



香港城市大學
City University of Hong Kong

專業 創新 胸懷全球
Professional · Creative
For The World

CityU Scholars

Study of a homoclinic canard explosion from a degenerate center

Qin, Bo-Wei; Chung, Kwok-Wai; Algaba, Antonio; Rodríguez-Luis, Alejandro J.

Published in:

Applied Mathematics Letters

Published: 01/10/2022

Document Version:

Final Published version, also known as Publisher's PDF, Publisher's Final version or Version of Record

License:

CC BY-NC-ND

Publication record in CityU Scholars:

[Go to record](#)

Published version (DOI):

[10.1016/j.aml.2022.108203](https://doi.org/10.1016/j.aml.2022.108203)

Publication details:

Qin, B.-W., Chung, K.-W., Algaba, A., & Rodríguez-Luis, A. J. (2022). Study of a homoclinic canard explosion from a degenerate center. *Applied Mathematics Letters*, 132, Article 108203.
<https://doi.org/10.1016/j.aml.2022.108203>

Citing this paper

Please note that where the full-text provided on CityU Scholars is the Post-print version (also known as Accepted Author Manuscript, Peer-reviewed or Author Final version), it may differ from the Final Published version. When citing, ensure that you check and use the publisher's definitive version for pagination and other details.

General rights

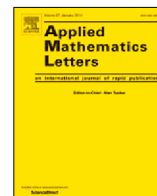
Copyright for the publications made accessible via the CityU Scholars portal is retained by the author(s) and/or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights. Users may not further distribute the material or use it for any profit-making activity or commercial gain.

Publisher permission

Permission for previously published items are in accordance with publisher's copyright policies sourced from the SHERPA RoMEO database. Links to full text versions (either Published or Post-print) are only available if corresponding publishers allow open access.

Take down policy

Contact lbscholars@cityu.edu.hk if you believe that this document breaches copyright and provide us with details. We will remove access to the work immediately and investigate your claim.



Study of a homoclinic canard explosion from a degenerate center



Bo-Wei Qin^{a,1}, Kwok-Wai Chung^{b,1}, Antonio Algaba^{c,1},
Alejandro J. Rodríguez-Luis^{d,*,1}

^a Research Institute of Intelligent Complex Systems, Fudan University, Shanghai 200433, China

^b Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong

^c Departamento de Ciencias Integradas, Centro de Estudios Avanzados en Física, Matemática y Computación, Universidad de Huelva, 21071 Huelva, Spain

^d Departamento de Matemática Aplicada II, E.T.S. Ingeniería, Universidad de Sevilla, 41092 Sevilla, Spain

ARTICLE INFO

Article history:

Received 16 February 2022

Received in revised form 15 May 2022

Accepted 15 May 2022

Available online 23 May 2022

Keywords:

Canard

Singularly perturbed system

Asymptotic expansion

Degenerate center

Homoclinic connection

ABSTRACT

Canard explosion is an appealing event occurring in singularly perturbed systems. In this phenomenon, upon variation of a parameter within an exponentially small range, the amplitude of a small limit cycle increases abruptly. In this letter we analyze the canard explosion in a limit cycle related to a degenerate center (with zero Jacobian matrix). We provide a second-order approximation of the critical value of the parameter for which the canard explosion occurs. Numerical results are compared with the analytical predictions and excellent agreements are found. As in this problem the canard explosion ends in a homoclinic connection, a very good approximation for the homoclinic curve in the parameter plane is also obtained.

© 2022 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

1. Introduction

In dynamical systems with multiple time scales, canard explosion alludes to a sudden transition between a small amplitude limit cycle (canard cycle) and a relaxation oscillation (see, for instance, [1–8] and references therein). An important problem is the determination of the critical value of the parameter for which the canard explosion occurs. One efficient way for its computation is the asymptotic expansion method [9–11].

When the canard cycle emerges from a Hopf bifurcation, the problem is well studied and understood under generic conditions (in this case there is an algebraic solution) [1,5,9]. However, when some non-generic conditions occur, there is no explicit expression for the first integral and the computations become harder [10]. The situation is more complicated when the canard cycle appears from a nilpotent center [12]. In this case, the period of the emerging orbit becomes unbounded [2]. A new situation corresponds to the

* Corresponding author.

E-mail addresses: boweiqin@fudan.edu.cn (B.-W. Qin), makchung@cityu.edu.hk (K.-W. Chung), algaba@uhu.es (A. Algaba), ajrluis@us.es (A.J. Rodríguez-Luis).

¹ All the authors contributed equally to this work.

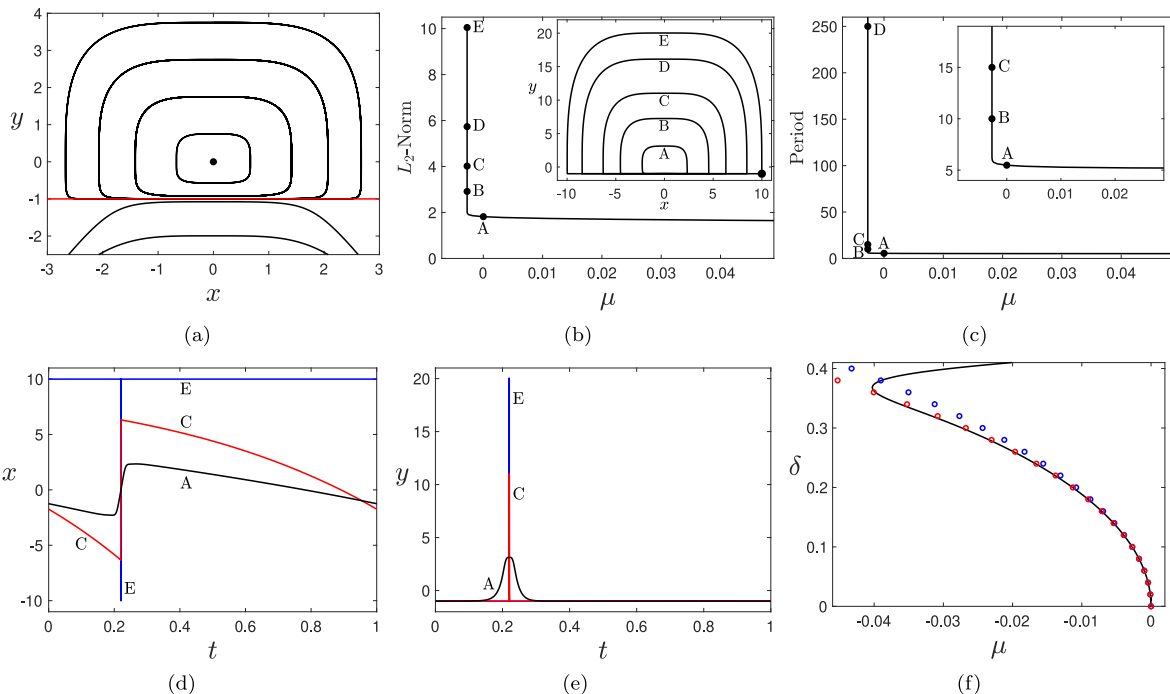


Fig. 1. (a) Phase portrait of the unperturbed system (1). The straight line $y = -1$ (in red) corresponds to a heteroclinic connection at infinity. For $\delta = 0.1$ and $a = 1$ in system (2): (b) bifurcation diagram L_2 -Norm versus μ near the canard explosion which occurs when $\mu \approx -2.73441 \cdot 10^{-3}$. Four limit cycles (labeled A-D) illustrate the very rapid growth of the amplitude which is limited by a homoclinic connection (E) to the equilibrium marked with a bullet; (c) bifurcation diagram Period versus μ . In this case, the period tends to infinity since a homoclinic orbit, E, exists; (d) and (e) temporal profiles of orbits A (black), C (red) and E (blue), where period is normalized to 1. (f) Curve of the homoclinic connections in the (μ, δ) -parameter plane, when $a = 1$. The circles correspond to the analytical approximations (first-order in blue, second-order in red) provided in Theorem 1.

presence of a degenerate center (with zero Jacobian matrix). The aim of this letter is to provide a first example in the literature where the unperturbed system has a degenerate center. Specifically, we will consider

$$\dot{x} = y^3, \quad \dot{y} = -x^3(1 + y). \tag{1}$$

This system has a degenerate center at the origin (surrounded by a continuum of periodic orbits which rotate clockwise) and the straight line $y = -1$ is a heteroclinic connection at infinity (see Fig. 1(a)).

We choose a simple perturbation of system (1) to analyze the canard explosion, namely

$$\dot{x} = y^3 + \delta x := P(x, y, \delta), \quad \dot{y} = -x^3(1 + y) + \delta\mu + \delta^2 ax := Q(x, y, \delta, \mu). \tag{2}$$

Note that the scaling

$$X = \epsilon^{-1/4}x, \quad Y = y, \quad \tau = \epsilon^{3/4}t, \tag{3}$$

where $\delta = \epsilon^{3/4}$, $0 < \delta \ll 1$, transforms system (2) into the singularly perturbed system

$$\epsilon \frac{dX}{d\tau} = Y^3 + \epsilon X, \quad \frac{dY}{d\tau} = -X^3(1 + Y) + \mu + \epsilon aX. \tag{4}$$

The main result of this work is the following theorem that provides an approximation to the critical parameter value for which the canard explosion occurs in system (2) (and also in the singularly perturbed system (4)).

Theorem 1. *A canard explosion occurs in system (2) when $\mu = \mu_2\delta^2 + \mu_4\delta^4 + \mathcal{O}(\delta^6)$, where μ_2 and μ_4 are given in Eqs. (11) and (25), respectively. In system (4) it occurs for $\mu = \mu_2\epsilon^{3/2} + \mu_4\epsilon^3 + \mathcal{O}(\epsilon^{9/2})$.*

The rest of the letter is organized as follows. Section 2 is devoted to prove Theorem 1. Numerical results, which illustrate very good agreements with the theoretical predictions, are shown in Section 3. Finally, some conclusions are included.

2. Asymptotic expansion

The goal of this section is to demonstrate Theorem 1. First, eliminating the time variable, we obtain $Q(x, y, \delta, \mu) - y'(x)P(x, y, \delta) = 0$, where $' := d/dx$. We look for a solution in the following form

$$y(x) = \sum_{i=0}^{\infty} \delta^i y_i(x), \quad \mu = \sum_{i=0}^{\infty} \delta^i \mu_i. \tag{5}$$

Then, for the zero-order solution, we have $y_0 = -1$, which is actually the critical manifold corresponding to the canard explosion.

In each order, we can obtain the corresponding linear equation as follows

$$y'_i(x) - x^3 y_i(x) + \mu_{i-1} = R_i(x), \tag{6}$$

of which $R_i(x)$ comprises all terms determined in preceding orders and $u(x) = e^{-x^4/4}$ is an integrating factor satisfying $u' = -ux^3$. Then, (6) can be rewritten as

$$(uy_i)' + u\mu_{i-1} = uR_i. \tag{7}$$

Thus, looking for solutions satisfying $\lim_{x \rightarrow \pm\infty} \frac{y_i(x)}{e^{|x|}} = 0$, the values of μ_{i-1} and $y_i(x)$ are uniquely found as

$$\mu_{i-1} = \frac{\int_{-\infty}^{\infty} u(x)R_i(x)dx}{\int_{-\infty}^{\infty} u(x)dx}, \quad y_i(x) = \frac{1}{u(x)} \int_{-\infty}^x u(s) [R_i(s) - \mu_{i-1}] ds. \tag{8}$$

Before finding the exact value of μ_{i-1} , we first define the following improper integral

$$I_n = \int_{-\infty}^{\infty} x^n u dx, \quad n \in \mathbb{N}, \tag{9}$$

that satisfies $I_n = [1 + (-1)^n] 2^{\frac{n-3}{2}} \Gamma(\frac{n+1}{4})$. Note that, $I_n = 0$ if n is an odd number.

From the first-order equation, $y'_1 - x^3 y_1 + \mu_0 = 0$, we obtain $\mu_0 = 0$ and $y_1(x) = 0$.

Considering the second-order equation, $y'_2 - x^3 y_2 + \mu_1 = -ax$, we deduce that $\mu_1 = 0$ and $(uy_2)' = -aux$. In later calculations it will be useful to define $T_{m,n} = \int_{-\infty}^{\infty} x^m uy_n dx$ for $m \geq 0$ and $n \geq 2$. For $n = 2$,

$$T_{m,2} = \frac{1}{m+1} \int_{-\infty}^{\infty} uy_2 dx^{m+1} = -\frac{1}{m+1} \int_{-\infty}^{\infty} x^{m+1} (-aux) dx \Rightarrow T_{m,2} = \frac{a}{m+1} I_{m+2}. \tag{10}$$

Therefore, $T_{m,2} = 0$ for odd m .

The analysis of the third-order equation allows to determine the first non-zero μ_i coefficient:

$$\begin{aligned} y'_3(x) - x^3 y_3(x) + \mu_2 = xy'_2 &\Rightarrow (uy_3)' = u(xy'_2 - \mu_2) \Rightarrow \mu_2 I_0 = \int_{-\infty}^{\infty} xu(x^3 y_2 - ax) dx \\ \Rightarrow \mu_2 = \frac{(T_{4,2} - aI_2)}{I_0} &= -a \frac{2\sqrt{2}}{5\pi} \left[\Gamma\left(\frac{3}{4}\right) \right]^2 \approx -0.2703912958a. \end{aligned} \tag{11}$$

Now, for $n = 3$,

$$\begin{aligned}
 T_{m,3} &= -\frac{1}{m+1} \int_{-\infty}^{\infty} x^{m+1} (uy_3)' dx = -\frac{1}{m+1} \int_{-\infty}^{\infty} x^{m+1} u [x(x^3y_2 - ax) - \mu_2] dx \\
 &= \frac{1}{m+1} (aI_{m+3} + \mu_2 I_{m+1} - T_{m+5,2}) = \frac{1}{m+1} \left(aI_{m+3} + \mu_2 I_{m+1} - \frac{a}{m+6} I_{m+7} \right).
 \end{aligned}
 \tag{12}$$

Considering the fourth-order equation, $y_4'(x) - x^3y_4(x) + \mu_3 = xy_3' + 3y_2y_2'$, we obtain $\mu_3 = 0$ and $(uy_4)' = u(xy_3' + 3y_2y_2')$.

Now we define $S_m = \int_{-\infty}^{\infty} x^m uy_2^2 dx$ with $m \geq 0$ for convenience. Then,

$$\begin{aligned}
 T_{m,4} &= -\frac{1}{m+1} \int_{-\infty}^{\infty} x^{m+1} (uy_4)' dx \\
 &= -\frac{1}{m+1} \int_{-\infty}^{\infty} x^{m+1} u [x(x^3y_3 - \mu_2 + x(x^3y_2 - ax)) + 3y_2(x^3y_2 - ax)] dx \\
 &= \frac{1}{m+1} (aI_{m+4} + \mu_2 I_{m+2} + 3aT_{m+2,2} - T_{m+6,2} - T_{m+5,3} - 3S_{m+4}) \\
 &= \frac{1}{m+1} \left[\frac{a}{(m+6)(m+11)} I_{m+12} - \frac{(2m+13)a}{(m+6)(m+7)} I_{m+8} - \frac{\mu_2}{m+6} I_{m+6} \right. \\
 &\quad \left. + \left(1 + \frac{3a}{m+3} \right) aI_{m+4} + \mu_2 I_{m+2} - 3S_{m+4} \right].
 \end{aligned}
 \tag{13}$$

The analysis of the fifth-order equation, after some laborious calculations, leads to the value of μ_4 .

$$\begin{aligned}
 y_5'(x) - x^3y_5(x) + \mu_4 = 3y_2y_3' + 3y_3y_2' + xy_4' &\Rightarrow (uy_5)' = u(3y_2y_3' + 3y_3y_2' + xy_4' - \mu_4) \\
 \Rightarrow \mu_4 I_0 = 3 \int_{-\infty}^{\infty} u d(y_2y_3) + \int_{-\infty}^{\infty} xu dy_4 &= 3 \int_{-\infty}^{\infty} x^3 uy_2y_3 dx + \int_{-\infty}^{\infty} (x^4 - 1) uy_4 dx = 3\alpha + \beta.
 \end{aligned}
 \tag{14}$$

Then, using (10), (11) and (12)–(14), the integrals α and β can be computed as

$$\begin{aligned}
 \alpha &= \int_{-\infty}^{\infty} x^3 uy_2y_3 dx = \int_{-\infty}^{\infty} (uy_2)(uy_3) d\left(\frac{1}{u}\right) = \int_{-\infty}^{\infty} u [axy_3 - y_2(x(x^3y_2 - ax) - \mu_2)] dx \\
 &= aT_{1,3} + aT_{2,2} + \mu_2 T_{0,2} - S_4 \Rightarrow \alpha = \frac{3a\mu_2}{2} I_2 + \frac{5a^2}{6} I_4 - \frac{a^2}{14} I_8 - S_4, \\
 \beta &= \int_{-\infty}^{\infty} (x^4 - 1) uy_4 dx = T_{4,4} - T_{0,4} = -\mu_2 I_2 - (1+a) aI_4 + \frac{11\mu_2}{30} I_6 \\
 &\quad + \frac{(107+18a)a}{210} I_8 - \frac{\mu_2}{50} I_{10} - \frac{4a}{75} I_{12} + \frac{a}{750} I_{16} + 3S_4 - \frac{3}{5} S_8.
 \end{aligned}
 \tag{15}$$

The following proposition is useful in finding S_4 and S_8 .

Proposition 1. For $m \geq 0$,

$$S_m = -\frac{1}{m+1} S_{m+4} + \frac{2a^2}{(m+1)(m+3)} I_{m+4}.
 \tag{16}$$

Proof. From the definition of S_m , we have

$$\begin{aligned}
 S_m &= -\int_{-\infty}^{\infty} x^{m-3} y_2^2 du = \int_{-\infty}^{\infty} u [(m-3)x^{m-4} y_2^2 + 2x^{m-3} y_2(x^3y_2 - ax)] dx \\
 &= (m-3)S_{m-4} + 2(S_m - aT_{m-2,2}) \Rightarrow S_m = -(m-3)S_{m-4} + \frac{2a^2}{m-1} I_m,
 \end{aligned}
 \tag{17}$$

from which (16) follows if m is replaced by $m + 4$ in (17). This completes the proof. \square

From (17), we have

$$S_8 = -5S_4 + \frac{2a^2}{7}I_8 \quad \text{and} \quad S_4 = -S_0 + \frac{2a^2}{3}I_4. \tag{18}$$

It follows from (16) that

$$S_0 = -\lim_{n \rightarrow \infty} \frac{(-1)^n S_{4n+4}}{\prod_{i=0}^n (4i+1)} + 2a^2 \sum_{n=0}^{\infty} \frac{(-1)^n I_{4n+4}}{(4n+3) \prod_{i=0}^n (4i+1)}. \tag{19}$$

The following proposition shows that the limit in (19) vanishes.

Proposition 2. $\lim_{n \rightarrow \infty} \frac{S_{4n+4}}{\prod_{i=0}^n (4i+1)} = 0.$

Proof. We first show that

$$x^2 [u(x)y_2(x)]^2 \leq a^2 [u(x)]^2. \tag{20}$$

We assume that $a < 0$ and define

$$f(x) = \frac{-au(x)}{x} - u(x)y_2(x), \quad x \in (-\infty, 0). \tag{21}$$

Differentiating (21) with respect to x , we have $f'(x) = a(x^4 + x^3 + 1)u(x)/x^2 < 0$, for all $x \in (-\infty, 0)$. Therefore, $f(x)$ is strictly decreasing. Moreover, $\lim_{x \rightarrow -\infty} f(x) = 0$. Hence, for $x \in (-\infty, 0)$, we have $f(x) < 0$, that is, $-u(x)y_2(x) < au(x)/x$. Furthermore, both sides are positive since $y_2(x) = -\frac{a}{u(x)} \int_{-\infty}^x su(s) ds$ and then $y_2(x) < 0$ for $x \in (-\infty, 0)$.

Therefore, we have proved (20) for $a < 0$ and $x \in (-\infty, 0)$. Taking into account that $y_2(-x) = y_2(x)$ and $u(-x) = u(x)$ for all $x \in (-\infty, \infty)$, the inequality (20) is verified for all $a \in \mathbb{R}$ and $x \in (-\infty, \infty)$.

Thus, using (20) we have $S_{4n+4} = \int_{-\infty}^{\infty} \frac{x^{4n+4}}{u(x)} [u(x)y_2(x)]^2 dx \leq a^2 \int_{-\infty}^{\infty} x^{4n+2} u dx = a^2 I_{4n+2}$. Since $I_{4n+2} = 4^n \sqrt{2} \Gamma(n + \frac{3}{4}) = \sqrt{2} \Gamma(\frac{3}{4}) \prod_{i=0}^{n-1} (4i+3)$, we have

$$0 \leq \lim_{n \rightarrow \infty} \frac{S_{4n+4}}{\prod_{i=0}^n (4i+1)} \leq \sqrt{2} a^2 \Gamma\left(\frac{3}{4}\right) \lim_{n \rightarrow \infty} \left[\frac{1}{4n+1} \prod_{i=0}^{n-1} \left(\frac{4i+3}{4i+1}\right) \right] = 0. \tag{22}$$

This completes the proof. \square

Furthermore,

$$I_{4n+4} = \frac{1}{\sqrt{2}} \Gamma\left(\frac{1}{4}\right) \prod_{i=0}^n (4i+1) \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)} = \frac{\sqrt{2}}{4} \left[\frac{\pi}{2} - \ln(1 + \sqrt{2}) \right]. \tag{23}$$

From (19), (22) and (23), we obtain

$$S_0 = \frac{a^2}{2} \Gamma\left(\frac{1}{4}\right) \left[\frac{\pi}{2} - \ln(1 + \sqrt{2}) \right]. \tag{24}$$

According to (14), $\mu_4 = (3\alpha + \beta) / I_0$. Thus, using (15), (18) and (24) we obtain

$$\begin{aligned} \mu_4 &= \left(\frac{16}{25} - 9a\right) \frac{2a}{5\pi^2} \left[\Gamma\left(\frac{3}{4}\right) \right]^4 + a^2 \left(2 - \frac{3\sqrt{2}\pi}{4} + \frac{3\sqrt{2}\ln(1 + \sqrt{2})}{2} \right) - \frac{38a}{525} \\ &\approx -a(0.01389179008 + 0.2849903285a). \end{aligned} \tag{25}$$

In this way we have proved the result stated in Theorem 1.

Table 1
Comparison between analytical predictions and numerical results for $a = 1$ and for $a = -1$.

δ	$a = 1$			$a = -1$		
	$\mu(\text{num.})$	$\mu(\text{1st})$	$\mu(\text{2nd})$	$\mu(\text{num.})$	$\mu(\text{1st})$	$\mu(\text{2nd})$
1E-03	-2.70393E-7	-2.703913E-7	-2.703916E-7	2.70392E-7	2.703913E-7	2.703910E-7
1E-02	-2.70421E-5	-2.703913E-5	-2.704212E-5	2.70364E-5	2.703913E-5	2.703642E-5
1E-01	-2.73441E-3	-2.703913E-3	-2.733801E-3	2.67717E-3	2.703913E-3	2.676803E-3
2E-01	-1.13361E-2	-1.081565E-2	-1.129386E-2	1.04046E-2	1.081565E-2	1.038189E-2
3E-01	-2.73186E-2	-2.433522E-2	-2.675616E-2	2.23815E-2	2.433522E-2	2.213932E-2

3. Numerical results

To validate the analytical approximations of the previous section we present some numerical results obtained with AUTO [13] for system (2).

In Figs. 1(b)-1(c), for $\delta = 0.1$ and $a = 1$, we show the bifurcation diagrams L_2 -Norm versus μ and Period versus μ , respectively, near the canard explosion. The sudden increase of amplitude and period occurs when $\mu \approx -2.73441 \cdot 10^{-3}$. To see the evolution of the limit cycles we draw several of them in Fig. 1(b). The smallest one (A) exists for $\mu = 0$ and three of them (B-D) are present along the explosion. Although it is usually the most frequent [1], in this system there are no relaxation oscillations but the explosion ends with a homoclinic orbit (E) which connects the equilibrium marked with a bullet, situated at $(x, y) \approx (9.99701, -0.99990)$ (the period tends to infinity since a homoclinic orbit exists).

The temporal profiles of orbits A, C and E (see Figs. 1(d)-1(e), where the period is normalized to 1) show that slow-fast motions are present: the slow motion appears when the orbit is close to $y = -1$ (where the unperturbed system (1) has a heteroclinic connection at infinity) and the fast motion in the semielliptical-like part. Indeed, the explosion occurs when the orbits are close enough to $y = -1$ (note that orbit A is close, but still outside the explosion zone).

In Fig. 1(f), for $a = 1$, we compare the curve of homoclinic connections (solid line) in the (μ, δ) -parameter plane with the analytical approximations for the canard explosion (circles; first-order in blue, second-order in red) stated in Theorem 1. We observe that the agreement of the second-order approximation is very good for δ values up to about 0.3.

Finally, in Table 1 for five values of δ , in the cases $a = 1$ and $a = -1$, we compare analytical predictions and numerical results. We observe that the first-order approximation only provides accurate results when δ is small enough. As expected, when δ is relatively large, the second-order approximation clearly improves the results.

Remark that, using scaling (3), all the numerical results can be easily translated to system (4). In this case, $\epsilon = 0.1^{4/3} \approx 0.0464$.

In summary, we are dealing with periodic orbits which experience a rapid growth in amplitude and period (in an extremely narrow range of the parameter μ) and have a slow-fast behavior (for δ or ϵ small enough). Due to its similarity with the situation that appears, for example, in the van der Pol system (see [11] and references therein), we call it “canard explosion” (with a certain abuse of language, since the standard conditions for a critical manifold with unstable and stable parts are not fulfilled in system (4), see [1]).

4. Conclusions

The principal goal of this letter is to study a system, as simple as possible, exhibiting a canard explosion related to a degenerate center (with zero Jacobian matrix). As far as we know, it is the first time in the literature that an asymptotic expansion is found for this kind of problem. Specifically, we consider an unperturbed system which has a heteroclinic connection at infinity. The perturbation of this curve allows to find the corresponding asymptotic expansions, namely we are able to find the exact values of the first

two terms of the critical value of the parameter μ . This analytical approximation agrees very well with the numerical results, even if the parameter δ is relatively large. Finally, it is worth noting that the canard explosion of the studied system ends in a homoclinic connection. In this way, the approximation obtained for the explosion is also valid for the corresponding curve of global connections.

Acknowledgments

This work has been partially supported by the *Ministerio de Economía y Competitividad*, Spain (MTM2017-87915-C2-1-P), by the *Ministerio de Ciencia, Innovación y Universidades*, Spain (PGC2018-096265-B-I00) and by the *Consejería de Economía, Innovación, Ciencia y Empleo de la Junta de Andalucía*, Spain (projects FQM-276, TIC-0130 and UHU-1260150). B.-W.Q. is also supported by the National Natural Science Foundation of China (No. 12001110).

References

- [1] M. Krupa, P. Szmolyan, Relaxation oscillation and canard explosion, *J. Differ. Equ.* 174 (2001) 312–368, <http://dx.doi.org/10.1006/jdeq.2000.3929>.
- [2] P. De Maesschalck, F. Dumortier, Canard solutions at non-generic turning points, *Trans. Amer. Math. Soc.* 358 (2006) 2291–2334, <http://dx.doi.org/10.1090/S0002-9947-05-03839-0>.
- [3] C. Kuehn, *Multiple Time Scale Dynamical Systems*, Springer, Berlin, 2015, <http://dx.doi.org/10.1007/978-3-319-12316-5>.
- [4] G.N. Gorelov, V.A. Sobolev, Duck-trajectories in a thermal explosion problem, *Appl. Math. Lett.* 5 (1992) 3–6, [http://dx.doi.org/10.1016/0893-9659\(92\)90002-Q](http://dx.doi.org/10.1016/0893-9659(92)90002-Q).
- [5] E. Freire, E. Gamero, A.J. Rodríguez-Luis, First-order approximation for canard periodic orbits in a van der Pol electronic oscillator, *Appl. Math. Lett.* 12 (1999) 73–78, [http://dx.doi.org/10.1016/S0893-9659\(98\)00152-9](http://dx.doi.org/10.1016/S0893-9659(98)00152-9).
- [6] E.K. Ersöz, M. Desroches, C.R. Mirasso, S. Rodrigues, Anticipation via canards in excitable systems, *Chaos* 29 (2019) 013111, <http://dx.doi.org/10.1063/1.5050018>.
- [7] K.H.M. Nyman, P. Ashwin, P.D. Ditlevsen, Bifurcation of critical sets and relaxation oscillations in singular fast-slow systems, *Nonlinearity* 33 (2020) 2853–2904, <http://dx.doi.org/10.1080/14689367.2019.1575337>.
- [8] E. Shchepakina, O. Korotkova, Canard explosion in chemical and optical systems, *Discrete Cont. Dyn. Syst.-B* 18 (2013) 495–512, <http://dx.doi.org/10.3934/dcdsb.2013.18.495>.
- [9] M. Brøns, K. Uldall Kristiansen, On the approximation of the canard explosion point in singularly perturbed systems without an explicit small parameter, *Dyn. Syst.* 33 (2018) 136–158, <http://dx.doi.org/10.1080/14689367.2017.1313390>.
- [10] B.W. Qin, K.W. Chung, A. Algaba, A.J. Rodríguez-Luis, Asymptotic expansions for a family of non-generic canards using parametric representation, *Appl. Math. Lett.* 106 (2020) 106355, <http://dx.doi.org/10.1016/j.aml.2020.106355>.
- [11] A. Algaba, K.W. Chung, B.W. Qin, A.J. Rodríguez-Luis, Analytical approximation of the canard explosion in a van der Pol system with the nonlinear time transformation method, *Physica D* 406 (2020) 132384, <http://dx.doi.org/10.1016/j.physd.2020.132384>.
- [12] B.W. Qin, K.W. Chung, A. Algaba, A.J. Rodríguez-Luis, Asymptotic expansions for a degenerate canard explosion, *Physica D* 418 (2021) 132841, <http://dx.doi.org/10.1016/j.physd.2020.132841>.
- [13] E.J. Doedel, A.R. Champneys, F. Dercole, T. Fairgrieve, Y. Kuznetsov, B. Oldeman, R. Paffenroth, B. Sandstede, X. Wang, C. Zhang, *AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations (with HomCont)*, Technical report, Concordia University, 2012.