) + H(S_2^3) + H(S_3^2))$$

$$\geq 2(H(S_1^2) + H(S_1^3)) + H(S_3^2) + H(S_2^3))$$

$$\geq 2(H(S_1^2) + H(S_2^3)) + H(S_3^2) + H(S_2^3)$$

This follows from the fact that all transmitted messages for repair are functions of corresponding stored (coded) messages at the encoders.

Next, we apply submodularity property (or equivalently, the non-negativity of conditional mutual information) to the three pairs of the terms in (37). Then, we have

$$12N \beta \geq H(W_1W_3X^{(N)}) + H(W_1W_2X^{(N)}) + H(W_2W_3X^{(N)})$$

$$\geq H(S_1^2) + H(S_3^2) + H(S_2^3) + H(S_3^2) + H(S_2^3).$$

From the decoding abilities, we can see that

$$H(W_1W_2X^{(N)}) + H(W_1W_3X^{(N)}) + H(W_2W_3X^{(N)})$$

$$\geq 3H(X^{(N)}Y^{(N)}) = 3NH(X) + 3NH(Y).$$

For the other three terms in (38), we will apply the submodularity property (or equivalently, the non-negativity of conditional mutual information) step to step and the decoding abilities. Then, we have

$$H(S_1^2S_3^1S_2^3X^{(N)}) + H(S_3^2S_2^3X^{(N)}) + H(S_2^3S_3^1X^{(N)})$$

$$\geq H(S_1^2S_3^1S_2^3S_3^1X^{(N)}) + H(X) + H(S_2^3S_3^1X^{(N)})$$

$$\geq H(S_1^2S_3^1S_2^3S_3^1X^{(N)}) + 2H(X^{(N)})$$

$$\geq 3NH(X) + NH(Y).$$

Then, from (38), (39), and (43), we have

$$12\beta \geq 6H(X) + 4H(Y),$$

which is equivalent to (34).

For (35), we have

$$6N \alpha + 6N \beta$$

$$\geq 2H(W_1) + H(W_2) + H(W_3) + H(S_1^2) + H(S_1^3) + H(S_3^2) + H(S_2^3)$$

$$\geq 2H(W_1X^{(N)}S_1^1S_2^3) + 2H(W_2X^{(N)}S_2^1S_3^1)$$

$$+ 2H(W_3X^{(N)}S_3^1S_1^3) + H(S_1^2S_1^3)$$

$$\geq H(W_1X^{(N)}S_1^1S_2^3) + H(W_2X^{(N)}S_2^1S_3^1)$$

$$+ H(W_3X^{(N)}S_3^1S_1^3) + H(S_1^2S_1^3W_1X^{(N)})$$

$$+ H(S_2^3S_3^1S_2^1W_2X^{(N)}) + H(S_3^2S_3^1S_3^1W_3X^{(N)})$$

where (46) follows from the encoding constraints for generating repair messages at each encoder and the fundamental property of joint entropies, while (47) follows from (36).

Note that, by applying the submodularity property (or equivalently, the non-negativity of conditional mutual information) twice, we will have

$$H(W_1W_3X^{(N)}S_1^1S_2^3) + H(S_1^2S_1^3W_1X^{(N)})$$

$$\geq 1/2H(W_1W_3X^{(N)}S_1^1S_2^3) + 1/2H(W_1W_3S_1^1S_2^3S_3^1W_1X^{(N)})$$

$$+ 1/2H(X^{(N)}S_1^1) + 1/2H(X^{(N)}S_2^3)$$

$$\geq H(X^{(N)}S_1^1) + 1/2H(X^{(N)}S_2^3) + 1/2H(X^{(N)}S_3^1).$$

(48)
Similarly, we can have
\[
\frac{1}{2} H(W_1 X^{(N)} S_1^3 S_3^1) + \frac{1}{2} H(W_3 X^{(N)} S_1^1 S_3^2) + H(S_1^1 S_3^2 W_2 X^{(N)}) \\
\geq H(X^{(N)} Y^{(N)}) + \frac{1}{2} H(X^{(N)} S_1^1) + \frac{1}{2} H(X^{(N)} S_3^2),
\]
and
\[
\frac{1}{2} H(W_1 X^{(N)} S_1^1 S_3^2) + \frac{1}{2} H(W_2 X^{(N)} S_1^3 S_3^1) + H(S_1^1 S_3^2 W_3 X^{(N)}) \\
\geq H(X^{(N)} Y^{(N)}) + \frac{1}{2} H(X^{(N)} S_1^1) + \frac{1}{2} H(X^{(N)} S_3^2).
\]

Further, by applying submodularity (or equivalently, the non-negativity of conditional mutual information) twice, we will have
\[
H(W_1 X^{(N)} S_1^3 S_3^1) + \frac{1}{2} H(X^{(N)} S_1^1 S_3^2) + \frac{1}{2} H(X^{(N)} S_3^2) \\
\geq \frac{1}{2} H(W_1 X^{(N)} S_1^2 S_3^2) + \frac{1}{2} H(W_1 X^{(N)} S_1^3 S_3^1) + \frac{1}{2} H(X^{(N)} S_1^1 S_3^2) \\
+ H(X^{(N)}) \\
\geq \frac{1}{2} H(W_1 W_2) + \frac{1}{2} H(W_1 W_3) + H(X^{(N)}) \\
\geq H(X^{(N)} Y^{(N)}) + H(X^{(N)}) \\
\geq 2H(X^{(N)}) + H(Y^{(N)}).
\]

Similarly, we will have
\[
H(W_2 X^{(N)} S_2^1 S_3^2) + \frac{1}{2} H(X^{(N)} S_1^3) + \frac{1}{2} H(X^{(N)} S_3^1) \\
\geq 2H(X^{(N)}) + H(Y_N),
\]
and
\[
H(W_3 X^{(N)} S_3^1 S_3^2) + \frac{1}{2} H(X^{(N)} S_1^1) + \frac{1}{2} H(X^{(N)} S_2^1) \\
\geq 2H(X^{(N)}) + H(Y^{(N)}).
\]

Applying (49)–(56) to (47), we will have
\[
6N\alpha + 6N\beta \geq 9H(X^{(N)}) + 6H(X^{(N)}) \\
= 9NH(X) + 6NH(Y),
\]
that is,
\[
2\alpha + 2\beta \geq 3H(X) + 2H(Y).
\]

From the achievability proofs in Table 1, we have the following corollary.

**Corollary 2:** Binary linear codes suffice for the 2-source 3-encoder IDSC problems with exact repair.

We see that though the network topologies are different for different instances, the tradeoff regions can be the same. This may be because the topology differences are not as crucial to constrain the region as the exact repair constraints. However, the tradeoff regions are different in general. For instance, we can let \( H(X) = H(Y) = 1 \) to compare the region \( \mathcal{R}_2 \) in Theorem 2 and the region \( \mathcal{R}_7 \) in Theorem 7, as shown in Fig. 3. It is shown that \( \mathcal{R}_7 \) is included in the region \( \mathcal{R}_2 \). By solving many similar instances and analyze the effects of the topologies, one may find the structural properties that are key for characterizing the tradeoff regions of the IDSCER instances.

**IV. CONCLUSION**

This paper investigates the independent distributed source coding with exact repair constraints, which modifies the reconstruction part of the traditional distributed storage exact repair problem to be a multi-source multicast model. As the first step towards solving the general case, all 33 non-isomorphic IDSCER instances with two sources and three encoders are solved and binary codes are shown to be optimal.
As an interesting direction for future research, one may try to solve more instances with relatively small sizes and then find the structural properties that are crucial in characterizing the tradeoff regions of the general IDSCER instances.

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REFERENCES